International Journal of Child-Computer Interaction 32 (2022) 100419

Contents lists available at ScienceDirect

International Journal of Child-Computer Interaction

journal homepage: www.elsevier.com/locate/ijcci

Action-based embodied design for mathematics learning: A decade of variations on a theme

Rosa Alberto *, Anna Shvarts, Paul Drijvers, Arthur Bakker

Utrecht University, The Netherlands

ARTICLE INFO

Article history:
Received 25 November 2020
Received in revised form 13 September 2021
Accepted 23 September 2021
Available online 2 October 2021

Keywords:
Design principles
Embodied cognition
Mathematics education
Educational technology

ABSTRACT

Embodied cognition theory emphasizes that bodily interaction with the environment is important for all forms of learning, including mathematics. This theoretical trend coincides well with developments in motion responsive technology, and has resulted in numerous embodied technologies for mathematics learning. This review aims to contribute to clarifying theoretically and empirically grounded design principles of action-based embodied designs for mathematics learning. We analyzed 79 publications between 2010 and 2019, containing 15 studies assessing 15 sensorimotor problems for five mathematical domains (proportion, angle, area, parabola, and sine function), and explicated the characteristics of the technologies, their learning sequences and elicited learning processes, and the influence of within-topic ask variations on students' learning. We found that action-based designs pose motor control problems using continuous motion feedback to facilitate learners to discover and practice a challenging new ways of moving their hand(s) in which to ground mathematical cognition. The state of discovery of the sensorimotor solution is important, and passive and readymade designs are cautioned. The learning sequence in which these technologies are embedded, elicit mathematical knowing through necessary and sequential phases in which personal idiosyncratic experiences increasingly converge into a culturally shared mathematical discourse. In the qualitative stage, an acting step elicits students to actively establish new motor coordination-patterns through the emergence of new perceptual structures known as attentional anchors. In the subsequent reflecting step, students' personal sensorimotor experiences and attentional anchors become the ground for referencing in (a shared) mathematical discourse through multimodal (words, gestures) collaboration with a tutor. In the quantitative stage, measuring artifacts (grids, protractors, numbers, variables) are included in students' field of promoted action, which discretize and formalize students' actions and subsequent reflections into culturally recognizable quantitative forms. Critically, task factors such as the type of objects students manipulate (cursors icons, bars, rectangle), and the direction these objects are moved (parallel, orthogonal), affect students' attentional anchors and subsequent reflections in the qualitative stage, but converge to similar mathematical insights in the quantitative stage. These insights help to better use (new) motion responsive technology in eliciting child–computer interaction that can lead to mathematical cognition and beyond.

Contents

* Correspondence to: Utrecht University, Freudenthal Institute, Buys Ballot Building, room 3.71, Princetonplein 5, 3584 CC Utrecht, The Netherlands.
E-mail address: r.a.alberto@uu.nl (R. Alberto).

https://doi.org/10.1016/j.ijcci.2021.100419
2212-8689/

1. Introduction

Counter to traditional views that locate knowledge inside the brain, the theory of embodied cognition emphasizes that we humans think through and with our bodies (Varela, Thompson, & Rosch, 1991). This broadens the locus of cognition from the brain to the interaction of one's body with the physical-material and social-cultural environment (Smith & Thelen, 1996; Wilson & Golonka, 2013). Sheets-Johnstone stated it elegantly: Movement is the 'mother of all cognition' (Sheets-Johnstone, 2011, p. xxii). This embodied movement aligns well with some types of learning, such as learning to walk or ride a bicycle, though for other types of learning, such as for instance mathematics, the body does not seem naturally implicated. Mathematics is seen as a collection of disembodied ideas (e.g., Boaler, Chen, Williams, & Cordero, 2016), and its learning mainly as an abstract and cognitive activity. Yet an expanding base of literature points to involvement of the body when learning mathematics, such as in reasoning (Barsalou, 1999), the interaction with tools and the material culture (de Freitas & Sinclair, 2014), and in gestures accompanying speech (Radford, 2009).

Centralizing bodily interaction as the source of (mathematical) knowledge offers an exciting perspective on the design of educational technology (Antle, 2013). This endeavor coincides fruitfully with advances in technology, such as motion sensors and touchscreens, which are better suited for an embodied take because of their responsiveness to multi-touch, body postures, and spatial positions. This has resulted in diverse and distinct manifestations of embodied inspired learning technologies for mathematics learning. Examples are walking along a number line for numerical estimation (Dackermann, Fischer, Nuerk, Cress, & Moeller, 2017), using hands on tablets for counting and basic arithmetic (TouchCounts in Sinclair & Heyd-Metzuyanim, 2014), multiplayer interaction for linear graphs (Nemirovsky, Kelton, & Rhodehamel, 2013) and whole body movement for early algebra (Nemirovsky, Ferrara, Ferrari, & Adamuz-Povedano, 2020). In this expanding landscape, the different efforts using movement as a source of mathematical cognition have been increasingly reviewed (see Duijzer, van den Heuvel-Panhuizen, Veldhuis, Doorman, & Leseman, 2019 for whole body movement activities for time distance graphs) and assessed in terms of design principles (see Abrahamson et al., 2020 for insights from for example graspable or playful math). These aid experimental and comparative studies to identify more precisely the possible effects of design features on students' learning (see Abrahamson & Abdu, 2020 for a proposed of such study), as well as help designers and educators to make informed choices when developing or using embodied child–computer interaction for mathematics learning.

In this paper, we continue this line by reviewing another embodied child–computer interaction in depth: the action-based embodied design genre (Abrahamson, 2014a). This is one of the largest and most extensively studied and theorized embodied framework for mathematics learning, by now containing a collection of similar designs for a variety of mathematical topics, both elementary and complex. Historically, the action-based research program was initiated in the domain of proportions and driven by the conjecture that "mundane activities do not afford the performance and practice of embodied coordinative routines that, with suitable guidance, could be signified quantitatively and symbolically as proportional" (Abrahamson, 2014a, p. 7). Since the general theory of embodiment did not constrain concrete design decisions of what students were to do (Bakker, Shvarts, & Abrahamson, 2014), design-based research was used (Bakker, 2018), in which iterative testing of embodied theoretical conjectures, through the design of embodied technologies, and empirical study on students' and teachers' embodied processes were intricately linked. The initial conjecture was that mathematical knowing could be elicited by inducing an "image" of proportionality (Abrahamson & Howison, 2008). As shown in Fig. 1a, a mechanical pulley design hand-held students' hands to move in a 1:2 proportion (Abrahamson & Howison, 2010a), but, despite students undergoing an embodied experience of proportionality, this passive design did not yield sufficient ground for mathematical knowing (Howison, Trninic, Reinholz, & Abrahamson, 2011). The genre became unique in its kind by adopting theoretical and practical findings of not only general embodied theories, but also from concrete motion sciences involved in facilitating skill acquisition in for example sports, and extending these towards mathematical learning (e.g., Abrahamson & Sánchez-García, 2016). Fig. 1b illustrates the final design for proportions (Abrahamson, 2014a; Howison et al., 2011): A motor problem is posed in which students are tasked with maintaining green feedback through bimanual movements. Unknown to the students, the feedback turns green only when the distances from the hands to the bottom of the screen are in a pre-set proportional relation, e.g., 1:2, otherwise it remains red, e.g., when the hands are at a 3:4 ratio. This 'keep-it-green' design invited students to struggle productively to discover and practice a new and challenging way of moving their hand(s), which, when described and nurtured in collaboration with a tutor, became a way of expressing proportionality (Abrahamson, 2014a; Howison et al., 2011). The final technological design (Abrahamson, 2014a; Howison et al., 2011) and theoretical foundations (e.g., Abrahamson & Sánchez-García, 2016) for proportions gave rise to a decade of studies into its working mechanisms (e.g., with multimodal learning analytics), as well as a rich field of variations of other action-based technologies.

These variations occurred within proportions, with different task versions (e.g., Abrahamson, Shayan, Bakker, & van der Schaaf, 2016a), but importantly also outside of proportions, as the design rationale was applied to four other mathematical domains: angle (Petrick Smith, King, & Hoyte, 2014), area (Shvarts, 2017), parabola (e.g., Shvarts & Abrahamson, 2019) and sine function (e.g., Alberto, Bakker, Walker-van Aalst, Boon, & Drijvers, 2019; Shvarts, Alberto, Bakker, Doorman, & Drijvers, 2019a).[1] Given the generality of the genre, it has interesting potential to be extended to other mathematics topics, as well as be generalized outside of mathematics learning (see Bos, Doorman, Drijvers, & Shvarts, 2021 for an initiation in geography).

As a desirable outcome of this review, we aim to provide guidance for researchers, designers, and educators in effectively eliciting mathematical knowing with action-based design in the form of an actionable set of theoretically and empirically grounded design principles that have been substantiated within the collective research program. While design principles have been formulated before, namely, principles of moving in a new way, signification, and dialog (Abrahamson, 2014a; Abrahamson et al., 2020), these are largely based on the initial study and design for proportions. This review is the first article that provides an overview of how those design principles worked out within a decade of variation within the design genre. To answer the central question, *How do action-based embodied designs elicit mathematical knowing*?, we analyzed papers which were inspired by the initial action-based design for proportions (Abrahamson, 2014a), and explicated the general rationale and characteristics of the five domain-specific action-based embodied designs (Section 4.1), the learning sequence of these action-based embodied designs and the elicited learning processes (Section 4.2), and the influence of design variations on students' perceptions, actions, and verbalizations (Section 4.3).

Design principles for education are never general step-by-step procedures to be followed (Bakker, 2018); we therefore rather aim for them to be generative, and to inspire by means of general principles exemplified in concrete situations. We adopt a view on generativity that emphasizes facilitating third parties to create and implement new content unique to that system without additional help or input from the system's original creators. By assisting in creating new action-based designs, as well as passing on valued insights and knowledge from the action-based tradition, we hope to establish a strong foundation for the next generation of design for mathematics and beyond. In this way, we intend to contribute both to scientific knowledge about how embodied designs work, and to provide concrete advice to educational designers.

2. Theoretical background

Embodied cognition is an umbrella term for a range of theoretical views (e.g., Hutto & Myin, 2017) each highlighting an intricate connection between the body and cognition. Embodiment proposes that humans have more cognitive resources than the brain alone, and that the body does much of the work to achieve our goals (Wilson & Golonka, 2013). In the early days, Varela, Thompson, and Rosch stated that "the enactive approach consists of two points: (1) perception consist in perceptually guided action, and (2) cognitive structures emerge from the recurrent sensorimotor pattern that enables action to be perceptually guided" (Varela et al., 1991, p. 173). The embodied approach thus acknowledges the role of perception, action and the environment as critical constituents of cognition.

[1] See https://embodieddesign.sites.uu.nl/activity/ for a subset of the action-based designs.

The action-based design genre has explored a number of different theories within embodied cognition and beyond. Reviewing this history of ideas is beyond the scope of this article. We rely on ecological dynamics as the central theoretical ground for explicating design principles of the action-based embodied design genre in its current form (Abrahamson & Sánchez-García, 2016). Ecological dynamics blends dynamical systems theory (Smith & Thelen, 1996) and Gibson's ecological psychology (Gibson, 1979), and is used by sport scientists as a theoretical approach to study skill acquisition. From an ecological dynamics approach, a learner is considered as a complex system that needs to organize itself to produce functional movements, which they do through a process of systematic, emergent, non-linear, distributed and adaptive self-organization (Abrahamson & Sánchez-García, 2016).

The following sections summarize three key aspects of ecological dynamics relevant for action-based embodied learning: (1) nonlinear dynamics, motor problem, emergent functional states of coordination, (2) fields of promoted actions, and (3) attentional anchors. These aspects, that originated mostly in sports sciences, have been applied and adapted to mathematics education research (see Abrahamson & Trninic, 2015, where these movement science aspects were first applied to mathematics education). In the fourth section, we highlight the need to view the cultural-historical approach of Vygotsky (1987) as adjacent to ecological dynamics in conceptualizing embodied learning. Taking this combined view enables to extend skill acquisition beyond sport performance in cultural and language-rich areas such as mathematics.

2.1. Nonlinear dynamics, motor problem, emergent functional states of coordination

Traditionally, motor behavior is understood as an outcome of a nervous system treated as an input–output mechanism, much like a mechanical system (Reed & Bril, 1996). The basic premise is that units of motor action are either reflexive or the result of central nervous system commands. From this perspective, the central outcome in action learning is the exact repetition of a movement. The sequence of disembodied symbolical propositions within the brain would regulate this target performance. Some have placed question marks on the feasibility of central control or an idealized movement mechanism. As described by Kugler, Kelso, and Turvey, the human body has 792 muscles and 100 mobile joints, leading to so many degrees of freedom that these cannot be regulated under a centrally-driven command (Kugler, Kelso, & Turvey, 1982). Instead, their research shows that the nervous system is more accurately modeled as a complex non-linear dynamic system. They propose that the organizational principles of movement as "control" or "co-ordination" of this complexity "systematically dissipate degrees of freedom (Kugler et al., 1982, p. 6). The notion of *functional synergy*, coming from the pioneering investigations of movement control by Russian physiologist Nikolai Bernstein, appears to be key in solving this puzzle of overcoming complexity (Bernstein, 1967a): Functional synergies are a relatively stable coordination that spontaneously emerges in response to a functional demand in the environment at a low level of muscle coordination without top-down control.

Whereas linear pedagogies emphasize repetition of idealized motor actions, Bernstein (1996) viewed motor learning as a form of solving a *motor problem*, in which one arrives at a target solution within the environmental constraints, ad hoc and with any available resources. As he described, "motor skill is not a movement formula and certainly not a formula of permanent muscle forces imprinted in some motor center. Motor skill is an ability to solve one or another motor problem" (Bernstein, 1996, p. 181). Learning a particular action (such as chopping wood)

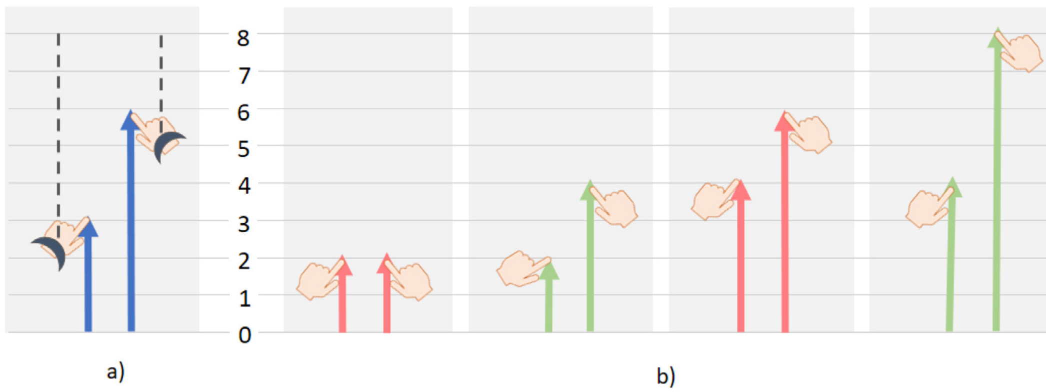


Fig. 1. Embodied tasks for proportions, with (a) passive inducement of a proportional movement (Abrahamson & Howison, 2010a) versus (b) discovering proportional movements with green and red color feedback (Abrahamson, 2014a; Howison et al., 2011). Green and red are shown as respectively light and dark grey in the black and white version. Numbers and lines are given for illustration purposes and are initially absent in students' interaction field.

evolves through *repetition without repetition*: One repeats "not the means for solving a given motor problem, but the process of its solution, the changing and improving means" (essay 6). Thus, one learns how to solve the motor problem, how to act properly across various instance, rather than which movements to perform (such as how the arms should move exactly). Therefore, instead of repetition of the same movement, variability of the movement needs to be fostered through variability in conditions. It is through this variability that *dexterity* develops— flexibility in adapting motor performance to diverse ad hoc contexts.

The constitution of a new sensorimotor coordination within the complexity of bodily and environmental constraints is based on transformations in a complex system of perception-action loops (Kelso & Schöner, 1988; Shvarts, Alberto, Bakker, Doorman, & Drijvers, 2021; Tancredi, Abdu, Abrahamson, & Balasubramaniam, 2021). The behavior unfolds in continuous interaction between the body and environment, constantly feeding forward anticipations of the sensorial input and feeding backward responses from the environment (Bernstein, 1967b). Tapping into this system of learning a new sensorimotor coordination through posing a sensorimotor problem with continuous interaction feedback is at the core of action-based embodied designs.

2.2. Field of promoted actions

While enactment in response to a motor problem is considered a self-organization process, this self-organization can be promoted in particular directions. In investigating the process of target-motor-performance facilitation in various cultures, Reed and Bril (1996) introduced the notion of *the field of promoted actions*. They describe the creation of environments and ecological conditions that promote culturally desirable performances. This concept was appropriated within action-based embodied design research to describe the process of designing interactive environments for children to promote embodied discoveries of mathematical patterns (Abrahamson & Sánchez-García, 2016; Abrahamson & Trninic, 2015), so called conceptual performances (Trninic, 2015; Trninic & Abrahamson, 2012, 2013). In line with continuous feedback that our bodies use to anticipate and receive input from our physical environment, designers incorporate continuous feedback on students' motions within an interactive activity, thus activating bodily resources and natural forms of exploration. A positive—usually green color—feedback is provided to motions that align with mathematical concepts, while negative feedback—usually red color—signals when motions are in misalignment (e.g., Abrahamson, 2014a). The (color) feedback is similar to the type of feedback learners receive while they learn, for instance, to ride a bicycle and maintain positive not-falling feedback while balancing. Aiming to maintain positive feedback, students discover a new sensorimotor coordination, which, with suitable guidance, can became a way of expressing mathematical concepts.

2.3. Attentional anchors as emergent perceptual structures

Given the active nature of action-based embodied designs, research from sport science helps to analyze what facilitates the learning of particular mathematical sensorimotor behavior. Of particular importance is the notion of *attentional anchors*, first proposed by Hutto and Sánchez-García (2015). While constraints are usually understood as embodied in concrete objects, they can also be immaterial, invisible, and even imaginary (Abrahamson & Sánchez-García, 2016). As sport science evidences, attentional anchors are imaginary perceptual structures or routines for orienting toward the environment that facilitate efficient performance of specific motor tasks. These mediating perceptual structures serve as a self-imposed motor constraint and emerge spontaneously in the development of motor control and coordination (Abrahamson & Sánchez-García, 2016). For example, while novice jugglers follow the trajectory of the balls, experts gaze at a central location within this pattern, leading to better throwing accuracy and correction of errors (Dessing, Rey, & Beek, 2012). These perceptual structures productively lower the degrees of freedom of the performance (Hutto, Kirchhoff, & Abrahamson, 2015): instead of moving multiple elements, learners now perceive task-critical elements of the environment as interconnected in the form of a new structure (Gestalt), which is easier to manipulate. Action-based embodied design research assumes that these emergent perceptual structures are key ontological achievements for mathematical understanding to be grounded in motor performance (Hutto et al., 2015). Students themselves install imaginary constraints within their field of promoted action, initially pragmatically, to facilitate motor performance. These attentional anchors then become the 'object to think with' (Abrahamson & Howison, 2010a) as students make these public through reflecting and discursive referencing and practices.

2.4. Cultural-historical approach

While ecological dynamics theory covers the understanding of how attentional anchors emerge in solving a motor problem, the further transition from fluent enactment to mathematical conceptualization requires bridging complex dynamic systems theory with a cultural-historical approach. While fields of promoted action trigger the emergence of sensorimotor coordination patterns these new forms need to be coordinated within cultural forms such as speaking, if they are to form scientific concepts and higher psychological functions (Vygotsky, 1987). Forming an inter-corporeal system with a more knowledgeable other in multimodal collaboration, a student *reflects* on and *describes* their sensorimotor experiences, thus gradually embedding and elaborating emerging perceptual structures and motor actions into mathematical discourse (Flood, 2018; Shvarts & Abrahamson, 2019).

3. Methods

3.1. Literature search and selection criteria

For this review, we intended to identify a complete set of publications on action-based embodied designs published from the initiation of the design genre until 2019 and written in the English language. Inclusion criteria were set for the *learning activities* within the literature corpus articles. Based on known design principles (Abrahamson, 2014a), learning activities were classified as action-based, if they would (1) pose a motor control problem (2) provide continuous sensory feedback on users' movement(s) (correct or incorrect) and (3) refrain initially from measuring artifacts (such as grids and numbers) in the action field. All three criteria needed to be met, to form a homogeneous set of action-based activities. The majority of other known embodied inspired learning genres did not meet these criteria (e.g., walking the number line Dackermann et al., 2017, TouchCounts Sinclair & Heyd-Metzuyanim, 2014, perception-based embodied designs Abrahamson et al., 2020). A subset met the criteria partially, namely the Cartesian graph task (Nemirovsky et al., 2013) (from which the action-based design genre emerged), and time-distance graphs (Duijzer et al., 2019). While these tasks also pose a motor control problem, they do not provide binary color feedback (correct or incorrect) on user's movement. We excluded these learning activities from the analysis, but classified them as closely related to action-based designs.

As a database, Google Scholar was used, facilitated with Harzing's Publish or Perish software (Harzing & van der Wal, 2008); this choice yielded the most extensive outcome, including conference proceedings, theses, and dissertations. No restriction was set on type of publication or methodology, to include qualitative and/or quantitative studies, designer reflections, frameworks, and theoretical work. Several keywords were considered. Keywords relating to the core characteristics of the learning activities, such as 'sensorimotor problem' and 'continuous feedback', yielded only a small subset of action-based papers because of the developing use of terminology within the design genre over time. In the end, the original name of the technological solution of the learning activity, "Mathematics Imagery Trainers" was the most useful identifying keywords, yielding the majority of publications reporting on action-based designs.

Specifically, a search with "Mathematical OR Mathematics AND Imagery Trainer" generated 176 unique results, after removal of duplicates (9). Publications not written in English (9), with no full text available (1), or only referencing action-based embodied designs (101) were excluded. After a split of one document reporting on two studies, 64 publications were deemed relevant for the purpose of the review. By snowballing the reference lists of these publications, 15 additional publications were found. The final selection for our analysis included 79 publications published between 2010 and 2019—a decade of variations on a theme.

3.2. Coding of designs

The coding process was facilitated by organizing all publications and coding information in an Excel database. Coding decisions and synthesis of the findings within the publications were frequently discussed with all authors, focusing on agreement between the authors. The 79 publications were coded for general information (authors, year of publication, journal, and type of paper (e.g., proceeding, peer-reviewed article), and for general content (empirical study, theory orientation, or design reflection). The majority of publications were conference symposia or proceedings (47) leading to peer-reviewed articles (18) or book chapters (7). The largest group reported on empirical findings with qualitative analysis; a smaller set were theoretically driven papers connecting to other fields or outlining the general pedagogy. Design reflections and principles, including unimplemented design ideas, formed the smallest group.

The 79 publications report on 15 empirical studies, with each assessing one or more action-based tasks. The empirical studies were generally coded for mathematical domain, sample characteristics, technology and multimodal trackers used, and duration of the learning activities. The collective data corpus involved over 400 students ranging from Grade 3 (age 8) to university, with interviews lasting between 15–70 min. The action-based tasks per study were further characterized by number of hands, motion pattern targeted, feedback elements, objects on the screen, and overlay of quantifying elements. This yielded a total of 15 different sensorimotor problems. These problems were operationalized using different technologies including PCs, tablets, and sensor-technologies, and studied using videography and hand and eye tracking. Per design, the empirical findings of aspects of students' or a tutor's multimodal behavior were summarized, covering action strategies, eye movements, verbal utterances, and gestures, (see Tables A.1 and A.2 in Appendix for an overview of studies, designs and empirical findings).

3.3. Corpus characteristics

Research into action-based embodied design was initiated in the domain of proportions. Design considerations for proportions were first described in 2008 (Abrahamson & Howison, 2008), in which small scale design-testing work was done towards the final design. The first empirical study of the type of design that is now known as action-based with students was conducted in 2010 (Abrahamson & Howison, 2010a). The study used Wii technology, and was situated in a controlled laboratory environment with one-to-one tutor guidance. While students generally worked as individuals, a few student pairs were also studied (eg., Abrahamson, Trninic, Gutiérrez, Huth and Lee, 2011). More than half of the publications (43) report exclusively on this first study in the domain of proportions, providing a substantial body of empirical findings, designers' reflections of prototyping towards the final design, and theoretical work (see Table A.1 in Appendix for a categorization of the publications reporting on this initial study).

Five studies repeated this study and implemented the design within different media (Abdullah et al., 2017; Ghasemaghaei, 2017; Rosen, Palatnik, & Abrahamson, 2018), added multimodal tracking technology such as eye trackers (Abrahamson et al., 2016a; Cuiper, 2015), or implemented it as one task within an embodied classroom intervention (Petrick Smith, 2012). Five studies remained within the domain of proportions, but varied on the original design in terms of the objects that were manipulated (Abrahamson et al., 2016a; Palatnik & Abrahamson, 2018) and/or in which direction these objects were moved (Duijzer, Shayan, Bakker, van der Schaaf, & Abrahamson, 2017), taking place in either a controlled laboratory environment with one-to-one tutoring and eye tracking (Duijzer et al., 2017) or in classroom settings (Lee, 2013; Negrete, 2013; Negrete, Lee, & Abrahamson, 2013).

Six studies applied the core design rationale from the action-based design for proportions to four other mathematical topics: angle (King & Petrick Smith, 2018; Petrick Smith et al., 2014), area (Shvarts, 2017), parabola (Shvarts, 2018; Shvarts & Abrahamson, 2018, 2019), and sine function: unit circle (Shvarts et al., 2019a) and sine graph (Alberto et al., 2019; Shvarts, Alberto, Bakker, Doorman, & Drijvers, 2019b). With the exception of angle, all mathematical domains have been studied with eye tracking in a controlled laboratory environment with one-to-one tutor guidance. For parabola (Shvarts, 2018; Shvarts & Abrahamson, 2018, 2019) and sine function (Shvarts et al., 2019a), dual eye tracking was used to capture perceptions of respectively student–tutor and dyadic interaction.

3.4. Synthesis

The formulation of design principles from a design-based research collective is a creative and abductive process. To guide this process we combined top-down and bottom-up approaches. High-level conjectures were organized in a conjecture map (Bakker, 2018; Sandoval, 2014) and initially based on those explicated from the initial study on proportions (namely, principles of moving in a new way, signification, and dialog (Abrahamson, 2014a)). Through critical reading, we collaboratively deliberated if and how each of the findings reported in the publications on design prototypes and final designs for proportions, angles, area, parabola and sine function, substantiated, enhanced and/or added onto these high level conjectures. The conjecture map was as such iteratively updated over several rounds (see the final stage of our conjecture map on Fig. 8). In Section 4.1 we focused on theoretically grounded principles and their implementation across domains (design), in Section 4.2 we conducted a more detailed analysis with the focus on the learning sequence as it appeared in initial designs and was developed for other domains, finally, addressing Section 4.3, we focused on variability of the findings across task variations, which could not be seen as universal and so aimed at explaining observed variations. The overall findings and design principles, were checked with Abrahamson, who initiated the action-based research program, and has been the main contributor to the field through the study of proportions. Abrahamson suggested a few alternative formulations (e.g., around the term ''visualisations''), but by and large underscored the findings. Discussions on both passivity and ready-made examples, and on the interaction with cultural artifacts (whether it bars or grid), brought about avenues for future research.

4. Results

Ways of mathematical knowing elicited by the various action-based embodied designs form the basis for formulating design principles. We start with explicating the characteristics of action-based embodied designs across all five mathematical topics (Section 4.1). Then we review the main findings on the learning sequence and student behavior elicited by these action-based embodied designs (Section 4.2). In Section 4.3, we move from the general characteristics of learning with action-based embodied design to an overview of variations of students' performances in response to motoric orientation and the type of objects that are manipulated.

4.1. Characteristics of action-based embodied designs

4.1.1. Mathematical concepts and sensorimotor coordination patterns

The initial action-based embodied design for proportions has been described as posing a motor problem whose solution is a sensorimotor coordination pattern that matches the target mathematical concept (Abrahamson, 2014a). The sensorimotor coordination pattern is a physical performance, in which students coordinate their two hands much like a choreographed bimanual gesture (Abrahamson, 2014a). To assess how this was operationalized beyond the original proportion task, we extracted and analyzed the sensorimotor coordination patterns used within the research program. Based on this, we composed Fig. 2, which illustrates seven sensorimotor coordination patterns matching the five mathematical concepts that have been studied thus far: proportion, angle, area, parabola, and sine function.

Proportions are traditionally presented numerically, such as in 1:2 = 2:4 = 3:6. The sensorimotor coordination pattern used in the action-based design for proportions consists of learners positioning their right hand twice as high as the left hand, or in continuous form, to move their right hand twice as fast as the left (see Fig. 2a, b). The sensorimotor coordination pattern thus adheres to the definition of proportions, albeit presenting it with unmeasured magnitudes rather than with numbers, shifting the pattern from a discrete to a continuous one (Boyer & Levine, 2015). This sensorimotor coordination pattern was initially operationalized along two vertical axes ↑↑, thereby mimicking a ratio table orientation (Howison et al., 2011) (Fig. 2a). Later, a similar performance pattern, which varied the right hand's direction to move along Cartesian axes ↑→, was used (Abrahamson et al., 2016a), thereby mimicking a linear graph representation of proportionality (Fig. 2b) (see Lee, Hung, Negrete, & Abrahamson, 2013 for the initiation of this movement direction).

In the action-based design for angle (Fig. 2c), students' arms and body enacted the cultural visualization of two rays with a common endpoint (King & Petrick Smith, 2018; Petrick Smith et al., 2014). In this task, students rotated their arms to form different magnitudes of angles. Whereas the task for proportion targets a single sensorimotor coordination pattern (e.g., 1:2), the design for angle targets multiple sensorimotor coordination patterns for several classes of angles within the same activity. That is, the sensorimotor coordination pattern for the class of acute angles was represented by arm rotations between (but not including) 0 and 90 degrees, the class of obtuse angles by arm rotations between (but not including) 90 and 180 degrees, and right and straight angles by arms positions of respectively 90 and 180 degrees. The four sensorimotor coordination patterns for angles were regardless of orientation: students could make the particular angles for example above their heads but also by their sides.

For the sine function, similar to the proportion task, two sensorimotor coordination patterns were developed: the unit circle and the sine graph variant. Both sensorimotor coordination patterns address the relation between distance traveled (the input of the sine function) and/or height (the output of the sine function). In the unit circle variant (Shvarts et al., 2019a) (Fig. 2f), the sensorimotor coordination pattern consisted of students moving two points along a unit circle such that they would be at the same height, representing an analog to equivalent sine values (output) for angles (input) in adjacent quarters. The sine graph variant (Alberto et al., 2019; Shvarts et al., 2019b) aimed to connect the unit circle with the sine graph (Fig. 2g), and used two consecutive sensorimotor coordination patterns (we will elaborate on this in the next section). The first sensorimotor coordination pattern (top part of Fig. 2g) matches the equivalence in input in both inscriptions –the left hand moves along the unit circle, while the right hand moves at the same speed or distance but along an x axis–, while the second sensorimotor coordination pattern (bottom part of Fig. 2g) matches the equivalence in output in both inscriptions –the left hand moves along the unit circle, while the right hand now has to move so that it is at the same height. Once the two sensorimotor coordination patterns are combined, the characteristic sinusoid form of the sine graph emerges.

Whereas in the aforementioned concepts the sensorimotor coordination patterns involved two hands, the action-based designs for area and parabola are unimanual. In these unimanual designs, students manipulated a partially fixed geometric object by moving one of its points freely in two dimensions, changing its form and size. In the task for rectangular area (Shvarts, 2017), students manipulated the tip of a rectangle whose sides and opposite vertex were fixed (Fig. 2d). The sensorimotor coordination pattern consisted of manipulating this rectangle so that its area is constant (e.g., 10 square units), which is done by manipulating the sides of the rectangle in such a way that the product of their magnitudes is constant (e.g., at 2.5 to 4, but also at 1 to 10, and every instance in between). Although that is beyond the scope of the design for primary school students, the trajectory of the manipulated point is hyperbolic with the formula $y = Area/x$.

The second unimanual design targets the concept of a parabola (Shvarts, 2018; Shvarts & Abrahamson, 2018, 2019), the basic quadratic function characterized by a U-shape. Its definition describes a collection of points that are equidistant from a line (the directrix vertically below) and a fixed point (the focus), forming a collection of isosceles triangles. In the action-based task for parabola (Fig. 2e), students manipulated a triangle's shape and size by controlling the tip of the triangle (C), while the second point (A) was fixed to the focus, and the third (B) ran along a vertical projection of the manipulated point. The sensorimotor coordination pattern that matches the notion of parabola was to manipulate the triangle so that CB (distance from point to directrix) = CA (distance from point to focus), or, put differently, that the triangle remains isosceles. The trajectory of the manipulated point is parabolic with the formula $y = x^2$.

4.1.2. Moving is not enough: posing a motor control problem

The designer's goal is *not* to have students produce these movements per se, but having them struggle to produce it (Abrahamson & Bakker, 2016). That is, the sensorimotor coordination patterns are not given through direct instruction or shown to the learners but are for them to discover or reinvent through indirect instruction (Abrahamson & Bakker, 2016; Abrahamson & Kapur, 2018). This is in contrast to many technological tools that directly constrain students' movements by providing them with ready-made solutions in the form of visualization (Abrahamson & Abdu, 2020). Empirical evidence for the importance of "solving dynamical interaction problems, rather than being taught directly how to move" (Abrahamson & Bakker, 2016, p. 4), comes from prior unsuccessful designs in the domain of proportion (see Trninic, Reinholz, Howison, & Abrahamson, 2010 for the most elaborate discussion on all prototypes).

In a first mechanical device (see Fig. 1a), students held on to two pulleys and were passively moved in a proportion of 2:3 (Abrahamson & Howison, 2010a, 2010b; Howison et al., 2011; Trninic et al., 2010). Their arms moved in a kinematic sense, but because students 'achieved' the sensorimotor task by simply hanging onto the pulleys, there was no need to initiate or monitor movements, thus providing limited ground to reflect.[2] As shown in Fig. 3a, tasking learners to *anticipate* the movement of the pulleys required initiation and yielded some feedback from the rope: a tug when the hands moved too slowly, and seeing the rope becoming loose when moving too fast (Abrahamson & Howison, 2010b). Findings showed that learners were tuning themselves to the "2:3 dance by synchronizing with the mechanism" (Abrahamson & Howison, 2010b, p. 3). Thus, passively inducing a proportional motor pattern did not elicit sufficient ground for mathematical knowing of proportions, but of synchronizing with rope mechanisms (proportional or not).

A key principle in the next iterations of action-based embodied designs was to have students move their hands freely to explore and take immediate agency (Trninic et al., 2010). Instead of passively inducing the motor patterns, the discovery of a pattern was operationalized by posing a motor problem with performance feedback. Motor problems require coordination, and it being a problem means there needs to be a goal for moving, that is, actions are goal-oriented (Bernstein, 1967a; Hutto & Sánchez-García, 2015; Wilson & Golonka, 2013). Continuous feedback from the environment is a core constitutive element for any enactment according to coordination dynamics approaches (Abrahamson & Sánchez-García, 2016) and lies at the core of imposing motor problems in the action-based designs. The design objective became for students to receive continuous sensory feedback from the technological system (such as a green color on the screen, or a particular sound) in response to their physical activity (Abrahamson, 2014a).

However, not all sensorimotor problems with continuous feedback seem equally convincing in eliciting specifically mathematical knowing. A digital technological analog to the mechanical pulley system was envisioned (Trninic et al., 2010). As shown in Fig. 3b, two targets would move proportionally on the screen, and students are tasked to follow these targets with two handheld cursors. Continuous accuracy feedback would be implemented, with green indicating the cursors being on the targets and red being off the targets. Because the hands were decoupled from the pulleys, students could no longer look on passively, and instead needed to actively and continuously initiate the hand-to-artifact spatial correspondence.[3] Although the digital pulley design was not empirically investigated, we expect that students would learn to coordinate their hand movements in correspondence with the ready-made trajectory. While such activity does result in the enactment of the target sensorimotor coordination patterns, students would likely achieve this by focusing on the distance between the targets and their hands: keeping this distance zero (see blue lines in Fig. 3b). Having students follow a target movement likely does not elicit sufficient ground for proportional reasoning, and instead trains students to become proficient in synchronizing with moving targets (proportional or not).

The key transformation towards the final action-based design was the insight to provide students with an opportunity to not only move freely but importantly, "without the restrictions introduced by moving targets" (Trninic et al., 2010, p. 1528). By removing the ready-made solution, but keeping students' hand cursors, continuous feedback now responded to students' absolute hand positions (Fig. 3c). Green feedback represents hand positions expressing a 1:2 proportion: all instances of the right hand being twice as high as the left, for example at 3 and 6 inches above the base, but also at 12″ and 24″ and all 1:2 proportional instances in between. Red feedback represents hand positions that are not part of the 1:2 proportion class, such as at 4″ and 6″ which matches with a 2:3 proportion. The interaction mechanism is thus set up so that many distinct physical inputs yield the same output, a many-to-one function (Abrahamson, 2014a). To maintain the target feedback state, the students need to discover and perform the sensorimotor coordination pattern matching

[2] Personal correspondence, August 19, 2021.

[3] Personal correspondence, August 19, 2021.

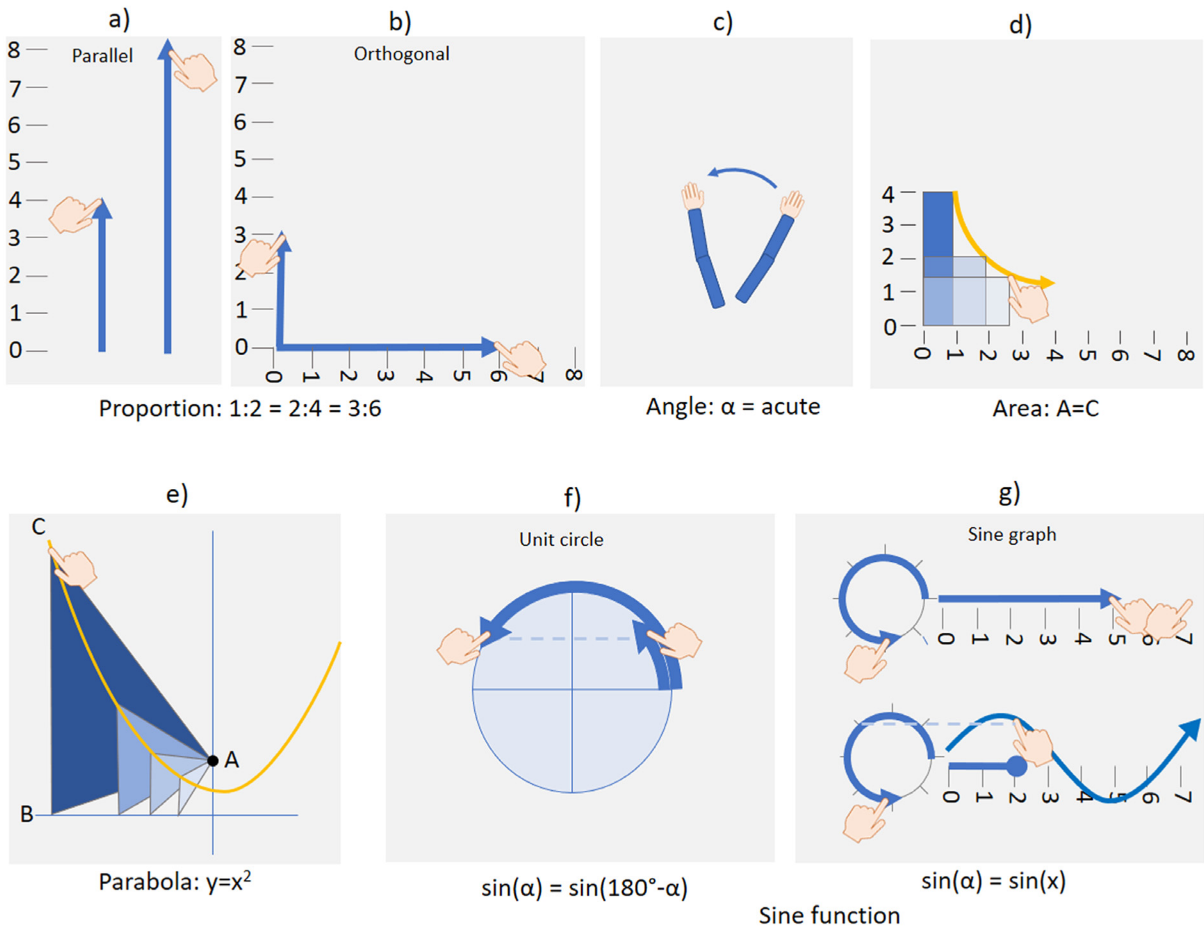


Fig. 2. Sensorimotor coordination patterns targetting the mathematical concepts of proportion (a,b), angle (c), area (d), parabola (e) and sine function (f,g) (Numbers, letters, yellow and dashed lines are given for illustration purposes; they are absent in the early qualitative stage of the designs) . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

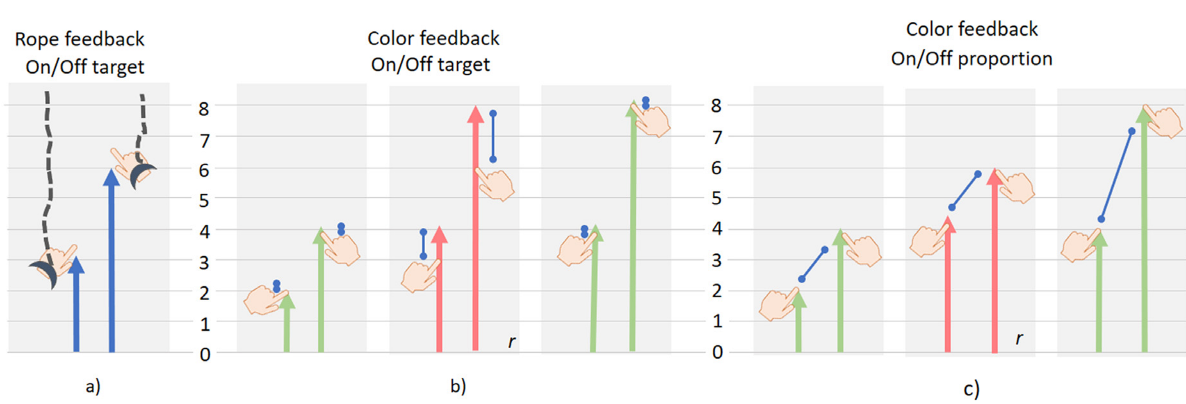


Fig. 3. Different interaction-designs for proportions; (a) a mechanical pulley system that moves students hands proportionaly; (b) a digital pulley system with green and red color feedback in which students follow a proportional movement and (c) an action-based design with green and red color feedback in which students discover a proportional movement. Green and red are shown as respectively light and dark grey in the black and white version.

proportionality; in a sense, the students enact an embodied mathematical function (Abrahamson, 2014a). As will be elaborated in Section 4.2, students started to realize that the relation between the hands' respective positions (not how they relate to a pulley or a target) was a critical factor for achieving positive green feedback (Abrahamson & Trninic, 2011), with statements like “they have to be a certain distance away from each other for it to turn green”, and later “keep doubling it” (Abrahamson & Trninic, 2011, p. 5). The motor problem thus elicited students' attention to a critical property or relation of proportionality itself. While green feedback serves as a goal state in sensorimotor problems, the emerging coordination grounds descriptions of the experience that can be expressed in mathematical terms. Thus, posing a motor problem with continuous feedback signaling the accuracy of students' sensorimotor coordination pattern elicited sufficient ground for mathematical knowing of proportions; this problem design was ultimately adopted for proportions and forms the key design feature of all action-based designs.

The ineffectiveness of “showing how to move” was further substantiated in the prototyping for the action-based design for the sine function. To recapitulate, the sine graph variant (Fig. 2g), aimed to elicit correspondence between the unit circle and the sine graph in terms of distance traveled (input of the sine function) and height (output of the sine function). In the first operationalization of the design, students moved their left hand along the unit circle and the right hand along the curve of the sine graph (see bottom part of Fig. 2g). The input of the sine function was enacted indirectly, by a segment that connected the origin of the graph with the *x* coordinate of the manipulated point on the curve (Alberto et al., 2019). Students were tasked with finding and keeping green, and similar to the proportion task they achieved success and sufficient ground for reflecting. However, while students reported they kept the two points at the same level –representing correspondence between the outputs in both inscriptions, they ignored the arc-length and the *x* coordinate segment on the sine graph –representing correspondence between the inputs in both inscriptions. Some students even erroneously reasoned that the arc length on the unit circle was equal to the length of the curve (Alberto et al., 2019). The students thus questioned only what was under their direct manual control, not the "visually outsourced" relations. Taking away enactments thus risks taking away opportunities to conceptualize relations. Therefore, in the final tasks for the sine graph (Fig. 4g), each of the two sensorimotor coordination patterns was targeted separately first and combined after (Shvarts et al., 2019b).

As condensed in Fig. 4, the action-based embodied designs across all five mathematical domains foster students' discovery of the sensorimotor coordination patterns matching the target concept through continuous feedback from the environment. In each of these action-based designs, the numerous distinct physical inputs yield the same positive feedback output of green or another color (the many-to-one function); to maintain the target positive feedback state, the students need to discover how to enact the embodied mathematical function (Abrahamson, 2014a). The positive green feedback thus links together a phenomenological class of equivalent hand positions (Trninic & Abrahamson, 2011), thereby expressing an embodied version of the equal sign, whether for different hand heights with the same proportional relationship (e.g., Trninic et al., 2010), rectangles with the same surface area (Shvarts, 2017) (Fig. 4d), or triangles which are all isosceles (Shvarts, 2018; Shvarts & Abrahamson, 2018, 2019) (Fig. 4e). Whereas most tasks use green-to-red gradient feedback, in the design for angle (Fig. 4c), four different mathematical functions were incorporated and four different colors were used, i.e., pink, yellow, light blue and dark blue for respectively acute, right, obtuse, and straight angles (King & Petrick Smith, 2018; Petrick Smith et al., 2014).

Although in the majority of action-based solutions, color is used as continuous feedback on students' movements, other options have been tried and considered. Ghasemaghaei and colleagues, for example, used smileys and applause as positive feedback (Ghasemaghaei, 2017; Ghasemaghaei, Arya and Biddle, 2015; Ghasemaghaei, Arya, & Biddle, 2016; Ghasemaghaei, Biddle and Arya, 2015). Others have proposed haptic instead of visual feedback, making the tasks eligible for students with visual impairments (Abrahamson, Flood, Miele, & Siu, 2019): a rattle indicates incorrect performances (equivalent to the red color), while no rattle means a correct performance (equivalent to the green color). While this widens the diversity of learners that could benefit from action-based designs, thus far no comparative studies have been conducted on the effect of different types or modalities of feedback on students' mathematical knowing.

4.2. The action-based learning sequence and findings

In this section, we focus on typical learning sequences and the elicited learning processes within action-based embodied designs, using illustration from all five mathematical domains. Despite targeting different sensorimotor coordination patterns, the learning sequences of action-based embodied designs generally use similar stages. Students first interact with the design in a qualitative stage and then in a quantitative one. Within each of these stages, students start with an acting step followed by a reflecting step. In this section, we focus first on the acting step in the qualitative stage and describe the main course of sensorimotor transformations as students develop a new coordination (4.2.1). We then showcase the interaction between the student and the tutor in a second reflecting step, in which students describe their newly found coordination with words and gestures (4.2.2). Last, we discuss the acting and reflecting step in the quantitative stage, in which students appropriate measuring artifacts into their sensorimotor problem solving and further conceptualizations with the tutor (4.2.3).

4.2.1. The acting step: perception guides action

Despite interpersonal variations, and some versions of tasks being more predictable than others (Abrahamson et al., 2016a), students solve the motor control problems along similar developmental paths. In all action-based embodied designs, students start by exploring the space, moving their hands or the object without a clear pattern. In the design for proportions, students wave their hands up and down, keeping their hands at the same height or alternating them (eg., Abrahamson & Howison, 2010a; Abrahamson, Lee, Negrete, & Gutiérrez, 2014), while in the design for parabola they move the triangle across the screen in all sorts of directions (Shvarts & Abrahamson, 2019). Through these explorations, students haphazardly find green feedback and lock their hands. According to a protocol, the tutor stimulates them to find other green positions, and with sufficient green positions, to move between two instances of green, also called a *dynamic conservation* task. Essentially, the tutor asks the student to move in a continuous way and enact the embodied mathematical function (Abrahamson, 2014a), which is a challenging task requiring a new coordination.

While students are perfectly able to perform the new sensorimotor coordination pattern, it generally does not occur spontaneously in the problem situation, and so initially students attempt a simpler or *default coordination* pattern that fails to solve the interaction problem, resulting in negative, red feedback (Abrahamson, 2014a). A common strategy in the design for proportions is that students find a small 1:2 proportion at the bottom, with one hand being for example "one unit" above the other (i.e., double as high; Fig. 4a). As students elevate their hands, they keep this unit fixed (e.g., Abrahamson et al., 2014). Interestingly, their qualitative enactment of a constant function ($R = L + k$) echoes a well-known quantitative error reported in the proportion literature: students reason additively and think that 1:2 = 2:3 = 3:4 (e.g., Lamon, 2007). This simpler or default interaction pattern is the physical analog of what some scholars call misconceptions about the content (Abrahamson, 2014a). In other action-based designs, simpler strategies were also tried first. In the design for area, students transformed the rectangle by moving its vertex horizontally or vertically manipulating either the rectangle's length or width (Shvarts, 2017). In the design for the sine function, students should move both their hands at the same speed for corresponding inputs in the unit circle and the sine graph (Fig. 4g). While the same-speed strategy is one of the first strategies students try in the design for proportions, it did not appear spontaneously in the sine function task. Students' default

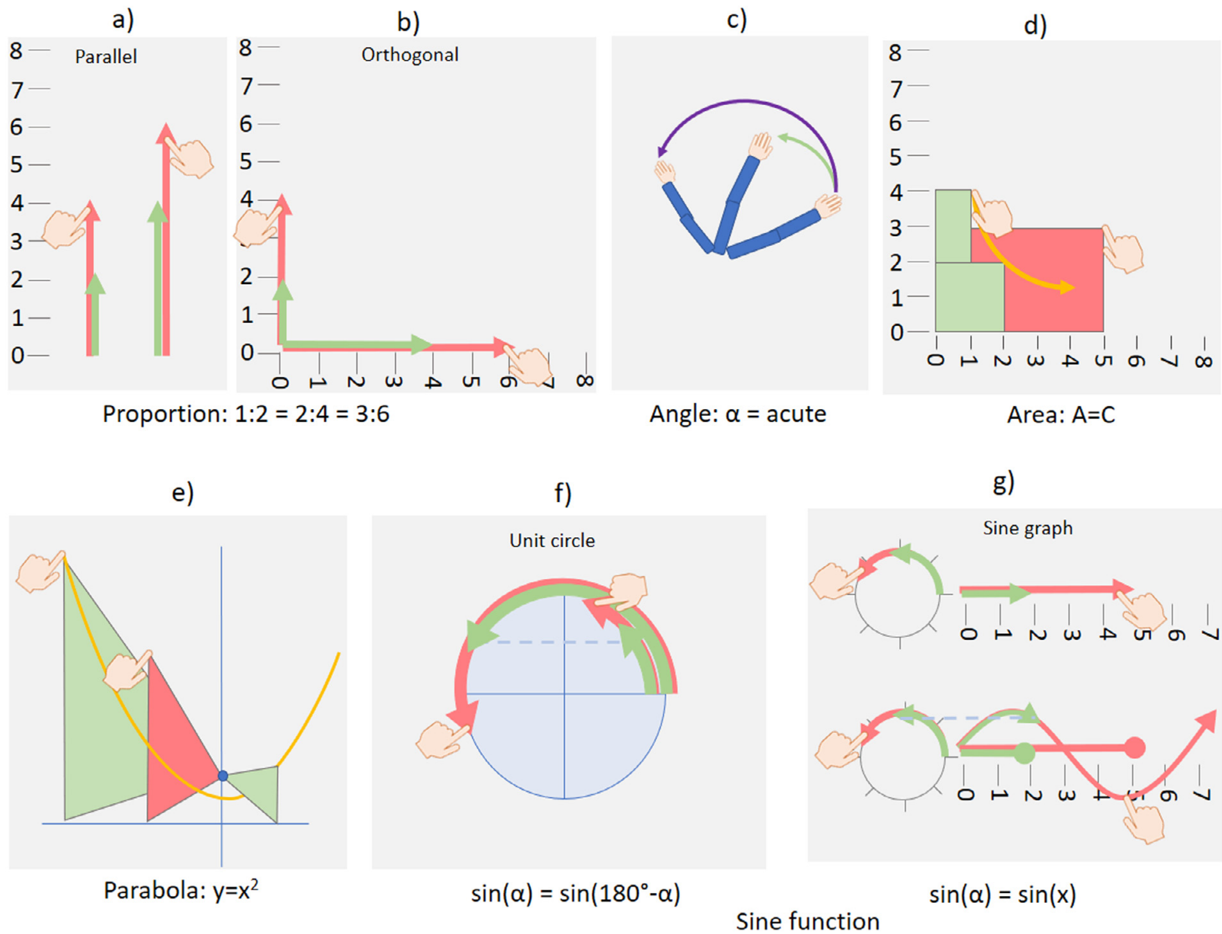


Fig. 4. Sensorimotor control problems with continuous green and red color feedback to assist discovery and practice of sensorimotor coordination patterns targetting mathematical concepts. Green and red are shown as respectively light and dark grey in the black and white version. Numbers, letters, yellow and dashed lines are given for illustration purposes; they are absent in the qualitative stage of the learning sequence.

tendency was instead to move the point on the *x* axis of the sine graph slower than their hand on the unit circle, as if affected by the horizontal displacement of the point along the circle (Shvarts et al., 2019b, 2021).

The negative red feedback in response to students' default and other incorrect strategies however, forces students to reconsider their strategy. In the end, all learners manage to solve the sensorimotor problem of keeping the screen green by discovering how to move in line with the targeted concept. While for an outside viewer, the students' hand motions might be salient, students actually attended not to their hand(s) but to the relation between their hands, in bimanual coordination, or to properties of the moving geometric objects on the screen, in unimanual problems. This was shown using eye-tracking, which was used to study sensorimotor transformation for all mathematical topics, with the exception of angles. Gaze data consistently showed that students gain control over the environment (maintaining green feedback) because they start to see in a new way (Abrahamson et al., 2016a; Alberto et al., 2019; Shvarts, 2017; Shvarts & Abrahamson, 2019; Shvarts et al., 2019a, 2019b). As described by Abrahamson and Bakker (2016), while during exploration students gazed at their fingertip(s), attention later includes "non-stimuli", locations with no discernable contours for an outside viewer. Students' gazing at these loci, better known as attentional anchors, facilitates their competency of controlling the environment. That is, through controlling this object students achieved sensorimotor coordination. Importantly, students describe particular properties of these attentional anchors they perceive (their length, angularity, etc.) as their approach to achieving green. Attentional anchors always occur prior to verbalizations (Shayan, Abrahamson, Bakker, Duijzer, & van der Schaaf, 2015, 2017).

We have condensed the attentional anchors emerging within the mathematical domains in Fig. 5. In the designs for proportions students initially focused on their hands, but then shifted attention towards a location between their hands, which improved their performance (Abrahamson et al., 2016a; Abrahamson, Shayan, Bakker, & van der Schaaf, 2016b; Cuiper, 2015; Shayan et al., 2017) (Fig. 5a, b). In the design for area (Shvarts, 2017), students first gazed in horizontal or vertical directions, controlling the width and length of the rectangle, and then shifted towards a location in the center of the rectangle (Fig. 5c, area). In the design for parabola, students initially gazed at the point of the triangle they were manipulating and then gained more control, as their attention shifted towards an attentional anchor located along the center line (median) of the triangle (Abrahamson, 2019; Bakker et al., 2014; Shvarts, 2018; Shvarts & Abrahamson, 2018, 2019; Shvarts et al., 2019a) (Fig. 5d). In both the unit circle as well as the sine graph task, students first alternated attention between their hands, and later shifted towards a location between their two hands (Alberto et al., 2019; Shvarts et al., 2019a) (Fig. 5e, f).

The notion of attentional anchors is one of the key behavioral processes emerging from action-based embodied interaction and has been described as a missing theoretical link between an action and a concept (Hutto et al., 2015). While the emergence of attentional anchors is well known within the sport sciences (Hutto & Sánchez-García, 2015), it was not known in mathematical problem solving. Connecting mathematical knowing with the evolutionary old system of perception–action control is beneficial, as insights from ecological dynamics and non-linear pedagogy developed and applied in sports might be generalized to learning mathematically relevant performances (Abrahamson & Sánchez-García, 2016). However, while in sports the physical performance itself is the ultimate goal, in the action-based embodied designs, the fluent enactment and the emerging attentional anchors are a necessary but essentially intermediary goal. They serve as the ground for the reflecting step which follows the acting step, in which mathematical discourse about these emerging structures develops.

4.2.2. The reflecting step: sharing performances and emergent perceptual structures

An essential part of the learning sequence in action-based embodied designs is that students reflect on their sensorimotor solutions in collaboration with a tutor. While attentional anchors improve students' performances in the acting stage, students might not be aware of these dynamical patterns, nor their significance. Describing embodied experiences is as important as the embodied experiences themselves, as it is in this reflective stage that students are steered towards expressing their solution in cultural ways. Empirical evidence of the importance of reflecting on sensorimotor solutions has been shown with an experimental study with the unit circle task for the sine function (Shvarts et al., 2019a). Activities limited to the acting step of the action-based embodied design sequence, without further collaborative reflecting, resulted in students not developing their sensorimotor coordination into mathematical notions for further problem solving (Shvarts et al., 2019a).

Within action-based designs, students are asked to think of a rule that keeps the screen green. Attentional anchors form the ground for students' reflections, but a key challenge in the transition from sensorimotor movement to mathematical discourse is that of expressing linguistically one's experiences and solutions. A student engaged in the parabola task searched for a way to describe the isosceles triangle that led to the parabolic shape; she felt the form but did not know how to express it: "The triangle is obviously… Oh, I am bad with geometrical terminology" (Shvarts & Abrahamson, 2019, p. 35). The tutor supported the student to continue, and she described the triangle as "it is not equilateral… but isosceles", "it means … that it has two sides of equal length" (Shvarts & Abrahamson, 2019, p. 35). Students thus use the attentional anchor located at the median of the triangle (see Fig. 5d) as a source to categorize the triangle as isosceles. In trying to find a way of expressing their embodied experiences, students might appeal to previous sensorimotor experiences outside mathematical practice, which they find similar to the emerging bimanual coordination. Making rectangles with the same area felt "as if I am making it from plasticine", "I can make it longer, but the amount of plasticine is still the same" (Shvarts, 2017, p. 268). In time, the description starts to take on a more formal shape, "it has a constant size, not size... constant volume, or no... how to call it, a constant area" (Shvarts, 2017, p. 268). The new perceptual structure thus enabled the student to see the rectangle as a whole figure, with area constancy as its prime feature. Similar patterns were found in the unit circle task for the sine function, in which two students had to work together: while one student expressed their solution in rather vague terms such as "move that way… just keep going that way, slowly", the other used mathematically relevant descriptions about angles, "we want this angle between the middle line and our point to be the same" (Shvarts et al., 2019a, p. 662). In the angles task, the various angle types each had their own description. While a straight angle was described as "I put my arms straight… It's almost like if you're doing something jazzy with your hands" (Petrick Smith et al., 2014, p. 105), for obtuse angles a reference was made to birds flapping their wings, and the obtuse angle being mid-flap. Within this angle task it was shown that the level of physical engagement influenced their reflections. Concretely, to describe a right angle, students who were themselves physically engaged in solving the task used more spatial language, "If I have one arm kind of straight and the other one pointing down", while students who only observed another student solving the task were more likely to use metaphors, "It looks like a person dancing" or "she is saying hi to somebody" (King & Petrick Smith, 2018, p. 585).

In the reflecting stage, students might also refer to their correct sensorimotor enactments in incorrect ways. Within the task for proportions, an attentional anchor emerges between students' hands, and they consistently refer to the distance between their hands as the object of their reflection. Students revisited their simpler or default strategy from the acting stage in their reflection with statements such as "I think what's going on is that they have to be the same far … the same distance away from each other" (Trninic, Gutiérrez, & Abrahamson, 2011, p. 276). Thus, even when students managed to maintain positive green feedback by moving their right hand twice as fast as the left, and in some cases even gestured this enactment correctly mid-air, they could nevertheless report that the distance between their hands remains constant with elevation (see examples in Charoenying & Trninic, 2011; Palatnik & Abrahamson, 2018; Reinholz, Trninic, Howison, & Abrahamson, 2010). In collaboration with the tutor, students in the end reflected that the "distance between their hands should increase with elevation". Similarly, mismatches between enactments and speech have been shown in the sine graph task (Alberto et al., 2019), in which a student correctly varied the speed on the sine graph curve to correspond horizontally with the point on the unit circle, but described her solution incorrectly: "If I move my hands at the same speed I will go the same distance" (Alberto et al., 2019, p. 3). Thus, physical performance can be ahead of conscious awareness in speech, and tutors are essential in providing feedback in such instances.

Tutors are a key factor in steering students' transition from embodied experience towards normative cultural expressions (Abrahamson, Gutiérrez, Charoenying, Negrete, & Bumbacher, 2012a). Collaborative work with a more knowledgeable other establishes shareable and culturally normative ways of describing sensorimotor experiences. Micro-ethnographic conversational analysis revealed the complexity of the multimodal processes between a student and a tutor, as the tutor uses several tactics (Abrahamson, Gutiérrez, Charoenying, Negrete, & Bumbacher, 2012b) aiming to connect everyday descriptions of the student's experience with scientific discourse relevant for mathematics (Flood, 2018). The participants manage "transforming an initially vague reference into a ratified, mutually established mathematical object" (Flood, Harrer, & Abrahamson, 2016, p. 122) by recognizing a student's expression as prospectively indexical in terms of referring to an object-in-progress that is yet to be established between participants. This transformation is happening through a process similar to bootstrapping: the tutor's development of an utterance in one modality relies on the repetition of the student's verbal or gestural utterance in another modality (Flood & Abrahamson, 2015), thus forming an example of multimodal re-voicing (Flood, 2018). Tutors might also respond to ambiguities in students' descriptions in creative ways; one tutor, for example, asked a student how to describe their solution "if you were speaking to someone on the phone and they can't see what's going on" (Flood et al., 2016, p. 124), thereby steering the students towards the disciplinary practice of describing mathematical phenomena in a context-independent manner.

Importantly, the tight collaboration between a student and a tutor is idiosyncratic and bidirectional, as participants mutually

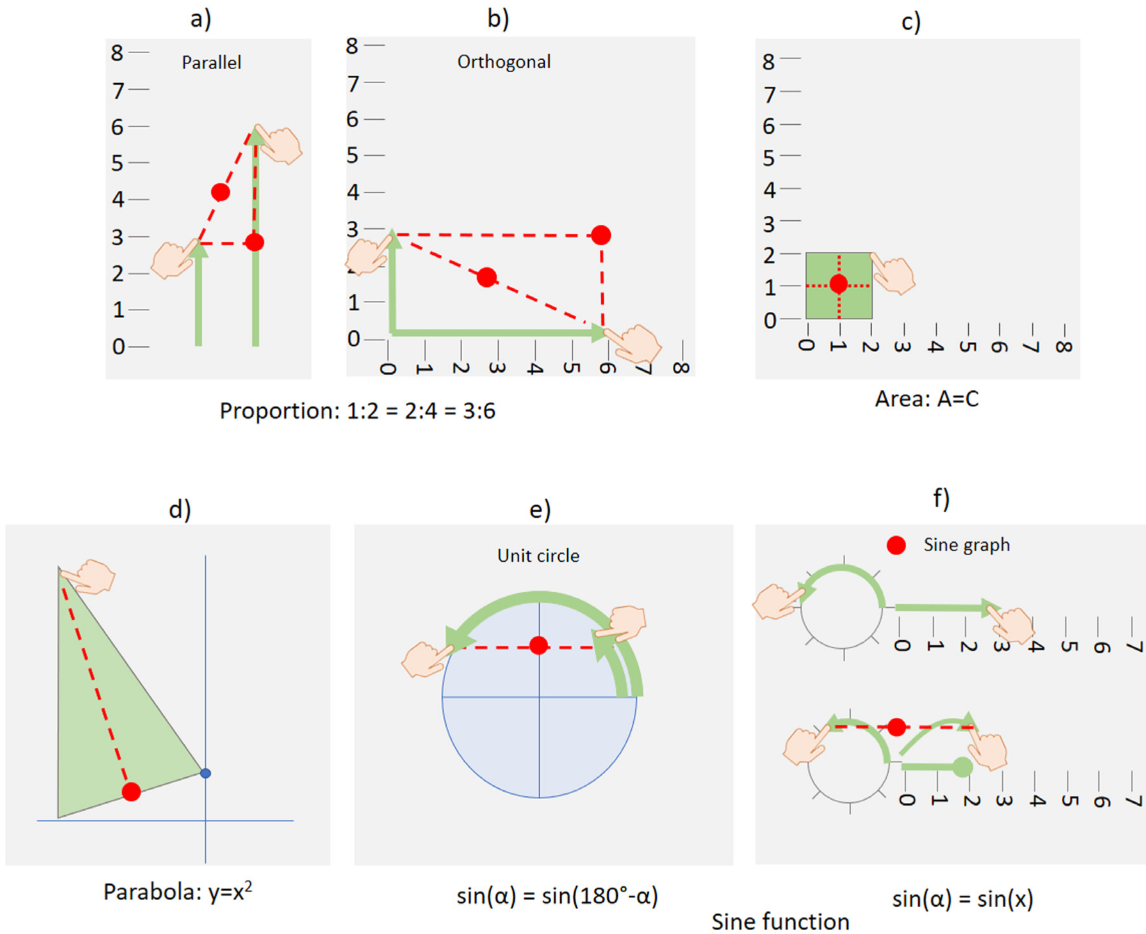


Fig. 5. Attentional anchors when solving sensorimotor control problems. (Numbers and axes are given for illustration purposes; they are absent in the qualitative stage of the learning sequence).

and contingently adjust to each other's behavior (Flood et al., 2016), as they gradually progress from the sensorimotor experience to everyday and then to scientific discourse (Flood, 2018). It is a multimodal conversation between participants about new perceptual objects (attentional anchors) which are invisible to the naked eye and so materialized in a rich network of pointing, gesturing and describing. Often, the tutor will repeat and enhance the student's gesture–speech reflections to steer them into culturally meaningful directions. Flood and Abrahamson (2015) exemplified how tutors select and refine students' verbal and gestural descriptions within the task for proportions. Asked to explain the relationship between her two functionally equivalent strategies, *one hand being always double the other* and *one hand moving twice as fast*, a student explained, "If you're going to do it like, at the same time, that one would have to go faster to like end at the same time; that one would have to lift higher" (Flood & Abrahamson, 2015, p. 6). While explaining, two gestures were used by the student, the first a same-height strategy (which would result in red feedback) and the second a changing-distance strategy (which would result in green feedback). The tutor re-enacted only the accurate gesture, and further elaborated on the student's description with a more generalized and culturally accepted description of "it has more ground to cover" (Flood & Abrahamson, 2015, p. 3).

Additional methodological means of synchronous eye tracking of a student's and tutor's foveal attention allowed for tracing the coupling of sensorimotor processes between two bodies. The contingency of interaction appeared to reach coincidence in the dynamics of visual behavior up to milliseconds, as the student would anticipate or re-ask the tutor's prompts and gestures, thus synchronizing visual attention between two participants (Shvarts, 2018). The tutor not only follows the student's performance but anticipates the emergence of new perceptual structures during the student's sensorimotor enactment. Repetitive eye-movements in the tutor's behavior reveal evidence of tutors' attentional anchors, even though they themselves do not solve the sensorimotor problem. These attentional anchors are coupled with the student's motor performance and precede the moment when students develop these new perceptual strategies. Apparently, such anticipation of students' strategies helps the tutors distinguish an optimal moment for intervention when a student has sufficient sensorimotor ground for moving towards describing an experience and thereby enculturating it (Shvarts & Abrahamson, 2018, 2019), a so-called *micro-zone of proximal development*. Such tight coupling between a student's and a tutor's sensorimotor performance was found in the studies for parabola (Shvarts, 2018; Shvarts & Abrahamson, 2018, 2019) and for the unit circle (Shvarts et al., 2019a). Theoretically, it allows considering a student and a tutor as joined into an inter-corporeal unified system, in which the teaching/learning task is distributed between two bodies (Shvarts & Abrahamson, 2019). This system gradually develops towards cultural forms of perception and discourse within this unified system; and then within a student's perception–action system, as it becomes differentiated from the tutor (Shvarts, 2018).

4.2.3. Measuring artifacts: grids, protractors, numbers, and variables

A key feature of action-based embodied designs is to empower students' problem solving by incrementally introducing cultural artifacts (including mathematical symbols) into the interaction space. In this quantification stage, students solve the same motor control problem, but in the presence of measuring artifacts, which shifts students' perceptions and reasoning "in nuanced yet conceptually critical ways" (Abrahamson & Howison, 2010a, p. 4). Depending on the mathematical domain, different types of artifacts were added, including a grid (proportions, area, sine graph), a protractor or circle marks (angles, sine graph and unit circle), projections towards the axes (parabola and unit circle) as well as mathematical symbols such as numbers (proportion, angle, area, sine graph, and unit circle) and/or letters (parabola, sine graph and sine function). Demonstrated first for the task for proportions, the utility of these artifacts depends on the student's goal, and a variety of uses has been reported (Abrahamson, Gutierrez, Lee, Reinholz and Trninic, 2011; Abrahamson, Trninic et al., 2011; Gutiérrez, Trninic, Lee, & Abrahamson, 2011). To categorize these various artifact behaviors, the dual construct of hooks-and-shifts was formulated (Abrahamson et al., 2012a; Abrahamson, Gutierrez et al., 2011; Abrahamson, Trninic et al., 2011; Gutiérrez et al., 2011), and later fine-tuned within the ecological dynamic framework (Abrahamson & Sánchez-García, 2016).

In the proportion task, a grid was added with the intention to support students' articulation of quantitative descriptions (see Fig. 6). Students were first attending to this artifact as a feature that optimized their grip on the world (Abrahamson & Sánchez-García, 2016), with the grid lines forming a collection of welcome location anchors.[4] Students tended to restrict their search for green to integer-unit locations upon the gridlines. Students' bimanual motor actions thus transitioned from simultaneous in the qualitative stage, to sequential in the quantification stage with turn-taking of the hands and elevating them to a discrete number or units. The grid is a pedagogically productive constraint introduced into the student–screen relation: it readily affords the existing motor-action coordination, yet it transforms the perception–action loops and ensuing descriptions (Abrahamson & Sánchez-García, 2016). That is, whereas in the qualitative stage, students used continuous–qualitative descriptions, such as "the higher you go … the bigger the distance", in the quantitative stage, this is formalized toward discrete–qualitative descriptions, such as "so maybe the higher you go, the more boxes it is apart" (Abrahamson, Trninic et al., 2011, p. 69) (Fig. 6a). As described by Abrahamson and Sánchez-García (2016), the "it", referring to the interval between their hands, is now being materialized or reified upon the gridlines. While the interval was first a personal and private attentional anchor, it now appears in the public domain in the form of an externally present, stable, and publicly inspectable entity (Abrahamson & Sánchez-García, 2015, 2016). Because of these properties, the grid also better affords to distinguish between the fixed-distance and changing-distance strategy, and especially in the case of dyadic work with contrasting views, the grid served as a means of arbitration (e.g., Abrahamson, Trninic et al., 2011).

Some students did not (immediately) appropriate the grid in their problem solving (Gutiérrez et al., 2011). There are several reasons why students might not appropriate measuring artifacts, some of which are student- and others which are tutor-dependent (see Gutiérrez et al., 2011 for the most elaborate description on reasons not to appropriate artifacts). First of all, students should be familiar with the added artifact to be able to benefit from its affordances. Further, students need to have discovered a working solution, which has been confirmed and is under evaluation the moment the artifact is overlaid onto the activity space. Last, the way students orient towards the task should align with the artifacts presented. For example, one of the students referred to the length of the line between her hands in diagonal terms, for which the horizontal gridlines offered did not afford discretization (Abrahamson et al., 2012a; Gutiérrez et al., 2011). Including measuring artifacts is thus a matter of timing (a working idea has to have formed), familiarity with the artifact, and a match between the artifact and students' strategies, all in which the tutor plays a key role (Abrahamson et al., 2012a).

While the prior strategies described how the grid smoothly enhanced students' qualitative sensorimotor solution (discretizing their attentional anchor shown in Fig. 6a), over time, other affordances of the grid become salient, and new sensorimotor solutions can emerge quite abruptly. With similar turn-taking strategies, students in the proportion task discursively shifted with the grid, and new, more established mathematical forms of speaking were elicited (Abrahamson, Gutierrez et al., 2011; Abrahamson, Trninic et al., 2011; Charoenying & Trninic, 2011; Gutiérrez et al., 2011). While, before, the relation between the hands was central, the grid now draws attention to the individual hand locations, motions, and respective distances (Fig. 6b). In the *a*-per-*b* strategy (Abrahamson et al., 2014; Abrahamson, Negrete, Lee and Gutiérrez, 2012; Abrahamson, Trninic et al., 2011), students described their hand-movements by their respective unit-rates: "for every box he goes up – you have to go up half" (Abrahamson, Trninic et al., 2011, p. 71) or "the right hand always goes up two and the left hand goes one" (Charoenying & Trninic, 2011, p. 4). Students thus shifted towards a discrete–quantitative description of the mathematical properties (Abrahamson, Gutierrez et al., 2011; Abrahamson, Trninic et al., 2011; Gutiérrez et al., 2011). Taken together the grid artifact discretizes students' actions, and in the process "became a frame of reference for establishing and articulating within quantitative systems (Abrahamson & Sánchez-García, 2016, p. 204). These *a*-per-*b* perceptions and descriptions have been well established and concretized within the mathematical culture, and so come close to cultural conventions.

When numbers were added to the grid in the task for proportions (Fig. 6c), students attended not to the hands' individual steps, but to the height of each hand above the baseline, thereby recruiting their arithmetical knowledge. This shifted the reflection of the solution strategy once again: Given one numerical position, the other could be determined by calculation, either through addition, "One plus one is two—two plus two is four" (Gutiérrez et al., 2011, p. 25), or multiplication, "You double the number the left one is on, and you put the right one on that number" (Gutiérrez et al., 2011, p. 25). Students realized "it's always half!" (Abrahamson, Trninic et al., 2011, p. 75), thereby describing the multiplicative constant of the 1:2 proportion. Others associated this multiplicative constant to personal experiences outside of the mathematical domain: "If it was a car race, then the one on the right would be twice as fast" (Charoenying & Trninic, 2011, p. 4).

Similar to the qualitative stage, collaborative reflecting with tutors also plays an important role in the quantification stage. Flood, Harrer, and Abrahamson illustrated how a student, working on the proportion task with the grid in place, provides an ambiguous speech–gestural demonstration of the insight that "each time it's increasing the square" (Flood et al., 2016, p. 124). Whereas the student referred to "it" as the distance between the cursors (changing interval strategy), the tutor interpreted it as meaning the cursors themselves (*a*-per-*b* strategy). In response to the misunderstanding, the student provided a richer and clearer performance in speech and gestures to direct the tutor's attention: The student manually carved out a space close to the ground, "this is one", after which the space was lifted and increased, "so that would be two", and again lifted and increased, "then it keeps increasing" (Flood et al., 2016, p. 127).

[4] Personal correspondence, August 19, 2021.

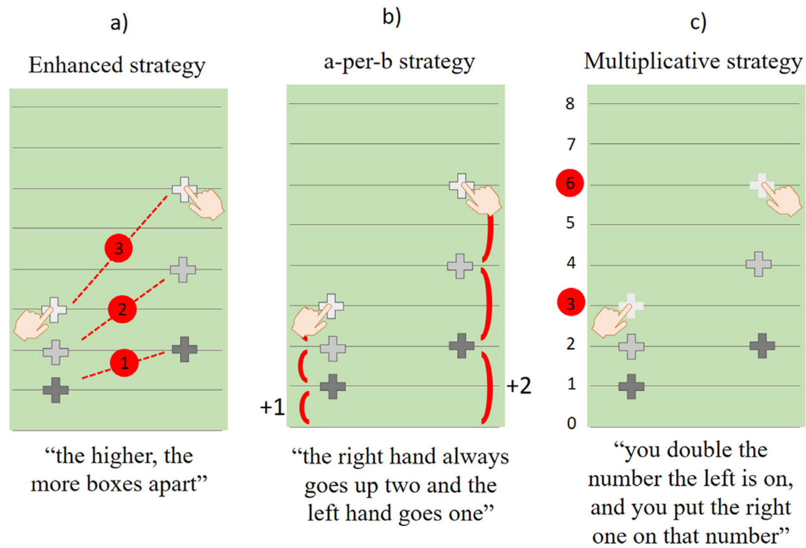


Fig. 6. Sensorimotor solution strategies in the quantitative stage of the proportion task, with the inclusion of a grid (a,b) and numbers (c).

As in the qualitative stage, the tutor elaborated on the student's description, this time by highlighting the empty space through tracing it, thereby also demonstrating that both student and tutor are attending to the same feature.

The dynamics of artifact adoption (improved grip, smooth transition and abrupt shift) have not been directly applied to artifact behavior in the other mathematical domains, possibly because generally the quantification stage has been studied less extensively. While no empirical findings on artifact behavior have been reported for the design for area, findings from the domains of angle, parabola and sine function do show changes in students' sensorimotor coordination patterns and their reflection in response to the included measuring artifacts in their interaction space. In the quantification stage of the task for angles for example, a protractor was added that automatically aligned to one ray of the angle as well as a number in degrees specifying the particular angle made (Petrick Smith et al., 2014) (Fig. 7a). Students enhanced their solution strategy by transitioning from describing locations of their arms, "You put them straight out to the side", towards using the angle measure, "You make it 180 degrees" (King & Petrick Smith, 2018, p. 584). Prior to the addition of the protractor, students enacted many discrete sensorimotor patterns, while with the protractor in place, they shifted towards attending to a range of arm movement, with utterances such as "light blue [obtuse]…it's 94 all the way up to about 178, then dark blue [straight] starts" (Petrick Smith et al., 2014, p. 103). Thus, whereas in the proportion task students strategies shifted from continuous to discrete, in the angle task students shifted from instances of angles towards the continuous range of angles between discrete points.

The parabola task was intended to quantify students' solutions strategy by connecting them to the parabola's definition with its algebraic expression. Students were tasked with finding the formula of the curve that could be drawn by the manipulated vertex C when the triangle was green (Shvarts, 2018). Instead of a grid, orthogonal projections from the manipulated point towards the x and y axes were added, accompanied by variable names (Fig. 7b). For the students, these artifacts highlighted a possibility to consider a point in the context of its coordinates and express y in terms of x, thus finding the trajectory of point C. In collaboration with the tutor, the students expressed the coordinate of point C in two ways and created an equation. Based on the isosceles quality of the green triangle—as it emerged for the students in embodied interaction—students found out that CA equals y. Further, a Pythagorean theorem was applied to the triangle CAY, and the y coordinate of point C was expressed as $y = \sqrt{y^2 - x^2} + d$, where d was the distance from the origin to the parabola's focus A. The algebraic solution of this equation led students to express parabola as a quadratic function. Overall, an embodied experience of enacting parabola as a movement was embedded into further mathematical reasoning and grounded the collaboration with the tutor in reaching formal algebraic expression of parabola.

In the sine function designs (Alberto et al., 2019; Shvarts et al., 2019a), measuring artifacts included circle marks (e.g., of 15 degrees each) for the unit circle (Shvarts et al., 2019a) and a radius-sized grid (or marks) for the sine graph (Alberto et al., 2019; Shvarts et al., 2019b), as well as numbers for either the input (distance traveled) or the output (height). Different from other action-based designs, the new artifacts were introduced together with posing new mathematical problems for the students. In the sine graph task (Alberto et al., 2019) (Fig. 7d), a constant line was introduced representing the value of sine together with the mathematical task of solving equations such as $\sin(\theta) = .86$. Students appropriated the new artifact by, for example, moving the constant to .86 and then moving the point on the unit circle so that it intersects, to read off the input value (Alberto et al., 2019). This problem solving stage is part of a new line of research focusing on the instrumental usage or embodied instrumentation of artifacts (see e.g., Drijvers, 2019; Shvarts et al., 2021).

4.3. The influence of within-task variations on students' learning trajectories

The previous sections discussed the general rationale of the design (motor control problems) and the elicited learning processes (coordination, emerging perceptual structures, reflection, and measuring artifacts). While the main structure of all action-based embodied design repeats these processes, studies show that even when the movement of the hands is identical, design alterations such as what objects students manipulate, influence how students orient toward the task, indicating the importance of attending to perceptions guiding these actions (see perceptual guidance of action in Mechsner, 2004). While within task variations have been used as a design heuristic in the sine function tasks (see Shvarts et al., 2019a in which students interact consecutively with cursors, angles, and bars in the unit circle), limited empirical data was presented. Therefore, to specify the

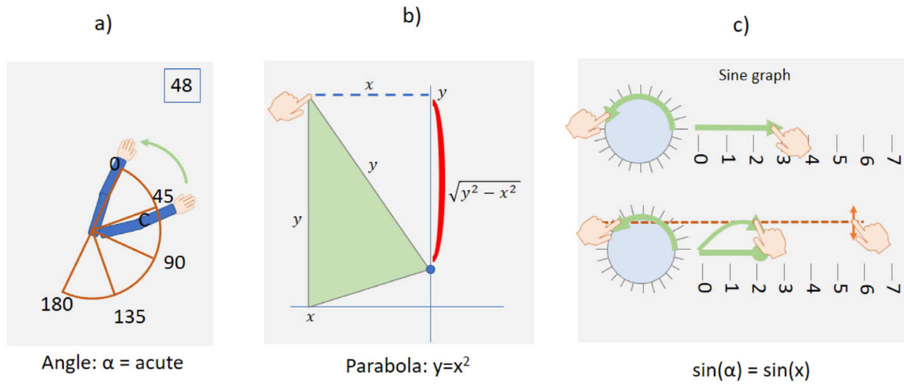


Fig. 7. Measuring artifacts in action-based designs (a) a protractor and angles in degrees for the angles task; (b) Algebraic expression in the quantitative stage for parabola (*y*-coordinate of point C and variable names), (c) equation solving with a constant, and angle and sine values in the sine graph task.

effect of within-topic variations, we compared students' learning processes across different action-based versions for proportions which have been extensively studied. These tasks varied along two dimensions: movement direction and objects to be manipulated. In three task versions, students coordinated their hands moving in parallel, manipulating either cursors, bars or icons (Table 1), while in another three tasks students coordinated their hands moving in orthogonal directions, manipulating either cursors, bars, or a rectangle (Table 2).

4.3.1. Parallel proportion tasks

Table 1 shows the three parallel sensorimotor problems, and students' solution processes. In all three versions students attended to the speeds of their hands and tried a fixed-interval strategy (e.g., "The bars turn green when they move in the same pace" (Abrahamson et al., 2016a, p. 30)). However, differences also emerged under the influence of what objects students were manipulating. To recapitulate Section 4.2, in the original full-screen task for proportions (e.g., Howison et al., 2011), students manipulated cursors, with color feedback filling the entire screen. In this parallel full-screen version, students' attention was drawn to the relation between their hands, with gazes hovering on a point in the space between their hands (Abrahamson et al., 2016a, 2016b; Cuiper, 2015; Shayan et al., 2017). A fictive diagonal interval between the hands, served as an attentional anchor that facilitated to coordinate movements and to ground further conceptualization. The students manipulated the interval's length with elevation. While at first, students thought the length of this line should remain fixed with elevation, they soon figured out that the distance changes, with statements such as "the higher, the bigger" (e.g., Howison et al., 2011). The grid first improved students' grip and then became a frame of reference in mathematical discourse (quantification of the interval and *a*-per-*b* strategies), while including numbers yielded multiplicative strategies (Abrahamson & Sánchez-García, 2016; Gutiérrez et al., 2011).

One study had students manipulate everyday-life icons, like hot air balloons or cars, either before or after the full-screen version (Palatnik & Abrahamson, 2017, 2018; Rosen, Palatnik, & Abrahamson, 2016; Rosen et al., 2018). When students manipulated everyday-life icons, less than half of the students referred to an interval between their hands as their attentional anchor (Rosen et al., 2016, 2018). Those that did, similar to the full-screen version, attended either to its length or angularity. One student described incorrectly that "It's the same angle… well, I mean the line connecting them is the same direction" (Rosen et al., 2018, p. 23). Whereas quantification of the interval in the full-screen version generally occurred when the grid was added (Fig. 6a), at least one student in the iconic version did so spontaneously already in the qualitative stage. The student explained his changing distance strategy as "…there's about a balloon between them… the length of the balloon... two balloons... it grows by one at a time" (Palatnik & Abrahamson, 2018, p. 302), thus exemplifying the spontaneous use of the balloon as a unit of measure (Palatnik & Abrahamson, 2017, 2018). What enables this measurement behavior is unclear, but it could be due to properties of the icons (balloons being objects and thus having a length), or the larger size of icons compared to the cursors.

Despite not having eye-tracking data available for the iconic version, students' gestures exposed that many students attended to the distances from each individual icon to the bottom of the screen. A student described his solution as, "One should stay in the middle while the other moves" (Rosen et al., 2016, p. 1514), where the middle referred to the balloon's location on its vertical axis, irrespective of the other balloon. Thus, while the parallel full-screen version elicited the hands to become connected in gaze and speech, and steered students' conceptualizations into mathematical directions, in the iconic version, the entities remained largely separate, hindering conceptualizations in some instances. Further, when students were asked to manipulate in full-screen mode after the iconic mode, some students became disoriented because of the absence of an "earth" (Rosen et al., 2016, 2018). This shows how pre-existing action perception loops come into play: balloons in a natural context have the ground as a starting point. The icons thus evoked a spatial–temporal narrative, forming an immediately available frame of reference (Rosen et al., 2018) which might not always be beneficial. Whether this is specifically for this context or all contexts, and whether some frames of reference might be beneficial mathematically is yet to be assessed.

Alongside contextual factors, three studies show that students' solution processes can also be steered by manipulating bars (Cuiper, 2015; Duijzer et al., 2017; Negrete, 2013). In the parallel bars versions, students' focus was on the relationship between the bars, drawing attention to their comparative heights. Studies with eye tracking show triangular gaze patterns formed between the top of the left bar, the top of the right bar, and midway the right bar, at the height of the left bar (Duijzer et al., 2017; Shayan et al., 2015) (Table 1). Students described self-adding strategies, "It is this piece here [left bar], that I hold with my hand, that should be added over there" (Duijzer et al., 2017, p. 12), or multiplicative strategies, "This piece [difference between left and right bar], actually is doubling the other one [left bar], so this one [left bar] is being doubled" (Duijzer et al., 2017, p. 13), "When you keep the left bar a little over halfway of the right bar" (Shayan et al., 2015, p. 5737). While for the full-screen variant, such multiplicative conceptualizations occur largely in the quantitative stage when a grid is added and attention is drawn to the height of the cursors, this bars layout already stimulated the shift in the qualitative stage. While this might be beneficial, it is at the cost of noticing the covariation between the length and the elevation foregrounded in the full-screen condition.

Despite differences between the full-screen and bars condition in the qualitative stage, including measuring artifacts such as grid and numbers, resulted in similar behavior across the task conditions: also in the bars task, students used the grid as a frame of reference and described the *a*-per-*b* strategy, "The left must always move a half, and the right must always move a whole" (Shayan et al., 2015, p. 5737). Interestingly, when the bars version was implemented in a classroom study, students spontaneously included measurement tools, such as their pen or ruler, already in the qualitative stage, "We want to measure how much it's going to take for this [gestures to left bar] to, you know, fill the rest of this [gestures to right bar]" (Negrete, 2013, p. 26). For the iconic version such a-per-b strategy could not be validated, as the study was restricted to the qualitative stage.

4.3.2. Orthogonal proportion tasks

Two studies investigated designs that changed the parallel orientation of the sensorimotor coordination pattern for proportions, and had students move their hands with orthogonal movements. Table 2 shows students' solution processes of the three orthogonal sensorimotor problems, in which they manipulated either full-screen cursors (Abrahamson et al., 2016a, 2016b; Shayan et al., 2017), bars (Abrahamson et al., 2016a, 2016b; Shayan et al., 2017), or a rectangle (Boven, 2017).

Compared to the parallel versions, students' attentional anchors and enactment fluency emerged quicker in orthogonal full-screen design (Abrahamson et al., 2016a, 2016b; Boven, 2017; Shayan et al., 2017). It is likely that students responded (unconsciously) to the spatial setting of the tablet (landscape), thereby priming the right hand to move further along the *x* axis (Abrahamson et al., 2016a). While this might seem advantageous, quicker motor fluency did not yield better conceptualizations. Students in the orthogonal condition took longer time to discover and articulate a relationship between their hands (Abrahamson et al., 2016a). Further, similar to the iconic task, some students focused on the positions of the cursors in their respective axes, "This [left finger] is exactly in the middle and this [right finger] almost in the middle" (Abrahamson et al., 2016a, p. 235). The middle refers to the hand's location with respect to the screen dimension, irrespective of the other hand's location, which potentially hinders conceptualizations.

Eye tracking studies showed a great variety of attentional anchors for the orthogonal full-screen design (Abrahamson et al., 2016a). As in the parallel full-screen task, some students fixated on a point hovering in the space between their hands, and described a line between their fingers. However, different properties of these lines were addressed: Whereas in parallel, the line's length was often considered (which increases with elevation), in the orthogonal version, its angularity was often noticed as a solution (which is kept constant with movement in the axes). Students gestured this imaginary line and expanded a fictive right triangle (Abrahamson et al., 2016a; Shayan et al., 2017), with utterances such as, "The line between the fingers only gets longer, not steeper" (Boven, 2017, p. 27). This phenomenological experience of "scaling triangles" matches well with mathematical definitions for proportions (Trninic et al., 2010). Other students' gazes included the Cartesian point $[x, y]$ with x and y respectively the position of the right and left hand, forming an imaginary rectangle. As students moved their hands, they tracked and updated this Cartesian point, resulting in a new sequentially formed structure: a diagonal line with the function $y = \frac{1}{2}x$ (Abrahamson & Bakker, 2018; Abrahamson et al., 2016a). Unfortunately, neither the student nor the tutor were aware of this mathematically relevant line, and thus no reflections have been reported. Yet again, other students mapped the position on the y axis onto the x axis, thereby gazing along a $y = x$ function.

Despite successful sensorimotor solutions, students' reflections in the full-screen condition often included a faulty strategy. Students described their hands as moving with the same speed or distance: "Ohh, I get it, you have to keep it the same, both length and width the same, otherwise it is red" (Abrahamson et al., 2016a, p. 232). As discussed before, successful reasoning can lag behind successful sensorimotor enactments. Such same-distance descriptions are absent in the parallel versions, as the height difference is visually very striking when compared side-by-side. A difficulty within the orthogonal variants is that of comparing horizontal with vertical lengths (Abrahamson et al., 2016a). Through the guidance of the tutor and the inclusion of the grid, the student realized that the hands were not equidistant, "On this [gestures to x-axis] I need to have more blocks" (Abrahamson et al., 2016a, p. 238); after this realization, most students continued with counting the blocks, thereby realizing the multiplicative rule (Abrahamson et al., 2016a).

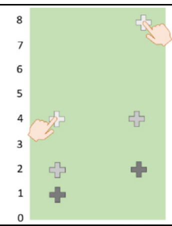
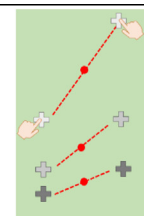
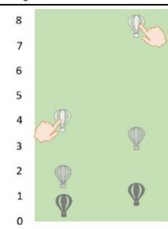
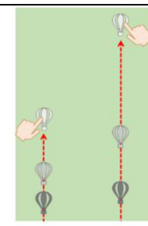
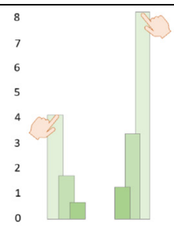
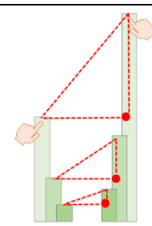
Whereas in the orthogonal full-screen variant a rectangle emerged as an attentional anchor for some students, in one study, students manipulated an actual rectangle by their movements along the *x* and *y* axis (Boven, 2017) (Table 2, row b). Findings showed that students' attention was drawn to the overall shape of the rectangle, with descriptions such as, "It's always the same shape" (Boven, 2017, p. 24), or, "The rectangle should not be too narrow or too thick" (Boven, 2017, p. 25). Similar to the orthogonal full-screen condition, tutor prompts or the introduction of a grid were needed to direct students' focus from the whole-shape towards the component-parts that make up the rectangle. Students first described the relation between the length and the width qualitatively with "the bottom is longer than the side", and later more quantitatively with "the rectangle is two squares next to each other", equivalent to "the bottom is twice the side" (Boven, 2017, p. 25).

The orthogonal full-screen and rectangle version thus generally shifted students' conceptualizations into geometry and scaling figures, and given the strong attention for the whole shape, attention needed to be actively drawn to the components making up these shapes with tutor guidance and measuring artifacts. An exception was the orthogonal bars version (Abrahamson et al., 2016a, 2016b; Shayan et al., 2017), which drew attention to the components in the qualitative stage. Similar to the parallel bars version, in the orthogonal bars variant, students focused on the relationship between the bars. Eye tracking showed gazes on a point marking the vertical bar onto the horizontal bar, forming the triangular pattern form (top vertical bar, mid horizontal bar, top horizontal bar) also found in the parallel version. Students' gestures included compass-like movements, such as rotating the left thumb from the top of the vertical bar into the horizontal axis, while describing a multiplicative rule, "Maybe this [vertical bar] is half of this [horizontal bar]" (Abrahamson et al., 2016a, p. 239). Bars thus consistently elicited comparison of length and multiplicative rules, regardless of moving in orthogonal or parallel.

5. Conclusion

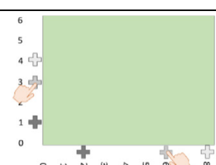
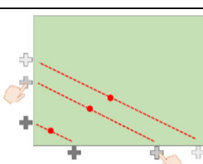
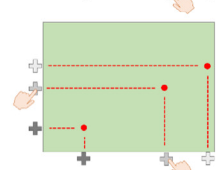
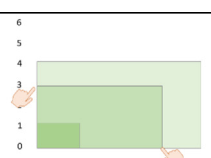
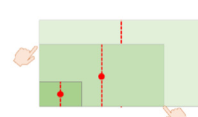
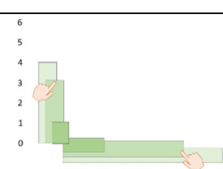
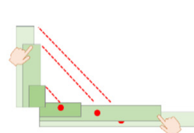
With this review paper, we aimed to facilitate educational researchers and designers to generate scientifically informed technological solutions for embodied design in mathematics education and beyond. We focused on well-studied action-based embodied designs and questioned how this particular embodied genre can elicit mathematical knowing. In particular, we explicated the characteristics of action-based embodied designs across

Table 1
Variations of the design for proportion with parallel hand movements.

Design	Sensorimotor problem	Attentional anchor(s)	Focus	Sample reflection
Parallel full screen			Speed, line-length	"the higher, the bigger"
Parallel icons		 *	Speed, height of each object within its vertical axis	"one should stay in the middle [of the axis] while the other moves"
Parallel bars			Speed, comparative heights	"one is always half the other"

* no eye tracking was used, so attentional anchors were inferred from (gestured) verbalizations.

Table 2
Variations of the design for proportion with orthogonal hand movements.

Design	Sensorimotor problem	Attentional anchor(s)	Focus	Sample reflection
Orthogonal full screen		 	Speed, line-angularity, position of each object within its axis	"The line has the same angle" Focus on angularity Unreflected upon
Orthogonal rectangle		 *	Similarity of shape	"always a rectangle", "Length is double the width"
Orthogonal bars			Speed, comparative heights	"one is always half the other"

* no eye tracking was used, so attentional anchors were inferred from (gestured) verbalizations.

five mathematical domains (Section 4.2), the typical learning sequence and elicited learning processes within action-based embodied designs (Section 4.2), and the influence of within-topic variations on students' perceptions, actions, and verbalizations (Section 4.3). The review offers a comprehensive reiteration of

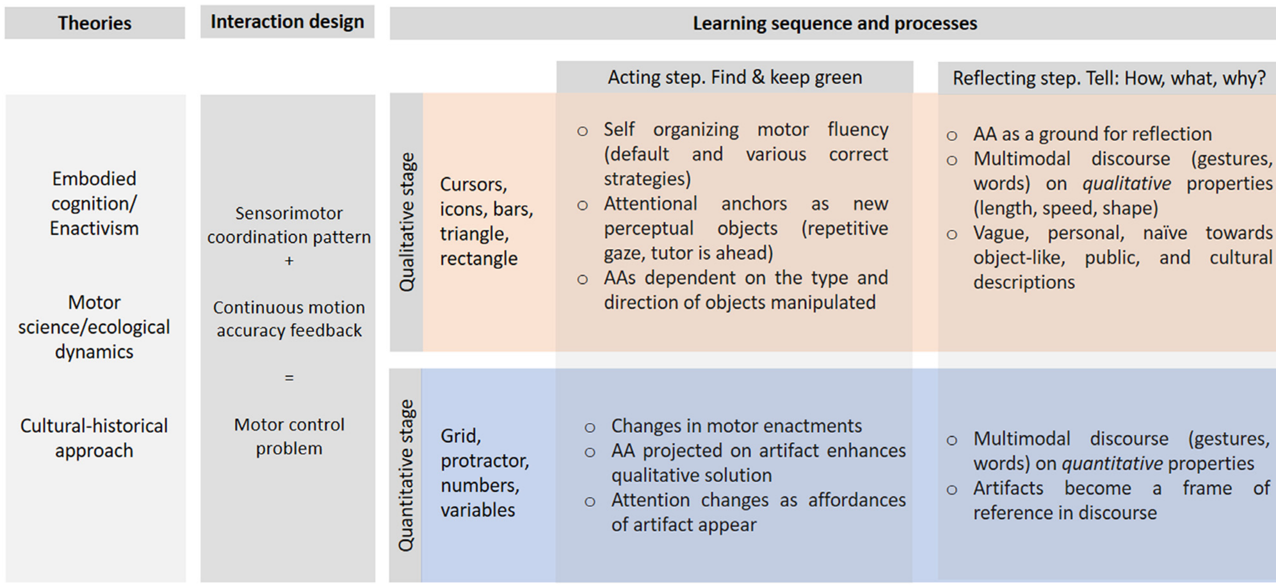


Fig. 8. Overview of design principles for the action-based design genre.

the action-based genre in the embodied-design framework (Abrahamson, 2014a, 2014b), bolstered by a broad inspection of multiple studies inspired by this framework (see Fig. 8 for a condensation of our findings in the form of a conjecture map).

In line with prior descriptions from the domain of proportions (Abrahamson, 2014a), we found that all action-based designs, regardless of topic and within topic variations, pose a motor problem whose solution is a sensorimotor coordination pattern that matches the target mathematical concept. These sensorimotor coordination patterns can be either bimanual, when students coordinate two hands, or unimanual, when students coordinate motor actions of geometric objects on the screen. The action-based embodied designs pose motor problems in which continuous positive (color, sound, vibration or other modality) feedback needs to be maintained. Learning a new movement in this way resembles the process of acquiring any new motor skill, in which functional coordination is shaped by continuous sensory feedback from the environment (Bernstein, 1967b). As designers and educators we might be tempted to induce or show these sensorimotor coordination patterns in a ready-made form. However, even though moving in correspondence with a ready-made trajectory does result in the target movement enactment, it does not per se elicit mathematical ideas. It is not the performance of the physical activity itself, but the sensorimotor coordination required to solve the motor problem (Bernstein, 1996) that has potential for mathematical conceptualizations. Therefore, we advise to be cautious with passive tasks where students need just gaze at technological elements (no coordination required), ready-made examples (students will just imitate them), and outsourcing elements of problem solving to the technology (students will ignore them).

Overall, mathematical knowing elicited by action-based embodied designs is based on the active establishing of new coordination-patterns that foster the emergence of new perceptual structures, which further become the ground for referencing in mathematical discourse. In line with non-linear pedagogy and dynamical systems theory (Abrahamson & Sánchez-García, 2016; Thelen, 2000), students' sensorimotor behaviors are undoubtedly diverse and determined by individual students' histories and task related features. The development of students' sensorimotor and discursive behavior passes through a phase of variability before arriving at a stable enculturated form of the target mathematical conceptualization. The review foregrounds two progressions in the learning sequence that determine the critical transition from sensorimotor enactment to describing mathematical concepts within enculturated mathematical discourse: from acting to reflecting and from a qualitative to a quantitative stage. These progressions are combined to form the necessary and sequential phases to elicit mathematical knowing through action-based embodied designs.

1. The progression from acting to reflecting.
   a. An acting step should be evoked by the design to foster new sensorimotor coordination in the form of embodied discoveries through solving motor-control problems. Students are provided ample opportunity to attempt a variety of (in)appropriate strategies, and finally establish a new coordination pattern. New perceptual structures — attentional anchors — emerge as a self-imposed constraint that limits the degrees of freedom, thereby facilitating enactment.
   b. A reflecting step should trigger students' reflections upon their performance. The attentional anchors that facilitated coordination in the acting step, become the point of reference in discourse, thereby making personal experiences public and open to scrutiny (speech–gesture mismatches) and enculturation. This stage's efficiency is determined by timely and responsive multimodal collaboration with a more knowledgeable other who helps in incorporating students' experiences in mathematical discourse.
2. The progression from a qualitative to a quantitative stage, in which the acting and reflecting steps are carried out in environments increasingly enriched with cultural artifacts.
   a. In the qualitative stage the motor problem is solved within an environment lacking any measurement tools, thus triggering new continuous motor coordination free from any mathematical structure. Students attend to and reflect on qualitative mathematical properties as they emerge in sensorimotor enactment.
   b. In the quantitative stage the motor problem is solved in environments enhanced with measurement artifacts (including grids, mathematical symbols and inscriptions). These artifacts embed the emerged coordination into mathematical constraints and facilitate enculturated descriptions of

sensorimotor experiences, as they become a frame of reference in quantitative systems of mathematical discourse.

The analysis of the within-topic variations foregrounded that even when the hand-movements are the same (e.g., moving one hand twice as fast as the other), different attentional anchors, subsequent reflections and trajectory of insights are elicited depending on the objects students are manipulating, and in what direction these objects are moved. Some versions might be at risk of eliciting attentional anchors for each individual hand, thereby hiding the relational properties (e.g., iconic conditions). Objects that students manipulate often elicit similar attentional anchor forms regardless of the orientation of manipulation (e.g., lines in both full-screen conditions), but students can attend to different qualitative properties (e.g., the line's length or angularity). Overall, despite being an easier coordination task, the orthogonal versions did not facilitate conceptualization better than the parallel conditions. The form of students' solutions however was different dependent on movement orientation, as the orthogonal versions generally shifted students' conceptualizations into geometry and scaling figures, while paralell versions elicited solutions in terms of lenght and covariation. Whereas, in most task versions the grid and numerals within the quantitative stage triggered conceptualizations of measurement (of the distance towards the ground or origin, or the sides of the shapes), some versions (e.g., the orthogonal and parallel bars variants, and the iconic version) elicited these perceptually critical features already in the qualitative stage. Elevating the top of the bar creates a longer object with greater area thus affording quantification at an earlier stage than free-floating (such as cursors) or geometric (rectangle) elements. But, regardless of when perceptually critical elements are elicted, students will eventually arrive at mathematically similar insights as artifacts are increasingly added.

Having articulated key features of the action-based embodied design genre, we want to highlight several limitations and future directions of study. First, while the learning processes within action-based designs have been studied extensively, learning gain assessments on standardized tests or intervention studies comparing different embodied and non- or less embodied are rare. Petrick Smith and colleagues showed that, after the angle task, students improved in estimating, but not drawing or ordering (Petrick Smith et al., 2014). The same researchers conducted a pre- and posttest analysis for an embodied classroom intervention for proportions (Petrick Smith, 2012). Although students in the embodied condition performed better on conceptual items than those only observing, this effect could not be ascribed to the action-based embodied intervention, as it was only one of multiple embodied tasks. Second, while some studies have been conducted in classrooms, the lion's share is conducted in controlled laboratory settings with one-to-one tutor guidance. While fruitful for the development of this embodied genre, it raises the question of applicability in classroom situations (Brown, 1992). Some classroom studies already showed that the action-based learning sequence and processes can collapse when students have more freedom (Negrete, 2013; Negrete et al., 2013). Further, research comparing individual enactment with observing another student acting, showed an influence on students' reflections (King & Petrick Smith, 2018; Petrick Smith, 2012; Petrick Smith & Martin, 2012), with those in the observing conditions using less mathematically relevant descriptions. Explorations of automatic classification of learners' behavior (Ou, Andrade, Alberto, Bakker, & Bechger, 2020; Pardos et al., 2018; Tancredi et al., 2021) and responsive and multimodal virtual tutoring (Abdullah et al., 2017; Flood et al., 2015, 2014) could serve, at least in part, as a solution. Last, a more comprehensive review could look more carefully at similarities and differences among the various (non) embodied child–computer interaction for mathematics learning. While some activities, such as walking the number line (Dackermann et al., 2017), TouchCounts (Sinclair & Heyd-Metzuyanim, 2014), and perception-based embodied designs (Abrahamson et al., 2020), apparently stand further from the interaction rationale of posing motor control (see criteria in methods), others, namely the Cartesian graph task (Nemirovsky et al., 2013) (from which the action-based design genre emerged), and time-distance graphs (Duijzer et al., 2019), were classified as closely related to action-based designs. Comparing students' and tutors' multimodal interactions empirically across different designs, could elucidate the potential generality of attentional anchors, reflections, and artifact dynamics beyond action-based embodied designs (also see Abrahamson & Abdu, 2020 for a proposal to compare an action-based design with a GeoGebra–based design).

With this review we wish to facilitate educational designers and human–computer interaction experts in developing new interactive activities for learning. Researchers and designers need to be ready for resolving a multiplicity of degrees of freedom in the design process and for observing the vast variability in students' performances. Initial variability is a mark of each student's genuine developmental process, and is needed to finally arrive at convergent understanding in the later stages. The review shows a variety of mathematical topics that can be taught in this way. We are convinced that many other mathematical topics could be embodied in a similar way, and that the established tasks could be adjusted to facilitate students with impairments (Abrahamson et al., 2019). The limits of generalizability of the action-based embodied design genre are currently being explored: Can action-based embodied designs facilitate learning in other scientific fields? And, to what extent can they be implemented in other interactive technologies such as 3D virtual reality technologies? Recent findings show potential with an action-based design using an interactive Sandbox for the topic of gradient in geography (Bos et al., 2021), and we suspect that concepts involving motion and graphing found in for example physics, chemistry or biology lend themselves well to a similar action-based approach.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

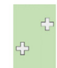
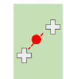
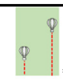
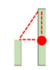
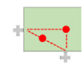
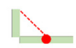
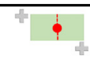
Acknowledgment

We acknowledge funding from The Netherlands Organisation for Scientific Research (NWO) under grant number 652.001.004

Appendix

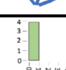
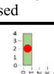
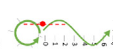
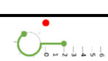
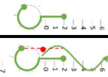
See Tables A.1 and A.2

Table A.1
Action based embodied studies, designs and empirical findings for proportions.

#	Setting	Device	N	Grade (age)	Duration	Design	Gaze	Reflections presymbolic	Artifacts added	Reflections symbolic
1	Laboratory	Nintendo Wii	24	4-6 (9-12)	70 min	Parallel full-screen 		Line-length: distance grows with elevation	Grid, numbers	Quantifying distance; Respective rates; Arithmetic
2	Classroom	Nintendo Wii	162	9-12 (14-19)	N.A.					
3	Laboratory	Tablet	11 (task 1)	7-8 (12-15)	45 min					
4	Laboratory	Large Tablet	10	4-6 (9-12)	20 min					
5	Laboratory	Large Tablet	14 (stark)	4-6 (9-12)	20 min					
6	Laboratory	Leap Motion	11	(under)graduates (21-60)	30 min					
5	Iconicity	Large Tablet	11 (iconic first)	4-6 (9-12)	20 min	Parallel icons 		Position of each balloons compared to the base	No artifacts added	Spontaneous quantification
7	Classroom	Tablet	34	Algebra support class	70 min	Parallel bars		Comparing Δ in bars' heights: half/double	Lines, numbers	Respective rates; Arithmetic
3	Laboratory	Tablet	11 (task 2)	7-8 (12-15)	45 min					
8	Laboratory	Tablet	30	5-6 (11.3)	20-30 min					
3	Laboratory	Tablet	10 (task 1)	7-8 (12-15)	45 min	Orthogonal full-screen 		Line-angularity: same angle	Grid, numbers	Shift towards components; Arithmetic
9	Laboratory	Tablet	43 (cond. 1)	4 (9-10)	N.A.					
3	Laboratory	Tablet	10 (task 2)	7-8 (12-15)	45 min	Orthogonal bars		Comparing Δ in bars' heights: half/double	Grid, numbers	Shift towards components; Arithmetic
9	Laboratory	Tablet	43 (cond. 2)	4 (9-10)	N.A.	Orthogonal rectangle 		Same shape	Grid, numbers	Shift towards components;

Study 1: Prototyping: Abrahamson and Howison (2008, 2010a, 2010b), Howison et al. (2011) and Trninic et al. (2010); Qualitative empirical: Abrahamson et al. (2014), Abrahamson, Negrete et al. (2012), Abrahamson, Trninic and Gutiérrez (2012), Reinholz et al. (2010), Trninic and Abrahamson (2011) and Trninic et al. (2011); Quantitative: Pardos, Hu, Meng, Neff, and Abrahamson (2018); collaborative reflection: Abrahamson et al. (2012b), Flood (2018), Flood and Abrahamson (2015) and Flood et al. (2016); Virtual tutor: Flood, Neff, and Abrahamson (2015), Flood, Schneider, and Abrahamson (2014); Artifacts: Abrahamson et al. (2012a), Abrahamson, Gutierrez et al. (2011), Abrahamson, Trninic et al. (2011), Charoenying and Trninic (2011) and Gutiérrez et al. (2011); Ecologic/REC: Abrahamson (2016), Abrahamson and Sánchez-García (2015, 2016), Abrahamson, Sánchez-García and Trninic (2016), Abrahamson and Shulman (2019), Abrahamson and Trninic (2015, 2016), Hutto and Abrahamson (0000) and Hutto et al. (2015); Dance/martial: Abrahamson and Shulman (2017), Duijzer et al. (2019), Trninic (2015) and Trninic and Abrahamson (2012, 2013, 2016); general embodied design/taxonomy: Abrahamson (2013, 2014a, 2014b, 2015, 2017), Abrahamson and Bakker (2016), Abrahamson and Lindgren (2014) and Charoenying, Gaysinsky, and Riyokai (2012); Study 3: Abrahamson and Bakker (2018), Abrahamson et al. (2016a, 2016b), Cuiper (2015) and Shayan et al. (2017); Study 4: Abdullah et al. (2017); Study 5: Palatnik and Abrahamson (2017, 2018) and Rosen et al. (2016, 2018); Study 6: Ghasemaghei (2017), Ghasemaghaei, Arya et al. (2015), Ghasemaghaei et al. (2016) and Ghasemaghaei, Biddle et al. (2015); Study 7: Negrete (2013) and Lee (2013), Negrete et al. (2013); Study 8: Abrahamson et al. (2016a, 2016b), Duijzer et al. (2017) and Shayan et al. (2015, 2017); Study 9: Boven (2017).

Table A.2
Action based embodied studies, designs and empirical findings for angles, area, parabola and trigonometric function.

#	Setting	Device	N	Grade (age)	Duration	Design	Gaze	Reflections presymbolic	Artifacts added	Reflections symbolic
10	Classroom	Kinect sensor	20	3-4 (8-10)	15-20 min	Angles, e.g., alpha = acute 	No eye tracking was used	Position of arms per angle type	Protractor, numbers	Range of arm-movements
11	Laboratory	PC	13	5 (10-11)	N.A.	Area, area = constant 		The same size, volume, area	Grid, numbers	N.A.
12	Laboratory	PC	4 pairs	Undergraduates (17-23)	Unknown, >10 min	Parabola, $y=x^2$ 		Isosceles triangle	Variable names	Right-angled triangle and formula
13	Laboratory	Tablet	5	University students	N.A.	Sine function, sin(a) = sin(x) 		Keeping the points level	Grid, numbers	Included in equation solving
14	Laboratory	Interactive whiteboard	3	University students	N.A.	Sine function, e.g., sin(a) = sin(2a) 		Points are level	Circle marks, numbers	Included in equation solving
								The angles are the same		
								The points match in height		
15	Laboratory	Interactive whiteboard	N.A.	University students	N.A.	Sine function, sin(a) = sin(x) 		Same speed	Circle marks, grid and numbers	Included in equation solving
								Same height; level		

Study 10: King and Petrick Smith (2018) and Petrick Smith et al. (2014); Study 11: Shvarts (2017); Study 12: Abrahamson (2019), Bakker et al. (2014), Drijvers (2019), Shvarts (2018) and Shvarts and Abrahamson (2018, 2019); Study 13: Abrahamson (2019), Alberto et al. (2019) and Bakker et al. (2014); Study 14: Shvarts et al. (2019a); Study 15: Shvarts et al. (2019b).

References

Abdullah, A., Adil, M., Rosenbaum, L., Clemmons, M., Abrahamson, D., & Neff, M. (2017). Pedagogical agents to support embodied, discovery-based learning. In *Int. conf. intell. virtual agents* (pp. 1–14). Cham: Springer, http://dx.doi.org/10.1007/978-3-319-67401-8.

Abrahamson, D. (2013). Toward a taxonomy of design genres: Fostering mathematical insight via perception-based and action-based experiences. In *Proc. 12th int. conf. interact. des. child* (pp. 218–227). http://dx.doi.org/10.1145/2485760.2485761.

Abrahamson, D. (2014a). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds.

International Journal of the Child-Computer Interactions, *2*, 1–16. http://dx.doi.org/10.1016/j.ijcci.2014.07.002.

Abrahamson, D. (2014b). The monster in the machine, or why educational technology needs embodied design. In *Learn. technol. body integr. implement. form. informal learn. environ* (pp. 21–38). http://dx.doi.org/10.4324/9781315772639.

Abrahamson, D. (2015). Reinventing learning: a design-research odyssey. *ZDM - Mathematical Education*, *47*, 1013–1026. http://dx.doi.org/10.1007/s11858-014-0646-3.

Abrahamson, D. (2016). The ecological dynamics of mathematics education: the emergence of proportional reasoning in fields of promoted action. In *Proc. 13th int. congr. math. educ.* (pp. 1–8).

Abrahamson, D. (2017). Embodiment and mathematics learning. In *SAGE encycl. out-of- sch. learn.* (pp. 248–252).

Abrahamson, D. (2019). A new world: Educational research on the sensorimotor roots of mathematical reasoning. In A. Shvarts (Ed.), *Proc. pme yandex russ. conf. technol. psychol. math. educ.* (pp. 48–68). Moscow, Russia: HSE Publishing House.

Abrahamson, D., & Abdu, R. (2020). *Towards an ecological-dynamics design framework for embodied-interaction conceptual learning: the case of dynamic mathematics environments*. Springer US, http://dx.doi.org/10.1007/s11423-020-09805-1.

Abrahamson, D., & Bakker, A. (2016). Making sense of movement in embodied design for mathematics learning. *Cognition Research Principles Implications*, *1*, 1–13. http://dx.doi.org/10.1186/s41235-016-0034-3.

Abrahamson, D., & Bakker, A. (2018). An ecological dynamics view on movement-based mathematics learning: On the emergence of sensorimotor schemes in sociocultural settings. In *Proc. int. conf. learn. sci.* (pp. 1244–1245).

Abrahamson, D., Flood, V. J., Miele, J. A., & Siu, Y. T. (2019). Enactivism and ethnomethodological conversation analysis as tools for expanding universal design for learning: The case of visually impaired mathematics students. *ZDM*, *51*, 291–303. http://dx.doi.org/10.1007/s11858-018-0998-1.

Abrahamson, D., Gutiérrez, J., Charoenying, T., Negrete, A. G., & Bumbacher, E. (2012a). Fostering hooks and shifts: Tutorial tactics for guided mathematical discovery. *Technology Knowledge Learning*, *17*, 61–86. http://dx.doi.org/10.1007/s10758-012-9192-7.

Abrahamson, D., Gutiérrez, J. F., Charoenying, T., Negrete, A. G., & Bumbacher, E. (2012b). Fostering mathematical discovery: one tutor's strategies for ushering the construction of proportional schemas via mediated embodied interaction. In *Annu. meet. am. educ. res. assoc.*

Abrahamson, D., Gutierrez, J., Lee, R. G., Reinholz, D. L., & Trninic, D. (2011). From tacit sensorimotor coupling to articulated mathematical reasoning in an embodied design for proportional reasoning. In *Annu. meet. am. educ. res. assoc* (pp. 1–29).

Abrahamson, D., & Howison, M. (2008). Kinemathics: Kinetically induced mathematical learning—Overview of rationale. In *UC berkeley gesture study gr* (pp. 1–6).

Abrahamson, D., & Howison, M. (2010a). Embodied artifacts: Coordinated action as an object-to-think-with. In *Embodied enactive approaches to instr. implic. innov* (pp. 1–18). AERA, Denver.

Abrahamson, D., & Howison, M. (2010b). Kinemathics: Exploring kinesthetically induced mathematical learning. In *Present. 2010 annu. meet. am. educ. res. assoc.*.

Abrahamson, D., & Kapur, M. (2018). Reinventing discovery learning: a field-wide research program. *Instrumental Sciences*, *46*, http://dx.doi.org/10.1007/s11251-017-9444-y.

Abrahamson, D., Lee, R. G., Negrete, A. G., & Gutiérrez, J. F. (2014). Coordinating visualizations of polysemous action: Values added for grounding proportion. *ZDM*, *46*, 79–93. http://dx.doi.org/10.1007/s11858-013-0521-7.

Abrahamson, D., & Lindgren, R. (2014). Embodiment and embodied design. In *Cambridge handb. learn. sci* (pp. 358–376).

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., et al. (2020). The future of embodied design for mathematics teaching and learning. *Frontiers on Education*, *5*, 147. http://dx.doi.org/10.3389/feduc.2020.00147.

Abrahamson, D., Negrete, A. G., Lee, R. G., & Gutiérrez, J. F. (2012). Adding up to multiplicative concepts: the role of embodied reasoning. In *Annu. meet. am. educ. res. assoc.* (pp. 1–28).

Abrahamson, D., & Sánchez-García, R. (2015). A call to action: Towards an ecological-dynamics theory of mathematics learning, teaching and design. In H. Bartell, T. G. Bieda, K. N. Putnam, R. T. Bradfield, & K. Dominguez (Eds.), *37th annu. meet. north am. chapter int. gr. psychol. math. educ.* (pp. 1261–1268). East Lansing, MI: Michigan State University.

Abrahamson, D., & Sánchez-García, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *Journal of the Learning Science*, *25*, 203–239. http://dx.doi.org/10.1080/10508406.2016.1143370.

Abrahamson, D., Sánchez-García, R., & Trninic, D. (2016). Praxes proxies: Revisiting educational manipulatives from an ecological dynamics perspective. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *38th annu. meet. north am. chapter int. gr. pyschology math. educ* (pp. 1565–1572).

Abrahamson, D., Shayan, S., Bakker, A., & van der Schaaf, M. F. (2016a). Eye-tracking piaget: Capturing the emergence of attentional anchors in the coordination of proportional motor action. *Human Developments*, *58*, 218–224. http://dx.doi.org/10.1159/000443153.

Abrahamson, D., Shayan, S., Bakker, A., & van der Schaaf, M. F. (2016b). Exposing Piaget's scheme: Empirical evidence for the ontogenesis of coordination in learning a mathematical concept. In *Proc. int. conf. learn. sci.* (pp. 466–473).

Abrahamson, D., & Shulman, A. (2017). Constructing movement in mathematics and dance: An interdisciplinary pedagogical dialogue on subjectivity and awareness. In *1st annu. meet. mov. brain, body, cogn.* Oxford, UK.

Abrahamson, D., & Shulman, A. (2019). Co-constructing movement in mathematics and dance: An interdisciplinary pedagogical dialogue on subjectivity and awareness. *Feldenkrais Research Journal*, *6*.

Abrahamson, D., & Trninic, D. (2011). Toward an embodied-interaction design framework for mathematical concepts. In *Proc. 10th int. conf. interact. des. child* (pp. 1–10). http://dx.doi.org/10.1145/1999030.1999031.

Abrahamson, D., & Trninic, D. (2015). Bringing forth mathematical concepts: signifying sensorimotor enactment in fields of promoted action. *ZDM*, *47*, 295–306. http://dx.doi.org/10.1007/s11858-014-0620-0.

Abrahamson, D., & Trninic, D. (2016). Working out: Mathematics learning as motor problem solving in instrumented fields of promoted action. In *Knowl. interact. a synth. agenda learn. sci.* (pp. 212–235). Routledge, Taylor & Francis Group.

Abrahamson, D., Trninic, D., & Gutiérrez, J. F. (2012). You made it! from action to object in guided embodied interaction design. In *10th int. conf. learn. sci.* (pp. 100–101).

Abrahamson, D., Trninic, D., Gutiérrez, J. F., Huth, J., & Lee, R. G. (2011). Hooks and shifts: A dialectical study of mediated discovery. *Technology and Knowledge Learning*, *16*, 55–85. http://dx.doi.org/10.1007/s10758-011-9177-y.

Alberto, R., Bakker, A., Walker-van Aalst, O., Boon, P., & Drijvers, P. (2019). Networking theories in design research: an embodied instrumentation case study in trigonometry embodied instrumentation: techno-physical mathematical learning. In *Elev. congr. eur. soc. res. math. educ*.

Antle, A. N. (2013). Exploring how children use their hands to think: An embodied interactional analysis. *Behavioral Information Technology*, *32*, 938–954. http://dx.doi.org/10.1080/0144929X.2011.630415.

Bakker, A. (2018). *Design research in education: a practical guide for early career researchers*. London: Taylor.

Bakker, A., Shvarts, A., & Abrahamson, D. (2014). Generativity in design research : the case of developing a genre of action-based mathematics learning activities. In *Elev. congr. eur. soc. res. math. educ*.

Barsalou, L. W. (1999). Perceptual symbol systems and emotion. *Behavioral Brain and Science*, *22*, 577–660. http://dx.doi.org/10.1017/S0140525X99252144.

Bernstein, N. A. (1967a). *The co-ordination and regulation of movements*. Pergamon Press, http://www.sciepub.com/reference/174846.

Bernstein, A. N. (1967b). *The Co-Ordination and Regulation of Movements*. Oxford etc: Pergamon Press.

Bernstein, N. A. (1996). Dexterity and its development. In M. L. Latash, & M. T. Turvey (Eds.), *Dexterity its dev* (pp. 1–244). Mahwah, NJ: Lawrence Erlbaum Associates.

Boaler, J., Chen, L., Williams, C., & Cordero, M. (2016). Seeing as understanding: The importance of visual mathematics for our brain and learning. *Journal of Applied Computational Mathematics*, *5*, 1–6. http://dx.doi.org/10.4172/2168-9679.1000325.

Bos, R., Doorman, M., Drijvers, P., & Shvarts, A. (2021). Embodiment through augmented reality: the case of gradient. *Teaching Mathematics and its Applications*.

Boven, L. (2017). Students' proportional reasoning elicited by two-dimensional embodied design.

Boyer, T. W., & Levine, S. C. (2015). Prompting children to reason proportionally: Processing discrete units as continuous amounts. *Development on Psychology*, *51*, 615–620. http://dx.doi.org/10.1037/a0039010.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challanges in creating complex interventions in classoom settings. *Journal of the Learning Sciences*, *2*, 141–178.

Charoenying, T., Gaysinsky, A., & Riyokai, K. (2012). The choreography of conceptual development in computer supported instructional environments. In *11th int. conf. interact. des. child* (pp. 162–167).

Charoenying, T., & Trninic, D. (2011). Evocation and enactment: Understanding trajectories of conceptual development in artifact-mediated situations. In *Proc. 19th int. conf. comput. educ.* (pp. 165–172).

Cuiper, A.-C. F. D. (2015). Vocational students' search patterns while solving a digital proportion task of the MIT-p app.

Dackermann, T., Fischer, U., Nuerk, H. C., Cress, U., & Moeller, K. (2017). Applying embodied cognition: from useful interventions and their theoretical underpinnings to practical applications. *ZDM*, *49*, 545–557. http://dx.doi.org/10.1007/s11858-017-0850-z.

Dessing, J. C., Rey, F. P., & Beek, P. J. (2012). Gaze fixation improves the stability of expert juggling. *Experimental Brain Research*, *216*, 635–644. http://dx.doi.org/10.1007/s00221-011-2967-6.

Drijvers, P. H. M. (2019). Embodied instrumentation: combining different views on using digital technology in mathematics education. In *Elev. congr. eur. soc. res. math. educ.*.

Duijzer, C. A. C. G., van den Heuvel-Panhuizen, M., Veldhuis, M., Doorman, M., & Leseman, P. (2019). Embodied learning environments for graphing motion: A systematic literature review. *Education Psychology Review*, *31*, 597–629.

Duijzer, C. A. C. G., Shayan, S., Bakker, A., van der Schaaf, M. F., & Abrahamson, D. (2017). Touchscreen tablets: Coordinating action and perception for mathematical cognition. *Frontiers on Psychology*, *8*, 144. http://dx.doi.org/10.3389/fpsyg.2017.00144.

Flood, V. J. (2018). Multimodal revoicing as an interactional mechanism for connecting scientific and everyday concepts. *Human Developments.*, *61*, 145–173. http://dx.doi.org/10.1159/000488693.

Flood, V. J., & Abrahamson, D. (2015). Refining mathematical meanings through multimodal revoicing interactions: the case of faster. In *Annu. meet. am. educ. res. assoc.* (pp. 1689–1699). Chicago, IL: http://dx.doi.org/10.1017/CBO9781107415324.004.

Flood, V. J., Harrer, B. W., & Abrahamson, D. (2016). The interactional work of configuring a mathematical object in a technology - enabled embodied learning environment. In *Int. conf. learn. sci.* (pp. 122–129).

Flood, V. J., Neff, M., & Abrahamson, D. (2015). Boundary interactions: Resolving interdisciplinary collaboration challenges using digitized embodied performances. In *Int. soc. learn. sci.*

Flood, V. J., Schneider, A., & Abrahamson, D. (2014). Gesture enhancement of a virtual tutor via investigating human tutor discursive strategies: Forms and functions for proportions. In *Proc. int. conf. learn. sci.* (pp. 1593–1594).

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: material entanglements in the classroom*. Cambridge University Press, http://dx.doi.org/10.1017/cbo9781139600378.

Ghasemaghaei, R. (2017). Multimodal software for affective education: User interaction design and evaluation.

Ghasemaghaei, R., Arya, A., & Biddle, R. (2015). Design practices for multimodal affective mathematical learning. In *Int. symp. comput. sci. softw. eng.*. http://dx.doi.org/10.1109/CSICSSE.2015.7369246.

Ghasemaghaei, R., Arya, A., & Biddle, R. (2016). Made ratio: Affective multimodal software for mathematical concepts. In *Int. conf. learn. collab. technol* (pp. 487–498). Cham: Springer, http://dx.doi.org/10.1007/978-3-319-39483-1_44.

Ghasemaghaei, R., Biddle, R., & Arya, A. (2015). The MADE framework: Multimodal software for affective education. In *EdMedia+ innov. learn* (pp. 1861–1871). Association for the Advancement of Computing in Education.

Gibson, J. J. (1979). *The ecological approach to visual perception*. Psychology Press, http://dx.doi.org/10.1075/lsse.2.03bli.

Gutiérrez, J. F., Trninic, D., Lee, R. G., & Abrahamson, D. (2011). Hooks and shifts in instrumented mathematics learning. In *Annu. Meet. Am. Educ. Res. Assoc.* New Orleans, LA.

Harzing, A. W. K., & van der Wal, R. (2008). Google scholar as a new source for citation analysis. *Ethics Science Environmental Politics*, *8*, 61–73. http://dx.doi.org/10.3354/esep00076.

Howison, M., Trninic, D., Reinholz, D. L., & Abrahamson, D. (2011). The mathematical imagery trainer: From embodied interaction to conceptual learning. In G. Fitzpatrick, C. Gutwin, B. Begole, W. Kellogg, & D. Tan (Eds.), *Proc. annu. meet. assoc. comput. mach. spec. interes. gr. comput. hum. interact. human factors comput. syst. (chi 2011)* (pp. 1989–1998). http://dx.doi.org/10.1145/1978942.1979230.

Hutto, D. D., & Abrahamson, D. (0000). Embodied, enactive education, (n.d.)..

Hutto, D. D., Kirchhoff, M. D., & Abrahamson, D. (2015). The enactive roots of STEM: Rethinking educational design in mathematics. *Education on Psychology Review*, *27*, 371–389. http://dx.doi.org/10.1007/s10648-015-9326-2.

Hutto, D. D., & Myin, M. (2017). *Evolving Enactivism: Basic Minds Meet Content*. MIT press.

Hutto, D. D., & Sánchez-García, R. (2015). Choking rectified: Embodied expertise beyond dreyfus. *Phenomenology Cognitive Sciences*, *14*, 309–331. http://dx.doi.org/10.1007/s11097-014-9380-0.

Kelso, J. A. S., & Schöner, G. (1988). Self-organization of coordinative movement patterns. *Human Movement Science*, *7*, 27–46. http://dx.doi.org/10.1016/0167-9457(88)90003-6.

King, B., & Petrick Smith, C. J. (2018). Mixed-reality learning environments: What happens when you move from a laboratory to a classroom? *International Journal of the Research on Educational Sciences*, *4*, 577–594. http://dx.doi.org/10.21890/ijres.428961.

Kugler, P. N., Kelso, J. A. S., & Turvey, M. T. (1982). On the control and coordination of naturally developing systems. In J. A. S. Kelso, & J. Clark (Eds.), *Dev. mov. control coord* (pp. 5–78). New York: Wiley.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework. In F. K. Lester (Ed.), *Second handb. res. math. teach. learn* (pp. 629–668). Charlotte: Information Age Publishing.

Lee, R. G. (2013). Negotiating mathematical visualizations in classroom group work: the case of a digital design for proportion.

Lee, R. G., Hung, M., Negrete, A. G., & Abrahamson, D. (2013). Rationale for a ratio-based conceptualization of slope: results from a design-oriented embodied-cognition domain analysis. In *Annu. meet. am. educ. res. assoc.*, San Francisco.

Mechsner, F. (2004). A psychological approach to human voluntary movements. *Journal of Motor Behavior*, *36*, 355–370. http://dx.doi.org/10.1080/00222895.2004.11007993.

Negrete, A. G. (2013). Toward didactical contracts for mathematics learning with digital media: Coordinating pedagogical design and classroom practices.

Negrete, A. G., Lee, R. G., & Abrahamson, D. (2013). Facilitating discovery learning in the tablet era: Rethinking activity sequences vis-à-vis digital practises. In *M. Martinez, A. Castro Superfine (Eds.), Proc. 35th annu. meet. north-american chapter int. gr. psychol. math. educ.*, Chicago, IL, (p. 1205).

Nemirovsky, R., Ferrara, F., Ferrari, G., & Adamuz-Povedano, N. (2020). Body motion, early algebra, and the colours of abstraction. *Educational Studies in Mathematics*, *104*, 261–283. http://dx.doi.org/10.1007/s10649-020-09955-2.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal of the Research on Mathematical Education*, *44*, 372–415. http://dx.doi.org/10.5951/jresematheduc.44.2.0372.

Ou, L., Andrade, A., Alberto, R. A., Bakker, A., & Bechger, T. (2020). Identifying qualitative between-subject and within-subject variability: A method for clustering regime-switching dynamics. *Frontiers on Psychology*, *11*, 1–13. http://dx.doi.org/10.3389/fpsyg.2020.01136.

Palatnik, A., & Abrahamson, D. (2017). Taking measures to coordinate movements: unitizing emerges as a means of building event structures for enacting proportions. In E. Galindo, & J. Newton (Eds.), *Synergy crossroads—proceedings 39th annu. conf. north-american chapter int. gr. psychol. math. educ* (pp. 1439–1442). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educator.

Palatnik, A., & Abrahamson, D. (2018). Rhythmic movement as a tacit enactment goal mobilizes the emergence of mathematical structures. *Education Studies in Mathematics*, *99*, 293–309. http://dx.doi.org/10.1007/s10649-018-9845-0.

Pardos, Z. A., Hu, C., Meng, P., Neff, M., & Abrahamson, D. (2018). Classifying learner behavior from high frequency touchscreen data using recurrent neural networks. In *26th conf. user model. adapt. pers* (pp. 317–322). http://dx.doi.org/10.1145/3213586.3225244.

Petrick Smith, C. J. (2012). Every body move: Learning mathematics through embodied actions.

Petrick Smith, C. J., King, B., & Hoyte, J. (2014). Learning angles through movement: Critical actions for developing understanding in an embodied activity. *Journal of the Mathematical Behaviour*, *36*, 95–108. http://dx.doi.org/10.1016/j.jmathb.2014.09.001.

Petrick Smith, C. J., & Martin, H. T. (2012). Learning mathematics: You're it vs. it's it. In *Proc. int. conf. learn. sci. futur. learn* (pp. 101–102). Sydney, NSW: University of Sydney/ISLS.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, *70*, 111–126. http://dx.doi.org/10.1007/s10649-008-9127-3.

Reed, E. S., & Bril, B. (1996). The primacy of action in development. In *Dexterity its dev* (pp. 431–451). http://dx.doi.org/10.4324/9781410603357-23.

Reinholz, D. L., Trninic, D., Howison, M., & Abrahamson, D. (2010). It's not easy being green: Embodied artifacts and the guided emergence of mathematical meaning. In *Proc. 32nd annu. meet. north am. chapter int. gr. psychol. math. educ.* (pp. 1488–1496).

Rosen, D., Palatnik, A., & Abrahamson, D. (2016). Tradeoffs of situatedness: iconicity constrains the development of content-oriented sensorimotor schemes. In M. B. Wood, E. E. Turner, M Civil, & J. A. Eli (Eds.), *38th annu. meet. north am. chapter int. gr. pyschology math. educ* (pp. 1509–1516).

Rosen, D., Palatnik, A., & Abrahamson, D. (2018). A better story: An embodied-design argument for generic manipulatives. In N. Calder, K. Larkin, & N. Sinclair (Eds.), *Using mob. technol. learn. math* (pp. 189–211). New York: Springer.

Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the Learning Sciences*, *23*, 18–36. http://dx.doi.org/10.1080/10508406.2013.778204.

Shayan, S., Abrahamson, D., Bakker, A., Duijzer, C. A. C. G., & van der Schaaf, M. F. (2015). The emergence of proportional reasoning from embodied interaction with a tablet application: an eye-tracking Study. In *Proc. 9th int. technol. educ. dev. conf.* (pp. 5732–5741).

Shayan, S., Abrahamson, D., Bakker, A., Duijzer, C. A. C. G., & van der Schaaf, M. F. (2017). Eye-tracking the emergence of attentional anchors in a mathematics learning tablet activity. In *Eye-tracking technol. appl. educ. res* (pp. 166–194). http://dx.doi.org/10.4018/978-1-5225-1005-5.ch009.

Sheets-Johnstone, M. (2011). *The primacy of movement, Vol. 82*. John Benjamins Publishing.

Shvarts, A. (2017). Eye movements in emerging conceptual understanding of rectangle area. In *Proc. 41st conf. int. gr. psychol. math. educ.*, 268.

Shvarts, A. (2018). A dual eye-tracking study of objectification as students-tutor joint activity appropriation. In *PME (Ed.), Proc. 42nd conf. int. gr. psychol. math. educ.* Umeå, Sweden, (pp. 171–178).

Shvarts, A., & Abrahamson, D. (2018). Towards a complex systems model of enculturation: A dual eye- tracking study. In *Annu. conf. am. educ. res. assoc* (pp. 13–17). NYC.

Shvarts, A., & Abrahamson, D. (2019). Dual-eye-tracking Vygotsky: A microgenetic account of a teaching/learning collaboration in an embodied-interaction technological tutorial for mathematics. *Learning Culture Social Interactions*, *22*, Article 100316. http://dx.doi.org/10.1016/j.lcsi.2019.05.003.

Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2019a). Embodied collaboration to foster instrumental genesis in mathematics. In *Int. conf. comput. support. collab. learn* (pp. 660–663). https://repository.isls.org//handle/1/1646.

Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2019b). Embodied instrumentation: Reification of sensorimotor activity into a mathematical artifact. In *Proc. 14th int. conf. technol. math. teach.* (pp. 127–128). http://dx.doi.org/10.17185/duepublico/70749.

Shvarts, A., Alberto, R. A., Bakker, A., Doorman, M., & Drijvers, P. H. M. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educational Studies in Mathematics*, 1–23. http://dx.doi.org/10.1007/s10649-021-10053-0.

Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning number with TouchCounts: The role of emotions and the body in mathematical communication. *Technology Knowledge Learning*, *19*, 81–99. http://dx.doi.org/10.1007/s10758-014-9212-x.

Smith, L. B., & Thelen, E. (1996). *A dynamic systems approach to the development of cognition and action*. MIT Press, http://dx.doi.org/10.7551/mitpress/2524.001.0001.

Tancredi, S., Abdu, R., Abrahamson, D., & Balasubramaniam, R. (2021). Modeling nonlinear dynamics of fluency development in an embodied-design mathematics learning environment with recurrence quantification analysis. *International Journal of the Child-Computer Interactions*, *29*, Article 100297. http://dx.doi.org/10.1016/j.ijcci.2021.100297.

Thelen, E. (2000). Motor development as foundation and future of developmental psychology. *International Journal of the Behavior Developments*, *24*, 385–397. http://dx.doi.org/10.1080/016502500750037937.

Trninic, D. (2015). Body of knowledge: Practicing mathematics in instrumented fields of promoted action. http://dx.doi.org/10.1016/j.physbeh.2008.04.026.

Trninic, D., & Abrahamson, D. (2011). Emergent ontology in embodied interaction: Automated feedback as conceptual placeholder. In *Proc. 33rd annu. meet. north am. chapter int. gr. psychol. math. educ.* (pp. 1777–1785).

Trninic, D., & Abrahamson, D. (2012). Embodied artifacts and conceptual performances. In *10th int. conf. learn. sci. futur. learn* (pp. 283–290).

Trninic, D., & Abrahamson, D. (2013). Embodied interaction as designed mediation of conceptual performance. In D. Martinovic (Ed.), *Vis. math. cyberlearning* (pp. 119–139). Dordrecht: Springer, http://dx.doi.org/10.1007/978-94-007-2321-4.

Trninic, D., & Abrahamson, D. (2016). Making direct instruction and discovery learning play along: Restoring the historical educational role of practice. In *Annu. meet. am. educ. res. assoc*.

Trninic, D., Gutiérrez, J. F., & Abrahamson, D. (2011). Virtual mathematical inquiry: Problem solving at the gestural-symbolic interface of remote-control embodied-interaction design. In *Int. comput. collab. learn. conf.* (pp. 272–279).

Trninic, D., Reinholz, D. L., Howison, M., & Abrahamson, D. (2010). Design as an object-to-think-with: Semiotic potential emerges through collaborative reflective conversation with material. In *Proc. 32nd annu. meet. north am. chapter int. gr. psychol. math. educ.* (pp. 1523–1530).

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: cognitive science and human experience*. Cambridge, MA: MIT Press.

Vygotsky, L. S. (1987). Collected works of l.s. vygotsky (volume 1): Problems of general psychology, including the volume thinking and speech.

Wilson, A. D., & Golonka, S. (2013). Embodied cognition is not what you think it is. *Frontiers of Psychology*, *4*, 58. http://dx.doi.org/10.3389/fpsyg.2013.00058.