Foundations of Physics (2023) 53:35
https://doi.org/10.1007/s10701-023-00676-z

de Broglie Normal Modes in the Madelung Fluid

Eyal Heifetz[1] · **Anirban Guha**[2] · **Leo Maas**[3]

Received: 9 May 2022 / Accepted: 17 February 2023

Abstract
In an attempt to explore further the Madelung fluid-like representation of quantum mechanics, we derive the small perturbation equations of the fluid with respect to its basic states. The latter are obtained from the Madelung transform of the Schrödinger equation eigenstates. The fundamental eigenstates of de Broglie monochromatic matter waves are then shown to be mapped into the simple basic states of a fluid with constant density and velocity, where the latter is the de Broglie group velocity. The normal modes with respect to these basic states are derived and found to also satisfy the de Broglie dispersion relation. Despite being dispersive waves, their propagation mechanism is equivalent to that of sound waves in a classical ideal adiabatic gas. We discuss the physical interpretation of these results.

Keywords De Broglie matter waves · Madelung fluid · Normal modes

1 Introduction

The hydrodynamic-like form of the Schrödinger equation has been suggested by Madelung [1] in 1927, less than a year after Schrödinger published his celebrated equation [2]. In this paper Madelung also showed that the eigenstates of the Schrödinger equation (hereafter SE) can be mapped into the steady states

✉ Eyal Heifetz
eyalh@tauex.tau.ac.il

Anirban Guha
anirbanguha.ubc@gmail.com

Leo Maas
L.R.M.Maas@uu.nl

1 Porter school of Environment and Earth Sciences, Tel Aviv University, Tel Aviv 69978, Israel

2 School of Science and Engineering Dundee, University of Dundee, Dundee DD1 4HN, United Kingdom

3 Institute for Marine and Atmospheric Research Utrecht, University of Utrecht, Utrecht 3584 CC, The Netherlands

Published online: 10 March 2023

(denoted as basic states) of the, now called, Madelung fluid. One of the most basic analysis of a fluid system is to find the normal mode solutions with respect to its basic states. These are the small amplitude wave-like perturbations superimposed on the basic states. Generally, the normal mode solutions form a complete set, hence any perturbation can be written as a linear combination of these normal modes. For instance, in the mid-latitudinal atmosphere, the large-scale basic state is the jet stream and the normal modes are the Rossby waves, whose interaction with the jet shapes the onset of weather systems [3]. Despite the ample literature on various physical and mathematical aspects of the Madelung fluid (e.g., Ref. [4] for a recent publication and a rich reference list therein), it seems that such a standard normal mode analysis has not been performed on the Madelung fluid.

Different basic states of flow accommodate different normal modes. The simplest basic states usually provide the natural normal modes of the flow which describe the fundamental propagation mechanism within the flow. For instance, in classical compressible adiabatic flows, the natural normal modes are sound waves - compressible non-dispersive pressure waves. These are most easily obtained when assuming an unbounded basic state with a constant density ρ_0 which is either at rest or moves with a constant speed u_0 (accounting for a Doppler shift in the sound waves' dispersion relation) [5]. For different basic states, e.g., in the presence of an external potential like gravity, physical boundaries or more complex velocity profiles, other set of normal modes emerge. Nevertheless, their underlying dynamics can be often explained by the fundamental propagation mechanism of sound waves.

This motivates us to provide an equivalent analysis of the natural normal modes of the Madelung fluid. Toward this end we first (in Sec. 2) summarize Madelung's derivation of his fluid equations and then show how the eigenstates solutions of SE are mapped into the basic states of the Madelung flow. Next (in Sec. 3) we derive the small perturbation equations with respect to these basic states. Then (in Sec. 4) we consider the fundamental eigenstate solutions of the de Broglie monochromatic matter waves. We show that they are mapped into the Madelung basic states of constant density and velocity, where the latter is the de Broglie waves' group velocity. Next, we derive their normal modes and show that they satisfy as well de Broglie dispersion relation. In Sec. 5 we analyze the normal modes propagation mechanism and show that they can be regarded as dispersive longitudinal pressure sound-like waves. In Sec. 6 we develop the normal modes wave energy-momentum relation and close in Sec. 7 by discussing our results.

2 Basic States of the Madelung Fluid

Madelung [1] showed that when writing the wave function Ψ in its polar form:

$$\Psi(\mathbf{x}, t) = \sqrt{\rho}(\mathbf{x}, t)e^{iS(x,t)/\hbar}, \tag{1}$$

the Schrödinger equation (using standard notation):

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x},t)\right]\Psi\,, \tag{2}$$

can be decomposed into two equations describing a fluid. The probability density function $\rho(\mathbf{x}, t)$, to find a quantum particle with mass m at position $\mathbf{x}$ and time t, is interpreted now as the density of a fluid whose hydrodynamic velocity is proportional to the gradient of the phase:

$$\mathbf{u}(\mathbf{x},t) \equiv \nabla\tilde{S} \tag{3}$$

(where hereafter the tilde superscript denotes a quantity divided by m). The real part of SE becomes the continuity equation of a compressible fluid:

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{u})\,, \tag{4}$$

and its imaginary part becomes the time-dependent Bernoulli equation of a barotropic flow (which can be regarded as a hydrodynamic Hamilton-Jacobi equation [6]):

$$\frac{\partial\tilde{S}}{\partial t} = -(\tilde{K} + \tilde{Q} + \tilde{V})\,. \tag{5}$$

Here $\tilde{K} = \frac{1}{2}\mathbf{u}^2$ is the fluid kinetic energy density, Q is the quantum potential (also known as the Bohm potential [7]):

$$\tilde{Q} = -\left(\frac{\hbar}{2m}\right)^2\left[\nabla^2\ln\rho + \frac{1}{2}(\nabla\ln\rho)^2\right], \tag{6}$$

where V is an external scalar potential. For a classical barotropic fluid $\tilde{Q}$ plays the role of its enthalpy in equation (5), while in the Madelung fluid its interpretation is more complex [6, 8]. Madelung assumed a simply connected domain so that the flow vorticity is zero ($\boldsymbol{\omega} \equiv \nabla\times\mathbf{u} = \nabla\times(\nabla\tilde{S}) = 0$), applied the nabla operator on equation (5) and used the identity $\nabla\tilde{K} \equiv \mathbf{u}\cdot\nabla\mathbf{u} - \boldsymbol{\omega}\times\mathbf{u} = \mathbf{u}\cdot\nabla\mathbf{u}$, which is the nonlinear advection term, to obtain the irrotational Euler fluid momentum equation:

$$\frac{\partial\mathbf{u}}{\partial t} = -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla(\tilde{Q} + \tilde{V}) = -\nabla(\tilde{K} + \tilde{Q} + \tilde{V})\,. \tag{7}$$

Basic states of the Madelung fluids are time-independent (steady states) solutions $(\rho, \mathbf{u}) = (\rho_0(\mathbf{x}), \mathbf{u}_0(\mathbf{x}))$. They satisfy the anelastic continuity version of equation (4):

$$\nabla\cdot(\rho_0\mathbf{u}_0) = 0\,, \tag{8}$$

and yield $\tilde{K}_0$ and $\tilde{Q}_0$ to be time independent as well. Therefore for a time-independent external potential V, the RHS of equation (7) vanishes and the combination $(\tilde{K}_0 + \tilde{Q}_0 + \tilde{V})$ becomes a constant, denoted in fluid dynamics as the Bernoulli potential:

$$\tilde{K}_0 + \tilde{Q}_0 + \tilde{V} = -\frac{\partial \tilde{S}_0}{\partial t} = Be_0 = \text{const}\,. \tag{9}$$

Consequently $\tilde{S}_0 = -Be_0\, t + F_0(\mathbf{x})$, where F_0 is the velocity potential of $\mathbf{u}_0$. Madelung noted that these basic state solutions are the mapped eigenstates of the time-independent Schrödinger equation:

$$E\,\Phi_0 = \hat{H}\Phi_0\,; \quad \Psi_0 = \Phi_0(\mathbf{x})e^{-iEt/\hbar}\,; \quad \Phi_0 = \sqrt{\rho_0}e^{imF_0/\hbar}\,; \quad Be_0 = \tilde{E}\,. \tag{10}$$

3 Small Perturbation Equations for the Madelung Fluid

We now consider a basic state of the Madelung fluid $(\rho_0(\mathbf{x}), \mathbf{u}_0(\mathbf{x}))$, that is being perturbed by $(\rho'(\mathbf{x},t), \mathbf{u}'(\mathbf{x},t))$, so that the total density and velocity fields are given by:

$$\rho = \rho_0 + \rho'\,, \qquad \mathbf{u} = \mathbf{u}_0 + \mathbf{u}'\,. \tag{11}$$

Linear combinations in the Madelung flow fields do not imply linear combinations in the wave function ($\Psi \neq \Psi_0 + \Psi'$). Writing $\mathbf{u}' = \nabla\tilde{S}'(\mathbf{x},t)$, the wave function associated with equation (11) reads:

$$\Psi = \left[\sqrt{\left(1+\frac{\rho'}{\rho_0}\right)}e^{iS'/\hbar}\right]\Psi_0 \neq \sqrt{\rho_0}e^{iS_0/\hbar} + \sqrt{\rho'}e^{i\tilde{S}'/\hbar}\,. \tag{12}$$

This should not come as a surprise because the polar form equation (1) of Ψ yields a nonlinear relation between its squared amplitude ρ and its phase S (and hence $\mathbf{u}$). This is the reason why the linear SE is mapped into a nonlinear fluid system. Consequently, linear combinations of the flow fields should not be mapped back into linear combinations of the wave function.

Next we follow the standard small perturbation linearization procedure applied in fluid dynamics, assuming the perturbations are small in comparison with the basic state: $\mathcal{O}\big(\rho'/\rho_0\big) = \mathcal{O}\big(\mid \mathbf{u}' \mid / \mid \mathbf{u}_0 \mid\big) = \epsilon \ll 1$, and omitting all $\mathcal{O}(\epsilon^2)$ terms from the equations [5]. Substitute $(\rho, \mathbf{u})$ in the continuity equation (4), subtract the basic steady state (8) and the nonlinear $\mathcal{O}(\epsilon^2)$ terms, we obtain the linearized Madelung continuity equation:

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}' + \rho' \mathbf{u}_0) \tag{13}$$

(where hereafter we use the common assumption of smoothness – that if a variable is small so are its spatial derivatives). To obtain the linearized momentum equation we first write $\ln\rho = \ln\rho_0 + \ln\left(1+\frac{\rho'}{\rho_0}\right) \approx \ln\rho_0 + \frac{\rho'}{\rho_0}$, and next write write $\tilde{Q} = \tilde{Q}_0 + \tilde{Q}'$, where the expression for $\tilde{Q}_0$ is the same as in the RHS of equation (6) when ρ_0 is replaced by ρ. Hereafter we drop the tilde superscripts, unless they are required for clarity. For Q' we obtain after linearization:

$$Q' = -\left(\frac{\hbar}{2m}\right)^2 \left[\nabla^2\left(\frac{\rho'}{\rho_0}\right) + \nabla \ln \rho_0 \cdot \nabla\left(\frac{\rho'}{\rho_0}\right)\right]. \tag{14}$$

After subtracting the basic state solution and the nonlinear small terms from equation (7), the linearized Madelung momentum equation reads:

$$\frac{\partial \mathbf{u}'}{\partial t} = -\nabla(\mathbf{u}_0 \cdot \mathbf{u}' + Q'). \tag{15}$$

Hence, given a basic state of the Madelung fluid $(\rho_0, \mathbf{u}_0)$, mapped from a SE eigenstate, equation-set (13) and (15) describe the evolution of their perturbed fields $(\rho', \mathbf{u}')$.

4 de Broglie Matter Waves as Both Basic States and Normal Modes of the Madelung Fluid

Consider the most fundamental eigenstate solution of SE - the monochromatic de Broglie matter wave, representing a free particle with momentum $\hbar\mathbf{k}$ and unspecified position:

$$E\Psi_0 = -\frac{\hbar^2}{2m}\nabla^2\Psi_0 \Rightarrow \Psi_0 = \sqrt{\rho_0}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega^{dB}t)}\,; E = \hbar\omega^{dB} = \frac{(\hbar\mathbf{k})^2}{2m}, \tag{16}$$

where ω^{dB} denotes the de Broglie frequency, $\mathbf{k}$ is the wavenumber vector and k is its magnitude. Writing $S_0 = \hbar(\mathbf{k}\cdot\mathbf{x} - \omega^{dB}t) \Rightarrow \mathbf{u}_0 = \nabla\tilde{S}_0 = \frac{\hbar}{m}\mathbf{k} = \mathbf{c}_g^{dB}$, which is the de Broglie group velocity. Thus, for a given wavenumber, the de Broglie plane wave solution of SE is mapped into a basic state of the Madelung fluid whose density and velocity $(\rho_0, \mathbf{u}_0)$ are both constant in both time and space. The anelastic equation (8) is trivially satisfied and since Q_0 vanishes and V is zero for a free particle, (9) yields:

$$K_0 = \frac{1}{2}\mathbf{u}_0^2 = \frac{1}{2}\mathbf{c}_g^2 = \frac{1}{2}\left(\frac{\hbar\mathbf{k}}{m}\right)^2 = \hbar\tilde{\omega}^{dB} = \tilde{E} = Be_0. \tag{17}$$

Perturbing this basic state by $(\rho', \mathbf{u}')$ and defining the linearized Lagrangian (material) derivative as $\frac{D}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{u_0}\cdot\nabla\right)$, the linearized continuity and momentum equations (13, 15) read:

$$\frac{D}{Dt}\left(\frac{\rho'}{\rho_0}\right) = -\nabla\cdot\mathbf{u}', \tag{18}$$

$$\frac{D}{Dt}\mathbf{u}' = \nabla\left[\left(\frac{\hbar}{2m}\right)^2\nabla^2\left(\frac{\rho'}{\rho_0}\right)\right] = -\frac{1}{\rho_0}\nabla p', \tag{19}$$

where

$$p' = -\left(\frac{\hbar}{2m}\right)^2 \nabla^2 \rho' \tag{20}$$

can be regarded as the pressure perturbation in the Madelung fluid [8]. Equations (18)-(19) are then identical to the linearized equations of a classical ideal adiabatic compressible gas, except that the pressure $p \propto \rho^\gamma$ (where $\gamma = C_p/C_v$ is the ratio between its isobaric and isochoric heat capacities). Applying then $\frac{D}{Dt}$ on Eq. (18) and using Eq. (19) we obtain:

$$\frac{D^2}{Dt^2}\rho' = \nabla^2 p' = -\left(\frac{\hbar}{2m}\right)^2 \nabla^4 \rho' \,. \tag{21}$$

Consider now the normal mode plane wave solutions: $(\rho', \mathbf{u}') = (\hat{\rho}, \hat{\mathbf{u}})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, so that $\frac{D}{Dt} \Rightarrow i(-\omega + \mathbf{k}\cdot\mathbf{u}_0)$ and $\nabla^2 \Rightarrow -\mathbf{k}^2$. Substituting back in Eq. (21), we obtain the normal mode dispersion relation:

$$\omega = \mathbf{k}\cdot(\mathbf{u}_0 \pm \mathbf{c}_p^{dB}) = \mathbf{k}\cdot(\mathbf{c}_g^{dB} \pm \mathbf{c}_p^{dB}) \,, \tag{22}$$

where $\mathbf{c}_p^{dB} = \frac{\hbar \mathbf{k}}{2m} = \frac{1}{2}\mathbf{c}_g^{dB}$ is the de Broglie phase speed. Hence, the normal modes satisfy the two solutions for waves propagating in the same or opposite direction as $\mathbf{u}_0$, respectively:

$$\hbar\omega_1 = 3\hbar\omega^{dB} \,; \qquad \hbar\omega_2 = \hbar\omega^{dB} \,. \tag{23}$$

5 Propagation Mechanism of the de Broglie Normal Modes

For an ideal adiabatic compressible gas, $\left(p'/p_0\right) = \gamma\left(\rho'/\rho_0\right)$, yielding $p' = \rho' c_s^2$, where $c_s = \sqrt{\gamma R T_0}$ is the magnitude of the non-dispersive speed of sound (and R is the ideal gas constant) [9]. From (20) we see that for the de Broglie normal modes $p' = \rho'(c_p^{dB})^2$, hence, as suggested by [8], c_p^{dB} plays the role of a dispersive quantum speed of sound.

Rewriting the dispersion relation (22) as: $\omega = (\omega_0 + \mathbf{k}\cdot\mathbf{c}_p)$, where $\omega_0 = \mathbf{k}\cdot\mathbf{u}_0$, is the Doppler shift and $\mathbf{c}_p = \pm\mathbf{c}_p^{dB}$ is the intrinsic phase velocity of the wave, relative to the basic state flow, (19-20) yield the normal mode structure:

$$\mathbf{u}' = \frac{\rho'}{\rho_0}\mathbf{c}_p = \frac{p'}{\rho_0 c_p^2}\mathbf{c}_p \,. \tag{24}$$

The longitudinal propagation mechanism of these waves are illustrated in Fig. 1. For positive direction of propagation relative to the basic flow ($\mathbf{c}_p = +\mathbf{c}_p^{dB}$), (u', p', ρ'), are all in phase (24), see Fig. 1a. The wave propagation mechanism is such that the pressure gradient force accelerates/decelerates the velocity perturbation in concert with the velocity convergence/divergence that increases/decreases the density and hence the pressure perturbation. As the pressure and density perturbations are always in phase, for $\mathbf{c}_p = -\mathbf{c}_p^{dB}$, the same mechanism is obtained when the velocity perturbation is in anti-phase with the density and the pressure perturbations,

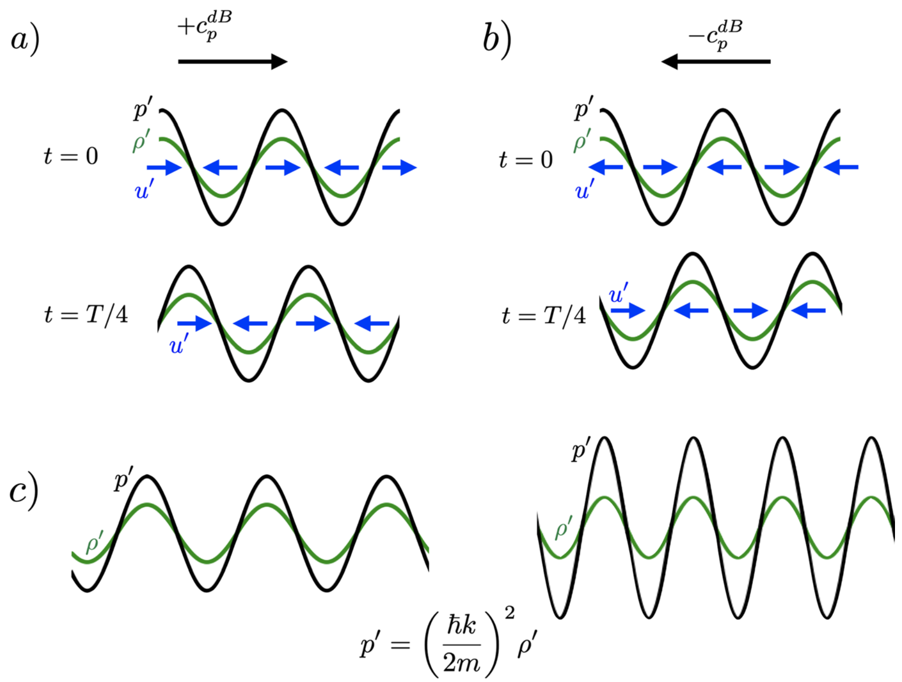

Fig. 1 de Broglie wave propagation mechanism: **a** positive (rightward) and **b** negative (leftward) propagation. In each case, the situations at time $t = 0$ and $t = T/4$ (where $T = 2\pi/\omega^{dB}$ denotes time period) are shown. **c** The dispersive nature of the wave leads to a pressure perturbation, which is amplified by the square of the wavenumber. Note that the wavy structure of the velocity field is indicated by the alternating horizontal arrows

see Fig. 1b. The de Broglie modes are dispersive since the pressure perturbation is proportional to the negative of the second spatial derivative of the density. Consequently, the generation of a density perturbation leads to a pressure perturbation that is amplified by the square of the wavenumber, see Fig. 1c. In contrast, for non-dispersive sound waves in ideal gas, the pressure and the density perturbations are proportional to each other by a constant (RT_0) which is independent of the wavenumber.

Recalling that $\mathbf{u}' = \nabla \tilde{S}'$, the superposition of the basic states Eq. (16) and the two normal modes yields the wave function whose amplitude and phase read:

$$\rho = \rho_0 \left[1 + \epsilon_1 \cos(\mathbf{k}\cdot\mathbf{x} - 3\omega^{dB} t) + \epsilon_2 \cos(kx - \omega^{dB} t)\right], \tag{25}$$

$$\frac{S}{\hbar} = (kx - \omega^{dB} t) + \frac{1}{2}\left[\epsilon_1 \sin(kx - \omega^{dB} t) + \epsilon_2 \sin(kx - \omega^{dB} t)\right], \tag{26}$$

where $\epsilon_{1,2} \equiv \hat{\epsilon}_{1,2}/\epsilon_0$.

6 Energy-Momentum Relations for de Broglie Normal Modes

SE conserves the energy expectation value [10]:

$$\langle H\rangle = \int \rho(K + Q + V)d\forall\,. \tag{27}$$

where $d\forall$ denotes an infinitesimal volume element. As been shown by Ref. [6], the quantum potential Q, which is not a positive definite quantity, can be written as:

$$Q = I + \frac{p}{\rho}\,;\; I = \frac{1}{2}\left(\frac{\hbar}{2m}\nabla \ln\rho\right)^2;\; p = -\left(\frac{\hbar}{2m}\right)^2\nabla^2\rho\,, \tag{28}$$

where I can be regarded as the positive definite internal energy of the flow and p is the pressure. Assuming the density and its derivatives vanish at the domain boundaries, then $\int pd\forall = 0$ and the Hamiltonian can be written as well as:

$$\langle H\rangle = \int \rho(K + I + V)d\forall\,. \tag{29}$$

For the superposition of a basic state and a small perturbation of the form of (11), $K = \frac{1}{2}(\mathbf{u}_0 + \mathbf{u}')^2 = K_0 + \mathbf{u}_0\cdot\mathbf{u}' + K'$, where $K_0 = \frac{1}{2}\mathbf{u}_0^2$ and $K' = \frac{1}{2}\mathbf{u}'^2$. Keeping terms in the Hamiltonian up to second order of the perturbations, then for periodic perturbations we obtain:

$$\langle K\rangle = \int (\rho_0 + \rho')(K_0 + \mathbf{u}_0\cdot\mathbf{u}' + K')d\forall = \int (\rho_0 K_0 + \rho'\mathbf{u}_0\cdot\mathbf{u}' + \rho_0 K')d\forall\,. \tag{30}$$

Similarly, for the internal energy $I = I_0 + \left(\frac{\hbar}{2m}\right)^2\nabla\ln\rho_0\nabla\frac{\rho'}{\rho_0} + I'$, where $I_0 = \frac{1}{2}\left(\frac{\hbar}{2m}\nabla\ln\rho_0\right)^2$ and $I' = \frac{1}{2}\left(\frac{\hbar}{2m}\nabla\frac{\rho'}{\rho_0}\right)^2$, so that:

$$\langle I\rangle = \int \left[\rho_0 I_0 + \frac{1}{2}\left(\frac{\hbar}{2m}\right)^2\nabla\ln\rho_0\nabla\left(\frac{\rho'}{\rho_0}\right)^2 + \rho_0 I'\right]d\forall\,. \tag{31}$$

Assuming $\int \rho' V d\forall = 0$, we can write the Hamiltonian as:

$$\langle H\rangle = \langle H_0\rangle + \left\langle H'\right\rangle\,,$$
$$\langle H_0\rangle = \int \rho_0(K_0 + I_0 + V)d\forall\,;$$

$$\left\langle H'\right\rangle = \int \left[\rho_0(K' + I') + \left(\rho'\mathbf{u}_0\cdot\mathbf{u}' + \frac{1}{2}\left(\frac{\hbar}{2m}\right)^2\nabla\ln\rho_0\nabla\left(\frac{\rho'}{\rho_0}\right)^2\right)\right]d\forall\,, \tag{32}$$

where the last term in the integrand of $\langle H'\rangle$ vanishes when the density basic state ρ_0 is constant.

In the absence of an external potential V, SE conserves as well the momentum expectation value of the vector [10]:

$$\langle \mathbf{P} \rangle = \int \rho \mathbf{u} d\forall \,, \tag{33}$$

where for small periodic perturbations:

$$\langle \mathbf{P} \rangle = \langle \mathbf{P}_0 \rangle + \left\langle \mathbf{P}' \right\rangle = \int \rho_0 \mathbf{u}_0 d\forall + \int \rho' \mathbf{u}' d\forall \,. \tag{34}$$

In fluid dynamics $\left\langle \mathbf{P}' \right\rangle$ and $\langle H' \rangle$ are denoted by (the somewhat confusing name of) pseudo-momentum and pseudo-energy respectively and they represent the small perturbation contribution to the momentum and energy of the flow [11]. As the basic state is assumed constant both the pseudo-momentum and pseudo-energy are independently conserved. Furthermore, for the case where the basic state flow $\mathbf{u}_0$ is also constant, Eqs. (32) and (34) indicate that:

$$\left\langle H' \right\rangle = \left\langle E' \right\rangle + \mathbf{u}_0 \cdot \left\langle \mathbf{P}' \right\rangle \,, \qquad \left\langle E' \right\rangle = \int \rho_0 (K' + I') d\forall \,, \tag{35}$$

where E' is the positive definite fluctuation (eddy) energy perturbation. Hence in the simple case where both ρ_0 and $\mathbf{u}_0$ are constant $\langle E' \rangle$ is as well a constant of motion of the small perturbation dynamics.

For the de Broglie normal modes Eq. (24) indicates that the eddy energy is equipartitioned between its kinetic and internal energy:

$$\left\langle K' \right\rangle = \left\langle I' \right\rangle = \frac{1}{4}\left(\frac{\hat{\rho}}{\rho_0}\right)^2 \left(c_p^{dB}\right)^2 \Rightarrow \left\langle E' \right\rangle = \frac{1}{2}\left(\frac{\hat{\rho}}{\rho_0}\right)^2 \left(c_p^{dB}\right)^2 = \left(\frac{\hat{\rho}}{2\rho_0}\right)^2 \hbar \tilde{\omega}^{dB} \,. \tag{36}$$

Furthermore, Eq. (24) as well that:

$$\left\langle \mathbf{P}' \right\rangle = \frac{1}{2}\left(\frac{\hat{\rho}}{\rho_0}\right)^2 \mathbf{c}_p = \pm\frac{1}{2}\left(\frac{\hat{\rho}}{\rho_0}\right)^2 \mathbf{c}_p^{dB} \qquad \Rightarrow \qquad \left\langle E' \right\rangle = \mathbf{c}_\mathbf{p} \cdot \left\langle \mathbf{P}' \right\rangle \,. \tag{37}$$

Hence Eq. (37) obeys the general wave relations of which the wave pseudo-momentum is proportional to the intrinsic phase speed and flows in the same direction of the intrinsic phase velocity (relative to the basic state flow). The latter is evident from Fig. 1. When the correlation between the density and the wave velocity perturbations is positive, as in Fig. 1a (negative, as in Fig. 1b), the longitudinal propagation mechanism yields positive (negative) phase speed of the modes. Moreover, it obeys as well the general relation in which the scalar product between the pseudo-momentum and its intrinsic phase speed (yielding a positive scalar) is equal to the total eddy energy of the wave [12]. Using Eq. (35) we obtain further the familiar relation between the wave pseudo-energy and the product between the phase speed, viewed from a frame of rest, and the pseudo-momentum:

$$\langle H' \rangle = \langle E' \rangle + \mathbf{u}_0 \cdot \langle \mathbf{P}' \rangle = (\mathbf{c}_p + \mathbf{u}_0) \cdot \langle \mathbf{P}' \rangle \ . \tag{38}$$

7 Discussion

The Madelung transformation of the Schrödinger equation into the equations of an irrotational compressible fluid provides an alternative semi-classical perspective on quantum phenomena. For instance, quantum tunneling, described by the Madelung fluid, has a straightforward interpretation—the pressure gradient force of the fluid balances the gradient of the external potential barrier, so that the sum of the kinetic and internal energy of the fluid remains unchanged when crossing the barrier [13]. This, together with other examples [14–16], suggest that it seems worthwhile to consider quantum phenomena from the Madelung fluid angle by implementing standard analytical tools of classical fluid dynamics.

As in every nonlinear dynamical system, the first step of the analysis is to find the equilibrium fixed points of the system. These are the basic states of the flow, which for the Madelung fluid are the eigenstates of the Schrödinger equation. We find it neat that the energy eigenvalues are mapped into the constant values of the Bernoulli potential of the Madelung basic states. The next step is analyzing the behavior of the system in the vicinity of the fixed points. These are described by the linearized small perturbation equations with respect to the basic states. The following step of the standard analysis is to span the small perturbations into their normal mode solutions and analyze their physical properties. These include their dispersion relation and physical structure as well as their propagation mechanism. Generally, in classical fluids, normal modes may be unstable and grow on the expense of the basic state energy. A necessary condition for that to happen is that both of the small amplitude conserved quantities of pseudo-energy and pseudo-momentum are zero (interestingly, these conditions for classical fluids were first formulated in the PhD thesis of Heisenberg, supervised by Sommerfeld [17]). The perturbations in the Madelung fluids however are stable by definition as they are composed of a superposition of stable eigenstates of the Schrödinger equation. Hence these two constant of motion of small perturbation dynamics cannot vanish simultaneously in the Madelung fluid.

The simplest and usually most fundamental normal modes are the ones obtained with respect to the simplest non-trivial basic states. In classical fluids the latter are usually represented by uniform constant density and velocity, in the absence of imposed boundaries and external potentials. It is elegant that the de Broglie non-relativistic monochromatic matter waves, representing quantum particles whose positions are unspecified but whose velocities are the wave group velocities, are mapped into Madelung basic states whose densities are uniform and whose constant velocities are equal to the ones of the particles. Nonetheless, from the Schrödinger equation itself we cannot come up with a mechanistic explanation for the propagation of these matter waves whose phase velocity is equal to half of their group velocity. To give a counter example, for surface gravity waves in deep water it is the other way around - their group velocity is equal to half of their phase velocity. There we can come up with a mechanistic explanation for the wave propagation in terms of the interplay between the horizontal pressure gradient force accelerating/decelerating water columns and the convergence/divergence of these water columns whose elevation differences determine, in turn, the changes in the pressure gradient force [5].

It is interesting that when performing the normal modes analysis with respect to these basic states we find that the normal modes themselves obey the same dispersion relation as the de Broglie matter waves. Their propagation mechanism can then be explained as longitudinal dispersive pressure waves, which can be regarded as dispersive sound-like waves, as suggested heuristically by [8]. In distinction to non dispersive sound waves in classical adiabatic compressible fluids, their dispersive properties result from the fact that the pressure perturbation in the Madelung fluid is proportional to minus the Laplacian of the density perturbation (rather than the density perturbation itself, as in classical adiabatic fluids). Hence, the fundamental de Broglie matter waves are "imprinted" in the Madelung fluid. Their particle representation yields its basic states whereas its wave dynamics yields its normal modes.

For future work we intend to map eigenstate solutions of the Schrödinger equation (such as in the presence of parabolic, rectangular barrier and square well potentials) to their corresponding Madelung basic states and then analyze their normal modes, respectively. We expect that such analyses will provide new insight on these familiar fundamental problems.

Acknowledgements Eyal Heifetz is grateful to Rachel Heifetz and Yair Zarmi for inspiring discussions.

Data availability The manuscript has no associated data.

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