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Graphs of low average degree without independent transversals

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Abstract

An independent transversal of a graph G with a vertex partition $\mathcal{P}$ is an independent set of G intersecting each block of $\mathcal{P}$ in a single vertex. Wanless and Wood proved that if each block of $\mathcal{P}$ has size at least t and the average degree of vertices in each block is at most $t/4$, then an independent transversal of $\mathcal{P}$ exists. We present a construction showing that this result is optimal: for any $\varepsilon > 0$ and sufficiently large t, there is a forest with a vertex partition into parts of size at least t such that the average degree of vertices in each block is at most $(\frac{1}{4} + \varepsilon)t$, and there is no independent transversal. This unexpectedly shows that methods related to entropy compression such as the Rosenfeld–Wanless–Wood scheme or the Local Cut Lemma are tight for this problem. Further constructions are given for variants of the problem, including the hypergraph version.

KEYWORDS
average degree, counterexample, hypergraph, independent transversal, maximum block average degree

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1 | INTRODUCTION

Graphs in this paper contain no parallel edges or loops. Let G be a graph and let $\mathcal{P}$ be a partition of its vertex set $V(G)$ (also called a *vertex partition* of G). The sets in $\mathcal{P}$ are called the *blocks* of $\mathcal{P}$. Following [13], we say that $\mathcal{P}$ is *t-thick* if the size of each block of $\mathcal{P}$ is at least t. An *independent transversal* of $\mathcal{P}$ in G is an independent set in G intersecting each block of $\mathcal{P}$ in a single vertex.

Conditions for the existence of independent transversals in G in relation to the maximum degree $\Delta(G)$ of G date back to a 1972 conjecture of Erdős [12]: if $\mathcal{P}$ is *t*-thick and $\Delta(G) \leq t - 1$, then there is an independent transversal of $\mathcal{P}$ in G. The conjecture was disproved by Seymour (cf. [8]).

Bollobás et al. [8] constructed graphs G with a *t*-thick partition $\mathcal{P}$ with n blocks (for any $n \geq 3$) such that $\Delta(G) = \frac{t}{2} + \frac{t}{n-2}$ and $\mathcal{P}$ has no independent transversal in G. They conjectured that for any $\varepsilon > 0$ and large enough $|\mathcal{P}|$, $\Delta(G) \leq (\frac{1}{2} - \varepsilon)t$ ensures the existence of an independent transversal of a *t*-thick partition $\mathcal{P}$.

Alon [4] and Fellows [13] independently proved that there is a constant $\alpha > 0$ such that any *t*-thick vertex partition of a graph G with $\Delta(G) \leq \alpha t$ admits an independent transversal (with $\alpha = \frac{1}{25}$ and $\alpha = \frac{1}{16}$, respectively). Both proofs are based on the Lovász Local Lemma. An improvement to $\alpha = \frac{1}{2e}$ using the same method was proved by Alon [5] (see also [7, chapter 5.5]).

Constructions in [18, 27] show that for any d which is a power of two, there is a graph G with a $(2d - 1)$-thick partition $\mathcal{P}$ such that $\Delta(G) = d$ and $\mathcal{P}$ has no independent transversal. (This was later extended to every d by Szabó and Tardos [25].) Consequently, the above constant α must be at least $\frac{1}{2}$.

Haxell [15] finally proved the optimal sufficient condition for the existence of an independent transversal in terms of maximum degree.

Theorem 1 (Haxell [15, theorem 2]). *Let $\mathcal{P}$ be a t-thick vertex partition of a graph G with $\Delta(G) \leq \frac{t}{2}$. Then $\mathcal{P}$ has an independent transversal in G.*

There is a close connection between independent transversals in a graph and topological properties of its 'independence complex'. These connections were first uncovered in [3] and further developed in, for example [1, 21, 22]. Another closely related notion is that of strong colouring of graphs which was introduced in [4, 13] (see [16] and the references therein).

Reed and Wood [23] observed that the Lovász Local Lemma implies a version of Theorem 1 in terms of average degree rather than maximum degree. Given a graph G and a partition $\mathcal{P}$ of $V(G)$, let us say that the degree $d(B)$ of a block $B \in \mathcal{P}$ is the number of edges with (precisely) one endvertex in B. Furthermore, the *maximum block average degree* of $\mathcal{P}$ of $V(G)$ is the maximum, over all blocks B, of $d(B)/|B|$. Reed and Wood's result then states that if $\mathcal{P}$ is a *t*-thick partition of $V(G)$ with maximum block average degree at most $\frac{t}{2e}$, then $\mathcal{P}$ has an independent transversal in G.

In [11], an equivalent problem related to the single-conflict chromatic number is considered, and it is noted that a stronger sufficient condition with $\frac{t}{2e}$ replaced by $\frac{t}{4}$ follows from the Local Cut Lemma [10].

Another approach which yields the same improvement is the one of Wanless and Wood [26], who apply the idea of Rosenfeld [24] to a number of problems related to hypergraph colouring, obtaining simpler proofs and sometimes stronger results compared to earlier applications of the Lovász Local Lemma. Their result on independent transversals is stated in a more general form for hypergraphs. Let us restrict our attention to the graph case which reads as follows.

Theorem 2 (Wanless and Wood [26, theorem 9]). *If G is a graph and $\mathcal{P}$ a t-thick partition of $V(G)$ with maximum block average degree at most $\frac{t}{4}$, then there exist $\left(\frac{t}{2}\right)^{|\mathcal{P}|}$ independent transversals of $\mathcal{P}$ in G.*

It is natural to ask if the factor $\frac{1}{4}$ can be improved, perhaps even to $\frac{1}{2}$ so as to obtain a genuine strengthening of Theorem 1 in terms of maximum block average degree. This is mentioned as an open problem in [26] as well as in [19].

The main result of this paper answers this question, showing that the bound in Theorem 2 cannot be improved. This has the unexpected consequence that methods related to entropy compression such as the Rosenfeld–Wanless–Wood scheme or the Local Cut Lemma perform optimally for the given problem. To the best of our knowledge, this is the first nontrivial instance of a problem where this is known to be the case.

Theorem 3. *For all $\varepsilon > 0$ and sufficiently large t, there exists a forest F and a t-thick vertex partition with maximum block average degree at most $\left(\frac{1}{4} + \varepsilon\right)t$ and no independent transversal in F.*

Thus, if we let ε decrease to 0, there is a sharp transition: while maximum block average degree of $(\frac{1}{4} + \varepsilon)t$ does not imply an independent transversal for arbitrarily small nonzero ε, exponentially many are guaranteed at $\varepsilon = 0$ by Theorem 2.

In Section 2 we prove Theorem 3. In Section 3 we show that the average degree bound of Theorem 3 can be combined with an upper bound on the maximum degree of the graph. Section 4 discusses an extension of Theorem 3 to hypergraphs, which again demonstrates the optimality of the general bound obtained in [26]. We finish with various open problems in Section 5.

2 | LOW AVERAGE DEGREE FORESTS

In this section, we prove Theorem 3. Let G be a graph with a vertex partition $\mathcal{P}$. We say a vertex of G is *forbidden* if it cannot appear in an independent transversal of $\mathcal{P}$. If we can find a block of $\mathcal{P}$ in which every vertex is forbidden, this means no independent transversal exists. We say a subset S of $V(G)$ is *forced* if any independent transversal contains (at least one) vertex from S. As an example, if $B \in \mathcal{P}$ and all vertices in $A \subseteq B$ are forbidden, then $B \backslash A$ is forced.

Any vertex v which is joined entirely to a forced subset S is forbidden: an independent transversal must contain a vertex from S, and v is adjacent to every vertex in S, so a transversal containing v cannot be independent.

A subset $B \in \mathcal{P}$ is always forced. Suppose that a vertex $v \in B' \in \mathcal{P}$ is joined entirely to some block B. Then v is forbidden and therefore $B' \backslash \{v\}$ is forced and of strictly smaller size. The idea of our construction is to iterate this phenomenon.

2.1 | The construction

Let $0 = n_1 \leq \cdots \leq n_k = t$ be a given sequence of integers for some $t > 0$. We inductively define forests $F_1, F_2, \ldots, F_k$ which consist of blocks of several 'grades'; these blocks define the desired vertex partition. Each block will consist of some number of *light* vertices of
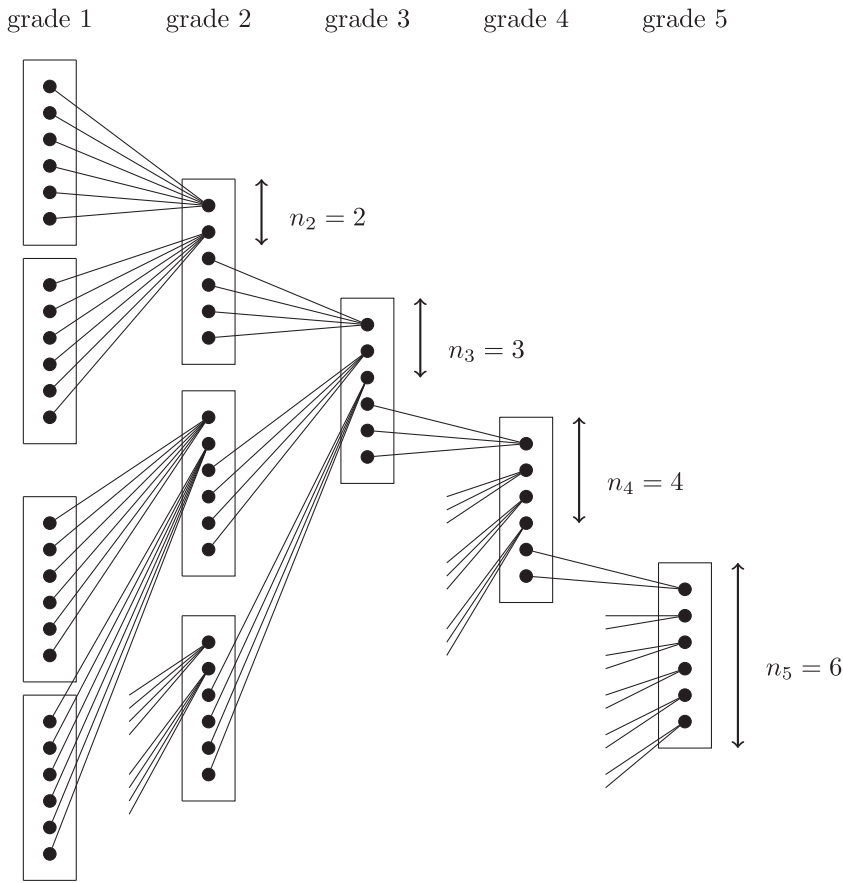
FIGURE 1 The construction of Section 2 is illustrated. The square boxes denote entire blocks of that grade

degree at most 1, and some number of *heavy* vertices of high degree. The construction is illustrated in Figure 1.

- Let F_1 be an independent set of size t. The vertex partition consists of a single block of grade 1 (consisting entirely of light vertices).
- Let F_2 consist of n_2 copies of F_1 plus t vertices in a single block of grade 2. The vertices in the block of grade 2 consist of $t - n_2$ light vertices and n_2 heavy vertices. Each heavy vertex is fully connected to a private copy of F_1.
- Having defined F_j for some $2 \leq j < k$, we define F_{j+1} as follows. Let F_{j+1} consist of n_{j+1} copies of F_j and a single block of grade $j + 1$, inducing an independent set of size t. The vertices in the block of grade $j + 1$ consist of $t - n_{j+1}$ light vertices and n_{j+1} heavy vertices, which are each fully adjacent to the $t - n_j$ light vertices in the unique block of grade j in a private copy of F_j. This defines a unique block of grade $j + 1$ in F_{j+1}, and it inherits the partition into blocks from the copies of F_j for the remaining vertices.

Within each forest F_i, every heavy vertex is forbidden. This can be proven by induction on the grade $j \in \{2, ..., i\}$; each heavy vertex of grade 2 is joined to an entire block of grade 1, so is forbidden. If the heavy vertices of grade j are forbidden, this means the light vertices of a grade j block form a

forced set, and so each heavy vertex of grade $j + 1$ is forbidden. In a block of grade k within F_k all vertices are heavy, and so no independent transversal exists.

2.2 | Analysis

To prove Theorem 3, it remains to show that there is a sequence of integers $(n_i)_{i=1}^k$ such that the construction of the previous subsection has low block average degree. This is shown in the following lemma.

Lemma 4. *For every $\varepsilon > 0$, there exists $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$, there is a sequence of integers $0 = n_1 < \cdots < n_k = t$ such that for all $j \in \{1, ..., k - 1\}$*

$$\frac{1}{t}(n_{j+1} \cdot (t - n_j) + (t - n_{j+1}) \cdot 1) \leq \left(\frac{1}{4} + \varepsilon\right)t. \tag{1}$$

Proof. Let $\varepsilon > 0$ be given. Let $\delta > 0$ and $t_1 \in \mathbb{N}$ be such that

$$\left(\frac{1}{4} + \delta\right)t + 1 \leq \left(\frac{1}{4} + \varepsilon\right)t \tag{2}$$

for all $t \geq t_1$. Let $t_0 \geq t_1$ such that $\frac{\delta t}{2} > 1$ for all $t \geq t_0$.

Let $t \geq t_0$ be given. We set $n_1 = 0$. For $j \geq 1$, while $n_j < \frac{3}{4}t$, we set n_{j+1} to be the minimum of $\left\lfloor \frac{1/4 + \delta}{1 - n_j/t}t \right\rfloor$ and t. If $n_j = t$ we set $k = j$ and abort the procedure. It remains to show (1) holds and that the procedure terminates.

We first show (1). For all integers $j \in \{1, ..., k - 1\}$, we find using the definition of n_j and (2) that

$$\frac{1}{t}(n_{j+1} \cdot (t - n_j) + (t - n_{j+1}) \cdot 1) \leq \frac{1}{t}\frac{(1/4 + \delta)t}{1 - n_j/t}(t - n_j) + 1$$

$$= \left(\frac{1}{4} + \delta\right)t + 1$$

$$\leq \left(\frac{1}{4} + \varepsilon\right)t.$$

We now show that $n_{j+1} > n_j$ if $n_j < t$, which implies that the procedure terminates. We need to be slightly careful in our analysis due to the floor in the definition of n_{j+1}. Since $\frac{\delta t}{2} > 1$, the interval from $\left(\frac{1}{4} + \frac{\delta}{2}\right)\frac{t}{1 - x}$ to $\left(\frac{1}{4} + \delta\right)\frac{t}{1 - x}$ has width greater than 1 for all $x < 1$, and so for $n_j < t$ we find

$$\frac{n_{j+1}}{t} \geq \frac{1/4 + \delta/2}{1 - n_j/t}.$$

We use the fact that $\frac{1/4 + 0}{1 - x} \geq x$ for $x < 1$ (which can be shown by proving that $x(1 - x)$ has a unique maximum of $\frac{1}{4}$ at $x = \frac{1}{2}$). For $n_j < t$, this fact implies that

$$\frac{1/4 + \delta/2}{1 - n_j/t} \geq \frac{n_j}{t} + \frac{\delta/2}{1 - n_j/t} > \frac{n_j}{t}.$$

Combining the two displayed inequalities above shows that $n_{j+1} > n_j$ if $n_j < t$. This implies that there exists a j for which $n_j \geq \frac{3}{4}t$, as desired. $\qquad\square$

Remark 5. One could instead take $n_j = j$ for each of $j = 1, ..., t$. Then $n_j > n_{j-1}$ and inequality (1) holds by the following sequence of inequalities for $t > 4/\varepsilon$:

$$\frac{1}{t}((j + 1)(t - j) + (t - j - 1)) \leq \frac{1}{t}(j(t - j) + 2t) \leq \frac{1}{4}t + 2 \leq \left(\frac{1}{4} + \varepsilon\right)t.$$

This gives a simplified proof of Lemma 4, however, we include the above argument since this generalises more easily to the maximum degree setting.

Proof of Theorem 3. Let $\varepsilon > 0$. Let $t_0 \in \mathbb{N}$ be given from Lemma 4. For every $t \geq t_0$, we obtain from the lemma a sequence of integers $0 = n_1 \leq \cdots \leq n_k = t$ such that (1) holds for all $j \in \{1, ..., k - 1\}$. We use this sequence of integers in the construction of Section 2.1 and claim that the graph F_k with its vertex partition satisfies the requirements. We already remarked that it is a forest and that it does not admit an independent transversal, so it suffices to verify that each block has average degree at most $(\frac{1}{4} + \varepsilon)t$. For $j \in \{1, ..., k - 1\}$, a block of grade $j + 1$ consists of n_{j+1} heavy vertices of degree $t - n_j$ and $t - n_{j+1}$ light vertices of degree 1. Therefore, the average degree of these blocks is at most $(\frac{1}{4} + \varepsilon)t$ by (1). A block of grade 1 consists of t vertices of degree $1 \leq (\frac{1}{4} + \varepsilon)t$ (for $t \geq 4$), so indeed all blocks have average degree at most $(\frac{1}{4} + \varepsilon)t$ for t sufficiently large. $\qquad\square$

The termination of the procedure in the proof of Lemma 4 can be predicted by studying the behaviour of the Möbius transformation $f: z \mapsto \frac{\alpha}{1-z}$ for $\alpha \in \left[\frac{1}{4}, 1\right)$. For $\alpha \in (\frac{1}{4}, 1)$ the transformation is conjugate to a rotation $z \mapsto kz$ for some $k \in \mathbb{C}$ with $|k| = 1$ and $k \neq 1$ (see [9, theorem 4.3.4] for more information). A Möbius transformation gets determined by the image of any three of its points and maps generalised circles to generalised circles. Since $0, 1, \infty$ get mapped to $\alpha, \infty, 0$, respectively, the orbit under iteration of f of any element on the real line must contain some element of $[1, \infty)$. The behaviour of the dynamical system determined by f changes dramatically at $\alpha = \frac{1}{4}$: here it has an attracting fixed point at $\frac{1}{2}$ in this case, and this is the reason that for such α we cannot find a sequence $(n_j)_{j=0}^{k}$ satisfying the assumptions of Lemma 4. This aligns nicely with the fact that when the average degree is at most $\frac{1}{4}t$, an independent transversal must always exist.

3 | BOUNDING THE MAXIMUM DEGREE

In our low block average degree forest construction in Section 2.1, the grade 2 blocks contained vertices joined entirely to a block, so these vertices have degree t. Is this the only obstruction to finding an independent transversal, or can we find an example of a construction where the

maximum degree of a vertex is strictly smaller? In this section, we resolve this question by presenting a slightly modified version of our earlier construction. We will only highlight the modifications.

Theorem 6. *For all $\varepsilon > 0$ and t sufficiently large, there is a graph with a t-thick vertex partition, maximum block average degree at most $(\frac{1}{4} + \varepsilon)t$ and maximum degree at most $0.854t$.*

Proof. In the construction of Section 2.1, the vertices of maximum degree arise because the forced sets at grade 1 form the entire block. If we can replace this with a smaller forced set, while maintaining the same average degree condition, we can decrease the maximum degree.

We define the new blocks of grade 1 as follows. Consider two vertex sets V_a, V_b of size t, which form two separate parts in our vertex partition but together form a block of grade 1. We create a complete bipartite graph between $A \subseteq V_a$ and $B \subseteq V_b$. The number of edges between blocks V_a and V_b is $|A||B|$, and no independent transversal can contain both a vertex from A and a vertex from B. Therefore, $V_a \cup V_b - (A \cup B)$ is a forced set of size $2t - |A| - |B|$.

This defines a graph F_1': a copy of $K_{|A|,|B|}$ (consisting of heavy vertices) together with $2t - |A| - |B|$ isolated (light) vertices. The associated block partition will be A, together with $t - |A|$ isolated vertices, and the remaining t vertices. In the previous construction, we will replace each copy of F_1 with a copy of F_1'. This is depicted in Figure 2.

We require $2t - |A| - |B| \leq 0.854t$ to satisfy the maximum degree condition for grade 2 blocks.

To ensure the new grade 1 blocks do not have too high average degree, we only need

$$\frac{|A||B|}{t} + 1 < \left(\frac{1}{4} + \varepsilon\right)t. \tag{3}$$

The maximum degree within this pair is $\max(|A|, |B|)$, so both $|A|$ and $|B|$ can be at most $0.854t$. In the following analysis we will omit floor signs; at the end, $|A|$ and $|B|$ can be rounded up (provided t is sufficiently large) without violating the conditions in the statement.

Suppose we set $|A| = \alpha t$ for some $\alpha \in (0, 1)$. Equation (3) is satisfied as long as $|B| \leq \frac{t}{4\alpha}$ (for t sufficiently large). Setting $|B| = \frac{t}{4\alpha}$, the maximum degree of grade 2 blocks is

$$2t - |A| - |B| \leq \left(2 - \alpha - \frac{1}{4\alpha}\right)t.$$

We take $\alpha \geq \frac{1}{2}$, in which case $|A| \geq |B|$. Then

$$\max\{|A|, |B|, 2t - |A| - |B|\} = \max\left\{\alpha, 2 - \alpha - \frac{1}{4\alpha}\right\}.$$

We minimise $\max\left\{\alpha, 2 - \alpha - \frac{1}{4\alpha}\right\}$ by setting the two functions to be equal, which yields $\alpha = \frac{1}{2} + \frac{1}{2\sqrt{2}} < 0.854$. This value now also upper bounds the maximum degree of
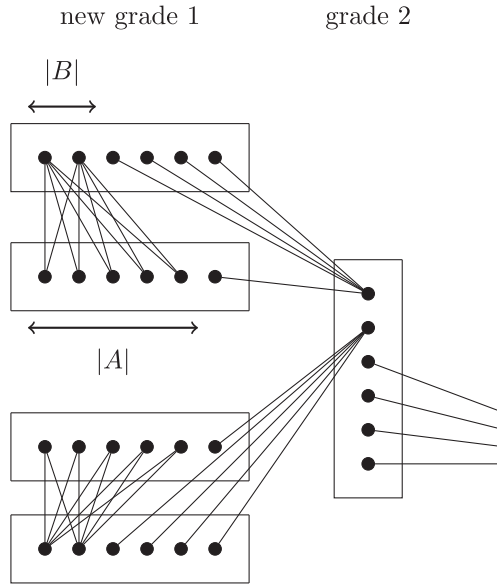
new grade 1   grade 2

FIGURE 2   Part of the construction of Theorem 6 is illustrated

grade 1 and grade 2 blocks. We set $n_1 = 2t - |A| - |B| < 0.854t$, and follow the construction from Lemma 4 otherwise. This will set $n_2 \geq 0.25t$, and $n_j \geq n_2$ for $j \geq 2$, and so the vertices in blocks of grade at least 3 all have maximum degree at most $0.75t$. □

Loh and Sudakov [20] introduced the notion of *local degree* for a graph G with a vertex partition $\mathcal{P}$ as the maximum number of edges from a vertex to some block. Instead of maximum degree we can look at local degree and get a slightly better upper bound in our construction.

Theorem 7. *For all $\varepsilon > 0$ and t sufficiently large, there is a graph with a t-thick partition $\mathcal{P}$ of $V(G)$, maximum block average degree at most $(\frac{1}{4} + \varepsilon)t$ and local degree at most $0.731t$.*

Proof sketch   We use the same construction as in the proof of Theorem 6, but we use $\alpha = 0.731$ instead. We still find $\alpha \geq \frac{1}{4\alpha}$, and the maximum number of edges a vertex in a block of grade 2 has to another block is hence $(1 - \frac{1}{4\alpha})t < 0.731t$. For t sufficiently large, with $n_2 = (2 - \alpha - \frac{1}{4\alpha})^{-1}\frac{t}{4}$, the vertices of in a block of grade 3 have maximum degree at most $t - n_2 \leq 0.731t$. For vertices in blocks of higher grade, the maximum degree will be even lower due to the fact that $n_j > n_{j-1}$ for $j \geq 3$. □

4 | EXTENSION TO HYPERGRAPHS

In this section, we illustrate how our results translate to the setting of r-uniform hypergraphs. In this setting, the edges are all of size r and an independent transversal must avoid containing all the vertices in an edge. Clearly, independent transversals are only obstructed by edges which are *stretched*—that is, each vertex of the edge lies in a different part of the partition. Again,

exponentially many independent transversals can be found if the block average degree is sufficiently small.

Theorem 8 (Wanless and Wood [26]). *Fix integers $r \geq 2$ and $t \geq 1$. Let G be an r-uniform hypergraph, and let $\mathcal{P} = \{V_1, ..., V_n\}$ be a t-thick partition of $V(G)$. Suppose that, for each $i \in \{1, ..., n\}$, at most $\frac{(r-1)^{r-1}}{r^r} t^{r-1}|V_i|$ stretched edges in G intersect V_i. Then there exist at least $\left(\frac{r-1}{r}\right)^n t^n$ independent transversals of $\mathcal{P}$.*

We show the result above is tight by extending the construction of Theorem 3.

Theorem 9. *Fix $r \geq 2$ and $\varepsilon > 0$. There is a function $C(\varepsilon, r)$ such that for all sufficiently large t, and any $n > t^{C(\varepsilon, r)}$ there exists an r-uniform hypergraph G with a t-thick partition $\mathcal{P} = \{V_1, ..., V_n\}$ of $V(G)$ such that for each $i \in \{1, ..., n\}$, at most $(\frac{(r-1)^{r-1}}{r^r} + \varepsilon)t^r$ (stretched) edges in G intersect V_i, but there is no independent transversal of $\mathcal{P}$.*

Proof. We mimic the proof of Theorem 3. Let $\varepsilon \in (0, 1)$ and $r \geq 2$ be given, reducing ε if necessary so that $\varepsilon < \frac{(r-1)^{r-1}}{2r^r}$. Let $\frac{2\varepsilon}{3} < \delta < \varepsilon$, and let t be sufficiently large that

$$t^{r-1} + (1 + \delta)\frac{(r-1)^{r-1}}{r^r}t^r \leq (1 + \varepsilon)\frac{(r-1)^{r-1}}{r^r}t^r. \tag{4}$$

Blocks of grade 1 will consist of t vertices, all of which will have degree at most t^{r-2}, and hence the block average degree is sufficiently low by (4).

Blocks of grade 2 consist of

$$n_2 = \left\lfloor (1 + \delta)\frac{(r-1)^{r-1}}{r^r}t \right\rfloor \geq \frac{\varepsilon t}{2}$$

heavy vertices. Each such vertex v has a private $(r-1)$-tuple of blocks $B_1, ..., B_{r-1}$ and incident edges

$$\{\{v, b_1, ..., b_{r-1}\}|b_1 \in B_1, ..., b_{r-1} \in B_{r-1}\}.$$

Each block of grade 2 will be part of an $(r-1)$-tuple of blocks of grade 2, and the remaining $t - n_2$ vertices will each be incident with at most $(t - n_2)^{r-2}$ edges. The number of edges incident to a block of grade 2 is

$$t^{r-1}n_2 + (t - n_2)^{r-1} \leq (1 + \varepsilon)\frac{(r-1)^{r-1}}{r^r}t^r$$

by (4). Having defined blocks of grade j for some $j \geq 2$, the blocks of grade $j + 1$ are defined as follows. Once $(t - n_j)^{r-1} \leq \frac{(r-1)^{r-1}}{r^r}t^{r-1}$, we create a single block of grade $j + 1$, with each vertex joined completely to a private $(r-1)$-tuple of blocks of grade j, and we terminate the construction. Otherwise, we set n_{j+1} to be the minimum of

$$\left\lfloor (1 + \delta) \frac{(r-1)^{r-1}}{r^r} \left(\frac{t}{t - n_j} \right)^{r-1} t \right\rfloor \tag{5}$$

and t. The edges will be defined as in the case $j = 2$, and the blocks will form $(r - 1)$-tuples.

By (4), the number of edges incident to a block of grade $j + 1$ is

$$n_{j+1}(t - n_j)^{r-1} + (t - n_{j+1})^{r-2} \leq (1 + \varepsilon) \frac{(r-1)^{r-1}}{r^r} t^r.$$

Again, it suffices to show that the construction terminates, for which it suffices to show that $n_j < n_{j+1}$ when $n_j, n_{j+1} < t$. For t sufficiently large,

$$\begin{aligned}
\frac{n_{j+1}}{t} &= \left\lfloor (1 + \delta) \frac{(r-1)^{r-1}}{r^r} \left(\frac{t}{t - n_j} \right)^{r-1} t \right\rfloor t^{-1} \\
&\geq \left(1 + \frac{\delta}{2} \right) \frac{(r-1)^{r-1}}{r^r} \left(\frac{1}{1 - n_j/t} \right)^{r-1} \\
&\geq \left(1 + \frac{\delta}{2} \right) \frac{n_j}{t},
\end{aligned}$$

using that $(1 - x)^{r-1} x \leq \frac{(r-1)^{r-1}}{r^r}$ for all $x \in (0, 1)$.

Observe that $\frac{n_2}{t} > \frac{\varepsilon}{2}$, and for $i \geq 3$, $\frac{n_i}{n_{i-1}} \geq \left(1 + \frac{\varepsilon}{3} \right)$ (or $n_i = t$). As such, for some $j \leq \log_{1+\varepsilon/3}(\frac{2}{\varepsilon} + 2)$, $n_j = t$, and the construction terminates at that grade.

Each block of grade j is joined to $(r - 1)n_j$ (private) blocks of grade $j - 1$. Each of these blocks in turn has private blocks of grade $j - 2$. Overall, if j is such that $n_j = t$, we require at most $((r - 1)t)^{j+1}$ blocks. We can obtain constructions for any larger n by adding additional blocks with no incident edges. □

We also offer the following analogue of Theorem 6.

Theorem 10. *In the construction of Theorem 9, we can also require*

$$\Delta(G) \leq \left(1 - \frac{(r-1)^{r-1}}{3r^r} \right) t^{r-1}$$

For brevity, we only provide an outline of how the above construction is modified, and do not provide a detailed proof. Let $c_r = \frac{(r-1)^{r-1}}{r^r}$.

We bounded the maximum degree for the forest construction by replacing each block of grade 1 with two blocks, and placed an unbalanced complete bipartite graph between them. The natural generalisation of this to hypergraphs is to replace each grade 1 block with r blocks, and place a

complete r-partite r-uniform hypergraph between them, with part sizes $m_1 \geq \cdots \geq m_r$. We cannot choose all vertices from this graph, and so the remaining vertices form a forced set, of size $rt - \sum m_i$.

As before, the maximum vertex degree must occur in either a grade 1 or a grade 2 block—beyond this point, since the n_i are increasing, the degrees of the heavy vertices are decreasing. We will ignore all edge contributions of order less than t^{r-1}: this is admissible, since we can if necessary make ε smaller, and take t sufficiently large. The degrees in the grade 1 block are at most $m_1 m_2 \dots m_{r-1}$ (arising from vertices in a part of minimal size), and the degrees in the grade 2 block are at most $(rt - \sum m_i)^{r-1}$. Our problem is thus to minimise the maximum of these two values, given the constraint that the product of the m_i is at most $c_r t^r$.

These parameters can be optimally picked by reduction to a three variable optimisation problem. Taking two (nonminimal) m_i and slightly increasing the larger, while decreasing the smaller will improve our objective function. We can therefore assume that the only values which appear are m_r, t and possibly one block with an intermediate size.

If we have multiple m_i which are equal and minimal, we cannot get a large average degree from the grade 1 blocks, and so we can increase all m_i without violating a constraint. Therefore, we can assume there is only one smallest block. We now have an optimisation problem in two variables m_r and m_{r-1}, and reduce this to one variable by letting $m_r m_{r-1} = c_r t^{r-1}$. One can check that substituting $m_{r-1} = (1 - \frac{c_r}{3})t$ gives the claimed upper bound; this gives a good approximation to the general exact solution.

5 | CONCLUSION

Let us conclude with some remarks, and open questions for further investigation.

5.1 | Local degree

Loh and Sudakov [20] proved that for every $\varepsilon > 0$ there exists $\gamma > 0$ such that if $\mathcal{P}$ is a t-thick partition of $V(G)$, the maximum degree of G is at most $(1 - \varepsilon)t$ and the local degree of G is at most γt, then $\mathcal{P}$ has an independent transversal. This was extended by Kang and Kelly [19] and independently by Glock and Sudakov [14] to the case where the maximum degree is replaced by the maximum block average degree.

As noted by Kang and Kelly [19], this could possibly be further strengthened by introducing a parameter they called 'maximum average colour multiplicity'. Let G be a graph with a partition $\mathcal{P}$ of $V(G)$. We define $\bar{\mu}(G)$ as the maximum, taken over pairs A, B of blocks of $\mathcal{P}$, of the average number of neighbours in B of a vertex $v \in A$. Kang and Kelly propose a problem which we formulate as follows.

Problem 11 (Kang and Kelly [19])**.** Is it true that for every $\varepsilon > 0$ there exists $\gamma > 0$ such that if t is sufficiently large, $\mathcal{P}$ is a t-thick partition of $V(G)$, the maximum block average degree of G is at most $(1 - \varepsilon)t$ and $\bar{\mu}(G) \leq \gamma t$, then $\mathcal{P}$ has an independent transversal? An affirmative answer would imply a conjecture of Loh and Sudakov [20] where the maximum block average degree is replaced by the maximum degree and the condition $\bar{\mu}(G) \leq \gamma t$ is replaced by the number of edges between any two blocks of $\mathcal{P}$ being at most γt^2.

5.2 | Maximum degree and maximum block average degree

It is natural to ask if the tightness example provided by Theorem 3 can be combined with the tightness examples for Theorem 1 discussed in Section 1.

> **Problem 12.** Is it true that for every $\varepsilon > 0$ and every sufficiently large t, there is a graph G with a t-thick partition $\mathcal{P}$ of $V(G)$ such that $\Delta(G) \leq (\frac{1}{2} + \varepsilon)t$, the maximum block average degree of G is at most $(\frac{1}{4} + \varepsilon)t$ and $\mathcal{P}$ admits no independent transversal in G?

Theorem 6 provides a step in this direction, but the maximum degree bound of $0.854t$ is a long way from $(\frac{1}{2} + \varepsilon)t$.

More generally, what can we say about the set of pairs $(\alpha, \beta) \in [0, 1]^2$ such that for sufficiently large t, there is a graph with a t-thick partition $\mathcal{P}$, maximum degree at most αt, maximum block average degree at most βt and no independent transversal of $\mathcal{P}$?

For r-uniform hypergraphs, we can ensure a 'tight example' while each vertex meets at most ηt^{r-1} edges, for $\eta(r) \sim (1 - \frac{1}{3er})$. Although we can ensure the number of edges meeting each part is $o_r(1)t^r$, we leave open whether this remains true under a 'maximum degree' condition of $o_r(1)t^{r-1}$, or even $0.99t^{r-1}$.

> **Problem 13.** Is there a constant $0 < \eta < 1$ such that for any r, ε there is an r-uniform hypergraph H and vertex partition $\mathcal{P} = (V_i)$ with at most $(\frac{(r-1)^{r-1}}{r^r} + \varepsilon)t^r$ stretched edges meeting each V_i, each vertex meeting at most ηt^{r-1} edges, and no independent transversal of $\mathcal{P}$? Can we do the same for some $\eta(r) \to 0$?

5.3 | Avoiding complete bipartite graphs

The examples with a bound on the maximum degree given by Theorem 6 contain complete bipartite subgraphs of order linear in the block size. We may ask if similar examples may be constructed if we exclude this, say by forbidding cycles of length 4 in the graph.

> **Problem 14.** Is there a $\delta > 0$ such that for every $\varepsilon > 0$ and all sufficiently large t, there is a graph G with a t-thick partition $\mathcal{P}$ of $V(G)$ such that G contains no 4-cycle, the maximum degree of G is at most $(1 - \delta)t$, the maximum block average degree of G is at most $(\frac{1}{4} + \varepsilon)t$, and $\mathcal{P}$ has no independent transversal in G?

5.4 | Number of blocks

The number of blocks of the vertex partitions of the graphs provided by our proof of Theorem 3 is large with respect to the block size. If we fix the number of blocks and let the block size vary, how will it affect the existence of independent transversals in relation to maximum block average degree?

We remark that the corresponding question for the maximum degree variant of the problem is well understood. Let $\Delta(n, t)$ be the largest integer such that for every graph G with maximum degree $\Delta(n, t)$, every t-thick partition of $V(G)$ with n blocks has an independent transversal. It follows from the results of [6, 17, 25] that

$$\Delta(n, t) = \left\lceil \frac{nt}{2(n-1)} \right\rceil.$$

5.5 | Unequal part sizes

We finish with a remark. One might be tempted to modify the average degree assumption in Theorem 2 as follows: If G is a graph and $\mathcal{P}$ a partition of $V(G)$ such that each block B is incident with at most $|B|^2/4$ edges, then there is an independent transversal of $\mathcal{P}$ in G. This fails, however—even if the constant $1/4$ is replaced with an arbitrary $\beta > 0$. To see this, take a large integer k and let G be the disjoint union of stars $S_1, ..., S_{k^2}$ of $k + 1$ vertices each. Let the set V_0 consist of the centers of the stars and let V_i $(1 \leq i \leq k^2)$ consist of the leaves of S_i. Then the partition $\{V_0, ..., V_{k^2}\}$ has no independent transversal in G, since for any choice of a vertex $v \in V_0$ there is a block consisting of neighbours of v. At the same time, any block B is incident with at most $|B|^2/k$ edges.

DATA AVAILABILITY STATEMENT
Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.

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