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Recognition of objects through symplectic capacities

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ABSTRACT

We prove that the generalized symplectic capacities recognize objects in symplectic categories whose objects are of the form (M, ω) , such that M is a compact and 1-connected manifold, ω is an exact symplectic form on M , and there exists a boundary component of M with negative helicity. The set of generalized symplectic capacities is thus a complete invariant for such categories. This answers a question by Cieliebak, Hofer, Latschev, and Schlenk. It appears to be the first result concerning this question, except for recognition results for manifolds of dimension 2, ellipsoids, and polydiscs in \mathbb{R}^4 . Strikingly, our result holds more generally for differential form categories. Recognition of objects is therefore not a symplectic phenomenon.

We also prove a version of the result for normalized capacities.

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1. Introduction and main results

The main results of this article are concerned with recognition through generalized capacities and normalized capacities on differential form categories. These generalize (generalized and normalized) symplectic

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capacities and symplectic categories. To explain our results, we define the notion of a capacity in the following even more general setup. Let $\widehat{\mathcal{C}}$ be a category determined by a formula. By this, we mean a category whose objects, morphisms, and pairs constituting the composition map² are determined by well-formed logical formulas.

Remark (*Categories and set theory*). This article is based on the Zermelo-Fraenkel axiomatic system with choice (ZFC). In most cases, the collection of objects and morphisms of a category are proper classes and not sets. Therefore, we need to be careful when talking about general categories. However, categories determined by a formula can be dealt with in ZFC. \diamond

We assume that $\widehat{\mathcal{C}}$ is locally small.³ Suppose also that there exists a set S of objects of $\widehat{\mathcal{C}}$, such that every object of $\widehat{\mathcal{C}}$ is isomorphic to an element of S . We denote $\mathbb{R}^+ := (0, \infty)$ and fix an \mathbb{R}^+ -action on $\widehat{\mathcal{C}}$ that is defined by a logical formula. (See Appendix A, Definition 18.) Let \mathcal{C} be an (\mathbb{R}^+) -invariant isomorphism-closed subcategory of $\widehat{\mathcal{C}}$. (See Appendix A, Definitions 16 and 19.)

Example (*Invariant isomorphism-closed subcategory*). As an example, $\mathcal{C} = \widehat{\mathcal{C}}$ is an \mathbb{R}^+ -invariant, isomorphism-closed subcategory of itself. This follows from the assumptions of local smallness and the existence of a set S as above. (See Appendix A, Remark 17.)

Definition 1 (*Generalized and normalized capacity*). A (*generalized*) *capacity*⁴ on \mathcal{C} is a pair (\mathcal{C}_0, c) , where:

- $(\mathcal{O}_0, \mathcal{M}_0) := \mathcal{C}_0$ is an invariant full small subcategory of \mathcal{C} , such that every object of \mathcal{C} is isomorphic to an object of \mathcal{C}_0 .
- $c : \mathcal{O}_0 \rightarrow [0, \infty]$ is a map with the following properties:
 - (i) (**monotonicity**) If A and B are two objects in \mathcal{O}_0 , such that there exists a \mathcal{C}_0 -morphism from A to B , then

$$c(A) \leq c(B).$$

- (ii) (**conformality**) For every $A \in \mathcal{O}_0$ and every $a \in \mathbb{R}^+$ we have

$$c(a_*A) = ac(A),$$

where a_*A denotes the action of $a \in \mathbb{R}^+$ on the object A .

Let X be a set of objects of \mathcal{C} . We say that a capacity (\mathcal{C}_0, c) is (X) -normalized iff

- (iii) (**normalization**) $c(A) = 1$ for every A in \mathcal{O}_0 that is isomorphic to an element of X .

In the following \mathcal{O}_0 always denotes the set of objects of \mathcal{C}_0 .

Remarks (*Capacity*).

- It follows from Definition 16, the local smallness of $\widehat{\mathcal{C}}$, and Remark 20 that there exists an invariant full small subcategory \mathcal{C}_0 of \mathcal{C} , such that every object of \mathcal{C} is isomorphic to an object of \mathcal{C}_0 . For every such \mathcal{C}_0 , the pair $(\mathcal{C}_0, c = 0)$ is a capacity. Hence, capacities on \mathcal{C} exist.

² By these we mean pairs $((f, g), h)$, where (f, g) is the input of the composition map and h is the output, i.e. $f \circ g = h$.

³ This means that for every pair of objects (A, B) of $\widehat{\mathcal{C}}$, the collection of morphisms from A to B is a set.

⁴ In [1], a *capacity* is defined to be a generalized capacity that is non-trivial. Since we do not use that condition in this article, we will use the word *capacity* in the sense of *generalized capacity*.

- It is not possible to define the class of all maps between two classes in ZFC, even if the target class is actually a set. For this reason, we restrict the domain of a capacity to a set of objects \mathcal{O}_0 as above.
- If A and B are isomorphic objects in \mathcal{C} , then by monotonicity, every capacity (\mathcal{C}_0, c) , such that A and B lie in \mathcal{O}_0 , satisfies $c(A) = c(B)$. \diamond

Definition (Recognition).

- Let \mathcal{C} be an invariant isomorphism-closed subcategory of $\widehat{\mathcal{C}}$. We say that the (generalized) capacities on \mathcal{C} recognize objects iff for every pair (A, B) of non-isomorphic objects of \mathcal{C} there exists a capacity (\mathcal{C}_0, c) , such that $A, B \in \mathcal{O}_0$ and $c(A) \neq c(B)$.
- Let X be a set of objects of \mathcal{C} . We say that the X -normalized capacities on \mathcal{C} recognize objects iff for every pair A, B of non-isomorphic objects of \mathcal{C} , such that A or B is not isomorphic to any element of X , there exists an X -normalized capacity (\mathcal{C}_0, c) , such that $A, B \in \mathcal{O}_0$ and $c(A) \neq c(B)$.

Remark (Recognition). By the normalization condition (iii) in Definition 1, there is no X -normalized capacity that distinguishes between two objects that are isomorphic to elements of X . \diamond

In order to define the notion of a form category, we need the following. Let $m, k \in \mathbb{N}_0 := \{0, 1, \dots\}$.

Definition (Universal form category). We define $\Omega^{m,k}$, the *universal (m, k) -form category* as follows:

- Its objects are the pairs (M, ω) , where M is a (smooth) manifold⁵ of dimension m and ω is a differential k -form on M .
- Its morphisms are (smooth) embeddings⁶ that intertwine the differential forms.

Remark (Universal form category). The category $\Omega^{m,k}$ is determined by a logical formula and locally small. For every $a \in \mathbb{R}^+$, we consider the functor that sends an object (M, ω) to $(M, a\omega)$ and a morphism $\varphi : (M, \omega) \rightarrow (M', \omega')$ to the same map viewed as a morphism $(M, a\omega) \rightarrow (M', a\omega')$. This defines an \mathbb{R}^+ -action on $\Omega^{m,k}$.

For every set X , we denote by $\mathcal{P}(X)$ its power set. Every object of $\Omega^{m,k}$ is isomorphic to an element of the set

$$\mathcal{O}_0^{m,k} := \{(M, \omega) \text{ object of } \Omega^{m,k} \mid \text{The set underlying } M \text{ is a subset of } \mathcal{P}(\mathbb{N}_0)\}. \tag{1}$$

This is explained in [4, p. 12, footnote 28]. Therefore, $\widehat{\mathcal{C}} := \Omega^{m,k}$ fits into the setting described on page 2 of this article. \diamond

Definition $((m, k)$ -form category, symplectic category and capacity). An (m, k) -*(differential) form category* is an invariant isomorphism-closed subcategory of $\Omega^{m,k}$. For every $n \in \mathbb{N}_0$, we call a $(2n, 2)$ -form category whose objects are symplectic manifolds, a *symplectic category*. We call a (generalized) capacity on a symplectic category also a *(generalized) symplectic capacity*.

Example 2 (Embedding capacity). Let $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be an (m, k) -form category, \mathcal{C}_0 be the invariant full small subcategory of \mathcal{C} whose set of objects is given by $\mathcal{O}_0 := \mathcal{O} \cap \mathcal{O}_0^{m,k}$, and let (M, ω) be an object of $\Omega^{m,k}$. We define the *(domain-)embedding capacity* for (M, ω) and \mathcal{C} to be the map

⁵ All the manifolds in this article are smooth, finite-dimensional, and are allowed to have boundary.

⁶ We do not impose any condition involving the boundaries of the manifolds.

$$\begin{aligned}
c_{(M,\omega)} &:= c_{(M,\omega)}^{\mathcal{C}} : \mathcal{O}_0 \rightarrow [0, \infty], \\
c_{(M,\omega)}(M', \omega') &:= \\
&\sup \{a \in (0, \infty) \mid \exists \Omega^{m,k}\text{-morphism } (M, a\omega) \rightarrow (M', \omega')\}.
\end{aligned} \tag{2}$$

The pair $(\mathcal{C}_0, c_{(M,\omega)})$ is a capacity.

Let $n \in \mathbb{N}_0$. We denote by $B := B^{2n}$ the $2n$ -dimensional open unit ball, by $Z := Z^{2n} := B^2 \times \mathbb{R}^{2n-2}$ the open unit cylinder, and by ω_{st} the standard symplectic form on \mathbb{R}^{2n} . Consider the case $k = 2$. We define the *Gromov width* for \mathcal{C} to be

$$w := w^{\mathcal{C}} := c_{(B, \omega_{\text{st}}|_B)}^{\mathcal{C}}.$$

If B and Z are objects of \mathcal{C} , then (\mathcal{C}_0, w) is a $\{B, Z\}$ -normalized capacity by Gromov's non-squeezing Theorem [3].

The goal of this article is to provide an answer to the following question by K. Cieliebak, H. Hofer, J. Latschew, and F. Schlenk for certain symplectic categories containing many objects.

Question 3 ([1], question 2, p. 20). Do the generalized symplectic capacities on a given symplectic category recognize objects?

This question appears to be widely open. To our knowledge, it has only been answered in dimension 2, for the category of ellipsoids in \mathbb{R}^{2n} (for every $n \in \mathbb{N}$), and for polydiscs in \mathbb{R}^4 . More precisely, a Moser-type argument classifies closed⁷ connected symplectic manifolds of dimension 2 by their total area. The Ekeland-Hofer capacities recognize ellipsoids in \mathbb{R}^{2n} . (See [1, Fact 10, p. 27].) The Gromov width and the volume capacity recognize polydiscs in \mathbb{R}^4 . (See [1, Example 6(iv), p. 21].)

A negative answer to a variant of Question 3 was recently provided by E. Kerman and Y. Liang in [5, Theorem 1.3]. These authors showed that the Gutt-Hutchings capacities and the volume capacity do not recognize starshaped compact smooth submanifolds (with boundary) in \mathbb{R}^4 . (This also follows from the proof of Proposition 13 below.)

To state our main results, we need the following definitions. Let $k, n \in \mathbb{N} := \{1, 2, \dots\}$.

Definition 4 (*Helicity*). Assume that $n \geq 2$. Let N be a closed $(kn - 1)$ -dimensional manifold with an orientation O , and let σ be an exact k -form on N . We define the helicity of (N, O, σ) to be the integral

$$h(N, O, \sigma) = \int_{N, O} \alpha \wedge \sigma^{\wedge(n-1)},$$

where α is an arbitrary primitive of σ and $\int_{N, O}$ denotes the integral on N with respect to the orientation O . By Lemma 21 below, this number is well-defined. We detail some basic properties of helicity in Section B below.

Definition (*Maxipotent*). A k -form ω on a kn -dimensional manifold M is called *maxipotent* iff $\omega^{\wedge n}$ is nowhere vanishing.

⁷ This means compact and without boundary.

Remark (*Maxipotent*).

- If $n \geq 2$ and k is odd, we have that $\omega \wedge \omega = 0$, and hence there are no maxipotent k -forms.
- A maxipotent k -form ω on a kn -dimensional manifold induces an orientation on the manifold. The orientation is induced by the kn -form $\omega^{\wedge n}$, which is nowhere-vanishing, and hence a volume form. \diamond

The first main result of this article is as follows.

Theorem 5 (*Recognition of objects*). *Assume that $n \geq 2$. Let \mathcal{C} be a (kn, k) -form category, such that for every object (M, ω) of \mathcal{C} , the manifold M is compact and 1-connected,⁸ ω is exact and maxipotent, and ω induces negative helicity on some connected component of the boundary ∂M .⁹ Then the generalized capacities on \mathcal{C} recognize objects.*

This result answers Question 3 affirmatively for every symplectic category satisfying the hypotheses of Theorem 5. To our knowledge, this is the first result concerning Question 3, except for the results mentioned above. Note that Theorem 5 treats the more general setting of (kn, k) -form categories. This shows that recognition of objects by capacities is not a symplectic phenomenon.

Remark (*Even degree*). Under the hypotheses of Theorem 5, the integer k is even, provided that \mathcal{C} is nonempty. This follows from the conditions that $n \geq 2$ and ω is maxipotent.

The compactness and the helicity hypotheses of Theorem 5 cannot be dropped. See Proposition 13 and Remark 14 below.

Example (*Recognition of objects*). A category with a large class of objects to which Theorem 5 applies, is provided by the following full subcategory $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ of $\Omega^{kn, k}$. Let M be a compact kn -dimensional manifold, ω an exact maxipotent k -form on M , and $U \subseteq M$ a nonempty, open subset of M whose boundary is a smooth submanifold of the interior of M and lies on one side of U . The class \mathcal{O} consists of all $(M \setminus U, \omega|_{M \setminus U})$ obtained in this way, such that $M \setminus U$ is 1-connected. (The objects satisfy the negative helicity assumption by Stokes' Theorem for helicity, Lemma 23 below.) Hence \mathcal{C} satisfies the hypotheses of Theorem 5. If $k = 2$ then \mathcal{O} consists of symplectic manifolds, and hence \mathcal{C} is a symplectic category.

Remark (*Capacities as a complete set of invariants*). Let $\widehat{\mathcal{C}}$ be a locally small category determined by a formula and such that there exists a set of objects of $\widehat{\mathcal{C}}$ with the property that every object of $\widehat{\mathcal{C}}$ is isomorphic to an element of that set. Let G be a group that is determined by a formula, and let ρ denote a G -action on $\widehat{\mathcal{C}}$ that is determined by a formula. Let \mathcal{C} be an invariant isomorphism-closed subcategory of $\widehat{\mathcal{C}}$.

By a *complete invariant* for (\mathcal{C}, ρ) , we mean a triple $(\mathcal{C}_0, \mathcal{C}_1, F)$, where \mathcal{C}_0 is an invariant full small subcategory of \mathcal{C} , such that every object of \mathcal{C} is isomorphic to an object of \mathcal{C}_0 , \mathcal{C}_1 is a small category equipped with a G -action determined by a formula, and $F : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ is an equivariant and essentially injective¹⁰ functor. For two sets X and Y , we denote by Y^X the set of maps from X to Y .

By a *complete set of invariants* for (\mathcal{C}, ρ) , we mean a triple $(\mathcal{C}_0, \mathcal{C}_1, \mathcal{F})$, where \mathcal{C}_0 and \mathcal{C}_1 are as above, and \mathcal{F} is a set of equivariant functors $\mathcal{C}_0 \rightarrow \mathcal{C}_1$, such that the following pair of maps is a complete invariant for (\mathcal{C}, ρ) :

$$\mathcal{C}_0 \rightarrow \mathcal{C}_1^{\mathcal{F}} := (\mathcal{O}_1^{\mathcal{F}}, \mathcal{M}_1^{\mathcal{F}}),$$

⁸ This means connected and simply connected.

⁹ By this we mean that the pullback of ω to the component has (strictly) negative helicity with respect to the orientation induced by ω .

¹⁰ This means that for each pair of objects A, B of \mathcal{C}_0 satisfying $F(A) \cong F(B)$, we have $A \cong B$.

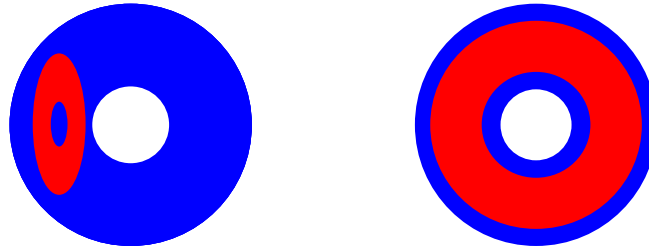


Fig. 1. The two blue annuli depict the target spherical shell M and the two red annuli the images of M under two form-intertwining (i.e., symplectic, if $k = 2$) embeddings $\chi = \psi \circ \varphi$. The embedding on the left only exists for small values of C , as the hole in the middle of $\chi(M)$ “takes space”. The embedding on the right wraps the interior hole of the red shell around the interior hole of the blue shell. Using the equality $\chi^*\omega = C^n\omega$ and Stokes’ Theorem for helicity, the volume of the thin blue shell around the central hole is $C^n - 1 \geq 0$. On the other hand, by assumption, we have $C \leq 1$, which implies that $C = 1$. (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)

$$\mathcal{O}_0 \ni A \mapsto (F(A))_{F \in \mathcal{F}} \in \mathcal{O}_1^{\mathcal{F}},$$

$$\mathcal{M}_0 \ni (A, B, \varphi) \mapsto (F(A), F(B), F(\varphi))_{F \in \mathcal{F}} \in \mathcal{M}_1^{\mathcal{F}}.$$

Now, we assume that \mathcal{C} is a (kn, k) -form category, with $n \geq 2$ and k even, and \mathcal{C}_0 is an invariant full small subcategory of \mathcal{C} , such that every object of \mathcal{C} is isomorphic to an object of \mathcal{C}_0 . We choose \mathcal{C}_1 to be the small category whose objects are the elements of $[0, \infty]$ and whose morphism set between $a, b \in [0, \infty]$ is $\{(a, b)\}$ iff $a \leq b$ and \emptyset otherwise, and we choose \mathcal{F} to be the set of capacities of the form (\mathcal{C}_0, c) , viewed as functors. We denote by ρ the restriction of the \mathbb{R}^+ -action on $\Omega^{kn, k}$ to \mathcal{C} . Theorem 5 means that $(\mathcal{C}_0, \mathcal{C}_1, \mathcal{F})$ is a complete set of invariants for (\mathcal{C}, ρ) . \diamond

The idea of the proof of Theorem 5 is the following. Let (M, ω) and (M', ω') be two objects of \mathcal{C} and assume that all capacities whose domain contains (M, ω) and (M', ω') agree on (M, ω) and (M', ω') . Since the embedding capacities $c_{(M, \omega)}$ and $c_{(M', \omega')}$ agree on (M, ω) and (M', ω') , there exist embeddings

$$\varphi : (M, a\omega) \hookrightarrow (M', \omega') \text{ and } \psi : (M', b\omega') \hookrightarrow (M, \omega)$$

for a, b in $(0, 1]$ that are arbitrarily close to 1. The composition $\chi := \psi \circ \varphi$ then yields an embedding $(M, C\omega) \hookrightarrow (M, \omega)$ with $C = ab$.

Our assumption that M has some negative helicity boundary component, Stokes’ Theorem for helicity (Lemma 23 below), and the fact that $C = ab$ is close to 1 imply that the constant C is equal to 1.¹¹ This implies that $a = b = 1$, which implies that φ and ψ are isomorphisms. This finishes the outline of the proof of Theorem 5. Fig. 1 illustrates the possible kinds of embeddings of $(M, C\omega)$ into (M, ω) for spherical shells, and provides a pictorial idea of the main step of the proof.

Let $n \geq 2$ and $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a $(2n, 2)$ -form category. Recall that B and Z denote the open unit ball and cylinder. From this point on, by a *normalized capacity*, we mean a $\{B, Z\}$ -normalized capacity. Our second main result states that the normalized capacities recognize objects under the hypotheses of Theorem 5 and one more assumption on \mathcal{C} :

¹¹ More precisely, these facts imply the following. We denote by I the set of connected components of the boundary of M . The embedding χ induces a partition of the set $I \sqcup I$. Two elements of $I \sqcup I$ lie in the same partition element iff they lie in the same path component of $M \setminus \chi(M)$. Here we identify each element of the first component of $I \sqcup I$ with its image under χ .

Every element of the partition containing a negative helicity component does not contain any positive helicity component. (See the Key Lemma 9.) Using Stokes’ Theorem for helicity again, it follows that $C = 1$.

Proposition 6 (Recognition of objects through normalized capacities). Assume that $B, Z \in \mathcal{O}$. Suppose also that every object $(M, \omega) \in \mathcal{O}$ satisfies the assumptions of Theorem 5 and the inequality¹²

$$w(M, \omega)c_{(M, \omega)}(Z) < 1, \tag{3}$$

except if (M, ω) is isomorphic to B or Z . Then the normalized capacities recognize objects in \mathcal{C} .

Remarks (Recognition of objects through normalized capacities).

- Examples of manifolds for which assumption (3) holds are thin spherical shells, as explained in [6, Corollary 5].
- The inequality (3) is invariant in the sense that if it holds for an object (M, ω) in $\Omega^{2n,2}$ then it also holds for $(M, a\omega)$, for every constant $a \in (0, \infty)$.
- By definition, no normalized capacity distinguishes between B and Z . The result above states that the normalized capacities distinguish between all other objects and every object in the category. \diamond

The idea of the proof of Proposition 6 is similar to that of Theorem 5. Here we replace the domain-embedding capacity $c_{(M, \omega)}$ by the maximum of a rescaling of $c_{(M, \omega)}$ and the Gromov width.

Organization of the article. In Sections 2 and 3, we prove the main results. In Section 4 we show that the compactness and the negative helicity hypotheses cannot be dropped. Appendix A deals with the set- and category-theoretic setting needed to defined capacities. In Appendix B we go over some basic properties of helicity, such as Stokes’ theorem for helicity. These properties are indirectly used in the proofs of the main results.

Acknowledgments. We would like to thank Vukašin Stojisavljević for making us aware of [2, Theorem 1.1], on which Proposition 13 below is based. We thank Jaap van Oosten for a discussion about categories and ZFC. We would like to thank Felix Schlenk for an interesting correspondence.

2. Proof of Theorem 5

Let $k, n \in \mathbb{N}$ with $n \geq 2$. We will prove Theorem 5 by reducing it to the following lemma.

Lemma 7 (Sufficient criterion for recognition). Let M be a compact, 1-connected, kn -dimensional manifold and ω an exact, maxipotent k -form on M , such that ω induces negative helicity on some connected component of ∂M . Let (M', ω') be another object with these properties. Assume that¹³

$$c_{(M, \omega)}(M', \omega') \geq 1 \quad \text{and} \quad c_{(M', \omega')}(M, \omega) \geq 1. \tag{4}$$

Then there is a diffeomorphism $\varphi : M \rightarrow M'$, such that $\varphi^*\omega' = \omega$.

For a proof, see page 9.

Proof of Theorem 5. Assume that (M, ω) and (M', ω') are two objects of \mathcal{C} , such that all capacities whose domains contain (M, ω) and (M', ω') , agree on (M, ω) and (M', ω') . We show that (M, ω) and (M', ω') are isomorphic. It follows from our assumption that

¹² Recall the definitions of the embedding capacity $c_{(M, \omega)}$ and the Gromov width w from Example 2. Here we evaluate $c_{(M, \omega)}$ on a general symplectic manifold (M', ω') by evaluating it on an object that is isomorphic to (M', ω') and lies in the set $\mathcal{O}_0^{m,k}$, defined in (1).

¹³ Compare to footnote 12.

$$c_{(M,\omega)}(M',\omega') = c_{(M,\omega)}(M,\omega),$$

$$c_{(M',\omega')}(M,\omega) = c_{(M',\omega')}(M',\omega').$$

Since $c_{(M,\omega)}(M,\omega) \geq 1$ and $c_{(M',\omega')}(M',\omega') \geq 1$, it follows that the hypothesis (4) of Lemma 7 holds. Applying this lemma, it follows that (M,ω) and (M',ω') are isomorphic in $\Omega^{kn,k}$. Since \mathcal{C} is isomorphism-closed, they are also isomorphic in \mathcal{C} . This proves Theorem 5. \square

For the proof of Lemma 7 we need the following. Let $\varphi : M \rightarrow M'$ be an embedding between two topological manifolds. We denote by $\text{Int } M$ the manifold interior of M and

$$I := I_M := \{\text{connected components of } \partial M\},$$

$$I' := I_{M'},$$

$$P := M' \setminus \varphi(\text{Int}(M)).$$

For every pair of sets X, Y we denote by

$$X \sqcup Y := \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}$$

the disjoint union of X and Y .¹⁴ We denote by

$$\psi : I \sqcup I' \rightarrow \mathcal{P}(P)$$

the map induced by φ on I and given by the inclusion on I' . We define a partition

$$\mathcal{P}^\varphi$$

on $I \sqcup I'$ by declaring that two boundary components $i, j \in I \sqcup I'$ lie in the same element of \mathcal{P}^φ if and only if there is a continuous path in P that starts in $\psi(i)$ and ends in $\psi(j)$.

Let M be a kn -dimensional (smooth) manifold with boundary, O an orientation on M , and ω an exact k -form on M . For every $i \in I_M$, we denote by O_i the induced orientation on i and by ω_i the pullback of ω to i .

Assume that M is compact. Recall Definition 4 of helicity.

Definition (*Boundary helicity*). We define the *boundary helicity* of (M, O, ω) to be the map $h_{M,O,\omega} : I_M \rightarrow \mathbb{R}$ given by

$$h_{M,O,\omega}(i) := h(i, O_i, \omega_i).$$

Lemma 8 (*Helicity inequality*). Let M and M' be compact kn -dimensional smooth manifolds, and ω and ω' be *maxipotent*¹⁵ exact k -forms on M and M' , respectively. Let C be a positive real number and $\varphi : M \rightarrow M'$ a smooth orientation-preserving embedding that intertwines $C\omega$ and ω' . We denote by O and O' the orientations of M and M' induced by ω and ω' . Then for every $J \in \mathcal{P}^\varphi$, the following inequality holds:

$$-C^n \sum_{i \in J \cap I} h_{M,O,\omega}(i) + \sum_{i' \in J \cap I'} h_{M',O',\omega'}(i') \geq 0. \tag{5}$$

¹⁴ When no confusion is possible, we identify $\{0\} \times X$ with X and $\{1\} \times Y$ with Y to ease the notation.

¹⁵ This implies that k is even.

Proof. This is Lemma 49 in [4]. \square

Remark. The idea of the proof of this lemma is that for every $J \in \mathcal{P}^\varphi$, the left-hand side of inequality (5) is the volume of a path-component of P . This follows from Stokes' theorem for helicity (Lemma 23) and Remark 22. \diamond

For the remainder of this section, we will use the notation

$$\varphi : (M, \omega) \hookrightarrow (M', \omega')$$

to denote an embedding of M into M' that intertwines ω and ω' . A crucial ingredient of the proof of Lemma 7 is the following.

Lemma 9 (*Key lemma, separation of positive and negative helicity components*). *Let M be a nonempty, compact, 1-connected kn -dimensional manifold with nonempty boundary, and ω be a maxipotent exact k -form on M . Then there exists a real number $C_0 \in (0, 1)$, such that for every $C \in (C_0, 1]$ and every embedding $\varphi : (M, C\omega) \hookrightarrow (M, \omega)$, the partition \mathcal{P}^φ of $I \sqcup I$ has the following property. Every element of \mathcal{P}^φ containing a negative helicity component does not contain any positive helicity component.*

The proof of this lemma is based on Lemma 8 (helicity inequality). We use that the positive helicity boundary components in the domain and the target come in pairs that lie in the same partition element and have the same helicity. (See page 10.)

In the following, we denote

$$\begin{aligned} I_+ &:= \{\text{element of } I \text{ with (strictly) positive helicity}\}, \\ I_- &:= \{\text{element of } I \text{ with negative helicity}\}, \\ I_0 &:= \{\text{element of } I \text{ with zero helicity}\}. \end{aligned}$$

We use the notation

$$h := h_{M,O,\omega} : I \rightarrow \mathbb{R}, \quad h := h_{M,O,\omega} \sqcup h_{M,O,\omega} : I \sqcup I \rightarrow \mathbb{R}$$

for both the boundary helicity map on I and the induced map on the disjoint union of two copies of I .

Proof of Lemma 7. We choose C_0 as in the conclusion of Lemma 9, applied to (M, ω) . By the inequalities in (4), there exist a and b in $(\sqrt{C_0}, 1]$ and embeddings

$$\varphi : (M, a\omega) \hookrightarrow (M', \omega'), \quad \psi : (M', b\omega') \hookrightarrow (M, \omega).$$

The map $\chi := \psi \circ \varphi$ is an embedding of M into itself, and it intertwines the forms $ab\omega$ and ω .

Since $C := ab > C_0$, Lemma 9 implies that each element of \mathcal{P}^χ that contains some negative helicity element does not contain any positive helicity element. Thus, summing inequality (5) from Lemma 8 over all these partition elements yields

$$-C^n \sum_{i \in \{0\} \times I_-} h(i) + \sum_{i' \in \{1\} \times I_-} h(i') \geq 0.$$

Each sum is equal to $\sum_{i \in I_-} h(i)$, hence we obtain

$$(1 - C^n) \sum_{i \in I_-} h(i) \geq 0. \quad (6)$$

Since, by assumption, M has some boundary component with negative helicity, we have that $\sum_{i \in I_-} h(i) < 0$. Using inequality (6), it follows that $C^n \geq 1$, and hence $C \geq 1$. Since $C \leq 1$ by definition, it follows that $C = ab = 1$. Because $a, b \in (0, 1]$, it follows that $a = b = 1$. Hence we have

$$\chi := \psi \circ \varphi : (M, \omega) \hookrightarrow (M, \omega).$$

Since M is compact, the image $\chi(M)$ is also compact and thus closed. Hence $M \setminus \chi(M)$ is open. Since χ is volume-preserving, the set $M \setminus \chi(M)$ has volume zero, which implies that it is empty. We conclude that χ and therefore ψ is surjective. Hence ψ is a diffeomorphism between M and M' that intertwines ω and ω' . This concludes the proof of Lemma 7. \square

In the proof of Lemma 9 we will use the following. Let M and M' be topological manifolds and $\varphi : M \rightarrow M'$ a topological embedding.

Lemma 10 (*Partition induced by an embedding*). Assume that M and M' are compact, connected, and of the same dimension, and that $M \neq \emptyset$. Suppose also that M' is 1-connected. Then for every $J \in \mathcal{P}^\varphi$ we have $|J \cap (\{0\} \times I)| = 1$.

Proof of Lemma 10. This follows from the proof of Lemma 47(i) in [4].¹⁶ \square

Definition 11 (*Partition element containing some boundary component*). For every $i \in I$, we denote by

$$J_i := J_i^\varphi$$

the element of \mathcal{P}^φ containing $(0, i)$.

Since \mathcal{P}^φ is a partition, for every $i \in I$, there is only one $J \in \mathcal{P}^\varphi$ containing it. Hence J_i is well-defined.

Proof of Lemma 9. Set

$$C_1 := \max \{C \in [0, 1] \mid \exists i_+, i'_+ \in I_+ : \quad (7)$$

$$h(i_+) > h(i'_+) \text{ and } h(i'_+) \geq C^n h(i_+)\},$$

$$C_2 := \max \{C \in [0, 1] \mid \exists i_+ \in I_+, i_- \in I_- : \quad (8)$$

$$(1 - C^n)h(i_+) \geq -h(i_-)\},$$

$$C_0 := \max \{C_1, C_2\}.$$

Since the set I is finite, these numbers are well-defined, $C_1 < 1$, $C_2 < 1$, and hence $C_0 < 1$. Let C be in $(0, 1]$ and

$$\varphi : (M, C\omega) \hookrightarrow (M, \omega).$$

We prove Lemma 9 by showing that if $C > C_0$ and $J \in \mathcal{P}^\varphi$ contains a positive element, then it does not contain any negative element.

¹⁶ In this lemma, it is assumed that $\partial M' \neq \emptyset$. However, this is not necessary.

Let $i_+ \in I_+$. By Lemma 10, we have $J_{i_+} \cap (\{0\} \times I) = \{(0, i_+)\}$. Using Lemma 8, it follows that

$$-C^n h(i_+) + \sum_{i \in I: (1, i) \in J_{i_+}} h(i) \geq 0. \tag{9}$$

We define

$$I_{i_+} := \{i'_+ \in I_+ \mid (1, i'_+) \in J_{i_+}\}. \tag{10}$$

Claim 1. We have $|I_{i_+}| = 1$.

Proof of Claim 1. Since the first term on the left-hand side of (9) is negative, the remaining sum is positive. It follows that there exists $i'_+ \in I_+$, such that $(1, i'_+) \in J_{i_+}$. This shows that

$$|I_{i_+}| \geq 1. \tag{11}$$

Since $I_+ \supseteq \bigcup_{i_+ \in I_+} I_{i_+}$, we have that

$$|I_+| \geq \left| \bigcup_{i_+ \in I_+} I_{i_+} \right|. \tag{12}$$

Since the elements of the partition \mathcal{P}^φ are disjoint, the J_{i_+} 's are disjoint, and hence the I_{i_+} 's are also disjoint. It follows that

$$\left| \bigcup_{i_+ \in I_+} I_{i_+} \right| = \sum_{i_+ \in I_+} |I_{i_+}|. \tag{13}$$

If there exists $i_+ \in I_+$, such that $|I_{i_+}| > 1$, then by (11), we have

$$\sum_{i_+ \in I_+} |I_{i_+}| > |I_+|.$$

Combining this with (12) and (13), it follows that there is no such i_+ . This proves Claim 1. \square

We define the map

$$f := f^\varphi : I_+ \rightarrow I_+, f(i_+) = i'_+, \text{ where } i'_+ \text{ is the unique element of } I_{i_+}.$$

By Claim 1 this map is well-defined. Since the I_{i_+} are disjoint, f is injective. Since I_+ is a finite set, it follows that f is also surjective. Let $i'_+ \in I_+$. Since f is surjective, there exists $i_+ \in I_+$, such that $f(i_+) = i'_+$. Using $f(i_+) \in I_{i_+}$ and (10), it follows that $(1, i'_+) \in J_{i_+}$. Therefore, we have

$$\{1\} \times I_+ \subseteq \bigcup_{i_+ \in I_+} J_{i_+}.$$

By Definition 11 we have $(0, i_+) \in J_{i_+}$, for every $i_+ \in I_+$. It follows that

$$\{0, 1\} \times I_+ \subseteq \bigcup_{i_+ \in I_+} J_{i_+}. \tag{14}$$

Let $i_+ \in I_+$. We define

$$K_{i_+} := K_{i_+}^\varphi := \{i \in I \setminus \{f(i_+)\} \mid (1, i) \in J_{i_+}\}.$$

Let $C \in (C_1, 1]$ and $\varphi : (M, C\omega) \hookrightarrow (M, \omega)$.

Claim 2. *If $C > C_2$, then $K_{i_+} \cap I_- = \emptyset$.*

Proof of Claim 2. It follows from the definitions of K_{i_+} and f that the elements of K_{i_+} have negative or zero helicity.

Claim 3. *We have*

$$h(f(i_+)) = h(i_+).$$

Proof of Claim 3. Using (9) and that the elements of K_{i_+} have negative or zero helicity, we have

$$-C^n h(i_+) + h(f(i_+)) \geq 0.$$

Considering $i'_+ := f(i_+)$ and using our assumption that $C > C_1$, by the definition of C_1 in (7), it follows that

$$h(i_+) \leq h(f(i_+)). \quad (15)$$

Since f is a permutation on the finite set I_+ , there exists $s_0 \in \mathbb{N}$, such that $f^{s_0}(i_+) = i_+$.¹⁷ From this and (15), it follows that

$$h(i_+) \leq h(f(i_+)) \leq \dots \leq h(f^{s_0}(i_+)) = h(i_+).$$

This implies that $h(i_+) = h(f(i_+))$. This concludes the proof of Claim 3. \square

Assume that $K_{i_+} \cap I_- \neq \emptyset$. We choose $i_- \in K_{i_+} \cap I_-$. We have

$$\begin{aligned} (1 - C^n)h(i_+) &= -C^n h(i_+) + h(f(i_+)) && \text{(using Claim 3)} \\ &\geq - \sum_{i \in K_{i_+}} h(i) && \text{(using (9))} \\ &\geq -h(i_-) && \text{(since } h(i) \leq 0, \text{ for every } i \in K_{i_+}\text{).} \end{aligned}$$

Using the definition (8) of C_2 , it follows that $C \leq C_2$. The statement of Claim 2 follows. \square

Let $C \in (C_0, 1]$ and $\varphi : (M, C\omega) \hookrightarrow (M, \omega)$. Let J be an element of \mathcal{P}^φ that contains some $i \in \{0, 1\} \times I_+$. By (14), there exists $i_+ \in I_+$, such that $i \in J_{i_+}$. Since the elements of \mathcal{P}^φ are disjoint, it follows that $J = J_{i_+}$. Hence by Lemma 10 and Claim 2, J does not contain any negative helicity element. The statement of Lemma 9 follows. \square

¹⁷ Here f^{s_0} denotes the composition $f \circ \dots \circ f$, in which f occurs s_0 times.

3. Proof of Proposition 6

In the proof of Proposition 6 we will use the following.

Remark 12 (*Normalization of capacity*). Let $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a $(2n, 2)$ -form category containing the open unit ball B and cylinder Z . Let \mathcal{C}_0 be the full subcategory of \mathcal{C} whose objects are given by $\mathcal{O}_0 := \mathcal{O} \cap \mathcal{O}_0^{2n,2}$, where $\mathcal{O}_0^{2n,2}$ is defined in (1). Let (M, ω) be an object of $\Omega^{2n,2}$. We define

$$\bar{c}_{(M,\omega)} := \max\{c_{(M,\omega)}, w\}. \tag{16}$$

Then $(\mathcal{C}_0, \bar{c}_{(M,\omega)})$ is a capacity on \mathcal{C} . Furthermore, if¹⁸

$$c_{(M,\omega)}(Z) \leq 1, \tag{17}$$

then $(\mathcal{C}_0, \bar{c}_{(M,\omega)})$ is a normalized capacity on \mathcal{C} . \diamond

Proof of Proposition 6.

Claim 1. *Let (M, ω) and (M', ω') be objects of \mathcal{C} . Assume that all normalized capacities, whose domain contains (M, ω) and (M', ω') , agree on (M, ω) and (M', ω') . Suppose also that (M, ω) and (M', ω') are not isomorphic to B nor Z . Then they are isomorphic to each other in \mathcal{C} .*

Proof of Claim 1. We define

$$A := c_{(M,\omega)}(Z).$$

Since M has a negative helicity boundary component, it is nonempty. Hence there exists $a \in (0, \infty)$ and a ball B' of area a that symplectically embeds into (M, ω) . We have

$$A = c_{(M,\omega)}(Z) \leq c_{B'}(Z) = \frac{\pi}{a} < \infty. \tag{18}$$

Claim 2. *We have*

$$c_{(M,\omega)}(M', \omega') \geq 1. \tag{19}$$

Proof of Claim 2. Case $A \neq 0$: Then (17) with (M, ω) replaced by $(M, A\omega)$ holds. (Here we use (18), in order to make sense of $A\omega$.) Hence by Remark 12 the map $\bar{c}_{(M,A\omega)}$ is a normalized capacity. Hence by our assumption we have that

$$\begin{aligned} \bar{c}_{(M,A\omega)}(M', \omega') &= \bar{c}_{(M,A\omega)}(M, \omega) \\ &\geq \frac{1}{A}, \end{aligned} \tag{20}$$

where in the second step we used (16), (2) and our assumption $A \neq 0$. Since the Gromov width w is a normalized capacity, we have by assumption that

¹⁸ Compare to footnote 12.

$$\begin{aligned}
 w(M', \omega') &= w(M, \omega) \\
 &< \frac{1}{A}, \qquad \text{(by our assumption (3)).}
 \end{aligned}$$

Combining this with (16) and (20), it follows that

$$c_{(M, A\omega)}(M', \omega') \geq \frac{1}{A}.$$

Inequality (19) follows.

Case $A = 0$: We define

$$\varepsilon := \frac{1}{2w(M, \omega)}.$$

Since M is nonempty, we have $w(M, \omega) > 0$, which implies that $\varepsilon < \infty$. Hence $\varepsilon\omega$ is well-defined. Since M is compact, we also have that $w(M, \omega) < \infty$ and hence $\varepsilon > 0$. Since $c_{(M, \omega)}(Z) = A = 0$ and $\varepsilon > 0$, inequality (17) holds with (M, ω) replaced by $(M, \varepsilon\omega)$. Hence by Remark 12 the map $\bar{c}_{(M, \varepsilon\omega)}$ is a normalized capacity. Using our assumption, it follows that

$$\begin{aligned}
 \bar{c}_{(M, \varepsilon\omega)}(M', \omega') &= \bar{c}_{(M, \varepsilon\omega)}(M, \omega) \\
 &\stackrel{\text{by (16),(2)}}{\geq} \frac{1}{\varepsilon}.
 \end{aligned} \tag{21}$$

Since w is a normalized capacity, by assumption, we have

$$\begin{aligned}
 w(M', \omega') &= w(M, \omega) \\
 &= \frac{1}{2\varepsilon} \qquad \text{(by definition of } \varepsilon \text{).}
 \end{aligned}$$

Using (21), it follows that

$$c_{(M, \varepsilon\omega)}(M', \omega') \geq \frac{1}{\varepsilon}.$$

Since $\varepsilon > 0$, inequality (19) follows.

We have shown that this inequality holds in all cases. This proves Claim 2. \square

Claim 2 with the rôles of (M, ω) and (M', ω') interchanged, implies that

$$c_{(M', \omega')}(M, \omega) \geq 1.$$

Combining this with Claim 2 and Lemma 7, it follows that (M, ω) and (M', ω') are isomorphic in $\Omega^{2n,2}$. (Here we used our assumption that these objects are not isomorphic to B or Z , and therefore they satisfy the hypotheses of Theorem 5.) Since \mathcal{C} is isomorphism-closed, it follows that (M, ω) and (M', ω') are isomorphic in \mathcal{C} . This proves Claim 1. \square

Claim 3. *Let (M, ω) be an object in \mathcal{C} that is not isomorphic to the ball nor the cylinder, such that $w(M, \omega) = 1$. Then there exists a normalized capacity on \mathcal{C} that distinguishes (M, ω) from B and Z .*

Proof of Claim 3. We define $A := c_{(M,\omega)}(Z)$. As in the proof of Claim 1, we have $A < \infty$.

Case $A \neq 0$: By our assumption that (M, ω) is not isomorphic to B nor Z , inequality (3) holds. We have

$$\begin{aligned} \bar{c}_{(M,A\omega)}(M, \omega) &\geq \frac{1}{A} && \text{(using (16) and } A \neq 0) \\ &> 1 && \text{(using (3) and our assumption that } w(M, \omega) = 1) \\ &= \bar{c}_{(M,A\omega)}(Z) && \text{(by our definition of } A). \end{aligned}$$

It follows that $\bar{c}_{(M,A\omega)}$ distinguishes (M, ω) from B and Z .

Case $A = 0$: Then we have

$$\bar{c}_{(M,\frac{1}{2}\omega)}(M, \omega) \geq 2 > 1 = \bar{c}_{(M,\frac{1}{2}\omega)}(Z),$$

where in the last step we used that $A = 0$ and our assumption $w(M, \omega) = 1$. It follows that $\bar{c}_{(M,\frac{1}{2}\omega)}$ distinguishes (M, ω) from B and Z . This proves Claim 3. \square

The statement of Proposition 6 follows from Claims 1 and 3 and the fact that in the case $w(M, \omega) \neq 1$, the Gromov width distinguishes (M, ω) from B and Z . \square

4. Hypotheses of the main result that cannot be dropped

The hypothesis in Theorem 5 that every object of \mathcal{C} has some negative helicity boundary component cannot be dropped. This is the content of the following proposition.

Proposition 13 (*negative helicity hypothesis cannot be dropped*). *Let $n \geq 2$ and consider the full subcategory \mathcal{C} of $\Omega^{2n,2}$ whose objects are the compact, 1-connected, exact symplectic manifolds. This category does not satisfy the conclusion of Theorem 5.¹⁹*

Proof of Proposition 13. By Theorem 1.1 in [2] there exist strongly starshaped²⁰ compact submanifolds M and M' of \mathbb{R}^{2n} of dimension $2n$ that are not symplectomorphic w.r.t. the standard symplectic form ω_{st} , but have symplectomorphic interiors.

The manifolds M and M' are contractible and therefore 1-connected.²¹ It follows that (M, ω_{st}) and (M', ω_{st}) are objects of \mathcal{C} .

Claim 1. *The capacities on \mathcal{C} do not distinguish these objects.*

Proof of Claim 1. Let c be a capacity on \mathcal{C} . Let $a \in (0, 1)$. We choose a symplectomorphism $\varphi : \text{Int } M \rightarrow \text{Int } M'$. Since M is strongly starshaped, aM is contained in $\text{Int } M$. Hence $\varphi(aM)$ is well-defined. We have

$$\begin{aligned} &a^2 c(M, \omega_{\text{st}}) \\ &= c(M, a^2 \omega_{\text{st}}) \\ &= c(aM, \omega_{\text{st}}) \end{aligned}$$

¹⁹ The objects of \mathcal{C} need not satisfy the helicity assumption of Theorem 5.

²⁰ We call a submanifold M of \mathbb{R}^m *strongly starshaped* iff for every point $x \in M$ the line segment from 0 to x is contained in M and every line emanating from 0 intersects the boundary of M transversely.

²¹ We may choose M and M' to be arbitrarily C^∞ -close to the closed unit ball.

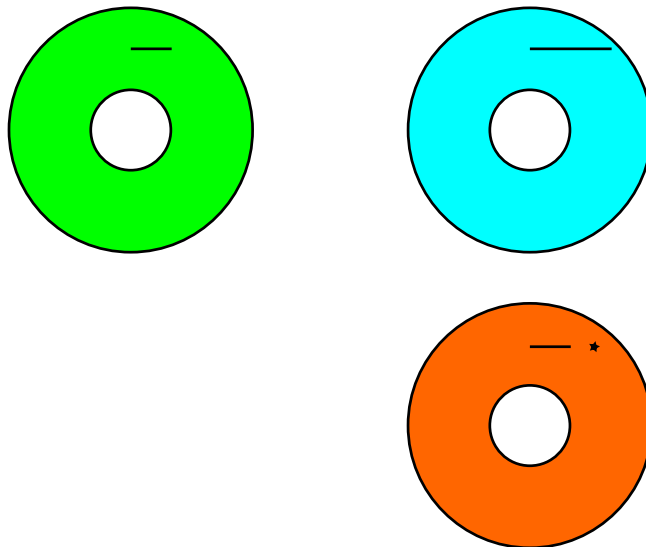


Fig. 2. The green slit shell is M , and the orange punctured and slit shell is M' , which is M with a point removed. The blue shell is the image of M under the Hamiltonian flow $\varphi^{1 \log 2}$. The line segment is stretched so much that it includes the puncture of the orange shell. Hence the orange punctured and slit shell includes the blue slit shell. (For better visibility we moved the puncture slightly to the left.)

$$\begin{aligned} & \text{(since } (M, a^2\omega_{\text{st}}) \text{ and } (aM, \omega_{\text{st}}) \text{ are symplectomorphic)} \\ & = c(\varphi(aM), \omega_{\text{st}}) \text{ (using that } \varphi : aM \rightarrow \varphi(aM) \text{ is a symplectomorphism)} \\ & \leq c(M', \omega_{\text{st}}) \text{ (using that } \varphi(aM) \subseteq \text{Int } M' \subseteq M'). \end{aligned}$$

Since this holds for every $a \in (0, 1)$, it follows that $c(M, \omega_{\text{st}}) \leq c(M', \omega_{\text{st}})$. Interchanging the rôles of M and M' , the opposite inequality follows. Hence we have

$$c(M, \omega_{\text{st}}) = c(M', \omega_{\text{st}}).$$

Since this holds for all capacities c , it follows that (M, ω_{st}) and (M', ω_{st}) cannot be distinguished by capacities. This proves Claim 1 and completes the proof of Proposition 13. \square

Remark 14 (*Compactness hypothesis cannot be dropped*). The compactness hypothesis in Theorem 5 cannot be dropped. To see this, we denote by $B_r^m(x)$ (resp. $\overline{B}_r^m(x)$) the open (resp. closed) ball of radius r around x in \mathbb{R}^m . We define

$$\begin{aligned} M & := \\ & \overline{B}_3^{2n}(0, -2, 0, \dots, 0) \setminus \left(B_1^{2n}(0, -2, 0, \dots, 0) \cup ([0, 1] \times \{(0, \dots, 0)\}) \right), \\ M' & := M \setminus \{(2, 0, \dots, 0)\}. \end{aligned}$$

See Fig. 2. We equip M and M' with the standard symplectic form. Since $M' \subseteq M$, it follows that $c(M') \leq c(M)$ for every capacity c . We consider the Hamiltonian function $H(q^1, p_1, \dots, q^n, p_n) := q^1 p_1$. We choose a smooth function $\rho : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ that equals 1 in some neighborhood of $[0, 2] \times \{(0, \dots, 0)\}$ and has compact support in $\overline{B}_3(0, -2, 0, \dots, 0) \setminus B_1(0, -2, 0, \dots, 0)$. The Hamiltonian flow of H in \mathbb{R}^{2n} is given by

$$\varphi_H^t(q^1, p_1, \dots, q^n, p_n) = (e^t q^1, e^{-t} p_1, q^2, p_2, \dots, q^n, p_n).$$

It follows that the restriction of the Hamiltonian time- t flow φ^t of ρH to $[0, 1] \times \{(0, \dots, 0)\}$ is given by the same formula. Hence the image of $[0, 1] \times \{(0, \dots, 0)\}$ under $\varphi^{\log(2)}$ includes $[0, 2] \times \{(0, \dots, 0)\}$. Therefore, $\varphi^{\log(2)}$ maps M to M' . See Fig. 2. It follows that $c(M) \leq c(M')$ for every capacity c . Hence all capacities agree on M and M' , although these manifolds are not diffeomorphic (and hence not symplectomorphic). \diamond

Appendix A. Isomorphism-closed subcategories

In this section, we address some set-theoretic issues that arise when considering categories and subcategories in the ZFC axiomatic system. In ZFC, there is no notion of a class. However, ZFC can deal with classes that are determined by a well-formed logical formula, such as the class of all sets, and the class of all smooth manifolds. This way, every statement about the class can be reformulated in terms of the logical formula. (Since there are only countably many logical formulas, they do not define all categories.)

Let $\widehat{\mathcal{C}}$ be a category determined by a formula, and let \mathcal{C}_0 be a small subcategory of $\widehat{\mathcal{C}}$, i.e. a pair $(\mathcal{O}_0, \mathcal{M}_0)$ of sets that satisfies the logical formula defining $\widehat{\mathcal{C}}$.

Definition 15 (*Isomorphism-closure*). We define the *isomorphism-closure* of \mathcal{C}_0 in $\widehat{\mathcal{C}}$ to be the category whose objects are the objects of $\widehat{\mathcal{C}}$ that are isomorphic in $\widehat{\mathcal{C}}$ to an object of \mathcal{C}_0 , and whose morphisms are obtained by composition of the isomorphisms of $\widehat{\mathcal{C}}$ with the morphisms of \mathcal{C}_0 . More specifically, the morphisms are of the form $\psi \circ f_0 \circ \varphi^{-1}$, where φ and ψ are isomorphisms in $\widehat{\mathcal{C}}$ and f_0 is a morphism in \mathcal{C}_0 .

Assume that $\widehat{\mathcal{C}}$ is locally small and that there exists a set S of objects of $\widehat{\mathcal{C}}$, such that every object of $\widehat{\mathcal{C}}$ is isomorphic to an element of S .

Definition 16 (*Isomorphism-closed subcategory*). The subcategories of $\widehat{\mathcal{C}}$ that we obtain as isomorphism-closures as in Definition 15 are called *isomorphism-closed subcategories* of $\widehat{\mathcal{C}}$.

Remark 17 (*Isomorphism-closed subcategory*). The category $\widehat{\mathcal{C}}$ is an isomorphism-closed subcategory of itself. This holds, because the full subcategory of $\widehat{\mathcal{C}}$ with the elements of S as objects is small by the local smallness of $\widehat{\mathcal{C}}$. This is the reason for imposing the conditions of the existence of S and the local smallness of $\widehat{\mathcal{C}}$. Without these conditions $\widehat{\mathcal{C}}$ need not be an isomorphism-closed subcategory of itself in the sense of Definition 16. As an example, the category with sets as objects and only the identity morphisms as morphisms is not an isomorphism-closed subcategory of itself in this sense. \diamond

Remark (*Essential bijection with a small subcategory*). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called *essentially injective* iff for every pair (A, B) of objects in \mathcal{C} , the condition that $F(A)$ is isomorphic to $F(B)$ implies that A is isomorphic to B . The functor F is *essentially surjective* iff for every object B of \mathcal{D} , there is an object A of \mathcal{C} , such that B is isomorphic to $F(A)$. A functor that is both essentially injective and essentially surjective is called *essentially bijective*. Let S be as described before Definition 16. The inclusion of the full small subcategory of $\widehat{\mathcal{C}}$ whose objects are the elements of S into $\widehat{\mathcal{C}}$ is an essentially bijective functor. \diamond

Let G be a group that is determined by a logical formula, and let $(\widehat{\mathcal{O}}, \widehat{\mathcal{M}}) := \widehat{\mathcal{C}}$ be a category determined by a formula. Formally, by a G -action on $\widehat{\mathcal{C}}$ we mean a map

$$\rho : G \times \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$$

such that for every g, h in G , every A in $\widehat{\mathcal{O}}$ and every f in $\widehat{\mathcal{M}}$, we have that

$$\begin{aligned} \rho_g &:= \rho(g, \cdot) \text{ is a functor } \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}} \\ \rho_e(A) &= A \quad \rho_e(f) = f \\ \rho_h(\rho_g(A)) &= \rho_{hg}(A) \quad \rho_h(\rho_g(f)) = \rho_{hg}(f), \end{aligned}$$

where e denotes the neutral element of G . The following definition makes this notion precise within ZFC.

Definition 18 (*Group action on a category*). A G -action on $\widehat{\mathcal{C}}$ is a logical formula that satisfies the properties corresponding to the above conditions.

Remark (*Group action on a category*). We think of a G -action ρ as the “subclass” of $(G \times \widehat{\mathcal{C}}) \times \widehat{\mathcal{C}}$ determined by ρ .

Definition 19 (*Invariant subcategory*). An isomorphism-closed or small subcategory \mathcal{C} of $\widehat{\mathcal{C}}$ is called G -invariant iff for every $g \in G$, the functor ρ_g leaves \mathcal{C} invariant.

Remark 20 (*Generating invariant small subcategory*). If \mathcal{C} is an invariant isomorphism-closed subcategory of $\widehat{\mathcal{C}}$, and if $\widehat{\mathcal{C}}$ is locally small, then there exists an invariant small subcategory \mathcal{C}_0 of $\widehat{\mathcal{C}}$, such that \mathcal{C} is the isomorphism-closure of \mathcal{C}_0 . Indeed, if $\widetilde{\mathcal{C}}_0 = (\widetilde{\mathcal{O}}_0, \widetilde{\mathcal{M}}_0)$ is a small subcategory whose isomorphism-closure is \mathcal{C} , then the small category $\mathcal{C}_0 = (\mathcal{O}_0, \mathcal{M}_0)$ given by

$$\mathcal{O}_0 = \{ \rho_g(A) \mid g \in G, A \in \widetilde{\mathcal{O}}_0 \} \quad \text{and} \quad \mathcal{M}_0 = \{ \rho_g(f) \mid g \in G, f \in \widetilde{\mathcal{M}}_0 \}$$

is invariant and has \mathcal{C} as isomorphism-closure.

Appendix B. Helicity

In this section, we list some properties of helicity and prove a version of Stokes’ Theorem for helicity. This is the main ingredient of the proof of Theorem 5, used indirectly through Lemma 8. Consider the setting of Definition 4 (helicity).

Lemma 21. *Helicity is well-defined, i.e. it is independent of the choice of a primitive.*

Proof of Lemma 21. Let α and α' be two primitives of σ . Then $\alpha - \alpha'$ is closed and hence $(-1)^{k-1}(\alpha - \alpha') \wedge \alpha \wedge \sigma^{\wedge(n-2)}$ is a primitive of $(\alpha - \alpha') \wedge \sigma^{\wedge(n-1)}$. Using that Stokes’ Theorem and our assumption that N does not have boundary, it follows that

$$\int_N (\alpha - \alpha') \wedge \sigma^{\wedge(n-1)} = 0.$$

The statement of Lemma 21 follows. \square

Remark (*Well-definedness of helicity, odd degree*).

- Helicity is only well-defined for $n \geq 2$. See [4, Remark 31].
- It vanishes if k is odd. See [4, Remark 31].

The following facts about helicity are used in the proof of Lemma 8, which is Lemma 49 in [4].

Remarks 22 (*helicity*).

- For every $C \in \mathbb{R}$, we have $h(N, O, C\sigma) = C^n h(N, O, \sigma)$.
- Reversing the orientation on N changes the sign of the helicity: $h(N, \bar{O}, \sigma) = -h(N, O, \sigma)$, where \bar{O} is the orientation opposite to O .
- Let N and N' be two closed $kn - 1$ -dimensional manifolds, O an orientation on N , σ an exact k -form on N , and $\varphi : N \rightarrow N'$ a smooth embedding. Then we have

$$h(\varphi(N), \varphi_*O, \varphi_*\sigma) = h(N, O, \sigma). \quad \diamond$$

The next lemma is one of the main ingredients in the proofs of the main results. It says that the helicity of the boundary of a manifold is equal to the volume of that manifold. Let M be a manifold, $N \subseteq M$ a submanifold and ω a differential form on M . We define

$$\omega_N := \text{pullback of } \omega \text{ by the inclusion of } N \text{ into } M.$$

Let ∂M denote the boundary of M and assume that $N \subseteq \partial M$. Let O be an orientation on M . We define:

$$O_N := \text{orientation of } N \text{ induced by } O.$$

Let $k, n \in \mathbb{N}$, such that $n \geq 2$. Let (M, O) be a compact, oriented manifold of dimension kn , and ω be an exact k -form on M .

Lemma 23 (*Stokes' Theorem for helicity*). *The following equation holds:*

$$\int_{M, O} \omega^{\wedge n} = h(\partial M, O_{\partial M}, \omega_{\partial M}).$$

Proof of Lemma 23. Let α be a primitive of ω . Then we have that $\omega^{\wedge n} = d(\alpha \wedge \omega^{\wedge(n-1)})$. Using Stokes' Theorem, we obtain

$$\int_{M, O} \omega^{\wedge n} = \int_{\partial M, O_{\partial M}} \alpha \wedge \omega^{\wedge(n-1)} = h(\partial M, O_{\partial M}, \omega_{\partial M}).$$

This completes the proof. \square

Remarks (*Stokes' Theorem for helicity*).

- If k is odd then both sides of the equality in the statement of Lemma 23 are zero. Here for the right hand side see [4, Remark 31].
- If ω is maxipotent and O is the orientation induced by ω , then Lemma 23 implies that the helicity of the boundary of M with respect to the orientation induced by O is positive, as it equals the volume of M .
- If the image of a negative helicity boundary component under a form-preserving embedding bounds a compact manifold (e.g. a ball), then the volume of this manifold equals minus the helicity of the boundary component. This means that the negative helicity component eats up some volume. Compare to the left hand side of Fig. 1 on page 6. We used this in the proofs of our main results.

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