

Polynomial Tau-Functions for the Multicomponent KP Hierarchy

by

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Abstract

In a previous paper we constructed all polynomial tau-functions of the 1-component KP hierarchy, namely, we showed that any such tau-function is obtained from a Schur polynomial $s_\lambda(t)$ by certain shifts of arguments. In the present paper we give a simpler proof of this result, using the (1-component) boson–fermion correspondence. Moreover, we show that this approach can be applied to the s -component KP hierarchy, using the s -component boson–fermion correspondence, finding thereby all its polynomial tau-functions. We also find all polynomial tau-functions for the reduction of the s -component KP hierarchy, associated to any partition consisting of s positive parts.

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§1. Introduction

In his paper [6], Sato introduced the (1-component) KP hierarchy of evolution equations of Lax type on the pseudo-differential operator

$$(1) \quad L = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2} + \cdots,$$

where $t = (t_1, t_2, \dots)$ and $\partial = \frac{\partial}{\partial t_1}$, and the corresponding tau-function is $\tau(t)$. Moreover, for any positive integer s , he also introduced the s -component KP hierarchy on the $(s \times s)$ -matrix pseudo-differential operator L of the form (1), where $u_i(t)$ are $(s \times s)$ -matrices and $t = (t_j^{(k)} \mid j = 1, 2, \dots; k = 1, 2, \dots, s)$ and $\partial = \frac{\partial}{\partial t_1^{(1)}} + \cdots + \frac{\partial}{\partial t_1^{(s)}}$, along with certain subsidiary equations (which are absent for $s = 1$).

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In the subsequent paper [2], the tau-function was introduced for the s -component KP hierarchy, generalizing the one by Sato for $s = 1$. The theory was further developed in [3]. Since any solution of the s -component KP hierarchy is explicitly expressed in terms of the tau-function, it is important to construct the latter.

In our paper [4], we constructed all polynomial tau-functions of the 1-component KP hierarchy, namely, we showed that any such tau-function is obtained from a Schur polynomial $s_\lambda(t)$ through certain shifts of arguments. In the present paper we give a simpler proof of this result (Theorem 2), using the (1-component) boson–fermion correspondence. Moreover, we show that this approach can be applied to the s -component KP hierarchy, using the s -component boson–fermion correspondence, thereby finding all its polynomial tau-functions (Theorem 4).

In [3] we studied the reduction of the s -component KP hierarchy associated to any s -part partition $\lambda = \{n_1 \geq n_2 \geq \cdots \geq n_s > 0\}$. A special case is the Gelfand–Dickey n th KdV, associated to the 1-part partition $\lambda = \{n\}$, for which we found in [4] all polynomial tau-functions. In the present paper we re-prove this result, using boson–fermion correspondence (Theorem 6). We use the same method to find all polynomial tau-functions for the λ -reduced s -component KP hierarchy (Theorem 7). In the conclusion of the paper we consider, as the simplest example beyond the 1-part partition, the 2-part partition $1 + 1$. This produces the AKNS (or nonlinear Schrödinger hierarchy; see [3]), and we find all its polynomial tau-functions.

§2. The fermionic formulation of the KP hierarchy

The group GL_∞ , consisting of all complex matrices $G = (g_{ij})_{i,j \in \mathbb{Z}}$ which are invertible and all but a finite number of $g_{ij} - \delta_{ij}$ are 0, acts on the vector space $\mathbb{C}^\infty = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}e_j$, via the formula $E_{ij}(e_k) = \delta_{jk}e_i$.

The semi-infinite wedge space (see e.g. [5, 3]) $F = \Lambda^{\frac{1}{2}\infty} \mathbb{C}^\infty$ is the vector space with a basis consisting of all semi-infinite monomials of the form $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots$, where $i_1 > i_2 > i_3 > \cdots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell \gg 0$. One defines the representation R of GL_∞ on F by

$$R(G)(e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots) = Ge_{i_1} \wedge Ge_{i_2} \wedge Ge_{i_3} \wedge \cdots,$$

and apply linearity and anticommutativity of the wedge product \wedge .

The corresponding representation r of the Lie algebra \mathfrak{gl}_∞ of GL_∞ can be described in terms of wedging and contracting operators ψ_j^+ and ψ_j^- ($j \in \mathbb{Z} + \frac{1}{2}$) on F :

$$\psi_j^+(e_{i_1} \wedge e_{i_2} \wedge \cdots) = e_{-j+\frac{1}{2}} \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots,$$

$$\psi_j^-(e_{i_1} \wedge e_{i_2} \wedge \cdots) = \begin{cases} 0 & \text{if } j - \frac{1}{2} \neq i_s \text{ for all } s, \\ (-1)^{s+1} e_{i_1} \wedge \cdots \\ \quad \wedge e_{i_{s-1}} \wedge e_{i_{s+1}} \wedge \cdots & \text{if } j = i_s - \frac{1}{2}. \end{cases}$$

These operators satisfy the relations of a Clifford algebra, which we denote by $\mathcal{Cl}(i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -)$:

$$\psi_i^\lambda \psi_j^\mu + \psi_j^\mu \psi_i^\lambda = \delta_{\lambda, -\mu} \delta_{i, -j}.$$

Let $(k \in \mathbb{Z})$

$$(2) \quad |k\rangle = e_k \wedge e_{k-1} \wedge e_{k-2} \wedge \cdots ;$$

then F is an irreducible \mathcal{Cl} -module, such that

$$\psi_j^\pm |0\rangle = 0 \quad \text{for } j > 0.$$

The representation r of \mathfrak{gl}_∞ in F , corresponding to the representation R of GL_∞ , is given by the formula $r(E_{ij}) = \psi_{-i+\frac{1}{2}}^+ \psi_{j-\frac{1}{2}}^-$. Define the *charge decomposition*

$$F = \bigoplus_{k \in \mathbb{Z}} F^{(k)}, \quad \text{where } \text{charge}(|k\rangle) = k \text{ and } \text{charge}(\psi_j^\pm) = \pm 1.$$

The space $F^{(k)}$ is an irreducible highest weight \mathfrak{gl}_∞ -module, with highest weight vector $|k\rangle$:

$$r(E_{ij})|k\rangle = 0 \text{ for } i < j, \quad r(E_{ii})|k\rangle = 0 \text{ (resp. } = |k\rangle) \text{ if } i > k \text{ (resp. if } i \leq k).$$

Let

$$\mathcal{O}_k = R(\mathrm{GL}_\infty)|k\rangle \subset F^{(k)}$$

be the GL_∞ -orbit of the highest weight vector $|k\rangle$.

Theorem 1 ([5, Thm. 5.1]). *Let $0 \neq g_k \in F^{(k)}$. Then $g_k \in \mathcal{O}_k$ if and only if*

$$(3) \quad \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ g_k \otimes \psi_{-i}^- g_k = 0.$$

Equation (3) is called the KP hierarchy in the fermionic picture.

Now recall that the orbit \mathcal{O}_k of $\mathbb{C}|k\rangle \in F^{(k)}$ of the group GL_∞ , for a fixed $k \in \mathbb{Z}$ can be decomposed as follows. Let P be the stabilizer of $\mathbb{C}|k\rangle$, let W be the subgroup of permutations of basis vectors of \mathbb{C}^∞ and let W_k be its subgroup, consisting of permutations permuting vectors with indices less than or equal to k between themselves. Then one has the Bruhat decomposition:

$$\mathrm{GL}_\infty = \bigcup_{w \in W/W_k} UwP \quad (\text{disjoint union}).$$

Applying this to $\mathbb{C}|k\rangle$, we obtain that the projectivized orbit $\mathbb{P}\mathcal{O}_k$ is a disjoint union of Schubert cells $C_w = Uw \cdot |k\rangle$, $w \in W/W_k$. It is well known (see e.g. [5]) that each $w \cdot |k\rangle$ corresponds to a partition $\lambda = \lambda(w)$, and the corresponding Schubert cell $C_\lambda = Uw(\lambda) \cdot |k\rangle$, where U is the subgroup of GL_∞ , consisting of upper triangular matrices with 1s on the diagonal, is an affine algebraic variety isomorphic to $\mathbb{C}^{|\lambda|}$.

§3. The bosonic formulation of KP

Define the fermionic fields by $\psi^\pm(z) = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^\pm z^{-i - \frac{1}{2}}$ and the bosonic field $\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1} =: \psi^+(z)\psi^-(z)$. Then there exists a unique vector space isomorphism, called the boson-fermion correspondence, $\sigma: F \rightarrow B = \mathbb{C}[q, q^{-1}] \otimes \mathbb{C}[t_1, t_2, \dots]$ such that $\sigma(|k\rangle) = q^k$, $\sigma\alpha_n\sigma^{-1} = \frac{\partial}{\partial t_n}$, $\sigma\alpha_{-n}\sigma^{-1} = nt_n$, for $n > 0$ and $\sigma\alpha_0\sigma^{-1} = q\frac{\partial}{\partial q}$. Moreover, one has

$$(4) \quad \sigma\psi^\pm(z)\sigma^{-1} = q^{\pm 1} z^{\pm q \frac{\partial}{\partial q}} \exp\left(\pm \sum_{k=1}^{\infty} t_k z^k\right) \exp\left(\mp \sum_{k=1}^{\infty} \frac{\partial}{\partial t_k} \frac{z^{-k}}{k}\right).$$

Note that $Q = \sigma^{-1}q\sigma$ is the following operator on F :

$$(5) \quad Q|k\rangle = |k+1\rangle \quad \text{and} \quad Q\psi_i^\pm = \psi_{i \mp 1}^\pm Q.$$

For $g_k \in \mathbb{P}\mathcal{O}_k \cup \{0\}$ we write $\sigma(g_k) = \tau_k(t)q^k$, where $t = (t_1, t_2, \dots)$. Such a τ_k is called a tau-function. It is well known (see e.g. [5]) that $\tau_k(t)$ is equal to the coefficient of $|k\rangle$ in $\exp(\sum_{i=1}^{\infty} t_i \alpha_i)g_k$. Under the isomorphism σ we can rewrite (3), using (4), to obtain a Hirota bilinear identity for tau-functions (see e.g. [2] or [5]). Let $[z] = (z, \frac{z^2}{2}, \frac{z^3}{3}, \dots)$, $y = (y_1, y_2, \dots)$, and $\mathrm{Res} \sum_i f_i z^i dz = f_{-1}$; then

$$(6) \quad \mathrm{Res} \tau_k(t - [z^{-1}])\tau_k(y + [z^{-1}]) \exp\left(\sum_{i=1}^{\infty} (t_i - y_i)z^i\right) dz = 0.$$

§4. Polynomial solutions of KP

Introduce the elementary Schur polynomials by

$$(7) \quad \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) = \sum_{j=0}^{\infty} s_j(t) z^j;$$

then in [4] we proved the following:

Theorem 2. *All polynomial tau-functions of the KP hierarchy are, up to a constant factor, of the form*

$$(8) \quad \begin{aligned} & \tau_{\lambda_1, \lambda_2, \dots, \lambda_m}(t; c_1, c_2, \dots, c_m) \\ &= \det(s_{\lambda_j + i - j}(t_1 + c_{1,j}, t_2 + c_{2,j}, t_3 + c_{3,j}, \dots))_{1 \leq i, j \leq m}, \end{aligned}$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is a partition and $c_j = (c_{1,j}, c_{2,j}, \dots) \in \mathbb{C}^{\lambda_j + m - j}$ are arbitrary.

Here we will give a simpler proof of this theorem.

Proof. Fix an integer k . Then the element $w(\lambda)|k\rangle$ in the Schubert cell C_λ of O_k , corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, is the element

$$(9) \quad e_{\lambda_1 + k} \wedge e_{\lambda_2 + k - 1} \wedge \dots \wedge e_{\lambda_m + k - m + 1} \wedge e_{k - m} \wedge e_{k - m - 1} \wedge \dots.$$

If we let an element $u \in U$, an upper triangular matrix with 1s on the diagonal, act on this element we obtain the element

$$(10) \quad f_1 \wedge f_2 \wedge \dots \wedge f_m \wedge e_{k - m} \wedge e_{k - m - 1} \wedge \dots,$$

where

$$(11) \quad f_j = e_{\lambda_j + k - j + 1} + \sum_{i=k-m+1}^{\lambda_j + k - j} a_{ij} e_i.$$

Note that we can use Gauss elimination with which we can eliminate certain a_{ij} , viz., we may assume that $a_{\lambda_i + k - i + 1, j} = 0$ for all $i > j$. This means that we can put $1 + 2 + \dots + m - 1 = \frac{1}{2}(m - 1)m$ constants to zero, all others are arbitrary. Counting all coefficients which are arbitrary, we find

$$-\frac{1}{2}(m - 1)m + \sum_{j=1}^m (\lambda_j + k - j) - (k - m) = \frac{1}{2}(m - 1)m + \sum_{j=1}^m \lambda_j + \sum_{j=1}^m (m - j) = \sum_{j=1}^m \lambda_j.$$

Hence, we indeed find that the dimension of the Schubert cell C_λ is equal to $|\lambda|$.

Assume from now on that all a_{ij} of (11) are again arbitrary; then we want to calculate the image $\tau_\lambda^{(k)}(t)$ of the element (10) under the isomorphism σ . Recall, from e.g. [5], that $\tau_\lambda^{(k)}(t)$ is the coefficient of $|k\rangle$ in

$$(12) \quad \begin{aligned} & \exp\left(\sum_{i=1}^{\infty} t_i \alpha_i\right) f_1 \wedge \dots \wedge f_m \wedge e_{k - m} \wedge e_{k - m - 1} \wedge \dots \\ &= f_1(t) \wedge \dots \wedge f_m(t) \wedge e_{k - m} \wedge e_{k - m - 1} \wedge \dots, \end{aligned}$$

where, since α_i is a derivation of the exterior product such that $\alpha_i(e_j) = e_{j-i}$, we have

$$\begin{aligned}
 f_j(t) &= \exp\left(\sum_{i=1}^{\infty} t_i \alpha_i\right) f_j \\
 (13) \quad &= \sum_{\ell=0}^{\infty} s_{\ell}(t) \left(e_{\lambda_j-j+1+k-\ell} + \sum_{i=k-m+1}^{\lambda_j-j+k} a_{ij} e_{i-\ell} \right) \\
 &= \sum_{p=0}^{\infty} \left(s_p(t) + \sum_{i=1}^p a_{\lambda_j-j+1+k-i,j} s_{p-i}(t) \right) e_{\lambda_j-j+1+k-p}.
 \end{aligned}$$

Since $e_{k-m}, e_{k-m-1}, \dots$ appear in (10), we can replace $f_j(t)$ by the element (13) where all e_s with $s \leq k-m$ are removed. In other words, we may assume that

$$(14) \quad f_j(t) = \sum_{p=0}^{\lambda_j-j+m} \left(s_p(t) + \sum_{i=1}^p a_{\lambda_j-j+1+k-i,j} s_{p-i}(t) \right) e_{\lambda_j-j+1+k-p}.$$

Since, by (7), $\frac{\partial s_a(t)}{\partial t_1} = s_{a-1}(t)$, equation (14) can be rewritten as

$$\begin{aligned}
 (15) \quad f_j(t) &= \sum_{r=0}^{\lambda_j-j+m} \frac{\partial^r \left(s_{\lambda_j-j+m}(t) + \sum_{i=1}^{\lambda_j-j+m} a_{\lambda_j-j+1+k-i,j} s_{\lambda_j-j+m-i}(t) \right)}{\partial t_1^r} \\
 &\quad \times e_{k-m+1+r}.
 \end{aligned}$$

Next, it follows from (7) that

$$(16) \quad s_j(t+c) = \sum_{i=0}^j s_{j-i}(c) s_i(t).$$

Note also that the map $(s_1, \dots, s_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$ is surjective. Hence there exist constants $c_j = (c_{1,j}, c_{2,j}, \dots)$, such that

$$(17) \quad a_{k-p+1,j} = s_{\lambda_j-j+p}(c_j).$$

It follows from (16) that

$$s_{\lambda_j+m-j}(t+c_j) = s_{\lambda_j+m-j}(t) + \sum_{\ell=0}^{\lambda_j+m-j-1} s_{\lambda_j+m-j-\ell}(c_j) s_{\ell}(t).$$

Thus, using (17), we obtain from (15),

$$\begin{aligned}
 f_j(t) &= \sum_{r=0}^{\lambda_j-j+m} \frac{\partial^r s_{\lambda_j-j+m}(t+c_j)}{\partial t_1^r} e_{k-m+r+1} \\
 (18) \qquad &= \sum_{r=0}^{\lambda_j-j+m} s_{\lambda_j-j+m-r}(t+c_j) e_{k-m+r+1}.
 \end{aligned}$$

Hence the coefficient of $|k\rangle$ in (12) is equal to (cf. [5])

$$\det(s_{\lambda_j+i-j}(t_1+c_{1,j}, t_2+c_{2,j}, t_3+c_{3,j}, \dots))_{1 \leq i, j \leq m},$$

which is the desired result. \square

Remark 3. Different $c_{i,j}$ can give the same polynomial solutions in C_λ . From the proof of Theorem 2, it is clear how to obtain a one-to-one correspondence in terms of the constants a_{ij} , viz. one has to assume that all $a_{\lambda_i+k-i+1,j} = 0$ for all $1 \leq j < i \leq m$. Using (17) this means that $s_{\lambda_j-j-\lambda_i+i}(c_j) = 0$, for all $1 \leq j < i \leq m$, or stated differently,

$$\begin{aligned}
 (19) \qquad &c_{\lambda_j-j-\lambda_i+i,j} \\
 &= -s_{\lambda_j-j-\lambda_i+i}(c_{1,j}, c_{2,j}, \dots, c_{\lambda_j-j-\lambda_i+i-1,j}, 0) \quad \text{for all } 1 \leq j < i \leq m.
 \end{aligned}$$

This means that for fixed j we find constants $c_{i,j}$ for $1 \leq i \leq \lambda_j - j + m$, of which we can eliminate all $c_{\lambda_j-j-\lambda_i+i,j}$, with $j < i \leq m$, by using formula (19) recursively. Note that for fixed j there are $\lambda_j - j + m$ constants $c_{i,j}$ of which $m - j$ can be eliminated. Thus we have λ_j constants $c_{i,j}$. Hence in total, we have $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_m$ constants. This agrees with the fact that $\dim C_\lambda = |\lambda|$.

§5. The multicomponent KP

In this section we introduce the s -component KP, where s is a positive integer. In [3] we introduced a relabeling of the e_i to obtain the multicomponent KP. One approach would be to use this relabeling and describe all vectors (11) of the semi-infinite wedges (10) in terms of these new relabeled e_i and determine its corresponding tau-functions, using the multicomponent bosonization. But in that case the solution depends on the relabeling and one does not obtain all possible solutions, but one obtains all solutions related to that relabeling. To obtain all solutions one has to calculate all solutions corresponding to all possible relabelings. Instead, we will introduce a new basis of \mathbb{C}^∞ , viz., $e_i^{(a)}$ where $1 \leq a \leq s$ and $i \in \mathbb{Z}$. We assume that $|0\rangle$ is the semi-infinite wedge vector consisting of all $e_i^{(a)}$ with $1 \leq a \leq s$ and $i \leq 0$. We introduce creation (+) and annihilation (−) operators,

$\psi_i^{\pm(a)}$, with, as before, $i \in \frac{1}{2} + \mathbb{Z}$, such that ($\lambda, \mu = +$ or $-$)

$$\psi_i^{\lambda(a)} \psi_j^{\mu(b)} + \psi_j^{\mu(b)} \psi_i^{\lambda(a)} = \delta_{\lambda, -\mu} \delta_{ab} \delta_{i, -j}$$

and

$$\psi_i^{\pm(a)}|0\rangle = 0 \text{ for } i > 0 \quad \text{and} \quad \psi_i^{+(a)}|0\rangle = e_{-i+\frac{1}{2}}^{(a)} \wedge |0\rangle.$$

We have the same charge decomposition, where now the charge of $|0\rangle$ is again 0 and the charge of $\psi_i^{\pm(a)}$ is ± 1 . The disadvantage of this approach is that we cannot describe $\sigma(|k\rangle)$, for $k \neq 0$, explicitly.

The KP hierarchy (3) thus turns into the s -component KP in the fermionic picture:

$$(20) \quad \sum_{a=1}^s \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^{+(a)} g_k \otimes \psi_{-i}^{-(a)} g_k = 0, \quad g_k \in F^{(k)}.$$

Define the fermionic fields by $\psi^{\pm(a)}(z) = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^{\pm(a)} z^{-i-\frac{1}{2}}$ and the bosonic fields by $\alpha^{(a)}(z) = \sum_{n \in \mathbb{Z}} \alpha_n^{(a)} z^{-n-1} =: \psi^{+(a)}(z) \psi^{-(a)}(z)$, where $a = 1, 2, \dots, s$. As in the 1-component case we need additional operators Q_a , $a = 1, 2, \dots, s$ to define the boson–fermion correspondence. They are uniquely defined by

$$(21) \quad Q_a |0\rangle = \psi_{-\frac{1}{2}}^{+(a)} |0\rangle \quad \text{and} \quad Q_a \psi_i^{\pm(b)} = (-1)^{1-\delta_{ab}} \psi_{i \mp \delta_{ab}}^{\pm(b)} Q_a.$$

It is straightforward to check that Q_a and Q_b anticommute if $a \neq b$ and that

$$Q_a \alpha_j^{(b)} = \alpha_j^{(b)} Q_a - \delta_{ab} \delta_{j0} Q_a, \quad 1 \leq a, b \leq s.$$

Then as before there exists a unique vector space isomorphism $\sigma: F \rightarrow B = \mathbb{C}[q_a, q_a^{-1}; 1 \leq a \leq s] \otimes \mathbb{C}[t_1^{(a)}, t_2^{(a)}, \dots; 1 \leq a \leq s]$ such that $\sigma(|0\rangle) = 1$, $\sigma \alpha_n^{(a)} \sigma^{-1} = \frac{\partial}{\partial t_n^{(a)}}$, $\sigma \alpha_{-n}^{(a)} \sigma^{-1} = n t_n^{(a)}$, for $n > 0$ and $\sigma \alpha_0^{(a)} \sigma^{-1} = q_a \frac{\partial}{\partial q_a}$. Then $\sigma Q_a \sigma^{-1} = \epsilon_a q_a$, where $\epsilon_a q_b = (-1)^{1-\delta_{ab}} q_b \epsilon_a$ and $\epsilon_a |0\rangle = |0\rangle$. Moreover, one has (cf. [3])

$$(22) \quad \sigma \psi^{\pm(a)}(z) \sigma^{-1} = \epsilon_a q_a^{\pm 1} (z)^{\pm q_a \frac{\partial}{\partial q_a}} \exp \left(\pm \sum_{k=1}^{\infty} t_k^{(a)} z^k \right) \exp \left(\mp \sum_{k=1}^{\infty} \frac{\partial}{\partial t_k^{(a)}} \frac{z^{-k}}{k} \right).$$

For $g_0 \in \mathcal{O}_0 \cup \{0\}$, where $\mathcal{O}_0 = R(\text{GL}_{\infty})|0\rangle \subset F^{(0)}$ we write

$$\sigma(g_0) = \sum_{\substack{m_1, \dots, m_s \in \mathbb{Z} \\ m_1 + \dots + m_s = 0}} \tau^{(m_1, m_2, \dots, m_s)}(t) q_1^{m_1} q_2^{m_2} \dots q_s^{m_s} \quad \text{and} \quad \sigma(|0\rangle) = 1,$$

where $t = (t_k^{(a)})_{a=1, \dots, s, k=1, 2, \dots}$. Then, as before,

$$(23) \quad \tau^{(m_1, m_2, \dots, m_s)}(t) = \text{coefficient of } Q_1^{m_1} Q_2^{m_2} \dots Q_s^{m_s} |0\rangle \text{ in } \exp \left(\sum_{a=1}^s \sum_{i=1}^{\infty} t_i^{(a)} \alpha_i^{(a)} \right) g_0.$$

Under the isomorphism σ we can rewrite (20) for $k = 0$ using (22), to obtain a Hirota bilinear identity for these tau-functions:

$$\begin{aligned}
 (24) \quad & \text{Res } dz \sum_{a=1}^s (-1)^{m_1 + \dots + m_{a-1} + q_1 + \dots + q_{a-1}} z^{m_a - q_a - 2} \exp \left(\sum_{i=1}^{\infty} (t_i^{(a)} - y_i^{(a)}) z^i \right) \\
 & \times \exp \left(\sum_{i=1}^{\infty} \frac{\frac{\partial}{\partial y_i^{(a)}} - \frac{\partial}{\partial t_i^{(a)}}}{i} z^{-i} \right) \tau^{(m_1, \dots, m_{a-1}, m_a - 1, m_{a+1}, \dots, m_s)}(t) \\
 & \times \tau^{(q_1, \dots, q_{a-1}, q_a + 1, q_{a+1}, \dots, q_s)}(y) = 0.
 \end{aligned}$$

Note that this equation also holds for $k \neq 0$. As in the 1-component case, we want to describe all polynomial tau-functions of this hierarchy. Of course in this case this is a collection of tau-functions. The approach is similar. First of all, in (20) we let $k = m$, a positive integer, and, as in the 1-component case, we would like to consider g_m instead of g_0 , where m is a positive integer such that a polynomial tau-function corresponds to an element

$$(25) \quad f_1 \wedge f_2 \wedge \dots \wedge f_m \wedge |0\rangle,$$

where

$$(26) \quad f_j = \sum_{a=1}^s \sum_{\ell=1}^{M_j^{(a)}} b_{\ell j}^{(a)} e_{\ell}^{(a)}, \quad j = 1, \dots, m, \text{ with } M_j^{(a)} \geq 1, b_{\ell j}^{(a)} \in \mathbb{C}.$$

Note that we may assume this without loss of generality. Indeed, instead of calculating (25), one could also calculate

$$f_1 \wedge f_2 \wedge \dots \wedge f_m \wedge Q_1^{r_1} Q_2^{r_2} \dots Q_s^{r_s} |0\rangle.$$

This gives the same polynomial tau-function but translated over the lattice

$$(m_1, m_2, \dots, m_s) \mapsto (m_1 + r_1, m_2 + r_2, \dots, m_s + r_s),$$

i.e. now

$$f_j = \sum_{a=1}^s \sum_{\ell=1}^{M_j^{(a)}} b_{\ell j}^{(a)} e_{\ell + r_a}^{(a)}, \quad \text{with } M_j^{(a)} \geq 1, b_{\ell j}^{(a)} \in \mathbb{C}.$$

The first step is to determine (cf. (13))

$$\begin{aligned}
 (27) \quad & f_j(t) = \exp \left(\sum_{a=1}^s \sum_{i=1}^{\infty} t_i^{(a)} \alpha_i^{(a)} \right) f_j \\
 & = \sum_{a=1}^s \sum_{\ell=1}^{M_j^{(a)}} \sum_{i=0}^{\infty} b_{\ell j}^{(a)} s_i(t^{(a)}) e_{\ell - i}^{(a)}.
 \end{aligned}$$

Note that we can remove all $e_i^{(a)}$ with $i \leq 0$ in the above expression, since these elements already appear in $|0\rangle$. Thus

$$(28) \quad f_j(t) = \sum_{a=1}^s \sum_{\ell=1}^{M_j^{(a)}} \sum_{i=0}^{M_j^{(a)}-\ell} b_{\ell+i,j}^{(a)} s_i(t^{(a)}) e_\ell^{(a)}.$$

As before, the coefficient of $e_\ell^{(a)}$, which is equal to $\sum_{i=0}^{M_j^{(a)}-\ell} b_{\ell+i,j}^{(a)} s_i(t^{(a)})$, is the ℓ th derivative of $\sum_{i=0}^{M_j^{(a)}} b_{i,j}^{(a)} s_i(t^{(a)})$ with respect to $t_1^{(a)}$. As in the 1-component case, we can find constants $c_j^{(a)} = (c_{1,j}^{(a)}, c_{2,j}^{(a)}, \dots)$ such that a sum of elementary Schur functions can be expressed as one elementary Schur function with shifted t . Thus,

$$\sum_{i=0}^{M_j^{(a)}} b_{i,j}^{(a)} s_i(t^{(a)}) = b_{M_j^{(a)},j}^{(a)} s_{M_j^{(a)}}(t^{(a)} + c_j^{(a)})$$

and

$$f_j(t) = \sum_{a=1}^s b_{M_j^{(a)},j}^{(a)} \sum_{\ell=1}^{M_j^{(a)}} \frac{\partial^\ell s_{M_j^{(a)}}(t^{(a)} + c_j^{(a)})}{\partial(t_1^{(a)})^\ell} e_\ell^{(a)}.$$

Next, define

$$(29) \quad h_j(t) = \sum_{a=1}^s b_{M_j^{(a)},j}^{(a)} s_{M_j^{(a)}}(t^{(a)} + c_j^{(a)});$$

then

$$(30) \quad f_j(t) = \sum_{a=1}^s \sum_{\ell=1}^{M_j^{(a)}} \frac{\partial^\ell h_j(t)}{\partial(t_1^{(a)})^\ell} e_\ell^{(a)}.$$

Now, $\tau^{(m_1, m_2, \dots, m_s)}(t)$ is the coefficient of

$$(31) \quad Q_1^{m_1} Q_2^{m_2} \dots Q_s^{m_s} |0\rangle = e_{m_1}^{(1)} \wedge \dots \wedge e_1^{(1)} \wedge e_{m_2}^{(2)} \wedge \dots \wedge e_1^{(2)} \wedge e_{m_3}^{(3)} \wedge \dots \wedge e_1^{(s)} \wedge |0\rangle$$

in (25) where $f_j(t)$ is given by (30). It follows from (25) and (26) that all $m_a \geq 0$ and $m_1 + m_2 + \dots + m_s = m$. Hence, all the labels of nonzero tau-functions lie in the convex polyhedron

$$\{(m_1, m_2, \dots, m_s) \in \mathbb{Z}^s \mid m_i \geq 0 \text{ and } m_1 + m_2 + \dots + m_s = m\}.$$

It is straightforward to calculate the coefficient of (31) in (25). As in the 1-component case it is the determinant of a matrix whose $(m_1 + m_2 + \dots + m_{a-1} + i, j)$ th entry, with $1 \leq i \leq m_a$, is the coefficient of $e_{m_a-i+1}^{(a)}$ of $f_j(t)$. By (30) this coefficient is equal to $\frac{\partial^{m_a-i+1} h_j(t)}{\partial(t_1^{(a)})^{m_a-i+1}}$. More explicitly, since the group orbits in the

1- and multicomponent cases are the same, we only have another realization of the module and hence the orbit. The calculations prove the following:

Theorem 4. *All polynomial tau-functions of the s -component KP hierarchy (24) are, up to a shift over the lattice, of the form*

$$\tau_m(q, t) = \sum_{\substack{m_i \geq 0, \\ m_1 + \dots + m_s = m}} \tau^{(m_1, m_1, \dots, m_s)}(t) q_1^{m_1} q_2^{m_2} \dots q_s^{m_s},$$

where m is a positive integer, and $\tau^{(m_1, m_1, \dots, m_s)}(t)$ is given by

$$(32) \quad \tau^{(m_1, m_1, \dots, m_s)}(t) = \det \begin{pmatrix} \frac{\partial^{m_1} h_1(t)}{\partial (t_1^{(1)})^{m_1}} & \frac{\partial^{m_1} h_2(t)}{\partial (t_1^{(1)})^{m_1}} & \dots & \frac{\partial^{m_1} h_m(t)}{\partial (t_1^{(1)})^{m_1}} \\ \frac{\partial^{m_1-1} h_1(t)}{\partial (t_1^{(1)})^{m_1-1}} & \frac{\partial^{m_1-1} h_2(t)}{\partial (t_1^{(1)})^{m_1-1}} & \dots & \frac{\partial^{m_1-1} h_m(t)}{\partial (t_1^{(1)})^{m_1-1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_1(t)}{\partial t_1^{(1)}} & \frac{\partial h_2(t)}{\partial t_1^{(1)}} & \dots & \frac{\partial h_m(t)}{\partial t_1^{(1)}} \\ \hline \frac{\partial^{m_2} h_1(t)}{\partial (t_1^{(2)})^{m_2}} & \frac{\partial^{m_2} h_2(t)}{\partial (t_1^{(2)})^{m_2}} & \dots & \frac{\partial^{m_2} h_m(t)}{\partial (t_1^{(2)})^{m_2}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_1(t)}{\partial t_1^{(2)}} & \frac{\partial h_2(t)}{\partial t_1^{(2)}} & \dots & \frac{\partial h_m(t)}{\partial t_1^{(2)}} \\ \hline \vdots & \vdots & & \vdots \\ \hline \frac{\partial^{m_s} h_1(t)}{\partial (t_1^{(s)})^{m_s}} & \frac{\partial^{m_s} h_2(t)}{\partial (t_1^{(s)})^{m_s}} & \dots & \frac{\partial^{m_s} h_m(t)}{\partial (t_1^{(s)})^{m_s}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_1(t)}{\partial t_1^{(s)}} & \frac{\partial h_2(t)}{\partial t_1^{(s)}} & \dots & \frac{\partial h_m(t)}{\partial t_1^{(s)}} \end{pmatrix}.$$

Here,

$$h_j(t) = \sum_{a=1}^s b_{M_j^{(a)}, j} s_{M_j^{(a)}}(t^{(a)} + c_j^{(a)}),$$

where $M_j^{(a)}$ are arbitrary positive integers, $c_j^{(a)} = (c_{1,j}^{(a)}, c_{2,j}^{(a)}, c_{3,j}^{(a)}, \dots)$ and $b_{M_j^{(a)}, j}$ for $1 \leq j \leq m$ and $1 \leq a \leq s$ are arbitrary constants.

§6. The n -KdV

Let n be an integer, $n \geq 2$. The n th Gelfand–Dickey hierarchy, or n -KdV, describes in the 1-component case the group orbit in a projective representation of the loop group of SL_n . This is not a subgroup of Gl_∞ . One has to take a bigger group,

$\widehat{\text{GL}}_\infty$, containing it, as e.g. in [5]. Then the representation R of GL_∞ extends to a projective representation, denoted by \widehat{R} , of $\widehat{\text{GL}}_\infty$. We obtain the loop algebra $\text{gl}_n(\mathbb{C}[x, x^{-1}])$ of gl_n as follows (see [5]). Let $\mathbb{C}[x, x^{-1}]$ be the algebra of Laurent polynomials L and denote by u_1, u_2, \dots, u_n a basis of \mathbb{C}^n . Identifying

$$x^{-k}u_j = e_{nk+j},$$

we can identify

$$x^k E_{ij} = \sum_{\ell \in \mathbb{Z}} E_{\ell n + i, (\ell + k)n + j}.$$

As explained in [5, Lem. 9.1], the centralizer in $\widehat{\text{GL}}_\infty$ of all operators $\Lambda_n^j = \sum_{\ell \in \mathbb{Z}} E_{\ell, \ell + jn}$, $j \in \mathbb{Z}$, is the central extension of the group of $\text{SL}_n(\mathbb{C}[t, t^{-1}])$ times the scalar operators. Furthermore, the orbit in $F^{(k)}$ of the vacuum vector $|k\rangle$ under the action of the latter group is the intersection of $\text{GL}_\infty|k\rangle$ with the kernels of all operators Λ_n^j , $j = 1, 2, \dots$. Hence all polynomial tau-functions of the n -KdV hierarchy correspond to the vectors (10) in $F^{(k)}$, satisfying the condition

$$(33) \quad \hat{r}(\Lambda_n^j)(f_1 \wedge f_2 \wedge \dots \wedge f_m \wedge e_{k-m} \wedge e_{k-m-1} \wedge \dots) = 0, \quad j = 1, 2, \dots$$

Since $\hat{r}(\Lambda_n^j) = \alpha_{jn}$, we find that the corresponding polynomial tau-function $\tau_k \in \mathcal{O}_k$ satisfies $\frac{\partial \tau_k}{\partial t_{jn}} = 0$ for $j \geq 1$. Using this, we can differentiate the KP hierarchy (6) by t_{jn} , which gives the n -KdV hierarchy of Hirota bilinear equations on tau-functions:

$$(34) \quad \text{Res } z^{jn} \tau_k(t - [z^{-1}]) \tau_k(y + [z^{-1}]) \exp \left(\sum_{i=1}^{\infty} (t_i - y_i) z^i \right) dz = 0, \\ j = 0, 1, 2, \dots$$

Let \widehat{G} be the central extension of the group of $\text{SL}_n(\mathbb{C}[t, t^{-1}])$ times the scalar operators. Then equation (34) indeed describes the \widehat{G} -orbit of $|k\rangle$ of nonzero tau-functions. Note that $\sigma(\widehat{R}(A)|k + jn) = \tau_k(t) q^{k+jn} = q^{jn} \sigma(\widehat{R}(A)|k)$, $j \in \mathbb{Z}$, and $A \in \widehat{G}$, since $A = (a_{ij})$ has the property that $a_{i+n, j+n} = a_{ij}$. Thus (see e.g. [5, Thm. 5.1]) τ_k indeed satisfies (34).

Next, (34) implies that

$$(35) \quad \text{Res } \frac{\partial \tau_k(t - [z^{-1}])}{\partial t_{jn}} \tau_k(y + [z^{-1}]) \exp \left(\sum_{i=1}^{\infty} (t_i - y_i) z^i \right) dz = 0, \quad j = 0, 1, 2, \dots$$

Now divide this equation by $\frac{\partial \tau_k(t)}{\partial t_{jn}} \tau_k(y)$ and let

$$Q(t, z) = \frac{1}{\frac{\partial \tau_k(t)}{\partial t_{jn}}} \frac{\partial \tau_k(t - [z^{-1}])}{\partial t_{jn}}, \quad P(t, z) = \frac{\tau_k(t - [z^{-1}])}{\tau_k(t)}.$$

Then, using pseudo-differential operators (see e.g. [3] or [4]), equation (35) implies $(Q(t, \partial) \circ P(t, \partial)^{-1})_- = 0$, which gives that $Q(t, \partial) \circ P(t, \partial)^{-1} = 1$. Thus $P(t, \partial) = Q(t, \partial)$, from which one can deduce that $\frac{\partial \tau_k(t)}{\partial t_{jn}}$ is proportional to $\tau_k(t)$. Since we assume that τ_k is a polynomial, this scalar must be 0. Hence, $\sigma^{-1}(\tau_k(t)q^k)$ is in the intersection of $\text{GL}_\infty|k\rangle$ with the kernels of all operators Λ_n^j , $j = 1, 2, \dots$. Therefore $\sigma^{-1}(\tau_k(t)q^k)$ is in the \widehat{G} -orbit of $|k\rangle$.

Now, equation (34) for $j = 1$ is the n th modified KP equation. We assume that $\tau_k(y)$, the second tau-function in (34), lies in the k th charge sector and corresponds to

$$F_k = f_1 \wedge f_2 \wedge \cdots \wedge f_m \wedge e_{k-m} \wedge e_{k-m-1} \wedge \cdots,$$

where we may assume that the f_j are a linear combination of e_i with $i > k - m$. Then $\tau_k(t)$, the first tau-function in (34), lies in the $(k + n)$ th charge sector (recall $j = 1$) and thus corresponds to

$$F_{k+n} = \Lambda_n^{-1} f_1 \wedge \Lambda_n^{-1} f_2 \wedge \cdots \wedge \Lambda_n^{-1} f_m \wedge e_{k-m+n} \wedge e_{k-m+n-1} \wedge \cdots.$$

Moreover, in the Grassmannian picture, this n -modified KP equation means that the linear span of the factors in F_k is a linear subspace, of codimension n , of the span of the factors in F_{k+n} ([5]). Thus every f_j is a linear combination of $\Lambda_n^{-1} f_1, \Lambda_n^{-1} f_2, \dots, \Lambda_n^{-1} f_m, e_{k-m+n}, e_{k-m+n-1}, \dots$. Thus $\Lambda_n f_j$ is a linear combination of $f_1, f_2, \dots, f_m, e_{k-m}, e_{k-m-1}, \dots, e_{k-m-n+1}$. Hence, one can choose g_i , $1 \leq i \leq r < s$ of the form

$$(36) \quad g_j = e_{\mu_j} + \sum_{i=k-m+1}^{\mu_j-1} c_{i,j} e_i$$

such that

$$(37) \quad \begin{aligned} & f_1 \wedge f_2 \wedge \cdots \wedge f_m \wedge e_{k-m} \wedge e_{k-m-1} \wedge \cdots \\ &= c g_1 \wedge \Lambda_n g_1 \wedge \cdots \wedge \Lambda_n^{\lceil \frac{\mu_1 - k + m - n}{n} \rceil} g_1 \wedge g_2 \wedge \cdots \wedge \Lambda_n^{\lceil \frac{\mu_2 - k + m - n}{n} \rceil} g_2 \wedge g_3 \\ & \quad \wedge \cdots \wedge \Lambda_n^{\lceil \frac{\mu_r - k + m - n}{n} \rceil} g_r \wedge e_{k-m} \wedge e_{k-m-1} \wedge \cdots, \end{aligned}$$

for certain $0 \neq c \in \mathbb{C}$. Here $\lceil x \rceil$ stands for the ceiling of x , i.e. the smallest integer greater than or equal to x . One finds these g_i as follows. In this construction we assume that m is minimal. Choose $g_1 = f_1$; then all $\Lambda_n^p g_1$, $p \geq 0$ are in the span of the factors in F_k . Moreover, $\Lambda_n^{\lceil \frac{\mu_1 - k + m - n}{n} \rceil} g_1$ still contains vectors e_i with $i > m - k$, but $\Lambda_n^{\lceil \frac{\mu_1 - k + m - n}{n} \rceil + 1} g_1$ is expressed in the e_i with $i \leq m - k$, hence it lies in the span of the factors in $|m - k\rangle$ and $\Lambda_n^{\lceil \frac{\mu_1 - k + m - n}{n} \rceil} g_1$ does not. Next, from the vectors f_2, f_3, \dots, f_m choose the first one, say f_{i_2} , that is not in the span of $\Lambda_n^p g_1$, with

$0 \leq p \leq \lceil \frac{\mu_1 - k + m - n}{n} \rceil$, and set $g_2 = f_{i_2}$. Again, all $\Lambda_n^p g_2$, $p \geq 0$ are in the span of the factors in F_k and $\Lambda_n^{\lceil \frac{\mu_2 - k + m - n}{n} \rceil + 1} g_2$ is in the span of the factors in $|m - k\rangle$. Then choose $g_3 = f_{i_3}$, where f_{i_3} is the first f that is not in the span of $\Lambda_n^p g_a$, with $0 \leq p \leq \lceil \frac{\mu_a - k + m - n}{n} \rceil$, and $a = 1, 2$, and continue in this way. After $r < n$ choices this stops.

This makes it possible to obtain all polynomial tau-functions for the n -KdV hierarchy. To be more precise, in [4] we introduced to each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ the set

$$V_\lambda = \{\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots, \lambda_m - m + 1, -m, -m - 1, -m - 2, \dots\},$$

and gave the following definition.

Definition 5. A partition λ is called n -periodic if the corresponding infinite sequence V_λ is mapped to itself when subtracting n from each term.

This reflects condition (33). Thus we obtained the following ([4]):

Theorem 6. All polynomial tau-functions of the n -KdV hierarchy are, up to a constant factor, of the form

$$(38) \quad \begin{aligned} & \tau_{\lambda_1, \lambda_2, \dots, \lambda_k}^n(t; c_{\overline{\lambda_1}}, c_{\overline{\lambda_2 - 1}}, \dots, c_{\overline{\lambda_k - k + 1}}) \\ &= \det(s_{\lambda_i + j - i}(t_1 + c_{1, \overline{\lambda_i - i + 1}}, t_2 + c_{2, \overline{\lambda_i - i + 1}}, \dots))_{1 \leq i, j \leq k}, \end{aligned}$$

where $\bar{i} \equiv i \pmod n$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is an n -periodic partition. Here the $c_{\bar{i}} = (c_{1, \bar{i}}, c_{2, \bar{i}}, \dots)$ for $i = 1, 2, \dots, n$ (where at most $n - 1$ such \bar{i} appear) are arbitrary constants.

§7. The (n_1, n_2, \dots, n_s) -KdV

In this section we want to consider a reduction of the s -component KP hierarchy, which describes again the loop group orbit of SL_n , where $n = n_1 + n_2 + \dots + n_s$, with $n_1 \geq n_2 \geq \dots \geq n_s \geq 1$. The case $s = 1$ is the n th Gelfand–Dickey hierarchy. The case $n = s = 2$, i.e. $n_1 = n_2 = 1$, is the AKNS (or nonlinear Schrödinger) hierarchy.

From now on let

$$n = n_1 + n_2 + \dots + n_s, \quad \text{where } n_1 \geq n_2 \geq \dots \geq n_s \geq 1.$$

The identification with $\mathrm{gl}_n(\mathbb{C}[x, x^{-1}])$ is via

$$x^{-k} u_j^{(a)} = e_{n_a k + j}^{(a)}.$$

Here the $u_j^{(a)}$, with $1 \leq a \leq s$ and $1 \leq j \leq n_a$, form a basis of \mathbb{C}^n . In this case, it is convenient to relabel the 1-component basis in a periodic way to obtain the s -component one, i.e. if $e_j^{(a)} = e_i$, then $e_{i+n} = e_{j+n_a}^{(a)}$. As explained in Section 6, an element of the central extension of the loop group of SL_n commutes with all the elements

$$\Lambda^j := \Lambda_n^j = \sum_{a=1}^s \Lambda_{n_a}^{(a)j},$$

where

$$\Lambda_{n_a}^{(a)j} e_\ell^{(b)} = \delta_{ab} e_{\ell-jn_a}^{(a)} \quad \text{and} \quad \hat{r}(\Lambda_{n_a}^{(a)j}) = \alpha_{jn_a}^{(a)} \quad \text{for } j \geq 1.$$

Note that

$$(39) \quad \sigma \Lambda^j \sigma^{-1} = D_j := \sum_{a=1}^s \frac{\partial}{\partial t_{jn_a}^{(a)}}, \quad j = 1, 2, \dots$$

This means that the collection of polynomial tau-functions $\tau^{(m_1, m_2, \dots, m_s)}$ of the s -component KP hierarchy satisfies the conditions

$$(40) \quad D_j(\tau^{(m_1, m_2, \dots, m_s)}(t)) = \sum_{a=1}^s \frac{\partial \tau^{(m_1, m_2, \dots, m_s)}(t)}{\partial t_{jn_a}^{(a)}} = 0 \quad \text{for } j = 1, 2, \dots$$

Thus, letting D_j (only in t not in y) act on equation (24) gives zero on the tau-functions, but acting on the exponential it produces in every component a power of z^{jn_a} . Hence the (n_1, n_2, \dots, n_s) -KdV hierarchy is given by the following equations:

$$(41) \quad \begin{aligned} & \mathrm{Res} \, dz \sum_{a=1}^s (-1)^{m_1 + \dots + m_{a-1} + q_1 + \dots + q_{a-1}} z^{m_a - q_a + jn_a - 2} \\ & \times \exp \left(\sum_{i=1}^{\infty} (t_i^{(a)} - y_i^{(a)}) z^i \right) \exp \left(\sum_{i=1}^{\infty} \frac{\frac{\partial}{\partial y_i^{(a)}} - \frac{\partial}{\partial t_i^{(a)}}}{i} z^{-i} \right) \\ & \times \tau^{(m_1, \dots, m_{a-1}m_a-1, m_{a+1}, \dots, m_s)}(t) \tau^{(q_1, \dots, q_{a-1}, q_a+1, q_{a+1}, \dots, q_s)}(y) = 0, \\ & j = 0, 1, 2, \dots \end{aligned}$$

In a similar way to the 1-component case, see Section 6, one can deduce from equation (41), for $j \neq 0$, that (40) holds for polynomial tau-functions.

It is now straightforward to construct all polynomial solutions of this hierarchy. Without loss of generality we may assume again that $m = k$ in (37). As the g_j in (36), we choose $r < n$ linearly independent functions g_j ($1 \leq j \leq r$) of the form (26). Let

$$(42) \quad k_j = \max \left\{ \left\lceil \frac{M_j^{(a)}}{n_a} \right\rceil - 1 \mid 1 \leq a \leq s \right\}, \quad j = 1, 2, \dots, r.$$

These k_j are determined by the properties

$$\Lambda^{k_j} g_j \wedge |0\rangle \neq 0, \quad \text{and} \quad \Lambda^{k_j+1} g_j \wedge |0\rangle = 0.$$

Then a polynomial tau-function corresponds to (cf. the right-hand side of (37))

$$(43) \quad g_1 \wedge \Lambda g_1 \wedge \cdots \wedge \Lambda^{k_1} g_1 \wedge g_2 \wedge \cdots \wedge \Lambda^{k_2} g_2 \wedge g_3 \wedge \cdots \wedge \Lambda^{k_r} g_r \wedge |0\rangle.$$

Note that $r + \sum_{i=1}^r k_i = m$. Define $h_j(t)$ (related to such a g_j) again by (29); then $\tau^{(m_1, m_1, \dots, m_s)}(t)$ is still given by (32), but with $h_1(t), h_2(t), h_3(t), \dots, h_m(t)$ replaced by (cf. (37) and (39))

$$\begin{aligned} & h_1(t), D_1 h_1(t), D_2 h_1(t), \dots, D_{k_1} h_1(t), h_2(t), D_1 h_2(t), \dots, D_{k_2} h_2(t), \\ & h_3(t), \dots, D_{k_r} h_r(t). \end{aligned}$$

More specifically, an element in the SL_n -loop group orbit, where $n = n_1 + n_2 + \cdots + n_s$ corresponds to a semi-infinite wedge (43). Then (cf. (27))

$$\begin{aligned} \exp \left(\sum_{a=1}^s \sum_{i=1}^{\infty} t_i^{(a)} \alpha_i^{(a)} \right) (\Lambda^p g_j) &= \sum_{a=1}^s \sum_{\ell=1}^{M_j^{(a)}} \sum_{i=0}^{\infty} b_{\ell j}^{(a)} s_i(t^{(a)}) \Lambda^p e_{\ell-i}^{(a)} \\ &= \sum_{a=1}^s \sum_{\ell=1}^{M_j^{(a)}} \sum_{i=0}^{\infty} b_{\ell j}^{(a)} s_i(t^{(a)}) e_{\ell-i-pn_a}^{(a)} \\ &= \sum_{a=1}^s \sum_{\ell=1}^{M_j^{(a)}} \sum_{i=-pn_a}^{\infty} b_{\ell j}^{(a)} D_p(s_{i+pn_a}(t^{(a)})) e_{\ell-i-pn_a}^{(a)} \\ &= D^p(h_j(t)). \end{aligned}$$

Note that in the above calculations we may replace D_p by D^p , where $D = D_1$, but this is only because this operator acts on linear combinations of elementary Schur functions in one set of variables $t^{(a)}$, with a fixed. Thus, $\tau^{(m_1, m_1, \dots, m_s)}(t)$ is the coefficient of (31) in

$$\begin{aligned} (44) \quad & \exp \left(\sum_{a=1}^s \sum_{i=1}^{\infty} t_i^{(a)} \alpha_i^{(a)} \right) g_1 \wedge \Lambda g_1 \wedge \cdots \wedge \Lambda^{k_1} g_1 \wedge g_2 \wedge \cdots \wedge \Lambda^{k_r} g_r \wedge |0\rangle \\ &= h_1(t) \wedge D h_1(t) \wedge \cdots \wedge D^{k_1} h_1(t) \wedge h_2(t) \wedge \cdots \wedge D^{k_r} h_r(t) \wedge |0\rangle. \end{aligned}$$

Again, as in the s -component KP case, $\tau^{(m_1, m_1, \dots, m_s)}(t)$ is the coefficient of

$$Q_1^{m_1} Q_2^{m_2} \cdots Q_s^{m_s} |0\rangle$$

of expression (44). In the same way as in the standard s -component KP case, one can calculate this coefficient, which is the determinant of a certain matrix. We thus obtain the following theorem.

Theorem 7. *All polynomial tau-functions of the (n_1, n_2, \dots, n_s) -KdV hierarchy (41) are, up to a shift over the lattice, of the form*

$$\tau(q, t) = \sum_{\substack{m_i \geq 0, \\ m_1 + \dots + m_s \\ = r + k_1 + \dots + k_r}} \tau^{(m_1, m_1, \dots, m_s)}(t) q_1^{m_1} q_2^{m_2} \dots q_s^{m_s},$$

where $\tau^{(m_1, m_1, \dots, m_s)}(t) =$

$$\left| \begin{array}{cccccc} \frac{\partial^{m_1} h_1(t)}{\partial (t_1^{(1)})^{m_1}} & \frac{\partial^{m_1} D h_1(t)}{\partial (t_1^{(1)})^{m_1}} & \dots & \frac{\partial^{m_1} D^{k_1} h_1(t)}{\partial (t_1^{(1)})^{m_1}} & \frac{\partial^{m_1} h_2(t)}{\partial (t_1^{(1)})^{m_1}} & \frac{\partial^{m_1} D h_2(t)}{\partial (t_1^{(1)})^{m_1}} & \dots & \frac{\partial^{m_1} D^{k_r} h_r(t)}{\partial (t_1^{(1)})^{m_1}} \\ \frac{\partial^{m_1-1} h_1(t)}{\partial (t_1^{(1)})^{m_1-1}} & \frac{\partial^{m_1-1} D h_1(t)}{\partial (t_1^{(1)})^{m_1-1}} & \dots & \frac{\partial^{m_1-1} D^{k_1} h_1(t)}{\partial (t_1^{(1)})^{m_1-1}} & \frac{\partial^{m_1-1} h_2(t)}{\partial (t_1^{(1)})^{m_1-1}} & \frac{\partial^{m_1-1} D h_2(t)}{\partial (t_1^{(1)})^{m_1-1}} & \dots & \frac{\partial^{m_1-1} D^{k_r} h_r(t)}{\partial (t_1^{(1)})^{m_1-1}} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial h_1(t)}{\partial t_1^{(1)}} & \frac{\partial D h_1(t)}{\partial t_1^{(1)}} & \dots & \frac{\partial D^{k_1} h_1(t)}{\partial t_1^{(1)}} & \frac{\partial h_2(t)}{\partial t_1^{(1)}} & \frac{\partial D h_2(t)}{\partial t_1^{(1)}} & \dots & \frac{\partial D^{k_r} h_r(t)}{\partial t_1^{(1)}} \\ \hline \frac{\partial^{m_2} h_1(t)}{\partial (t_1^{(2)})^{m_2}} & \frac{\partial^{m_2} D h_1(t)}{\partial (t_1^{(2)})^{m_2}} & \dots & \frac{\partial^{m_2} D^{k_1} h_1(t)}{\partial (t_1^{(2)})^{m_2}} & \frac{\partial^{m_2} h_2(t)}{\partial (t_1^{(2)})^{m_2}} & \frac{\partial^{m_2} D h_2(t)}{\partial (t_1^{(2)})^{m_2}} & \dots & \frac{\partial^{m_2} D^{k_r} h_r(t)}{\partial (t_1^{(2)})^{m_2}} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial h_1(t)}{\partial t_1^{(2)}} & \frac{\partial D h_1(t)}{\partial t_1^{(2)}} & \dots & \frac{\partial D^{k_1} h_1(t)}{\partial t_1^{(2)}} & \frac{\partial h_2(t)}{\partial t_1^{(2)}} & \frac{\partial D h_2(t)}{\partial t_1^{(2)}} & \dots & \frac{\partial D^{k_r} h_r(t)}{\partial t_1^{(2)}} \\ \hline \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \hline \frac{\partial^{m_s} h_1(t)}{\partial (t_1^{(s)})^{m_s}} & \frac{\partial^{m_s} D h_1(t)}{\partial (t_1^{(s)})^{m_s}} & \dots & \frac{\partial^{m_s} D^{k_1} h_1(t)}{\partial (t_1^{(s)})^{m_s}} & \frac{\partial^{m_s} h_2(t)}{\partial (t_1^{(s)})^{m_s}} & \frac{\partial^{m_s} D h_2(t)}{\partial (t_1^{(s)})^{m_s}} & \dots & \frac{\partial^{m_s} D^{k_r} h_r(t)}{\partial (t_1^{(s)})^{m_s}} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial h_1(t)}{\partial t_1^{(s)}} & \frac{\partial D h_1(t)}{\partial t_1^{(s)}} & \dots & \frac{\partial D^{k_1} h_1(t)}{\partial t_1^{(s)}} & \frac{\partial h_2(t)}{\partial t_1^{(s)}} & \frac{\partial D h_2(t)}{\partial t_1^{(s)}} & \dots & \frac{\partial D^{k_r} h_r(t)}{\partial t_1^{(s)}} \end{array} \right|,$$

where $D = D_1$ is given by (39) and

$$h_j(t) = \sum_{a=1}^s b_{M_j^{(a)}, j} s_{M_j^{(a)}}(t^{(a)} + c_j^{(a)}), \quad j = 1, \dots, r.$$

Here $M_j^{(a)}$ are arbitrary positive integers, the $c_j^{(a)} = (c_{1,j}^{(a)}, c_{2,j}^{(a)}, c_{3,j}^{(a)}, \dots)$, $b_{M_j^{(a)}, j}$ for $1 \leq j \leq r$ and $1 \leq a \leq s$ are arbitrary constants, and k_j are nonnegative integers defined by (42).

§8. The AKNS hierarchy

The (1,1)-KdV hierarchy is the famous AKNS hierarchy. Making the change of variables

$$x_i = \frac{1}{2}(t_i^{(1)} - t_i^{(2)}), \quad y_i = \frac{1}{2}(t_i^{(1)} + t_i^{(2)}),$$

then the tau-functions become independent of all y_i . Setting

$$q = -\frac{\tau^{(1,-1)}}{\tau^{(0,0)}}, \quad r = \frac{\tau^{(-1,1)}}{\tau^{(0,0)}}, \quad x = 2x_1, \quad t = -4ix_2,$$

one obtains the Ablowitz–Kaup–Newell–Segur (AKNS) system ([1])

$$i \frac{\partial q}{\partial t} = -\frac{1}{2} \frac{\partial^2 q}{\partial x^2} - q^2 r, \quad i \frac{\partial r}{\partial t} = -\frac{1}{2} \frac{\partial^2 r}{\partial x^2} + r^2 q,$$

as one of the simplest equations in the hierarchy (see e.g. [3] for more details). In this case there is only one function (29), viz.

$$h(t) = h_1(t) = b_1 s_{M_1}(t^{(1)} + c^{(1)}) + b_2 s_{M_2}(t^{(2)} + c^{(2)}),$$

$$\frac{\partial^k D^j h(t)}{\partial (t_1^{(a)})^k} = b_a s_{M_a - k - j}(t^{(a)} + c^{(a)}).$$

Expressing this in the variables x_i , one has

$$\frac{\partial^k D^j h(t)}{\partial (t_1^{(a)})^k} = b_a s_{M_a - k - j}(-(-1)^a x + c^{(a)}).$$

Now let $K = k_1 + 1$, where k_1 is as in Theorem 7; then all nonzero polynomial tau-functions $\tau^{(p, K-p)}(x)$, where p is an integer, $0 \leq p \leq K$, are as follows:

$$\tau^{(p, K-p)}(x) = b_1^p b_2^{K-p}$$

$$\times \begin{vmatrix} s_{M_1-1}(x + c^{(1)}) & s_{M_1-2}(x + c^{(1)}) & \cdots & s_{M_1-K}(x + c^{(1)}) \\ s_{M_1-2}(x + c^{(1)}) & s_{M_1-3}(x + c^{(1)}) & \cdots & s_{M_1-K-1}(x + c^{(1)}) \\ \vdots & \vdots & & \vdots \\ s_{M_1-p}(x + c^{(1)}) & s_{M_1-p-1}(x + c^{(1)}) & \cdots & s_{M_1-p-K+1}(x + c^{(1)}) \\ \hline s_{M_2-1}(-x + c^{(2)}) & s_{M_2-2}(-x + c^{(2)}) & \cdots & s_{M_2-K}(-x + c^{(2)}) \\ s_{M_2-2}(-x + c^{(2)}) & s_{M_2-3}(-x + c^{(2)}) & \cdots & s_{M_2-K-1}(-x + c^{(2)}) \\ \vdots & \vdots & & \vdots \\ s_{M_2-K+p}(-x + c^{(2)}) & s_{M_2-K+p-1}(-x + c^{(2)}) & \cdots & s_{M_2-2K+p+1}(-x + c^{(2)}) \end{vmatrix}.$$

Here x stands for $x = (x_1, x_2, \dots)$. Note that $\tau^{(p, K-p)}(x) = 0$ if $K > \max(M_1, M_2)$.

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