Incomplete Information and Justifications

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Abstract. We present a logic for reasoning about higher-order upper and lower probabilities of justification formulas. We provide sound and strongly complete axiomatization for the logic. Furthermore, we show that the introduced logic generalizes the existing probabilistic justification logic PPJ.

Keywords: Justifcation logic · Probabilistic logic · Upper and lower probabilities · Strong completeness

1 Introduction

Since the seminal paper about justification logics was published, [3], a whole family of justification logics has been established, including logics with uncertain justifications, see [2,4,9,10,12,15,17]. However, justification logics in which uncertainty originates from incompleteness of information is still not provided.

The main feature of justification logic is that evidence is representable directly in the object language, i.e. the language of justification logic includes formulas of the form $t : A$ meaning that t *justifies* A. In this paper, we distinguish the following two types of incomplete information within $t : A$:

1) *"t" is incomplete.*
A friend tells me that she saw in *some* weather forecast that tomorrow is going to rain. I know which are possible forecasts she could have checked. As a consequence of an incomplete justification t (she read in *some* weather forecast and did not specify in which one) and since each forecast provides a probability for the rain, my degree of belief that "tomorrow is going to rain" is true lies in an interval $[r, s]$, where r represents the lowest probability according to the possible forecasts she checked, and s the highest.

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2) *":" is incomplete.*
   After taking a medication x, a patient gets a symptom S. It is known that the symptom S is a side effect of the medication x and that there exist old and new series of the medication x. The chance that the side effect occurs is smaller when taking medication of the new series.
   In this case both t: "the patient took the medication" and A: "the patient got the symptom" are certain, but we do not know if t is the reason for A or there exists another reason. Also, we do not know if the patient took the medication of the old or the new series. In the former case, the chance that x caused S is bigger than in the later case. Thus, our degree of belief that $t : A$ lies in some interval.

In this paper we formalize both types of uncertainty illustrated above. To capture uncertainty about probabilities we use the lower and upper probability measures. For an arbitrary set of probability measures P, the former assigns to an event X the infimum of the probabilities assigned to X by the measures from the set P, while the later returns their supremum.

We provide a new logic, ILUPJ[1], as an extension of the justification logic J with two families of unary operators $L_{\geq s}$ and $U_{\geq s}$, for $s \in \mathbb{Q} \cap [0, 1]$. That idea comes from some of our previous papers, see e.g., [6,18]. The intended meanings of these operators are that 'the lower (upper) probability is greater or equal to s'. Therefore, saying that our degree of belief lies in an interval $[r, s]$ is represented by saying that the lower probability is equal to r and the upper probability is equal to s.

The first case, when "t" is incomplete and therefore our degree of belief that A is true belongs to an interval $[r, s]$ we can represent in the logic ILUPJ with

$$t : L_{=r}A \wedge t : U_{=s}A.$$

The second case, when ":" is incomplete, i.e., situations in which we are not sure if t is the justification for A, can be represented by

$$L_{=r}(t : A) \wedge U_{=s}(t : A).$$

The corresponding semantics of our logic consists of special types of possible world models, where every world is equipped with a space that consists of the non-empty set of accessible worlds, algebra of subsets and a set of probability measures.

We propose a sound and complete axiomatization of the logic. In order to prove the strong completeness theorem, which is the main technical result of the paper, we use a Henkin-like construction modifying our previous techniques for probabilistic and temporal logic [5,13,14,16]. Also, the proofs in our logic can be infinite although all the formulas of the logic are finite.

We also compare our logic with the probabilistic justification logic PPJ from [10], and we prove that ILUPJ properly generalizes it. The logic PPJ is obtained

[1] I stands for iterations, LUP for lower and upper probabilities and J for the justification logic J.

by extending the justification logic J by a list of standard unary operators, $P_{\geq s}$, whose intended meaning is 'the probability is greater or equal to s'. In that approach there is no uncertainty about probabilities, and a unique probability value is assigned to an event. In our example from the first case, this would correspond to the situation where only one forecast is available. In the general case, where we consider several forecasts, we need to assign sets of probabilities to events, which lead to our more general semantics and, consequently, to different probability operators. Indeed, if r is the lowest probability according to the possible forecasts, and s the highest, we cannot always assign a truth value to the sentence "t justifies that our degree of belief that tomorrow will rain with probability at least ℓ" (in the language of PPJ: $t : P_{\geq \ell}A$), where $\ell \in (r, s)$ – according to some forecasts the sentence is true, and according to others it is false. On the other hand, in ILUPJ we can distinguish two cases: $t : L_{\geq \ell}A$ is false and $t : U_{\geq \ell}A$ is true.

The content of this paper is as follows. In Sect. 2 we define the basic notions needed for defining our logic. In Sect. 3 we propose the logic ILUPJ, whereas in Sect. 4 we prove the soundness and strong completeness theorem. In Sect. 5 we prove that our logic generalizes the logic PPJ and we conclude the paper in Sect. 6.

2 Preliminaries

We start with preliminary notions that will be used in the definition of the semantics of the logic ILUPJ.

Definition 1 (Algebra Over a Set). *Let $W \neq \emptyset$ and let $\emptyset \neq H \subseteq \mathcal{P}(W)$. H is called* algebra over W *if:*

1) $W \in H$,
2) For $X, Y \in H$, $\quad W \setminus X \in H$ and $X \cup Y \in H$.

Definition 2 (Finitely Additive Probability Measure). *For an algebra H over W, a function $\mu : H \longrightarrow [0, 1]$ is called* finitely additive probability measure, *if:*

1) $\mu(W) = 1$,
2) For $X, Y \in H$, $\quad \mu(X \cup Y) = \mu(X) + \mu(Y)$, whenever $X \cap Y = \emptyset$.

Definition 3 (Lower and Upper Probability Measures). *Let H be an algebra over W and P be a set of finitely additive probability measures defined on H. For $X \in H$, the* lower probability measure P_* *and the* upper probability measure P^* *are defined as follows:*

1) $P_*(X) = \inf\{\mu(X) \mid \mu \in P\}$,
2) $P^*(X) = \sup\{\mu(X) \mid \mu \in P\}$.

Now we state three properties which are used in our proof of soundness and completeness theorem for ILUPJ. The proof of these basic properties of P_* and P^* follows directly from the properties of infimum and supremum.

1) $P_*(X) \leq P^*(X)$,
2) $P_*(X) = 1 - P^*(X^c)$,
3) $P^*(X \cup Y) \leq P^*(X) + P^*(Y)$, whenever $X \cap Y = \emptyset$.

Complete characterization of P_* and P^* is needed in order to axiomatize upper and lower probabilities. We use the characterization used by Anger and Lembcke [1], which was also used by Halpern and Pucella [8, Theorem 2.3]. For that characterization we need a notion of (n, k)-cover.

Definition 4 ((n, k)-cover). *A set X is* covered *n times by a multiset*

$$\{\{X_1, \ldots, X_m\}\}$$

of sets if every element of X appears in at least n sets from $X_1, \ldots, X_m$ meaning that for all $x \in X$, there exist $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that for all $j \leq n$, $x \in X_{i_j}$.

An (n, k)-cover *of (X, W) is a multiset $\{\{X_1, \ldots, X_m\}\}$ that covers the set W k times and covers the set X $n + k$ times.*

With the notion of (n, k)-cover we are ready to define the characterization theorem:

Theorem 1 (Anger and Lembcke [1]). *Let $W \neq \emptyset$, H an algebra over W, and f a function $f : H \longrightarrow [0, 1]$. There exists a set P of probability measures such that $f = P^*$ iff the function f satisfies the following three conditions:*

(1) $f(\emptyset) = 0$,
(2) $f(W) = 1$,
(3) for all $m, n, k \in \mathbb{N}$ and all $X, X_1, \ldots, X_m \in H$, if $\{\{X_1, \ldots, X_m\}\}$ is an (n, k)-cover of (X, W), then

$$k + nf(X) \leq \sum_{i=1}^{m} f(X_i).$$

3 The logic ILUPJ

In this section we describe the syntax and semantics of the logic ILUPJ and provide an axiomatization.

3.1 Syntax

We will use the following notation:

$\mathsf{Con} = \{c_0, c_1, \dots, c_n, \dots\}$ for a countable set of constants,
$\mathsf{Var} = \{x_0, x_1, \dots, x_n, \dots\}$ for a countable set of variables, and
$\mathsf{Prop} = \{p_0, p_1, \dots, p_n, \dots\}$ for a countable set of atomic propositions.

Definition 5 (Justification Terms). Terms *are built from the sets* Con *and* Var *with the following grammar:*

$$t ::= c \mid x \mid t \cdot t \mid t + t \mid !t,$$

where $c \in \mathsf{Con}$ *and* $x \in \mathsf{Var}$. *The set of all terms will be denoted by* Tm.

For a term t and non-negative integer n we use the following notation:

$$!^0 t := t \quad \textit{and} \quad !^{n+1} t := !(!^n t).$$

Terms represent justifications for an agent's belief (or knowledge). In the original justification logic, the *Logic of Proofs* [3], terms represent formal proofs in e.g. Peano arithmetic [11]. In possible world models for justifcation logic, first developed by Fitting [7], terms may represent arbitrary justifications like direct observation, public announcements, private communication, and so on.

Let us discuss the role of a given justification term depending on its main connective [12]:

- Constants are used in situations where the justification is not further analyzed, e.g. to justify axioms, see rule (IR1).
- Variables are used to represent arbitrary justifications.
- The operation $\cdot$ represents the agent's ability to reason by modus ponens. Assume that s justifies the agent's belief in A and t justifies the agent's belief in $A \to B$, then $t \cdot s$ will justify her belief in B, see axiom (Ax2).
- The operation $+$ combines two justifcations to a justification with broader scope, see axiom (Ax3). Often this is illustrated as follows. Let s and t be two volumes of an encyclopedia and $s + t$ be the set of those two volumes. Suppose that one of the volumes, say s, contains justification for a proposition A. Then also the larger set $s + t$ contains justification for A.
- The operation ! represents the agent's ability to perform positive introspection. In our logic ILUPJ, we only include positive introspection for axioms and iterated belief of axioms, see rule (IR1). Assume an agent believes an axiom A and c is a justification for that belief. By positive introspection the agent believes that she believes A and that A is justified by c. The term $!c$ will justify the result of the positive introspection act.

Definition 6 (Formulas of the Logic ILUPJ). Formulas *of the logic* ILUPJ *are defined with the following grammar:*

$$\mathsf{For} \qquad A ::= p \mid U_{\geq s} A \mid L_{\geq s} A \mid \neg A \mid A \wedge A \mid t : A$$

where $p \in \mathsf{Prop}$ *and* $s \in \mathbb{Q} \cap [0, 1]$.

Other connectives, $\vee, \rightarrow, \leftrightarrow$, are defined as usual. The following abbreviations will be used for introducing other types of inequalities:

$$U_{<s}A \equiv \neg U_{\geq s}A$$

$$L_{<s}A \equiv \neg L_{\geq s}A$$

$$U_{\leq s}A \equiv L_{\geq 1-s}\neg A$$

$$L_{\leq s}A \equiv U_{\geq 1-s}\neg A$$

$$U_{=s}A \equiv U_{\leq s}A \wedge U_{\geq s}A$$

$$L_{=s}A \equiv L_{\leq s}A \wedge L_{\geq s}A$$

$$U_{>s}A \equiv \neg U_{\leq s}A$$

$$L_{>s}A \equiv \neg L_{\leq s}A.$$

We set $A \wedge \neg A \equiv \bot$ and $A \vee \neg A \equiv \top$.

3.2 Axiomatization

Axioms of the logic ILUPJ:

(Ax1) $\vdash A$, where A is a propositional tautology
(Ax2) $\vdash t : (A \rightarrow B) \rightarrow (s : A \rightarrow (t \cdot s) : B)$
(Ax3) $\vdash t : A \vee s : A \rightarrow (t + s) : A$
(Ax4) $\vdash U_{\leq 1}A \wedge L_{\leq 1}A$
(Ax5) $\vdash U_{\leq r}A \rightarrow U_{<s}A$, $s > r$
(Ax6) $\vdash U_{<s}A \rightarrow U_{\leq s}A$
(Ax7) $\vdash (U_{\leq r_1}A_1 \wedge \cdots \wedge U_{\leq r_m}A_m) \rightarrow U_{\leq r}A$, if $A \rightarrow \bigvee_{J \subseteq \{1,\ldots,m\},|J|=k+n} \bigwedge_{j \in J} A_j$ and $\bigvee_{J \subseteq \{1,\ldots,m\},|J|=k} \bigwedge_{j \in J} A_j$ are propositional tautologies, where $r = \frac{\sum_{i=1}^{m} r_i - k}{n}$, $n \neq 0$
(Ax8) $\vdash \neg(U_{\leq r_1}A_1 \wedge \cdots \wedge U_{\leq r_m}A_m)$, if $\bigvee_{J \subseteq \{1,\ldots,m\},|J|=k} \bigwedge_{j \in J} A_j$ is a propositional tautology and $\sum_{i=1}^{m} r_i < k$
(Ax9) $\vdash L_{=1}(A \rightarrow B) \rightarrow (U_{\geq s}A \rightarrow U_{\geq s}B)$

Before we state the inference rules of the ILUPJ logic, we define a constant specification:

Definition 7 (Constant Specification). Constant specification CS *is any set that satisfies:*

$$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an instance of some axiom of } \mathsf{ILUPJ}\}.$$

The constant specification is used to control an agent's reasoning capabilities, i.e. to specify which axioms the agent has a justification of. So we can model agents that are not logically omniscient. Assume that the constant specification includes (c, A) for some axiom A and some constant c. Then using rule (IR1), see below, we can infer $c : A$, i.e. the agent beliefs A and c justifies that belief. However, if for no constant c we have that $(c, A) \in \mathsf{CS}$, then the agent does not have an atomic justification for A, i.e. she may not have justified belief of the axiom A.

Inference Rules of the logic ILUPJ:

(IR1) $\vdash !^n c :!^{n-1} c : \cdots :!c : c : A$ where $(c, A) \in \mathsf{CS}$ and $n \in \mathbb{N}$
(IR2) If $T \vdash A$ and $T \vdash A \to B$ then $T \vdash B$
(IR3) If $\vdash A$ then $\vdash L_{\geq 1} A$
(IR4) If $T \vdash A \to U_{\geq s - \frac{1}{k}} B$, for every $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \to U_{\geq s} B$
(IR5) If $T \vdash A \to L_{\geq s - \frac{1}{k}} B$, for every $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \to L_{\geq s} B$

Axioms (Ax7) and (Ax8) together are the logical representation of the third condition from Theorem 1. Equivalent to saying that $\{\{X_1, \ldots, X_m\}\}$ covers a set X n times is to say that:

$$X \subseteq \bigcup_{J \subseteq \{1,\ldots,m\}, |J|=n} \bigcap_{j \in J} X_j.$$

Hence, the condition that the formula

$$A \to \bigvee_{J \subseteq \{1,\ldots,m\}, |J|=k+n} \bigwedge_{j \in J} A_j$$

is a tautology states that $[A]_{M,w}$[2] is covered $n + k$ times by a multiset

$$\{\{[A_1]_{M,w}, \ldots, [A_m]_{M,w}\}\},$$

while the condition that

$$\bigvee_{J \subseteq \{1,\ldots,m\}, |J|=k} \bigwedge_{j \in J} A_j$$

is a propositional tautology states that the set W is covered k times by a multiset $\{\{[A_1]_{M,w}, \ldots, [A_m]_{M,w}\}\}$.

Formula A is deducible from a set of formulas T, denoted by $T \vdash A$, if there exists at most countable sequence of formulas $A_0, A_1, \ldots, A$, where every A_i is an axiom or a formula that belongs to the set T, or is derived from the preceding formulas by some inference rule (exception is that the Rule (IR3) can be applied on the theorems only). Formula A is a theorem, denoted by $\vdash A$, if it can be deduced from the empty set.

[2] $[A]_{M,w}$ represents the set of all worlds from $W(w)$ in a model M where A holds and will be defined later.

3.3 Semantics

For sets of formulas X and Y, we will use the following notation:

$$X \cdot Y := \{A \mid B \to A \in X \text{ and } B \in Y, \text{ for some formula } B\}.$$

In order to provide semantics for the logic ILUPJ, we start with the notion of a basic evaluation.

Definition 8 (Basic Evaluation). *Let* CS *be a constant specification. A* basic CS-evaluation *is a function* $*$, *such that*

$$* : \mathsf{Prop} \to \{true, false\} \quad \textit{and} \qquad * : \mathsf{Tm} \to \mathcal{P}(\mathsf{For}),$$

and for $s, t \in \mathsf{Tm}$, $c \in \mathsf{Con}$ *and* $A \in \mathsf{For}$ *we have:*

1) $s^* \cdot t^* \subseteq (s \cdot t)^*$
2) $s^* \cup t^* \subseteq (s + t)^*$
3) if $(c, A) \in \mathsf{CS}$ *then*
  a) $A \in c^*$
  b) $!^n c :!^{n-1} c : \cdots :!c : c : A \in (!^{n+1} c)^*$, *for* $n \in \mathbb{N}$.

We will write t^* and p^* instead of $*(t)$ and $*(p)$ respectively.

Definition 9 ($\mathsf{ILUPJ}_{\mathsf{CS}}$-Model). *Let* CS *be any constant specification. An* $\mathsf{ILUPJ}_{\mathsf{CS}}$-model *(or simply* model*) is a tuple* $\langle W, LUP, * \rangle$, *where:*

- *W is a nonempty set of worlds.*
- *LUP assigns to every $w \in W$ a space, such that $LUP(w) = \langle W(w), H(w), P(w) \rangle$, where:*
  - $\emptyset \neq W(w) \subseteq W$,
  - *$H(w)$ is an algebra of subsets of $W(w)$ and*
  - *$P(w)$ is a set of finitely additive probability measures defined on $H(w)$.*
- *$*$ is a function from W to the set of all basic CS-evaluations, i.e. $*(w)$ is a basic CS-evaluation for each world $w \in W$.*

We will denote $*(w)$ by $*_w$.

Definition 10 (Truth in a Model). *Let* CS *be any constant specification. and let* $M = \langle W, LUP, * \rangle$ *be a model. We define what does it mean for a formula* $A \in \mathsf{For}_{\mathsf{ILUPJ}}$ *to hold in* M *at the world* w *by:*

- $M, w \models p$ *iff* $p^*_w = true$, *for* $p \in \mathsf{Prop}$
- $M, w \models U_{\geq s} A$ *iff* $P^*(w)([A]_{M,w}) \geq s$,
- $M, w \models L_{\geq s} A$ *iff* $P_*(w)([A]_{M,w}) \geq s$,
- $M, w \models \neg A$ *iff* $M \not\models A$,
- $M, w \models A \wedge B$ *iff* $M \models A$ *and* $M \models B$,
- $M, w \models t : A$ *iff* $A \in t^*_w$,

where[3] $[A]_{M,w} = \{u \in W(w) \mid M, u \models A\}$ *and* $W(w)$ *and* $P(w)$ *are given by* $LUP(w)$. *The functions* $P^*(w)$ *and* $P_*(w)$ *are defined as in Definition 3.*

Definition 11 (Measurable Model). *Let* CS *be a constant specification and let* M *be a model.* M *is said to be* measurable *if* $[A]_{M,w} \in H(w)$ *for every* $A \in \mathsf{For}$. *The class of all measurable* $\mathsf{ILUPJ_{CS}}$*-models will be denoted by* $\mathsf{ILUPJ}_{\mathsf{CS},Meas}$.

For a model M, we write $M \models A$ if for every $w \in W$, $M, w \models A$. For $T \subseteq \mathsf{For}$, $M \models T$ means that $M \models A$ for every $A \in T$. Finally, $T \models A$ means that $M \models T$ implies $M \models A$.

Definition 12 (Satisfiability). *Formula* A *is satisfiable if there exists a measurable model* M *and* $w \in W$ *such that* $M, w \models A$. *A set of formulas* T *is satisfiable if every formula in* T *is satisfiable.*

As usual, we have the deduction theorem.

Theorem 2 (Deduction Theorem). *Let* $A, B \in \mathsf{For}$, T *a set of formulas and* CS *be any constant specification. Then* $T \cup \{A\} \vdash B$ *iff* $T \vdash A \to B$.

Proof The proof is completely standard. We only show the case in the direction from left to right where the last rule application is an instance of (IR4). In this case $B = C \to U_{\geq s}B'$. We have:

(1) $T, A \vdash C \to U_{\geq s-\frac{1}{k}}B'$, for all $k \geq \frac{1}{s}$
(2) $T \vdash A \to (C \to U_{\geq s-\frac{1}{k}}B')$, for all $k \geq \frac{1}{s}$ by induction hypothesis
(3) $T \vdash (A \wedge C) \to U_{\geq s-\frac{1}{k}}B'$, for all $k \geq \frac{1}{s}$
(4) $T \vdash (A \wedge C) \to U_{\geq s}B'$ by (IR4)
(5) $T \vdash A \to (C \to U_{\geq s}B')$,

which is $T \vdash A \to B$. □

We also need the following technical lemma.

Lemma 1

(a) $\vdash U_{\geq s}A \to U_{>r}A$, $s > r$
(b) $\vdash U_{>s}A \to U_{\geq s}A$
(c) If $\vdash A \leftrightarrow B$ *then* $\vdash U_{\geq s}A \leftrightarrow U_{\geq s}B$

Proof From (Ax5) and (Ax6), using contraposition we obtain proofs for (a) and (b), while (c) is a direct consequence of (IR3) and (Ax9). □

4 Soundness and Completeness

The soundness theorem can be proved as usual by transfinite induction on the depth of the derivation $T \vdash A$.

Theorem 3 (Soundness). *Let* CS *be a constant specification. For* $T \subseteq \mathsf{For}$ *and* $A \in \mathsf{For}$ *we have:*

$$T \vdash A \quad \Rightarrow \quad T \models A.$$

[3] When M is clear from the context we will write $[A]_w$.

4.1 Completeness

Definition 13 ($\mathsf{ILUPJ_{CS}}$-Consistent Set). *For an arbitrary constant specification* CS *and* $T \subseteq \mathsf{For}$ *we say that:*

(a) T *is* $\mathsf{ILUPJ_{CS}}$-consistent *if and only if* $T \nvdash \bot$. *Otherwise,* T *is* $\mathsf{ILUPJ_{CS}}$-inconsistent.
(b) T *is* maximal *if and only if for all* $A \in \mathsf{For}$, *either* $A \in T$ *or* $\neg A \in T$.
(c) T *is* maximal $\mathsf{ILUPJ_{CS}}$-consistent *if and only if it is maximal and* $\mathsf{ILUPJ_{CS}}$-*consistent.*

Lemma 2. *Let* CS *be an arbitrary constant specification and* T *an* $\mathsf{ILUPJ_{CS}}$-*consistent set of formulas.*

(1) For any $A \in \mathsf{For}$, *either* $T \cup \{A\}$ *is* $\mathsf{ILUPJ_{CS}}$-*consistent or* $T \cup \{\neg A\}$ *is* $\mathsf{ILUPJ_{CS}}$-*consistent.*
(2) If $\neg(A \to U_{\geq s}B) \in T$, *then there exists some* $n > \frac{1}{s}$ *such that*

$$T \cup \{A \to \neg U_{\geq s-\frac{1}{n}}B\}$$

is $\mathsf{ILUPJ_{CS}}$-*consistent.*
(3) If $\neg(A \to L_{\geq s}B) \in T$, *then there exists some* $n > \frac{1}{s}$ *such that*

$$T \cup \{A \to \neg L_{\geq s-\frac{1}{n}}B\}$$

is $\mathsf{ILUPJ_{CS}}$-*consistent.*

Proof. (1) Suppose that both $T \cup \{A\} \vdash \bot$ and $T \cup \{\neg A\} \vdash \bot$ hold. From the Deduction Theorem, we get $T \vdash \neg A$ and $T \vdash A$ which contradicts the assumption that the set T is $\mathsf{ILUPJ_{CS}}$-consistent.
(2) Assume that for all $n > \frac{1}{s}$ we have:

$$T, A \to \neg U_{\geq s-\frac{1}{n}}B \vdash \bot.$$

From Deduction Theorem and propositional reasoning, we obtain

$$T \vdash A \to U_{\geq s-\frac{1}{n}}B,$$

and from Inference Rulle 4 $T \vdash A \to U_{\geq s}B$. Contradiction with the assumption that $\neg(A \to U_{\geq s}B) \in T$.
(3) Similar to the previous case. □

Theorem 4 (Lindenbaum). *Let* CS *be an arbitrary constant specification. Every* $\mathsf{ILUPJ_{CS}}$-*consistent set can be extended to a maximal* $\mathsf{ILUPJ_{CS}}$-*consistent set.*

Proof. Consider a $\mathsf{ILUPJ_{CS}}$-consistent set T and let $A_0, A_1, A_2, \ldots$ be an enumeration of all the formulas from For. We define a sequence of sets T_i, $i = 0, 1, 2, \ldots$ in the following way:

(1) $T_0 = T$,
(2) for every $i \geq 0$,
  (a) if $T_i \cup \{A_i\}$ is $\mathsf{ILUPJ_{CS}}$-consistent, then $T_{i+1} = T_i \cup \{A_i\}$, otherwise
  (b) if A_i is of the form $B \rightarrow U_{\geq s}C$, then $T_{i+1} = T_i \cup \{\neg A_i, B \rightarrow \neg U_{\geq s-\frac{1}{n}}C\}$, for some $n > 0$, so that T_{i+1} is $\mathsf{ILUPJ_{CS}}$-consistent, otherwise
  (c) if A_i is of the form $B \rightarrow L_{\geq s}C$, then $T_{i+1} = T_i \cup \{\neg A_i, B \rightarrow \neg L_{\geq s-\frac{1}{n}}C\}$, for some $n > 0$, so that T_{i+1} is $\mathsf{ILUPJ_{CS}}$-consistent, otherwise
  (d) $T_{i+1} = T_i \cup \{\neg A_i\}$,
(3) $T^{\spadesuit} = \bigcup_{i=0}^{\infty} T_i$.

Using induction on i, we prove that for every $i \in \mathbb{N}$, T_i is $\mathsf{ILUPJ_{CS}}$-consistent.

(i) T_0 is $\mathsf{ILUPJ_{CS}}$-consistent because T is.
(ii) Suppose that T_i is $\mathsf{ILUPJ_{CS}}$-consistent. We prove that also T_{i+1} is:
  – T_{i+1} is constructed using the step (2)(a). Trivially.
  – T_{i+1} is constructed using the step (2)(b). From Lemma 2((1) and (2)).
  – T_{i+1} is constructed using the step (2)(c). From Lemma 2((1) and (3)).
  – T_{i+1} is constructed using the step (2)(d). Since $T_i \cup \{A_i\}$ is $\mathsf{ILUPJ_{CS}}$-inconsistent, we know that $T_i \cup \{\neg A_i\}$ is $\mathsf{ILUPJ_{CS}}$-consistent.

Now let us show that $T^{\spadesuit}$ is maximal $\mathsf{ILUPJ_{CS}}$-consistent set. From the construction above we know that for any $A \in \mathsf{For}$ either $A \in T^{\spadesuit}$ or $\neg A \in T^{\spadesuit}$, i.e., $T^{\spadesuit}$ is maximal.

In order to prove that $T^{\spadesuit}$ is ILUPJ-consistent, we prove that:

(i) It does not contain all the formulas from For;
(ii) It is deductively closed.

It is clear from the construction that $T^{\spadesuit}$ does not contain all the formulas from For, so the only thing left to prove is that $T^{\spadesuit}$ is deductively closed. Assume $T^{\spadesuit} \vdash A$. Using transfinite induction on a depth of derivation we prove that $A \in T^{\spadesuit}$.

1) $A \in T^{\spadesuit}$. Trivially.
2) A is an instance of some of the axioms (Ax1)–(Ax9). There exists $k \in \mathbb{N}$ with $A = A_k$. Assuming that $\neg A_k \in T_{k+1}$, we get a contradiction from:

$$T_{k+1} \vdash A_k \quad \text{and} \quad T_{k+1} \vdash \neg A_k.$$

3) A is obtained from $T^{\spadesuit}$ by an application of (IR1), i.e.,

$$A = !^n c : !^{n-1} c : \cdots : !c : c : B,$$

for some $n \in \mathbb{N}$, axiom B and $(c, B) \in \mathsf{CS}$. There exists k such that $A = A_k$ and if $\neg A \in T_{k+1}$, then

$$T_{k+1} \vdash A \quad \text{and} \quad T_{k+1} \vdash \neg A$$

which gives us a contradiction.

4) A is obtained from $T^{\spadesuit}$ by an application of (IR2). Induction hypothesis tells us that there exists l, such that both premises belong to T_l. Since there exists k such that $A = A_k$, if $\neg A \in T_{max(k,l)+1}$, then

$$T_{max(k,l)+1} \vdash A \quad \text{and} \quad T_{max(k,l)+1} \vdash \neg A$$

which gives us a contradiction.

5) A is obtained from $T^{\spadesuit}$ by an application of (IR3), i.e., $A = L_{\geq 1}B$ and $\vdash B$. Since there exists some k such that $A = A_k$, same reasoning as in 2) gives us the claim.

6) A is obtained from $T^{\spadesuit}$ by an application of (IR4). That means, $A = B \rightarrow U_{\geq s}C$ and for every $k \geq \frac{1}{s}$,

$$T^{\spadesuit} \vdash B \rightarrow U_{\geq s-\frac{1}{k}}C.$$

Assuming that $A \notin T^{\spadesuit}$, i.e., $\neg(B \rightarrow U_{\geq s}C) \in T^{\spadesuit}$, we find a number m, such that

$$\neg(B \rightarrow U_{\geq s}C) \in T_m.$$

Also, from the construction of $T^{\spadesuit}$ we know that for some l,

$$\neg(B \rightarrow U_{\geq s-\frac{1}{l}}C) \in T_l.$$

Further, from inductive hypothesis,

$$B \rightarrow U_{\geq s-\frac{1}{l}}C \in T^{\spadesuit}.$$

Hence, there exists m' with

$$B \rightarrow U_{\geq s-\frac{1}{l}}C \in T_{m'}.$$

Contradiction with a consistency of $T_{max(l,m')+1}$, since both

$$B \rightarrow U_{\geq s-\frac{1}{l}}C \in T_{max(l,m')+1}, \quad \neg(B \rightarrow U_{\geq s-\frac{1}{l}}C) \in T_{max(l,m')+1}.$$

7) The case when A is obtained from $T^{\spadesuit}$ by an application of (IR5) can be proved similarly to the previous case.

We conclude that $T^{\spadesuit}$ is deductively closed set which does not contain all formulas meaning that $T^{\spadesuit}$ is consistent. □

Definition 14 (Canonical Model). *Let* CS *be an arbitrary constant specification. The* canonical model *is the tuple* $M_{can} = \langle W, LUP, * \rangle$ *, where:*

1) $W = \{w \mid w$ *is a maximal* $\mathsf{ILUPJ_{CS}}$*-consistent set of formulas*$\}$,

2) $LUP(w) = \langle W(w), H(w), P(w) \rangle$ *is defined as follows:*
$W(w) = W,$
$H(w) = \{\{u \mid u \in W(w), A \in u\} \mid A \in \mathsf{For}\},$

$P(w)$ is any set of probability measures such that

$$P^*(w)(\{u \mid u \in W(w), A \in u\}) = sup\{s \mid U_{\geq s}A \in w\}.$$

3) for every world $w \in W$, the basic CS*-evaluation is defined with:*
1. For $p \in$ Prop*:*

$$p^*_w = \begin{cases} true \;\; if \;\; p \in w \\ false \; if \;\; \neg p \in w \end{cases}$$

2. For $t \in$ Tm*:*

$$t^*_w = \{A \mid t : A \in w\}$$

Lemma 3. *Let $M_{can} = \langle W, LUP, * \rangle$ be the canonical model. For every $u \in W$ and every formula A,*

$$\{u \mid u \in W, A \in u\} = [A]_{M_{can},u}.$$

Proof. We prove the statement by proving that $A \in u$ iff $u \models A$ by induction on the length of A. If $A = p$ the claim follows by definition of the canonical model. Cases when $A = \neg B$ or $A = B \wedge C$ are trivial.

1. Let $A = U_{\geq s}B$. First, let $U_{\geq s}B \in u$. Then

$$\sup\{r \mid U_{\geq r}B \in u\} = P^*(u)\{w \mid w \in W, B \in w\} = P^*(u)([B]_u) \geq s,$$

so $u \models U_{\geq s}B$.
Now, suppose that $u \models U_{\geq s}B$, i.e.

$$P^*(u)([B]_u) = \sup\{r \mid U_{\geq r}B \in u\} \geq s.$$

If $P^*(u)([B]_u) > s$, then we have (properties of a supremum and monotonicity) $U_{\geq s}B \in u$.
If $P^*(u)([B]_u) = s$, then as a direct consequence of (IR4), we have that $U_{\geq s}B \in u$.

2. Now, let $A = L_{\geq s}B$ or equivalently $A = U_{\leq 1-s}\neg B$. Suppose $U_{\leq 1-s}\neg B \in u$. Our goal is to show that

$$\sup\{r \mid U_{\geq r}\neg B \in u\} \leq 1 - s,$$

hence, suppose towards contradiction that

$$\sup\{r \mid U_{\geq r}\neg B \in u\} > 1 - s.$$

Then, there exists a rational number $q \in (1 - s, 1 - s + \epsilon]$, for some $\epsilon > 0$, such that $U_{\geq q}\neg B \in u$. Therefore, $U_{>1-s}\neg B \in u$. Contradiction. That means

$$\sup\{r \mid U_{\geq r}\neg B \in u\} \leq 1 - s,$$

i.e., $P^*(u)([\neg B]_u) \leq 1 - s$ and therefore we obtain $u \models L_{\geq s}B$.
For the other direction, assume that $u \models U_{\leq 1-s}\neg B$, i.e.

$$\sup\{r \mid U_{\geq r}\neg B \in u\} \leq 1 - s.$$

We distuingish the following cases:

(1) $\sup\{r \mid U_{\geq r}\neg B \in u\} < 1-s$. In this case, if $U_{>1-s}\neg B \in u$, we would have also that $U_{\geq 1-s}\neg B \in u$, so $\sup\{r \mid U_{\geq r}\neg B \in u\} \geq 1-s$. Contradiction.

(2) $\sup\{r \mid U_{\geq r}\neg B \in u\} = 1-s$. We want to show that then it must hold

$$\inf\{r \mid U_{\leq r}\neg B \in u\} = 1-s.$$

Suppose first towards contradiction that

$$\inf\{r \mid U_{\leq r}\neg B \in u\} < 1-s.$$

Then there exists a rational number $q_1 \in [1-s-\epsilon, 1-s)$ such that $U_{\leq q_1}\neg B \in u$, and so $U_{<1-s}\neg B \in u$. Contradiction with $U_{\geq 1-s}\neg B \in u$ (this follows directly from Inference Rule 4). Now, suppose that

$$\inf\{r \mid U_{\leq r}\neg B \in u\} > 1-s,$$

i.e.,

$$\inf\{r \mid U_{\leq r}\neg B \in u\} = 1-s+\varepsilon.$$

Taking an arbitrary rational number $q_2 \in (1-s, 1-s+\varepsilon)$, we obtain that both

$$U_{\leq q_2}\neg B \in u \quad \text{and} \quad U_{\geq q_2}\neg B \in u$$

which contradicts properties of an infimum and supremum. Hence

$$\inf\{r \mid U_{\leq r}\neg B \in u\} = 1-s,$$

or equivalently

$$\inf\{r \mid L_{\geq 1-r}B \in u\} = 1-s$$

and as a consequence of an Inference Rule 5, we get $L_{\geq s}B \in u$.

3. Finally let $A = t : B$. Since $\{u \mid u \in W, u \models t : B\} = [A]_{M_{can},u}$ and

$$\{u \mid u \in W, t : B \in u\} = \{u \mid u \in W, B \in t^*_u\} = \{u \mid u \in W, u \models t : B\},$$

the proof is finished. □

Theorem 5. *Let* CS *be an arbitrary constant specification.* M_{can} *is a* $\mathsf{ILUPJ}_{\mathsf{CS},Meas}$*-model.*

Proof. Since there exists a maximal $\mathsf{ILUPJ}_{\mathsf{CS}}$-consistent set, we know that $W \neq \emptyset$ and $W(w) \neq \emptyset$. Proof that $H(w)$ is an algebra is straightforward. Also note that for every $w \in W$, $*_w$ is a basic CS-evaluation by the construction of the canonical model.

Let us prove the existence of a set of probability measures $P(w)$ claimed in 2) and that $P^*(w)$ is well defined.

1) *There exists a set of finitely additive probability measures* $P(w)$ *and* $P^*(w)$ *is an upper probability measure for* $P(w)$*:*

We prove the three conditions from Theorem 1 and since the first two conditions, $P^*(w)(\emptyset) = 0$ and $P^*(w)(W) = 1$, are trivial, we prove only the third, i.e., if

$$\{\{[A_1], \ldots, [A_m]\}\}$$

is an (n, k)-cover of $([A], W)$, then

$$k + nP^*(w)([A]) \leq \sum_{i=1}^{m} P^*(w)([A_i]).$$

Let $P^*(w)([A_i]) = \sup\{r \mid U_{\geq r}A_i \in w\} = a_i$, for $i = 1, \ldots, m$. For an arbitrary fixed $\varepsilon > 0$, there exist rational numbers $q_i \in (a_i, a_i + \varepsilon)$ with $U_{\leq q_i}A_i \in w$. If that would not be the case, then $U_{>q_i}A_i \in w$ which contradicts with the fact that a_i is supremum. As a consequence we get

$$w \vdash U_{\leq q_1}A_1 \wedge \cdots \wedge U_{\leq q_m}A_m,$$

and by (Ax7)

$$w \vdash U_{\leq q}A,$$

where $q = \frac{\sum_{i=1}^{m} q_i - k}{n}$, $n \neq 0$. Thus, $\sup\{r \mid U_{\geq r}A_i \in w\} \leq q$ or equivalently $P^*(w)([A]) \leq q$. Thus, we have

$$P^*(w)([A]) \leq \frac{\sum_{i=1}^{m} q_i - k}{n} = \frac{\sum_{i=1}^{m} a_i + m\varepsilon - k}{n}.$$

Because this holds for every $\varepsilon > 0$ we obtain $k + nP^*(w)([A]) \leq \sum_{i=1}^{m} P^*(w)([A_i])$. If $n = 0$, we have to show that $k \leq \sum_{i=1}^{m} P^*(w)([A_i])$. Reasoning as above, we obtain

$$w \vdash U_{\leq q_1}A_1 \wedge \cdots \wedge U_{\leq q_m}A_m,$$

for some $q_i \in (a_i, a_i + \varepsilon)$. From (Ax8), how

$$\bigvee_{J \subseteq \{1,\ldots,m\}, |J|=k} \bigwedge_{j \in J} A_j$$

is a propositional tautology, we have that $\sum_{i=1}^{m} q_i \geq k$. Again, from the fact that it holds for every $\varepsilon > 0$, we obtain $\sum_{i=1}^{m} a_i \geq k$.

2) $P^*(w)$ *is well defined*: that Lemma 1(c) tells us that a value of the supremum does not depend on a choice of an element from $[A]$. Hence $P^*(w)([A])$ is well defined.
Finally, note that as a direct consequence of the Lemma 3 we have that this model is measurable. □

Theorem 6 (Strong Completeness for ILUPJ). *For an arbitrary constant specification* CS, $T \subseteq$ For *and* $A \in$ For *we have:*

$$T \models A \quad \Rightarrow \quad T \vdash A.$$

Proof. Suppose that $T \nvdash A$ or equivalently $T \nvdash \neg A \to \bot$. From Deduction Theorem we get $T, \neg A \nvdash \bot$ meaning that the set $T \cup \{\neg A\}$ is $\mathsf{ILUPJ}_{\mathsf{CS}}$-consistent. From Theorem 4 we know that there exists a maximal $\mathsf{ILUPJ}_{\mathsf{CS}}$-consistent set $T^{\spadesuit}$ with $T \cup \{\neg A\} \subseteq T^{\spadesuit}$. Finally, since $T^{\spadesuit}$ is a world in the canonical model, we get $M_{can}, T^{\spadesuit} \models T$ and $M_{can}, T^{\spadesuit} \models \neg A$ and thus $T \not\models A$. □

5 ILUPJ as a Generalization of the Logic PPJ

In this section we prove that the logic ILUPJ generalizes the logic PPJ from [10]. The strategy we use relies heavily on the strategy used in [6]. Let us briefly recall the logic PPJ.

The language of the logic PPJ extends the language of the justification logic J with the list of operators $P_{\geq s}$, where s is a rational number from the $[0, 1]$. For example,

$$p \wedge P_{\leq \frac{1}{2}}(t : q) \quad \text{and} \quad P_{= \frac{1}{3}} P_{\geq 1}(s : (p \vee r))$$

are well defined formulas. PPJ-models are defined as triples $M = \langle W, Prob, * \rangle$, where:

- W is a non empty set of worlds
- $Prob$ is an assignment which assigns to every $w \in W$ a probability space, such that $Prob(w) = \langle W(w), H(w), \mu(w) \rangle$, where:
  - $W(w)$ is a non empty subset of W,
  - $H(w)$ is an algebra of subsets of $W(w)$ and
  - $\mu(w) : H(w) \to [0, 1]$ is a finitely additive probability measure.
- $*_w$ is a basic CS-evaluation.

Satisfiability of a formula is defined as expected for the justification logic formulas and

$$M, w \models P_{\geq s} A \text{ iff } \mu(w)(\{v \in W(w) \mid v \models A\}) \geq s.$$

Axiomatization of the logic PPJ is the following:

(P1) $\vdash A$, where A is a propositional tautology
(P2) $\vdash t : (A \to B) \to (s : A \to (t \cdot s) : B)$
(P3) $\vdash t : A \vee s : A \to (t + s) : A$
(P4) $P_{\geq 0} A$,
(P5) $P_{\leq r} A \to P_{< s} A$, $s > r$,
(P6) $P_{< s} A \to P_{\leq s} A$,
(P7) $(P_{\geq t} A \wedge P_{\geq s} B \wedge P_{\geq 1}(\neg A \vee \neg B)) \to P_{\geq min\{1, t+s\}}(A \vee B)$,
(P8) $(P_{\leq t} A \wedge P_{< s} B) \to P_{< t+s}(A \vee B)$, $t + s \leq 1$.

Inference Rules

(1) $\vdash !^n c : !^{n-1} c : \cdots : !c : c : A$ where $(c, A) \in \mathsf{CS}$ and $n \in \mathbb{N}$
(2) If $T \vdash A$ and $T \vdash A \to B$ then $T \vdash B$
(3) If $\vdash A$ then $\vdash P_{\geq 1} A$

(4) If $T \vdash A \rightarrow P_{\geq s-\frac{1}{k}}B$, for every $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \rightarrow P_{\geq s}B$

Soundness and strong completeness theorems for the logic PPJ are proved (see [10], Theorems 11 and 22).

The ILUPJ logic has the similar semantical structure as the logic PPJ. Also, it is clear that the semantics of the logic ILUPJ is more general, since reasoning about upper and lower probabilities requires *sets* of probability measures, while in the logic PPJ one measure per possible world is sufficient (thus they are isomorphic to the "sets of" probability measures which are singletons).

However, the axiomatic systems are quite different. We focus on the two proof theoretical aspects of the generalization:

1. which axioms should be added to the logic ILUPJ to reduce the proposed class of models to the class of models isomorphic to the models for the logic PPJ
2. how can we use the added axioms to formally obtain the axiomatization of PPJ.

As already stated, the subclass of the ILUPJ-models that contains only those structures where the set of probability measures is a singleton set is isomorphic to the class of PPJ-models. Thus, we add the following axiom which guarantees that it is the case:

$$(Ax10) \quad U_{\geq r}A \rightarrow L_{\geq r}A. \tag{1}$$

We will denote ILUPJ+Axiom $(Ax10)$ by ILUPJ^{Ext}.

It can easily be proved that the following holds (see the proof of Proposition 1 in [18]):

$$\vdash U_{\leq r}A \rightarrow L_{\leq r}A. \tag{2}$$

From (1) and (2) follows that operators U and L have the same behavior in the sense that for every formula A and every $r \in \mathbb{Q} \cap [0,1]$

$$\vdash U_{\geq r}A \leftrightarrow L_{\geq r}A. \tag{3}$$

As a consequence we have that in ILUPJ^{Ext} one type of operators is sufficient, since changing one type of operator with other will lead to an equivalent formula. For example, if we replace all the operators for lower probability with the operators of upper probability in $A \equiv L_{\geq\frac{1}{3}}U_{\leq\frac{1}{2}}L_{=1}(t:p)$, we will obtain the formula B equivalent to A $B \equiv U_{\geq\frac{1}{3}}U_{\leq\frac{1}{2}}U_{=1}(t:p)$. It can be proved in a straightforward manner by the induction on the complexity of a formula that this holds for any formula. This fact allows us, without loss of generality, to consider only formulas with the U operators in ILUPJ^{Ext}.

Our aim is to prove that the set of theorems of the logic PPJ is a subset of the set of theorems of the logic ILUPJ^{Ext}. In order to prove that, we show that all the axioms and inference rules of the logic PPJ can be inferred in the logic ILUPJ^{Ext}, where an operator P is replaced by U.

First not that the axioms (P1)–(P6) correspond to the axioms (Ax1)–(Ax6) and inference rules coincide as well. Our goal is to prove that the appropriate counterparts of the axioms (P7) and (P8), i.e.,

(U7) $(U_{\geq t}B \wedge U_{\geq s}C \wedge U_{\geq 1}(\neg B \vee \neg C)) \rightarrow U_{\geq min\{1,t+s\}}(B \vee C)$,
(U8) $(U_{\leq t}B \wedge U_{<s}C) \rightarrow U_{<t+s}(B \vee C)$, $t + s \leq 1$,

follow from the axiomatization of ILUPJ^{Ext}, where in that inference the essential role is played by the axioms (Ax7) and (Ax8).

In order to prove that we need the following Lemma:

Lemma 4. $\mathsf{ILUPJ}^{Ext} \vdash (U_{\leq t}B \wedge U_{\leq s}C) \rightarrow U_{\leq t+s}(B \vee C)$, $t + s \leq 1$.

Proof. We will show that the claim can be inferred from the axiom (Ax7). Consider the axiom (Ax7) for:

$m = 2;\ n = 1, k = 0;\ r_1 = t;\ r_2 = s;$
$A_1 = B;\quad A_2 = C;\quad A = B \vee C.$

In this case we get $r = t + s$ and therefore the Axiom (Ax7) has exactly the shape of the required formula. We also have to check whether the formulas

$$A \rightarrow \bigvee_{J\subseteq\{1,2\},|J|=1} \bigwedge_{j\in J} A_j$$

and

$$\bigvee_{J\subseteq\{1,2\},|J|=0} \bigwedge_{j\in J} A_j$$

are tautologies. The first formula has the form $B \vee C \rightarrow B \vee C$ which is clearly a tautology, while the second formula has the form $\bigwedge_{j\in\emptyset} A_j$, and $\bigwedge_{j\in\emptyset} A_j = \top$ by definition and hence a tautology. □

Theorem 7. *The set of theorems of the logic* PPJ *is a subset of the set of theorems of the logic* ILUPJ^{Ext}.

Proof. As already mentioned, we only need to prove that:

(a) $\mathsf{ILUPP}^{Ext} \vdash (U_{\geq t}B \wedge U_{\geq s}C \wedge U_{\geq 1}(\neg B \vee \neg C)) \rightarrow U_{\geq min\{1,t+s\}}(B \vee C)$,
(b) $\mathsf{ILUPP}^{Ext} \vdash (U_{\leq t}B \wedge U_{<s}C) \rightarrow U_{<t+s}(B \vee C)$, $t + s \leq 1$.

Proof of (a). First recall that the formula

$$(U_{\geq t}B \wedge U_{\geq s}C \wedge U_{\geq 1}(\neg B \vee \neg C)) \rightarrow U_{\geq min\{1,t+s\}}(B \vee C)$$

can be written as:

$$(U_{\leq 1-t}\neg B \wedge U_{\leq 1-s}\neg C \wedge U_{\leq 0}(B \wedge C)) \rightarrow U_{\leq 1-min\{1,t+s\}}\neg(B \vee C).$$

Now, consider the axiom (Ax7) for:

$m = 3;\ n = k = 1;\ r_1 = 1 - t;\ r_2 = 1 - s;\ r_3 = 0;$
$A_1 = \neg B; \quad A_2 = \neg C; \quad A_3 = B \wedge C; \quad A = \neg(B \vee C).$

We obtain that $r = 1 - (t + s)$.

(i) If $t + s > 1$ then (Axiom (Ax8), $\sum_{i=1}^{m} r_i < k$)

$$\vdash \neg(U_{\leq 1-t}\neg B \wedge U_{\leq 1-s}\neg C \wedge U_{\leq 0}(B \wedge C)),$$

so $\vdash (U_{\leq 1-t}\neg B \wedge U_{\leq 1-s}\neg C \wedge U_{\leq 0}(B \wedge C)) \rightarrow U_{\leq 1-min\{1,t+s\}}\neg(B \vee C))$.
(ii) If $t+s \leq 1$, then $1 - min\{1, t+s\} = 1-(t+s) = r$ and it is left to check if

$$A \rightarrow \bigvee_{J \subseteq \{1,2,3\}, |J|=2} \bigwedge_{j \in J} A_j$$

and

$$\bigvee_{J \subseteq \{1,2,3\}, |J|=1} \bigwedge_{j \in J} A_j$$

are tautologies. Namely, in this case, the first formula has the following form:

$$\neg(B \vee C) \rightarrow ((\neg B \wedge \neg C) \vee (\neg B \wedge B \wedge C) \vee (\neg C \wedge B \wedge C)),$$

and the second formula:

$$\neg B \vee \neg C \vee (B \wedge C).$$

It is obvious that both of these formulas are tautologies and therefore this part is proved.

Proof of (b). Let us show equivalently that $\mathsf{ILUPJ}^{Ext} \vdash (U_{\leq t}B \wedge U_{\geq t+s}(B \vee C)) \rightarrow U_{\geq s}C$:

$\vdash U_{\geq t+s}(B \vee C) \rightarrow U_{>t+s'}(B \vee C)$, for all $s' < s$ (contraposition (Ax5))
$U_{\leq t}B \wedge U_{\geq t+s}(B \vee C) \vdash U_{\leq t}B \wedge U_{>t+s'}(B \vee C)$, for all $s' < s$
$U_{\leq t}B \wedge U_{\geq t+s}(B \vee C) \vdash U_{\leq t}B \wedge U_{>s'}C$, for all $s' < s$ (by Lemma 4)
$U_{\leq t}B \wedge U_{\geq t+s}(B \vee C) \vdash U_{\geq s}C$ (by (IR4))
$\vdash (U_{\leq t}B \wedge U_{\geq t+s}(B \vee C)) \rightarrow U_{\geq s}C$ (by Deduction theorem) □

6 Conclusion

We present a logic which allows making statements about upper and lower probabilities of the justification formulas. In this framework, we can represent information like: "t is justification that probability of A lies in the interval..." and our formalism, the logic ILUPJ, can be used for reasoning not only about lower and upper probabilities of a certain justification formula, but also about uncertain belief about other imprecise probabilities. The language of our logic is modal language which extends justification logic language with the unary operators $U_{\geq r}$ and $L_{\geq r}$, where r ranges over the unit interval of rational numbers. The corresponding semantics consist of the measurable Kripke models with sets of finitely additive probability measures attached to each possible world, as well as a function $*$ from the set of worlds to the set of all basic CS-evaluations. We prove that the proposed axiomatic system is strongly complete with respect to the class of measurable models.

We also provided an extension of the proposed axiomatization in order to prove that the logic ILUPJ is a generalization of the logic PPJ for reasoning about sharp probabilities of justification formulas from [10].

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