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A relative Hofer estimate and the asymptotic Hofer-Lipschitz constant

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A R T I C L E I N F O

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A B S T R A C T

Let (M, ω) be a symplectic manifold and $U \subseteq M$ an open subset. We study the natural inclusion of the compactly supported Hamiltonian group of U in the compactly supported Hamiltonian group of M. The main result is an upper bound for this map in terms of the Hofer norms for U and M.
Applications are upper bounds on the asymptotic Hofer-Lipschitz constant and the relative Hofer diameter of U. The first bound is often sharp and the second one is often sharp up to a factor of 2.

Contents

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1. Results

1.1. The main result

The main result of this article is concerned with the following question. For simplicity, in this article *manifold* means *manifold without boundary.* Let (M, ω) be a symplectic manifold. For every $F \in C^\infty(M)$ we denote by X_F the Hamiltonian vector field generated by F, which is determined by the condition $\omega(X_F, \cdot) = dF$. We denote by $C_c^\infty\big([0,1] \times M\big)$ the set of compactly supported real-valued functions on $[0,1] \times M$. For every $H \in C_c^\infty\big([0,1] \times M\big)$ we denote $H_t := H(t, \cdot)$ and by $\varphi_H = (\varphi_H^t)_{t \in [0,1]}$ the Hamiltonian flow of H w.r.t. ω. We define the *compactly supported Hamiltonian group of* (M, ω) and the *Hofer norms* on the sets of functions and the Hamiltonian group by

$$
\begin{aligned}
\mathrm{Ham_c}(M) := \mathrm{Ham_c}(M, \omega) &:= \big\{ \varphi_H^1 \,\big|\, H \in C_c^\infty\big([0,1] \times M\big) \big\}, \\
\|\!|\cdot|\!\| := \|\!|\cdot|\!\|_c^M &: C_c^\infty\big([0,1] \times M\big) \to \mathbb{R}, \\
\|\!|H|\!\| &:= \int_0^1 \Big(\max_M H_t - \min_M H_t \Big)\, dt, \\
\|\cdot\|_c^M := \|\cdot\|_c^{M,\omega} &: \mathrm{Ham_c}(M) \to \mathbb{R}, \\
\|\varphi\|_c^M &:= \inf \big\{ \|\!|H|\!\| \,\big|\, H \in C_c^\infty\big([0,1] \times M\big) :\, \varphi_H^1 = \varphi \big\}.
\end{aligned}
\tag{1}
$$

Let $U \subseteq M$ be an open subset. Consider the natural inclusion

$$
\mathrm{Ham_c}(U) \ni \varphi \mapsto \widetilde{\varphi} \in \mathrm{Ham_c}(M), \quad \widetilde{\varphi}(x) := \begin{cases} \varphi(x), & \text{if } x \in U, \\ x, & \text{otherwise,} \end{cases} \tag{2}
$$

Question. *How much does this map fail to be an isometry with respect to the Hofer norms for U and M?*

The main result of this article is the following theorem, which implies that the answer to this question is "a lot", if U is small compared to M in a suitable sense. To state this result, for $a > 0$, we denote by $B^2(a), \overline{B}^2(a) \subseteq \mathbb{R}^2$ the open and closed balls of radius $\sqrt{a/\pi}$, around 0. We denote by ω_{st} the standard symplectic form on $\mathbb{R}^{2n}$.

Theorem 1 *(relative Hofer estimate). For every $\varphi \in \mathrm{Ham_c}(U)$ we have*

$$
\|\widetilde{\varphi}\|_c^M \le \inf \left(2a + \frac{\|\varphi\|_c^U}{N} \right), \tag{3}
$$

where $a \in (0, \infty)$ and $N \in \mathbb{N} := \{1, 2, \dots\}$ run over all numbers for which there exists a symplectic manifold (M', ω') and a symplectic embedding

$$
\psi : B^2(Na) \times M' \to M
$$

(with respect to $\omega_{\mathrm{st}} \oplus \omega'$ and ω), satisfying

$$
U \subseteq \psi(B^2(a) \times M').
$$

The weaker version of estimate (3) with the additive constant $2a$ replaced by $2Na$ and a factor of 2 in front of the second term can be deduced from the argument in the proof of the recent result by Polterovich and Shelukhin [18, Theorem C] on p. 19.[1] The interest of Theorem 1 lies in the facts that the additive constant $2a$ does not depend on N and there is no extra factor of 2.

The estimate (3) is often asymptotically sharp as the Hofer norm of φ on U tends to infinity. See Corollaries 2 and 4 below. On top of this, the additive constant is often sharp up to a factor of 2. See Corollary 5 and Proposition 6. This improves the result of J.-C. Sikorav for $\mathbb{R}^{2n}$ [20, Proposition, p. 62] by a factor of 8.

The strategy of proof of Theorem 1 is to adapt a version of Sikorav's method that was used by M. Brandenbursky and Kędra in [2] to estimate the autonomous norm. This version of the method uses an algebraic identity of D. Burago, S. Ivanov, and L. Polterovich [1].

To show our estimate, for a given Hamiltonian H that generates φ we choose a compact subset K of M, such that $[0,1]\times K$ contains the support of H. The trick is to choose a Hamiltonian diffeomorphism ψ with Hofer norm less than a, such that the sets $\psi^i(K)$, $i=0,\ldots,N-1$, are disjoint. We now cut φ into time-pieces, which we transport to the regions $\psi^i(K)$. The resulting Hamiltonian diffeomorphism differs from $\widetilde{\varphi}$ by some commutator with ψ. The estimate (3) follows from this and the fact that H can be chosen in such a way that

$$c_- \leq H \leq c_- + c, \tag{4}$$

where c_-, c are constants, with c arbitrarily close to the Hofer norm of φ.

To show this fact, we choose a Hamiltonian for φ whose Hofer norm is close to that of φ. We reparametrize the Hamiltonian in such a way that at each time its oscillation is less than c. The idea is now to shift the Hamiltonian by the product of a suitable function of time and some cut-off function on M, in such a way that the resulting Hamiltonian satisfies (4).

1.2. Application to the asymptotic Hofer-Lipschitz constant

Theorem 1 has the following direct application. Let (M,ω) be a symplectic manifold and U an open subset of M. We define the *asymptotic Hofer-Lipschitz constant of* (M,U,ω) to be

$$\begin{gathered}\mathrm{Lip}^\infty(M,U) := \mathrm{Lip}^\infty(M,U,\omega) := \\ \lim_{C\to\infty}\sup\left\{\frac{\|\widetilde{\varphi}\|_c^M}{\|\varphi\|_c^U}\,\middle|\,\varphi\in\mathrm{Ham}_c(U):\ \|\varphi\|_c^U > C\right\}.\end{gathered} \tag{5}$$

(Here our convention is that $\sup\emptyset := 0$.) This number can be understood as the asymptotic (for large distances) Lipschitz constant of the inclusion (2), with respect to the Hofer distances for U and M. It is the simplest interesting quantity comparing the two Hofer geometries, if M is closed. (Compare to Subsection 5.1.)

Corollary 2 *(upper bound on the asymptotic Hofer-Lipschitz constant). Assume that there exists $a>0$, $N\in\mathbb{N}\cup\{\infty\}$, and a symplectic manifold (M',ω'), such that, defining $c := Na$, we have*

[1] This is a consequence of the following (in-)equalities, which are stated in the proof of [18, Theorem C] on p. 19 (with the setting and notation of that article):

$$d_{\mathrm{Hofer}}(\varphi_f^{lt}, \varphi_{f_1}^t\cdot\ldots\cdot\varphi_{f_l}^t) \leq C \leq 2\big(d_{\mathrm{Hofer}}(\varphi_1,\mathrm{id})+\ldots+d_{\mathrm{Hofer}}(\varphi_l,\mathrm{id})\big),$$

$$\varphi_{f_1}^t\cdot\ldots\cdot\varphi_{f_l}^t = \varphi_F^t,\quad |F|_{C^0}=1.$$

The extra factor 2 in front of the second term in (3) comes from the fact that a generating Hamilton function may be both positive and negative. In the situation of [18, Theorem C] this factor disappears, since the generating Hamilton function f is positive.

$$M = B^2(c) \times M', \quad \omega = \omega_{\mathrm{st}} \oplus \omega', \quad U = B^2(a) \times M', \tag{6}$$

where for $c = \infty$ we define $B^2(c) := \mathbb{R}^2$. Then we have

$$\mathrm{Lip}^\infty(M, U) \leq \frac{1}{N} = \frac{a}{c}. \tag{7}$$

Proof. This follows immediately from Theorem 1. □

In particular, we have $\mathrm{Lip}^\infty(M, U) = 0$, if $N = \infty$. Extending the estimate (7) to a general triple (M, ω, U) consisting of a finite volume symplectic manifold and an open subset, one may guess that the inequality

$$\mathrm{Lip}^\infty(M, U) \leq C \frac{\int_U \omega^n}{\int_M \omega^n}$$

holds for such a triple, where C is a constant depending only on the dimension $2n$ of M. This guess is false. Hence the hypothesis that M, ω and U are products, cannot be dropped. This is a consequence of the following example.

Example *(big asymptotic Hofer-Lipschitz constant).* Let (M, ω) be a two-dimensional symplectic manifold and $U \subseteq M$ an open neighbourhood of some non-contractible embedded circle in M. Then the equality

$$\mathrm{Lip}^\infty(M, U) = 1$$

holds. See Proposition 12 in Subsection 5.2, which also provides another example, in which the above equality holds.

The next result provides a sufficient criterion under which the estimate (7) is sharp. We call ω *aspherical* if

$$\int_{S^2} u^*\omega = 0, \quad \forall u \in C^\infty(S^2, M). \tag{8}$$

In this case we call the pair (M, ω) *symplectically aspherical.* We denote $2n := \dim M$.

Proposition 3 *(lower bound on the asymptotic Hofer-Lipschitz constant). The inequality*

$$\mathrm{Lip}^\infty(M, U) \geq \frac{\int_U \omega^n}{\int_M \omega^n} \tag{9}$$

holds if one of the following conditions is satisfied:

(a) The form ω is exact and the symplectic volume of M is finite.
(b) The manifold M is closed, ω is aspherical, and U is displaceable in a Hamiltonian way.

In the case (a) the proof of this result is based on the fact that in this situation the Calabi invariant descends to the Hamiltonian group. In the case (b) the proof of this result is based on an argument by Y. Ostrover used in the proof of [16, Theorem 1.1]. Its key ingredient is a result of M. Schwarz about action selectors (spectral invariants). We will deduce the following corollary from Proposition 3.

Corollary 4 *(lower bound on the asymptotic Hofer-Lipschitz constant). Assume that there exist numbers $a > 0$ and $c \geq 2a$, and a closed symplectically aspherical symplectic manifold (M', ω'), such that (6) holds. Then we have*

$$\mathrm{Lip}^\infty(M, U) \geq \frac{a}{c}. \tag{10}$$

It follows that under the hypotheses of this corollary, the inequality (7) is sharp.

Remark. In the case (b) the proof of Proposition 3 given below can be extended to the more general settings of [12, Theorems 1.1 and 1.3], which provide conditions under which the (asymptotic) spectral invariants descend to $\mathrm{Ham}(M)$.

1.3. Application to the relative Hofer diameter

Another immediate consequence of Theorem 1 is the following. Let (M, ω) be a symplectic manifold and U an open subset of M. We define the *(extension) relative Hofer diameter of U in M* to be

$$\mathrm{Diam}(U, M) := \mathrm{Diam}(U, M, \omega) := \sup \left\{ \|\widetilde{\varphi}\|_c^M \,\middle|\, \varphi \in \mathrm{Ham}_c(U) \right\}. \tag{11}$$

This is the diameter of the distance function induced by the composition of the canonical extension homomorphism $\mathrm{Ham}_c(U) \to \mathrm{Ham}_c(M)$ with the Hofer norm, see Subsection 5.3.

Corollary 5 *(upper bound on the relative Hofer diameter). Assume that there exists a symplectic manifold (M', ω') and a number $a > 0$, such that*

$$(M, U, \omega) = \left(\mathbb{R}^2 \times M', B^2(a) \times M', \omega_{\mathrm{st}} \oplus \omega'\right).$$

Then we have

$$\mathrm{Diam}(U, M) \leq 2a.$$

Proof. This follows immediately from Theorem 1. □

In the case in which $(M', \omega') = \left(\mathbb{R}^{2n-2}, \omega_{\mathrm{st}}\right)$ for some $n \in \mathbb{N}$ J.-C. Sikorav proved this estimate with the right hand side replaced by $16a$, see [20, Proposition, p. 62].[2]

Remark. The *absolute* Hofer diameter $\mathrm{Diam}(M, M)$ has been calculated for many symplectic manifolds. In all known examples it is infinite. For an overview and references, see [11].

The next result provides sufficient conditions under which Corollary 5 is sharp up to a factor of 2. Let (M, ω) be a symplectic manifold. We call a symplectic manifold (M, ω) *(geometrically) bounded* if there exist an almost complex structure J on M and a complete Riemannian metric g such that the following conditions hold:

- The sectional curvature of g is bounded and $\inf_{x \in M} \iota_x^g > 0$, where ι_x^g denotes the injectivity radius of g at the point $x \in M$.

[2] [20, Proposition, p. 62] states that for every bounded subset B of $\mathbb{R}^{2n}$ and every Hamiltonian isotopy φ with support in B we have $\|\varphi^1\|_c^M \leq 8\|\psi\|_c^M$, where ψ ranges over all compactly supported Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$, such that B and $\psi(B)$ are separated by some hyperplane. However, the proof only shows the estimate with a factor 16 instead of 8. See also [7, Theorem 10, Section 5.6], where the mistake is corrected.

- There exists a constant $C \in (0, \infty)$ such that

$$|\omega(v,w)| \leq C|v|\,|w|, \quad \omega(v,Jv) \geq C^{-1}|v|^2,$$

for all $v, w \in T_xM$ and $x \in M$. Here $|v| := \sqrt{g(v,v)}$.

Proposition 6 *(lower bound on the relative Hofer diameter). Assume that there exist (M', ω') and a as in Corollary 5. Suppose also that (M', ω') is symplectically aspherical and geometrically bounded, and there exists a nonempty closed symplectic manifold (X, σ), such that*

$$n := \frac{1}{2}\big(\dim M' - \dim X - 2\big) \geq 0, \quad B^2(2a) \times (B^2(a))^n \times X \subseteq M'.$$

Then we have

$$\mathrm{Diam}(U, M) \geq a. \tag{12}$$

The proof of this result is based on a leafwise fixed point theorem for coisotropic submanifolds proved by the second author in [22].

1.4. Organization of the article and notation

In Section 2 we prove our main result, Theorem 1. In Section 3 we prove the lower bounds on the asymptotic Hofer-Lipschitz constant stated in Proposition 3 and Corollary 4. In Section 4 we prove the lower bound on the relative Hofer diameter stated in Proposition 6. Section 5 contains some remarks and examples. In the appendix we prove Proposition 11, which is used in the proof of Proposition 3.

In the rest of this article we will use the abbreviated notation

$$\|\cdot\| := \|\cdot\|_c^M : \mathrm{Ham_c}(M) \to \mathbb{R}.$$

1.5. Acknowledgements

We thank Felix Schlenk for making us aware that the Hofer-Lipschitz constant defined in formula (60) below satisfies $\mathrm{Lip}(M, U) \geq 1$. We are grateful to Leonid Polterovich for sharing Proposition 12 (under the assumption (b)) with us. Finally, we would like to thank Dusa McDuff for valuable feedback and Peter Spaeth for useful discussions.

2. Proof of Theorem 1 (relative Hofer estimate)

For the proof of Theorem 1 we need the following. Let (M, ω) be a symplectic manifold, U an open subset of M with compact closure, and φ a Hamiltonian diffeomorphism on M that is generated by a function with support in $[0, 1] \times U$.

Lemma 7 *(pinching the generating Hamiltonian). For every real number $c > \|\varphi|_U\|$ there exists a real number c_- and a smooth function $H : [0, 1] \times M \to \mathbb{R}$ that has compact support and Hamiltonianly generates φ, such that*

$$c_- \leq H \leq c_- + c \quad (\textit{on } [0, 1] \times M). \tag{13}$$

In order to prove this lemma we choose a Hamiltonian for φ whose Hofer norm is close to that of φ. We reparametrize the Hamiltonian in such a way that at each time its oscillation is less than c. The idea is now to shift the Hamiltonian by the product of a suitable function of time and some cut-off function on M. We need the following.

Lemma 8. *Let $c \in (0, \infty)$ and $f \in C\big([0,1],[0,\infty)\big)$, such that $\int_0^1 f(t)dt < c$. There exists a function $\tau \in C^\infty\big([0,1],[0,1]\big)$, such that*

$$\tau' > 0, \tag{14}$$

$$\tau(s) = s, \quad \forall s \in \{0,1\}, \tag{15}$$

$$f \circ \tau(s)\tau'(s) < c, \quad \forall s \in [0,1]. \tag{16}$$

Proof of Lemma 8. By approximating f from above with some smooth positive function, we may assume without loss of generality that $f > 0$ and f is smooth. We define

$$\sigma(t) := \frac{\int_0^t f(t')dt'}{\int_0^1 f(t')dt'}. \tag{17}$$

This is an orientation preserving diffeomorphism of $[0,1]$. We define

$$\tau := \sigma^{-1} : [0,1] \to [0,1].$$

This map satisfies (14), (15). Furthermore, for every $s \in [0,1]$, we have

$$f \circ \tau(s)\tau'(s) = f \circ \tau(s) \frac{\int_0^1 f(t')dt'}{f \circ \tau(s)} < c,$$

where in the last step we used our hypothesis. Hence (16) holds. This proves Lemma 8. $\square$

Proof of Lemma 7. Since U has compact closure, by the smooth version of Urysohn's lemma there exists a smooth function $\rho : M \to [0,1]$ that has compact support and equals 1 on U. By our hypothesis $c > \|\varphi|_U\|$ there exists a smooth function $\widetilde{H} : [0,1] \times M \to \mathbb{R}$ that satisfies

$$\varphi^1_{\widetilde{H}} = \varphi, \tag{18}$$

and has support in $[0,1] \times U$ and Hofer norm less than c. The functions $t \mapsto \min \widetilde{H}_t, \max \widetilde{H}_t$ are continuous. Therefore, by Lemma 8 with

$$f(t) := \max \widetilde{H}_t - \min \widetilde{H}_t$$

there exists a function τ is in the statement of that lemma. We define

$$\widehat{H} : [0,1] \times M \to \mathbb{R}, \quad \widehat{H}(s,x) := \tau'(s)\widetilde{H}\big(\tau(s),x\big).$$

By (14), (16) this function satisfies the inequality

$$\max \widehat{H}_s - \min \widehat{H}_s < c, \quad \forall s \in [0,1].$$

We choose a smooth function $g : [0,1] \to \mathbb{R}$, such that

$$\max \widehat{H}_s - c < g(s) < \min \widehat{H}_s, \quad \forall s \in [0,1]. \tag{19}$$

We define

$$c_- := \int_0^1 g(s)ds,\ G : [0,1] \times M \to \mathbb{R},\ G_s := (-g(s) + c_-)\rho,\ H := \widehat{H} + G.$$

To see that the function H satisfies (13), recall that $\widehat{H}$ vanishes outside of $[0,1] \times U$, and ρ equals 1 on U and takes values in $[0,1]$. The inequality $c_- \le H$ follows from these facts and the inequality $g(s) < \min \widehat{H}_s$ in (19). The inequality $H \le c_- + c$ follows from these facts and the inequality $\max \widehat{H}_s - c < g(s)$ in (19). (This inequality implies that $c + c_- > \max \widehat{H}_s \ge 0$ and therefore, $G_s \le (c + c_-)\rho \le c + c_-$.) Hence H satisfies (13).

For every $s \in [0,1]$ the derivative $d\widehat{H}_s$ has support inside of U, and the derivative dG_s has support outside of U. It follows that for every $s \in [0,1]$, we have

$$\varphi_H^s = \varphi_G^s \circ \varphi_{\widehat{H}}^s.$$

Since $\int_0^1 \big(-g(s) + c_-\big)ds = 0$, we have $\varphi_G^1 = \mathrm{id}$. It follows that $\varphi_H^1 = \varphi_{\widehat{H}}^1 = \varphi$, where we used (18). Hence H has the desired properties. This proves Lemma 7. □

Proof of Theorem 1. Without loss of generality we may assume that there exist $a \in (0,\infty)$, $N \in \mathbb{N}$, and a symplectic manifold (M',ω'), such that

$$M = B^2(Na) \times M', \tag{20}$$

$$\omega = \omega_{\mathrm{st}} \oplus \omega',$$

$$U = B^2(a) \times M'. \tag{21}$$

Let $\varphi \in \mathrm{Ham}_c(U)$ and

$$c > \|\varphi\|$$

be a real number. We choose a function $H' \in C_c^\infty\big([0,1] \times U\big)$, such that

$$\varphi_{H'}^1 = \varphi, \quad |||H'||| < c.$$

We choose an open subset $V \subseteq M$ whose closure is compact and contained in U, such that $[0,1] \times V$ contains the support of H'. By Lemma 7 with M, U replaced by U, V, there exists a real number c_-, a compact subset K of U, and a smooth function $H : [0,1] \times U \to \mathbb{R}$ that has support contained in $[0,1] \times K$, such that

$$\varphi_H^1 = \varphi$$

and the inequalities (13) holds. We define $\widetilde{H} : [0,1] \times M \to \mathbb{R}$ to be equal to H on $[0,1] \times U$ and 0 outside of this set.

An elementary argument using (20), (21), shows that there exists $\psi \in \mathrm{Ham}_c(M)$, such that

$$\|\psi\| < a, \tag{22}$$

$$K, \psi(K), \ldots, \psi^{N-1}(K) \text{ are (pairwise) disjoint,} \tag{23}$$

where $\psi^i := \psi \circ \cdots \circ \psi$ (i factors). We abbreviate

$$\varphi_i := \varphi_{\widetilde{H}}^{\frac{i}{N}}, \quad \varphi_{i,j} := \psi^j \varphi_i \psi^{-j}, \quad \forall i, j \in \{0, \ldots, N-1\}.$$

We define

$$\chi := \varphi_{N-1,0}\varphi_{N-2,1} \cdots \varphi_{1,N-2}, \tag{24}$$

where for simplicity we leave out the composition signs. We define $F : [0,1] \times M \to \mathbb{R}$ by

$$F(t,x) := \begin{cases} \dfrac{H_{\frac{t+N-i-1}{N}} \circ \psi^{-i}(x)}{N}, & \text{on } \psi^i(K),\ \forall i \in \{0, \ldots, N-1\}, \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

We denote by $\widetilde{\varphi} : M \to M$ the map given by φ on U and the identity outside U.

Claim 1. *We have*

$$\widetilde{\varphi} = \varphi_F^1 \chi \psi \chi^{-1} \psi^{-1}. \tag{26}$$

Proof of Claim 1. We have

$$\begin{aligned} \psi\chi^{-1}\psi^{-1} &= \psi\psi^{N-2}\varphi_1^{-1}\psi^{2-N}\psi^{N-3}\varphi_2^{-1}\psi^{3-N} \cdots \varphi_{N-1}^{-1}\psi^{-1} \\ &= \psi^{N-1}\varphi_1^{-1}\psi^{1-N}\psi^{N-2}\varphi_2^{-1}\psi^{2-N} \cdots \psi\varphi_{N-1}^{-1}\psi^{-1} \\ &= \varphi_{1,N-1}^{-1} \cdots \varphi_{N-1,1}^{-1}. \end{aligned} \tag{27}$$

Since φ_i equals the identity outside K, it follows from (23) that $\varphi_{i,j}$ and $\varphi_{i',j'}$ commute, if $j \neq j'$. Combining this with (27), (24), it follows that

$$\chi\psi\chi^{-1}\psi^{-1} = \begin{cases} \varphi_{N-1,0} & \text{on } K, \\ \varphi_{N-i-1,i}\varphi_{N-i,i}^{-1} & \text{on } \psi^i(K),\ \forall i \in \{1, \ldots, N-1\}, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Using (25), equality (26) follows. This proves Claim 1. □

Using Claim 1, we have

$$\|\widetilde{\varphi}\| \leq \|\varphi_F^1\| + \|\chi\psi\chi^{-1}\| + \|\psi^{-1}\|. \tag{28}$$

By the inequalities (13) we have

$$\max F - \min F \leq \frac{c}{N}$$

and therefore

$$\|\varphi_F^1\| \leq \frac{c}{N}.$$

Combining this with (28), the equalities

$$\|\chi\psi\chi^{-1}\| = \|\psi\|, \quad \|\psi^{-1}\| = \|\psi\|,$$

and (22), it follows that

$$\|\widetilde{\varphi}\| < \frac{c}{N} + 2a.$$

Since this holds for all $c > \|\varphi\|$, the desired inequality (3) follows. This proves Theorem 1. $\square$

Remark. The idea of writing $\widetilde{\varphi}$ as in (26) comes from [2, proof of the theorem on p. 64], in which a given Hamiltonian diffeomorphism of $\mathbb{R}^{2n}$ is written as a product of autonomous pieces. The identity (26) corresponds to the algebraic identity provided by the proof of [1, Lemma 2.4].

3. Proofs of Proposition 3 and Corollary 4 (lower bound on the asymptotic Hofer-Lipschitz constant)

In this section we prove Proposition 3 and Corollary 4. To treat the case (a) in Proposition 3, we need the following. Let (M, ω) be an exact symplectic manifold.[3] We denote $2n := \dim M$ and define the *Calabi homomorphism for* (M, ω) to be the map

$$\mathrm{Cal} := \mathrm{Cal}_{(M,\omega)} : \mathrm{Ham_c}(M) \to \mathbb{R}, \quad \mathrm{Cal}(\varphi) := \int_0^1 \left(\int_M H_t \omega^n \right) dt, \tag{29}$$

where $H \in C_c^\infty\big([0,1] \times M\big)$ is an arbitrary function, whose Hamiltonian time-1 flow equals φ. This map is well-defined, i.e., it does not depend on the choice of H. This follows from the definition of the Calabi homomorphism on $\mathrm{Ham_c}(M)$ as in [14, (10.3.2), p. 407], and from [14, Lemma 10.3.4, p. 408], which links that definition with the above definition of Cal. In the proof of Proposition 3 in the case (a) we will use the following remark.

Remark 9. Let M be a (smooth) manifold, Ω a volume form on M, and $F : M \to \mathbb{R}$ a continuous function, such that $0 \in F(M)$. Then the following inequality holds:

$$\left(\sup_M F - \inf_M F \right) \int_M \Omega \geq \int_M F\, \Omega.$$

Proof of Proposition 3 in the case (a). Let $\varphi \in \mathrm{Ham_c}(M)$. Let $c \in \big(\|\varphi\|, \infty\big)$. We choose $H \in C_c^\infty\big([0,1] \times M\big)$, such that $\varphi_H^1 = \varphi$ and $c \geq |||H|||$. For every measurable subset $X \subseteq M$ we write $|X| := \int_X \omega^n$. We have

$$\begin{aligned} c &\geq |||H||| \\ &= \int_0^1 \big(\max_M H_t - \min_M H_t \big) dt \\ &\geq \frac{1}{|M|} \int_0^1 \left(\int_M H_t \omega^n \right) dt \\ &\qquad \text{(using Remark 9 and our hypothesis that } |M| \text{ is finite)} \end{aligned}$$

[3] Together with our standing assumption that M has no boundary, this implies that each connected component of M is noncompact.

$$=\frac{\mathrm{Cal}(\varphi)}{|M|}.$$

Since $c > \|\varphi\|$ is arbitrary, it follows that

$$\|\varphi\| \geq \frac{\mathrm{Cal}(\varphi)}{|M|}. \tag{30}$$

Let now $C \in [1, \infty)$. We choose a function $H \in C^\infty\big(U, [0, C]\big)$ with compact support, such that

$$\int_U H\omega^n \geq (C-1)|U|. \tag{31}$$

We denote $\varphi := \varphi_H^1$ and by $\widetilde{\varphi} : M \to M$ the map given by φ on U and the identity outside U. We have

$$\begin{aligned} \|\widetilde{\varphi}\| &\geq \frac{\mathrm{Cal}(\widetilde{\varphi})}{|M|} \qquad \text{(by (30))} \\ &= \frac{\int_U H\omega^n}{|M|} \qquad \text{(by (29))} \\ &\geq (C-1)\frac{|U|}{|M|} \qquad \text{(using (31))} \end{aligned} \tag{32}$$

Since $0 \leq H \leq C$ (on U), we have

$$\|\varphi\| \leq |||H||| \leq C.$$

Combining this with (32), it follows that

$$\frac{\|\widetilde{\varphi}\|}{\|\varphi\|} \geq \frac{C-1}{C}\frac{|U|}{|M|}.$$

Using that C is arbitrarily big, the inequality $\|\varphi\| \geq \|\widetilde{\varphi}\|$, and again (32), it follows that

$$\mathrm{Lip}^\infty(M, U) \geq \frac{|U|}{|M|}.$$

This proves Proposition 3 in the case (a). □

To prove Proposition 3 in the case (b), we will now adapt the proof of [16, Theorem 1.1], which is based on a result of M. Schwarz.

Let (M, ω) be a symplectically aspherical symplectic manifold (i.e., (8) holds) and $H \in C^\infty\big([0, 1] \times M\big)$. We define the *action spectrum of H* as follows. We denote by $\mathbb{D} \subseteq \mathbb{C}$ the closed unit disk, and define the set of *contractible H-periodic points* to be

$$\mathcal{P}^\circ(H) := \big\{x_0 \in M \,\big|\, \exists u \in C^\infty(\mathbb{D}, M) : \varphi_H^t(x_0) = u(e^{2\pi i t}),\ \forall t \in [0, 1]\big\}.$$

We define the *H-twisted symplectic action of $x_0 \in \mathcal{P}^\circ(H)$* to be

$$\mathcal{A}_H(x_0) := -\int_{\mathbb{D}} u^*\omega - \int_0^1 H\big(t, \varphi_H^t(x_0)\big)dt, \tag{33}$$

where $u \in C^\infty(\mathbb{D}, M)$ is any map satisfying $\varphi_H^t(x_0) = u(e^{2\pi i t})$, for every $t \in [0,1]$. It follows from asphericity of ω that this number does not depend on the choice of u and hence is well-defined. We define the *action spectrum of H* to be

$$\Sigma_H := \mathcal{A}_H(\mathcal{P}^\circ(H)) \subseteq \mathbb{R}.$$

The proof of Proposition 3 in the case (b) is based on the following result, which is a consequence of an argument of M. Schwarz.

Proposition 10 *(lower bound on Hofer-norm). Assume that M is closed and connected, and that ω is aspherical. Then for every $H \in C^\infty\big([0,1] \times M\big)$ we have*

$$\|\varphi_H^1\| \geq \min \Sigma_H + \frac{\int_0^1 \left(\int_M H_t \omega^n\right) dt}{\int_M \omega^n}. \tag{34}$$

Remark. By [19, Proposition 3.7] the action spectrum Σ_H is compact. It follows that the minimum $\min \Sigma_F$ exists. Hence the right hand side of (34) makes sense.

We call $F \in C^\infty\big([0,1] \times M\big)$ *mean normalized (w.r.t. ω)* if

$$\int\limits_M F_t \omega^n = 0, \quad \forall t \in [0,1]. \tag{35}$$

We denote

$$\mathcal{H} := \left\{ F \in C^\infty\big([0,1] \times M\big) \,\middle|\, F \text{ is mean normalized} \right\}.$$

Proof of Proposition 10. It follows from [13, Theorem 12.4.4] and [12, Proposition 3.1(i)] that there exists a map

$$c : \mathcal{H} \to \mathbb{R},$$

such that for every $F \in \mathcal{H}$, we have

$$c(F) \in \Sigma_F, \tag{36}$$

$$\|\varphi_F^1\| \geq c(F). \tag{37}$$

Namely, in the notation of [13, Theorem 12.4.4] the spectral invariant $c(F) := \rho(\widetilde{\varphi}_F; 1)$ satisfies (36) by [13, Theorem 12.4.4, (Spectrality)] and the inequality

$$|||F||| \geq c(F), \quad \forall F \in \mathcal{H}, \tag{38}$$

by [13, Theorem 12.4.4, (Continuity), (12.4.5)]. Here we used the definition (1) and the assumptions that M is closed and ω is aspherical, and therefore strongly semi-positive and rational on $\pi_2(M)$ (conditions [13, (8.5.1), (12.4.1)]). Using again our hypothesis that ω is aspherical, it follows from [12, Proposition 3.1(i)] that for all $F, F' \in \mathcal{H}$ satisfying $\varphi_F^1 = \varphi_{F'}^1$ we have $c(F) = c(F')$. (This means that the spectral invariant c descends from the universal cover of $\mathrm{Ham}(M)$ to $\mathrm{Ham}(M)$.) Combining this with (38), inequality (37) follows.

Let now $H \in C^\infty\big([0,1] \times M\big)$. We define

$$f : [0,1] \to \mathbb{R}, \quad f(t) := \frac{\int_M H_t \omega^n}{\int_M \omega^n},$$

and $F : [0,1] \times M \to \mathbb{R}$ by $F_t(x) := F(t,x) := H_t(x) - f(t)$. By straight-forward arguments this function is mean normalized, generates φ_H^1, and satisfies

$$\Sigma_F = \Sigma_H + \int_0^1 f(t)dt.$$

Inequality (34) follows from this and (36), (37). This proves Proposition 10. □

In the proof of Proposition 3 in the case (b) we will also use the following result, which is due to Y. Ostrover. Let (M,ω) be a symplectic manifold and $H, H' \in C_c^\infty([0,1] \times M)$. We denote by $H\#H' : [0,1] \times M \to \mathbb{R}$ the time-concatenation of H and H', given by

$$(H\#H')_t := \begin{cases} 2H^{2t}, & \text{if } t \in [0, \frac{1}{2}], \\ 2{H'}^{2t-1}, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Proposition 11 *(action for concatenated Hamiltonian). Assume that $H_t, H'_t = 0$ for t in some neighbourhood of $\{0,1\}$, and defining $X := \bigcup_{t\in[0,1]} \operatorname{supp} H_t$,*[4] *we have*

$$\varphi_{H'}^1(X) \cap X = \emptyset. \tag{39}$$

Then the following holds:

(i)

$$\mathcal{P}^\circ(H\#H') = \mathcal{P}^\circ(H'). \tag{40}$$

(ii) If ω is aspherical then we have

$$\mathcal{A}_{H\#H'}(x_0) = \mathcal{A}_{H'}(x_0), \quad \forall x_0 \in \mathcal{P}^\circ(H'). \tag{41}$$

This result follows from the proof of [16, Proposition 2.2]. For the convenience of the reader we prove it on page 25.

Proof of Proposition 3 in the case (b). Without loss of generality, we may assume that M is connected and $U \neq \emptyset$. For every measurable subset $X \subseteq M$ we write $|X| := \int_X \omega^n$. Let $C > 0$ and

$$c < c_0 := \frac{|U|}{|M|} \tag{42}$$

be a positive constant. We denote by $\widetilde{\varphi} : M \to M$ the map given by φ on U and the identity outside U.

[4] Here supp denotes the support of a function.

Claim 1. *There exists* $\varphi \in \mathrm{Ham}_c(U)$ *such that*

$$\|\widetilde{\varphi}\| \geq \max\{C, c\|\varphi\|\}. \tag{43}$$

Proof of Claim 1. By hypothesis there exists a function $F \in C^\infty\big([0,1] \times M\big)$ such that

$$\varphi_F^1(U) \cap U = \emptyset. \tag{44}$$

Reparametrizing F, we may assume that $F_t = 0$ for t in some neighbourhood of $\{0,1\}$. Furthermore, replacing F_t by $F_t - \int_M F_t\omega^n/|M|$, we may assume that F is mean normalized, i.e., it satisfies (35). We choose a compact subset $K \subseteq U$ such that

$$\frac{|K|}{|M|} > c. \tag{45}$$

Furthermore, we choose a smooth function $H_0 : U \to [0,1]$ with compact support, such that

$$H_0|_K = 1. \tag{46}$$

We define

$$t_0 := \max\left\{\frac{|||F||| - \min \Sigma_F}{\frac{|K|}{|M|} - c}, \frac{C}{c}\right\}. \tag{47}$$

It follows from (45) that $t_0 < \infty$. We define

$$\varphi := \varphi^1_{t_0 H_0}.$$

Claim 2. *This map satisfies inequality* (43).

Proof of Claim 2. We choose a function $f \in C^\infty\big([0,1],[0,1]\big)$ such that $f = i$ in a neighbourhood of i, for $i = 0, 1$. We define

$$H : [0,1] \times M \to \mathbb{R}, \quad H_t(x) := \begin{cases} f'(t)t_0H_0(x), & \text{if } x \in U, \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

We have

$$\varphi_H^1 = \widetilde{\varphi}. \tag{49}$$

Using (44), the fact that the support of H_0 is contained in U, and asphericity of ω, the hypotheses of Proposition 11 with $H' := F$ are satisfied. Hence applying this proposition, it follows that

$$\Sigma_{H\#F} = \Sigma_F. \tag{50}$$

Applying Proposition 10, we have

$$\|\varphi^1_{H\#F}\| \geq \min \Sigma_{H\#F} + \frac{\int_0^1 \left(\int_M (H\#F)_t\omega^n\right) dt}{|M|}. \tag{51}$$

Using the triangle inequality and the fact $\|\varphi_F^1\| \leq |||F|||$, we have

$$\|\varphi_H^1\| \geq \|\varphi_{H\#F}^1\| - |||F|||. \tag{52}$$

We have

$$\begin{aligned}
\int_0^1 \left(\int_M (H\#F)_t \omega^n \right) dt &= \int_0^1 \left(\int_M H_t \omega^n \right) dt + \int_0^1 \left(\int_M F_t \omega^n \right) dt \\
&\geq t_0|K| + 0 \qquad \text{(using (48), (46), (35))},
\end{aligned} \tag{53}$$

$$\begin{aligned}
\|\widetilde{\varphi}\| &= \|\varphi_H^1\| \qquad \text{(using (49))} \\
&\geq \min \Sigma_{H\#F} - |||F||| + t_0 \frac{|K|}{|M|} \qquad \text{(using (52), (51), (53))} \\
&\geq ct_0 \qquad \text{(using (50), (47))}.
\end{aligned} \tag{54}$$

Using again (47), it follows that

$$\|\widetilde{\varphi}\| \geq C. \tag{55}$$

Condition (46), the fact $K \neq \emptyset$, and the inequality $H_0 \leq 1$ imply that $\max_U H_0 = 1$. Since H_0 has compact support and satisfies $H_0 \geq 0$, we have $\min_U H_0 = 0$. Using (48), the fact $f(i) = i$, for $i = 0, 1$, and the Fundamental Theorem of Calculus, it follows that

$$|||H|_{[0,1]\times U}||| = t_0.$$

Since $\varphi = \varphi^1_{H|_{[0,1]\times U}}$, it follows that

$$\|\varphi\| \leq t_0.$$

Combining this with (54) and (55), inequality (43) follows. This proves Claim 2 and hence Claim 1. □

We choose a map φ as in Claim 1. The fact $\|\varphi\| \geq \|\widetilde{\varphi}\|$ and inequality (43) imply that $\|\varphi\| \geq C$. Inequality (43) also implies that $\|\widetilde{\varphi}\|/\|\varphi\| \geq c$. It follows that

$$\mathrm{Lip}^\infty(M, U) \geq c.$$

Since $c < c_0$ (as defined in (42)) is arbitrary, the estimate (9) follows. This completes the proof of Proposition 3 in the case (b). □

Proof of Corollary 4. We choose an area form σ on the two-torus $\mathbb{T}^2$ such that $\int_{\mathbb{T}^2} \sigma = c$, and a symplectic embedding $\psi : B^2(c) \to \mathbb{T}^2$. Let $\varepsilon > 0$. We define

$$(M, U, \omega) := \left(\mathbb{T}^2 \times M', \psi(B^2(a - \varepsilon)) \times M', \sigma \oplus \omega'\right).$$

Then the hypotheses of Proposition 3 are satisfied. (That the subset $U \subseteq M$ is displaceable in a Hamiltonian way, follows from our hypothesis $c \geq 2a$ and the fact that every open two-dimensional ball of area less than a is displaceable inside every ball of area $2a$.) Therefore, applying this theorem, it follows that

$$\mathrm{Lip}^\infty(M,U) \geq \frac{\int_U \omega^n}{\int_M \omega^n} = \frac{(a-\varepsilon)\int_{M'} {\omega'}^{n'}}{c\int_{M'} {\omega'}^{n'}},$$

where $2n' := \dim M'$. Since $\varepsilon > 0$ is arbitrary, the claimed inequality (10) follows. This proves Corollary 4. $\square$

4. Proof of Proposition 6 (lower bound on the relative Hofer diameter)

In the proof of Proposition 6 we will use the following definition. Let (M,ω) be a symplectic manifold and $N \subseteq M$ a coisotropic submanifold. We define the *action spectrum* and the *minimal area* of (M,ω,N) as

$$S(M,\omega,N) :=$$

$$\left\{ \int_{\mathbb{D}} u^*\omega \,\middle|\, u \in C^\infty(\mathbb{D},M) : \exists \text{ isotropic leaf } F \subseteq N : u(S^1) \subseteq F \right\},$$

$$A(M,\omega,N) := \inf\big(S(M,\omega,N) \cap (0,\infty)\big) \in [0,\infty].$$

Furthermore, for $n \in \mathbb{N}$ and $a > 0$ we denote by $S^{2n-1}(a) \subseteq \mathbb{R}^{2n}$ the sphere of radius $\sqrt{a/\pi}$, around 0.

Proof of Proposition 6. Let $\varepsilon > 0$. We define

$$N := S^1(a-\varepsilon) \times S^1(a-\varepsilon) \times S^{2n-1}(a-\varepsilon) \times X.$$

This is a closed and regular coisotropic submanifold of U. We choose a map $\varphi_0 \in \mathrm{Ham}_c(B^2(2a))$ such that

$$\varphi_0(S^1(a-\varepsilon)) \cap S^1(a-\varepsilon) = \emptyset. \tag{56}$$

(That there exists such a map follows the fact that every open two-dimensional ball of area less than a is displaceable inside every ball of area $2a$.) Since N is compact, by a cutoff argument there exists a map $\varphi \in \mathrm{Ham}_c(U)$ such that $\varphi = \big(\mathrm{id}_{\mathbb{R}^2} \times \varphi_0 \times \mathrm{id}_{(B^2(a))^n \times X}\big)$ on N. (See for example [21, Lemma 35].) It follows from (56) that

$$\varphi(N) \cap N = \emptyset. \tag{57}$$

We define $V := \mathbb{R}^2 \times B^2(2a) \times (B^2(a))^n \times X$. It follows from the proof of [22, Proposition 1.3]

$$S\big(W,\omega_{\mathrm{st}},S^{2m-1}(b)\big) = b\mathbb{Z} \tag{58}$$

for every $m \in \mathbb{N}$, $b \in (0,\infty)$, and open subset W of $\mathbb{R}^{2m}$ containing $\overline{B}^{2m}(b)$. Since by hypothesis, ω' is aspherical, the same holds for σ. Combining this with [21, Lemma 30], it follows that

$$S\big(X,\sigma,X\big) = \{0\}.$$

Using (58) and [21, Remark 31], it follows that

$$A\big(V,\omega|_V,N\big) = a - \varepsilon. \tag{59}$$

Using again that ω' is aspherical and [21, Lemma 33], we have

$$A(M,\omega,N) = A\big(V,\omega|_V,N\big).$$

We denote by $\widetilde{\varphi} : M \to M$ the map given by φ on U and the identity outside U. Combining this with (59) and using (57) and geometric boundedness of (M',ω'), it follows from [22, Theorem 1] that $\|\widetilde{\varphi}\| \geq a - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$\|\widetilde{\varphi}\| \geq a.$$

The inequality (12) follows from this. This proves Proposition 6. □

5. Remarks on Hofer-Lipschitz constants and Corollary 2, examples with big asymptotic Hofer-Lipschitz constant, remarks on the relative Hofer diameter

5.1. Hofer-Lipschitz constants

Let (M,ω) be a symplectic manifold and $U \subseteq M$ an open subset. Instead of $\mathrm{Lip}^\infty(M,U)$ (as defined in (5)), consider the *Hofer-Lipschitz constant* of (M,U,ω), which we define as

$$\mathrm{Lip}(M,U) := \mathrm{Lip}(M,U,\omega) := \sup\left\{ \frac{\|\widetilde{\varphi}\|_c^M}{\|\varphi\|_c^U} \;\middle|\; \mathrm{id} \neq \varphi \in \mathrm{Ham_c}(U) \right\}. \tag{60}$$

This is the Lipschitz constant of the natural inclusion (2) w.r.t. the Hofer norms for U and M. By [8, Theorem 1.1] every $\varphi \in \mathrm{Ham_c}(U)$ other than id has positive Hofer norm. Hence this definition makes sense. However, if M is closed and $U \neq \emptyset$ then

$$\mathrm{Lip}(M,U) = 1, \tag{61}$$

hence this number is uninteresting. To see that (61) holds, note that without loss of generality, we may assume that M is connected. By definition, we have $\mathrm{Lip}(M,U) \leq 1$. Furthermore,[5] let $H \in C_c^\infty(U)$ be a non-constant function. We define $\widetilde{H} : M \to \mathbb{R}$ by $\widetilde{H}(x) := H(x)$, if $x \in U$, and $\widetilde{H}(x) := 0$, otherwise. It follows from [10, Theorem 1.6(i) and the definition of a ν-geodesic on p. 203] that there exists $t_0 > 0$, such that

$$\|\varphi_{\widetilde{H}}^{t_0}\|_c^M = t_0 \left(\max_M \widetilde{H} - \min_M \widetilde{H} \right).$$

The right hand side is bounded below by $\|\varphi_H^{t_0}\|_c^U$. It follows that $\mathrm{Lip}(M,U) \geq 1$, and therefore, equality (61) holds.

5.2. Corollary 2 (upper bound on the asymptotic Hofer-Lipschitz constant)

In view of the estimate (7), it is natural to ask the following question.

Question. *Does there exist a constant $C > 0$, such that for every symplectic manifold (M,ω) of finite volume and every open subset $U \subseteq M$, the estimate*

$$\mathrm{Lip}^\infty(M,U) \leq C\frac{\int_U \omega^n}{\int_M \omega^n} \tag{62}$$

[5] We were made aware of the following argument by F. Schlenk.

holds, where $2n := \dim M$*?*

The answer is negative, even if we allow the constant C to depend on the symplectic manifold. This follows from the next result, which in the case (a) is based on a technique by F. Lalonde and D. McDuff [9, proof of Lemma 5.7, p. 64], and in case (b) is due to L. Polterovich (private communication).

Proposition 12 *(big asymptotic Hofer-Lipschitz constant). The equality*

$$\mathrm{Lip}^{\infty}(M, U) = 1 \tag{63}$$

holds if (M, ω, U) *is given by one of the following:*

(a) (M, ω) *is a two-dimensional symplectic manifold and* $U \subseteq M$ *an open neighbourhood of some non-contractible embedded circle in* M*.*
(b) M *is the complex projective space* $\mathbb{C}\mathrm{P}^n$ *for some* $n \in \mathbb{N}$*,* ω *the Fubini-Studi form, and* $U \subseteq M$ *an open neighbourhood of the real projective space* $\mathbb{R}\mathrm{P}^n$ *(embedded in* $\mathbb{C}\mathrm{P}^n$ *in the standard way).*

Remarks.

- Since we may choose U to have arbitrary small volume in these examples, it follows that the bound (62) does not hold.
- The equality (63) means that there are arbitrarily Hofer-large Hamiltonian diffeomorphisms on U whose Hofer norm does almost not shrink when trivially extending the diffeomorphism to M. This equality is optimal, since $\mathrm{Lip}^{\infty}(M, U)$ is always bounded above by 1.
- The set U in these examples is non-displaceable, since the same holds for the circle and $\mathbb{R}\mathrm{P}^n$, respectively. Hence the statement of Proposition 3 continues to hold for some non-symplectically-aspherical symplectic manifolds and some small non-displaceable subsets U.

In the proof of Proposition 12 in the case (a) we will use the following standard fact.

Lemma 13 *(fundamental group of surface). If a connected real surface* M *is not diffeomorphic to the real projective space* $\mathbb{R}\mathrm{P}^2$ *then its fundamental group is torsionfree.*

Proof of Proposition 12 in the case (a). Let (M, ω, U) be as in (a). In order to prove equality (63), it suffices to prove the inequality

$$\mathrm{Lip}^{\infty}(M, U) \geq 1. \tag{64}$$

We choose a noncontractible embedded circle L in M that is contained in U. We also choose a universal cover $\pi : \widetilde{M} \to M$. We equip $\widetilde{M}$ with the symplectic form $\widetilde{\omega} := \pi^*\omega$. Let $C \in [4, \infty)$.

Claim 1.

(i) The total area of $\widetilde{\omega}$ *is infinite.*
(ii) There exists a function $H \in C^{\infty}\big(M, [0, C]\big)$ *with compact support contained in* U*, and compact submanifolds*[6] $\widetilde{K}^{\pm}$ *of* $\widetilde{M}$ *that are symplectomorphic to* $\overline{B}^2(C-3)$*, such that the following holds. If* $\widetilde{\varphi} : \widetilde{M} \to \widetilde{M}$ *is a continuous lift of* φ_H^1 *in the sense that* $\pi \circ \widetilde{\varphi} = \varphi_H^1 \circ \pi$*, then* $\widetilde{\varphi}$ *displaces* $\widetilde{K}^+$ *or* $\widetilde{K}^-$*.*

[6] with boundary.

Proof of Claim 1. We equip $\mathbb{R} \times \mathbb{R}$ and $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ with the canonical symplectic forms. By Weinstein's Lagrangian neighbourhood theorem there exist $a \in (0, \infty)$, an open neighbourhood V of L that is contained in U, and a symplectomorphism $\psi : (\mathbb{R}/\mathbb{Z}) \times (-a, a) \to V$, such that $\psi\big((\mathbb{R}/\mathbb{Z}) \times \{0\}\big) = L$.

We denote by $\pi' : \mathbb{R} \times (-a, a) \to (\mathbb{R}/\mathbb{Z}) \times (-a, a)$ the canonical projection. We choose a map $\widetilde{\psi} \in C^\infty\big(\mathbb{R} \times (-a, a), \widetilde{M}\big)$ satisfying

$$\pi \circ \widetilde{\psi} = \psi \circ \pi'. \tag{65}$$

By our hypothesis that L is noncontractible, the condition $\psi\big((\mathbb{R}/\mathbb{Z}) \times \{0\}\big) = L$, and Lemma 13, the map $\widetilde{\psi}$ is injective and therefore a symplectic embedding. It follows that the image of $\widetilde{\psi}$ has infinite area. Statement (i) follows.

To prove (ii), we choose a function $f \in C^\infty\big((-a, a), [0, C]\big)$ with compact support, such that

$$f(p) = \frac{C|p|}{a} \text{ on } \left(-a\frac{C-1}{C}, -\frac{a}{C}\right) \cup \left(\frac{a}{C}, a\frac{C-1}{C}\right). \tag{66}$$

We denote by $\mathrm{pr} : (\mathbb{R}/\mathbb{Z}) \times (-a, a) \to (-a, a)$ the canonical projection. We define the function $H : M \to \mathbb{R}$ by

$$H := \begin{cases} f \circ \mathrm{pr} \circ \psi^{-1} & \text{on } V, \\ 0 & \text{otherwise.} \end{cases} \tag{67}$$

We denote

$$U^+ := \left(0, \frac{C}{a}\right) \times \left(\frac{a}{C}, a\frac{C-1}{C}\right), \quad U^- := -U^+ = \big\{ -(q,p) \,\big|\, (q,p) \in U^+ \big\}, \tag{68}$$

and choose a compact submanifold

$$K^\pm \subseteq U^\pm \tag{69}$$

that is symplectomorphic to $\overline{B}^2(C-3)$. We define

$$\widetilde{K}^\pm := \widetilde{\psi}(K^\pm). \tag{70}$$

Since $\widetilde{\psi}$ is a symplectic embedding, $\widetilde{K}^\pm$ is symplectomorphic to $\overline{B}^2(C-3)$.

Let $\widetilde{\varphi} : \widetilde{M} \to \widetilde{M}$ be a continuous lift of φ_H^1. For every $c \in \mathbb{R}$ we define the shift map $s_c : \mathbb{R} \times (-a, a) \to \mathbb{R} \times (-a, a)$ by $s_c(q, p) := (q + c, p)$. We denote by $\mathrm{pr}' : \mathbb{R} \times (-a, a) \to (-a, a)$. It follows from the equality $\pi \circ \widetilde{\varphi} = \varphi_H^1 \circ \pi$ and (65), (67) that there exists an integer N, such that[7]

$$\widetilde{\psi}^{-1} \circ \widetilde{\varphi} \circ \widetilde{\psi} = s_N \circ \varphi^1_{f \circ \mathrm{pr}'}.$$

Using (66), (68), (69), (70), it follows that $\widetilde{\varphi}$ displaces $\widetilde{K}^+$ or $\widetilde{K}^-$. This proves (ii) and completes the proof of Claim 1. □

We choose H and $\widetilde{K}^\pm$ as in part (ii) of this claim. Let $F \in C_c^\infty\big([0, 1] \times M\big)$ be such that

$$\varphi_F^1 = \varphi_H^1.$$

[7] This follows from the fact that $\tilde{\varphi}$ and $\varphi^1_{f \circ \mathrm{pr}'}$ are continuous lifts of the same map, namely φ^1_H, modulo conjugation by $\tilde{\psi}$.

The time-1 flow of $F \circ \pi$ is well-defined on $\widetilde{M}$ and lifts the flow $\varphi_F^1 = \varphi_H^1$. Therefore, by the conclusion of part (ii) of Claim 1 the map $\varphi_{F\circ\pi}^1$ displaces $\widetilde{K}^+$ or $\widetilde{K}^-$.

Since M admits a noncontractible embedded circle (namely L), it is not diffeomorphic to S^2. Using that M is orientable, it follows that its universal cover is diffeomorphic to $\mathbb{R}^2$. Using Claim 1(i) and a result of R. Greene and K. Shiohama [6, Theorem 1],[8] it follows that $(\widetilde{M}, \widetilde{\omega})$ is symplectomorphic to $\mathbb{R}^2$ with the standard form. Therefore $(\widetilde{M}, \widetilde{\omega})$ is geometrically bounded.

Since $\widetilde{K}^\pm$ is symplectomorphic to $\overline{B}^2(C-3)$ and $\varphi_{F\circ\pi}^1$ displaces $\widetilde{K}^+$ or $\widetilde{K}^-$, the sharp energy-Gromov-width inequality[9] therefore implies that

$$|||F \circ \pi||| \geq C - 3.$$

Since $|||F||| = |||F \circ \pi|||$, it follows that

$$\|\varphi_H^1\| \geq C - 3.$$

Since $0 \leq H \leq C$, we have

$$\left\|\varphi_{H|_U}^1\right\| \leq |||H|_U||| \leq C.$$

(The flow $\varphi_{H|_U}^1$ is well-defined on U, since H has support contained in U.) Since $C \geq 4$ is arbitrary, the inequality (64) follows. This proves Proposition 12 in the case (a). $\square$

Remarks.

- The above proof technique is based on the proof of [9, Lemma 5.7, p. 64].
- In the above proof the time-t flow of $F \circ \pi$ need not lift φ_H^t for a general $t \in [0,1]$. Therefore, in Claim 1(ii) we may not assume that $\widetilde{\varphi}$ is the time-1 map of a continuous lift of the flow of H (for all times).
- In the proof of Claim 1(ii) the flow $\varphi_{f\circ \mathrm{pr}'}^1$ moves K^+ in positive q-direction and K^- in negative q-direction. Since the map $\widetilde{\varphi}$ differs from this flow by a shift in q-direction (and conjugation by $\widetilde{\psi}$), $\widetilde{\varphi}$ may displace only one of the sets $\widetilde{K}^\pm$, not necessarily both.

In the proof of Proposition 12 in the case (b) with $n \geq 2$ we will use the following. For every topological space X, abelian group A, and integer k we denote by $H_k(X;A)$ the k-th homology of X with coefficients in A.

Lemma 14 *(map on homology induced by inclusion of real projective space). For every $n \in \mathbb{N}$ and $k \in \left\{0, \dots, \lfloor \frac{n}{2} \rfloor\right\}$ the map*

$$H_{2k}\big(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2\mathbb{Z}\big) \to H_{2k}\big(\mathbb{C}\mathrm{P}^n; \mathbb{Z}/2\mathbb{Z}\big) \tag{71}$$

induced by the canonical inclusion $\mathbb{R}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ does not vanish.

[8] This result is based on Moser isotopy.

[9] See e.g. [21, Corollary 3].

Proof of Lemma 14. We denote by $[\cdot] : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n$ the canonical projection. We denote by $\mathbf{0}$ the origin in $\mathbb{C}^{n-2k}$ and define

$$\begin{aligned} X &:= \left\{[x, \mathbf{0}] \,\middle|\, x \in \mathbb{R}^{2k+1} \setminus 0\right\} \\ Y &:= \left\{\left[z_0, z_1, iz_1, \ldots, z_k, iz_k, z_{k+1}, z_{k+2}, \ldots, z_{n-k}\right] \,\middle|\right. \\ &\qquad \left. (z_0, \ldots, z_{n-k}) \in \mathbb{C}^{n-k+1} \setminus 0\right\}. \end{aligned}$$

These sets are closed real submanifolds of $\mathbb{C}\mathrm{P}^n$[10] Denoting by $\mathbf{0}$ the origin in $\mathbb{C}^n$, we have

$$X \cap Y = \{[1, \mathbf{0}]\}.$$

This intersection is transverse, as follows from a calculation in standard charts. It follows that X represents a nonzero $\mathbb{Z}/2\mathbb{Z}$-homology class. Since it is the image of the submanifold $\mathbb{R}\mathrm{P}^{2k}$ of $\mathbb{R}\mathrm{P}^n$ under the canonical inclusion $\mathbb{R}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$, the statement of Lemma 14 follows. □

Proof of Proposition 12 in the case (b). Let (M, ω, U) be as in (b). We denote $L := \mathbb{R}\mathrm{P}^n \subseteq \mathbb{C}\mathrm{P}^n$, respectively. Let $C \in (0, \infty)$. We choose a function $H \in C_c^\infty(U)$ such that

$$\int_U H\omega^n = 0, \quad -1 \leq H \leq C, \quad H = C \text{ on } L.$$

It follows that

$$\|\varphi_H^1\| \leq |||H||| = \max_M H - \min_M H \leq C + 1. \tag{72}$$

(Recall that we use the abbreviated notation

$$\|\cdot\| := \|\cdot\|_c^U : \mathrm{Ham}_c(U) \to \mathbb{R}.)$$

Claim 1. *If $n = 1$ then we have*[11]

$$\|\widetilde{\varphi_H^1}\| \geq C - \pi. \tag{73}$$

Otherwise we have

$$\|\widetilde{\varphi_H^1}\| \geq C. \tag{74}$$

Since $C > 0$ is arbitrary, the inequality (64) and therefore the equality (63) follow from this claim and (72).

Proof of Claim 1. Consider the **case** in which $n = 1$. The submanifold $L = \mathbb{R}\mathrm{P}^1$ of $\mathbb{C}\mathrm{P}^1$ is a stem in the sense of the definition on p. 775 in [5]. Therefore, by [5, Theorems 1.8 and 1.4], L is stably non-displaceable. By [17, 7.2.A] the fundamental group of $\mathrm{Ham}(\mathbb{C}\mathrm{P}^1)$[12] is isomorphic to $\mathbb{Z}_2$, and its nontrivial element γ is induced by the 1-turn rotation, where we view $\mathbb{C}\mathrm{P}^1$ as the sphere S^2. This element has norm $\nu(\gamma)$ (defined as

[10] They are diffeomorphic to $\mathbb{R}\mathrm{P}^{2k}$ and $\mathbb{C}\mathrm{P}^{n-k}$, respectively.
[11] Here the tilde is defined as in (2).
[12] $\mathrm{Ham}(M)$ denotes the group of Hamiltonian diffeomorphisms of M, which agrees with $\mathrm{Ham}_c(M)$ if M is closed.

in [17, Definition 7.3.A]) equal to π. Hence inequality (73) follows from [17, Theorem 7.4.A], using Definition 7.3.A in that book and the facts $\int_U H\omega^n = 0$ and $H = C$ on L.

Consider now the **case** in which $n \geq 2$. To see that (74) holds, we denote by

$$\mu : \mathrm{Ham}(\mathbb{C}\mathrm{P}^n) \to \mathbb{R}$$

the Floer homological Calabi quasi-morphism of $\mathbb{C}\mathrm{P}^n$ associated with the fundamental class $[\mathbb{C}\mathrm{P}^n]$. See [3, Sections 3.4 and 4.3].[13] By [3, Corollary 3.6] μ satisfies the bound

$$\|\varphi\| \geq \frac{|\mu(\varphi)|}{\int_{\mathbb{C}\mathrm{P}^n} \omega^n}, \quad \forall \varphi \in \mathrm{Ham}(\mathbb{C}\mathrm{P}^n). \tag{75}$$

We define

$$\zeta : C^\infty(\mathbb{C}\mathrm{P}^n) \to \mathbb{R}, \quad \zeta(F) := \frac{\int_{\mathbb{C}\mathrm{P}^n} F\omega^n - \mu(\varphi_F^1)}{\int_{\mathbb{C}\mathrm{P}^n} \omega^n}. \tag{76}$$

Real projective space $\mathbb{R}\mathrm{P}^n$ is a closed monotone Lagrangian submanifold of $\mathbb{C}\mathrm{P}^n$ with minimal Maslov number $N_{\mathbb{R}\mathrm{P}^n}$ equal to $n+1$. This follows from [15, Examples. (i), p. 954]. Since $n \geq 2$, by Lemma 14 the map (71) does not vanish for $k = 1$. Since $2 > \dim(\mathbb{R}\mathrm{P}^n) + 1 - N_{\mathbb{R}\mathrm{P}^n} = 0$, this means that $\mathbb{R}\mathrm{P}^n$ satisfies the Albers condition, as defined in [5, p. 785]. Therefore, by [5, Theorem 1.17] $\mathbb{R}\mathrm{P}^n$ is $[\mathbb{C}\mathrm{P}^n]$-heavy.[14] We define $\widetilde{H} : \mathbb{C}\mathrm{P}^n \to \mathbb{R}$ by $\widetilde{H}(x) := H(x)$, if $x \in U$, and $\widetilde{H}(x) := 0$, otherwise. Since $\mathbb{R}\mathrm{P}^n$ is $[\mathbb{C}\mathrm{P}^n]$-heavy and $\widetilde{H} = C$ on $\mathbb{R}\mathrm{P}^n$, it follows that

$$\zeta(\widetilde{H}) \geq C.$$

Combining this with the equality $\int \widetilde{H}\omega^n = 0$ and the definition (76) of ζ, we obtain

$$-\frac{\mu(\varphi_{\widetilde{H}}^1)}{\int_{\mathbb{C}\mathrm{P}^n} \omega^n} \geq C.$$

Combining this with the bound (75), inequality (74) follows. This proves Claim 1. □

This completes the proof of inequality (64) and hence of equality (63). This proves Proposition 12 in the case (b). □

Remarks.

- The above proof was suggested to us by L. Polterovich (private communication).
- The map ζ defined in (76), is a symplectic quasi-state. See [4], definition (4) on p. 84 and the discussion afterwards.

[13] The construction of the map μ as in [3, Sections 3.4 and 4.3] involves a unity of a factor in a splitting of $\mathrm{QH}_{\mathrm{ev}}(M)$, the even-dimensional quantum homology. (See [3, p. 1654].) Since $\mathrm{QH}_{\mathrm{ev}}(\mathbb{C}\mathrm{P}^n)$ is a field, $[\mathbb{C}\mathrm{P}^n]$ is indeed such a unity.

[14] Heavyness is defined in [5, Definition 1.3, p. 779]. For the convenience of the reader we recall this definition here. Let (M, ω) be a closed symplectic manifold for which the spectral invariants are well defined and enjoy the standard list of properties given by [13, Theorem 12.4.4]. We denote by $\mathrm{QH}_\bullet(M, \omega)$ the (even-degree part of the) quantum homology of (M, ω) as in [5, Subsection 1.3.2, p. 778]. Let $a \in \mathrm{QH}_\bullet(M, \omega)$ be a nonzero idempotent and $H \in C^\infty([0,1] \times M, \mathbb{R})$ be a Hamiltonian that is 1-periodic in time. We denote by $c(a, H)$ the spectral invariant of (a, H), as in [13, (12.4.3), p. 508]. We define the functional $\zeta_a : C^\infty(M) \to \mathbb{R}$ by $\zeta_a(H) := \lim_{l\to\infty} \frac{c(a, lH)}{l}$. We call a closed subset X of M *a-heavy* if $\zeta_a(H) \geq \inf_X H$, for every $H \in C^\infty(M)$.

5.3. Relative Hofer diameter

In this subsection we explain the remark made after (11). The diameter of a pseudo-distance function d on a set X is by definition the number

$$\operatorname{diam}(d) := \sup\left\{d(x,y) \,\middle|\, x,y \in X\right\}.$$

Let (M,ω) be a symplectic manifold and $U \subseteq M$ an open subset. We can view $\operatorname{Diam}(U,M)$ (defined in (11)) as such a diameter, as follows. Let G be a group. By a *semi-norm* on G we mean a map $\|\cdot\| : G \to [0,\infty]$ such that

$$\|\mathbf{1}\| = 0,$$
$$\|g^{-1}\| = \|g\|,$$
$$\|gh\| \leq \|g\| + \|h\|,$$

for every $g,h \in G$. We call the last of these conditions the *triangle inequality*. We call $\|\cdot\|$ a *norm* if it is also nondegenerate, i.e., for every $g \in G$ it satisfies

$$\|g\| = 0 \Longrightarrow g = \mathbf{1}.$$

Every semi-norm $\|\cdot\|$ on G gives rise to a pseudo-distance function $d(\|\cdot\|)$ on G via

$$d(\|\cdot\|)(g,h) := \|g^{-1}h\|.$$

The diameter of $d(\|\cdot\|)$ is given by

$$\operatorname{diam}(d(\|\cdot\|)) = \sup_{g\in G}\|g\|.$$

Consider now the canonical extension homomorphism $E : \mathrm{Ham}_\mathrm{c}(U) \to \mathrm{Ham}_\mathrm{c}(M)$ given by (2). The map $\|\cdot\|_c^M \circ E : \mathrm{Ham}_\mathrm{c}(U) \to [0,\infty)$ is a norm. (Nondegeneracy follows from Theorem 1.1 in the article [8] of D. McDuff and F. Lalonde.) The relative Hofer-diameter of U in M is given by the diameter of the distance function induced by this norm,

$$\operatorname{Diam}(U,M) = \operatorname{diam}\left(d\left(\|\cdot\|_c^M \circ E\right)\right).$$

Appendix A. Proof of Proposition 11 (action for concatenated Hamiltonian)

For the proof of Proposition 11, we need the following. Let (M,ω) be a symplectic manifold. For a function $H \in C_c^\infty([0,1]\times M)$ and $x_0 \in M$ we denote

$$\mathbb{D}^H_{x_0} := \left\{u \in C^\infty(\mathbb{D},M) \,\middle|\, u(e^{2\pi it}) = \varphi^t_H(x_0),\ \forall t \in [0,1]\right\}. \tag{77}$$

Lemma 15 *(concatenated Hamiltonian and symplectic action). Let $H, H' \in C_c^\infty\big([0,1]\times M\big)$ be such that $H_t, H'_t = 0$ for t in some neighbourhood of $\{0,1\}$, and defining $X := \bigcup_{t\in[0,1]} \operatorname{supp} H_t$, condition (39) is satisfied. For every $x_0 \in M$ there exists a bijection*

$$\Phi : \mathbb{D}^{H'}_{x_0} \to \mathbb{D}^{H\#H'}_{x_0}, \tag{78}$$

such that

$$\int_{\mathbb{D}} \Phi(u')^*\omega = \int_{\mathbb{D}} u'^*\omega, \quad \forall u' \in \mathbb{D}^{H'}_{x_0}. \tag{79}$$

Proof of Lemma 15. We define the map Φ as follows. By hypothesis, there exists $\varepsilon > 0$ such that $H_t, H'_t = 0$ for $t \in [0, 2\varepsilon] \cup [1 - 2\varepsilon, 1]$. We choose a diffeomorphism $\varphi : \mathbb{D} \to \mathbb{D}$ such that

$$\varphi(e^{2\pi it}) = e^{2\pi i(2t-1)}, \quad \forall t \in \left[\frac{1}{2} + \varepsilon, 1 - \varepsilon\right]. \tag{80}$$

Assume that $u' \in \mathbb{D}^{H'}_{x_0}$. We define

$$v := \Phi(u') := u' \circ \varphi : \mathbb{D} \to M. \tag{81}$$

Claim 1. *We have $v \in \mathbb{D}^{H\#H'}_{x_0}$.*

Proof of Claim 1. Since $u'(e^{2\pi it}) = \varphi^t_{H'}(x_0)$, for all $t \in [0, 1]$, and $H'_t = 0$ for $t \in [0, 2\varepsilon] \cup [1 - 2\varepsilon, 1]$, we have

$$u'(e^{2\pi it}) = \varphi^t_{H'}(x_0) = x_0, \quad \forall t \in [0, 2\varepsilon] \cup \big[1 - 2\varepsilon, 1\big]. \tag{82}$$

It follows from (80) that

$$\varphi(e^{2\pi it}) \in \big\{e^{2\pi it'} \,\big|\, t' \in [0, 2\varepsilon] \cup \big[1 - 2\varepsilon, 1\big]\big\}, \quad \forall t \in \left[0, \frac{1}{2} + \varepsilon\right] \cup \big[1 - \varepsilon, 1\big].$$

Combining this with (81), (82), it follows that

$$v(e^{2\pi it}) = x_0, \quad \forall t \in \left[0, \frac{1}{2} + \varepsilon\right] \cup [1 - \varepsilon, 1]. \tag{83}$$

The equalities $\varphi^1_{H'}(x_0) = u'(e^{2\pi i} = 1) = \varphi^0_{H'}(x_0) = x_0$ and (39) imply that

$$x_0 \notin X = \bigcup_{t\in[0,1]} \operatorname{supp} H_t.$$

It follows that

$$\varphi^t_H(x_0) = x_0, \quad \forall t \in [0, 1].$$

Combining this with (83) and using the definition of $H\#H'$, it follows that

$$v(e^{2\pi it}) = \varphi^{2t}_H(x_0) = \varphi^t_{H\#H'}(x_0), \quad \forall t \in \left[0, \frac{1}{2}\right]. \tag{84}$$

Combining (82), (83), we have

$$v(e^{2\pi it}) = \varphi^{2t-1}_{H'}(x_0) = \varphi^t_{H\#H'}(x_0), \quad \forall t \in \left[\frac{1}{2}, \frac{1}{2} + \varepsilon\right] \cup [1 - \varepsilon, 1]. \tag{85}$$

Finally, it follows from (80), (81) and the fact $u' \in \mathbb{D}^{H'}_{x_0}$, that

$$v(e^{2\pi it}) = \varphi^{2t-1}_{H'}(x_0) = \varphi^t_{H\#H'}(x_0), \quad \forall t \in \left[\frac{1}{2} + \varepsilon, 1 - \varepsilon\right].$$

Combining this with (85), (84), it follows that $v \in \mathbb{D}^{H\#H'}_{x_0}$. This proves Claim 1. $\square$

A similar argument shows that

$$v \circ \varphi^{-1} \in \mathbb{D}_{x_0}^{H'}, \quad \forall v \in \mathbb{D}_{x_0}^{H\#H'}.$$

It follows that the map Φ is a bijection.

Equality (79) follows from (81), using that φ is orientation preserving. This proves Lemma 15. $\square$

Proof of Proposition 11. Statement (i) follows from Lemma 15. We prove statement (ii). Assume that ω is aspherical, and that $x_0 \in \mathcal{P}^\circ(H')$. It follows from the definition of $H\#H'$ that

$$\int_0^1 (H\#H')_t \circ \varphi_{H\#H'}^t(x_0)dt = \int_0^1 H_t \circ \varphi_H^t(x_0)dt + \int_0^1 H'_t \circ \varphi_{H'}^t(x_0)dt. \tag{86}$$

Furthermore, it follows from (39) that

$$x_0 \notin X = \bigcup_{t\in[0,1]} \operatorname{supp} H_t. \tag{87}$$

This implies that $\varphi_H^t(x_0) = x_0$, for every $t \in [0,1]$. Hence, using (87) again, it follows that

$$H_t \circ \varphi_H^t(x_0) = 0, \quad \forall t \in [0,1]. \tag{88}$$

We choose a map Φ as in Lemma 15 and a map $u' \in \mathbb{D}_{x_0}^{H'}$. (defined as in (77)). The claimed equality (41) is a consequence of (33), (86), (88), (79). This proves (ii) and completes the proof of Proposition 11. $\square$

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