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A postulate-driven study of logical argumentation

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ABSTRACT

Logical argumentation is a well-known approach to modeling non-monotonic reasoning with conflicting information. In this paper we provide a comprehensive postulate-based study of properties of logical argumentation frameworks and a full characterization of their semantics and inference relations. In this way we identify well-behaved formal argumentative models of drawing logically justified inferences from a given set of possibly conflicting defeasible, as well as strict assumptions. Given some desiderata in terms of rationality postulates, we consider the conditions that an argumentation framework should fulfill for the desiderata to hold. One purpose of this approach is to assist designers to "plug-in" pre-defined formalisms according to actual needs. To this end, we present a classification of argumentation frameworks relative to the types of attacks they implement. In turn, for each class we determine which desiderata are satisfied. Our study is highly abstract, supposing only a minimal set of requirements on the considered underlying deductive systems, and in this way covering a broad range of formalisms, including classical, intuitionistic and modal logics.

1. Introduction

Logical argumentation is a common AI-based method for making inferences in the presence of arguments and counter-arguments (see [8] for an overview). Its setting, called an *argumentation framework*, consists of two ingredients:

- *arguments*, which are pairs $\langle \Gamma, \psi \rangle$ of a set of formulas (the argument's support, Γ) and a formula (the argument's conclusion, ψ) in some propositional language, such that ψ follows from Γ according to some underlying logic, and
- *attacks*, which are instances of a binary relation on the set of arguments, relating arguments and counter-arguments.

Given such a framework, an *argumentation semantics* [40] determines what arguments can be mutually accepted, and so what conclusions can be drawn from this setting.

The nature of an argumentation framework thus depends on several factors. This includes, among others, the following elements:

- The underlying *languages and logics* (consequence relations), on top of which the arguments are specified: For instance, the works in [24,25,45,48] are based on classical logic as the underlying logic, while those in [4,13,15] consider ar-

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gumentation frameworks that are based on any Tarskian logics and arbitrary propositional languages (see Definition 1 below).

- The nature of the *attack relations*, that is: what triggers conflicts between arguments (see, e.g., [24,48,74] for some proposals of attack relations in different contexts).
- The *semantics* of the framework, namely: criteria for selecting *extensions* [19,40] of the argumentation framework, which are sets of arguments that can be collectively accepted (without, e.g., attacking each other).
- The induced *entailment relation* that indicates what conclusions can be drawn, based on the arguments at hand, the attacks among them, and the chosen semantics.

The fact that there are so many possibilities to define logical argumentation frameworks raises the question how to choose the most appropriate framework for specific needs. The purpose of this work is to put some order in this 'jungle' of argumentation frameworks and to provide some guidelines on how to construct robust frameworks for particular purposes. For this, we first specify some criteria for choosing among the candidate frameworks. A common way to do so is by checking the satisfiability of *rationality postulates*, that is, to consider formal properties that the intended framework should satisfy. In the sequel we consider several types of such postulates, like those that refer to the properties of the extensions of the logical argumentation framework at hand (see, e.g., [29]), and postulates that refer to the properties of the induced entailment relations (see [30,54,55]).

The essence of this work is, therefore, to investigate the interplay between the basic ingredients of logical argumentation frameworks on one hand, and the properties of the frameworks and their entailment relations on the other hand. This allows us to assemble logical argumentation frameworks according to the desired properties that they and their entailment relations should have. As a result, we form an argumentative basis for what Prakken and Vreeswijk [66,77] (inspired by Rescher [70]) call *plausible reasoning*, namely: "sound (i.e., deductive) reasoning from uncertain premises" [66, p. 286].[1]

In our study, logical argumentation is investigated in the context of *sequent-based argumentation* [13], a simple and modular deductive argumentation setting, borrowing the proof-theoretic notion of *sequents* [47] for representing arguments. The incorporation of such notions provides a solid abstract representation of logical argumentation (in the sense that will be described in Section 2, see e.g., Note 1), and furthermore allows to properly construct and reason with arguments (see also [15]).

This paper is a revised and largely extended version of the conference papers in [11,12]. It provides a different perspective to earlier works on the subject (e.g., [2,5,48]) in several senses.

1. More postulates are considered[2] and their compatibility (i.e., their mutual satisfaction) is shown.
2. We cover many known Dung-based semantics for argumentation frameworks, including those (like stage or eager semantics) whose postulate-based behavior with respect to some attack relations has not been investigated before.[3]
3. Unlike some previous studies on logical argumentation (e.g., [11,48]), we do make a distinction between two types of premises. Strict premises are considered to be certain and therefore they cannot be attacked. Defeasible premises, on the other hand, are plausible yet uncertain assumptions. As such, they can be attacked. Distinguishing between these two types of assumptions has a significant effect on the form of the attack rules and their consequences.
4. We provide new results on how the nature of the attack rules (subset attacks versus direct attacks) affects the properties of the framework. This is done while avoiding some conditions that are used elsewhere, but which are problematic since they are hard to verify or too restrictive (see Notes 12 and 15).
5. We provide some new characterizations of the form of extensions for different Dung-style semantics (Theorems 1 and 3), as well as representations of the induced entailment relations in terms of inferences by maximally consistent subsets of the premises (Theorems 4 and 5).
6. Several assumptions that are made elsewhere are lifted in our case. For instance, in [2,5] it is assumed that the supports of the arguments are minimal and consistent, (that is, $\langle \Gamma, \psi \rangle$ is an argument only if Γ is consistent and there is no set $\Gamma' \subsetneq \Gamma$ such that $\langle \Gamma', \psi \rangle$ is a valid argument), and in [48] it is further assumed that the base logic is classical logic. None of these assumptions are made here. See [13,16,39] for a discussion on the advantages of avoiding these assumptions.

It is important to note that each one of the above 6 items is already (partially) addressed in the literature. This paper fills several gaps in the known results. Furthermore, against the background of a uniform setting, it provides an in-depth coverage of properties (in terms of different rationality postulates) and characterizations (of extensions and entailment relations) of logical argumentation frameworks. This is done with respect to a variety of base logics, different kinds of

[1] This is contrasted with *defeasible reasoning* (in the technical sense) as "unsound (i.e., defeasible) reasoning from firm premises" [65, p. 262] that is for instance modeled in Reiter's default logic [69] or in instantiations of ASPIC$^+$, in which genuinely defeasible rules are used (see [57,64] for a comparative study of default logic and ASPIC$^+$).

[2] Exhaustiveness (Definition 12), for example, was only defined and justified so far, but it was not characterized elsewhere, in the sense that the precise conditions for its satisfaction were not known (cf. Propositions 18–20 and Corollaries 15–17). Moreover, we define variants of some postulates (dubbed "limited" versions), fine-tuned to our generalized setting with strict *and* defeasible assumptions.

[3] The number of Dung-style semantics for argumentation frameworks is continuously growing, thus some semantics of lesser centrality, or those that are heavily based on graph-theoretic intuitions, such as the recursive CF2 [21], are left out.

assumptions (strict or defeasible), arguments (whose supports need not be minimal or consistent), several types of attack rules gathered according to their nature (direct attack, subset attacks, etc.), and all the main Dung-style semantics, where the corresponding extensions may be considered as individuals (credulous approach) or as a collective (skeptical or weakly skeptical approach). This allows us to compare logic-based argumentation frameworks with respect to different contexts and settings, demonstrating the advantages and shortcomings in each case.

The rest of the paper is organized as follows. The next section contains some basic notions and notations that are related to (sequent-based) logical argumentation, as well as further definitions concerning these frameworks, needed to configure the setting of this work and obtain its results. We then turn to the heart of the paper, which is divided to three parts:

1. Section 3 is an evaluation of logical argumentation frameworks from a postulate-based perspective. We distinguish between postulates concerning individual extensions (Section 3.1), and those that are concerned with sets of extensions (Section 3.2). At the end of each section we provide a table that summarizes the results in the section and shows what postulates are satisfied by which frameworks. These tables clearly indicate the large diversity among different argumentation frameworks and the crucial role of their ingredients in determining their properties.

   Section 4 provides some further examples that demonstrate the results of the above-mentioned study in case of argumentation frameworks based on non-classical logics.
2. Section 5 contains a sequence of characterization theorems that are obtained from the propositions in Section 3. These theorems are of two kinds: those that characterize the extensions according to different semantics of the logical argumentation frameworks (Section 5.1), and those that characterize the entailment relations that are induced by the frameworks (Section 5.2).
3. Section 6 further relies on insights gained in the previous sections for providing some results on the behavior of the argumentation-based entailments. Again, we conduct a postulate-driven study that mainly considers two types of properties: those that are related to the non-monotonic nature of the inferences (Section 6.1) and those that are related to the way inconsistency is tolerated (Section 6.2). As a by-product, we also provide in this section some new results on reasoning with maximal consistency with strict and defeasible premises and abstract base logics.

In Section 7 we discuss related work, and in Section 8 we conclude.

2. Preliminaries

In this section we describe the setting of our work and its context. In Section 2.1 we review some basic notions concerning logical argumentation frameworks, and in Section 2.2 we express some minimal assumptions on these frameworks that are needed for our results.

2.1. Logical argumentation

In what follows we shall assume that the underlying language $\mathcal{L}$ is propositional. Atomic formulas in $\mathcal{L}$ are denoted by lower-case letters (p, q, r), formulas are denoted by lower-case Greek letters $(\phi, \psi, \delta, \gamma)$, sets of formulas are denoted by upper-case (calligraphic) letters $(\mathcal{S}, \mathcal{T}, \mathcal{X})$, and finite sets of formulas are denoted by upper-case Greek letters $(\Gamma, \Delta, \Pi, \Theta)$, all of which can be primed or indexed. The set of atomic formulas of $\mathcal{L}$ (respectively, the set of atomic formulas appearing in the formulas of $\mathcal{S}$) is denoted $\mathsf{Atoms}(\mathcal{L})$ (respectively, $\mathsf{Atoms}(\mathcal{S})$). The set of the (well-formed) formulas of $\mathcal{L}$ is denoted $\mathsf{WFF}(\mathcal{L})$. The power set of a set S is denoted $\wp(\mathsf{S})$.

Definition 1 *(logic).* A *logic* for a language $\mathcal{L}$ is a pair $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a *(Tarskian) consequence relation* for $\mathcal{L}$, i.e., it is a relation on $\wp(\mathsf{WFF}(\mathcal{L})) \times \mathsf{WFF}(\mathcal{L})$, satisfying:

- *reflexivity:* if $\phi \in \mathcal{S}$ then $\mathcal{S} \vdash \phi$;
- *transitivity:* if $\mathcal{S} \vdash \phi$ and $\mathcal{S}', \phi \vdash \psi$, then $\mathcal{S}, \mathcal{S}' \vdash \psi$[4];
- *monotonicity:* if $\mathcal{S}' \vdash \phi$ and $\mathcal{S}' \subseteq \mathcal{S}$, then $\mathcal{S} \vdash \phi$.

Given a logic L, it is usual to assume the following properties:

- *non-trivialilty:* $\mathcal{S} \nvdash \phi$ for some nonempty $\mathcal{S}$ and formula ϕ;
- *structurality (closure under substitutions):* if $\mathcal{S} \vdash \phi$ then $\{\theta(\psi) \mid \psi \in \mathcal{S}\} \vdash \theta(\phi)$ for every $\mathcal{L}$-substitution θ;
- *compactness:* if $\mathcal{S} \vdash \phi$ then $\Gamma \vdash \phi$ for a finite $\Gamma \subseteq \mathcal{S}$.

[4] Following the usual convention, here and in what follows commas in the context of $\vdash$ stand for the union operator and singletons on the left hand side of $\vdash$ are written without set brackets.

In what follows, we shall assume that $\mathcal{L}$ contains at least a $\vdash$-negation operator ($\neg$), satisfying $p \nvdash \neg p$ and $\neg p \nvdash p$ (for atomic p),[5] and a $\vdash$-conjunction operator ($\wedge$), for which $\mathcal{S} \vdash \psi \wedge \phi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \phi$.[6] Also, we denote by $\bigwedge \Gamma$ the conjunction of all the formulas in Γ. Besides $\neg$ and $\wedge$, $\mathcal{L}$ may contain other logical operators, such as disjunction, implication, modal operators, logical primitives, etc.

Definition 2 *(closure and consistency).* Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas. The $\vdash$-*closure of* $\mathcal{S}$ is the set $\mathsf{CN}_{\mathsf{L}}(\mathcal{S}) = \{\phi \mid \mathcal{S} \vdash \phi\}$. We say that $\mathcal{S}$ is $\vdash$-*consistent*, if it does not have any subset whose negation follows from an empty set of assumptions, namely: there are no formulas $\phi_1, \ldots, \phi_n \in \mathcal{S}$ for which $\vdash \neg(\phi_1 \wedge \cdots \wedge \phi_n)$.[7]

Having a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ at our disposal, we can now define the notion of an *argument*, based on that logic. In its most general form, such an argument is just a pair $\langle \Gamma, \psi \rangle$, the first component of which (Γ) is a finite set of $\mathcal{L}$-formulas which are the argument's premises (or supports), and the other component (ψ) is an $\mathcal{L}$-formula, called the argument's conclusion. A minimal requirement from an argument is that it should be *logically valid*, that is: its conclusion should follow, according to L, from the premises. In the above notations, this means that $\Gamma \vdash \psi$. Using a well-known terminology from proof theory, this is what is called there a *sequent* [47]. In our case, the set of premises of a sequent may consist of two kinds of formulas: strict and defeasible ones. Intuitively, the former are formulas that are 'taken for granted' while the latter are formulas that may be challenged (i.e., attacked by other arguments).

Definition 3 *(arguments).* Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, $\mathcal{X}$ a $\vdash$-consistent set of $\mathcal{L}$-formulas (the strict assumptions), and $\mathcal{S}$ an arbitrary set of $\mathcal{L}$-formulas (the defeasible assumptions), such that $\mathcal{X} \cap \mathcal{S} = \emptyset$.

- An $\mathcal{L}$-*sequent* (*sequent* for short) is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas in $\mathcal{L}$ and $\Rightarrow$ is a symbol that does not appear in $\mathcal{L}$.
- An L-*argument* (*argument* for short) is an $\mathcal{L}$-sequent of the form $\Gamma \Rightarrow \psi$,[8] where $\Gamma \vdash \psi$. We say that Γ is the *support set* of $\Gamma \Rightarrow \psi$ (also denoted by $\mathsf{Supp}(\Gamma \Rightarrow \psi)$) and that ψ is its *conclusion* (also denoted $\mathsf{Conc}(\Gamma \Rightarrow \psi)$). For a set S of arguments, we let $\mathsf{Supps}(\mathsf{S}) = \bigcup \{\mathsf{Supp}(a) \mid a \in \mathsf{S}\}$ and $\mathsf{Concs}(\mathsf{S}) = \{\mathsf{Conc}(a) \mid a \in \mathsf{S}\}$.
- An L-*argument based on* $\mathcal{S}$ *and* $\mathcal{X}$ is an L-argument $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq \mathcal{S} \cup \mathcal{X}$. We denote by $\mathsf{Arg}_{\mathsf{L}}^{\mathcal{X}}(\mathcal{S})$ the set of all the L-arguments based on $\mathcal{S}$ and $\mathcal{X}$.
- An argument $\Gamma' \Rightarrow \psi'$ is a *sub-argument* of $\Gamma \Rightarrow \psi$ if $\Gamma' \subseteq \Gamma$. The set of all the sub-arguments of $\Gamma \Rightarrow \psi$ is denoted $\mathsf{Sub}(\Gamma \Rightarrow \psi)$.

Note 1. Sequents are a general representation of arguments in the sense that the only requirement on these expressions is that their conclusions will logically follow from their supports (according to the underlying logic). In addition to this requirement, it is sometimes also assumed that the argument's support is $\vdash$-consistent and/or that none of its proper subsets $\vdash$-entails the arguments' conclusion (see, e.g., [5,25]). As our goal here is to keep the discussion as general as possible, we do not make such restrictions. We refer to [13,16] and the discussion below for further justifications of this choice. Other works on sequent-based argumentation without these restrictions appear, e.g., in [10,15,26,74].

Formal systems for constructing sequents (and so arguments) for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ are called *sequent calculi* [47], denoted here by C. The construction of arguments from simpler arguments is done by means of derivations via the *inference rules* of the sequent calculus. A sequent is *provable* (or *derivable*) in C if there is a derivation for it in C.[9] In what follows we shall assume that the calculus C is sound and complete for its logic (i.e., $\Gamma \Rightarrow \psi$ is provable in C iff $\Gamma \vdash \psi$). Note that this implies, in particular, that for given sets $\mathcal{S}$ and $\mathcal{X}$, all the elements in $\mathsf{Arg}_{\mathsf{L}}^{\mathcal{X}}(\mathcal{S})$ are C-provable.

Just as arguments are constructed by inference rules in C, conflicts (attacks) between arguments are represented in a rule-like manner. Such an attack (or, sequent-elimination) rule consists of an attacking argument (the first condition of the rule), an attacked argument (the last condition of the rule), conditions for the attack (the other conditions of the rule) and a conclusion (the eliminated attacked sequent). The outcome of an application of such a rule is that the attacked sequent is 'eliminated' (or 'invalidated', see below the exact meaning of this). The elimination of a sequent $\Gamma \Rightarrow \phi$ is denoted by $\Gamma \not\Rightarrow \phi$. In sum, the general scheme for attack rules is as follows:

5 In the context of structurality and monotonicity, the presence of a negation renders the logic non-trivial.

6 By the definition of $\wedge$ we have that $\phi \wedge \psi \vdash \phi$; $\phi \wedge \psi \vdash \psi$ and $\phi, \psi \vdash \phi \wedge \psi$, so $\mathcal{S}, \phi, \psi \vdash \gamma$ iff $\mathcal{S}, \phi \wedge \psi \vdash \gamma$.

7 Following the usual convention, we write $\vdash \phi$ as an abbreviation of $\emptyset \vdash \phi$.

8 Set signs in arguments are omitted.

9 This usually means a finite sequence of sequent-based tuples, constructed according to the inference rules of C, that culminates with a tuple that contains the derived sequent. The exact details vary from one calculus to another.

$$\frac{\overbrace{\Gamma_1 \Rightarrow \phi_1}^{\text{attacker}} \quad \overbrace{\ldots}^{\text{conditions}} \quad \overbrace{\Gamma_2 \Rightarrow \phi_2}^{\text{attacked}}}{\underbrace{\Gamma_2 \not\Rightarrow \phi_2}_{\text{eliminated argument}}} \quad \text{where [additional conditions].}$$

We now define the attack rules that are central to our study.

Definition 4 *(attack rules).* Given a set $\mathcal{X}$ of strict (non-attacked) formulas, we consider the following attack rules:

- Defeat ($\mathsf{Def}_{\mathcal{X}}$): $\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$ where $\Gamma_2 \neq \emptyset$, $\Gamma_2 \cap \mathcal{X} = \emptyset$
- Direct Defeat ($\mathsf{DDef}_{\mathcal{X}}$): $\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg\gamma \quad \Gamma_2, \gamma \Rightarrow \psi_2}{\Gamma_2, \gamma \not\Rightarrow \psi_2}$ where $\gamma \notin \mathcal{X}$
- Undercut ($\mathsf{Ucut}_{\mathcal{X}}$): $\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \neg \bigwedge \Gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$ where $\Gamma_2 \neq \emptyset$, $\Gamma_2 \cap \mathcal{X} = \emptyset$
- Direct Undercut ($\mathsf{DUcut}_{\mathcal{X}}$): $\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg\gamma \quad \neg\gamma \Rightarrow \psi_1 \quad \Gamma_2, \gamma \Rightarrow \psi_2}{\Gamma_2, \gamma \not\Rightarrow \psi_2}$ where $\gamma \notin \mathcal{X}$
- Consistency Ucut ($\mathsf{ConUcut}_{\mathcal{X}}$): $\dfrac{\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi}$ where $\Gamma_2 \neq \emptyset$, $\Gamma_2 \cap \mathcal{X} = \emptyset$, $\Gamma_1 \subseteq \mathcal{X}$

The rules above indicate different cases in which the attacker challenges the attacked argument. For instance, when $\{p, \neg p\} \subseteq \mathcal{S}$ and classical logic (CL) is the core logic, the sequents $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ attack each other according to (Direct) Defeat and (Direct) Undercut. In contrast, the tautological sequent $\Rightarrow \psi \vee \neg\psi$ is not (direct) defeated or (direct) undercut by any argument in $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{CL}}(\mathcal{S})$, since it has an empty support set. We also note that the attack rules differentiate between defeasible and strict assumptions by only allowing attacks in the former. E.g., when $\{\neg p\} \subseteq \mathcal{S}$ and $\mathcal{X} = \{p\}$ the attack is uni-directional from $p \Rightarrow p$ to $\neg p \Rightarrow \neg p$, since an argument cannot be attacked in its strict premises.

Note 2. In the particular case where Γ_1 is empty, Consistency Undercut indicates that arguments with an inconsistent set of premises are attacked by tautological arguments. In relation to Note 1, then, support inconsistency may be handled by attack rules of the frameworks rather than simply ruled out (we refer to [16] for a further discussion on this).

Note 3. When $\mathcal{X} = \emptyset$, the rules in Definition 4 coincide with those of [13,74]. There, the reader can also find many other attack rules. In [26] sequent-based argumentation has been generalized along similar lines, where the left side of a sequent is a pair $\Pi \mid \Gamma$ consisting of a set of defeasible premises Π and a set of strict premises Γ.

An argumentation framework is now defined as follows:

Definition 5 *(argumentation frameworks).* A (sequent-based) *argumentation framework* (AF), based on a logic L and a set A of attack rules, for a set of defeasible premises $\mathcal{S}$ and a $\vdash$-consistent set of strict premises $\mathcal{X}$ such that $\mathcal{S} \cap \mathcal{X} = \emptyset$, is a pair $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S}) = \left\langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \right\rangle$, where $\mathcal{A} \subseteq \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \times \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ and $(a_1, a_2) \in \mathcal{A}$ iff there is an $\mathcal{R} \in \mathsf{A}$ such that a_1 $\mathcal{R}$-attacks a_2 (that is, (a_1, a_2) is an instance of the relation $\mathcal{R}$).[10] A pair (a_1, a_2) is an instance of an attack rule $\mathcal{R}$ in case that $\mathcal{R}$ can be instantiated in such a way that a_1 is the attacking sequent, a_2 is the attacked sequent, and the sequents in the condition of $\mathcal{R}$ (if any) can be derived in the underlying calculus C (in particular, they need not be in $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$).

The subscripts L, A and/or the superscript $\mathcal{X}$ will be omitted when they are clear from the context or arbitrary. Note that the attacking and the attacked arguments must be elements of $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, to prevent "irrelevant attacks", in which, e.g., $\neg p \Rightarrow \neg p$ attacks $p \Rightarrow p$ although $p \in \mathcal{S}$ and $\neg p \notin \mathcal{S} \cup \mathcal{X}$.

Example 1. Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{CL},\mathsf{A}}(\mathcal{S}) = \left\langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{CL}}(\mathcal{S}), \mathcal{A} \right\rangle$ be an argumentation framework for $\mathcal{S} = \{q, \neg p \vee \neg q, r\}$ and $\mathcal{X} = \{p\}$, classical logic (CL) as the base logic, and $\mathcal{A}$ is obtained from the attack rules in A, where $\{\mathsf{ConUcut}\} \subsetneq \mathsf{A} \subseteq \{\mathsf{DDef}, \mathsf{DUcut}, \mathsf{ConUcut}\}$. Among others, the following sequents are in $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{CL}}(\mathcal{S})$:

[10] Thus, A consists of the rule names, and $\mathcal{A}$ is their applications on $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$.

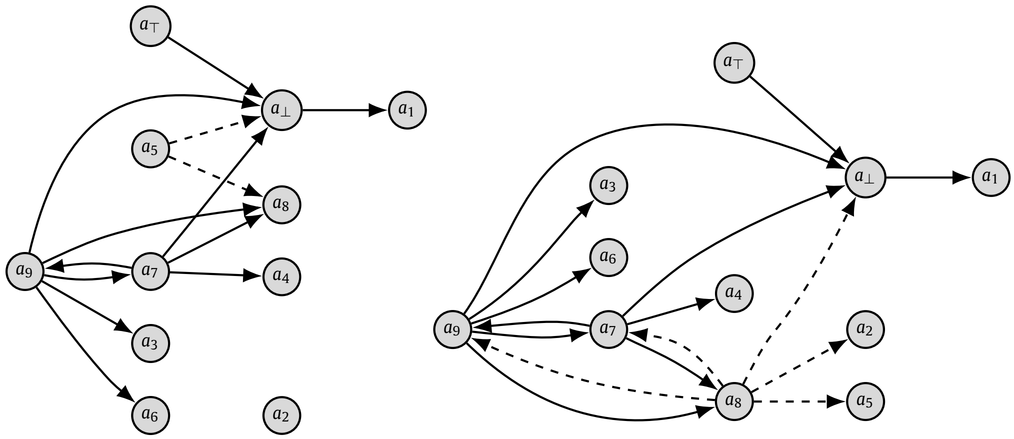

Fig. 1. Part of the frameworks from Example 1. In both frameworks $\mathsf{A} = \{\text{DDef}, \text{ConUcut}\}$. On the left side $\mathcal{S} = \{q, \neg p \vee \neg q, r\}$ and $\mathcal{X} = \{p\}$, on the right side $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$ and $\mathcal{X} = \{\}$. The dashed arrows denote the differences between the two frameworks.

$$
\begin{array}{llll}
a_1 = r \Rightarrow r & & a_7 = p, q \Rightarrow p \wedge q \\
a_2 = p \Rightarrow p & & a_8 = \neg p \vee \neg q, q \Rightarrow \neg p \\
a_3 = q \Rightarrow q & & a_9 = \neg p \vee \neg q, p \Rightarrow \neg q \\
a_4 = \neg p \vee \neg q \Rightarrow \neg p \vee \neg q & & a_\top = \Rightarrow \neg(p \wedge q \wedge (\neg p \vee \neg q)) \\
a_5 = p \Rightarrow \neg((\neg p \vee \neg q) \wedge q) & & a_\bot = p, q, \neg p \vee \neg q \Rightarrow \neg r \\
a_6 = q \Rightarrow \neg((\neg p \vee \neg q) \wedge p) & &
\end{array}
$$

The left part of Fig. 1 is a graphical representation of the argumentation framework $\mathcal{AF}^{\mathcal{X}}_{\mathsf{CL},\mathsf{A}}(\mathcal{S})$ restricted to the arguments above, and where direct defeat and consistency undercut are the attack rules. The right part of the figure represents the framework with the same attack rules, but when p is a defeasible assumption rather than a strict assumption (i.e., when $\mathcal{S} = \{p, q, \neg p \vee \neg q, r\}$ and $\mathcal{X} = \emptyset$). In the figure, nodes represent arguments, and directed edges represent attacks from the attacking to the attacked arguments. Dashed arrows designate the differences between the two parts of the figure.

Given an argumentation framework $\mathcal{AF}$, Dung-style semantics [19,40] can be applied to it, to determine what combinations of arguments (called *extensions*) can collectively be accepted from $\mathcal{AF}$.

Definition 6 *(extensions).* Let $\mathcal{AF} = \mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S}) = \left\langle \mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \right\rangle$ be an argumentation framework and let $\mathsf{S} \subseteq \mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ be a set of arguments. It is said that:

- S *attacks* a if there is an $a' \in \mathsf{S}$ such that $(a', a) \in \mathcal{A}$. The set of all arguments attacked by S is denoted by S^+. The set $\mathsf{S} \cup \mathsf{S}^+$ is called the *range* of S.
- S *defends* a if S attacks every a' such that $(a', a) \in \mathcal{A}$.
- S is *conflict-free* if for no $a_1, a_2 \in \mathsf{S}$ it holds that $(a_1, a_2) \in \mathcal{A}$.
- S is *naive* (nav) if it is a $\subseteq$-maximal conflict-free set.
- S is a *stage* (stg) *extension* of $\mathcal{AF}$ if it is conflict-free and $\mathsf{S} \cup \mathsf{S}^+$ is $\subseteq$-maximal among the ranges of the conflict-free sets.
- S is *admissible* in $\mathcal{AF}$ if it is conflict-free and defends all of its elements.
- S is a *complete* (cmp) *extension* of $\mathcal{AF}$ if it is an admissible set that contains all the arguments that it defends.
- S is the *grounded* (grd) *extension* of $\mathcal{AF}$ if it is the $\subseteq$-minimal complete extension of $\mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$.
- S is a *preferred* (prf) *extension* of $\mathcal{AF}$ if it is a $\subseteq$-maximal complete extension of $\mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$.
- S is the *ideal* (idl) *extension* of $\mathcal{AF}$ if it is the $\subseteq$-maximal admissible set that is included in each preferred extension.
- S is a *stable* (stb) *extension* of $\mathcal{AF}$ if it is a conflict-free set in $\mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ that attacks every argument not in it (that is, the range of S is the whole set of arguments, $\mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$).
- S is a *semi-stable* (sstb) *extension* of $\mathcal{AF}$ if it is a complete extension whose range is $\subseteq$-maximal.
- S is the *eager* (egr) *extension* of $\mathcal{AF}$ if it is the $\subseteq$-maximal admissible set that is included in every semi-stable extension.

We denote by $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ the set of all the extensions of $\mathcal{AF}$ of type sem for some sem $\in$ {nav, stg, cmp, grd, prf, idl, stb, sstb, egr}.

Note 4. As shown in [40], the grounded extension is unique for a given framework, and as shown in [19], the ideal and the eager extensions are unique complete extensions for a framework. Furthermore, a stable extension is also a semi-stable

$$
\begin{array}{llll}
[\text{Ref}] & \dfrac{}{\phi \Rightarrow \phi} & [\text{Cut}] & \dfrac{\Gamma_1 \Rightarrow \psi, \Pi_1 \quad \Gamma_2, \psi \Rightarrow \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Pi_1, \Delta} \\[2ex]
[\text{LMon}] & \dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta} & [\text{RMon}] & \dfrac{\Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi, \phi} \\[2ex]
[\neg\Rightarrow] & \dfrac{\Gamma \Rightarrow \Pi, \varphi}{\neg\varphi, \Gamma \Rightarrow \Pi} & [\Rightarrow\neg] & \dfrac{\varphi, \Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi, \neg\varphi} \\[2ex]
[\wedge\Rightarrow] & \dfrac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} & [\Rightarrow\wedge] & \dfrac{\Gamma_1 \Rightarrow \Pi_1, \varphi \quad \Gamma_2 \Rightarrow \Pi_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Pi_1, \Pi_2, \varphi \wedge \psi}
\end{array}
$$

Fig. 2. Rules that are part of (or admissible in) the calculus C (in case that C is single-conclusion, thus sequents may have at most one formula in their right-hand sides, Π, Π_1 and Π_2 should be empty, and Δ contains at most one formula).

extension, which in turn is a preferred extension. Other extensions and their properties are discussed, e.g., in [18–20]. A graphical representation of the relations among alternative extension-based semantics can be found, e.g., in [19, Figure 13].

On the basis of argumentation frameworks and their semantics we can define two general types of entailment relations: skeptical and credulous ones.

Definition 7 *(entailments).* Given an argumentation framework $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})$ and a semantics sem for it, the following *entailment relations* are induced from them:

- *Skeptical entailment:* $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{L,A},\mathcal{X},\mathsf{sem}} \phi$ if there is an argument $a \in \bigcap \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S}))$ such that $\mathsf{Conc}(a) = \phi$.
- *Weakly skeptical entailment:* $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{L,A},\mathcal{X},\mathsf{sem}} \phi$ if for every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S}))$ there is an argument $a \in \mathcal{E}$ such that $\mathsf{Conc}(a) = \phi$.
- *Credulous entailment:* $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{L,A},\mathcal{X},\mathsf{sem}} \phi$ iff there is an argument $a \in \bigcup \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S}))$ such that $\mathsf{Conc}(a) = \phi$.

For fixed $\mathsf{L}, \mathsf{A}, \mathcal{X}$ and sem, we clearly have that $\mathrel{|\!\sim}^{\cap}_{\mathsf{L,A},\mathcal{X},\mathsf{sem}} \subseteq \mathrel{|\!\sim}^{\Cap}_{\mathsf{L,A},\mathcal{X},\mathsf{sem}} \subseteq \mathrel{|\!\sim}^{\cup}_{\mathsf{L,A},\mathcal{X},\mathsf{sem}}$. The subscripts $\mathsf{L}, \mathsf{A}, \mathcal{X}$ and sem (or some of them) are omitted when they are clear from the context or arbitrary. Since the grounded, ideal and eager extensions are unique, $\mathrel{|\!\sim}^{\cap}_{\mathsf{sem}}$, $\mathrel{|\!\sim}^{\Cap}_{\mathsf{sem}}$ and $\mathrel{|\!\sim}^{\cup}_{\mathsf{sem}}$ coincide for a fixed $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{idl}, \mathsf{egr}\}$, so all three are denoted by $\mathrel{|\!\sim}_{\mathsf{sem}}$.

Example 2. Let $\mathcal{AF}^{\emptyset}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S})$ be an argumentation framework for $\mathcal{S} = \{p, \neg p, q\}$, based on CL and Undercut as the sole attack rule. As noted in the paragraph below Definition 4, the sequent $\Rightarrow\ p \vee \neg p$ belongs to every complete extension of $\mathcal{AF}^{\emptyset}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S})$, since it cannot be Undercut-attacked.[11] Similarly, $q \Rightarrow q$ belongs to every complete extension of the framework, since $\Rightarrow p \vee \neg p$ counter-attacks any attacker of $q \Rightarrow q$ that belongs to $\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\mathcal{S})$ (given that any attacker of $q \Rightarrow q$ has an inconsistent support set). This implies that $p, \neg p, q \mathrel{|\!\sim}^{\star}_{\mathsf{sem}} q$ for every $\mathsf{sem} \in \{\mathsf{stg}, \mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{idl}, \mathsf{stb}, \mathsf{sstb}, \mathsf{egr}\}$ and $\star \in \{\cap, \Cap, \cup\}$. On the other hand, for each $\mathsf{sem} \in \{\mathsf{nav}, \mathsf{stg}, \mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{idl}, \mathsf{stb}, \mathsf{sstb}, \mathsf{egr}\}$ and $\star \in \{\cap, \Cap\}$ it holds that $p, \neg p, q \mathrel{|\!\not\sim}^{\star}_{\mathsf{sem}} p$ and $p, \neg p, q \mathrel{|\!\not\sim}^{\star}_{\mathsf{sem}} \neg p$.

Example 3. Consider again the excerpt of the argumentation framework $\mathcal{AF}^{\mathcal{X}}_{\mathsf{CL,A}}(\mathcal{S})$ of Fig. 1 (left), i.e., when $\mathcal{X} = \{p\}$. In this figure, the grounded extension consists only of the arguments a_1, a_2, a_5 and $a_\top$, and the preferred/stable extensions are $\mathcal{E}_1 = \{a_\top, a_1, a_2, a_3, a_5, a_6, a_7\}$ and $\mathcal{E}_2 = \{a_\top, a_1, a_2, a_4, a_5, a_9\}$. It follows that $a_1 = r \Rightarrow r$ and $a_2 = p \Rightarrow p$ belong to every complete extension of the argumentation framework of Example 1, and so r and p are concluded in that case by $\mathrel{|\!\sim}^{\star}_{\mathcal{X},\mathsf{sem}}$ for every $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{idl}, \mathsf{stb}, \mathsf{sstb}, \mathsf{egr}\}$ and $\star \in \{\cap, \Cap, \cup\}$. Note that this is possible by the availability in Example 1 of ConUcut, otherwise a_1 would not be defended from the attack by $a_\perp$.

2.2. The scope of the study

Interestingly, despite the diversity of logics and their sequent calculi covered in this work, only a few assumptions on the sequent calculi are necessary for our results. In fact, we only need to assume that the rules of the basic calculus from Fig. 2 are part of (or admissible in) C.

The first four (structural) rules correspond to the properties of consequence relations (Definition 1): reflexivity [Ref], transitivity [Cut] and monotonicity [LMon, RMon]; the other four (logical) rules refer to the behavior of the negation in the left-hand side of the sequents [$\neg\Rightarrow$], in the right-hand side [$\Rightarrow\neg$], and similar rules for the conjunction ([$\wedge\Rightarrow$] and [$\Rightarrow\wedge$], respectively).

[11] We note that $\Rightarrow p \vee \neg p$ does not belong to every naive extension. Although an argument a such as $p, \neg p \Rightarrow q$ cannot be defended from the Undercut attack by $\Rightarrow p \vee \neg p$, since a is not self-attacking, it belongs to a conflict-free set and so also to a maximal conflict-free (i.e., naive) set.

Example 4. Gentzen's calculus LK for classical logic, its single-conclusion variation LJ for intuitionistic logic, as well as their extensions to modal logics, are some well-known calculi for base logics that are covered by our study.

We start with some simple lemmas that will be needed for the proofs in what follows.

Lemma 1. *For a formula ϕ and a finite set of formulas Γ, the sequents $\phi \Rightarrow \neg\neg\phi$ and $\Gamma \Rightarrow \bigwedge\Gamma$ are* C*-derivable.*

Proof. For $\phi \Rightarrow \neg\neg\phi$, note that by reflexivity $\phi \Rightarrow \phi$ is C-derivable, and by $[\neg\Rightarrow]$ so is $\phi, \neg\phi \Rightarrow$. Then, by $[\Rightarrow\neg]$, we get $\phi \Rightarrow \neg\neg\phi$. The sequent $\Gamma \Rightarrow \bigwedge\Gamma$ is derivable by reflexivity on every formula in Γ and then applications of $[\Rightarrow\wedge]$. □

Lemma 2. *If $\Gamma \Rightarrow \neg\bigwedge\Delta$ is* C*-derivable, then also $\Delta \Rightarrow \neg\bigwedge\Gamma$, $\Gamma \Rightarrow \neg\bigwedge(\Delta \cup \Delta')$, $\Gamma, \Delta \Rightarrow$ and $\Gamma, \Delta \setminus \{\delta\} \Rightarrow \neg\delta$ (for every $\delta \in \Delta$) are* C*-derivable.*

Proof. By $[\neg\Rightarrow]$, $\Gamma, \neg\neg\bigwedge\Delta \Rightarrow$ is C-derivable. By Lemma 1 and [Cut], $\Gamma, \bigwedge\Delta \Rightarrow$ is C-derivable. Since $\wedge$ is a $\vdash$-conjunction, by the completeness of C and by [LMon], the sequents $\Gamma, \Delta \Rightarrow$; $\Gamma, \Delta, \Delta' \Rightarrow$; $\Gamma, \bigwedge(\Delta \cup \Delta') \Rightarrow$; and $\bigwedge\Gamma, \Delta \Rightarrow$ are C-derivable. By $[\Rightarrow\neg]$, the sequents $\Delta \Rightarrow \neg\bigwedge\Gamma$; $\Gamma \Rightarrow \neg\bigwedge(\Delta \cup \Delta')$; and $\Gamma, \Delta \setminus \{\delta\} \Rightarrow \neg\delta$ are all C-derivable (where $\delta \in \Delta$). □

Lemma 3. *For every $\mathcal{L}$-formulas ϕ, ψ, the sequent $\phi, \neg\phi \Rightarrow \psi$ is* C*-derivable.*

Proof. By [Ref], $\phi \Rightarrow \phi$ is C-derivable. By $[\neg\Rightarrow]$ we get $\phi, \neg\phi \Rightarrow$, and by [RMon], $\phi, \neg\phi \Rightarrow \psi$ is C-derivable. □

Since C is sound and complete for the underlying logic, by the last two lemmas we have the following corollary:

Corollary 1. *All the logics $\mathsf{L} = \langle\mathcal{L}, \vdash\rangle$ considered in what follows are explosive ($\Gamma, \psi, \neg\psi \vdash \phi$) and contrapositive ($\Gamma, \psi \vdash \phi$ iff $\Gamma, \neg\phi \vdash \neg\psi$).*

By Lemma 2, we also have the following result:

Lemma 4. *(i) Γ is inconsistent iff $\Gamma \Rightarrow$ is derivable. (ii) Γ is inconsistent iff $\Gamma \setminus \{\gamma\} \Rightarrow \neg\gamma$ is derivable for any $\gamma \in \Gamma$.*

Proof. To see (i), note that Γ is inconsistent iff $\vdash \neg\bigwedge\Gamma'$ for some $\Gamma' \subseteq \Gamma$, iff (by the adequacy of the sequent calculus) $\Rightarrow \neg\bigwedge\Gamma'$ is C-derivable for some $\Gamma' \subseteq \Gamma$, iff (by Lemma 2) $\Gamma' \Rightarrow$ is C-derivable for some $\Gamma' \subseteq \Gamma$, iff (by [LMon]) $\Gamma \Rightarrow$ is C-derivable.

The proof of (ii) is left to the reader. □

The rules of Fig. 2 also imply the following necessary requirement for an attack between two arguments:

Lemma 5. *For any attack rule in Definition 4 it holds that $\Gamma \Rightarrow \gamma$ attacks $\Delta \Rightarrow \delta$ only if $\Delta \cup \Gamma$ is inconsistent.*

Proof. Suppose that $\Gamma \Rightarrow \gamma$ attacks $\Delta \Rightarrow \delta$. By [Cut] together with the condition of any of the attack rules in Definition 4, it is easy to see that there is a $\Delta' \subseteq \Delta$ for which $\Gamma \Rightarrow \neg\bigwedge\Delta'$ is C-derivable. By Lemma 2, $\Rightarrow \neg\bigwedge(\Gamma \cup \Delta')$ is also C-derivable. Hence, by the soundness of C, $\Gamma \cup \Delta$ is inconsistent. □

In what follows we distinguish between three types of attacks in argumentation frameworks, denoted set, dir and con.

Definition 8 *(types of attack rules).* Attack rules are categorized in the sequel as follows:

- **set:** $\mathsf{A} \cap \{\text{Def}, \text{Ucut}\} \neq \emptyset$ (i.e., attack rules in which an argument is attacked on a subset of its support, and where at least one of the rules is Undercut or Defeat),
- **dir:** $\emptyset \neq \mathsf{A} \subseteq \{\text{DDef}, \text{DUcut}\}$ (that is, nonempty sets of direct attack rules),
- **con:** $\{\text{ConUcut}\} \subsetneq \mathsf{A} \subseteq \{\text{ConUcut}, \text{DDef}, \text{DUcut}\}$ (that is, nonempty dir-type sets of attack rules that also contain ConUcut).

Note 5. Any framework with the attack rules of Definition 4, for which $\mathsf{A} \setminus \{\text{ConUcut}\} \neq \emptyset$, falls in one of the three categories above. Moreover, these categories are disjoint.

Note 6. In the presence of the inference rules in Table 2, Consistency Undercut is admissible if Defeat or Undercut are part of the attack rules. To see this, consider the following two derivations, where we assume that ConUcut can be applied and therefore that $\Gamma_1 \Rightarrow \neg\bigwedge\Gamma_2$ and $\Gamma_2, \Gamma_2' \Rightarrow \psi$ can be derived for some $\Gamma_1 \subseteq \mathcal{X}$.

$$\dfrac{\dfrac{\text{ConUcut Assumption}}{\Gamma_1 \Rightarrow \neg\bigwedge\Gamma_2} \qquad \dfrac{\dfrac{}{\neg\bigwedge\Gamma_2 \Rightarrow \neg\bigwedge\Gamma_2}\,\text{Ref}}{\neg\bigwedge\Gamma_2 \Rightarrow \neg(\bigwedge\Gamma_2 \wedge \bigwedge\Gamma_2')}\,\text{Lemma 2} \qquad \dfrac{\text{ConUcut Assumption}}{\Gamma_2, \Gamma_2' \Rightarrow \psi}}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi}\,\text{Def}$$

$$\dfrac{\dfrac{\text{ConUcut Assumption}}{\Gamma_1 \Rightarrow \neg\bigwedge\Gamma_2} \qquad \dfrac{}{\neg\bigwedge\Gamma_2 \Rightarrow \neg\bigwedge\Gamma_2}\,\text{Ref} \qquad \dfrac{}{\neg\bigwedge\Gamma_2 \Rightarrow \neg\bigwedge\Gamma_2}\,\text{Ref} \qquad \dfrac{\text{ConUcut Assumption}}{\Gamma_2, \Gamma_2' \Rightarrow \psi}}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi}\,\text{Ucut}$$

Concerning the categorization of the semantics in our setting, many of the results in what follows will apply to *completeness-based semantics*, as defined next:

Definition 9 *(completeness-based semantics).* A semantics sem (e.g., one of those in Definition 8) is *completeness-based relative to attack type* †, iff for every argumentation framework $\mathcal{AF}$ with attack rules of type †, it holds that $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF})$.

Definition 10 *(semantics classes).* In what follows we denote by CMP the set of completeness-based semantics, by ME the subset of multiple-extension semantics, and by SE the subset of single-extension semantics, namely: CMP = {cmp, prf, stb, sstb, stg, grd, idl, egr}, ME = {prf, stb, sstb, stg},[12] and SE = {grd, idl, egr}.

Note 7. By their definitions, complete, grounded, and preferred semantics are all completeness-based relative to any type of attacks. Since it can be shown that every stable, semi-stable, ideal, and eager extension is always complete (see, e.g., [19]), the same holds for these semantics. In general, naive and stage extensions need not be complete. We will show below (Proposition 1) that for the three types of attacks in Definition 8 (i.e., for every † ∈ {dir, con, set}) also stage semantics is completeness-based.[13]

Definition 11 *($\mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, $\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$).* For a logic $\mathsf{L} = \langle \mathcal{L}, \vdash\rangle$ a set $\mathcal{S}$ of $\mathcal{L}$-formulas, and a $\vdash$-consistent set $\mathcal{X}$ of $\mathcal{L}$-formulas, we say that $\mathcal{S}$ is $\vdash_{\mathcal{X}}$-consistent if $\mathcal{S} \cup \mathcal{X}$ is $\vdash$-consistent. We denote:

- $\mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$: the set of the $\vdash_{\mathcal{X}}$-consistent subsets of $\mathcal{S}$,
- $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$: the $\subseteq$-maximal sets in $\mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$,
- $\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$: the formulas in $\mathcal{S}$ that are not part of any $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent subset of $\mathcal{S}$.[14]

Lemma 6. *Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})$ be a framework with attack rules of one of the types in Definition 8 and let $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Then $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S}))$.*

Proof. The conflict-freeness of $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$ follows from Lemma 5. Consider now an argument $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \setminus \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$. If there is no such argument then $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$ is clearly stable. We show that a is attacked by $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$. Indeed, by the choice of a, there is a formula $\phi \in \mathsf{Supp}(a) \setminus \mathcal{T}$. By the maximal $\vdash_{\mathcal{X}}$-consistency of $\mathcal{T}$, by the compactness of L and by Lemma 4 (ii), there is a $\Gamma \subseteq \mathcal{T} \cup \mathcal{X}$ for which $\Gamma \Rightarrow \neg\phi \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$. The latter attacks a according to (D)Ucut and DDef (note that $\neg\phi \Rightarrow \neg\phi$ is derivable by Reflexivity). □

Proposition 1. *If $\mathcal{AF}$ is a framework with attacks of one of the types in Definition 8, then $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{stg}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF})$.*

Proof. The directions $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{stg}}(\mathcal{AF})$ and $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF})$ are trivial. For the converse, let sem ∈ {stg, sstb} and suppose that $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$. By Lemma 6, there is a $\mathcal{E}' \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$.[15] Since $\mathcal{E}'$ is conflict-free and $\mathcal{E}' \cup \mathcal{E}'^{+} = \mathsf{Arg}(\mathcal{AF})$, also $\mathcal{E} \cup \mathcal{E}^{+} = \mathsf{Arg}(\mathcal{AF})$, and so $\mathcal{E} \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$. □

Fig. 3 summarizes the relations between Dung-style semantics (Definition 6) and their classes (Definition 10). Strict lines indicate inclusion relations that hold in general (see, e.g., [19,40]). Dashed arrows indicate additional relations that hold in our logic-based setting (as shown in Theorem 1 below). As we shall see in the sequel, the main insight concerning

12 The exclusion of cmp from ME is explained in Note 22.

13 We note that naive semantics is not completeness-based (see, for instance, Footnote 11 for a counter-example).

14 It is well-known (and easily verified) that $\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ consists exactly of those formulas in $\mathcal{S}$ that are contained in every maximal $\vdash_{\mathcal{X}}$-consistent subset of $\mathcal{S}$, namely: $\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) = \bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ (see, for instance, [71, p. 186], [23, p. 24]).

15 It is easy to verify that every consistent subset of some $\mathcal{S}$ (including $\emptyset$) has a maximal $\vdash$-consistent superset $\mathcal{T} \subseteq \mathcal{S}$.

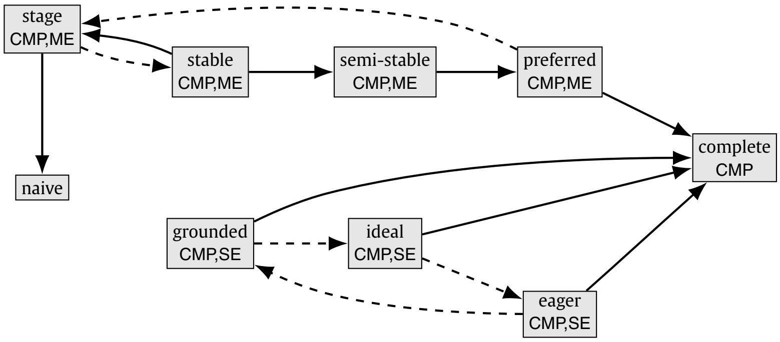

Fig. 3. Semantics, their classes, and inclusion relations among them. Strict lines indicate inclusion relations that hold in general and dashed arrows indicate additional relations that hold in the logic-based setting.

extension-based semantics in logic-based argumentation, is that the standard multiple-extension semantics (respectively, the standard single-extension semantics) collapses to a single type.

To summarize, in what follows we consider argumentation frameworks based on any propositional logic L with a sound and complete sequent calculus C, in which the rules in Fig. 2 are admissible, the set of premises may contain both strict ($\mathcal{X}$) and defeasible ($\mathcal{S}$) formulas, the set of attack rules A may be of any of the three types given in Definition 8, and the semantics sem may be any completeness-based ones. To the best of our knowledge, this variety has not been considered previously in systematic studies of meta-theoretic properties of logical argumentation. In the next sections we evaluate these frameworks with respect to different rationality postulates, divided according to their nature.

3. Evaluation of logical argumentation frameworks

As indicated previously, the definitions of sequent-based argumentation frameworks and the entailment relations induced by them leave plenty of choices to be made in their construction, as the base logic L, the attack rules A, the underlying semantics sem, and the inference method (skeptical, weakly skeptical, and credulous) may vary from one case to another. In this section we check how these choices affect the properties of the frameworks that are obtained and their entailment relations. For this, we consider several desirable properties and then check in what setting they can be warranted. In this section, we consider rationality postulates for the extensions of the frameworks, in Section 5 we show some characterization results, and in Section 6 the properties of the induced entailment relations are investigated.

3.1. Postulates concerning individual extensions

First, we consider rationality postulates for completeness-based semantics that are concerned with the properties of individual extensions.

Definition 12 *(rationality postulates I).* Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S}) = \left\langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \right\rangle$ be an argumentation framework, sem a semantics for it, $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S}))$, and $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. In Table 1 we list properties that $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})$ may have.[16,17]

Table 1
Rationality postulates for individual extensions.

Property	Definition
closure of extensions	$\mathsf{CN_L}(\mathsf{Concs}(\mathcal{E})) = \mathsf{Concs}(\mathcal{E})$.
closure under support	If $\mathsf{Supp}(a) \subseteq \mathsf{Supps}(\mathcal{E})$ then $a \in \mathcal{E}$.
sub-argument closure	If $a \in \mathcal{E}$ then $\mathsf{Sub}(a) \subseteq \mathcal{E}$.
support inclusion	$\mathsf{Supps}(\mathcal{E}) \subseteq \mathsf{Concs}(\mathcal{E})$.
(conclusion) consistency	$\mathsf{Concs}(\mathcal{E})$ is $\vdash_{\mathcal{X}}$-consistent.
support consistency	$\mathsf{Supps}(\mathcal{E})$ is $\vdash_{\mathcal{X}}$-consistent.
pairwise support consistency	For each $a, b \in \mathcal{E}$, $\mathsf{Supp}(a) \cup \mathsf{Supp}(b)$ is $\vdash_{\mathcal{X}}$-consistent.
exhaustiveness	If $\mathsf{Supp}(a) \cup \{\mathsf{Conc}(a)\} \subseteq \mathsf{Concs}(\mathcal{E})$ then $a \in \mathcal{E}$.
strong exhaustiveness	If $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$ then $a \in \mathcal{E}$.
free precedence	$\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})) \subseteq \mathcal{E}$.
strong free precedence	$\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}))$.
limited [free prec. / exhaus. / str. exhaus.]	[Free precedence / exhaustiveness / strong exhaustiveness] restricted to extensions $\mathcal{E}$ with $\bigcup \mathsf{Supps}(\mathcal{E}) \setminus \mathcal{X} \neq \emptyset$.

[16] Each of the properties in the table is defined with respect to sem. In what follows sem will be clear for the context.
[17] With the exceptions of pairwise support consistency and the limited versions of the postulates, all the other postulates are taken from [2,28,29].

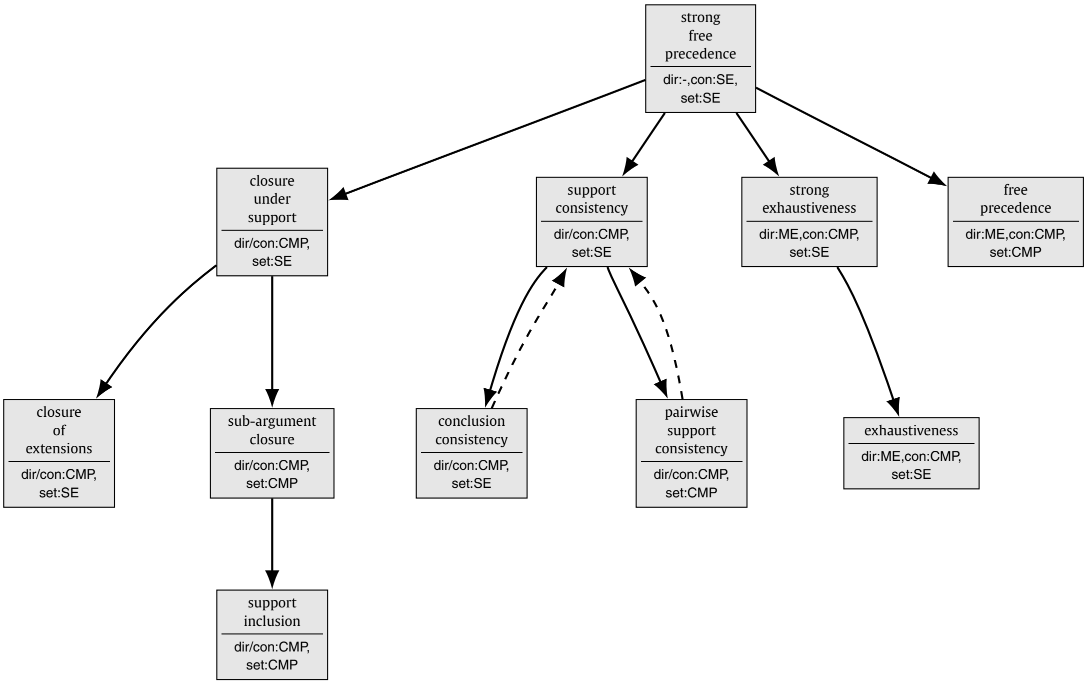

Fig. 4. Some relations between the postulates for individual extensions in our setting. If a property at the origin of an arrow in the diagram holds relative to classes of argumentation frameworks for a given semantics sem and a given attack type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$, then also the property at the end of the same arrow holds (for the specified semantics and attack type). For the dashed arrows sub-argument closure, resp. closure under support, is supposed. In each node label, the lower part indicates for which combinations of attack types † and semantics sem the property holds (as is proved in the propositions in this section).

Note 8. The postulates in Table 1 indicate some desirable properties of a framework's extensions.

- The first three postulates state different closure properties. For instance, the first postulate expresses that the set of conclusions of the arguments in an extension is logically closed.
- Support inclusion is a kind of reflexivity condition, demanding that anything that is assumed can be inferred.
- The next three postulates are consistency requirements regarding the supports and the conclusions of an extension. Violations of these postulates mean that contradictory conclusions may be inferred. Pairwise support consistency is a weaker version of support consistency, assuring that the support sets of each pair of arguments in an extension are mutually consistent. We shall see that only this form of consistency can be guaranteed in all circumstances.
- Exhaustiveness and strong exhaustiveness, like the closure postulates, state that what is expected to be in the extension is indeed there and is not left out. In [2], exhaustiveness is justified by the aspiration that 'if each step in an argument [namely, each sub-argument][18] is good enough to be in a given extension, then so is the argument itself'.
- Free precedence and strong free precedence refer to the containments of the 'safe arguments', that is: those that are based on the formulas in (the intersection of) the consistent subsets of the premises. In other words, these postulates state that any argument that is supported only by formulas that are not involved in any minimal conflict, is included in the extension. The stronger postulate, requiring an absolute identity between the elements of an extension and the safe arguments, might be regarded as too strong, but when it is satisfied all the other postulates are guaranteed (see Fig. 4).
- Finally, we also consider limited versions of some of the postulates, restricted to 'non-tautological' extensions.

The postulates in Table 1 are particularly useful in credulous reasoning, which is based on the content of specific extensions. In such cases these postulates assure the plausibility of the conclusion making process (being consistent, closed under logical inferences, etc.). For further descriptions of the postulates in Table 1 and discussions on the intuition behind them, we refer e.g. to [2,28,29].

Note 9. Clearly, some postulates in Definition 12 are related to each other. For instance, sub-argument closure follows from closure under support, pairwise support consistency follows from support consistency, exhaustiveness follows from strong exhaustiveness, free precedence follows from strong free precedence, and the limited version of a postulate follows from

[18] The text in square brackets is our clarification and not part of the original quotation.

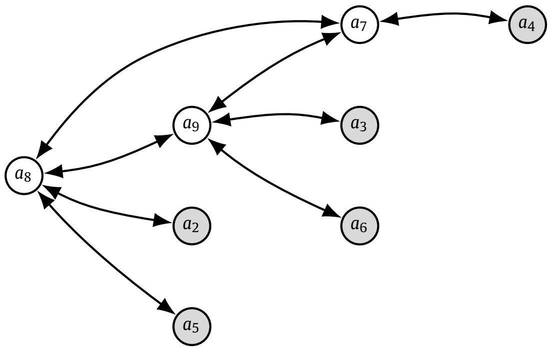

Fig. 5. The argumentation framework for Example 11. Highlighted is a stable extension with inconsistent support.

the non-limited version of the same postulate. These relations, as well as some other relations between the postulates are presented in Fig. 4. Further conditions for relating some of the postulates in Definition 12 are given in [2].

Let us now check the satisfiability of the postulates in Definition 12 (Table 1), in relation to the underlying semantics and the type of the attack rules. In what follows $\mathcal{E}$ denotes a sem-extension, for some sem $\in$ CMP. When the results are restricted to specific types of attacks or semantics, we assume that $\mathcal{E}$ is a sem-extension of frameworks satisfying such restrictions.

Proposition 2 *(support consistency). Frameworks with attacks of type* dir *or* con *satisfy* support consistency *for all the completeness-based semantics:* $\mathsf{Supps}(\mathcal{E})$ *is* $\vdash_{\mathcal{X}}$*-consistent for every complete extension* $\mathcal{E} \in \mathsf{Ext_{cmp}}(\mathcal{AF})$.

Proof. Assume for a contradiction that there is a $\subseteq$-minimal set $\Theta = \{\phi_1, \ldots, \phi_n\} \subseteq \mathsf{Supps}(\mathcal{E})$ for which $\mathcal{X} \vdash \neg\bigwedge\Theta$. Note that $\Theta \setminus \mathcal{X} \neq \emptyset$, since $\mathcal{X}$ is assumed to be $\vdash$-consistent. By the completeness of C, $\mathcal{X} \Rightarrow \neg\bigwedge\Theta$ is C-provable and by Lemma 2, $a = \mathcal{X}, \phi_1, \ldots, \phi_{n-1} \Rightarrow \neg\phi_n$ is C-provable, where without loss of generality, $\phi_n \notin \mathcal{X}$. By the minimality of Θ and the soundness of C, a is not ConUcut-attacked. Since for each $\phi_i \in \Theta$ there is an $a_i \in \mathcal{E}$ for which $\phi_i \in \mathsf{Supp}(a_i)$, any attacker of a is an attacker of some $a_i \in \mathcal{E}$. By the admissibility of $\mathcal{E}$, a is defended by $\mathcal{E}$ and by the completeness of $\mathcal{E}$, $a \in \mathcal{E}$. This contradicts the conflict-freeness of $\mathcal{E}$ since a attacks a_n (according to both DDef and DUcut). $\square$

Note 10. Proposition 2 does not hold for the naive semantics. To see this, note that in the argumentation framework from Example 1, part of which is shown in Fig. 1 (left), the set of arguments $\mathsf{S} = \{a_2, a_3, a_4, a_5, a_6\}$ is a conflict-free set (and is therefore also part of a $\subseteq$-maximal conflict-free set). However, $\mathsf{Supps}(\mathsf{S}) = \{p, q, \neg p \vee \neg q\}$ is not consistent. We will see that the same counterexample also applies to Corollary 2, Proposition 12, Corollary 13, and Proposition 19.

Note 11. A variation of Example 1 can also be used to show that support consistency does not hold for frameworks with attack rules of type set, thus in Proposition 2 it is essential to consider direct attacks. Indeed, let CL be the base logic, $\mathcal{S} = \{p, q, \neg p \vee \neg q\}$, $\mathcal{X} = \emptyset$, and Defeat or Undercut rather than their direct versions is the attack rule. A graphical representation of the framework with the attack rule Defeat (using the same notations for the arguments as in Example 1) is presented in Fig. 5.

Note that $\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{p\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{q\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\neg p \vee \neg q\})$ is a stable (and hence also semi-stable, stage, and preferred) extension, yet $\mathsf{Concs}(\mathcal{E})$ and $\mathsf{Supps}(\mathcal{E})$ are not consistent, neither is $\mathsf{CN_{CL}}(\mathsf{Concs}(\mathcal{E})) \subseteq \mathsf{Concs}(\mathcal{E})$ nor $\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\mathsf{Supps}(\mathcal{E})) \subseteq \mathcal{E}$.

For frameworks with attack rules of type set we get a weaker version of support consistency.

Proposition 3 *(pairwise support consistency). Frameworks with* set*-type attacks satisfy* pairwise support consistency *for all complete extensions: for all* $a, b \in \mathcal{E}$, $\mathsf{Supp}(a) \cup \mathsf{Supp}(b)$ *is* $\vdash_{\mathcal{X}}$*-consistent.*

Proof. Suppose that $\mathcal{E} \in \mathsf{Ext_{cmp}}(\mathcal{AF})$. Assume for a contradiction that there are $a, b \in \mathcal{E}$ for which $\mathsf{Supp}(a) \cup \mathsf{Supp}(b)$ is $\vdash_{\mathcal{X}}$-inconsistent. Hence, by Lemma 2, $a' = \mathsf{Supp}(a) \Rightarrow \neg\bigwedge\mathsf{Supp}(b)$ is derivable. Note that a' has the same attackers as a, and is therefore defended by $\mathcal{E}$. By the completeness of $\mathcal{E}$, $a' \in \mathcal{E}$ as well. Since a' Def-attacks and Ucut-attacks b, this contradicts the conflict-freeness of $\mathcal{E}$. $\square$

Next we show that with respect to single-extension semantics, support consistency holds for any framework in our setting (including those with attack rules of type set).

Lemma 7. *Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})$ be an $\mathcal{S}$-based framework with any type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ of attacks considered in Definition 8, and let $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) = \{\mathcal{E}\}$ for $\mathsf{sem} \in \mathsf{SE}$. Then $\mathcal{E} \subseteq \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}))$.*

Proof. Assume for a contradiction that $\mathsf{Supps}(\mathcal{E})$ contains a minimally $\vdash_{\mathcal{X}}$-inconsistent subset $\{\phi_1, \ldots, \phi_n\}$ of $\mathcal{S}$ (by $\vdash$-compactness, this set is finite). Let $\mathcal{T}$ be a maximal $\vdash_{\mathcal{X}}$-consistent subset of $\mathcal{S}$. Then there is an $1 \leq i \leq n$ for which $\phi_i \notin \mathcal{T} \cup \mathcal{X}$. Note that (a): there is an argument $a \in \mathcal{E}$ with $\phi_i \in \mathsf{Supp}(a)$, and (b): by Lemma 6, $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$. Since $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF})$ and $a \notin \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$, necessarily $a \notin \mathcal{E}$, a contradiction. $\square$

Proposition 4 *(support consistency II). Any framework $\mathcal{AF}$ with* set*-type attacks satisfies* support consistency *for every* $\mathsf{sem} \in \mathsf{SE}$*: If $\mathcal{E}$ is a* sem*-extension of $\mathcal{AF}$, then $\mathsf{Supps}(\mathcal{E})$ is $\vdash_{\mathcal{X}}$-consistent.*

Proof. Follows immediately from Lemma 7. $\square$

Note 12. The condition $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Supps}(\mathcal{E})) = \mathcal{E}$ (for every sem-extension $\mathcal{E}$) is frequently assumed in, e.g., [2] for assuring some rationality postulates (in the case where $\mathcal{X} = \emptyset$). Yet, this condition is not easily verified, and, as Note 11 indicates, it is rather strict, since, in general, frameworks with set-type attacks do not satisfy it for multi-extension semantics.[19] We thus do not assume it for our results (see also Note 15 below).

For showing further results concerning the consistency postulates, we next relate some postulates.

Proposition 5 *(support consistency $\rightarrow$ consistency). Any framework $\mathcal{AF}$ with any type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ of attacks and any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$ *satisfies* consistency *if it satisfies* support consistency*.*

Proof. We show the contraposition. Suppose that consistency does not hold for $\mathcal{AF}$. Then there is a $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ for which $\mathsf{Concs}(\mathcal{E})$ is $\vdash_{\mathcal{X}}$-inconsistent. By compactness, there are $a_1, \ldots, a_n \in \mathcal{E}$ for which $\Theta = \{\mathsf{Conc}(a_1), \ldots, \mathsf{Conc}(a_n)\} \subseteq \mathsf{Concs}(\mathcal{E})$, and $\vdash \neg \bigwedge \Theta$. By the completeness of C, the sequent $\Rightarrow \neg \bigwedge \Theta$ is provable, and by Lemma 4, so is the sequent $\Theta \Rightarrow$. By [Cut], we have that $\mathsf{Supp}(a_1), \ldots, \mathsf{Supp}(a_n) \Rightarrow$ is also provable in C. By Lemma 4 and the soundness of C, $\mathsf{Supps}(\mathcal{E})$ is $\vdash$-inconsistent. $\square$

Proposition 6 *(consistency and closure under support $\rightarrow$ support consistency). Any framework $\mathcal{AF}$ with any type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ of attacks and any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$ *satisfies* support consistency *if it satisfies* consistency *and* closure under support*.*

Proof. We show a contraposition of the statement. Suppose that closure under support is satisfied for $\mathcal{AF}$ and that for some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$, $\mathsf{Supps}(\mathcal{E})$ is $\vdash_{\mathcal{X}}$-inconsistent. By $\vdash$-compactness, there are $a_1, \ldots, a_n \in \mathcal{E}$ for which $\bigcup\{\mathsf{Supp}(a_1), \ldots, \mathsf{Supp}(a_n)\}$ is $\vdash_{\mathcal{X}}$-inconsistent. By closure under support and Lemma 1, $a = \mathsf{Supp}(a_1), \ldots, \mathsf{Supp}(a_n) \Rightarrow \bigwedge_{i=1}^{n} \mathsf{Supp}(a_i) \in \mathcal{E}$. Thus, $\mathsf{Concs}(\mathcal{E})$ is $\vdash_{\mathcal{X}}$-inconsistent. $\square$

The following corollaries are a consequence of Propositions 2, 4 and 5.

Corollary 2 *(consistency). Frameworks with attack rules of type* dir *or* con *satisfy* consistency *for any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$*: For every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$, $\mathsf{Concs}(\mathcal{E})$ is $\vdash_{\mathcal{X}}$-consistent.*

As the example in Note 10 shows, Corollary 2 does not hold for naive semantics.

Corollary 3 *(consistency II). Frameworks with attack rules of type* set *satisfy* consistency *for every semantics* $\mathsf{sem} \in \mathsf{SE}$*: $\mathsf{Concs}(\mathcal{E})$ is $\vdash_{\mathcal{X}}$-consistent for every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$.*

As shown in Note 11, consistency does not hold for frameworks with attack rules of type set under multiple extension semantics. By Note 10, it does not hold for naive semantics as well.

We turn now to (strong) free precedence. First, we need the following lemma.

Lemma 8. *Let $\Gamma \subseteq \mathcal{S}$ and $\Delta \subseteq \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Then Γ is $\vdash_{\mathcal{X}}$-inconsistent iff $\Delta \cup \Gamma$ is $\vdash_{\mathcal{X}}$-inconsistent.*

Proof. Suppose that $\Delta \cup \Gamma$ is $\vdash_{\mathcal{X}}$-inconsistent. Thus, there is a minimally $\vdash_{\mathcal{X}}$-inconsistent $\Theta \subseteq \Delta \cup \Gamma$, and since $\Delta \subseteq \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, $\Theta \subseteq \Gamma$. Thus, Γ is $\vdash_{\mathcal{X}}$-inconsistent. The converse holds by $\vdash$-monotonicity. $\square$

[19] See Corollaries 8 and 9 below for some cases in which this condition is satisfied.

We first show that a *limited version* of free precedence is satisfied by frameworks with direct attacks under any complete semantics:

Proposition 7 *(limited free precedence). Frameworks with attack rules of type* dir *or* con *satisfy* limited free precedence *for any complete semantics* sem $\in$ CMP*: If $\mathcal{E}$ is an extension of such a framework and* $\mathsf{Supps}(\mathcal{E}) \setminus \mathcal{X} \neq \emptyset$*, then* $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})) \subseteq \mathcal{E}$.

Proof. Let $\mathcal{AF}$ be an $\mathcal{S}$-based framework, and let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF})$ with $\mathsf{Supps}(\mathcal{E}) \setminus \mathcal{X} \neq \emptyset$. Then there is an $a' \in \mathcal{E}$ for which there is a $\sigma \in \mathsf{Supp}(a') \setminus \mathcal{X}$. Let $a = \Gamma \Rightarrow \phi \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}))$. Suppose that either $b = \Delta \Rightarrow \psi$ attacks a or a attacks b. Then $\Delta \cup \Gamma$ is $\vdash_{\mathcal{X}}$-inconsistent by Lemma 5 and so Δ is $\vdash_{\mathcal{X}}$-inconsistent by Lemma 8. By Proposition 2, $b \notin \mathcal{E}$. Thus, $\mathcal{E} \cup \{a\}$ is conflict-free. Also, since Δ is $\vdash_{\mathcal{X}}$-inconsistent, by Lemma 2 and [RMon], $c = \Delta \Rightarrow \neg\sigma \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, and by its definition, c attacks a'. Since $a' \in \mathcal{E}$ and $\mathcal{E}$ is admissible, $\mathcal{E}$ defends a' by attacking c, and so $\mathcal{E}$ also attacks b. As a consequence, $\mathcal{E}$ defends a. Thus, $a \in \mathcal{E}$ by the completeness of $\mathcal{E}$. $\square$

To check (unlimited) free-precedence, the next lemma is useful.

Lemma 9. *Let $\mathcal{AF} = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework with attack rules of any type* $\dagger \in \{$dir, con, set$\}$ *considered in Definition 8, and let* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$*. If* $\mathsf{Supps}(\mathcal{E}) \subseteq \mathcal{X}$ *then* $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\emptyset)$ *and all the formulas in $\mathcal{S}$ are $\vdash_{\mathcal{X}}$-inconsistent.*

Proof. By Lemma 6, for any maximal $\vdash_{\mathcal{X}}$-consistent subset $\mathcal{T}$ of $\mathcal{S}$, $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$ and thus $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$. Hence, since $\mathsf{Supps}(\mathcal{E}) \subseteq \mathcal{X}$, by the $\subseteq$-maximality of $\mathcal{E}$, the only maximal $\vdash_{\mathcal{X}}$-consistent subset $\mathcal{T}$ of $\mathcal{S}$ is empty, and so $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\emptyset)$. $\square$

Proposition 8 *(free precedence). Any framework $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ with* dir- *or* con-*type attacks satisfies* free precedence *with respect to every* sem $\in$ ME*: for every* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S}))$, $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})) \subseteq \mathcal{E}$.

Proof. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$. By Proposition 7, we only need to consider the case that $\mathsf{Supps}(\mathcal{E}) \subseteq \mathcal{X}$. By Lemma 9, $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\emptyset)$ and $\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) = \emptyset$. Thus, $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})) = \mathcal{E}$. We have shown, then, that $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) = \{\mathcal{E}\}$ and $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})) = \mathcal{E}$. Hence, we also have that $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})) \subseteq \mathcal{E}'$ for every $\mathcal{E}' \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ (where sem $\in$ ME), since by Proposition 1 it holds that $\mathsf{Ext}_{\mathsf{stg}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$. $\square$

Note 13. For frameworks with only dir-type attack rules (that is, when ConUcut is not one of the attack rules), free precedence does *not* hold for grounded, ideal and eager semantics (thus, strong free precedence does not hold for these semantics either in this case). To see this, let $\mathcal{S} = \{p \vee \neg p, p, \neg p\}$, $\mathcal{X} = \emptyset$, $\mathsf{A} = \{\text{DDef}\}$, and suppose that the underlying logic is CL. In that case, the grounded (which is also the ideal and eager) extension is $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{CL}}(\emptyset)$. Note that $p \vee \neg p \Rightarrow p \vee \neg p$ is attacked by $p, \neg p \Rightarrow \neg(p \vee \neg p)$. In the absence of ConUcut the latter is not attacked by the grounded argument $\Rightarrow \neg(p \wedge \neg p)$. Yet, $\mathsf{Free}^{\emptyset}_{\mathsf{CL}}(\mathcal{S}) = \{p \vee \neg p\}$.

We now show that (unlike the case where there are only dir-type attack rules) frameworks with set or con attack rules satisfy *unlimited* free precedence with respect to every complete semantics.

Proposition 9 *(free precedence II). Any framework $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ whose attack rules are of type* set *or* con *satisfies* free precedence *for every complete semantics* sem $\in$ CMP*, that is:* $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})) \subseteq \mathcal{E}$ *for all* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S}))$.

Proof. Consider an argument $a = \Gamma \Rightarrow \phi$ in $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{AF}))$ and an argument $b = \Delta \Rightarrow \psi$ in $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ that attacks a or is attacked by a. By Lemmas 5 and 8, Δ is $\vdash_{\mathcal{X}}$-inconsistent and so by Lemma 2 (and [RMon] and $\vdash$-compactness), $c = \Theta \Rightarrow \neg \bigwedge \Delta$ is in $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\emptyset)$ for some $\Theta \subseteq \mathcal{X}$. Since c has no attackers it is defended by $\mathcal{E}$ and by the completeness of $\mathcal{E}$, $c \in \mathcal{E}$. Also, c ConUcut-attacks and Def-attacks b. Thus, $\mathcal{E}$ defends a and again by the completeness of $\mathcal{E}$, $a \in \mathcal{E}$. $\square$

By Lemma 7 and Proposition 9, we obtain the following corollary:

Corollary 4 *(strong free precedence). Any framework $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ whose attack rules are of type* set *or* con *satisfies* strong free precedence *with respect to every single extension semantics* sem $\in$ SE*. That is: for every* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S}))$, $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}))$.

Corollary 5 *(SE collapse I). For any $\mathcal{S}$-based framework $\mathcal{AF}$ whose attack rules are of type* set *or* con*, we have:* $\mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{idl}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{egr}}(\mathcal{AF})$.

We now consider a special case under which strong free precedence holds for all complete semantics and any attack-types.

Lemma 10. *Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})$ be a framework with attack rules of type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$. If there are no $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent subsets of $\mathcal{S}$ of size greater than 1 (namely, every formula in $\mathcal{S}$ is either free or inconsistent with $\mathcal{X}$), then $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}))\}$ for all $\mathsf{sem} \in \mathsf{CMP}$.*

Proof. By the supposition we have two types of arguments: (1) those that contain $\vdash_{\mathcal{X}}$-inconsistent formulas in their support, and (2) those whose support is a subset of $\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \cup \mathcal{X}$. Note that where $\Delta, \phi \Rightarrow \psi$ is an argument of type (1) with ϕ being a $\vdash_{\mathcal{X}}$-inconsistent formula, it is attacked by $\Theta \Rightarrow \neg\phi$ of type (2) for some $\Theta \subseteq \mathcal{X}$. We also note that $\Theta \Rightarrow \neg\phi$ has no attackers. That means that all arguments of type (2) are defended by arguments without attackers, while arguments of type (1) cannot be defended. That means every completeness-based semantics has a unique extension consisting of all arguments of type (2). The rest follows immediately. $\square$

Corollary 6 *(strong free precedence II). Let $\mathcal{AF}$ be an $\mathcal{S}$-based framework with attack rules of type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$. $\mathcal{AF}$ satisfies strong free precedence for any complete semantics, if there are no $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent subsets of $\mathcal{S}$ of size greater than 1.*

When strong free precedence is satisfied, several other postulates are met.

Proposition 10 *(strong free precedence $\rightarrow$ closure under support, support consistency, free precedence). Any framework $\mathcal{AF}$ with attack type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ and any complete semantics $\mathsf{sem} \in \mathsf{CMP}$ satisfies* closure under support, support consistency *and* free precedence, *if it satisfies* strong free precedence.

Proof. Immediate from the definitions of the postulates. $\square$

Proposition 11 *(strong free precedence $\rightarrow$ strong exhaustiveness). Any framework $\mathcal{AF}$ with attack type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ and any complete semantics $\mathsf{sem} \in \mathsf{CMP}$ satisfies* strong exhaustiveness *if it satisfies* strong free precedence.

Proof. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and suppose that $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$. By strong free precedence, $\mathsf{Supp}(a) \in \mathsf{CN}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}))$. We note that $\mathsf{CN}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})) \cap \mathcal{S} = \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ and therefore $a \in \mathcal{E}$. $\square$

Next, we consider the closure properties.

Proposition 12 *(closure under support). Any framework $\mathcal{AF} = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A}\rangle$ with attack rules of type* dir *or* con *is* closed under supports *for all the complete semantics $\mathsf{sem} \in \mathsf{CMP}$: If $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$, $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, and $\mathsf{Supp}(a) \subseteq \mathsf{Supps}(\mathcal{E})$, then $a \in \mathcal{E}$.*

Proof. Assume that for $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, $\mathsf{Supp}(a) \subseteq \mathsf{Supps}(\mathcal{E})$. If a is not attacked then obviously $a \in \mathcal{E}$. Suppose that some $b \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ attacks a. By Proposition 2, a is not ConUcut-attacked. Thus, b either DUcut- or DDef-attacks a, and so there is a $\phi \in \mathsf{Supp}(a)$ for which $\mathsf{Conc}(b) \Rightarrow \neg\phi$ is derivable in C. Since $\mathsf{Supp}(a) \subseteq \mathsf{Supps}(\mathcal{E})$, there is a $c \in \mathcal{E}$ for which $\phi \in \mathsf{Supp}(c)$ and so b also attacks c. Since $\mathcal{E}$ is complete, it defends c, thus $\mathcal{E}$ must attack b. It follows that a is also defended by $\mathcal{E}$, and by the completeness of $\mathcal{E}$, $a \in \mathcal{E}$. $\square$

Note 14. Note 10 provides a counterexample to Proposition 12 in case of naive semantics.

By Corollary 4 and Proposition 10 we have:

Corollary 7 *(closure under support II). Any framework $\mathcal{AF} = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A}\rangle$ with attack rules of type* set *is closed under supports for every $\mathsf{sem} \in \mathsf{SE}$: If $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$, $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, and $\mathsf{Supp}(a) \subseteq \mathsf{Supps}(\mathcal{E})$, then $a \in \mathcal{E}$.*

Two other immediate corollaries are the following:

Corollary 8. *For any framework $\mathcal{AF} = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A}\rangle$ with attack rules of type* dir *or* con *and any complete semantics $\mathsf{sem} \in \mathsf{CMP}$, we have: If $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ then $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Supps}(\mathcal{E}))$.*

Corollary 9. *For any framework $\mathcal{AF} = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A}\rangle$ with attack rules of type* set *and any $\mathsf{sem} \in \mathsf{SE}$, we have: If $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ then $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Supps}(\mathcal{E}))$.*

Closure under support implies some other postulates:

Proposition 13 *(closure under support $\rightarrow$ sub-argument closure). Any framework $\mathcal{AF}$ for any attack type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ and any complete semantics $\mathsf{sem} \in \mathsf{CMP}$ satisfies* sub-argument closure, *if it satisfies* closure under support.

Proof. By the definitions of closure under support and sub-argument closure. □

Proposition 14 *(closure under support → closure of extensions). Any framework $\mathcal{AF}$ for any attack type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ and any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$ *satisfies* closure of extensions, *if it satisfies* closure under support.

Proof. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$. To see that $\mathsf{Concs}(\mathcal{E}) \subseteq \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}))$, suppose that $\phi \in \mathsf{Concs}(\mathcal{E})$. By the reflexivity of $\vdash$, $\phi \in \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}))$.

For the converse, suppose that $\phi \in \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}))$. Then there are $a_1, \ldots, a_n \in \mathcal{E}$ with $\Gamma_i = \mathsf{Supp}(a_i)$ and $\phi_i = \mathsf{Conc}(a_i)$ $(1 \leq i \leq n)$ such that $\phi_1, \ldots, \phi_n \vdash \phi$. (Note that n is finite by the compactness of L (Definition 1)). By the completeness of C, $\phi_1, \ldots, \phi_n \Rightarrow \phi$ is C-derivable, and by [Cut] so is $a = \bigcup_{i=1}^{n} \Gamma_i \Rightarrow \phi$. By closure under support, $a \in \mathcal{E}$. □

Corollary 10 *(sub-argument closure). Any framework $\mathcal{AF} = \langle \mathrm{Arg}_{\mathsf{L}}^{\mathcal{X}}(\mathcal{S}), \mathcal{A} \rangle$ with attack rules of type* dir *or* con *is* closed under sub-arguments *for any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$*: For every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and $a \in \mathcal{E}$ it holds that $\mathsf{Sub}(a) \subseteq \mathcal{E}$.*

Proof. By Propositions 12 and 13. □

Proposition 15 *(sub-argument closure II). Any framework $\mathcal{AF} = \langle \mathrm{Arg}_{\mathsf{L}}^{\mathcal{X}}(\mathcal{S}), \mathcal{A} \rangle$ with attack rules of type* set *is* closed under sub-arguments *for any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$*: For every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and $a \in \mathcal{E}$ it holds that $\mathsf{Sub}(a) \subseteq \mathcal{E}$.*

Proof. Let $a \in \mathcal{E}$ and $b \in \mathsf{Sub}(a)$. Suppose that c attacks b. Note that every attacker of b is an attacker of a. (Indeed, the only non-trivial case is Defeat. In this case, let $a = \Delta, \Delta' \Rightarrow \delta'$, $b = \Delta \Rightarrow \delta$ and $c = \Gamma \Rightarrow \gamma$ where $\gamma \Rightarrow \neg \bigwedge \Delta$. By Lemma 2, $\gamma \Rightarrow \neg \bigwedge(\Delta \cup \Delta')$ is C-derivable. Thus, c attacks a.) Thus, b is defended by $\mathcal{E}$ and by the completeness of $\mathcal{E}$, $b \in \mathcal{E}$. □

Corollary 11. *Any framework $\mathcal{AF} = \langle \mathrm{Arg}_{\mathsf{L}}^{\mathcal{X}}(\mathcal{S}), \mathcal{A} \rangle$ with attack rules of type* set *or* dir *or* con *is* closed under sub-arguments *for any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$. *For every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and $a \in \mathcal{E}$ it holds that $\mathsf{Sub}(a) \subseteq \mathcal{E}$.*

Proof. Follows from Corollary 10 and Proposition 15. □

Note 15. Sub-argument closure does not hold for $\mathsf{sem} = \mathsf{nav}$ and any of the types of attacks in Definition 8. To see this, consider an argumentation framework constructed from classical logic and the set $\mathcal{S} = \{p \wedge \neg q, q\}$. Some of the arguments in this case are:

$$a_1 = p \wedge \neg q \Rightarrow p \qquad a_2 = p \wedge \neg q \Rightarrow \neg q \qquad a_3 = p \wedge \neg q \Rightarrow p \wedge \neg q \qquad a_4 = q \Rightarrow q.$$

One possible conflict-free set of arguments is $\{a_1, a_4\}$. Note that $a_2, a_3 \in \mathsf{Sub}(a_1)$. However, a_2 and a_3 (Direct) Defeat a_4, thus $\{a_1, a_2, a_3, a_4\}$ would no longer be conflict-free. By similar arguments, support inclusion (Proposition 12) does not hold either for maximal conflict-free semantics.

In [2] sub-argument closure for naive semantics is shown, under the assumption that the support set of each extension is consistent (note that $\mathcal{S} = \mathsf{Supps}(\{a_1, a_4\})$ is not consistent). To keep our results as general as possible, we do not make that assumption here (see also Note 12).

Proposition 16 *(sub-argument closure → support inclusion). Any framework with any attack type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ and any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$ *satisfies support inclusion if it satisfies sub-argument closure.*

Proof. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and suppose that sub-argument closure holds for $\mathcal{AF}$. Let $\phi \in \mathsf{Supps}(\mathcal{E})$. Then $\phi \in \mathsf{Supp}(b)$ for some $b \in \mathcal{E}$. By Reflexivity, $a = \phi \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}^{\mathcal{X}}(\mathcal{S}) \cap \mathsf{Sub}(b)$. By sub-argument closure, $a \in \mathcal{E}$. Thus, $\phi \in \mathsf{Concs}(\mathcal{E})$. □

Corollary 12 *(support inclusion). Let $\mathcal{AF}$ be an argumentation framework with attack rules of any type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ considered in Definition 8. Then $\mathcal{AF}$ is* closed under support inclusion *for every complete semantics* $\mathsf{sem} \in \mathsf{CMP}$*: If $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ then* $\mathsf{Supps}(\mathcal{E}) \subseteq \mathsf{Concs}(\mathcal{E})$.

Proof. By Corollary 10 and Propositions 15 and 16. □

Corollary 13 *(closure of extensions). Any framework $\mathcal{AF}$ with attack rules of type* dir *or* con *satisfies* closure of extensions *for any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$*: If $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ then* $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}))$.

Proof. By Propositions 12 and 14. □

Note 16. Again, by Note 10, the last corollary does not hold for naive semantics.

Corollary 14 *(closure of extensions II). Any framework $\mathcal{AF}$ with attack rules of type* set *satisfies* closure of extensions *for every* $\mathsf{sem} \in \mathsf{SE}$*: If $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ then $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}))$.*

Proof. By Proposition 14 and Corollary 7. □

Note 17. As shown in Note 11, closure of extensions does not hold for complete semantics and frameworks with attack rules of type set. By Note 10, it also does not hold for naive semantics.

Next, we consider (strong) exhaustiveness. Like free precedence, for assuring strong exhaustiveness for frameworks with dir-type attack rules and completeness-based extensions, one has to consider the limited version of the postulate:

Proposition 17 *(limited strong exhaustiveness). Any framework $\mathcal{AF} = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A}\rangle$ with attack rules of type* dir *or* con *satisfies* limited strong exhaustiveness *for every complete semantics* $\mathsf{sem} \in \mathsf{CMP}$*: For every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, if $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$, and $\mathsf{Supps}(\mathcal{E}) \setminus \mathcal{X} \neq \emptyset$, then $a \in \mathcal{E}$.*

Proof. Let $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ and $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$. Suppose that $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$ and $\mathsf{Supps}(\mathcal{E}) \setminus \mathcal{X} \neq \emptyset$. Suppose further that some $b = \Gamma \Rightarrow \gamma \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ attacks a. Since $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$ and by Corollary 2, b does not ConUCut-attack a. Hence, there is a $\phi \in \mathsf{Supp}(a)$ for which $\gamma \Rightarrow \neg\phi$ and by [Cut], $\Gamma \Rightarrow \neg\phi$ are derivable. Since $\phi \in \mathsf{Concs}(\mathcal{E})$, there is a $c = \Delta \Rightarrow \phi \in \mathcal{E}$. By [Cut] and Lemma 2, $\Gamma, \Delta \Rightarrow$ is derivable. By Proposition 2, $\Gamma \neq \emptyset$.

As $\mathsf{Supps}(\mathcal{E}) \setminus \mathcal{X} \neq \emptyset$, there are $d \in \mathcal{E}$ and $\delta \in \mathsf{Supp}(d) \setminus \mathcal{X}$. By [LMon], $\Gamma, \Delta, \delta \Rightarrow$ is derivable. Let:

$$(\Delta \cup \{\delta\}) \setminus \mathcal{X} \supsetneq \Delta' = \begin{cases} \emptyset & \text{if } \Gamma, \Delta \cap \mathcal{X} \Rightarrow \text{ is derivable,} \\ \text{a } \subseteq\text{-maximal subset of } (\Delta \cup \{\delta\}) \setminus \mathcal{X} & \\ \text{such that } \Gamma, \Delta \cap \mathcal{X}, \Delta' \Rightarrow \text{ is not derivable} & \text{otherwise.} \end{cases}$$

By the definition of Δ', $(\Delta \cup \{\delta\}) \setminus (\Delta' \cup \mathcal{X}) \neq \emptyset$, and $\Gamma, \Delta \cap \mathcal{X}, \Delta', \delta' \Rightarrow$ for any $\delta' \in (\Delta \cup \{\delta\}) \setminus (\Delta' \cup \mathcal{X})$. By [$\Rightarrow\neg$], $b' = \Gamma, \Delta \cap \mathcal{X}, \Delta' \Rightarrow \neg\delta'$. Since b' attacks c or d, some $e \in \mathcal{E}$ attacks b'.

Suppose first that $\Delta' = \emptyset$ and so $b' = \Gamma, \Delta \cap \mathcal{X} \Rightarrow \neg\delta$. If e ConUcut-attacks b', $\mathsf{Conc}(b) = \neg \bigwedge \Gamma'$ for some $\Gamma' \subseteq \Gamma \setminus \mathcal{X}$ and so e also attacks b. If e DDef- or DUcut-attacks b' then $\mathsf{Conc}(b) \Rightarrow \neg\gamma$ for some $\gamma \in \Gamma \setminus \mathcal{X}$ and so e also attacks b.

Otherwise, $\Delta' \neq \emptyset$. Thus, $\Gamma \cup \Delta' \cup (\Delta \cap \mathcal{X})$ is $\vdash$-consistent, and hence there is a $\beta \in (\Gamma \setminus \mathcal{X}) \cup \Delta'$ such that $\mathsf{Conc}(e) \Rightarrow \neg\beta$ is derivable. Hence, e DDef- resp. DUcut-attacks b'. By the conflict-freeness of $\mathcal{E}$, $\beta \in (\Gamma \setminus \mathcal{X})$, since otherwise e attacks c or d. But then e attacks also b.

In both cases $\mathcal{E}$ defends a, and by completeness, $a \in \mathcal{E}$. □

Proposition 18 *(strong exhaustiveness). Any framework $\mathcal{AF} = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A}\rangle$ with attack rules of type* dir *satisfies* strong exhaustiveness *with respect to any semantics* $\mathsf{sem} \in \mathsf{ME}$*: For every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, if $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$ then $a \in \mathcal{E}$.*

Proof. Since $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{stg}}(\mathsf{AF}) \subseteq \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$, we need to show the proposition only for $\mathsf{sem} = \mathsf{prf}$. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$ and let $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ with $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$. By Proposition 17 we only need to consider the case in which $\mathsf{Supps}(\mathcal{E}) \setminus \mathcal{X} = \emptyset$. By Lemma 9, $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\emptyset)$ and all $\phi \in \mathcal{S}$ are $\vdash_{\mathcal{X}}$-inconsistent. Since by Corollary 2, $\mathsf{Concs}(\mathcal{E})$ is $\vdash$-consistent, $\mathsf{Supp}(a) \subseteq \mathcal{X}$, and hence $a \in \mathcal{E}$. □

By Proposition 18 and Note 9, we get:

Corollary 15 *(exhaustiveness). Frameworks with attack relations of type* dir *satisfy* exhaustiveness *with respect to any multiple-extension semantics.*

Note 18. The example in Note 13 illustrates also the failure of exhaustiveness (thus also the failure of strong exhaustiveness) for grounded, complete, ideal and eager extensions for frameworks with dir-type attack relations. Indeed, let $\mathcal{AF}$ be such a framework with the base logic CL. Then $p \vee \neg p \in \mathsf{Concs}(\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\emptyset))$, but $p \vee \neg p \Rightarrow p \vee \neg p \notin \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\emptyset)$. For $\mathsf{sem} \in \{\mathsf{idl}, \mathsf{egr}\}$, note that $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}) = \{\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{p, p \vee \neg p\}), \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\neg p, p \vee \neg p\})\}$. It follows that, for $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$, $\mathcal{E} \subseteq \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{p \vee \neg p\})$. However, the argument $p \vee \neg p \Rightarrow p \vee \neg p$ is not defended against the argument $p, \neg p \Rightarrow \neg(p \vee \neg p)$ in $\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{p \vee \neg p\})$.

Proposition 19 *(strong exhaustiveness II). Frameworks with attack rules of type* con *satisfy* strong exhaustiveness *for any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$*: If $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$ then $a \in \mathcal{E}$.*

Proof. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ with $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$. By Proposition 17 we only need to consider the case in which $\mathsf{Supps}(\mathcal{E}) \subseteq \mathcal{X}$. Consider an argument $b = \Gamma \Rightarrow \gamma \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ that attacks a. Since $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$ and by

Corollary 2, b does not ConUCut-attack a. By [Cut], $\Gamma \Rightarrow \neg\phi$ is derivable for a $\phi \in (\mathsf{Supp}(a) \setminus \mathcal{X})$. Since $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$, we have that $\phi \in \mathsf{Concs}(\mathcal{E})$, and so there is a $c = \Theta \Rightarrow \phi \in \mathcal{E}$ for some finite $\Theta \subseteq \mathcal{X}$. By [Cut] and Lemma 2, $\Theta, \Gamma \Rightarrow$ and $d = \Theta, \Gamma \cap \mathcal{X} \Rightarrow \neg\bigwedge(\Gamma \setminus \mathcal{X})$ are derivable, and since $\mathcal{X}$ is $\vdash$-consistent, we get that $(\Gamma \setminus \mathcal{X}) \neq \emptyset$. Since d has no attackers, by the completeness of $\mathcal{E}$, $d \in \mathcal{E}$. Since d ConUcut-attacks b, it defends a from b. Altogether, $\mathcal{E}$ defends a and by completeness, $a \in \mathcal{E}$. $\square$

Note 19. Note 10 provides a counterexample to Proposition 19 in case of naive semantics.

By Proposition 19 and Note 9 we get:

Corollary 16 *(exhaustiveness II). Frameworks with attack rules of type* con *satisfy* exhaustiveness *with respect to every complete semantics* sem $\in$ CMP*: If* $\mathsf{Supp}(a) \cup \{\mathsf{Conc}(a)\} \subseteq \mathsf{Concs}(\mathcal{E})$ *for some* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ *and* $a \in Arg_{\mathsf{L}}^{\mathcal{X}}(\mathcal{S})$*, then* $a \in \mathcal{E}$.

By Corollary 4 and Proposition 11 we have:

Proposition 20 *(strong exhaustiveness III). Frameworks with attack rules of type* set *satisfy* strong exhaustiveness *for every semantics* sem $\in$ SE*: If* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ *and* $\mathsf{Supp}(a) \subseteq \mathsf{Concs}(\mathcal{E})$ *then* $a \in \mathcal{E}$.

Corollary 17 *(exhaustiveness III). Frameworks with attack rules of type* set *satisfy* exhaustiveness *with respect to every semantics* sem $\in$ SE*: If* $\mathsf{Supp}(a) \cup \{\mathsf{Conc}(a)\} \subseteq \mathsf{Concs}(\mathcal{E})$ *for some* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ *and* $a \in Arg_{\mathsf{L}}^{\mathcal{X}}(\mathcal{S})$*, then* $a \in \mathcal{E}$.

As shown in Note 11 above, (limited) exhaustiveness does not hold for argumentation frameworks with non-direct attacks.

Table 2 summarizes the results in this section. It indicates the properties of individual extensions of argumentation frameworks in relation to the type of the attack rules and the semantics classes. For instance, it shows that (strong) exhaustiveness of extensions is satisfied with respect to every completeness-based extension when using con-type attack rules, but for dir-type (respectively, set-type) attack rules, this property is met only with respect to multiple-extension (respectively, single-extension) semantics.

Table 2
Properties of extensions according to the type of attack rules and the semantics classes.

Properties of individual extensions	dir-attacks	con-attacks	set-attacks
closure of extensions	CMP	CMP	SE
closure under support	CMP	CMP	SE
sub-argument closure	CMP	CMP	CMP
consistency	CMP	CMP	SE
support consistency	CMP	CMP	SE
pairwise support consistency	CMP	CMP	CMP
exhaustiveness	ME	CMP	SE
limited exhaustiveness	CMP	CMP	SE
strong exhaustiveness	ME	CMP	SE
limited strong exhaustiveness	CMP	CMP	SE
support inclusion	CMP	CMP	CMP
free precedence	ME	CMP	CMP
strong free precedence[a]	–	SE	SE
limited free precedence	CMP	CMP	CMP

[a] Strong free precedence holds for every sem $\in$ CMP and attack rules of type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$, if there are no $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent subsets of the defeasible premises of size greater than 1.

The results in Table 2 show that in general the set-based attack rules are inferior to the direct attack rules (with or without ConUcut), since for frameworks with set-based attack rules some important properties, like consistency, closure, and exhaustiveness of the extensions, cannot be guaranteed unless the semantics is a single extension one. In contrast, all the properties of Definition 12 (excluding strong free precedence) can be assured for frameworks with direct attacks under either the stable, semi-stable, preferred, or stage semantics. When ConUcut is also part of the attack rules, these properties can be guaranteed for all the complete semantics.

The analysis above should be taken with care, though, for several reasons. First, as we shall see in what follows, the entailment relations induced by direct attacks are preferential and rational (Definition 18) for a larger set of semantics than the entailments induced by subset attacks (see Section 6.1), while for some postulates (e.g., non-interference) subset-based attacks have some advantages over direct attacks (see Section 6.2). Second, this analysis *cannot necessarily* be applied to base logics that are not explosive or not contrapositive (recall Corollary 1). For such logics we note the following:

1. Not all the attack rules considered here (in Definition 4) may be applicable. For instance, while reflexivity, [$\Rightarrow\neg$] and [$\Rightarrow\wedge$] assure that ConUcut is applicable in case that all the rules in Fig. 2 are admissible, this is not guaranteed in other cases. For instance, in Dunn-Belnap's logic of first-degree entailments (FDE) [22,42] no sequent with an empty left-hand side is valid (i.e., derivable in a sound and complete sequent calculus for FDE), thus ConUcut is not applicable in a setting with FDE as the base logic and $\mathcal{X} = \emptyset$.
2. Even in cases that the attack rules are applicable, they may yield results that are considerably different than those that are obtained with logics that meet the conditions in Section 2.2. For instance, in Example 1, using the same set $\mathcal{S}$ and the same attack rule, but when Priest's 3-valued logic LP [67,68] is the base logic (instead of CL), both a_2 and a_3 are not attacked, simply because a_8 and a_9 (respectively) are not derivable (the Disjunctive Syllogism does not hold in LP).

3.2. Postulates concerning sets of extensions and the collapse of the classes

We now turn to properties that refer to *sets* of extensions of the same type (that is, sets of the same $\mathsf{sem} \in \{\mathsf{nav}, \mathsf{stg}, \mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{idl}, \mathsf{stb}, \mathsf{sstb}, \mathsf{egr}\}$). As a byproduct of establishing these properties we obtain the following theorem, which expresses that all multiple-extension, respectively all single-extension semantics, coincide in the logic-based setting studied in this paper (recall Fig. 3).

Theorem 1. *Let $\mathcal{AF}$ be a sequent-based argumentation framework with attack rules of type* dir, con, *or* set. *Then:*

1. $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{stg}}(\mathcal{AF})$, *and*
2. $\mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{idl}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{egr}}(\mathcal{AF})$.

Proof. Item 1 follows by Corollary 19 (below) and Item 2 by Corollaries 5 and 20 (below). $\square$

Definition 13 *(rationality postulates II).* Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S}) = \left\langle \mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \right\rangle$ be an argumentation framework and sem a semantics for it. In the remainder of this section we consider the postulates in Table 3.[20]

Table 3
Rationality postulates for sets of extensions.

Property	Definition
maximal consistency	$\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) = \{\mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \mid \mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\}$.
weak maximal consistency	$\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) \supseteq \{\mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \mid \mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\}$.
stability	$\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) \neq \emptyset$.
strong stability	$\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S}))$.

Note 20. The postulates in Table 3 refer to some properties of the extensions of a framework, given some semantics.

- Maximal consistency requires that the set of the sem-extensions coincides with the set of the maximal consistent subsets of the premises. Although the link between logical argumentation and reasoning with maximal consistency is well-known (see, e.g. [6,7,10,32,76]) we note that this requirement is a rather strong demand (see a discussion on this in [7, Section 7.1], pointing out that this requirement is violated in several settings). This is one of the reasons for introducing in Definition 13 a weaker postulate, weak maximal consistency.
- Stability and strong stability refer to the existence of stable extensions, a property that does not hold for every argumentation framework.[21] These properties will be useful for the crash-resistance result in Section 6.
- When strong stability is satisfied for preferred semantics, we have that

$$\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) = \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) = \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})),$$

since (see [30, Theorems 2 and 3]) it always holds that

$$\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) \subseteq \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) \subseteq \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})).$$

Stability follows immediately from Lemma 6. However, we will show that the considered frameworks also satisfy strong stability. For this we need the following lemma and proposition.

[20] For these postulates we need some notions from Definition 11.
[21] When sem = prf, strong stability is sometimes called coherence; see [44, Def. 31].

Lemma 11. *Any argumentation framework $\mathcal{AF}$, based on any attack type* $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ *and any complete semantics* $\mathsf{sem} \in \mathsf{CMP}$, *satisfies* $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}) \subseteq \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \mid \mathcal{T} \in \mathsf{CS}^{\mathcal{X}}_{\mathcal{L}}(\mathcal{S})\}$, *if it satisfies* support consistency *and* closure under support.

Proof. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$. By closure under support, $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Supps}(\mathcal{E}))$. By support consistency, $\mathsf{Supps}(\mathcal{E}) \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. □

Proposition 21 *(support consistency & closure under support → strong stability, maximal consistency). Any $\mathcal{AF}$ based on any attack type* $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ *and any semantics* $\mathsf{sem} \in \mathsf{ME}$ *satisfies* strong stability *and* maximal consistency, *if it satisfies* support consistency *and* closure under support.

Proof. By Lemma 6, for any $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$. Since $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ for all $\mathsf{sem} \in \{\mathsf{stg}, \mathsf{stb}, \mathsf{sstb}, \mathsf{prf}\}$, also $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$.

Let now $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$. By Lemma 11, $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$, where $\mathcal{T} = \mathsf{Supps}(\mathcal{E}) \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. By Lemma 6, $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}') \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$, where $\mathcal{T}' \supseteq \mathcal{T}$ is a maximal $\vdash_{\mathcal{X}}$-consistent superset of $\mathcal{T}$. By the $\subseteq$-maximality of $\mathcal{E}$, $\mathcal{T}' = \mathcal{T}$. Since $\mathsf{Ext}_{\mathsf{stg}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$ this completes the proof. □

Corollary 18 *(strong stability and maximal consistency). If $\mathcal{AF} = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{A}\rangle$ is an argumentation framework whose attack rules are of type* dir *or* con, *then for every semantics* $\mathsf{sem} \in \mathsf{ME}$ *it holds that* $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \mid \mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\}$.

Proof. Follows directly from Propositions 2, 12 and 21. □

Concerning frameworks with set-type attacks, we have strong stability (but not maximal consistency – see Note 11):

Proposition 22 *(strong stability II). Frameworks with attack rules of type* set *satisfy* strong stability *for any* $\mathsf{sem} \in \mathsf{ME}$: $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$.

Proof. By [40, Lemma 15], every stable extension is preferred. Moreover, every stable extension is semi-stable (trivially) and by Proposition 1 every stable extension is also stage. Thus, we only need to show that every preferred extension is stable. For this, let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF})$. If $\mathcal{E} \notin \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$ then there is an $a = \Delta \Rightarrow \delta \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \setminus \mathcal{E}$ and $\mathcal{E}$ does not attack a. We show towards a contradiction that $\mathcal{E}' = \mathcal{E} \cup \mathsf{Sub}(a)$ is admissible. Note first that since $\mathcal{E}$ does not attack a it also does not attack any argument in $\mathsf{Sub}(a)$ and no argument in $\mathsf{Sub}(a)$ attacks an argument in $\mathcal{E}$. So $\mathcal{E}'$ is conflict-free. Suppose that some argument b attacks some $a' \in \mathsf{Sub}(a)$. Thus, $\mathsf{Conc}(b) \Rightarrow \neg\bigwedge\Delta'$ is C-derivable for some $\Delta' \subseteq \mathsf{Supp}(a')$. By [Cut], $\mathsf{Supp}(b) \Rightarrow \neg\bigwedge\Delta'$ is C-derivable. By Lemma 2, $\Delta' \Rightarrow \neg\bigwedge\mathsf{Supp}(b)$ is C-derivable and so $\mathcal{E}'$ attacks b. But then $\mathcal{E}'$ is admissible which is a contradiction to our assumption. Thus, $\mathcal{E} \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$. □

In view of the strong stability shown in Corollary 18 and Proposition 22, we get:

Corollary 19 *(ME collapse). If $\mathcal{AF}$ is an argumentation framework whose attack rules are of type* dir, con *or* set, *then* $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) = \mathcal{E}_{\mathsf{sstb}}(\mathcal{AF}) = \mathcal{E}_{\mathsf{stg}}(\mathcal{AF}) = \mathcal{E}_{\mathsf{stb}}(\mathcal{AF})$.

We now generalize SE-collapse (Corollary 5) to all attack-forms (completing Fig. 3 and the proof of Theorem 1). For this, we first need the next lemma.

Lemma 12. *Let $\mathcal{AF}$ be an $\mathcal{S}$-based framework with attack rules of type* dir. *If there is a $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent subset of $\mathcal{S}$ of size greater than 1, then* $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\emptyset)\}$ *for all* $\mathsf{sem} \in \mathsf{SE}$.

Proof. Let $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}) = \{\mathcal{E}\}$ and $\mathsf{sem} \in \{\mathsf{egr}, \mathsf{idl}, \mathsf{grd}\}$. Assume there is a $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent subset $\Theta = \{\phi_1, \ldots, \phi_n\}$ of $\mathcal{S}$, where $n \geq 2$. (By compactness this set is finite). Suppose also that for some $\Delta \subseteq \mathcal{S} \cup \mathcal{X}$ for which $\Delta \setminus \mathcal{X} \neq \emptyset$, it holds that $\Delta \Rightarrow \phi \in \mathcal{E}$. Let $\delta \in \Delta \setminus \mathcal{X}$. Then, $\Theta \Rightarrow \neg\delta$ attacks $\Delta \Rightarrow \phi$. Thus, there is a $\Gamma \Rightarrow \psi \in \mathcal{E}$ that attacks $\Theta \Rightarrow \neg\delta$, where $\psi \Rightarrow \neg\phi_i$ is C-derivable for some $i \in \{1, \ldots, n\}$. By [Cut], $\Gamma \Rightarrow \neg\phi_i$ is C-derivable. So, $(\Gamma \setminus \mathcal{X}) \cup \{\phi_i\}$ is $\vdash_{\mathcal{X}}$-inconsistent. Note that $\{\phi_i\}$ is consistent since Θ is $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent and $n \geq 2$. So, there is a $\subseteq$-maximal $\vdash_{\mathcal{X}}$-consistent subset $\Gamma' \cup \{\phi_i\}$ of $(\Gamma \setminus \mathcal{X}) \cup \{\phi_i\}$, where $\Gamma' = \{\gamma_1, \ldots, \gamma_m\}$. Note that there is a $\gamma \in \Gamma \setminus (\mathcal{X} \cup \Gamma')$ and $\Gamma', \phi_i \Rightarrow \neg\gamma$ attacks $\Gamma \Rightarrow \psi$. By Corollary 18, there is a $\mathcal{E}' \in \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF})$ with $\Gamma', \phi_i \Rightarrow \neg\gamma \in \mathcal{E}'$. But then, by the definition of eager and ideal semantics, $\Gamma \Rightarrow \psi \notin \mathcal{E}$. This is a contradiction to our assumption on $\Gamma \Rightarrow \psi$. Hence, $\mathcal{E} = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\emptyset)$. □

Corollary 20 *(SE collapse II). For any $\mathcal{S}$-based framework $\mathcal{AF}$ whose attack rules are of type* dir *we have* $\mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{idl}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{egr}}(\mathcal{AF})$.

Proof. If there is a $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent subset of $\mathcal{S}$ of size greater than 1, the corollary follows by Lemma 12. Otherwise it follows by Lemma 10. □

Note 21. Maximal consistency does not hold for set-type attack rules (recall Note 11) and also it is violated in frameworks with the other types of attack rules when the semantics is not preferred, stage, stable, or semi-stable:

- For $\mathsf{sem} = \mathsf{nav}$ this follows from the example in Note 10, where the set of argument S considered there is contained in a naive extension, and $\mathsf{Supps}(\mathsf{S}) = \{p, q, \neg p \vee \neg q\}$.
- For $\mathsf{sem} \in \mathsf{SE}$ this follows immediately since these semantics result in a single extension, while $\mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$ may contain several maximally consistent subsets.

In the next proposition we therefore consider a weaker version of maximal consistency.

Proposition 23 *(weak maximal consistency). Frameworks with attack rules of type* $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ *satisfy* weak maximal consistency *with respect to any semantics* $\mathsf{sem} \in \mathsf{ME} \cup \{\mathsf{cmp}\}$*: It holds that* $\{\mathit{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \mid \mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\} \subseteq \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$.

Proof. Follows from Lemma 6, since $\mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}) = \mathsf{Ext}_{\mathsf{stg}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{sstb}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) \subseteq \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF})$. □

The results of this section are summarized in Table 4 (The sign ✓ indicates that the property always holds).

Table 4
Summary of the results in Section 3.2.

Properties of extensions sets	dir-attacks	con-attacks	set-attacks
maximal consistency	ME	ME	–
weak maximal consistency	$\mathsf{ME} \cup \{\mathsf{cmp}\}$	$\mathsf{ME} \cup \{\mathsf{cmp}\}$	$\mathsf{ME} \cup \{\mathsf{cmp}\}$
stability	✓	✓	✓
strong stability	ME	ME	ME

We conclude this section by an interesting corollary of the results in Sections 3.1 and 3.2:

Theorem 2. *All the postulates in Definition 12 (Table 1) and Definition 13 (Table 3), except for strong free precedence, are compatible (i.e., they can be mutually satisfied).*

Proof. As shown in Tables 2 and 4, when $\mathsf{sem} \in \mathsf{ME}$, all of these postulates, excluding strong free precedence, are satisfied by *every* framework whose attack rules are of type con. □

4. Some illustrations

In this section we consider two further examples, based on non-classical core logics, that illustrate some of the results in the previous section.

Example 5. Consider again the argumentation framework in Example 2, but this time where the base logic is intuitionistic logic (IL). For this, one has to replace the sequent calculus accordingly, e.g., trade LK by its single-conclusion counterpart LJ (see [47, page 192]). Clearly, this has far-reaching consequences on the arguments that can be constructed from the premises $\mathcal{S} = \{p, \neg p, q\}$. Yet, this change *does not affect* the properties of the extensions of the underlying framework, neither the properties of the induced entailments relations. For instance, in Example 2 we have argued that $q \Rightarrow q$ belongs to every complete extension of the (original) framework, since it is defended by $\Rightarrow p \vee \neg p$. Now, while the latter is not derivable in LJ anymore, we still can derive $\Rightarrow \neg(p \wedge \neg p)$, which in turn defends $q \Rightarrow q$ against an attack from $p, \neg p \Rightarrow \neg q$. Moreover, in this case we have that $\mathsf{MCS}^{\emptyset}_{\mathsf{IL}}(\mathcal{S}) = \{\{p, q\}, \{\neg p, q\}\}$ and $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\emptyset}_{\mathsf{IL}}(\mathcal{S})) = \{\mathsf{Arg}^{\emptyset}_{\mathsf{IL}}(\{p, q\}), \mathsf{Arg}^{\emptyset}_{\mathsf{IL}}(\{\neg p, q\})\} = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\emptyset}_{\mathsf{IL}}(\mathcal{S}))$, thus properties such as strong stability remain valid despite the change of the base logic.

The next example (a variation of [74, Example 3]) demonstrates the use of the modal logic S4 as the core logic of a framework, and – more generally – the use of modal languages, which allow to qualify statements with modal operators for expressing alethic arguments (concerning *necessity* and *possibility*), epistemic ones (concerning *knowledge* and *belief*) and deontic assertions (concerning *obligation* and *permission*).[22] A sequent calculus for S4 may be obtained by adding the following rules to LK:

$$[\Box\Rightarrow] \quad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Box\phi \Rightarrow \Delta} \qquad\qquad [\Rightarrow\Box] \quad \frac{\Box\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}$$

[22] In all of these contexts there are various applications in which distinguishing between strict and defeasible assumptions is useful. For instance, prima facia norms in deontic logic are often modeled as defeasible assumptions (e.g., in constrained input-output logic [60]), alethic arguments may be based on ceteris paribus regularities, beliefs on default assumptions, etc.

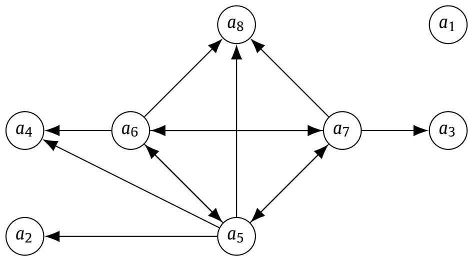

Fig. 6. A representation of the framework of Example 6. Note that the argument a_8 with conclusion $\neg p$ does *not* directly defeat arguments that contain p as a support (such as a_1, a_3, etc.), since p is a strict assumption.

Example 6. Let $\mathcal{S} = \{q,\ p \supset \Box r,\ q \supset \Box\neg r\}$ and $\mathcal{X} = \{p\}$. Some of the arguments in $\mathrm{Arg}^{\mathcal{X}}_{\mathsf{S4}}(\mathcal{S})$ are the following:

$$\begin{array}{ll} a_1 = p \Rightarrow p & a_4 = q, q \supset \Box\neg r \Rightarrow \Box\neg r \\ a_2 = q \Rightarrow q & a_5 = p, p \supset \Box r, q \supset \Box\neg r \Rightarrow \neg q \\ a_3 = p, p \supset \Box r \Rightarrow \Box r & a_6 = p, q, p \supset \Box r \Rightarrow \neg(q \supset \Box\neg r) \\ a_7 = p, q, q \supset \Box\neg r \Rightarrow \neg(p \supset \Box r) & a_8 = q, p \supset \Box r, q \supset \Box\neg r \Rightarrow \neg p \end{array}$$

Fig. 6 is a graphical representation of (part of) the argumentation framework for the above setting with direct defeat as the sole attack rule. Fig. 6 is a graphical representation of (part of) the argumentation framework for the above setting with direct defeat as the sole attack rule.

The preferred extensions in this case are:

$$\mathrm{Arg}^{\{p\}}_{\mathsf{S4}}(\{q, p \supset \Box r\}), \quad \mathrm{Arg}^{\{p\}}_{\mathsf{S4}}(\{q, q \supset \Box\neg r\}), \quad \mathrm{Arg}^{\{p\}}_{\mathsf{S4}}(\{p \supset \Box r, q \supset \Box\neg r\}).$$

These extensions are also the stable extensions. Also,

$$\mathsf{MCS}^{\{p\}}_{\mathsf{S4}}(\mathcal{S}) = \{\{q, p \supset \Box r\}, \{q, q \supset \Box\neg r\},\ \{p \supset \Box r, q \supset \Box\neg r\}\}.$$

This corresponds to Proposition 2, Corollary 2, and Proposition 18.

5. Characterization results

We now use the results in Section 3 for characterizing the semantic extensions of the argumentation frameworks from Definition 5 (Section 5.1) and the entailment relations induced by them (Section 5.2).

5.1. Characterizations of the semantic extensions

During the investigations of rationality postulates for different argumentation frameworks, we have characterized the extensions of some types of attacks and particular semantics (see Corollaries 5, 19 and 20). In this section we recall these results and extend them to all the attack types in Definition 8 and all the semantics in Definition 6.

The next definition is required for characterizing all the extensions of a logic-based argumentation framework and the entailment relations that are induced from it, which are not represented by maximally consistent subsets (MCS) of the premises (see Proposition 24 and Theorem 5 below).

Definition 14 *($\Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$).* Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ and $\mathcal{X}$ be two disjoint sets of $\mathcal{L}$-formulas, such that $\mathcal{X}$ is $\vdash$-consistent. We denote by $\Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \subseteq \wp(\wp(\mathcal{S}))$ the set of subsets of $\wp(\mathcal{S})$, where for every $\omega \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ the following two requirements are satisfied:

1. the elements of ω are pairwise $\vdash_{\mathcal{X}}$-consistent: $\mathcal{T}_i \cup \mathcal{T}_j$ is $\vdash_{\mathcal{X}}$-consistent for every $\mathcal{T}_i, \mathcal{T}_j \in \omega$.
2. for every finite set $\Theta \in \wp(\mathcal{S})$ there is a set $\mathcal{T} \in \omega$ such that either $\Theta \subseteq \mathcal{T}$ or $\Theta \cup \mathcal{T}$ is $\vdash_{\mathcal{X}}$-inconsistent.

For $\omega \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, we let $\mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega) = \bigcup_{\mathcal{T} \in \omega} \mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$.

Intuitively, $\Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ contains all subsets ω of $\wp(\wp(\mathcal{S}))$ which only contain pairwise consistent subsets of $\mathcal{S}$ and that satisfy the following closure property: for any finite subset Θ of $\mathcal{S}$ for which no superset is contained in ω, there is a set Θ' in ω that is inconsistent with Θ.

We are ready to characterize extension-based semantics in logical argumentation frameworks. The following theorem states that for nearly all combinations of multiple and single extension semantics with attack rules in one of the three classes $\mathsf{set}, \mathsf{dir}$ and con, the resulting extensions of an argumentation framework can be characterized by a set $\mathrm{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$

where $\mathcal{T}$ is an appropriate consistent subset of the given defeasible assumptions. The exception is the combination of multiple extension semantics with attack rules in set. In this case, extensions are characterized by sets $\bigcup_{\mathcal{T}\in\omega} \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$ where for each $\mathcal{T}, \mathcal{T}' \in \omega$, $\mathcal{T} \cup \mathcal{T}'$ is a consistent set of defeasible assumptions.

Theorem 3 *(characterization of extensions). Given an argumentation framework $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ in which attack rules that are of type $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$. It holds that:*

1. $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \mid \mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\}$ *where* $\dagger \in \{\mathsf{dir}, \mathsf{con}\}$ *and* $\mathsf{sem} \in \mathsf{ME}$.
2. $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega) \mid \omega \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\}$ *where* $\dagger = \mathsf{set}$ *and* $\mathsf{sem} \in \mathsf{ME}$.
3. $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}))\}$ *where* $\dagger \in \{\mathsf{con}, \mathsf{set}\}$ *and* $\mathsf{sem} \in \mathsf{SE}$.
4. $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})\}$ *where* $\dagger = \mathsf{dir}$, $\mathsf{sem} \in \mathsf{SE}$, $\mathcal{S}_{\top} = \{\phi \in \mathcal{S} \mid \phi \text{ is } \vdash_{\mathcal{X}}\text{-consistent}\}$*, and where* $\mathcal{T} = \mathcal{S}_{\top}$ *in case that* $\mathcal{S}_{\top}$ *is* $\vdash_{\mathcal{X}}$*-consistent and* $\mathcal{T} = \emptyset$ *otherwise.*

Proof. Item 1 is maximal consistency (Corollary 18), Item 2 is shown in Proposition 24 (below), and Item 3 is strong free precedence (Corollary 4). If there is a $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent subset of $\mathcal{S}$ of size greater than 1, Item 4 follows by Lemma 12. Otherwise it follows by Lemma 10. □

Proposition 24. *Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ be an argumentation framework in which the attack rules are of type* set*. Then, for every* $\mathsf{sem} \in \mathsf{ME}$ *it holds that* $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega) \mid \omega \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\}$.

Proof. By strong stability (Proposition 22) we only need to show this for stable extensions.

Let $\omega \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. We first show that $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega)$ is conflict-free. Suppose for a contradiction that $a = \Gamma \Rightarrow \gamma, b = \Delta \Rightarrow \delta \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega)$, and a attacks b. By Lemma 5, $\Gamma \cup \Delta$ is $\vdash$-inconsistent. Since there are $\mathcal{T}_i, \mathcal{T}_j \in \omega$ for which $\Gamma \subseteq \mathcal{T}_i$ and $\Delta \subseteq \mathcal{T}_j$, this implies that $\mathcal{T}_i \cup \mathcal{T}_j$ is $\vdash$-inconsistent, which contradicts the first condition in Definition 14. Thus, $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega)$ is conflict-free.

Let now $a = \Gamma \Rightarrow \gamma \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})) \setminus \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega)$. Then there is some $\mathcal{T} \in \omega$ and some $\Theta \subseteq \mathcal{T}$ such that $\vdash_{\mathcal{X}} \neg \bigwedge(\Gamma \cup \Theta)$, and hence $b = \Theta', \Theta \Rightarrow \neg \bigwedge \Gamma \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega)$ for some $\Theta' \subseteq \mathcal{X}$. Thus b Def-attacks or Ucut-attacks a. This shows that $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$.

Let now $\mathcal{E} \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$. By sub-argument closure (Corollary 11), there is a set $\omega = \{\mathcal{T}_i \mid i \in I\} \subseteq \wp(\mathcal{S})$ for which $\mathcal{E} = \bigcup_{i\in I} \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}_i) = \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\omega)$. We show Items 1 and 2 of Definition 14.

1. Assume for a contradiction that there are $i, j \in I$ for which $\mathcal{T}_i \cup \mathcal{T}_j$ is $\vdash_{\mathcal{X}}$-inconsistent. Thus, there are $\Theta_i \subseteq \mathcal{T}_i$ and $\Theta_j \subseteq \mathcal{T}_j$ such that $\vdash_{\mathcal{X}} \neg \bigwedge(\Theta_i \cup \Theta_j)$. By Lemma 2, $a = \Theta_i, \Theta \Rightarrow \bigwedge \Theta_j \in \mathcal{E}$ for some $\Theta \subseteq \mathcal{X}$, in contradiction to pairwise support consistency (Proposition 3).
2. Suppose for a contradiction that there is a set $\Theta \in \wp(\mathcal{S})$ such that $\Theta \nsubseteq \mathcal{T}_i$ and $\Theta \cup \mathcal{T}_i$ is consistent for all $i \in I$. Let $a = \Theta \Rightarrow \bigwedge \Theta \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Note that $a \notin \mathcal{E}$. By the stability of $\mathcal{E}$, there is an argument $b = \Delta \Rightarrow \delta \in \mathcal{E}$ that attacks a. Since $b \in \bigcup_{i\in I} \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}_i)$, there is an $i \in I$ for which $\Delta \subseteq \mathcal{T}_i$. But then $\Theta \cup \mathcal{T}_i$ is $\vdash_{\mathcal{X}}$-inconsistent by Lemma 5, a contradiction to our assumption. □

5.2. Characterizations of the induced entailments

We now turn to characterizations of the entailment relations that are induced by logical argumentation frameworks according to Definition 7. Like before, we consider the three types of argumentation frameworks from Definition 8, based on a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ with a corresponding sound and complete sequent calculus C, in which the rules from Fig. 2 are admissible.

Since the results in this section refer to *families* of argumentation frameworks and not to specific frameworks, in what follows we shall somewhat modify the notations of the entailments in Definition 7 and sometimes use the abbreviation $\mid\!\sim^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$, where $\dagger \in \{\mathsf{set}, \mathsf{dir}, \mathsf{con}\}$, $\mathsf{sem} \in \mathsf{CMP} = \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{idl}, \mathsf{stb}, \mathsf{sstb}, \mathsf{egr}, \mathsf{stg}\}$ and $\star \in \{\cap, \Cap, \cup\}$. Thus, for instance, a result that is applied to $\mid\!\sim^{\cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ (that is, when $\star = \cap$ and $\dagger = \mathsf{dir}$) covers, in terms of Definition 7, any skeptical entailment relation of the form $\mid\!\sim^{\cap}_{\mathsf{L},\mathsf{A},\mathcal{X},\mathsf{sem}}$ in which $\emptyset \neq \mathsf{A} \subseteq \{\mathsf{DDef}, \mathsf{DUcut}\}$ (recall Definition 8). The latter, in turn, covers any instance of $\mathsf{sem} \in \mathsf{CMP}$.

In these notations, Theorem 1 shows that for every fixed $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ and $\star \in \{\cap, \Cap, \cup\}$, we actually have two different argumentative entailments of the form $\mid\!\sim^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$: one for $\mathsf{sem} \in \mathsf{ME}$ and the other one for $\mathsf{sem} \in \mathsf{SE}$. This is formalized in the next corollary.

Corollary 21. *Let $\mathcal{X}$ be a set of formulas,* $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$ *and* $\star \in \{\cap, \Cap, \cup\}$*. For every* $\mathsf{sem} \in \mathsf{ME}$ *the entailments* $\mid\!\sim^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ *are the same. Likewise, for all* $\mathsf{sem} \in \mathsf{SE}$, $\mid\!\sim^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ *coincide.*

Note 22. To simplify the presentation, i.e., in order to avoid too many case distinctions, we have not listed cmp among the semantics in SE and in ME. However, we note that $\mid\!\sim^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{cmp}} = \mid\!\sim^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ either when $\star = \cup$ and $\mathsf{sem} \in \mathsf{ME}$, or when $\star \in \{\cap, \Cap\}$ and $\mathsf{sem} \in \mathsf{SE}$. Thus, implicitly, reasoning with $\mathsf{sem} = \mathsf{cmp}$ is covered (here and in what follows) as well.

Our next result (Theorem 4) shows a correspondence between many argumentative entailments and inference by maximally consistent sets of the premises ([71], see also [7,10]). For this, we recall the following entailments known from the area of inconsistency-tolerant (non-monotonic) logics.

Definition 15 *(MCS-based entailments).* Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, $\mathcal{S}$ a set of $\mathcal{L}$-formulas and $\mathcal{X}$ a $\vdash$-consistent set of $\mathcal{L}$-formulas such that $\mathcal{S} \cap \mathcal{X} = \emptyset$. Recall that $\mathsf{CN}_{\mathsf{L}}(\mathcal{S})$ denotes the $\vdash$-closure of $\mathcal{S}$ (Definition 2) and that $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ denotes the set of the $\subseteq$-maximally $\vdash_{\mathcal{X}}$-consistent subsets of $\mathcal{S}$ (Definition 11). The following entailments are defined in a way similar to those in Definition 7:

- $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$ iff $\phi \in \mathsf{CN}_{\mathsf{L}}(\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \cup \mathcal{X})$
- $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$ iff $\phi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})} \mathsf{CN}_{\mathsf{L}}(\mathcal{T} \cup \mathcal{X})$
- $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$ iff $\phi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})} \mathsf{CN}_{\mathsf{L}}(\mathcal{T} \cup \mathcal{X})$

We now relate the entailments in Definitions 7 and 15. A similar result to the next proposition was shown in [7] (see also [8, Section 2.3.1]), but only for direct undercut with consistency undercut as attack rules and for $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$ and without the distinction between strict and defeasible assumptions. Here we generalize the setting to sequent calculi in which the rules from Fig. 2 are admissible, every complete semantics $\mathsf{sem} \in \mathsf{CMP} = \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{idl}, \mathsf{stb}, \mathsf{sstb}, \mathsf{egr}, \mathsf{stg}\}$, and for the three different types of attack relations.

Theorem 4 *(characterization of entailments I). The following equivalences hold:*

1. $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$ *for every* $\dagger \in \{\mathsf{con}, \mathsf{set}, \mathsf{dir}\}$ *and* $\mathsf{sem} \in \mathsf{ME}$.
2. $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$ *for every* $\dagger \in \{\mathsf{con}, \mathsf{set}\}$ *and* $\mathsf{sem} \in \mathsf{SE}$.
3. $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$ *for* $\dagger \in \{\mathsf{con}, \mathsf{dir}\}$ *and* $\mathsf{sem} \in \mathsf{ME}$.
4. $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$ *for every* $\dagger \in \{\mathsf{con}, \mathsf{dir}\}$ *and* $\mathsf{sem} \in \mathsf{ME}$.
5. $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$ *for every* $\dagger \in \{\mathsf{con}, \mathsf{set}\}$ *and* $\mathsf{sem} \in \mathsf{SE}$.

Proof. We first show Item 1. Let $\mathsf{sem} \in \mathsf{ME}$.

[$\Leftarrow$] Suppose that $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$. Thus, there is a maximal $\vdash_{\mathcal{X}}$-consistent subset $\mathcal{T}$ of $\mathcal{S}$ for which $\mathcal{T} \cup \mathcal{X} \vdash \psi$. By Lemma 6, $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF})$. By Theorem 1, $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}) \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$. Since L is finitary and by the completeness of C, there is an argument $\Gamma \Rightarrow \psi \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T})$. Thus, $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \psi$.

[$\Rightarrow$] Suppose that $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \psi$. Thus, there is a set $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ and an argument $\Gamma \Rightarrow \psi \in \mathcal{E}$. By (pairwise) support consistency (Proposition 2 resp. Proposition 3), $\Gamma \setminus \mathcal{X}$ is a $\vdash_{\mathcal{X}}$-consistent subset of $\mathcal{S}$. By the soundness of C, $\Gamma \vdash_{\mathcal{X}} \psi$. Thus, there is a maximal $\vdash_{\mathcal{X}}$-consistent subset $\mathcal{T}$ of $\mathcal{S}$ for which $\mathcal{T} \vdash_{\mathcal{X}} \psi$, and so $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$.

The other items directly follow from previous results: Item 2 for $\mathsf{sem} \in \mathsf{SE}$ follows by strong free precedence (Corollary 4). Item 3 follows by maximal consistency (Corollary 18) and since $\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) = \bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Item 4 follows by maximal consistency (Corollary 18). Item 5 follows from Item 2 and since the entailments $\mid\!\sim^{\cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ and $\mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ coincide for single extension semantics sem. $\square$

To complete the characterization in Theorem 4 to the other argumentative entailments considered here, we need the following notations (see also Definition 14):

Definition 16 *(Ω–based entailments).* Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, $\mathcal{S}$ a set of $\mathcal{L}$-formulas and $\mathcal{X}$ a $\vdash$-consistent set of $\mathcal{L}$-formulas such that $\mathcal{S} \cap \mathcal{X} = \emptyset$. We denote:

- $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},\mathcal{X},\Omega} \phi$ iff $\phi \in \mathsf{CN}_{\mathsf{L}}\big(\bigcap_{\omega \in \Omega} \bigcap_{\mathcal{T} \in \omega} (\mathcal{T} \cup \mathcal{X})\big)$.
- $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\Omega} \phi$ iff $\phi \in \bigcap_{\omega \in \Omega} \bigcup_{\mathcal{T} \in \omega} \mathsf{CN}_{\mathsf{L}}(\mathcal{T} \cup \mathcal{X})$.
- $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\mathcal{X},\Omega} \phi$ iff $\phi \in \bigcup_{\omega \in \Omega} \bigcup_{\mathcal{T} \in \omega} \mathsf{CN}_{\mathsf{L}}(\mathcal{T} \cup \mathcal{X})$.

The following result immediately follows from Definition 16 and Proposition 24.

Theorem 5 *(characterization of entailments II). For* $\dagger = \mathsf{set}$ *and every* $\mathsf{sem} \in \mathsf{ME}$ *and* $\star \in \{\cap, \Cap, \cup\}$*, it holds that* $\mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathsf{L},\mathcal{X},\Omega} \psi$

Note 23. The only case that is not covered by Theorems 4 and 5 is when $\mathsf{sem} = \mathsf{SE}$ and $\dagger = \mathsf{dir}$. The corresponding entailment relation is directly obtained by Item 4 in Theorem 3.

Note 24. By Theorem 4, Theorem 5, and Note 23, it follows that *all* the entailment relations that are obtained in this paper can be described in terms of consistent subsets of the premises (albeit not necessarily the maximal ones).

6. Evaluation of the reasoning process

We now examine the properties of the entailment relations which were characterized in the previous section with respect to the setting described in Section 2.2. In what follows we divide the entailment properties to two kinds: those that are concerned with non-monotonic reasoning, and those for inconsistency handling.

6.1. Inference principles for non-monotonic reasoning

By Definition 1, consequence relations are monotonic, that is: the set of conclusions cannot decrease when the set of assumption is increased. Human reasoning, on the other hand, is often non-monotonic, as previously drawn conclusions are sometimes retracted in light of new data. In this section we examine the argumentation-based entailment relations in Definition 7 relative to the monotonicity property. As it turns out, credulous entailments $\mathrel{|\!\sim}^{\cup}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ with respect to multi-extension semantics (i.e., when $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}, \mathsf{sstb}, \mathsf{stg}\}$) for $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$, are monotonic (Proposition 25), while skeptical entailments are non-monotonic. The latter are further divided according to the general patterns for non-monotonic reasoning introduced in [46,54,55,59,72].

We start with some results concerning the (non-)monotonicity of argumentation-based entailments. For this, we need the next lemma that generalizes the result in [23] from classical logic to any compact Tarksian logic L.

Lemma 13. *Let* $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ *be a compact Tarskian logic. Then* $\mathrel{|\!\sim}^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ *is monotonic.*

Proof. Suppose that $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$. Then, there is a maximal $\vdash_{\mathcal{X}}$-consistent subset $\mathcal{T}$ of $\mathcal{S}$ for which $\mathcal{T} \cup \mathcal{X} \vdash \psi$. Consider now $\mathcal{S} \cup \mathcal{S}'$. Clearly, $\mathcal{T}$ is a $\vdash_{\mathcal{X}}$-consistent subset of $\mathcal{S} \cup \mathcal{S}'$, and so there is a maximal $\vdash_{\mathcal{X}}$-consistent subset $\mathcal{T}'$ of $\mathcal{S} \cup \mathcal{S}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$. By the monotonicity of $\vdash$, we have that $\mathcal{T}' \cup \mathcal{X} \vdash \psi$, thus $\mathcal{S}, \mathcal{S}' \mathrel{|\!\sim}^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$ as well. □

Proposition 25 *(monotonicity). Every entailment of the form* $\mathrel{|\!\sim}^{\cup}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$*, where* $\mathsf{sem} \in \mathsf{ME}$ *and* $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$*, is monotonic.*

Proof. This follows from Lemma 13 and Theorem 4 (Item 1). □

Not all the credulous entailments (i.e., those of the form $\mathrel{|\!\sim}^{\cup}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ for some $\dagger$ and sem) are monotonic. This is demonstrated in the next example.

Example 7. Let $\mathsf{L} = \mathsf{CL}$, $\mathcal{S} = \{p\}$ and $\mathcal{X} = \emptyset$. Clearly, for every A and sem, $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\emptyset}_{\mathsf{CL},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(p)\}$. Now, let $\mathcal{S}' = \mathcal{S} \cup \{\neg p\} = \{p, \neg p\}$. Let $\mathcal{AF}$ be any of the considered argumentation frameworks. We have that there are two preferred (and (semi-)stable) extensions: $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}) = \{\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(p), \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\neg p)\}$. Moreover, $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}) = \{\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\emptyset)\}$ for $\mathsf{sem} \in \mathsf{SE}$. This shows that for any $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$, $\star \in \{\cap, \Cap\}$ and any semantics sem in Definition 6, $\mathrel{|\!\sim}^{\star}_{\mathsf{CL},\dagger,\emptyset,\mathsf{sem}}$ is non-monotonic. Furthermore, for every $\star \in \{\cap, \Cap, \cup\}$ and $\mathsf{sem} \in \mathsf{SE}$, $\mathrel{|\!\sim}^{\star}_{\mathsf{CL},\dagger,\emptyset,\mathsf{sem}}$ is non-monotonic as well. Indeed, in all the above-mentioned cases, we have that $p \mathrel{|\!\sim}^{\star}_{\mathsf{CL},\dagger,\emptyset,\mathsf{sem}} p$, since $p \Rightarrow p$ is in the relevant extension(s), however, $p, \neg p \not\mathrel{|\!\sim}^{\star}_{\mathsf{CL},\dagger,\emptyset,\mathsf{sem}} p$, since only tautological arguments are in $\mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\emptyset)$.

Note 25. The last example also shows that skeptical entailments (those of the form $\mathrel{|\!\sim}^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ for $\star \in \{\cap, \Cap\}$) are non-monotonic.

It is common to examine non-monotonic entailment relations according to the following properties:

Definition 17 *(rationality postulates III).* Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic, $\mathcal{X}$ a $\vdash$-consistent set of $\mathcal{L}$-formulas, and $\mathrel{|\!\sim} \subseteq \wp(\mathsf{WFF}(\mathcal{L}) \setminus \mathcal{X}) \times \mathsf{WFF}(\mathcal{L})$.[23] We say that $\mathrel{|\!\sim}$ satisfies:

[23] The reason for excluding the strict assumptions from the premises is to keep our supposition that the strict and the defeasible premises are disjoint, and to preserve the similarity to Definition 11, where consistency is taken with respect to subsets of $\mathcal{S}$.

- $\vdash_{\mathcal{X}}$*-cautious reflexivity* ($\vdash_{\mathcal{X}}$-cREF): If ϕ is $\vdash_{\mathcal{X}}$-consistent, then $\phi \mid\sim \phi$.
Absent other information, consistent formulas are entailed.
- $\vdash_{\mathcal{X}}$*-right weakening* ($\vdash_{\mathcal{X}}$-RW): If $\mathcal{S} \mid\sim \phi$ and $\phi \vdash_{\mathcal{X}} \psi$, then $\mathcal{S} \mid\sim \psi$.
Logical consequences of entailed formulas are entailed.
- $\vdash_{\mathcal{X}}$*-left logical equivalence* ($\vdash_{\mathcal{X}}$-LLE): If $\psi \vdash_{\mathcal{X}} \phi$ and $\phi \vdash_{\mathcal{X}} \psi$ and $\mathcal{S}, \phi \mid\sim \sigma$, then $\mathcal{S}, \psi \mid\sim \sigma$.
Substitution of logically equivalent formulas holds on the left side of the entailment relation.
- *cautious monotonicity* (CM): If $\mathcal{S} \mid\sim \phi$ and $\mathcal{S} \mid\sim \psi$, then $\mathcal{S}, \phi \mid\sim \psi$.
Entailments are preserved under the addition of entailed formulas to the premises.
- *cautious cut* (CC): If $\mathcal{S} \mid\sim \psi$ and $\mathcal{S}, \psi \mid\sim \phi$, then $\mathcal{S} \mid\sim \phi$.
Transitivity holds for entailments.
- *introduction of disjunction* (OR): If $\mathcal{S}, \phi \mid\sim \sigma$ and $\mathcal{S}, \psi \mid\sim \sigma$, then $\mathcal{S}, \phi \vee \psi \mid\sim \sigma$.
Reasoning by cases is validated by entailments.
- *rational monotonicity* (RM): If $\mathcal{S} \mid\sim \psi$ and $\mathcal{S} \not\mid\sim \neg\phi$, then $\mathcal{S}, \phi \mid\sim \psi$.
Entailments are preserved under the addition of information that is $\mid\sim$-consistent with the given premises.

We refer to [54,59] for a detailed discussion on $\vdash_{\mathcal{X}}$-cREF, $\vdash_{\mathcal{X}}$-RW, $\vdash_{\mathcal{X}}$-LLE, CM and OR, and to [46] for a discussion on CC. The postulate RM was introduced in [55].[24] All of these properties are well-known and have been extensively examined in different contexts and for different purposes involving inference in a non-monotonic way.

The properties in Definition 17 are often gathered for defining systems for non-monotonic inference.

Definition 18 *(systems for non-monotonic inference).* We say that an entailment $\mid\sim$ is:

- *cumulative* (w.r.t. $\vdash_{\mathcal{X}}$), if it satisfies $\vdash_{\mathcal{X}}$-cREF, $\vdash_{\mathcal{X}}$-RW, $\vdash_{\mathcal{X}}$-LLE, CM and CC.
- *preferential* (w.r.t. $\vdash_{\mathcal{X}}$), if it is $\vdash_{\mathcal{X}}$-cumulative and satisfies OR.
- *rational* (w.r.t. $\vdash_{\mathcal{X}}$), if it is $\vdash_{\mathcal{X}}$-preferential and satisfies RM.

We first give results concerning cumulativity and preferentiality. Given our characterization of the argumentation-induced entailments in Theorem 4, the following two lemmas will be useful.

Lemma 14 *(mcs cumulativity). For every propositional logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and a $\vdash$-consistent set $\mathcal{X}$ of $\mathcal{L}$-formulas, the entailments $\mid\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ and $\mid\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ are $\vdash_{\mathcal{X}}$-cumulative.*

Proof. The properties $\vdash_{\mathcal{X}}$-cREF, $\vdash_{\mathcal{X}}$-RW and $\vdash_{\mathcal{X}}$-LLE follow directly from the definitions of $\mid\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ and $\mid\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$. Note that the restriction of $\vdash_{\mathcal{X}}$-cREF to $\vdash_{\mathcal{X}}$-consistent formulas is necessary, since for an $\vdash_{\mathcal{X}}$-inconsistent formula ϕ it holds that $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\{\phi\}) = \{\emptyset\}$. We show CM and CC together for $\mid\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ (the proof for $\mid\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ is similar): Suppose that $\mathcal{S} \mid\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$. Then the following equivalences hold: $\mathcal{S} \mid\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$, iff $\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}), \mathcal{X} \vdash \phi$, iff (using the fact that $\mathcal{S} \mid\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$ thus it is easy to verify that $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S} \cup \{\psi\}) = \{\mathcal{T} \cup \{\psi\} \mid \mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\}$) it holds that $\bigcap \mathsf{MCS}_{\mathsf{L}}(\mathcal{S} \cup \{\psi\}), \mathcal{X} \vdash \phi$, iff $\mathcal{S}, \psi \mid\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$. $\square$

The following example, from [23], shows that the entailment $\mid\sim^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ does not satisfy CC and is therefore neither cumulative, preferential nor rational.

Example 8. Let $\mathcal{S} = \{p \wedge q, \neg p \wedge r\}$ and $\mathcal{X} = \emptyset$. Then $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) = \{\{p \wedge q\}, \{\neg p \wedge r\}\}$ and therefore $\mathcal{S} \not\mid\sim^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} q \wedge r$, however, $\mathcal{S} \mid\sim^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} q$. Let $\mathcal{S}' = \mathcal{S} \cup \{q\}$. Then $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}') = \{\{p \wedge q, q\}, \{\neg p \wedge r, q\}\}$ and therefore $\mathcal{S}' \mid\sim^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} q \wedge r$.

In view of this counterexample and Theorem 4 (Item 1), it follows that $\mid\sim^{\cup}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ is not cumulative, and therefore not preferential nor rational, for every $\dagger \in \{\mathsf{con}, \mathsf{set}, \mathsf{dir}\}$ and $\mathsf{sem} \in \mathsf{ME}$.

Lemma 15. *For a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, a $\vdash$-consistent set $\mathcal{X}$ of $\mathcal{L}$-formulas, and $\mathsf{sem} \in \mathsf{SE}$, the entailments $\mid\sim^{\cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ and $\mid\sim^{\Cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ are $\vdash_{\mathcal{X}}$-cumulative.*

[24] RM is sometimes considered more controversial than other postulates. For instance, in [73] Stalnaker claims by a counter-example that RM is not a desirable property in some application contexts of defeasible reasoning. Nevertheless, also other reasoning principles have been criticized (e.g., CC in [63]), and some are not valid in central approaches of non-monotonic inference (e.g., CM does not hold in Reiter's default logic, see [59, Observation 3.2.4]).

Proof. Note that for sem $\in$ SE, $\mid\!\sim^{\cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}} = \mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}} = \mid\!\sim^{\cup}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ since sem only gives rise to a single extension. Let $\mid\!\sim \,=\, \mid\!\sim^{\star}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ where sem $\in$ SE and $\star \in \{\cap, \Cap, \cup\}$. For a set $\mathcal{T}$ of $\mathcal{L}$-formulas, let $\mathcal{T}_{\top} = \{\sigma \in \mathcal{T} \mid \sigma$ is $\vdash_{\mathcal{X}}$ consistent$\}$. Let also $\pi(\mathcal{T})$ be $\mathcal{T}_{\top}$ in case that $\mathcal{T}_{\top}$ is $\vdash_{\mathcal{X}}$-consistent, and $\emptyset$ otherwise. By Theorem 4 (Item 4), (†) $\mathcal{T} \mid\!\sim \sigma$ iff $\sigma \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\pi(\mathcal{T}))$.

First, $\vdash_{\mathcal{X}}$-cREF trivially follows from (†).

For CM and CC, suppose that $\mathcal{S} \mid\!\sim \phi$ and let $\mathcal{S}' = \mathcal{S} \cup \{\phi\}$. By (†), $\phi \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\pi(\mathcal{S}))$. Since $\vdash$ is Tarskian, $\mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_{\top}) = \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}'_{\top})$. So, $\pi(\mathcal{S}) = \pi(\mathcal{S}')$. By (†), we have: $\mathcal{S} \mid\!\sim \psi$ iff $\psi \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\pi(\mathcal{S}))$ iff $\psi \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\pi(\mathcal{S}'))$ iff $\mathcal{S}' \mid\!\sim \psi$.

For $\vdash_{\mathcal{X}}$-RW, suppose that $\mathcal{S} \mid\!\sim \phi$ and $\phi \vdash_{\mathcal{X}} \psi$. By (†), $\phi \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\pi(\mathcal{S}))$ and by $\vdash$-transitivity, $\psi \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\pi(\mathcal{S}))$. By (†), $\mathcal{S} \mid\!\sim \psi$.

For $\vdash_{\mathcal{X}}$-LLE, suppose that $\mathcal{S}, \phi \mid\!\sim \sigma$, $\phi \vdash_{\mathcal{X}} \psi$ and $\psi \vdash_{\mathcal{X}} \phi$. Then, by (†), $\sigma \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\pi(\mathcal{S} \cup \{\phi\}))$. Since $\vdash$ is Tarskian, $\sigma \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\pi(\mathcal{S} \cup \{\psi\}))$. By (†), $\mathcal{S}, \psi \mid\!\sim \sigma$. $\square$

Proposition 26 *(cumulativity). Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic satisfying the conditions in Section 2.2.*

1. *For every $\dagger \in \{\mathsf{con}, \mathsf{set}, \mathsf{dir}\}$ and sem $\in$ CMP it holds that $\mid\!\sim^{\cap}_{\mathsf{L},\dagger,\mathcal{X}\mathsf{sem}}$ is $\vdash$-cumulative.*
2. *For every $\dagger \in \{\mathsf{con}, \mathsf{dir}\}$ and sem $\in$ CMP it holds that $\mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ is $\vdash$-cumulative.*
3. *For $\dagger = \mathsf{set}$ and every sem $\in$ SE it holds that $\mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ is $\vdash$-cumulative.*

Proof. We divide the proof according to the different items:

1. For $\dagger \in \{\mathsf{con}, \mathsf{set}\}$ this item follows from Lemma 14 and Item 3 of Theorem 4. For $\dagger = \mathsf{dir}$ and sem $\in$ ME it follows from Lemma 14 and Item 4 of Theorem 4, while for sem $\in$ SE it follows from Lemma 15.
2. This item follows from Lemma 14 and Item 2 of Theorem 4 for sem $\in$ ME. For $\dagger = \mathsf{con}$ and sem $\in$ SE it follows from Lemma 14 and Item 5 of Theorem 4. For $\dagger = \mathsf{dir}$ and sem $\in$ SE it follows from Lemma 15.
3. This item follows from Lemma 14 and Item 5 of Theorem 4. $\square$

While Item 2 of Proposition 26 holds for frameworks with attack rules of type con or dir, it fails for frameworks with attack rules of type set. Before giving a counter-example, we present a partially positive result concerning CM, for which the following lemma will be useful.

Lemma 16. *Let $\mathcal{S}$ and $\mathcal{X}$ be disjoint sets of $\mathcal{L}$-formulas and suppose that $\mathcal{X}$ is $\vdash$-consistent.*

1. *If $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ then $\{\mathcal{T}\} \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$.*
2. *If $\Delta \cup \{\phi\} \subseteq \mathcal{S}$ and $\Delta \cup \{\phi\}$ is $\vdash_{\mathcal{X}}$-inconsistent, then Δ is $\vdash_{\mathcal{X}}$-inconsistent or $\phi \notin \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$.*
3. *If $\phi \in \mathcal{S} \setminus \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, there is a $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \cap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S} \setminus \{\phi\})$ such that $\mathcal{T} \vdash_{\mathcal{X}} \neg\phi$.*
4. *Let sem $\in$ ME and $\mathcal{S}' = \mathcal{S} \cup \{\phi\}$. If $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{set},\mathcal{X},\mathsf{sem}} \phi$ then $\phi \in \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}')$.*

Proof. Item 1. Let $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. We check the two requirements of Definition 16. The first requirement is clearly satisfied. For the second requirement, let $\Theta \in \wp(\mathcal{S})$ such that $\Theta \nsubseteq \mathcal{T}$. By the maximal consistency of $\mathcal{T}$, $\mathcal{T} \cup \Theta$ is $\vdash_{\mathcal{X}}$-inconsistent.

Item 2. Suppose that $\phi \in \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Since $\Delta \cup \{\phi\}$ is $\vdash_{\mathcal{X}}$-inconsistent, there is a $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent set $\Delta' \subseteq \Delta \cup \{\phi\}$. Since $\phi \in \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, $\phi \notin \Delta'$, and so $\Delta' \subseteq \Delta$. Thus, Δ is $\vdash_{\mathcal{X}}$-inconsistent.

Item 3. If $\phi \in \mathcal{S} \setminus \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, there is a $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent set $\Delta \subseteq \mathcal{S}$ with $\phi \in \Delta$. Since $\Delta \setminus \{\phi\}$ is $\vdash_{\mathcal{X}}$-consistent and $\Delta \setminus \{\phi\} \vdash_{\mathcal{X}} \neg\phi$, there is a $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ such that $\Delta \setminus \{\phi\} \subseteq \mathcal{T}$. Clearly, also $\mathcal{T} \subseteq \mathcal{S} \setminus \{\phi\}$ and $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S} \setminus \{\phi\})$. By $\vdash_{\mathcal{X}}$-monotonicity, $\mathcal{T} \vdash_{\mathcal{X}} \neg\phi$.

Item 4. Let $\mid\!\sim \,=\, \mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{set},\mathcal{X},\mathsf{sem}}$. We show the contraposition. Suppose that there is a $\subseteq$-minimal $\vdash_{\mathcal{X}}$-inconsistent set $\{\phi_1, \ldots, \phi_n, \phi\}$ of $\mathcal{S}'$. So, $\{\phi_1, \ldots, \phi_n\} \vdash_{\mathcal{X}} \neg\phi$ and by Item 3 there is a maximal consistent set $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ for which $\{\phi_1, \ldots, \phi_n\} \subseteq \mathcal{T}$ and $\mathcal{T} \nvdash_{\mathcal{X}} \phi$. By Item 1, $\{\mathcal{T}\} \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, and so by Theorem 5, $\mathcal{S} \not\mid\!\sim \phi$. $\square$

Proposition 27. *For a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, a $\vdash$-consistent set $\mathcal{X}$ of $\mathcal{L}$-formulas, and every sem $\in$ ME, the entailments $\mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{set},\mathcal{X},\mathsf{sem}}$ satisfy CM.*

Proof. Let $\mid\!\sim \,=\, \mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{set},\mathcal{X},\mathsf{sem}}$ and $\mathcal{S}' = \mathcal{S} \cup \{\phi\}$. Suppose that $\mathcal{S} \mid\!\sim \phi$. In case that $\phi \in \mathcal{S}$ the statement is trivial, so we assume that $\phi \notin \mathcal{S}$. We have to show that $\mathcal{S} \mid\!\sim \psi$ implies $\mathcal{S}' \mid\!\sim \psi$. We show the contraposition.

Suppose that $\mathcal{S}' \not\mid\!\sim \psi$. By Theorem 5 there is a $\omega = \{\mathcal{T}_i \mid i \in I\} \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}')$ such that for all $\mathcal{T} \in \omega$, $\mathcal{T} \nvdash_{\mathcal{X}} \psi$. Let $\omega' = \{\mathcal{T} \setminus \{\phi\} \mid \mathcal{T} \in \omega\}$. We show that $\omega' \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, which by Theorem 5 proves that $\mathcal{S} \not\mid\!\sim \psi$.

$$[\vee\Rightarrow]\ \frac{\Gamma,\phi\Rightarrow\Delta \quad \Gamma,\psi\Rightarrow\Delta}{\Gamma,\phi\vee\psi\Rightarrow\Delta} \qquad [\Rightarrow\vee]\ \frac{\Gamma\Rightarrow\phi,\Delta'}{\Gamma\Rightarrow\phi\vee\psi,\Delta'} \qquad [\Rightarrow\vee]\ \frac{\Gamma\Rightarrow\phi,\Delta'}{\Gamma\Rightarrow\psi\vee\phi,\Delta'}$$

Fig. 7. Rules for $\vee$ that are part of (or admissible in) the calculus C (where Δ is empty or a singleton and Δ' is empty, in case that C is a single-conclusion calculus).

Assume for a contradiction that there is a $\Theta \in \wp(\mathcal{S})$ such that for all $i \in I$, (i) $\Theta \nsubseteq \mathcal{T}_i \setminus \{\phi\}$ and (ii) $\Theta \cup \mathcal{T}_i \setminus \{\phi\}$ is $\vdash_{\mathcal{X}}$-consistent. By (i), and since $\phi \notin \mathcal{S}$, $\Theta \nsubseteq \mathcal{T}_i$ for all $i \in I$. Since $\omega \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}')$, there is an $i \in I$ such that $\mathcal{T}_i \cup \Theta$ is $\vdash_{\mathcal{X}}$-inconsistent. By (ii), $\phi \in \mathcal{T}_i$ and $\mathcal{T}_i \setminus \{\phi\}, \Theta \vdash_{\mathcal{X}} \neg\phi$. By (ii) and Lemma 16 (Item 2), $\phi \notin \mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}')$. By Lemma 16 (Item 3) there is a $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}') \cap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ for which $\mathcal{T} \vdash_{\mathcal{X}} \neg\phi$. By Lemma 16 (Item 1), $\{\mathcal{T}\} \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, and so $\mathcal{S} \not\mid\!\sim \phi$, a contradiction. $\square$

We now show that CC is violated for $\mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{set},\mathcal{X},\mathsf{sem}}$ when $\mathsf{sem} \in \mathsf{ME}$.

Example 9. Let $\mathcal{S} = \{\psi_1, \psi_2, \psi_3\}$, where $\psi_1 = p \wedge s$, $\psi_2 = q \wedge (s \supset t)$, and $\psi_3 = \neg(p \wedge q) \wedge (q \supset s) \wedge (s \supset t)$ and let $\mathcal{X} = \emptyset$. For every $\mathcal{S}$ and $\mathcal{X}$-based framework with set-attacks, we have the following stable (preferred, semi-stable, stage) extensions (recall Proposition 24): $\mathcal{E}_1 = \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_1, \psi_2\})$, $\mathcal{E}_2 = \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_2, \psi_3\})$, $\mathcal{E}_3 = \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_1, \psi_3\})$, and $\mathcal{E}_4 = \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_1\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_2\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_3\})$. Therefore, $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{CL},\mathsf{set},\emptyset,\mathsf{sem}} s$ but $\mathcal{S} \not\mid\!\sim^{\Cap}_{\mathsf{CL},\mathsf{set},\emptyset,\mathsf{sem}} t$.

We now let $\mathcal{S}' = \mathcal{S} \cup \{s\}$. This time, the stable (preferred, semi-stable, stage) extensions are: $\mathcal{E}'_1 = \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_1, \psi_2, s\})$, $\mathcal{E}'_2 = \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_2, \psi_3, s\})$, $\mathcal{E}'_3 = \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_1, \psi_3, s\})$, and $\mathcal{E}'_4 = \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_1, s\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_2, s\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{CL}}(\{\psi_3, s\})$. Therefore, $\mathcal{S}' \mid\!\sim^{\Cap}_{\mathsf{CL},\mathsf{set},\emptyset,\mathsf{sem}} t$.

We now turn to checking preferentiality. For this we need to consider the property OR, and thus a disjunction connective $\vee$. In what follows we suppose that, in addition to the rules in Fig. 2, the calculus C also has two admissible rules characterizing $\vee$, as shown in Fig. 7.

Lemma 17. *$\Delta, \neg(\phi \vee \psi) \Rightarrow \neg\phi$ and $\Delta, \neg(\phi \vee \psi) \Rightarrow \neg\psi$ are* C*-derivable.*

Proof. By [Ref] and [LMon], $\Delta, \phi \Rightarrow \phi$ and by $[\Rightarrow\vee]$ $\Delta, \phi \Rightarrow \phi \vee \psi$ are C-derivable. By Lemma 2 also $\Delta, \neg(\phi \vee \psi) \Rightarrow \neg\phi$ and $\Delta, \neg(\psi \vee \phi) \Rightarrow \neg\phi$ are C-derivable. $\square$

Lemma 18 *(mcs preferentiality). For every propositional logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ for which the rules in Figs. 2 and 7 hold,*[25] *the entailment $\mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ is $\vdash_{\mathcal{X}}$-preferential with respect to every $\vdash$-consistent $\mathcal{X}$.*

Proof. By Lemma 14 it just remains to show that $\mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ satisfies OR. Suppose that $\mathcal{S}, \phi_1 \mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$ and $\mathcal{S}, \phi_2 \mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$. Let $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$ and $\mathcal{T}' = \mathcal{T} \cap \mathcal{S}$. We have to show that $\mathcal{T} \vdash_{\mathcal{X}} \psi$. We consider two cases: (1) $\phi_1 \vee \phi_2 \notin \mathcal{T}$ and (2) $\phi_1 \vee \phi_2 \in \mathcal{T}$.

Case 1. In that case $\mathcal{T}' = \mathcal{T}$ and $\mathcal{T} \cup \{\phi_1 \vee \phi_2\}$ is $\vdash_{\mathcal{X}}$-inconsistent. By Lemma 4, $\Gamma \Rightarrow \neg(\phi_1 \vee \phi_2)$ is derivable for some finite $\Gamma \subseteq \mathcal{T}$. By Lemma 17 and Cut, $\Gamma \Rightarrow \neg\phi_1$. So, $\mathcal{T} \cup \{\phi_1\}$ is $\vdash_{\mathcal{X}}$-inconsistent. So, $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S} \cup \{\phi_1\})$. By the supposition $\mathcal{T} \vdash_{\mathcal{X}} \psi$.

Case 2. Assume for a contradiction that $\mathcal{T} \cup \{\phi_1\}$ and $\mathcal{T} \cup \{\phi_2\}$ are $\vdash_{\mathcal{X}}$-inconsistent sets. Then $\mathcal{T} \vdash_{\mathcal{X}} \neg\phi_1$ and $\mathcal{T} \vdash_{\mathcal{X}} \neg\phi_2$. By Corollary 1, $\mathcal{T} \setminus \{\phi_1 \vee \phi_2\}, \phi_1 \vdash_{\mathcal{X}} \neg(\phi_1 \vee \phi_2)$ and $\mathcal{T} \setminus \{\phi_1 \vee \phi_2\}, \phi_2 \vdash_{\mathcal{X}} \neg(\phi_1 \vee \phi_2)$. So, there are finite $\Gamma_1, \Gamma_2 \subseteq \mathcal{T} \setminus \{\phi_1 \vee \phi_2\}$ and $\Theta \subseteq \mathcal{X}$ for which $\Gamma_1, \Theta, \phi_1 \Rightarrow \neg(\phi_1 \vee \phi_2)$ and $\Gamma_2, \Theta, \phi_2 \Rightarrow \neg(\phi_1 \vee \phi_2)$ are derivable. By $[\vee\Rightarrow]$ and $\vdash$-monotonicity, $\Gamma_1, \Gamma_2, \Theta, \phi_1 \vee \phi_2 \Rightarrow \neg(\phi_1 \vee \phi_2)$, in view of which $\mathcal{T}$ is $\vdash_{\mathcal{X}}$-inconsistent. This is a contradiction. Without loss of generality we thus suppose that $\mathcal{T} \cup \{\phi_1\}$ is $\vdash_{\mathcal{X}}$-consistent. Let $\mathcal{T}'_1 = \mathcal{T}' \cup \{\phi_1\}$. By $\vdash$-monotonicity, $\mathcal{T}'_1$ is also $\vdash_{\mathcal{X}}$-consistent.

We show that (†) $\mathcal{T}'_1 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S} \cup \{\phi_1\})$ and (therefore by the main supposition) $\mathcal{T}'_1 \vdash_{\mathcal{X}} \psi$. Assume for a contradiction that there is a $\sigma \in \mathcal{S} \setminus \mathcal{T}'_1$ such that $\mathcal{T}'_1 \cup \{\sigma\}$ is $\vdash_{\mathcal{X}}$-consistent. Since $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$ and $\sigma \notin \mathcal{T}$, necessarily $\mathcal{T} \vdash_{\mathcal{X}} \neg\sigma$. So, $\mathcal{T}', \phi_1 \vee \phi_2 \vdash_{\mathcal{X}} \neg\sigma$. By Corollary 1, $\mathcal{T}', \sigma \vdash_{\mathcal{X}} \neg(\phi_1 \vee \phi_2)$ and by Lemma 17, $\mathcal{T}', \sigma \vdash_{\mathcal{X}} \neg\phi_i$. So, $\mathcal{T}'_1 \cup \{\sigma\}$ is $\vdash_{\mathcal{X}}$-inconsistent. This is a contradiction. So, $\mathcal{T}'_1 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S} \cup \{\phi_1\})$ and (†) holds. By the main supposition, $\mathcal{T}', \phi_1 \vdash_{\mathcal{X}} \psi$.

If $\mathcal{T}' \cup \{\phi_2\}$ is $\vdash_{\mathcal{X}}$-consistent, by the same reasoning $\mathcal{T}' \cup \{\phi_2\} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S} \cup \{\phi_2\})$ and $\mathcal{T}', \phi_2 \vdash_{\mathcal{X}} \psi$. Otherwise, if $\mathcal{T}' \cup \{\phi_2\}$ is $\vdash_{\mathcal{X}}$-inconsistent, it trivially holds that $\mathcal{T}', \phi_2 \vdash_{\mathcal{X}} \psi$.

So, $\mathcal{T}', \phi_i \vdash_{\mathcal{X}} \psi$ for $i \in \{1, 2\}$. By compactness and monotonicity of $\vdash$ there are $\Theta \subseteq \mathcal{T}'$ and $\Gamma \subseteq \mathcal{X}$ for which $\Theta, \phi_1, \Gamma \Rightarrow \psi$ and $\Theta, \phi_2, \Gamma \Rightarrow \psi$ are C-derivable. By $[\vee\Rightarrow]$, $\Theta, \phi_1 \vee \phi_2, \Gamma \Rightarrow \psi$ and by the soundness of C and $\vdash$-monotonicity, $\mathcal{T} \vdash_{\mathcal{X}} \psi$. $\square$

[25] We say that inference rules *hold in a logic* L if the respective rules are admissible in the underlying (presupposed) calculus C.

Lemma 19. *For every* $\star \in \{\cap, \Cap\}$ *and* sem $\in$ SE, *OR holds for* $\mid\!\sim^{\star}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$.

Proof. Let $\mid\!\sim = \mid\!\sim^{\star}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$, where sem $\in$ SE and $\star \in \{\cap, \Cap\}$. Suppose that $\mathcal{S}, \phi \mid\!\sim \sigma$ and $\mathcal{S}, \psi \mid\!\sim \sigma$. We have to show that $\mathcal{S}, \phi \vee \psi \mid\!\sim \sigma$. Using the notations in Item 4 of Theorem 3, we distinguish between the following two cases: (a) $(\mathcal{S} \cup \{\phi\})_\top$ or $(\mathcal{S} \cup \{\psi\})_\top$ is $\vdash_\mathcal{X}$-inconsistent, and (b) $(\mathcal{S} \cup \{\phi\})_\top$ and $(\mathcal{S} \cup \{\psi\})_\top$ are $\vdash_\mathcal{X}$-consistent. This case distinction mirrors the one in Theorem 3 (Item 4). We now utilize this fact.

Case (a). By Theorem 3 (Item 4), $\vdash_\mathcal{X} \sigma$ and by $\vdash_\mathcal{X}$-monotonicity, $\mathcal{S}, \phi \vee \psi \vdash_\mathcal{X} \sigma$. So, by Theorem 3 (Item 4), $\mathcal{S}, \phi \vee \psi \mid\!\sim \sigma$.

Case (b). By Theorem 3 (Item 4), $\mathcal{S}, \phi \vdash_\mathcal{X} \sigma$ and $\mathcal{S}, \psi \vdash_\mathcal{X} \sigma$. By $\vdash$-compactness, $\vdash$-monotonicity, and the completeness of C, there are $\Theta \subseteq \mathcal{S}$ and $\Gamma \subseteq \mathcal{X}$ for which $\Theta, \phi, \Gamma \Rightarrow \sigma$ and $\Theta, \psi, \Gamma \Rightarrow \sigma$ are C-derivable. So, by $[\vee\Rightarrow]$, $\Theta, \phi \vee \psi, \Gamma \Rightarrow \sigma$ in also C-derivable. We now show that $\mathcal{S} \cup \{\phi \vee \psi\}$ is $\vdash_\mathcal{X}$-consistent. Then, by Theorem 3 (Item 4), $\mathcal{S}, \phi \vee \psi \mid\!\sim \sigma$.

Assume for a contradiction that $\mathcal{S} \cup \{\phi \vee \psi\}$ is $\vdash_\mathcal{X}$-inconsistent. Then, $\mathcal{S} \vdash_\mathcal{X} \neg(\phi \vee \psi)$. By $\vdash$-compactness and the completeness of C, there are $\Theta \subseteq \mathcal{S}$ and $\Gamma \subseteq \mathcal{X}$ for which $\Theta, \Gamma \Rightarrow \neg(\phi \vee \psi)$ is C-derivable. By [Cut] and Lemma 17, $\Theta, \Gamma \Rightarrow \neg\phi$. Hence, $\mathcal{S} \cup \{\phi\}$ is $\vdash_\mathcal{X}$-inconsistent. This is a contradiction to (b). □

Proposition 28 *(preferentiality). Let* $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ *be a logic satisfying the conditions in Section 2.2, as well as the disjunction rules in Fig. 7.*

1. *For every* $\dagger \in \{\mathsf{con}, \mathsf{dir}\}$ *and* sem $\in$ ME, *the entailment* $\mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ *is* $\vdash$*-preferential.*
2. *For* $\dagger = \mathsf{dir}$ *and every* sem $\in$ SE, *the entailments* $\mid\!\sim^{\cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ *and* $\mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ *are* $\vdash$*-preferential.*

Proof. Item 1 follows from Theorem 4 (Item 2), and Lemma 18. Item 2 follows from Lemmas 15 and 19. □

We now show that OR fails for the cases not mentioned in Proposition 28 (see Table 5 for an overview). In all the examples here we consider classical logic as the base logic L. In Example 10 we give counter-examples for those entailments that are with Theorem 4 identical to $\mid\!\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$, namely $\mid\!\sim^{\cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ where $\dagger \in \{\mathsf{con}, \mathsf{set}\}$ and sem $\in$ CMP (Item 3), $\mid\!\sim^{\cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ where sem $\in$ ME (Item 4), and $\mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ for $\dagger \in \{\mathsf{set}, \mathsf{con}\}$ and sem $\in$ SE (Item 5). In Example 11, we consider $\mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{set},\mathcal{X},\mathsf{sem}}$ where sem $\in$ ME.

Example 10. We show that OR fails for $\mid\!\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ (and therefore, by Theorem 3, also for $\mid\!\sim^{\cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ where $\dagger \in \{\mathsf{con}, \mathsf{set}\}$ and sem $\in$ CMP (Item 3), for $\mid\!\sim^{\cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ where sem $\in$ ME (Item 4), and for $\mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$, where $\dagger \in \{\mathsf{con}, \mathsf{set}\}$ and sem $\in$ SE (Item 5)).

Let for this $\mathcal{S} = \{\neg p, \neg q, \neg p \supset r, \neg q \supset r\}$ and $\mathcal{X} = \emptyset$. Then:

- $\mathcal{S}, p \mid\!\sim^{\cap}_{\mathsf{CL},\emptyset,\mathsf{mcs}} r$,
  since $\mathsf{MCS}^{\emptyset}_{\mathsf{CL}}(\mathcal{S} \cup \{p\}) = \{\{p, \neg q, \neg p \supset r, \neg q \supset r\}, \{\neg p, \neg q, \neg p \supset r, \neg q \supset r\}\}$ and so
  $\bigcap \mathsf{MCS}^{\emptyset}_{\mathsf{CL}}(\mathcal{S} \cup \{p\}) = \{\neg q, \neg p \supset r, \neg q \supset r\}$,
- $\mathcal{S}, q \mid\!\sim^{\cap}_{\mathsf{CL},\emptyset,\mathsf{mcs}} r$,
  since $\mathsf{MCS}^{\emptyset}_{\mathsf{CL}}(\mathcal{S} \cup \{q\}) = \{\{\neg p, q, \neg p \supset r, \neg q \supset r\}, \{\neg p, \neg q, \neg p \supset r, \neg q \supset r\}\}$ and so
  $\bigcap \mathsf{MCS}^{\emptyset}_{\mathsf{CL}}(\mathcal{S} \cup \{q\}) = \{\neg p, \neg p \supset r, \neg q \supset r\}$, while
- $\mathcal{S}, p \vee q \not\mid\!\sim^{\cap}_{\mathsf{CL},\emptyset,\mathsf{mcs}} r$,
  since $\mathsf{MCS}^{\emptyset}_{\mathsf{CL}}(\mathcal{S} \cup \{p \vee q\}) = \{\{p \vee q, \neg p, \neg p \supset r, \neg q \supset r\}, \{p \vee q, \neg q, \neg p \supset r, \neg q \supset r\}, \{\neg p, \neg q, \neg p \supset r, \neg q \supset r\}\}$, and so
  $\bigcap \mathsf{MCS}^{\emptyset}_{\mathsf{CL}}(\mathcal{S} \cup \{p \vee q\}) = \{\neg p \supset r, \neg q \supset r\}$.

Example 11. Consider an argumentation framework $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$, where A consists of set-type attack rules, and let $\mathcal{S} = \{p \wedge u, q \wedge v\}$ and $\mathcal{X} = \emptyset$. Then:

- Consider $\mathcal{S}_1 = \mathcal{S} \cup \{\neg p \wedge u\}$. We have the following stable (preferred, semi-table and stage) extensions: $\mathcal{E}_1 = \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{p \wedge u, q \wedge v\})$ and $\mathcal{E}_2 = \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{\neg p \wedge u, q \wedge v\})$. Thus, $\mathcal{S}_1 \mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{set},\emptyset,\mathsf{sem}} u \wedge v$.
- Consider $\mathcal{S}_2 = \mathcal{S} \cup \{\neg q \wedge v\}$. For an analogous reason, $\mathcal{S}_2 \mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{set},\emptyset,\mathsf{sem}} u \wedge v$.
- Consider now $\mathcal{S}_\vee = \mathcal{S} \cup \{(\neg p \wedge u) \vee (\neg q \wedge v)\}$. We now also have the stable (preferred, semi-table and stage) extension $\mathcal{E}_\vee = \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{p \wedge u\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{q \wedge v\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{(\neg p \wedge u) \vee (\neg q \wedge v)\})$, which is why $\mathcal{S}_\vee \not\mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{set},\emptyset,\mathsf{sem}} u \wedge v$.

We now turn to rationality. First, we show that RM fails for $\mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ and in view of Theorem 4 it also fails for all the entailments $\mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ where $\dagger \in \{\mathsf{con}, \mathsf{dir}\}$ and sem $\in$ ME.

Example 12. To see that RM fails for $\mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$, consider again classical logic as the base logic and let $\mathcal{S} = \{r,\ p \wedge q \wedge \neg r,\ (p \wedge r) \supset \neg q,\ \neg p \wedge q\}$. We have $\mathsf{MCS}^{\emptyset}_{\mathsf{CL}}(\mathcal{S}) = \{\{r, (p \wedge r) \supset \neg q, \neg p \wedge q\}, \{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q\}\}$. Only one of the two sets in $\mathsf{MCS}^{\emptyset}_{\mathsf{CL}}(\mathcal{S})$ implies $\neg p$, while both of them imply q. Thus, $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\emptyset,\mathsf{mcs}} q$ and $\mathcal{S} \not\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\emptyset,\mathsf{mcs}} \neg p$. Now, $\mathsf{MCS}^{\emptyset}_{\mathsf{CL}}(\mathcal{S} \cup \{p\}) = \{\{r, (p \wedge r) \supset \neg q, \neg p \wedge q\}, \{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q, p\}, \{r, p, (p \wedge r) \supset \neg q\}\}$. It follows that $\mathcal{S}, p \not\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\emptyset,\mathsf{mcs}} q$, and so RM is violated.

We continue with a counter-example for $\mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ which, in view of Theorem 4, is also a counterexample for the following cases: $\mathrel{|\!\sim}^{\cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ where $\dagger \in \{\mathsf{con}, \mathsf{set}\}$ and $\mathsf{sem} \in \mathsf{CMP}$ (Item 3), $\mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ where $\mathsf{sem} \in \mathsf{ME}$ (Item 4), and $\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ for $\dagger \in \{\mathsf{set}, \mathsf{con}\}$ and $\mathsf{sem} \in \mathsf{SE}$ (Item 5).

Example 13. Let $\mathcal{S} = \{r, p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q, \neg p \wedge q\}$. Note that $\mathsf{Free}^{\emptyset}_{\mathsf{L}}(\mathcal{S}) = \{(p \wedge r) \supset \neg q\}$ (recall from Footnote 14 that $\mathsf{Free}^{\mathcal{X}}_{\mathsf{L}}(\cdot)$ coincides with $\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\cdot)$). Clearly, $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{L},\emptyset,\mathsf{mcs}} (p \wedge r) \supset \neg q$ and $\mathcal{S} \not\mathrel{|\!\sim}^{\cap}_{\mathsf{L},\emptyset,\mathsf{mcs}} \neg(p \wedge q)$. However, where $\mathcal{S}' = \mathcal{S} \cup \{p \wedge q\}$, $\mathsf{Free}^{\emptyset}_{\mathsf{L}}(\mathcal{S}') = \emptyset$ and so $\mathcal{S}' \not\mathrel{|\!\sim}^{\cap}_{\mathsf{L},\emptyset \mathsf{mcs}} (p \wedge r) \supset \neg q$.

Finally, we give a counter-example for $\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathsf{set},\emptyset,\mathsf{sem}}$ where $\mathsf{sem} \in \mathsf{ME}$.

Example 14. Let $\mathcal{S} = \{p \wedge s, q \wedge s, \neg(p \wedge q) \wedge (s \supset t)\}$. We have the following stable (which are also preferred, semi-stable and stage) extensions: $\mathcal{E}_1 = \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{p \wedge s, q \wedge s\})$, $\mathcal{E}_2 = \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{p \wedge s, \neg(p \wedge q) \wedge (s \supset t)\})$, $\mathcal{E}_3 = \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{q \wedge s, \neg(p \wedge q) \wedge (s \supset t)\})$, and $\mathcal{E}_4 = \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{p \wedge s\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{q \wedge s\}) \cup \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{\neg(p \wedge q) \wedge (s \supset t)\})$. Note that $\mathcal{S} \not\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathsf{set},\emptyset,\mathsf{sem}} \neg\neg t$ and $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathsf{set},\emptyset,\mathsf{sem}} s$.

Let now $\mathcal{S}' = \mathcal{S} \cup \{\neg t\}$. Note that we now also have the stable (which is also a preferred, semi-stable and stage) extension $\mathcal{E} = \mathsf{Arg}^{\emptyset}_{\mathsf{L}}(\{\neg(p \wedge q) \wedge (s \supset t), \neg t\})$. So, $\mathcal{S}' \not\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathsf{set},\emptyset,\mathsf{sem}} s$.

Yet, some entailments *are* rational. The next proposition lists them.

Proposition 29 *(rationality). Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic satisfying the conditions in Section 2.2 and in Fig. 7, and let $\mathsf{sem} \in \mathsf{SE}$. Then the entailments $\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ and $\mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ are rational.*

Proof. We note that $\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}} = \mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ since $\mathsf{sem} \in \mathsf{SE}$. Let now $\mathrel{|\!\sim} = \mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$. In view of Proposition 28 (Item 2) we just have to show that RM holds for $\mathrel{|\!\sim}$. Suppose then that $\mathcal{S} \mathrel{|\!\sim} \phi$ and $\mathcal{S} \not\mathrel{|\!\sim} \neg\psi$.

Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})$ be an $\mathcal{S}$ and $\mathcal{X}$-based framework whose core logic is L and the set A of attack rules consists only of dir-type attacks. Based on Theorem 3 (Item 4), and using its notations, we consider two cases: (a) $\mathcal{S}_\top$ is $\vdash_{\mathcal{X}}$-consistent, (b) $\mathcal{S}_\top$ is $\vdash_{\mathcal{X}}$-inconsistent. Consider (a). Then $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_\top)\}$. Thus, $\mathcal{S}_\top \cup \mathcal{X} \nvdash \neg\psi$ and so $\mathcal{S}_\top \cup \{\psi\} = (\mathcal{S} \cup \{\psi\})_\top$ is also $\vdash_{\mathcal{X}}$-consistent. Also, by the main supposition, there is an argument $\Gamma \Rightarrow \phi \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_\top)$. Again by Theorem 3 (Item 4), $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_\top \cup \{\psi\})\}$. Since $\Gamma \Rightarrow \phi \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_\top \cup \{\psi\})$, $\mathcal{S}, \psi \mathrel{|\!\sim} \phi$.

Consider (b). Then, by Theorem 3 (Item 4), $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\emptyset)\}$. Thus, also $(\mathcal{S} \cup \{\phi\})_\top$ is $\vdash_{\mathcal{X}}$-inconsistent, and so $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S} \cup \{\psi\})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\emptyset)\} = \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S}))$. Thus, by the main supposition, $\mathcal{S} \cup \{\psi\} \mathrel{|\!\sim} \phi$. □

Table 5 summarizes the results in this section.

Table 5
Summary of the results in Section 6.1 (the results concerning $\mathsf{sem} = \mathsf{cmp}$ follow from Note 22).

Postulates for non-monotonic inference	$\mathrel{|\!\sim}^{\cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$	$\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$	$\mathrel{|\!\sim}^{\cup}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$
cumulativity, $\dagger \in \{\mathsf{con}, \mathsf{dir}\}$	CMP	CMP	SE
cumulativity, $\dagger = \mathsf{set}$	CMP	SE ∪ {cmp}	SE
preferentiality, $\dagger = \mathsf{con}$	–	ME	–
preferentiality, $\dagger = \mathsf{dir}$	SE ∪ {cmp}	CMP	SE
preferentiality, $\dagger = \mathsf{set}$	–	–	–
rationality, $\dagger \in \{\mathsf{set}, \mathsf{con}\}$	–	–	–
rationality, $\dagger = \mathsf{dir}$	SE ∪ {cmp}	SE ∪ {cmp}	SE
monotonicity, $\dagger \in \{\mathsf{dir}, \mathsf{con}, \mathsf{set}\}$	–	–	ME ∪ {cmp}

Recall from Table 2 that, in almost all the cases, properties of extensions are satisfied by settings with con-attacks. In that respect, con-attacks may be preferred. However, as Table 5 indicates, as far as principles of non-monotonic reasoning are concerned, settings with dir-attacks may be superior.

6.2. Inference principles for inconsistency handling

We now consider properties that are related to reasoning with inconsistent assumptions.

Definition 19 *(rationality postulates IV).* Let $\mathcal{X} \subseteq \mathsf{WFF}(\mathcal{L})$ be a $\vdash$-consistent set of $\mathcal{L}$-formulas, $\mathrel{|\!\sim}_{\mathcal{X}} \subseteq \wp(\mathsf{WFF}(\mathcal{L}) \setminus \mathcal{X}) \times \mathsf{WFF}(\mathcal{L})$ and let $\vdash \subseteq \wp(\mathsf{WFF}(\mathcal{L})) \times \mathsf{WFF}(\mathcal{L})$ be a Tarskian consequence relation.

- We denote by $\mathcal{S}_1 \parallel \mathcal{S}_2$ that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *syntactically disjoint*, i.e., $\mathsf{Atoms}(\mathcal{S}_1) \cap \mathsf{Atoms}(\mathcal{S}_2) = \emptyset$.
- We say that a set $\mathcal{S}$ such that $\mathsf{Atoms}(\mathcal{S} \cup \mathcal{X}) \subsetneq \mathsf{Atoms}(\mathcal{L})$ is *contaminating* (w.r.t. $\mathrel{|\!\sim}_{\mathcal{X}}$), if for every $\mathcal{S}'$ such that $\mathcal{S} \cup \mathcal{X} \parallel \mathcal{S}'$ and for every $\mathcal{L}$-formula ϕ, it holds that $\mathcal{S} \mathrel{|\!\sim}_{\mathcal{X}} \phi$ iff $\mathcal{S}, \mathcal{S}' \mathrel{|\!\sim}_{\mathcal{X}} \phi$.

A contaminating set of formulas renders every syntactically disjoint set uninformative. For instance, in the context of classical logic the singleton $\{\phi \wedge \neg\phi\}$ is a contaminating set.

We say that $\mathrel{|\!\sim}_{\mathcal{X}}$ satisfies:

- *conservative $\vdash$-consistency:* for every $\vdash_{\mathcal{X}}$-consistent set $\mathcal{S}$ of $\mathcal{L}$-formulas and every $\mathcal{L}$-formula ψ it holds that $\mathcal{S} \mathrel{|\!\sim}_{\mathcal{X}} \psi$ iff $\mathcal{X}, \mathcal{S} \vdash \psi$.
- *paraconsistency:* for every distinct $p, q \in \mathsf{Atoms}(\mathcal{L})$ for which $q \notin \mathsf{Atoms}(\mathcal{X})$, it holds that $p, \neg p \not\mathrel{|\!\sim}_{\mathcal{X}} q$.
- *non-interference:* for every two sets $\mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{L}$-formulas, and every $\mathcal{L}$-formula ϕ such that $\mathcal{S}_1 \cup \{\phi\} \cup \mathcal{X} \parallel \mathcal{S}_2$, it holds that $\mathcal{S}_1 \mathrel{|\!\sim}_{\mathcal{X}} \phi$ iff $\mathcal{S}_1, \mathcal{S}_2 \mathrel{|\!\sim}_{\mathcal{X}} \phi$.
- *crash-resistance:* there is no $\mathrel{|\!\sim}_{\mathcal{X}}$-contaminating set of $\mathcal{L}$-formulas.

Conservative $\vdash$-consistency means that $\mathrel{|\!\sim}_{\mathcal{X}}$ is a conservative extension of $\vdash$ that coincides with the latter with respect to $\vdash_{\mathcal{X}}$-consistent premises. Paraconsistency [35] is a well-investigated logical property, disallowing the inference of every conclusion whatsoever from a single contradiction (see, e.g., [17,31] for some surveys on this subject with many references). Non-interference and the related notion of crash-resistance were introduced in [30] for similar purposes, and are mainly investigated in the context of logical argumentation frameworks. Non-interference expresses that a formula ϕ should be $\mathrel{|\!\sim}_{\mathcal{X}}$-entailed by $\mathcal{S}_1$ if and only if it is entailed by $\mathcal{S}_1 \cup \mathcal{S}_2$ where $\mathcal{S}_2$ contains only information irrelevant to $\mathcal{S}_1 \cup \mathcal{X}$ and ϕ (in the sense that $\mathcal{S}_2$ is syntactically disjoint from $\mathcal{S}_1 \cup \mathcal{X} \cup \{\phi\}$).

We start with conservative $\vdash$-consistency:

Proposition 30 *(mcs conservative $\vdash$-consistency). Given a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$. For every $\vdash$-consistent set $\mathcal{X}$ and $\star \in \{\cap, \Cap, \cup\}$, the entailment $\mathrel{|\!\sim}^{\star}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ satisfies conservative $\vdash$-consistency: For every $\vdash_{\mathcal{X}}$-consistent set $\mathcal{S}$ and formula ψ, we have: $\mathcal{X}, \mathcal{S} \vdash \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$.*

Proof. Suppose that $\mathcal{S}$ is a $\vdash_{\mathcal{X}}$-consistent set of $\mathcal{L}$-formulas. Then $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) = \{\mathcal{S}\}$ and, as a result, $\mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$, $\mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ and $\mathrel{|\!\sim}^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ coincide. To show that $\mathrel{|\!\sim}^{\star}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ coincides with $\vdash$ as well, note that: $\mathcal{X}, \mathcal{S} \vdash \psi$ iff $\psi \in \mathsf{CN}_{\mathsf{L}}(\mathcal{S} \cup \mathcal{X})$ iff $\psi \in \mathsf{CN}_{\mathsf{L}}(\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \cup \mathcal{X})$ iff $\mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \psi$. $\square$

Proposition 31 *(conservative $\vdash$-consistency). Given a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$. For every $\vdash$-consistent set $\mathcal{X}$, $\star \in \{\cap, \Cap, \cup\}$, $\mathsf{sem} \in \mathsf{CMP}$, and $\dagger \in \{\mathsf{set}, \mathsf{dir}, \mathsf{con}\}$, the entailment $\mathrel{|\!\sim}^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ satisfies conservative $\vdash$-consistency: For every $\vdash_{\mathcal{X}}$-consistent set $\mathcal{S}$ and formula ψ, we have: $\mathcal{X}, \mathcal{S} \vdash \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \psi$.*

Proof. Suppose that $\mathcal{S}$ is a $\vdash_{\mathcal{X}}$-consistent set of $\mathcal{L}$-formulas. Suppose first that there is some $a \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ such that $b \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ attacks a. Thus $\mathsf{Conc}(b) \Rightarrow \neg\phi$ is derivable, where $\phi \in \mathsf{Supp}(a)$ for frameworks with dir or con attacks, and $\phi = \bigwedge \mathsf{Supp}(a)$ for frameworks with set attacks (note that b cannot be a ConUcut-attacker, since $\mathcal{S}$ is supposed to be $\vdash_{\mathcal{X}}$-consistent). Thus, by [Cut], we derive $\mathsf{Supp}(b) \Rightarrow \neg\phi$. By [$\neg\Rightarrow$], Lemma 1 and [Cut], we derive $\mathsf{Supp}(b), \phi \Rightarrow$. Hence, by [$\wedge\Rightarrow$] and [$\Rightarrow\neg$], $\Rightarrow \neg(\bigwedge \mathsf{Supp}(b) \wedge \phi)$ is derivable, a contradiction to the consistency of $\mathcal{S}$. Thus $\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ is conflict-free. Therefore, $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\}$ for every $\mathsf{sem} \in \mathsf{CMP}$.

Let now $\Gamma \Rightarrow \phi \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Then $\Gamma \vdash \phi$ by the soundness of C, and by the monotonicity of $\vdash$ (Definition 1), $\mathcal{S}, \mathcal{X} \vdash \phi$. Hence, $\mathrel{|\!\sim}^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \subseteq \vdash$. Now, suppose that $\mathcal{S}, \mathcal{X} \vdash \phi$. Thus, by the completeness of C for L, and since L is finitary, there is a $\Gamma \subseteq \mathcal{S} \cup \mathcal{X}$, such that $\Gamma \Rightarrow \phi \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Since for every $\mathsf{sem} \in \{\mathsf{nav}, \mathsf{stg}, \mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{idl}, \mathsf{stb}, \mathsf{sstb}, \mathsf{egr}\}$ $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\mathcal{S})) = \{\mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})\}$, we have that $\mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}} \phi$. Thus $\vdash \subseteq \mathrel{|\!\sim}^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$. Altogether, $\vdash = \mathrel{|\!\sim}^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$. $\square$

We now turn to paraconsistency. We will show that it holds for uniform logics.

Definition 20 *(uniformity).* A logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ is said to be *uniform* [58,75], if for every two sets of $\mathcal{L}$-formulas $\mathcal{S}_1, \mathcal{S}_2$ and a formula ϕ such that $\mathcal{S}_2$ is both $\vdash$-consistent and syntactically disjoint from $\mathcal{S}_1 \cup \{\phi\}$, it holds that $\mathcal{S}_1 \vdash \phi$ iff $\mathcal{S}_1, \mathcal{S}_2 \vdash \phi$.

Note 26. By Łos-Suzsko Theorem [58], a compact propositional logic is uniform and structural, iff, it has a single characteristic matrix. Thus, classical logic, as well as many other logics, are uniform.

Lemma 20 *(mcs paraconsistency). Given a uniform logic* $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$*. For every* $\star \in \{\cap, \Cap, \cup\}$*, the entailment* $\mid\!\sim^{\star}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ *satisfies paraconsistency: for every distinct* $p, q \in \mathsf{Atoms}(\mathcal{L})$ *for which* $q \notin \mathsf{Atoms}(\mathcal{X})$ *it holds that* $p, \neg p \not\mid\!\sim^{\star}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} q$.

Proof. Let $p, q \in \mathsf{Atoms}(\mathcal{L})$, $p \neq q$, $q \notin \mathsf{Atoms}(\mathcal{X})$, and $\mathcal{S} = \{p, \neg p\}$. Then $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \subseteq \{\emptyset, \{p\}, \{\neg p\}\}$. We note that $\nvdash q$ by the structurality and non-triviality of L. Therefore, for any $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, by the uniformity of L and the $\vdash_{\mathcal{X}}$-consistency of $\mathcal{T}$, $\mathcal{T} \nvdash_{\mathcal{X}} q$. Therefore, $\mathcal{S} \not\mid\!\sim^{\star}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} q$. $\square$

Proposition 32 *(paraconsistency). Let* $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ *be uniform. For every complete semantics* $\mathsf{sem} \in \mathsf{CMP}$*,* $\star \in \{\cap, \Cap, \cup\}$ *and* $\dagger \in \{\mathsf{set}, \mathsf{con}\}$*, the entailment* $\mid\!\sim^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ *is paraconsistent.*

Proof. Let $p, q \in \mathsf{Atoms}(\mathcal{L})$, $p \neq q$, $q \notin \mathsf{Atoms}(\mathcal{X})$, sem be a complete semantics $\mathsf{sem} \in \mathsf{CMP}$ and $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\{p, \neg p\}))$. Suppose that there is an argument $a = \Gamma \Rightarrow q \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\{p, \neg p\})$. By the uniformity of L and since $q \notin \mathsf{Atoms}(\Gamma)$, Γ is $\vdash$-inconsistent or $\vdash q$. The latter is excluded by the structurality and non-triviality of L. Thus, a is attacked by an argument $b = \Delta \Rightarrow \neg \bigwedge \Gamma'$ where $\Gamma' \subseteq \Gamma \setminus \mathcal{X}$ and $\Delta \subseteq \mathcal{X}$. Since b has no attackers (given that $\Delta \subseteq \mathcal{X}$), $a \notin \mathcal{E}$. Altogether, this shows that there is no $a \in \mathcal{E}$ with $\mathsf{Conc}(a) = q$ and therefore $p, \neg p \not\mid\!\sim^{\star}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}} q$. $\square$

Proposition 33 *(paraconsistency II). Let* $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ *be uniform. For every complete semantics* $\mathsf{sem} \in \mathsf{CMP}$ *and* $\star \in \{\cap, \Cap, \cup\}$*, the entailment* $\mid\!\sim^{\star}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}}$ *is paraconsistent.*

Proof. Let $p, q \in \mathsf{Atoms}(\mathcal{L})$, $p \neq q$, $q \notin \mathsf{Atoms}(\mathcal{X})$. Let sem be a complete semantics and $\mathcal{E}$ a sem-extension of $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L},\mathsf{A}}(\{p, \neg p\})$. Suppose that there is an argument $a = \Gamma \Rightarrow q \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\{p, \neg p\})$. Thus, $\Gamma \setminus \mathcal{X} \vdash_{\mathcal{X}} q$. By the uniformity of L and since $q \notin \mathsf{Atoms}(\Gamma)$, $\Gamma \setminus \mathcal{X}$ is $\vdash_{\mathcal{X}}$-inconsistent or $\vdash q$. The latter is excluded by the structurality and non-triviality of L. By Proposition 2, $a \notin \mathcal{E}$. This shows that $p, \neg p \not\mid\!\sim^{\star}_{\mathsf{L},\mathsf{dir},\mathcal{X},\mathsf{sem}} q$. $\square$

We turn now to non-interference and crash-resistance. In the remainder of this section, we suppose that $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ is a uniform logic (as before, with a corresponding sound and complete calculus C in which the rules of the basic calculus from Fig. 2 are admissible). We also suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are syntactically disjoint sets of $\mathcal{L}$-formulas ($\mathcal{S}_1 \parallel \mathcal{S}_2$), and that both of them are disjoint from the set of strict assumptions (i.e., $(\mathcal{S}_1 \cup \mathcal{S}_2) \parallel \mathcal{X}$). In what follows we denote $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$.

Lemma 21. *Suppose that* $\mathcal{S}_1 \cup \{\phi\} \parallel \mathcal{S}_2$*. If* $\mathcal{S}_2$ *is* $\vdash_{\mathcal{X}}$*-consistent, we have that* $\mathcal{S}_1 \vdash_{\mathcal{X}} \phi$ *iff* $\mathcal{S}_1, \mathcal{S}_2 \vdash_{\mathcal{X}} \phi$*.*

Proof. The direction [⇒] follows from $\vdash$-monotonicity. The direction [⇐] holds since $\mathcal{S}_1, \mathcal{S}_2 \vdash_{\mathcal{X}} \phi$ and by $\vdash$-compactness, there is a finite $\Theta \subseteq \mathcal{X}$ for which $\mathcal{S}_1, \mathcal{S}_2, \Theta \vdash \phi$. By uniformity, $\mathcal{S}_1, \Theta \vdash \phi$, (recall that $\mathcal{S}_1 \cup \mathcal{S}_2 \parallel \mathcal{X}$, thus $\mathcal{S}_1 \cup \mathcal{S}_2 \parallel \Theta$), and so, by $\vdash$-monotonicity again, $\mathcal{S}_1, \mathcal{X} \vdash \phi$. Thus, $\mathcal{S}_1 \vdash_{\mathcal{X}} \phi$. $\square$

Lemma 22. *If* $\mathcal{T}_1 \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$ *and* $\mathcal{T}_2 \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)$ *then* $\mathcal{T}_1 \cup \mathcal{T}_2 \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$*.*

Proof. Suppose for a contradiction that $\mathcal{T}_1 \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$ and $\mathcal{T}_2 \in \mathsf{CN}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)$, however $\mathcal{T}_1 \cup \mathcal{T}_2$ is $\vdash_{\mathcal{X}}$-inconsistent. Then, there are $\Gamma_1 \subseteq \mathcal{T}_1$ and $\Gamma_2 \subseteq \mathcal{T}_2$ for which $\vdash_{\mathcal{X}} \neg \bigwedge(\Gamma_1 \cup \Gamma_2)$. By Lemma 2, $\Gamma_1 \vdash_{\mathcal{X}} \neg \bigwedge \Gamma_2$ and by Lemma 21, either $\mathcal{X} \vdash \neg \bigwedge \Gamma_2$ or Γ_1 is $\vdash_{\mathcal{X}}$-inconsistent. But then either $\mathcal{T}_1$ or $\mathcal{T}_2$ is $\vdash_{\mathcal{X}}$-inconsistent, contradicting our assumption. $\square$

Lemma 23. $\mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) = \{\mathcal{T}_1 \cup \mathcal{T}_2 \mid \mathcal{T}_1 \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1), \mathcal{T}_2 \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)\}$.

Proof. We show inclusions in both directions. [⊇]: This follows from Lemma 22. [⊆]: Suppose that $\mathcal{T} \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, and let $\mathcal{T}_i = \mathcal{T} \cap \mathcal{S}_i$ for $i \in \{1, 2\}$. By $\vdash$-monotonicity, if $\mathcal{T}_i$ were $\vdash_{\mathcal{X}}$-inconsistent then also $\mathcal{T}$ would be $\vdash_{\mathcal{X}}$-inconsistent. Thus, $\mathcal{T}_i \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_i)$. $\square$

Lemma 24. $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) = \{\mathcal{T}_1 \cup \mathcal{T}_2 \mid \mathcal{T}_1 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1), \mathcal{T}_2 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)\}$.

Proof. We show inclusions in both directions.

[⊇]: Suppose that $\mathcal{T}_i \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_i)$ for $i \in \{1, 2\}$. By Lemma 22, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Assume for a contradiction that there is a $\mathcal{T}' \supsetneq \mathcal{T}$ such that $\mathcal{T}' \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. By Lemma 23, $\mathcal{T}'_i = \mathcal{T}' \cap \mathcal{S}_i \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_i)$ for $i \in \{1, 2\}$. Since $\mathcal{T}' \supsetneq \mathcal{T}$, also $\mathcal{T}'_1 \supsetneq \mathcal{T}_1$ or $\mathcal{T}'_2 \supsetneq \mathcal{T}_2$, which contradicts the ⊆-maximal $\vdash_{\mathcal{X}}$-consistency of $\mathcal{T}_1$ and $\mathcal{T}_2$.

[⊆]: Suppose that $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, and let $\mathcal{T}_i = \mathcal{T} \cap \mathcal{S}_i$ for $i \in \{1, 2\}$. By Lemma 23, $\mathcal{T}_i \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_i)$. Suppose that for some $i \in \{1, 2\}$ there is a $\mathcal{T}'_i \supsetneq \mathcal{T}_i$ for which $\mathcal{T}'_i \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_i)$. By Lemma 22, $\mathcal{T}'_i \cup \mathcal{T} \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Since $\mathcal{T}'_i \cup \mathcal{T} \supsetneq \mathcal{T}$, this contradicts the ⊆-maximality of $\mathcal{T}$. $\square$

By Lemma 24 and Item 1 of Theorem 3, we get:

Corollary 22. *Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})$ be an argumentation framework with attack rules of type $\dagger \in \{\mathsf{dir}, \mathsf{con}\}$, and let $\mathsf{sem} \in \mathsf{ME}$. Then $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) = \{Arg^{\mathcal{X}}_{\mathsf{L}}(\mathcal{T}_1 \cup \mathcal{T}_2) \mid \mathcal{T}_1 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1), \mathcal{T}_2 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)\}$.*

Lemma 25. *Let $\{\mathcal{T}_i \mid i \in I_1\} \Cup \{\mathcal{T}_i \mid i \in I_2\} = \{\mathcal{T}_i \cup \mathcal{T}_j \mid (i,j) \in I_1 \times I_2\}$. Then $\Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) = \{\omega_1 \Cup \omega_2 \mid \omega_1 \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1), \omega_2 \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)\}$.*

Proof. We show containments in both directions.

[$\supseteq$]: Let $\omega_i = \{\mathcal{T}_j \mid j \in I_i\} \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_i)$ for $i \in \{1,2\}$ and let $\omega = \omega_1 \Cup \omega_2$. We show that ω satisfies the two requirements from Definition 14.

For Item 1, let $(i_1, j_1), (i_2, j_2) \in I_1 \times I_2$. Then $\mathcal{T}_{i_1}, \mathcal{T}_{i_2} \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$ and $\mathcal{T}_{j_1}, \mathcal{T}_{j_2} \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)$. By Item 1 in Definition 14, $\mathcal{T}_{i_1} \cup \mathcal{T}_{i_2} \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$ and $\mathcal{T}_{j_1} \cup \mathcal{T}_{j_2} \in \mathsf{CS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)$. Consider $\mathcal{T} = \mathcal{T}_{i_1} \cup \mathcal{T}_{i_2} \cup \mathcal{T}_{j_1} \cup \mathcal{T}_{j_2}$. By Lemma 23 $\mathcal{T}$ is $\vdash_{\mathcal{X}}$-consistent.

For Item 2 let Θ be a finite subset of $\mathcal{S}$ and let $\Theta_i = \Theta \cap \mathcal{S}_i$ for $i \in \{1,2\}$. Suppose that $\Theta \not\subseteq \mathcal{T}_i \cup \mathcal{T}_j$ for any $(i,j) \in I_1 \times I_2$. Then, there is an $i \in \{1,2\}$ such that $\Theta_i \not\subseteq \mathcal{T}_j$ for all $j \in I_i$. Since $\omega_i \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_i)$, $\Theta_i \cup \mathcal{T}_j$ is $\vdash_{\mathcal{X}}$-inconsistent for some $j \in I_i$. Thus, $\Theta \cup \mathcal{T}$ is $\vdash_{\mathcal{X}}$-inconsistent for any $\mathcal{T} \supseteq \mathcal{T}_j$ in ω.

[$\subseteq$]: Let $\{\mathcal{T}_i \mid i \in I\} \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. We show that $\omega_j = \{\mathcal{T}_i \cap \mathcal{S}_j \mid i \in I\} \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_j)$ for $j \in \{1,2\}$. Without loss of generality, $j = 1$. We again show that the two conditions of Definition 14 hold for ω_1.

For Item 1 let $i, k \in I$. Then $(\mathcal{T}_i \cap \mathcal{S}_1) \cup (\mathcal{T}_k \cap \mathcal{S}_1) \subseteq \mathcal{T}_i \cup \mathcal{T}_k$. Since $\mathcal{T}_i \cup \mathcal{T}_k$ is $\vdash_{\mathcal{X}}$-consistent, also $(\mathcal{T}_i \cap \mathcal{S}_1) \cup (\mathcal{T}_k \cap \mathcal{S}_1)$ is $\vdash_{\mathcal{X}}$-consistent.

For Item 2 let $\Theta \subseteq \mathcal{S}_1$. Suppose that $\Theta \not\subseteq (\mathcal{T}_i \cap \mathcal{S}_1)$ for all $i \in I$. Since $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, also $\Theta \not\subseteq \mathcal{T}_i$ for all $i \in I$. Thus, $\Theta \cup \mathcal{T}_i$ is $\vdash_{\mathcal{X}}$-inconsistent for some $i \in I$. By Lemma 21, $\Theta \cup (\mathcal{T}_i \cap \mathcal{S}_1)$ is $\vdash_{\mathcal{X}}$-inconsistent. □

By Lemma 25 and Item 2 of Theorem 3, we have:

Corollary 23. *Let $\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})$ be an $\mathcal{S}$ and $\mathcal{X}$-based framework with attack rules of type set, and let $\mathsf{sem} \in \mathsf{ME}$. Then $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\mathcal{X}}_{\mathsf{L,A}}(\mathcal{S})) = \{Arg^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})(\omega_1 \Cup \omega_2) \mid \omega_1 \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1), \omega_2 \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)\}$.*

Now we are ready to show non-interference. First, we show this property for MCS-based and Ω-based entailments, from which we can conclude non-interference with respect to argumentative entailments.

Proposition 34 *(non-interference for mcs-based entailments). Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a uniform logic satisfying the conditions in Section 2.2. Then for every $\vdash$-consistent set $\mathcal{X}$ and $\star \in \{\cup, \cap, \Cap\}$, the entailment $\mathrel{|\!\sim}^{\star}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$ satisfies non-interference.*

Proof. For fixed L and $\mathcal{X}$, we distinguish between the three cases where $\star \in \{\cup, \cap, \Cap\}$.

[$\Rightarrow$] Suppose that $\mathcal{S}_1 \mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$. Thus, for all $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$, $\mathcal{T} \vdash_{\mathcal{X}} \phi$. Let $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. By Lemma 24, $\mathcal{T}_1 = \mathcal{T} \cap \mathcal{S}_1 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$. Thus, $\mathcal{T}_1 \vdash_{\mathcal{X}} \phi$ and by monotonicity, $\mathcal{T} \vdash_{\mathcal{X}} \phi$. So $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$.

[$\Leftarrow$] Suppose that $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$. Thus, for all $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$, $\mathcal{T} \vdash_{\mathcal{X}} \phi$. Let $\mathcal{T}_i \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_i)$ for $i \in \{1,2\}$. Then, by Lemma 24, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. It follows that $\mathcal{T}_1, \mathcal{T}_2 \vdash_{\mathcal{X}} \phi$ and by Lemma 21, $\mathcal{T}_1 \vdash_{\mathcal{X}} \phi$. Since $\mathcal{T}_1$ is arbitrary in $\mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$, $\mathcal{S}_1 \mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$.

[$\Rightarrow$] Suppose that $\mathcal{S}_1 \mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$. Thus, $\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1) \vdash_{\mathcal{X}} \phi$. By Lemma 24, $\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) = \bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1) \cup \bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)$. Hence, by monotonicity, $\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \vdash_{\mathcal{X}} \phi$. So, $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$.

[$\Leftarrow$] Suppose that $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$. Thus, $\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \vdash_{\mathcal{X}} \phi$. By Lemma 21, $\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \cap \mathcal{S}_1 \vdash_{\mathcal{X}} \phi$. By Lemma 24, $\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1) = \bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}) \cap \mathcal{S}_1$. Thus, $\bigcap \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1) \vdash_{\mathcal{X}} \phi$, and so $\mathcal{S}_1 \mathrel{|\!\sim}^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$.

[$\Rightarrow$] Suppose that $\mathcal{S}_1 \mathrel{|\!\sim}^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$. Thus, there is a $\mathcal{T}_1 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$ for which $\mathcal{T}_1 \vdash_{\mathcal{X}} \phi$. Let $\mathcal{T}_2 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)$. By Lemma 24, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. By monotonicity $\mathcal{T} \vdash_{\mathcal{X}} \phi$, and so $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$.

[$\Leftarrow$] Suppose that $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}} \phi$. Thus, there is a $\mathcal{T} \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ for which $\mathcal{T} \vdash_{\mathcal{X}} \phi$. By Lemma 24, $\mathcal{T}_1 = \mathcal{T} \cap \mathcal{S}_1 \in \mathsf{MCS}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$. By Lemma 21, $\mathcal{T}_1 \vdash_{\mathcal{X}} \phi$. Thus, $\mathcal{S}_1 \mathrel{|\!\sim}^{\cup}_{\mathsf{L},\mathcal{X}\mathsf{mcs}} \phi$. □

Proposition 35 *(non-interference for Ω-based entailments). Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a uniform logic satisfying the conditions in Section 2.2. Then for every $\vdash$-consistent set $\mathcal{X}$ and $\star \in \{\cap, \Cap, \cup\}$, the entailment $\mathrel{|\!\sim}^{\star}_{\mathsf{L},\mathcal{X},\Omega}$ satisfies non-interference.*

Proof. We show the claim for $\star = \Cap$. The other two cases ($\star \in \{\cap, \cup\}$) are similar (using Lemmas 21 and 25) and are left to the reader.

[$\Rightarrow$] Suppose that $\mathcal{S}_1 \mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathcal{X},\Omega} \phi$. Thus, for every $\omega_1 \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$ there is a set $\mathcal{T} \in \omega_1$ for which $\mathcal{T} \vdash_{\mathcal{X}} \phi$. Let $\omega \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. By Lemma 25, there are $\omega_1 \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$ and $\omega_2 \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)$, for which $\omega = \omega_1 \Cup \omega_2$. Thus, there is a set $\mathcal{T}_1 \in \omega_1$ for which $\mathcal{T}_1 \vdash_{\mathcal{X}} \phi$. Let $\mathcal{T}_2 \in \omega_2$ be arbitrary. Then $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \in \omega$, and by monotonicity, $\mathcal{T} \vdash_{\mathcal{X}} \phi$. Thus, $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{L},\mathcal{X},\Omega} \phi$.

[⇐] Suppose that $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\Omega} \phi$. Thus, for all $\omega \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$ there is a set $\mathcal{T} \in \omega$ for which $\mathcal{T} \vdash_{\mathcal{X}} \phi$. Let $\omega_1 \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_1)$. We have to show that there is a set $\mathcal{T}_1 \in \omega_1$ for which $\mathcal{T}_1 \vdash_{\mathcal{X}} \phi$. Let ω_2 be arbitrary in $\Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}_2)$. By Lemma 25, $\omega = \omega_1 \Cup \omega_2 \in \Omega^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S})$. Thus, there is a $\mathcal{T} \in \omega$ for which $\mathcal{T} \vdash_{\mathcal{X}} \phi$. Again, by Lemma 25, $\mathcal{T} \cap \mathcal{S}_1 \in \omega_1$ and by Lemma 21, $(\mathcal{T} \cap \mathcal{S}_1) \vdash_{\mathcal{X}} \phi$. Thus, $\mathcal{S}_1 \mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\Omega} \phi$. □

Corollary 24 *(Non-Interference). Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a uniform logic satisfying the conditions in Section 2.2, and let $\mathcal{X}$ be a $\vdash$-consistent set of $\mathcal{L}$-formulas. Then entailments of the form $\mid\!\sim^{\star}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$ satisfy non-interference in the following cases:*

1. $\dagger \in \{\mathsf{con}, \mathsf{set}\}$, $\star \in \{\cap, \Cap\}$, $\mathsf{sem} \in \mathsf{CMP}$.
2. $\dagger \in \{\mathsf{con}, \mathsf{set}, \mathsf{dir}\}$, $\star = \cup$, $\mathsf{sem} \in \mathsf{ME}$.
3. $\dagger = \mathsf{dir}$, $\star \in \{\cap, \Cap\}$, $\mathsf{sem} \in \mathsf{ME}$.

Proof. The items are immediate consequences of the results listed in the following table:

Item	$\cap$	$\Cap$	$\cup$
1, $\dagger = \mathsf{con}$	Theorem 4 (3), Proposition 34	Theorem 4 (2,5), Proposition 34	–
1, $\dagger = \mathsf{set}$	Theorem 4 (3), Proposition 34	SE: Theorem 4 (5), Proposition 34 ME: Theorem 5, Proposition 35	–
2	–	–	Theorem 4 (1), Proposition 34
3	Theorem 4 (4), Proposition 34	Theorem 4 (2), Proposition 34	–

□

Proposition 36 *(crash-resistance). All the entailments relations in Corollary 24 satisfy crash-resistance.*

Proof. Let $\mid\!\sim_{\mathcal{X}}$ be any of the entailments relations in Corollary 24 (given a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ as in the corollary and a $\vdash$-consistent set $\mathcal{X}$), and let $\mathcal{S}'$ be any set of $\mathcal{L}$-formulas for which there is a atom $p \in \mathsf{Atoms}(\mathcal{L}) \setminus (\mathsf{Atoms}(\mathcal{S}') \cup \mathsf{Atoms}(\mathcal{X}))$. By the non-triviality of $\vdash$, necessarily $\nvdash p$. Suppose first that $\mathcal{S}'$ is $\vdash_{\mathcal{X}}$-consistent. Then by Lemma 21, $\mathcal{S}' \nvdash_{\mathcal{X}} p$ and therefore $\mathcal{S}' \not\mid\!\sim_{\mathcal{X}} p$. Suppose now that $\mathcal{S}'$ is $\vdash_{\mathcal{X}}$-inconsistent. Then, by Lemma 21, if there is an argument $a = \Gamma \Rightarrow p \in \mathsf{Arg}^{\mathcal{X}}_{\mathsf{L}}(\mathcal{S}')$, Γ is $\vdash_{\mathcal{X}}$-inconsistent. By Proposition 2 and 3, $\mathcal{S}' \not\mid\!\sim_{\mathcal{X}} p$.

On the other hand, since $p \mid\!\sim_{\mathcal{X}} p$ (by Proposition 26), by non-interference (shown in Corollary 24), $p, \mathcal{S}' \mid\!\sim_{\mathcal{X}} p$. Therefore, $\mathcal{S}'$ cannot be contaminating. Since $\mathcal{S}'$ was arbitrary this completes the proof. □

Note 27. The only case missing in Corollary 24 and in Proposition 36, which involves skeptical entailments (i.e., when $\star \in \{\cap, \Cap\}$), is for $\dagger = \mathsf{dir}$ and $\mathsf{sem} \in \mathsf{SE}$. Indeed, we give a counter-example for this case. Let $\mathcal{X} = \emptyset$, $\mathcal{S}_1 = \{p\}$ and $\mathcal{S}_2 = \{q, \neg q\}$. It is easy to see that $\mathcal{S}_1 \mid\!\sim^{\star}_{\mathsf{L},\dagger,\emptyset,\mathsf{sem}} p$ while $\mathcal{S}_1, \mathcal{S}_2 \not\mid\!\sim^{\star}_{\mathsf{L},\dagger,\emptyset,\mathsf{sem}} p$.[26] Note also that $\{q, \neg q\}$ is a contaminating set in this case.

We are now ready to summarize the results in this section.

Note 28. Throughout this section we have shown the results first for mcs-based entailments and then applied them to argumentation-based entailments. Some of the mcs-based results are novel (also due to the distinction between strict and defeasible premises). An overview of the results concerning mcs-based reasoning provided in this paper is given in Table 6. (These results may be compared with those regarding simple contrapositive ABFs, provided in [52, Section 6.4], and concerning ASPIC and ABA in general, provided in [8, Section 2.3.3].)

Table 6
Summary of the results on mcs-based entailments. For paraconsistency, non-interference and crash-resistance L is assumed to be uniform.

Postulates	$\mid\!\sim^{\cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$	$\mid\!\sim^{\Cap}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$	$\mid\!\sim^{\cup}_{\mathsf{L},\mathcal{X},\mathsf{mcs}}$
cumulativity	✓	✓	–
preferentiality	–	✓	–
rationality	–	–	–
monotonicity	–	–	✓
conservative $\vdash$-consistency	✓	✓	✓
paraconsistency	✓	✓	✓
non-interference	✓	✓	✓
crash-resistance	✓	✓	✓

Table 7 summarizes the results on inconsistency-tolerance by argumentative entailments.

[26] See similar considerations in Example 7.

Table 7
Summary of the results in Section 6.2. For paraconsistency, non-interference and crash-resistance L is also assumed to be uniform. (The results concerning sem = cmp follow from Note 22.)

Postulates for inconsistency handling	$\mid\!\sim^{\cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$	$\mid\!\sim^{\Cap}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$	$\mid\!\sim^{\cup}_{\mathsf{L},\dagger,\mathcal{X},\mathsf{sem}}$
conservative $\vdash$-consistency	CMP	CMP	CMP
paraconsistency	CMP	CMP	CMP
non-interference, $\dagger \in \{\mathsf{con}, \mathsf{set}\}$	CMP	CMP	$\mathsf{ME} \cup \{\mathsf{cmp}\}$
non-interference, $\dagger = \mathsf{dir}$	ME	ME	$\mathsf{ME} \cup \{\mathsf{cmp}\}$
crash-resistance, $\dagger \in \{\mathsf{con}, \mathsf{set}\}$	CMP	CMP	$\mathsf{ME} \cup \{\mathsf{cmp}\}$
crash-resistance, $\dagger = \mathsf{dir}$	ME	ME	$\mathsf{ME} \cup \{\mathsf{cmp}\}$

7. Related work

In the context of formal argumentation, postulate-based investigations of argumentation frameworks play a primary role, allowing not only to indicate how the ingredients of the framework affect its properties, but also to compare related approaches to argumentation-based reasoning. In [48], for instance, Gorogiannis and Hunter study the properties of attack relations in logic-based argumentation frameworks. In particular, they consider various necessary and sufficient conditions on attack relations similar to those in Definition 4. Unlike our case, however, the discussion in [48] is concentrated on classical logic as the base logic of the frameworks, where the supports of the arguments are assumed to be classically consistent and the minimal ones that entail the argument's conclusion (recall Note 1).

Studies on requirements on the attack relations to fulfill rationality postulates are also presented in [3,76], where the conditions are somewhat different than the ones presented here, and include, among others,

- *conflict-dependence*: for each $(a, b) \in \mathcal{A}$, $\mathsf{Supp}(a) \cup \mathsf{Supp}(b) \vdash \mathsf{F}$,[27]
- *conflict-sensitivity*: for each $a, b \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$, if $\mathsf{Supp}(a) \cup \mathsf{Supp}(b) \vdash \mathsf{F}$ then $(a, b) \in \mathcal{A}$, and
- *validity*: for each $\mathcal{E} \subseteq \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$, if $\mathcal{E}$ is conflict-free, then $\mathsf{Supp}(\mathcal{E})$ is consistent.

Again, the discussion is limited to classical logic as the base logic and to arguments whose support sets are both consistent and minimal in the sense discussed previously.

The interplay between logical principles about argumentation, on the one hand, and inference principles as studied in proof theory, on the other hand, is also considered in [33]. In that paper a series of logical principles of attack relations in argumentation frameworks is stated, and their collection leads to a characterization of classical logical consequence relations that only involves argumentation frameworks. We refer to [33] and [34] for further details.

Properties of extensions of logic-based argumentation frameworks are studied in, e.g., [2,3,29,48], again with respect to restricted supports of arguments. For checking the postulates, the following two properties of attack relations are assumed in [2]:

- if $\mathsf{Supp}(a) \subseteq \mathsf{Supp}(b)$, then $(a, c) \in \mathcal{A}$ implies $(b, c) \in \mathcal{A}$,
- if $\mathsf{Supp}(a) \subseteq \mathsf{Supp}(b)$, then $(c, a) \in \mathcal{A}$ implies $(c, b) \in \mathcal{A}$.

Postulates that are related to reasoning with maximally consistent subsets [71] play a primary role in several works and may be traced back to Cayrol [32]. For detailed discussions and surveys on this subject we refer to [7,10]. Rationality postulates for other forms of structured argumentation, such as ASPIC$^+$ and ABA systems, can be found, e.g., in [38,61,62] (for ASPIC$^+$ systems), in [37,53] (for ABA systems), and in [41] (for logic-associated abstract argumentation frameworks). We refer to the survey in [8] for some comparisons of the rationality postulates that these approaches to structured argumentation satisfy. Moreover, a variety of ABA frameworks (such as the simple contrapositive ABAs in [51] and (non-prioritized) ASPIC systems) can be embedded in sequent-based argumentation frameworks,[28] thus the results provided here may be carried on to those systems.

Our study involves some ideas and notions from proof theory.[29] The main contribution of this work in relation to related works such as the ones mentioned above is that it provides a comprehensive presentation of the semantical as well as the inferential properties of logic-based argumentation frameworks, where only minimal (proof-theoretical) requirements are made on the base logic and very little is assumed on the form of the arguments. This allows to capture a wide range of core logics and to base arguments only on deducibility in the core logic. In our study, we avoid the use of further conditions (such as conflict dependence, conflict sensitivity and the condition discussed in Notes 12 and 15) that are computationally demanding, and so are difficult to verify.

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[27] Where F is the propositional constant for falsity, satisfying $\mathsf{F} \vdash \psi$ for every formula ψ.

[28] We refer to [26] and [27] for some translations to sequent-based frameworks and a general approach to structured argumentation, respectively.

[29] The incorporation of proof theoretical concepts and techniques in order to investigate and implement specific logical argumentation frameworks is not new (see, for instance, [14,15,27,43,49,50]).

Studies of inferential behavior of logical argumentation, and in particular its relation to non-monotonic reasoning, can be found in [15, Section 5], in the context of dynamic proof systems. Similar studies for ABA and ASPIC systems appear, respectively, in [36,52,53] and [56]. A comprehensive survey that relates these disciplines and mentions further results regarding logic-based approaches to formal argumentation appears in [8]. This survey also cites (without proofs) some results from our paper. The main results that were formulated after the writing of the survey (and therefore do not appear in it) are related to the extensions characterization according semantics classes (Theorem 1), the evaluation of the induced entailments (Section 5.2 and several parts in Section 6), the incorporation of strict premises in addition to the defeasible ones, and the coverage of all the Dung-style completeness-based semantics (including stage, eager, and ideal semantics, which are not considered in [8]).

8. Conclusion

Postulate-based studies are a common approach to evaluate and compare different formalisms sharing a similar purpose. In some cases (like the AGM postulates for belief revision [1]) this is a cornerstone of a formal discipline that serves as a standard setting and a trigger for a variety of related works and formalisms. In this work we have provided a comprehensive postulate-based study of logical (sequent-based) argumentation frameworks, based on propositional languages and Tarskian logics that satisfy some very basic assumptions. This study covers all the central postulates for argumentation frameworks in this context (as well as some new ones), and refers to all the completeness-based Dung-style semantics, as well as the main (support-based) attack rules between arguments. It therefore lays the foundations for logical argumentation frameworks allowing for the making of logically justified inferences in the presence of possibly conflicting defeasible and of strict assumptions.

The results of this paper are summarized in Tables 2–7. Our findings also allow us to provide full characterization results concerning the extensions of sequent-based frameworks (Theorems 1 and 3) and the induced entailment relations (Theorems 4 and 5).

Future work involves, among other, the study of more expressive formalisms, like those that are based on first-order logics, or formalisms that incorporate priorities among the arguments (see, e.g., [9] for a description of how this can be done in the context of sequent-based argumentation).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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