# Probabilistic Deontic Logics for Reasoning about Uncertain Norms

Vincent de Wit
*Department of Computer Science; University of Luxembourg, Luxembourg*
vincent.j.wit@gmail.com

Dragan Doder
*Department of Information and Computing Sciences; Utrecht University, The Netherlands*
d.doder@uu.nl

John Jules Meyer
*Department of Information and Computing Sciences; Utrecht University, The Netherlands*
J.J.C.Meyer@uu.nl

**Abstract**

In this article, we present a proof-theoretical and model-theoretical approach to probabilistic logic for reasoning about uncertainty about normative statements. We introduce two logics with languages that extend both the language of monadic deontic logic and the language of probabilistic logic. The first logic allows statements like "the probability that one is obliged to be quiet is at least 0.9". The second logic allows iteration of probabilities in the language. We axiomatize both logics, provide the corresponding semantics and prove that the axiomatizations are sound and complete. We also prove that both logics are decidable. In addition, we show that the problem of deciding satisfiability for the simpler of our two logics is in PSPACE, no worse than that of deontic logic.

**Keywords:** MDL; Normative reasoning; Probabilistic logic; Completeness; Decidability.

This paper is a revised and extended version of our conference paper [20] presented at the Sixteenth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2021), in which we introduced a logic for reasoning about probabilities of a deontic statement, provided a complete axiomatization for the logic and proved its decidability. In this paper, we extend those results, by providing the complexity result for the satisfiability problem. Additionally, we also present in this paper another, richer logic with nesting of probability operators. For that novel logic we also provide an axiomatization, prove its completeness, and show that the logic is decidable.

# 1 Introduction

Norms govern many parts of human life and these norms need to be learned at some point. This means that before a norm is learned there is uncertainty about whether a norm holds or not. To formalize this notion a probabilistic deontic logic is developed in this paper. The seminal work of von Wright from 1951 [21] initiated a systematic study on the formalization of normative reasoning in terms of deontic logic. The latter is a branch of modal logic that deals with obligation, permission, and related normative concepts. A plethora of deontic logics have been developed for various application domains like legal reasoning, argumentation theory, and normative multi-agent systems [1, 5, 11].

Some recent work studied learning of behavioral norms from data [16, 18]. In [16] the authors pointed out that human norms are context-specific and laced with uncertainty, which poses challenges to their representation, learning, and communication. They gave an example of a learner that might conclude from observations that talking is prohibited in a library setting, while another learner might conclude the opposite when seeing people talking at the checkout counter. They represented uncertainty about norms using deontic operators equipped with probabilistic boundaries that capture the subjective degree of certainty.

In this paper, we study uncertain norms from a logical point of view. We use probabilistic logic [6, 7, 8, 10, 17, 19] to represent uncertainty, and we present the proof-theoretical and model-theoretical approach to a logic which allows reasoning about uncertain normative statements. We take two well-studied logics, monadic deontic logic (MDL) [14] and probabilistic logic of Fagin, Halpern, and Magido (FHM) [7], as the starting point, and combine them in a rich formalism that generalizes each of them. The resulting language makes it possible to adequately model different degrees of belief in norms; for example, we can express statements like "the probability that one is obliged to be quiet is at least 0.9".

The semantics for our main logic $\mathcal{PMDL}$ consists of specific Kripke-like structures, where each model contains a probability space whose sample space is the set of states, and with each state carrying enough information to evaluate a deontic formula. We consider so-called *measurable* models, which allow us to assign a probability value to every deontic statement. We also propose another, richer language $\mathcal{PMDL}^2$ which allows nesting of probability operators. In this case, the semantics is naturally generalized i.e. that all states are equipped with probability spaces. In addition to the nesting of operators, we modify the language in such a way that we allow different agents to place (possibly different) probabilities on norms and events. So the formulas can express one's uncertainty about another person's uncertainty about norms.

The main result of this article is a sound and complete axiomatization for our logics. Like any other real-valued probabilistic logic, it is not compact, so any finitary axiomatic system would fail to be strongly complete ("every consistent set of formulas has a model") [10]. We prove weak completeness ("every consistent formula has a model") by combining and modifying completeness techniques for MDL and FHM. We also show that our logics are decidable; combining the method of filtration and a reduction to a system of inequalities. In addition, we show that the problem of deciding satisfiability for the logic $\mathcal{PMDL}$ is in PSPACE, no worse than that of deontic logic.

The rest of the paper is organized as follows: In Section 2, the proposed syntax and semantics of the logic will be presented together with other needed definitions. In Section 3, the axiomatization of the logic is given; in section 4, soundness and completeness is proven. In Section 5, we show that our logic is decidable; in Section 6, the probability structure of the logic is changed such that iterations of probabilities are possible, and completeness and decidability is proven. Lastly, in Section 7, a conclusion is given together with future research topics.

## 2 Syntax and Semantics

In this section, we present the syntax and semantics of our probabilistic deontic logic. This logic, which we named $\mathcal{PMDL}$, contains two types of formulas: standard deontic formulas of MDL, and probabilistic formulas. Let $\mathbb{N}$ denote the set of integers.

**Definition 1** (Formula)**.** *Let $\mathbb{P}$ be a set of atomic propositions. The language $\mathcal{L}$ of probabilistic MDL is generated by the following two sentences of BNF (Backus Naur Form):*

$$
\begin{array}{lll}
[\mathcal{L}_{MDL}] & \phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid O\phi & p \in \mathbb{P} \\
[\mathcal{L}_{\mathcal{PMDL}}] & f ::= a_1 w(\phi_1) + \cdots + a_n w(\phi_n) \geq \alpha \mid \neg f \mid (f \wedge f) & a_i, \alpha \in \mathbb{N}
\end{array}
$$

*The set of all formulas $\mathcal{L}$ is $\mathcal{L}_{MDL} \cup \mathcal{L}_{\mathcal{PMDL}}$. We denote the elements of $\mathcal{L}$ with $\theta$ and $\theta$', possibly with subscripts.*

The construct $O\phi$ reads as "It is obligatory that $\phi$", while $w(\phi)$ stands for "probability of $\phi$". An expression of the form $a_1 w(\phi_1) + \cdots + a_n w(\phi_n)$ is called *term*. We denote terms with $x$ and $t$, possibly with subscripts. The propositional connectives, $\vee$, $\rightarrow$ and $\leftrightarrow$, are introduced as abbreviations, in the usual way. There are also two additional deontic operators that denote the following: forbidden, $F\phi \equiv O\neg\phi$ ; and permitted $P\phi \equiv \neg F\phi \wedge \neg O\phi$. We abbreviate $\theta \wedge \neg\theta$ with $\perp$, and $\neg\perp$ with $\top$. We also use abbreviations to define other types of inequalities; for example, $w(\phi) \geq w(\phi')$ is an abbreviation for $w(\phi) - w(\phi') \geq 0$, $w(\phi) = \alpha$ for $w(\phi) \geq \alpha$ and $-w(\phi) \geq -\alpha$, $w(\phi) < \alpha$ for $\neg w(\phi) \geq \alpha$.

It is very important to mention that we can also use abbreviations that allow us to see *rational numbers* as coefficients of terms i.e. they can be eliminated from any formula by clearing the denominator. For example, the formula

$$\frac{2}{3}t_1 + \frac{3}{4}t_2 \geq 1$$

is simply an abbreviation for $8t_1 + 9t_2 \geq 12$.

**Example 1.** *Following our informal example from the introduction about behavioral norms in a library, the fact that a person has become fairly certain that it is normal to be quiet might be expressed by the probabilistic statement "the probability that one is obliged to be quiet is at least 0.9". This sentence could be formalized using the introduced language as*

$$w(Oq) \geq 0.9.$$

Note that we do not allow mixing of the formulas from $\mathcal{L}_{MDL}$ and $\mathcal{L}_{\mathcal{PMDL}}$. For example, $O(p \vee q) \wedge w(Oq) \geq 0.9$ is not a formula of our language. Before we introduce the semantics of $\mathcal{PMDL}$ we will introduce MDL models.

**Definition 2** (MDL model)**.** *An MDL model $D$ is a tuple $D = (W, R, V)$ where:*

- *$W$ is a (non-empty) set of "possible worlds"; $W$ is called the universe of the model.*
- *$R \subseteq W \times W$ is a binary relation over $W$, such that*

  $$(\forall w \in W)(\exists u \in W)(wRu). \qquad \textit{(seriality)}$$

  *If $(w, u) \in R$, we say that $u$ is an $R-$successor of $w$.*
- *$V : \mathbb{P} \to 2^W$ is a valuation function assigning to each atom $p$ a set $V(p) \subseteq W$ (intuitively the set of worlds at which $p$ is true.)*

We denote the set of all MDL models with $\mathbb{D}$. As formalized in the following definition, the relation $R$ relates worlds to worlds, with the intention that everything obligatory at a world holds in its $R-$successors.

Next, we define the satisfiability relation of MDL.

**Definition 3** (Satisfaction in MDL)**.** *Let $D = (W, R, V)$ be an MDL model, and let $w \in W$. We define the satisfiability of a deontic formula $\phi \in \mathcal{L}_{MDL}$ in the world $w$, denoted by $D, w \models_{MDL} \phi$, recursively as follows:*

- $D, w \models_{MDL} p$ *iff* $w \in V(p)$.
- $D, w \models_{MDL} \neg\phi$ *iff* $D, w \not\models_{MDL} \phi$.
- $D, w \models_{MDL} \phi \wedge \psi$ *iff* $D, w \models_{MDL} \phi$ *and* $D, w \models_{MDL} \psi$.
- $D, w \models_{MDL} O\phi$ *iff for all* $u \in W$, *if* $wRu$ *then* $D, u \models_{MDL} \phi$.

Now we introduce the semantics of $\mathcal{PMDL}$.

**Definition 4** ($\mathcal{PMDL}$ Model)**.** *A probabilistic deontic model is a tuple $M = \langle S, \mathscr{X},$ $\mu, \tau\rangle$, where*

- $S$ *is a non-empty set of* states
- $\mathscr{X}$ *is a $\sigma$-algebra of subsets of $S$*
- $\mu : \mathscr{X} \to [0, 1]$ *is a probability measure, i.e.,*
  - $\mu(X) \geq 0$ *for all* $X \in \mathscr{X}$
  - $\mu(S) = 1$
  - $\mu(\bigcup_{i=1}^{\infty} X_i) = \sum_{i=1}^{\infty} \mu(X_i)$, *if the $X_i$'s are pairwise disjoint members of* $\mathscr{X}$
- $\tau$ *is a function that assigns to each state in $S$ a pair consisting of an MDL model and a world of that model, i.e., $\tau : s \mapsto (D_s, w_s)$, where:*
  - $D_s = (W_s, R_s, V_s) \in \mathbb{D}$
  - $w_s \in W_s$

Let us illustrate this definition.

**Example 1.** ***(continued)*** *Assume a finite set of atomic propositions $\{p, q\}$. Let us consider the model $M = \langle S, \mathscr{X}, \mu, \tau\rangle$, where*

- $S = \{s_1, s_2, s_3, s_4\}$
- $\mathscr{X}$ *is the set of all subsets of $S$*

- *$\mu$ is characterized by: $\mu(\{s_1\}) = 0.5$, $\mu(\{s_2\}) = \mu(\{s_3\}) = 0.2$, $\mu(\{s_4\}) = 0.1$ (other values follow from the properties of probability measures)*
- *$\tau$ is a mapping which assigns to the state $s_1$, $D_{s_1} = (W_{s_1}, R_{s_1}, V_{s_1})$ and $w_{s_1}$ such that*
  - $W_{s_1} = \{w_1, w_2, w_3, w_4\}$
  - $R_{s_1} = \{(w_1, w_2), (w_1, w_3), (w_2, w_2), (w_2, w_3), (w_3, w_2), (w_3, w_3), (w_4, w_2), (w_4, w_3), (w_4, w_4)\}$
  - $V_{s_1}(p) = \{w_1, w_3\}$, $V_s(q) = \{w_2, w_3\}$
  - $w_{s_1} = w_1$

    *Note that the domain of $\tau$ is always the whole set $S$, but in this example we only explicitly specify $\tau(s_1)$ for illustration purposes.*

*This model is depicted in Figure 1. The circle on the right contains the four states of the model, which are measured by $\mu$. Each of the states is equipped with a standard pointed model of MDL. In this picture, only one of them is shown, the one that corresponds to $s_1$. It is represented within the circle on the left. Note that the*

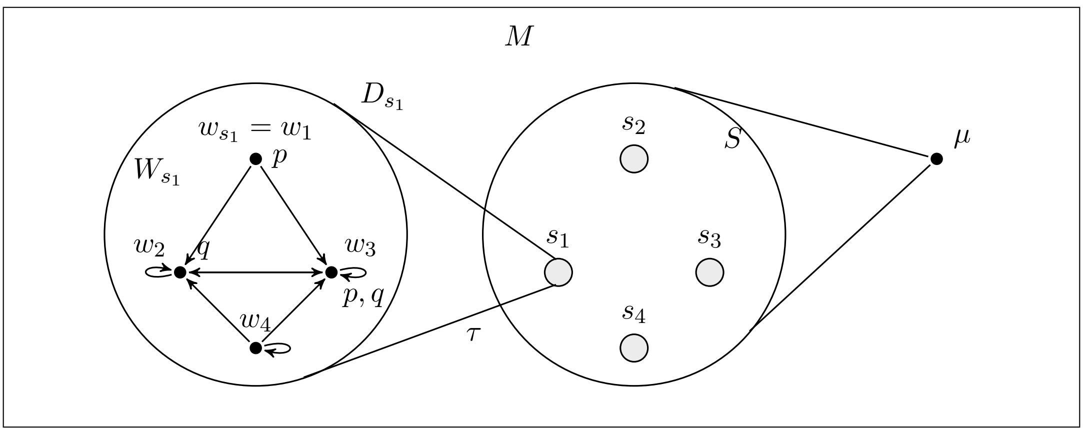


Figure 1: Model $M = \langle S, \mathscr{X}, \mu, \tau \rangle$.

*arrows depict the relation $R$. If we assume that $q$ stands for "quiet", like in the previous example, in all $R-$successors of $w_1$ the proposition $q$ holds. Note that, according to Definition 3, this means that in $w_1$ people are obliged to be quiet in the library.*

For a model $M = \langle S, \mathscr{X}, \mu, \tau \rangle$ and a formula $\phi \in \mathcal{L}_{MDL}$, let $\|\phi\|_M$ denote the set of states that satisfy $\phi$, i.e., $\|\phi\|_M = \{s \in S \mid D_s, w_s \models_{MDL} \phi\}$. We omit the subscript $M$ from $\|\phi\|_M$ when it is clear from context. The following definition introduces an important class of probabilistic deontic models, the so-called *measurable* models.

**Definition 5** (Measurable model). *A probabilistic deontic model is* measurable *if*

$$\|\phi\|_M \in \mathscr{X}$$

*for every $\phi \in \mathcal{L}_{MDL}$. Denote the class of all measurable models of $\mathcal{PMDL}$ by $\mathcal{PMDL}^{Meas}$.*

In this paper, we focus on measurable structures, and we prove soundness & completeness, and decidability results for this class of structures.

**Definition 6** (Satisfaction). *Let $M = \langle S, \mathscr{X}, \mu, \tau \rangle \in \mathcal{PMDL}^{Meas}$ be a measurable probabilistic deontic model. We define the satisfiability relation $\models$ recursively as follows:*

- $M \models \phi$ *iff* $D_s, w_s \models_{MDL} \phi$ *holds for every* $s \in S$, *where* $\tau(s) = (D_s, w_s)$
- $M \models a_1 w(\phi_1) + \cdots + a_k w(\phi_k) \geq \alpha$ *iff* $a_1\mu(\|\phi_1\|) + \cdots + a_k\mu(\|\phi_k\|) \geq \alpha$.
- $M \models \neg f$ *iff* $M \not\models f$
- $M \models f \wedge g$ *iff* $M \models f$ *and* $M \models g$

**Example 1.** ***(continued)*** *Continuing the previous example, it is now also possible to speak about the probability of the obligation to be quiet in a library. First, according to Definition 3 it holds that $D_{s_1}, w_{s_1} \models_{MDL} Oq$. Furthermore, assume that $\tau$ is defined in the way such that $D_{s_2}, w_{s_2} \models_{MDL} Oq$ and $D_{s_3}, w_{s_4} \models_{MDL} Oq$, but $D_{s_4}, w_{s_4} \not\models_{MDL} Oq$. Then $\mu(\|Oq\|) = \mu(\{s_1, s_2, s_3\}) = 0.5 + 0.2 + 0.2 = 0.9$. According to Definition 6, $M \models w(Oq) \geq 0.9$.*

Note that, according to Definition 6, a deontic formula is true in a model iff it holds in every state of the model. This is a consequence of our design choice that those formulas represent undisputable deontic knowledge, while probabilistic formulas express uncertainty about norms. At the end of this section, we define some standard semantical notions.

**Definition 7** (Semantical consequence). *Given a set $\Gamma$ of formulas, a formula $\theta$ is a semantical consequence of $\Gamma$ (notation: $\Gamma \models \theta$) whenever all the states of the model have, if $M, s \models \theta'$ for all $\theta' \in \Gamma$, then $M, s \models \theta$.*

**Definition 8** (Validity). *A formula $\theta$ is valid (notations: $\models \theta$) whenever for $M = \langle S, \mathscr{X}, \mu, \tau \rangle$ and every $s \in S$: $M, s \models \theta$ holds.*

# 3 Axiomatization

The following axiomatization contains 13 axioms and 3 inference rules. It combines the axioms of proof system D of MDL [14] with the axioms of probabilistic logic. The axioms for reasoning about linear inequalities are taken from [7].

### The Axiomatic System: $AX_{\mathcal{PMDL}}$

**Tautologies and Modus Ponens**

Taut. All instances of propositional tautologies.

MP. From $\theta$ and $\theta \to \theta'$ infer $\theta'$.

**Reasoning with $O$:**

O-K. $O(\phi \to \psi) \to (O\phi \to O\psi)$

O-D. $O\phi \to P\phi$

O-Nec. *From $\phi$ infer $O\phi$*

**Reasoning about linear inequalities:**

I1. $x \geq x$ (identity)

I2. $(a_1x_1 + ... + a_kx_k \geq c) \leftrightarrow (a_1x_1 + ... + a_kx_k + 0x_{k+1} \geq c)$ (adding and deleting 0 terms)

I3. $(a_1x_1 + ... + a_kx_k \geq c) \to (a_{j_1}x_{j_1} + ... + a_{j_k}x_{j_k} \geq c)$, if $j_1, ..., j_k$ is a permutation of $1, ..., k$ (permutation)

I4. $(a_1x_1 + ... + a_kx_k \geq c) \wedge (a'_1x_1 + ... + a'_kx_k \geq c') \to ((a_1 + a'_1)x_1 + ... + (a_k + a'_k)x_k \geq (c + c'))$ (addition of coefficients)

I5. $(a_1x_1 + ... + a_kx_k \geq c) \leftrightarrow (da_1x_1 + ... + da_kx_k \geq dc)$ if $d > 0$ (multiplication of non-zero coefficients)

I6. $(t \geq c) \vee (t \leq c)$ if $t$ is a term (dichotomy)

I7. $(t \geq c) \to (t > d)$ if $t$ is a term and $c > d$ (monotonicity)

**Reasoning about probabilities:**

W1. $w(\phi) \geq 0$ (non-negativity).

W2. $w(\phi \vee \psi) = w(\phi) + w(\psi)$, if $\neg(\phi \wedge \psi)$ is an instance of a classical propositional tautology (finite additivity).

W3. $w(\top) = 1$

P-Dis. From $\phi \leftrightarrow \psi$ infer $w(\phi) = w(\psi)$ (probabilistic distributivity)

The axiom Taut allows all $\mathcal{L}_{MDL}$-instances and $\mathcal{L}_{\mathcal{PMDL}}$-instances of propositional tautologies. For example, $w(Oq) \geq 0.9 \vee \neg w(Oq) \geq 0.9$ is an instance of Taut, but $w(Oq) \geq 0.9 \vee \neg w(Oq) \geq 1$ is not. Note that Modus Ponens (MP) can be applied to both types of formulas, but only if $\theta$ and $\theta'$ are both from $\mathcal{L}_{MDL}$ or both from $\mathcal{L}_{\mathcal{PMDL}}$. O-Nec is a deontic variant of necessitation rule. P-Dis is an inference rule which states that two equivalent deontic formulas must have the same probability values.

**Definition 9** (Syntactical consequence)**.** *A derivation of $\theta$ is a finite sequence $\theta_1, \ldots, \theta_m$ of formulas such that $\theta_m = \theta$, and every $\theta_i$ is either an instance of an axiom, or it is obtained by the application of an inference rule to formulas in the sequence that appear before $\theta_i$. If there is a derivation of $\theta$, we say that $\theta$ is a* theorem *and write $\vdash \theta$. We also say that $\theta$ is derivable from a set of formulas $\Gamma$, and write $\Gamma \vdash \theta$, if there is a finite sequence $\theta_1, \ldots, \theta_m$ of formulas such that $\theta_m = \theta$, and every $\theta_i$ is either a theorem, a member of $\Gamma$ or the result of an application of MP or P-Dis to formulas in the sequence that appear before $\theta_i$.*

Note that this definition restricts the application of O-Nec. to theorems only. This is a standard restriction for modal necessitations, which enables one to prove the Deduction Theorem using induction on the length of the inference. Also, note that only deontic formulas can participate in a proof of another deontic formula, thus derivations of deontic formulas in our logic coincide with their derivations in MDL.

**Definition 10** (Consistency)**.** *A set $\Gamma$ is consistent if $\Gamma \nvdash \bot$, and inconsistent otherwise.*

Now we prove some basic consequences of $AX_{\mathcal{PMDL}}$. The first one is the probabilistic variant of necessitation. It captures the semantical property that a deontic formula represents undisputable knowledge, and therefore it must have a probability value of 1. The second point states that we can derive from the axiomatization that the weight of falsum equals zero. The third part of the lemma shows that a form of additivity proposed as an axiom in [7] is provable in $AX_{\mathcal{PMDL}}$.

**Lemma 1.** *The following rules are derivable from our axiomatization:*

1. *From $\phi$ infer $w(\phi) = 1$*
2. $\vdash w(\bot) = 0$
3. $\vdash w(\phi \wedge \psi) + w(\phi \wedge \neg\psi) = w(\phi)$.

*Proof.*

1. Let us assume that a formula $\phi$ is derived. Then, using propositional reasoning (Taut and MP), one can infer $\phi \leftrightarrow \top$. Consequently, $w(\phi) = w(\top)$ follows from the rule P-Dis. Since we have that $w(\top) = 1$ (by W3), we can employ the axioms for reasoning about inequalities to infer $w(\phi) = 1$.

2. Then to show that $w(\bot) = 0$ using finite additivity (W2) $w(\top \vee \neg\top) = w(\top) + w(\neg\top) = 1$ and so $w(\neg\top) = 1 - w(\top)$. Since $w(\top) = 1$ and $\neg\top \leftrightarrow \bot$ we can derive $w(\bot) = 0$.

3. To derive additivity we begin with the propositional tautology, $\neg((\phi \wedge \psi) \wedge (\phi \wedge \neg\psi))$ then the following equation is given by W2 $w(\phi \wedge \psi) + w(\phi \wedge \neg\psi) = w((\phi \wedge \psi) \vee (\phi \wedge \neg\psi))$. The disjunction $(\phi \wedge \psi) \vee (\phi \wedge \neg\psi)$ can be rewritten to, $\phi \wedge (\psi \vee \neg\psi)$ which is equivalent to $\phi$. From $\phi \leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \neg\psi)$, using P-Dis, we obtain $w(\phi) = w(\phi \wedge \psi) + w(\phi \wedge \neg\psi)$.

□

# 4 Soundness & Completeness

In this section, we prove that our logic is sound and complete with respect to the class of measurable models; combining, adapting and following the approaches from [7, 3].

**Theorem 1** (Soundness & Completeness)**.** *The axiom system $AX_{\mathcal{PMDL}}$ is sound and complete with respect to the class of measurable models $\mathcal{PMDL}^{Meas}$, i.e., $\vdash \theta$ iff $\models \theta$.*

*Proof.* The proof of soundness is straightforward. To prove completeness, we need to show that every consistent formula $\theta$ is satisfied in a measurable model. Since we have two types of formulas, we distinguish two cases.

If $\theta \in \mathcal{L}_{MDL}$ we write $\theta$ as $\phi$. Since $\phi$ is consistent and MDL is complete [14], we know that there is an MDL model $(W, R, V)$ and $w \in W$ such that $(W, R, V), w \models \phi$.

Then, for any probabilistic deontic model $M$ with only one state $s$ and $\tau(s) = ((W, R, V), w)$ we have $M, s \models \phi$, and therefore $M \models \phi$ (since $s$ is the only state); so the formula is satisfiable.

When $\theta \in \mathcal{L}_{\mathcal{PMDL}}$ we write $\theta$ as $f$. Then $f$ is consistent and we prove that $f$ is satisfiable. First notice that $f$ can be equivalently rewritten as a formula in disjunctive normal form,

$$f \leftrightarrow g_1 \vee \cdots \vee g_n \tag{1}$$

this means that satisfiability of $f$ can be proven by showing that one of the disjuncts $g_i$ of the disjunctive normal form of $f$ is satisfiable. Note that every disjunct is of the form:

$$g_i = \bigwedge_{j=1}^{r} (\sum_k a_{j,k} w(\phi_{j,k}) \geq c_j) \wedge \bigwedge_{j=r+1}^{r+s} \neg(\sum_k a_{j,k} w(\phi_{j,k}) \geq c_j). \tag{2}$$

To show that $g_i$ is satisfiable we will substitute each weight term $w(\phi_{j,k})$ by a sum of weight terms that take as arguments formulas from the set $\Delta$ that will be constructed below. For any formula $\theta$, let us denote the set of subformulas of $\theta$ by $Sub(\theta)$. Then, for considered, $g_i$ we introduce the set of all deontic subformulas $Sub_{DL}(g_i) = Sub(g_i) \cap \mathcal{L}_{MDL}$. We create the set $\Delta$ as the set of all possible formulas that are conjunctions of formulas from $Sub_{DL}(g_i) \cup \{\neg e \mid e \in Sub_{DL}(g_i)\}$, such that for every $e$ either $e$ or $\neg e$ is taken as a conjunct (but not both). Then we can prove the following two claims about the set $\Delta$:

- The conjunction of any two different formulas $\delta_k$ and $\delta_l$ from $\Delta$ is inconsistent: $\vdash \neg(\delta_k \wedge \delta_l)$. This is the case because for each pair of $\delta$'s at least one subformula $e \in Sub_{DL}(g_i)$ such that $\delta_k \wedge \delta_l \vdash e \wedge \neg e$ and $e \wedge \neg e \vdash \bot$. If there is no such $e$, then by construction $\delta_k = \delta_l$.
- The disjunction of all $\delta$'s in $\Delta$ is a tautology: $\vdash \bigvee_{\delta \in \Delta} \delta$. Indeed, it is clear from the way the set $\Delta$ is constructed, that the disjunction of all formulas is an instance of a propositional tautology.

As noted earlier, we will substitute each term of each weight formula of $g_i$ with a sum of weight terms. This can be done by using the just introduced set $\Delta$ and the set $\Phi$, which we define as the set containing all deontic formulas $\phi_{j,k}$ that occur in the weight terms of $g_i$. In order to get all the relevant $\delta$'s to represent a weight term, we construct for each $\phi \in \Phi$ the set $\Delta_\phi = \{\delta \in \Delta \mid \delta \vdash \phi\}$ which contains all $\delta$'s that imply $\phi$. Then we can derive the following equivalence:

$$\vdash \phi \leftrightarrow \bigvee_{\delta \in \Delta_\phi} \delta.$$

From the rule P-Dis we obtain

$$\vdash w(\phi) = w(\bigvee_{\delta \in \Delta_\phi} \delta).$$

Since any two elements of $\Delta$ are inconsistent, from W2 and axioms about inequalities we obtain

$$\vdash w(\bigvee_{\delta \in \Delta_\phi} \delta) = \sum_{\delta \in \Delta_\phi} w(\delta).$$

Consequently, we have

$$\vdash w(\phi) = \sum_{\delta \in \Delta_\phi} w(\delta). \tag{3}$$

Note that some of the formulas $\delta$'s might be inconsistent (for example, a formula from $\Delta$ might be a conjunction in which both $O(p \wedge q)$ and $Fp$ appear as conjuncts). For an inconsistent formula $\delta$, we have $\vdash \delta \leftrightarrow \bot$ and, consequently $\vdash w(\delta) = 0$, by the inference rule P-Dis. This provably filters out the inconsistent $\delta$'s from each weight formula, using the axioms about linear inequalities. Thus, without any loss of generality, we can assume in the rest of the proof that all the formulas from $\Delta$ are consistent[2].

Lets us consider a new formula $f'$, created by substituting each term of each weight formula of $g_i$ from (1), thus transforming each conjunct (2) into

$$g_i' = \left( \bigwedge_{j=1}^{r} (\sum_k a_{j,k} \sum_{\delta \in \Delta_{\phi_{j,k}}} w(\delta) \geq c_j) \right) \wedge \left( \bigwedge_{j=r+1}^{r+s} \neg (\sum_k a_{j,k} \sum_{\delta \in \Delta_{\phi_{j,k}}} w(\delta) \geq c_j) \right)$$

Since consistency of the formula $f$ is equivalent to consistency of one of its disjuncts $g_i$ from (1), in the rest of the proof we will focus on one such disjunct, $g_i$. Note that (3) implies that $g_i$ and $g_i'$ are two provably equivalent formulas (and the same holds for $f$ and $f'$). Then we will construct $g_i''$ by adding to $g_i'$: a non-negativity constraint and an equality that binds the total probability weight of $\delta$'s to 1. In other words, $g_i''$ is the conjunction of the following formulas:

[2] We might introduce $\Delta^c$ and $\Delta^c_\phi$ as the sets of all consistent formulas from $\Delta$ and $\Delta_\phi$, respectively, but since we will still have $\vdash w(\phi) = \sum_{\delta \in \Delta^c_\phi} w(\delta)$, we prefer not to burden the notation with the superscripts in the rest of the proof, and we assume that we do not have inconsistent formulas in $\Delta$.

$$
\begin{array}{lr}
 & \sum_{\delta\in\Delta} w(\delta) = 1 \\
\forall \delta \in \Delta & w(\delta) \geq 0 \\
\forall l \in \{1,\ldots,r\} & \sum_k a_{l,k} \sum_{\delta\in\Delta_{\phi_{l,k}}} w(\delta) \geq c_l \\
\forall l \in \{r+1,\ldots,r+s\} & \sum_k a_{l,k} \sum_{\delta\in\Delta_{\phi_{l,k}}} w(\delta) < c_l
\end{array}
$$

Since the weights can be attributed independently while respecting the system of equations and inequalities, the formula $g_i''$ is satisfiable iff the corresponding system of equations and inequalities, that we denote by $Sys(g_i'')$ is solvable:

$$
\begin{array}{lr}
 & \sum_{i=1}^{|\Delta|} x_i = 1 \\
\forall i \in \{1,\ldots,|\Delta|\} & x_i \geq 0 \\
\forall l \in \{1,\ldots,r\} & \sum_k a_{l,k} \sum_{i=1}^{|\Delta_{\phi_{l,k}}|} x_i \geq c_l \\
\forall l \in \{r+1,\ldots,r+s\} & \sum_k a_{l,k} \sum_{i=1}^{|\Delta_{\phi_{l,k}}|} x_{r+i} < c_l
\end{array}
$$

Initially we considered a consistent formula $g_i$ and transformed it to a provably equivalent formula $g_i''$. Proving satisfiability of $g_i''$ is equivalent to proving satisfiability of $g_i$; since the set of models of $g_i''$ coincides with the set of models of $g_i'$, which in turn has the same models as $g_i$.

Using proof from the incongruous we assume $g_i''$ to be unsatisfiable and show that this leads to a contradiction. Since $g_i''$ is assumed unsatisfiable this means that the system of linear inequalities $Sys(g_i'')$ does not have a solution. This further means that in the process of solving the system $Sys(g_i'')$ (using any procedure for solving linear inequalities, e.g. we can use Fourier–Motzkin elimination) we would obtain an equivalent system containing an equation or inequality without solutions. Without any loss of generality, assume that the obtained formula is $0 = 1$. Now, since we have the axioms I1-I7 as a part of our $AX_{\mathcal{PMDL}}$, we can "syntactically" derive all those corresponding steps (of transforming inequalities using the procedure for solving linear inequalities) from $g_i''$ using our axiomatization, and therefore we obtain that $0 = 1$ is a formula (of our logic $\mathcal{PMDL}$) that can be derived from $g_i''$. That means $g_i''$ is inconsistent, which is a contradiction because we started with $g_i$ as a consistent formula. □

# 5 Decidability

In this section, we prove that our logic $\mathcal{PMDL}$ is decidable, and we show that there is a decidability procedure for the problem that runs in polynomial space. First, let us recall the satisfiability problem: given a formula $\theta$, we want to determine if there exists a model $M$ such that $M \models \theta$.

**Theorem 2** (Decidability)**.** *Satisfiability problem for $\mathcal{PMDL}$ is decidable.*

*Proof.* Since we have two types of formulas, we will consider two cases. First, let us assume that $\theta \in \mathcal{L}_{MDL}$. We start with the well-known result that the problem of whether a formula from $\mathcal{L}_{MDL}$ is satisfiable in an MDL model is decidable [14]. It is sufficient to show that each $\theta \in \mathcal{L}_{MDL}$ is satisfiable in an MDL model iff it is satisfiable under our semantics. First, if $(W', R', V'), w' \models \theta$ for some deontic model $(W', R', V')$ and $w' \in W'$, let us construct the model $M = \langle S, \mathscr{X}, \mu, \tau \rangle$, with $S = \{s\}$, $\mathscr{X} = \{\emptyset, S\}$, $\mu(S) = 1$ and $\tau(s) = ((W', R', V'), w')$. Since $(W', R', V'), w' \models \theta$, then $M, s \models \theta$. From the fact that $s$ is the unique state of $M$, we conclude that $M \models \theta$. On the other hand, if $\theta$ is not satisfiable in MDL, then for every $M = \langle S, \mathscr{X}, \mu, \tau \rangle$ and $s \in S$ we will have $M, s \not\models \theta$, so $M \not\models \theta$.

Now, let us consider the case $\theta \in \mathcal{L}_{\mathcal{PMDL}}$. In the proof, we use the method of filtration [12, 3], and reduction to finite systems of inequalities. We only provide a sketch of the proof, since we use similar ideas as in our completeness proof. We will also use the notation introduced in the proof of completeness. In the first part of the proof, we show that a formula is satisfiable iff it is satisfiable in a model with a finite number of (1) states and (2) worlds.

(1) First we show that if $\theta \in \mathcal{L}_{\mathcal{PMDL}}$ is satisfiable, then it is satisfiable in a model with a finite set of states, whose size is at most $2^{|Sub_{DL}(\theta)|}$ (where $Sub_{DL}(\theta)$ is the set of deontic subformulas of $\theta$, as defined in the proof of Theorem 1). Let $M = \langle S, \mathscr{X}, \mu, \tau \rangle$ be a model such that $M \models \theta$. Let us define by $\sim$ the equivalence relation over $S \times S$ in the following way: $s \sim s_2$ iff for every $\phi \in Sub_{DL}(\theta)$, $M, s \models \phi$ iff $s_2 \models \phi$. Then the corresponding quotient set $S_{/\sim}$ is finite and $|S_{/\sim}| \leq 2^{|Sub_{DL}(\theta)|}$. Note that every $C_i$ belongs to $\mathscr{X}$, since it corresponds to a formula $\delta_i$ of $\Delta$ (from the proof of Theorem 1), i.e., $C_i = \|\delta_i\|$. Next, for every equivalence class, $C_i$ we choose one element and denote it $s_i$. Then we consider the model $M' = \langle s_2, \mathscr{X}', \mu', \tau' \rangle$, where:

- $s_2 = \{s_i \mid C_i \in S_{/\sim}\}$,
- $\mathscr{X}'$ is the power set of $s_2$,
- $\mu'(\{s_i\}) = \mu(C_i)$ such that $s_i \in C_i$ and for any $X \subseteq s_2$,
  $\mu'(X) = \sum_{s_i \in X} \mu'(\{s_i\})$,
- $\tau'(s_i) = \tau(s_i)$.

Then it is straightforward to verify that $M' \models \theta$. Moreover, note that, by definition of $M'$, for every $s_i \in s_2$ there is $\delta_i \in \Delta$ such that $M', s_i \models \delta_i$, and that for every $s_j \neq s_i$ we have $M', s_j \not\models \delta_i$. We therefore say that $\delta_i$ is the *characteristic formula* of $s_i$.

(2) Even if $s_2$ is finite, some sets of worlds attached to a state might be infinite. Now we will modify $\tau'$, to ensure that every $W(s_i)$ is finite, and of the size which is bounded by a number that depends on the size of $\theta$. In this part of the proof, we refer to the filtration method used to prove completeness of MDL [3], which shows that if a deontic formula $\phi$ is satisfiable, it is satisfied in a world of a model $D(\psi) = (W, R, V)$ where the size of $W$ is at most exponential wrt. the size of the set of subformulas of $\phi$. Then we can replace $\tau'$ with a function $\tau''$ which assigns to each $s_i$ one such $D(\delta_i)$ and the corresponding world, where $\delta_i$ is the characteristic formula of $s_i$. We also assume that each $V(s_i)$ is restricted to the propositional letters from $Sub_{DL}(\theta)$. Finally, let $M'' = \langle s_2, \mathscr{X}', \mu', \tau'' \rangle$ It is easy to check that for every $\phi \in Sub_{DL}(\theta)$ and $s_i \in s_2$, $M', s_i \models \phi$ iff $M'', s_i \models \phi$. Therefore, $M'' \models \theta$.

From the steps (1) and (2) it follows that in order to check if a formula $\theta \in \mathcal{L}_{\mathcal{PMDL}}$ is satisfiable, it is enough to check if it is satisfied in a model $M = \langle S, \mathscr{X}, \mu, \tau \rangle$ in which $S$ and each $W_s$ (for every $s \in S$) are of finite size, bounded from above by a fixed number depending on the size of $|Sub_{DL}(\theta)|$. Then there are finitely many options for the choice of $S$ and $\tau$ (i.e., $(D_s, w_s)$, for every $s \in S$), and our procedure can determine in finite time whether there is a probability measure $\mu$ for some of them, such that $\theta$ holds in the model. We convert our formula $f$ into a formula in the complete disjunctive form as in (1). We guess $S$ and $\tau$ and check whether we can assign probability values to the states from $S$, considering each disjunct $g_i$ and using translation to a system of linear inequalities, in the same way as we have done in the proof of Theorem 1. This finishes the proof since the problem of checking whether a linear system of inequalities has a solution is decidable. $\square$

Moreover, we show that there is a procedure that decides the satisfiability of any formula of $\mathcal{PMDL}$ in $PSPACE$.

**Theorem 3.** *There is a procedure that decides whether a formula of the logic $\mathcal{PMDL}$ is satisfiable in a measurable structure from $\mathcal{PMDL}^{Meas}$ which runs in polynomial space.*

*Proof.* Let us first consider the formulas from $MDL$. Recall that in the proof of Theorem 2 we have shown that each $\theta \in \mathcal{L}_{MDL}$ is satisfiable in an MDL model iff it is satisfiable under our semantics. Thus we can use the result that there is a procedure for deciding whether a formula $\theta \in \mathcal{L}_{MDL}$ is satisfiable, that runs in polynomial space [13].

For probabilistic formulas, we want to use some parts of the proof of Theorem 2 (which in turn uses the proof of Theorem 1). Here we will use all the notations introduced in the proofs: Theorem 1 and Theorem 2. First, note that in the proof of Theorem 2 we proved, using filtration, that if a formula $f$ is satisfiable, then it is satisfied in a model $M' = \langle s_2, \mathscr{X}', \mu', \tau' \rangle$ with $m$ states, where $m$ at most $2^{|Sub_{DL}(f)|}$, i.e., $s_2 = \{s_1, \dots, s_m\}$, and where each state $s_i \in s_2$ is represented by its characteristic formula $\delta_i \in \Delta$. Now we will show that we can reduce the size of the set of states even more. Let $DS(f)$ denote the set of all deontic formulas $\phi$ such that $w(\phi)$ is a term that appears in the formula $f$ (i.e., $w(\phi)$ is a sub-expression of $f$), Let us consider the set of equations and inequalities over the variables $x_1, \dots, x_m$:

$$x_1 + \cdots + x_m = 1, \tag{4}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ \dots\ x_m \geq 0, \tag{5}$$

and, for each $\phi \in DS(f)$, the equation

$$\sum_{\delta_i \in \Delta_\phi} x_i = \mu'(\|\phi\|_{M'}), \tag{6}$$

where $\Delta_\phi = \{\delta \in \Delta \mid \delta \vdash \phi\}$. Here we employ the result form linear algebra which states that if a system of $k$ linear equations has a *non-negative* solution, then it has a non-negative solution where at most $k$ values are different than zero [4]. Since the above system has one solution, namely

$$(x_1, \dots, x_m) = (\mu'(\{\delta_1\}), , \dots, \mu'(\{\delta_m\})),$$

then the system of equations (4) and (6) has a non-negative solution with at most $k(f) = |DS(f)| + 1$ values different than zero (note that when we calculate the number of equations, we ignore (5), since it simply states non-negativity, which is already assumed). Without any loss of generality, assume that this solution assigns the values $x_i = d_i$, where $d_i = 0$ for $i > k(f)$. Then we can define $M = \langle S, \mathscr{X}, \mu, \tau \rangle$, where $S = \{s_1, \dots, s_{k(f)}\}$, $\tau$ is the restriction of $\tau'$ to the set $S \subseteq s_2$, and for every $s_i \in S$, $\mu(\{s_i\} = d_i$. Obviously $M' \models f$ implies $M \models f$, so it is sufficient to consider the structures with $k(f)$ worlds.

Now we describe our procedure which runs as follows: it systematically cycles through sets of characteristic formulas $\overline{\Delta} \subseteq \Delta$ of cardinality $k(f)$. Fixing such subsets can be obtained in polynomial space. Indeed, recall that each element of $\Delta$ is a conjunction of elements of $Sub_{DL}(f)$ and their negations, and the satisfiability of each conjunction in $MDL$ can be checked in polynomial space [13]. Then, for each such $\overline{\Delta}$, we check if we can assign the probability values $x_1, \dots, x_{k(f)}$ to its elements such that $f$ is satisfied. We consider the formula which is the conjunction of the following formulas:

$$x_1 + \cdots + x_{k(f)} = 1,$$

$$x_1 \geq 0,\ x_2 \geq 0,\ \dots\ ,\ x_{k(f)} \geq 0,$$

and the formula

$$Trans_{RCF}(f),$$

where $Trans_{RCF}(f)$ is obtained from $f$ by applying the following transformations:

- we replace in $f$ each occurrence of every $w(\phi)$ (for every $\phi \in DS(f)$) with
$$\sum_{\delta_i \in \Delta_\phi \cap \overline{\Delta}} x_i.$$
- We rewrite every integer coefficient from $f$ with an expression that uses only 1, 0, and $-1$, using the binary representation of the numbers, and the powers are represented using multiplication. For example, number 9 is rewritten as $(1+1)(1+1)(1+1)+1$.

In this way, the size of obtained conjunction stays polynomial wrt. length of $f$. With this transformation, we directly follow the approach of [7]. The idea is that the obtained formula is a quantifier-free formula in the language of real closed fields (RCF). Then Canny's procedure [2], which decides satisfiability of quantifier-free formulas of RCF in polynomial space, can be applied. It is clear that $f$ is satisfiable in $\mathcal{PMDL}^{Meas}$ iff for some $\overline{\Delta}$ the formula above is satisfied in RCF. This completes our proof. $\square$

# 6 The logic $\mathcal{PMDL}^2$

In this section, we present the logic $\mathcal{PMDL}^2$ whose language extends the language of $\mathcal{PMDL}$. This new logic assumes a fixed finite set of agents $Ag$, and it allows nesting of probabilities, enabling formulas that can express the uncertainty of one agent about some other agent's uncertainty about norms. Consequently, the logic $\mathcal{PMDL}^2$ will have a different probability structure, compared to the previous logic. Instead of having one measure $\mu$ over the states, we will have a function $\mathscr{P}$ that assigns a probability space to each agent and state ranging over a subset of all states. In the following sections, we will introduce the changes made to $\mathcal{PMDL}$ in order to construct $\mathcal{PMDL}^2$.

## 6.1 Syntax and Semantics

**Definition 11** (Formulae)**.** *Let $\mathbb{P}$ be a set of atomic propositions, and let $Ag$ be a set of agents. The language $\mathcal{L}_{\mathcal{PMDL}^2}$ is generated by the following two sentences of BNF:*

$$\begin{array}{lll} [\mathcal{L}_{MDL}] & \phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid O\phi & p \in \mathbb{P} \\ [\mathcal{L}_{\mathcal{PMDL}^2}] & \theta ::= \phi \mid a_1 w_i(\theta_1) + \cdots + a_n w_i(\theta_n) \geq \alpha \mid \neg\theta \mid \theta \wedge \theta & a_j, \alpha \in \mathbb{N}, i \in Ag \end{array}$$

*The expression $w_i(\phi) \geq \alpha$ stands for "according to the agent $i$, the probability of $\phi$ is at least $\alpha$".*

Note that the formula $a_1 w_i(\theta_1) + \cdots + a_n w_i(\theta_n) \geq \alpha$ contains exclusively one agent $i$; such a formula is called $i-$probability formula. Although we do not allow combination of agents within one linear combination, the formulas within the scope of $w_i$ might contain probabilities of other agents than $i$, as illustrated by the following example.

**Example 2.** *Following our previous example about behavioral norms in a library, we can now express the certainty of a person about another person's certainty. For example, the fact that a person has become fairly certain that another person is certain about it not being normal to be quiet in a library. This might be expressed by the probabilistic statement "agent i attributes the probability that agent j attributes the probability that one is obliged to be quiet to be at most 0.2 is at least 0.9". This sentence could be formalized using the introduced language as*

$$w_i(w_j(Oq) \leq 0.2) \geq 0.9.$$

For the formulas of $\mathcal{PMDL}^2$, we introduce the same types of abbreviations as we have done for $\mathcal{PMDL}$.

Now we introduce the semantics of $\mathcal{PMDL}^2$.

**Definition 12** (Model)**.** *A $\mathcal{PMDL}^2$ model is a tuple $M = \langle S, \tau, \mathscr{P} \rangle$, where:*

- *$S$ is a non-empty set of states*
- *$\tau$ associates with each state $s \in S$ a tuple containing an MDL model and one of its worlds: $\tau(s) = (D_s, w_s)$ where:*
  - $D_s = (W_s, R_s, V_s) \in \mathbb{D}$
  - $w_s \in W_s$
- *$\mathscr{P}(i, s)$ is a function assigning to each combination of agent $(i)$ and state $(s)$ a probability space $\mathscr{P}(i, s) = (S_{i,s}, \mathscr{X}_{i,s}, \mu_{i,s})$ where:*
  - *$S_{i,s} \subseteq S$ an arbitrary subset of $S$ that can be interpreted as the set of states that agent $i$ has conceptions about in state $s$.*
  - *$\mathscr{X}_{i,s}$ is a $\sigma$-algebra of subsets of $S_{i,s}$*
  - *$\mu_{i,s} : \mathscr{X}_{i,s} \mapsto [0, 1]$ is a probability measure.*

Let us illustrate this definition.

**Example 2.** ***(continued)*** *Assume a finite set of atomic propositions $\{p, q\}$. Let us consider the model $M = \langle S, \tau, \mathscr{P} \rangle$, where*

- $S = \{s_1, s_2, s_3, s_4\}$
- *$\mathscr{P}$ We will set the probability measures explicitly for each state-agent pair while the respective set $S_{i,s}$ will be set to $S$ and the respective sigma-algebra $\mathscr{X}_{i,s}$ will be the power set of $S$.*
  - *$\mu_{i,s_1}$ is characterized by: $\mu_{i,s_1}(\{s_1\}) = 0.5$, $\mu_{i,s_1}(\{s_2\}) = \mu_{i,s_1}(\{s_3\}) = 0.2$, $\mu_{i,s_1}(\{s_4\}) = 0.1$*
  - *$\mu_{j,s_1}$ is characterized by: $\mu_{j,s_1}(\{s_1\}) = \mu_{j,s_1}(\{s_2\}) = 0.1$, $\mu_{j,s_1}(\{s_3\}) = 0.0$, $\mu_{j,s_1}(\{s_4\}) = 0.8$.*
  - *$\mu_{j,s_2}$ is characterized by: $\mu_{j,s_2}(\{s_1\}) = \mu_{j,s_2}(\{s_2\}) = 0.0$, $\mu_{j,s_2}(\{s_3\}) = 0.1$, $\mu_{j,s_2}(\{s_4\}) = 0.9$.*
  - *$\mu_{j,s_3}$ is characterized by: $\mu_{j,s_3}(\{s_1\}) = \mu_{j,s_3}(\{s_2\}) = \mu_{j,s_3}(\{s_3\}) = 0.0$, $\mu_{j,s_3}(\{s_4\}) = 1$.*
  - *$\mu_{j,s_4}$ is characterized by: $\mu_{j,s_4}(\{s_1\}) = 0.5$, $\mu_{j,s_4}(\{s_2\}) = \mu_{j,s_4}(\{s_3\}) = 0.1$, $\mu_{j,s_4}(\{s_4\}) = 0.3$.*

- *$\tau$ maps each state in $S$ to a pointed deontic model; specifically for our interest is the assignment of state $s_1$, $D_{s_1} = (W_{s_1}, R_{s_1}, V_{s_1})$ and $w_{s_1}$ such that*
  - $W_{s_1} = \{w_1, w_2, w_3, w_4\}$
  - $R_{s_1} = \{(w_1, w_2), (w_1, w_3), (w_2, w_2), (w_2, w_3), (w_3, w_2), (w_3, w_3), (w_4, w_2), (w_4, w_3), (w_4, w_4)\}$
  - $V_{s_1}(p) = \{w_1, w_3\}$, $V_{s_1}(q) = \{w_2, w_3\}$
  - $w_{s_1} = w_1$

  *Note that the domain of $\tau$ is always the whole set $S$; but in this example, we only explicitly specify $\tau(s_1)$ for illustration purposes.*

*This model is depicted in Figure 2. The circle on the right contains the four states of the model. The dotted lines represent probability measure $\mu_{i,s_1}$ the others are not drawn to reduce cluttering. Each of the states is equipped, by $\tau$, with a standard pointed model of MDL. In this picture, only one of them is shown, the one that corresponds to $s_1$. It is represented within the circle on the left. Note that the arrows depict the relation $R$. If we assume that $q$ stands for "quiet", like in the previous example, in all $R-$successors of $w_1$ the proposition $q$ holds. Note that, according to Definition 3, this means that in $w_1$ people are obliged to be quiet in the library.*

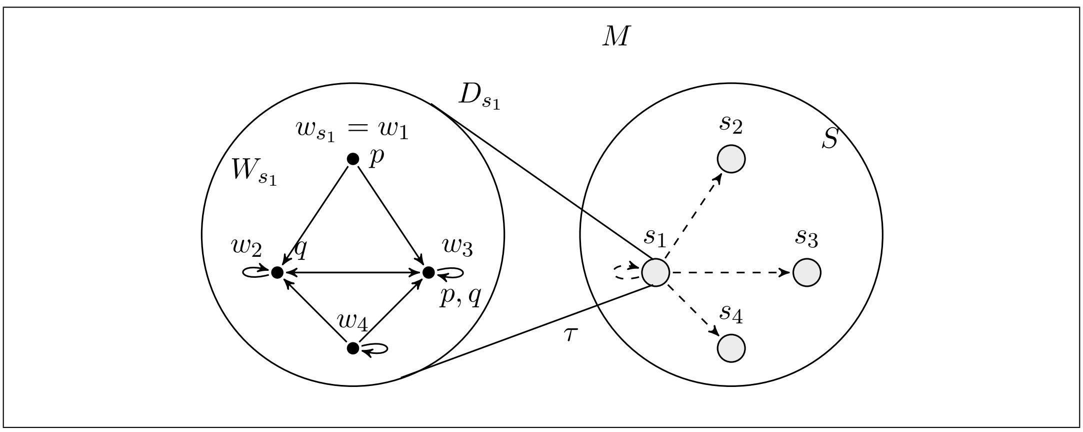


Figure 2: Model $M = \langle S, \tau, \mathscr{P} \rangle$.

Next, the satisfiability of a formula in a model can be defined. First, the truth of a deontic formula in a state of a $\mathcal{PMDL}^2$ model is given. This definition is in accordance with the standard satisfiability relation of MDL $\models_{MDL}$.

**Definition 13** (Satisfaction)**.** *Let $M = \langle S, \tau, \mathscr{P} \rangle$ be a $\mathcal{PMDL}^2$ model, and let $s \in S$. We define the satisfiability of formula $\theta \in \mathcal{L}_{\mathcal{PMDL}^2}$, in state $s$ of model $M$ denoted by $M, s \models \theta$ recursively as follows with $\phi \in \mathcal{L}_{MDL}$:*

- *$M, s \models \phi$ iff $D_s, w_s \models_{MDL} \phi$, where $\tau(s) = (D_s, w_s)$.*
- *$M, s \models a_1 w_i(\theta_1) + \cdots + a_n w_i(\theta_n) \geq \alpha$ iff $a_1 \mu_{i,s}(\|\theta_1\|^M_{i,s}) + \cdots + a_n \mu_{i,s}(\|\theta_n\|^M_{i,s}) \geq \alpha$.*
- *$M, s \models \neg\theta$ iff $M, s \not\models \theta$.*
- *$M, s \models \theta_l \wedge \theta_k$ iff $M, s \models \theta_l$ and $M, s \models \theta_k$.*

*For a model $M = \langle S, \tau, \mathscr{P} \rangle$, a formula $\theta \in \mathcal{L}_{\mathcal{PMDL}^2}$, state $s$ and agent $i$, let $\|\theta\|^M_{i,s}$ denote the set of states that satisfy $\theta$, from the perspective of agent $i$ in state $s$ i.e.,*

$$\|\theta\|^M_{i,s} = \{s_2 \in S_{i,s} \mid M, s_2 \models \theta\}.$$

*We omit the super- and subscripts from $\|\theta\|^M_{i,s}$ when it is clear from context. The satisfaction relation shows that in this model construction formulas $\theta$ can occur as the argument to a weight formula $w_i$, this means that weight formulas can be arguments of weight operators.*

Since the focus is on measurable structures and completeness is proven for this class of structures, this class is redefined for $\mathcal{PMDL}^2$ models.

**Definition 14** (Measurable model)**.** *A probabilistic deontic model is* measurable *if*

$$\|\phi\|^M_{i,s} \in \mathscr{X}_{i,s}$$

*for every $\phi \in \mathcal{L}_{MDL}$.*

**Example 2.** ***(continued)*** *Continuing the previous example, according to Definition 13 it holds that $M, s_1 \models Oq$. At this point it is also possible to speak of the uncertainty of agent $i$ about the uncertainty of agent $j$ of the obligation to be quiet in the library. Assume that $\tau$ is defined in the way such that $M, s_2 \models Oq$ and $M, s_3 \models Oq$, but $M, s_4 \not\models Oq$. Then $\mu_{j,s_1}(\|Oq\|) = \mu_{j,s_1}(\{s_1, s_2, s_3\}) = 0.1 + 0.1 + 0.0 = 0.2$; $\mu_{j,s_2}(\|Oq\|) = \mu_{j,s_2}(\{s, s_2, s_3\}) = 0.0 + 0.0 + 0.1 = 0.1$; $\mu_{j,s_3}(\|Oq\|) = \mu_{j,s_3}(\{s, s_2, s_3\}) = 0.0 + 0.0 + 0.0 = 0.0$; $\mu_{j,s_4}(\|Oq\|) = \mu_{j,s_4}(\{s_1, s_2, s_3\}) = 0.5 + 0.1 + 0.1 = 0.7$. From this follows that $\mu_{i,s_1}(\|w_j(Oq) \leq 0.2\|) = \mu_{i,s_1}(\{s_1, s_2, s_3\}) = 0.5 + 0.2 + 0.2 = 0.9$. According to Definition 6, $M, s_1 \models w_i(w_j(Oq) \leq 0.2) \geq 0.9$. Describing the uncertainty of agent $i$ about the uncertainty of agent $j$'s obligation to be quiet in the library.*

### 6.2 Axiomatization

The following axiomatization $AX_{\mathcal{PMDL}^2}$ combines –like $AX_{\mathcal{PMDL}}$– the axioms of proof system D of MDL [14] with the axioms of the probabilistic logic. In this case, the probabilistic axioms come from [6].

**The Axiomatic System:** $AX_{\mathcal{PMDL}^2}$

**Tautologies and Modus Ponens**

Taut. All instances of propositional tautologies.

MP. From $\theta$ and $\theta \rightarrow \theta'$ infer $\theta'$.

**Reasoning with $O$:**

O-... see axiomatization in Section 3

**Reasoning about linear inequalities:**

I1.-I7. see axiomatization in Section 3

**Reasoning about probabilities:**

W1. $w_i(\theta) \geq 0$ (non negativity).

W2. $w_i(\theta \vee \theta') = w_i(\theta) + w_i(\theta')$, if $\neg(\theta' \wedge \theta')$ is an instance of a classical propositional tautology (finite additivity).

W3. $w_i(\top) = 1$

P-Dis. From $\theta \leftrightarrow \theta'$ infer $w_i(\theta) = w_i(\theta')$ (probabilistic distributivity)

As before the axiom Taut allows all propositional tautologies. Though since $\mathcal{L}_{MDL}$ is included in $\mathcal{L}_{\mathcal{PMDL}^2}$ the distinction for Modus Ponens (MP) dissolves and can be applied to both types of formulas. P-Dis is an inference rule which states that two equivalent deontic formulas must have the same probability values.

### 6.3 Soundness and Completeness

In this section it is proven that the construction $\mathcal{PMDL}^2$ is sound and complete with respect to the class of measurable models, combining and adapting the approaches from [14, 6].

**Theorem 4** (Soundness & Completeness)**.** *The axiom system $AX_{\mathcal{PMDL}^2}$ is sound and complete with respect to the class of measurable probabilistic deontic models. i.e., $\vdash \theta$ iff $\models \theta$.*

*Proof.* The proof is a modification of the corresponding proof for $\mathcal{PMDL}$. To prove completeness, we need to show that every consistent formula $\theta$ is satisfiable in a measurable model. The modification of the logic gives iterations of weight formulas of arbitrary depth, also instead of one measure there is a measure for each agent and state pair $(i, s)$; for this, the proof needs to be adjusted.

For any formula $\theta$ we will denote the set of sub-formulas closed under negation as follows $Sub^+(\theta) = Sub(\theta) \cup \{\neg\theta' \mid \theta' \in Sub(\theta)\}$. We say that a set of formulas $A \subseteq B$ is maximal with regards to $B$ when $\forall\theta \in B$, $A$ contains either $\theta$ or $\neg\theta$.

Let $\theta$ be a consistent formula of $\mathcal{L}_{\mathcal{PMDL}^2}$. Then let $S$ denote the set of maximal consistent subsets of $Sub^+(\theta)$. And define for each $s \in S$ the element $\xi_s = \bigwedge_{\theta' \in s} \theta'$ to be the conjunction of elements in $s$. Denote the set of elements $\xi_s$ as follows $\Xi = \{\xi_s \mid s \in S\}$. $S$ will be the set of states of our model of the formula $\theta$. Furthermore, in order to define $\tau$ we construct for each state $s \in S$ the MDL context as $s_{MDL} = \{\phi \in \mathcal{L}_{MDL} \mid \phi \in s\}$ and its conjunction as $\delta_s = \bigwedge_{\phi \in s_{MDL}} \phi$. Then we can define $\tau$ in the following way. By completeness of MDL, for each $\delta_s$ there is a deontic model $D_s$ and a world $w_s$ in it such that $D_s, w_s \models_{MDL} \delta_s$. Then we define $\tau(s) = (D_s, w_s)$.

Since our probabilistic deontic model is of the form $M = (S, \tau, \mathscr{P})$ this leaves the task of defining the probability assignment $\mathscr{P}$. We chose $S_{i,s} = S$ and always assume that every subset of $S$ is measurable. The rest of the proof is essentially the same as the corresponding proof for defining probability assignment for probabilistic epistemic logic from [6]. $\mathscr{P}$ has to be defined in such a way that when we consider the model $M$, then for every $s \in S$ and every formula $\psi \in Sub^+(\theta)$ we have $M, s \models \psi$ iff $\psi \in s$. To do this we will make use of additivity using the following equivalence:

$$\vdash \psi \leftrightarrow \bigvee_{\{s \in S \mid \psi \in s\}} \xi_s.$$

Then using the axiom system, for every $i \in Ag$ we can derive in a similar way as in proof of Theorem 1 the following equation:

$$\vdash w_i(\psi) = \sum_{\{s \in S \mid \psi \in s\}} w_i(\xi_s).$$

By I1-I7, this can be extended to show that any $i-$probability formula $\theta' \in Sub^{+}(\theta)$ can be equivalently rewritten as a formula of the form $\sum_{s_2 \in S} c_{s_2}\mu_{i,s}(s_2) \geq \alpha$.

Similarly as in the proof of Theorem 1 we encode the problem to a set of linear equations and inequalities over variables of the form $x_{iss_2}$, where $x_{iss_2}$ represents $\mu_{i,s}(\{s_2\})$. For each $i-$probability formula $\psi \in Sub^{+}(\theta)$ we have a corresponding inequality. Using the conclusion et the end of the previous paragraph, When $\psi \in s$ then the corresponding inequality is: $\sum_{s_2 \in S_{i,s}} c_{s_2} x_{iss_2} \geq \alpha$. When $\neg\psi \in s$, then we have $\sum_{s_2 \in S_{i,s}} c_{s_2} x_{iss_2} < \alpha$. We also add the non-negativity constraints, and the condition that $\sum_{s_2 \in S_{i,s}} x_{iss_2} = 1$, as in the proof of Theorem 1. Furthermore, following that proof and [7], one can show that this system of inequalities has a solution $x^{*}_{iss_2}$ for all $s_2 \in S_{i,s}$; since each $\xi_s$ is consistent. The solution of this large system determines probability values of each agent in each state.

What is left to show is that for every formula $\psi \in Sub^{+}(\theta)$ and every state in $S$, we have $M, s \models \psi$ iff $\psi \in s$. The proof proceeds by induction on $\psi$. If $\psi$ is a deontic formula the result is immediate from the definition of $\tau$. The cases where $\psi$ is a negation or conjunction are straightforward. The case where $\psi$ is an $i$-probability formula follows from the construction above. Therefore if the formula $\theta$ is consistent then it is satisfiable in a model. $\square$

### 6.4 Decidability

Finally, we show that the logic $\mathcal{PMDL}^2$ is decidable.

**Theorem 5.** *Satisfiability problem for $\mathcal{PMDL}^2$ is decidable.*

*Proof.* Similarly, as in the proof of Theorem 2, we combine the method of filtration and reduction to finite systems of inequalities. Because of the similarity, we omit some details. In the proof, we will use some notation already introduced in the paper. Let us assume that the formula $\theta$ has a model $M = \langle S, \tau, \mathscr{P} \rangle$, where $\mathscr{P}(i,s) = (S_{i,s}, \mathscr{X}_{i,s}, \mu_{i,s})$. We will use filtration to construct the model of $\theta$ with finitely many states. By $\sim$ we denote the equivalence relation over $S \times S$, where $s \sim s_2$ iff for every $\theta' \in Sub(\theta)$, $s \models \theta'$ iff $s_2 \models \theta'$. Then the quotient set $S_{/\sim}$ is of the size $|S_{/\sim}| \leq 2^{|Sub(\theta)|}$. As before, for every class $C_j$ we choose an element and denote it $s_j$. We consider the model $M^{*} = \langle S^{*}, \tau^{*}, \mathscr{P}^{*} \rangle$, in which:

- $S^{*} = \{s_j \mid C_j \in S_{/\sim}\}$,
- $\mathscr{P}^{*}(i, s_j) = (S^{*}_{i,s_j}, \mathscr{X}^{*}_{i,s_j}, \mu^{*}_{i,s_j})$ such that:
  - $S^{*}(i, s_j) = \{s_k \in S^{*} \mid (\exists s_2 \in C_{s_k}) s_2 \in S(s_j)\}$,
  - $\mathscr{X}^{*}_{i,s_j}$ is the power set of $S^{*}(i, s_j)$,

– $\mu^*_{i,s_j}(\{s_k\}) = \mu_{i,s_j}(C_{w_k})$ (and $\mu^*_{i,s_j}$ extends to $\mathscr{X}^*_{i,s_j}$ by additivity),

- $\tau^*(s_j) = \tau(s_j)$.

It can be shown that $M^*$ is a measurable model. Moreover, using straightforward induction on the complexity of the formula, one can show that for any $\theta' \in Sub(\theta)$, $M, s \models \theta'$ iff $M^*, s_i \models \theta'$ where $s_i$ represents $C_s$ in $M^*$. Additionally in the same way, as in the proof of Theorem 2, we can show that the number of worlds in a deontic model of each state is finite and at most exponential wrt. size of $Sub(\theta)$. As the number of propositional letters and agents from $\theta$ is also finite, it turns out that we have to check only finitely many options for the choice of $S$ and $\tau$.

Let us describe the procedure which checks the satisfiability of a formula $\theta$. First, we transform $\theta$ to a disjunction of the formulas of the form $\bigwedge_{k=1}^{|Sub(\theta)|} \psi_k$, where $\psi_k \in Sub^+(\theta)$ and each subformula of $\theta$ appears exactly once in each conjunction (either negated or not). The conjunctions whose sub-conjunction consisting of deontic formulas is unsatisfiable can be eliminated using the decidability of MDL, as we have done in the proof of decidability of $\mathcal{PMDL}$. In each state $s \in S^*$ exactly one formula of the form $\bigwedge_{k=1}^{|Sub(\theta)|} \psi_k$ holds. Denote that (characteristic) formula by $\delta_s$ as before. Here we slightly abuse the notation, and we write $\psi \in \delta_s$ if $\psi$ is a conjunct in $\delta_s$. For every $\ell \leq 2^{Sub(\theta)}$ we will consider $\ell$ formulas of the above form such that the following three conditions hold:

- Those formulas $\delta_s$ are not necessarily different, but each formula does not contain both $\psi$ and $\neg\psi$ in the top conjunction.
- The conjunction of all deontic formulas from the top conjunction is consistent.
- At least one $\delta_s$ must contain $\theta$ in the top conjunction.

Then for every agent $i$, every state $s_j$, $j < \ell$, we consider the following set of equations and inequalities, with the set of variables $x_{iss_2}$, where $x_{iss_2}$ represents $\mu_{i,s}(\{s_2\})$ (as in the proof of completeness).

$$\sum_{s_2} x_{iss_2} = 1,$$

$$x_{iss_2} \geq 0,$$

$$a_1 \sum_{s_k : \theta_1 \in \delta_{s_k}} x_{is_js_k} + \cdots + a_n \sum_{s_k : \theta_n \in \delta_{s_k}} x_{is_js_k} \geq \alpha,$$

whenever $(a_1 w_i(\theta_1) + \cdots + a_n w_i(\theta_n) \geq \alpha) \in \delta_{s_j}$,

$$a_1 \sum_{s_k:\theta_1\in\delta_{s_k}} x_{is_js_k} + \cdots + a_n \sum_{s_k:\theta_n\in\delta_{s_k}} x_{is_js_k} < \alpha,$$

whenever $\neg(a_1 w_i(\theta_1) + \cdots + a_n w_i(\theta_n) \geq \alpha) \in \delta_{s_j}$,

Thus we have translated the problem of satisfiability of $\theta$ to a decidable problem of solving systems of linear inequalities, as before. Since we have finitely many possibilities for the choice of $\ell$, and for each $\ell$ finitely many possibilities to choose $\ell$ characteristic formulas, our logic $\mathcal{PMDL}^2$ is decidable. □

# 7 Conclusion

In this article, we introduced two probabilistic deontic logics. Each of them extends both monadic deontic logic and probability logic from [7]. The language of the first logic, $\mathcal{PMDL}$ is designed for reasoning about the probability of deontic statements. We axiomatized that language and proved soundness and completeness with respect to corresponding semantics. We also proved that our logic is decidable in PSPACE. The second proposed language allows nested probability operators, and it allows to express the uncertainty of one agent about the uncertainty that another agent places on deontic statements.

To the best of our knowledge, we are the first to propose logical frameworks of probabilistic deontic logics for reasoning with uncertainty about norms. It is worth mentioning that there is a recent knowledge representation framework about probabilistic uncertainty in deontic reasoning obtained by merging deontic argumentation and probabilistic argumentation frameworks [15].

Our logic $\mathcal{PMDL}$ used MDL as the underlying framework, we used this logic simply because it is one of the most studied deontic logics. On the other hand, MDL is also criticized because of some issues [9], like the representation of contrary-to-duty obligations. It is important to point out that the axiomatization technique developed in this work can also be applied if we replace MDL with, for example, dyadic deontic logic, simply by changing the set of deontic axioms and the function $\tau$ in the definition of the model, which would lead to a more expressive framework for reasoning about uncertain norms. Another avenue for future research is to extend the language by allowing conditional probabilities. In such a logic, it would be possible to express that one uncertain norm becomes more certain if another norm is accepted or learned.

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