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ABSTRACT

We study an inverse scattering problem for the Helmholtz equation on the whole line. The goal of this paper is to obtain a Gelfand–Levitan–Marchenko (GLM)-type equation for the Jost solution that corresponds to the 1D Helmholtz differential operator. We assume for simplicity that the refraction index is of compact support. Using the asymptotic behavior of the Jost solutions with respect to the wave-number, we derive a generalized Povzner–Levitan representation in the space of tempered distributions. Then, we apply the Fourier transform on the scattering relation that describes the solutions of the Helmholtz scattering problem and we derive a generalized GLM equation. Finally, we discuss the possible application of this new generalized GLM equation to the inverse medium problem.

I. INTRODUCTION

The so-called inverse medium problem appears in many applications, where one aims to reconstruct unknown coefficients of a (partial) differential equation from traces of its solution. This way, one can study materials and internal structures of media that are not directly accessible. In geophysics, for example, a well-known inverse problem is the estimation of the medium parameters of a subsurface domain from seismic measurements; see, for example, Ref. 1. Similar inverse problems can also be found in ultrasound and ultrasonic non-destructive testing; see, for example, Ref. 2. The governing equation in these applications is a variable coefficient wave equation, and the measurements consist of boundary traces of the wavefield for a number of frequencies. A variety of methods for solving the resulting inverse medium problem have been developed over the previous decades. We roughly divide these methods into two classes. On the one end of the spectrum, one can find "modern" variational formulations (that are often solved iteratively), such as full waveform inversion; see, for example, Ref. 3. On the other end, one can find classical inverse scattering methods; see, for example, Refs. 1 and 4. Recently, classical results from inverse scattering have found renewed interest in the geophysical community and hybrid methods have been proposed.[5,6]

In this paper, we revisit one such classical method based on the Gelfand–Levitan–Marchenko (GLM) equation. In particular, we focus on the derivation of a GLM equation corresponding to the scattering problem for the Helmholtz equation with the following form:

$$\left\{-\frac{d^2}{dx^2}+k^2 n(x)\right\} y(k,x)=k^2 y(k,x), \quad x\in\mathbb{R}, \tag{1}$$

$$y(k,\cdot)=y^i(k,\cdot)+y^s(k,\cdot), \tag{2}$$

$y^i(k,x)=e^{-\imath kx}, \quad x\in\mathbb{R}$, and asymptotic boundary conditions

$$\lim_{x\to\pm\infty}\left\{\frac{dy^s(k,x)}{dx}\mp \imath k y^s(k,x)\right\}=0. \tag{3}$$

Originally, the GLM equation is a fundamental relation that can be used to solve the Schrödinger inverse scattering problem on the line.[7] It relates scattering data (reflection measurements) to the so-called Jost solutions of the Schrödinger operator (plane wave-like, right or left propagating solutions). It is also well known that the Helmholtz equation can be transformed to the Schrödinger equation using the travel time coordinate transform. Therefore, by changing the setting, it is sufficient to just focus on the study of the transformed system governed by the Schrödinger equation; see, for example, Refs. 8–10. This equivalence allows us to use a wealth of mathematical tools available from the Schrödinger scattering theory, including, of course, the GLM equation.

One advantage of this approach is that the path leading from the measurements (scattering data) to the parameter of the differential operator involves linear steps (as opposed to full waveform inversion, for example). Although this approach is known for very long time, it is only limited to the 1D case. In higher-dimensional media, it is impossible to transform the Helmholtz equation to the Schrödinger equation. The only exception to this is when the medium is laterally stratified; see, e.g., Ref. 11. This creates the need for the development of a GLM framework that avoids the use of a transformation to the Schrödinger setting. The first step toward developing this new inversion method is given as the 1D case in this paper.

The following basic observation is the starting point of our analysis. Let u^+ be the Jost solution of the Helmholtz equation [Eq. (1)]. Contrary to the Schrödinger case, the function of the wave number

$$k\mapsto |e^{-\imath kx}u^+(k,x)-1|$$

might not have a growing behavior that allows for the use of the classical Paley–Wiener theory, but still grows in a controlled way as $|k|$ grows. This allows us to use a distributional setting, which also permits the definition of the Fourier transform.

Our main contributions are as follows:

- The extension of the so-called Povzner–Levitan representation to Jost solutions of the Helmholtz scattering problem.
- The derivation of a generalized GLM equation for the Helmholtz problem in the space of tempered distributions.

This paper is organized as follows. In Sec. II, we formulate the direct scattering problem and we review basic properties of the solutions of the forward problem. Then follows Sec. III that contains the main result of the paper and breaks down its proof in multiple lemmas. We continue with Sec. IV where we propose a practical way for solving the inverse medium problem using the main result, and we conclude this paper in Sec. V.

II. PRELIMINARIES

In this section, we present some well-known results regarding the forward scattering problem for the Helmholtz equation on the line. In Subsection II A, we formulate the forward scattering problem and we give the definition of Jost solutions of the Helmholtz equation. In Subsection II B, we recall some fundamental properties of the solutions of the forward scattering problem and we define reflection and transmission coefficients.

A. Forward scattering problem

We consider the forward scattering problem for the 1D Helmholtz operator on the real line described in Eqs. (1)–(3). We assume that the real valued coefficient, n, is sufficiently smooth, having a compact support in the form

$$\operatorname{supp}(n)=(0,b), \tag{4}$$

for some $b>0$, and that $1-n>0$. In addition, the incident wave field is a plane wave [incoming from the right $(+\infty)$], $y^i(k,x)=e^{-\imath kx}$, $x\in\mathbb{R}$. Working the same way as in Ref. 7, we reduce the Helmholtz differential equation to the following Volterra integral equations.

Proposition II.1. The solutions of the integral equations,

$$u(x)=e^{\imath kx}-\int_x^\infty k\,\sin(k(x-t))n(t)u(t)dt, \tag{5}$$

$$u(x)=e^{-\imath kx}+\int_{-\infty}^x k\,\sin(k(x-t))n(t)u(t)dt, \tag{6}$$

satisfy the Helmholtz differential equation (1). The solutions of these integral equations are the Jost solutions of the Helmholtz equation. In addition, the following asymptotic behavior holds true for the unique solution of (5), say $u^+(k,\cdot)$. For $k\in\mathbb{R}$, we get

$$|u^{+}(k,x) - e^{\imath kx}| \leq c(k,n)\frac{k^2}{1+|k|}(1+\max\{-x,0\})\int_x^{\infty}(1+|t|)|n(t)|dt \tag{7}$$

and

$$|\partial_x u^{+}(k,x) - \imath k e^{\imath kx}| \leq c(k,n)\frac{k^2}{1+|k|}\int_x^{\infty}(1+|t|)|n(t)|dt.$$

We get similar asymptotic behavior for the left-going Jost solution, u^-, which solves Eq. (6).

We obtain the above result working the same way as in Ref. 7 (Chap. 4) for the scattering potential $Q := k^2 n$. In addition, $c(k,n)$ grows faster than an exponential function as a function of k. Finally, we define the Fourier transform as

$$\widehat{f}(t) = (\mathscr{F}f)(t) = \frac{1}{\pi}\int_{\mathbb{R}} f(k)e^{2\imath kt}dk, \quad t\in\mathbb{R},$$

$$f(k) = \left(\mathscr{F}^{-1}\widehat{f}\right)(k) = \int_{\mathbb{R}}\widehat{f}(t)e^{-2\imath kt}dt, \quad k\in\mathbb{R},$$

for $f\in\mathscr{S}(\mathbb{R})$ (Schwartz functions).

B. Basic properties of the solutions of the forward problem

In this subsection, we present essential properties of the Jost solutions and, in turn, of the solutions of the forward problem. Again, the next result is classic and we refer to Ref. 7 for its proof.

Proposition II.2. Let $k\in\mathbb{R}\backslash\{0\}$ and $f,g\in H^2_{loc}(\mathbb{R})$ be solutions to the Helmholtz differential equation. Then, the Wronskian of the two solutions $W(f,g)$ is constant. In particular, we get for the Jost solutions,

$$W(u^{+}(k,\cdot),u^{+}(-k,\cdot)) = -2\imath k$$

and

$$W(u^{-}(k,\cdot),u^{-}(-k,\cdot)) = 2\imath k.$$

Remark II.1. Since the solution space for the Helmholtz differential equation is two-dimensional, we obtain that

$$u^{-}(k,x) = a_k^{+}u^{+}(k,x) + b_k^{+}u^{+}(-k,x) \tag{8}$$

and

$$u^{+}(k,x) = a_k^{-}u^{-}(k,x) + b_k^{-}u^{-}(-k,x) \tag{9}$$

for $x\in\mathbb{R}$. This implies that

$$2\imath k = W(u^{-}(k,\cdot),u^{-}(-k,\cdot)) = W(a_k^{+}u^{+}(k,\cdot)+b_k^{+}u^{+}(-k,\cdot),a_{-k}^{+}u^{+}(-k,\cdot)+b_{-k}^{+}u^{+}(k,\cdot)) \Rightarrow 1 = |b_k^{+}|^2 - |a_k^{+}|^2. \tag{10}$$

Remark II.2. For $k\in\mathbb{R}\backslash\{0\}$, we define reflection and transmission coefficients as

$$T(k) = \frac{1}{b_k^{+}} \tag{11}$$

and

$$R^{+}(k) = \frac{a_k^{+}}{b_k^{+}}, \tag{12}$$

respectively. The + superscript in the reflection coefficient denotes reflection caused by an incoming plane wave from the right. Similarly, we can define R^-; see Figs. 1 and 2 (assuming $k>0$ to make sense of "right" and "left"). The transmission coefficient is the same regardless of right or left sides of incidence. This is known as transmission reciprocity; see Ref. 12. Therefore, using (10), we obtain the conservation of energy,

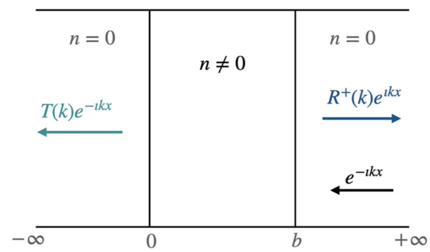

FIG. 1. $y^i = e^{-ik\cdot}$ enters from $+\infty$ and creates reflection and transmission responses R^+, T, respectively.

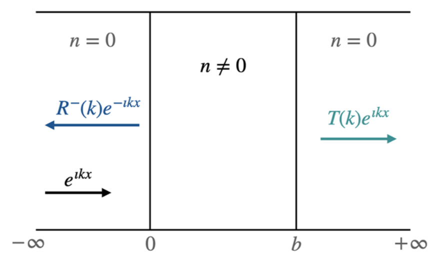

FIG. 2. $y^i = e^{ik\cdot}$ enters from $-\infty$ and creates reflection and transmission responses R^-, T, respectively.

$$|T(k)|^2 + |R^{\pm}(k)|^2 = 1, \quad k \in \mathbb{R}\backslash\{0\}. \tag{13}$$

Now, similar to the Schrödinger equation case, the solution of forward problems, (1)–(3), for $k \in \mathbb{R}\backslash\{0\}$ can be decomposed as

$$y(k,x) = T(k)u^-(k,x) = u^+(-k,x) + R^+(k)u^+(k,x) \tag{14}$$

for $x \in \mathbb{R}$. Finally, we get the following relations for $k \in \mathbb{R}\backslash\{0\}$:

$$R^+(k) = \frac{k}{2\imath}\int_{\mathbb{R}} n(x)y(k,x)e^{-\imath kx}dx \tag{15}$$

and

$$T(k) = 1 + \frac{k}{2\imath}\int_{\mathbb{R}} n(x)y(k,x)e^{\imath kx}dx. \tag{16}$$

III. MAIN RESULT

In this section, we present our main finding, which is a generalized Gelfand–Levitan–Marchenko equation for the 1D Helmholtz scattering problem. The difference between our finding and the classical Gelfand–Levitan–Marchenko equation for the 1D Schrodinger scattering is the mathematical setting. In the latter case, we are working in an L^2-setting due to the sufficiently regular asymptotic behavior of the Schrödinger–Jost solution as a function of the wave-number. In our Helmholtz case, we have a "less regular" asymptotic behavior that requires the use of tempered distributions to do calculations. The following theorem is the main result of this paper.

Theorem III.1. *Let the Jost solution, u^+, of the Helmholtz problem. Also let an $x \in \mathbb{R}$ be fixed, and define*

$$v^+(k,x) := e^{-\imath kx}u^+(k,x).$$

There exists a unique kernel $B_x^+ \in \mathscr{S}'(\mathbb{R})$ such that

$$v^+(k,x) = 1 + \pi\mathscr{F}B_x^+(k), \ \ k \in \mathbb{R}. \tag{17}$$

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This kernel can be decomposed as

$$B_x^+ = B_x^{+,L^p} + B_x^{+,\mathscr{E}'} \tag{18}$$

with $B_x^{+,L^p} \in \cap_{p=1}^{\infty} L^p(\mathbb{R})$ *and* $B_x^{+,\mathscr{E}'} \in \mathscr{E}'(\mathbb{R})$. *Furthermore, the following GLM-like relation holds true in* $\mathscr{S}'(\mathbb{R})$:

$$B_x^+ + K^+(x+\cdot) + K^+(x+\cdot) \star B_x^+ = \mathscr{F}\{k \mapsto T(k)v^-(k,x) - 1\}, \tag{19}$$

with $\star$ *denoting the correlation and*

$$K^+ = \mathscr{F}R^+ \in L^2(\mathbb{R}). \tag{20}$$

Remark III.1. It is clear that relation (19) has exactly the same form as in the Schrödinger case. In Sec. IV, we will discuss the use of this GLM-type relation. We can derive a similar GLM expression for B^- *using* $K^- = \mathscr{F}R^-$.

We will break down the Proof of Theorem III.1 in multiple lemmas. At the end of Sec. III, we will combine the results with the proof of the main theorem.

A. Generalized Povzner–Levitan representation

Similar to the Schrödinger equation case, in this section, we obtain a generalized Povzner–Levitan representation for the right going Jost solution of the Helmholtz equation. The major difference between the Schrödinger and Helmholtz cases is that in the latter equation case, we must work in a distributional setting. We show that the Jost solutions $u^{\pm}(\cdot,x)$ behave "nicely" at infinity as a function of k. However, before elaborating more on our theory, for the sake of completeness, we show how one can transform the Helmholtz equation to the Schrodinger equation using the travel-time transform. We only use the equivalence between Helmholtz and Schrödinger problems to show the Povzner–Levitan representation and to show regularity properties of the kernel and the scattering data. It is important to note that the travel-time transformation is the mean to show the generalized Povzner–Levitan representation. Obviously, this particular writing for Jost solutions holds true independently of how one shows it. However, possibly, the simplest way to prove it is the one we follow.

We define the new travel-time variable as

$$z(x) = \int_0^x \sqrt{m(y)}dy, \quad x \in \mathbb{R}, \tag{21}$$

with $m = 1 - n > 0$. The scattering potential, q, only depends on m, and it is defined similarly as in the acoustic case; see Refs. 9 and 10 In addition, q is compactly supported on $[0, z(b)]$; see, e.g., Ref. 13. The new equation now is given by

$$\left\{-\frac{d^2}{dz^2} + q(z)\right\} f(k,z) = k^2 f(k,z), \quad z \in \mathbb{R}. \tag{22}$$

Assuming that u solves the Helmholtz differential equation, f is connected with u via the formula

$$f(k,z(x)) = \theta(x)u(k,x) \tag{23}$$

with $\theta(x) = (m(x))^{1/4}$. The following result follows.

Proposition III.1. Let the Jost solutions be $u^{\pm}(k,\cdot)$ *of (1) and* $f^{\pm}(k,\cdot)$ *of (22). We get that*

$$f^+(k,z(x)) = \theta(x)e^{\imath k(I_b - b)}u^+(k,x), \ x \in \mathbb{R}, \tag{24}$$

and

$$f^-(k,z(x)) = \theta(x)u^-(k,x), \quad x \in \mathbb{R}, \tag{25}$$

with $I_b = \int_0^b \sqrt{m(y)}dy$.

Proof. Let the Jost solution of the Helmholtz equation be $u^+(k,\cdot)$ with

$$u^+(k,x) = e^{\imath kx}, \quad x > b.$$

We want to show that the Jost solutions of Eqs. (1) and (22), respectively, are related. By relation (23), we get that $u^+(k,\cdot)$ is related to a solution $f(k,\cdot)$ of the Schrödinger equation with $f(k,z(x)) = \theta(x)u^+(k,x), \quad x \in \mathbb{R}$. This gives

$$f(k,z(x)) = e^{\imath kx}, \quad x > b\ (z(x) > z(b)). \tag{26}$$

Keep in mind that $\theta(x) = 1$ if $x > b$. Now, for $x > b$, we also get

$$z(x) = \int_0^b \sqrt{m(y)}dy + \int_b^x dy = I_b + (x-b).$$

Combining the above equation, (26) gives

$$f(k,z(x))e^{\imath k(I_b-b)} = e^{\imath kx}e^{\imath k(I_b-b)} = e^{\imath kz(x)}, \quad x > b.$$

Therefore, $e^{\imath k(I_b-b)}f(k,\cdot)$ solves the Schrödinger equation and behaves as a plane wave when $z > z(b)$. Now, since the solution space of the Schrödinger equation is spanned by the Jost solutions $f^+(k,\cdot), f^+(-k,\cdot)$, we get

$$e^{\imath k(I_b-b)}f(k,z) = a_1f^+(k,z) + b_1f^+(-k,z), \quad z \in \mathbb{R} \Rightarrow e^{\imath k(I_b-b)}f(k,z) = a_1e^{\imath kz} + b_1e^{-\imath kz}, \quad z > z(b). \tag{27}$$

Therefore, $a_1 = 1$ and $b_1 = 0$. Thus,

$$e^{\imath k(I_b-b)}f(k,z) = f^+(k,z), \quad z \in \mathbb{R}. \tag{28}$$

Similarly, for the left propagating Jost solution, we get

$$u^-(k,x) = e^{-\imath kx}, \quad x < 0. \tag{29}$$

Since $z = x, \theta(x) = 1$ for $x < 0$, we obtain that $g = \theta u^-$ solves the Schrödinger equation, and for $x < 0$,

$$e^{-\imath kx} = e^{-\imath kz} = g(k,z) \Rightarrow \tag{30}$$

$$g(k,z) = f^-(k,z(x)) = \theta(x)u^-(k,x), \quad x \in \mathbb{R}. \tag{31}$$

□

Remark III.2. In view of relations (24) and (25), the Jost solutions $u^\pm(\cdot,x)$ are continuous as functions of k; see Ref. 7 (Corollary 4.1.4, Theorem 4.1.8). We can also define complex analytic extensions of the Jost solutions.

Using the above results, we show the following distributional Povnzer–Levitan representation for the right-going Jost solution, u^+, of the Helmholtz equation. Before proceeding to the result, we remind the reader a basic result from the theory of distributions.

Lemma III.1. Let a function $f \in L^1_{loc}(\mathbb{R};\mathbb{C})$ such that $|f(x)| = O(1+|x|), |x| \to \infty$. Then, the map $\mathscr{S}(\mathbb{R}) \ni \phi \mapsto \int_{\mathbb{R}} f(x)\phi(x)dx$ defines an element of $\mathscr{S}'(\mathbb{R})$.

Proof. See Ref. 14 (p. 105). □

Lemma III.2. Let $x \in \mathbb{R}$ be fixed. Then, there exists a tempered distribution $V_x^+ = V^+(x,\cdot)$ such that

$$\mathbb{R} \ni k \mapsto v^+(k,x) = 1 + \mathscr{F}^{-1}V_x^+(k) \ \text{ is in } \mathcal{S}'(\mathbb{R}) \tag{32}$$

as an $L^1_{loc}(\mathbb{R})$ function that defines distribution through integration.

Proof. Let x be fixed and $k \in \mathbb{R}$. We can change the spatial variable using relation (21). Since

$$f^+(k,z) = \theta(x)e^{\imath k(I_b - b)}u^+(k,x), \quad x \in \mathbb{R}, \tag{33}$$

we set $\widetilde{\theta} = \theta e^{ik(I_b - b)}$ and we obtain

$$|u^+(k,x) - e^{\imath kx}| = \left|\frac{f^+(k,z)}{\widetilde{\theta}(x)} - e^{\imath kx}\right| \leq \tag{34}$$

$$\left|\frac{f^+(k,z)}{\widetilde{\theta}(x)} - \frac{e^{\imath kz}}{\widetilde{\theta}(x)}\right| + \left|\frac{e^{\imath kz}}{\widetilde{\theta}(x)} - e^{\imath kx}\right| \leq \tag{35}$$

$$\frac{1}{|\theta(x)|}|f^+(k,z) - e^{ikz}| + 1 + \frac{1}{|\theta(x)|}. \tag{36}$$

Now, we know that for $k \in \mathbb{R}$, the Jost solution of the Schrödinger problem has the following behavior:

$$|f^+(k,z) - e^{\imath kz}| \leq \frac{C(q)(1 + \max\{-z, 0\})}{1 + |k|}\int_z^\infty (1 + |z|)q(z)dz. \tag{37}$$

See Ref. 7 (Chap. 4). Therefore, for fixed x, the map

$$\mathbb{R} \ni k \mapsto |u^+(k,x) - e^{\imath kx}| \tag{38}$$

behaves asymptotically at most as a constant $A \in \mathbb{R}$. Similarly,

$$\mathbb{R} \ni k \mapsto |u^+(k,x)e^{-\imath kx} - 1| \tag{39}$$

behaves at most as a constant; therefore, it defines a tempered distribution as a locally integrable function.

Now, since the Fourier transform is a homeomorphism,

$$\mathscr{F} : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R}),$$

and, similarly,

$$\mathscr{F}^{-1} : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$$

is also a homeomorphism (onto), there exists a tempered distribution $V^+(x,\cdot) \in \mathscr{S}'(\mathbb{R})$ such that

$$\mathbb{R} \ni k \mapsto u^+(k,x)e^{-\imath kx} = 1 + \mathscr{F}^{-1}V_x^+(k) \qquad \text{in } \mathcal{S}'(\mathbb{R}). \tag{40}$$

□

Corollary III.1. Let $x \in \mathbb{R}$ be fixed. The function

$$\mathbb{R} \ni k \mapsto u^\pm(k,x)e^{\mp \imath kx} \tag{41}$$

defines a tempered distribution as a locally integrable function.

Remark III.3. In the Schrödinger equation case, it is well known that the Fourier kernel of the Povzner–Levitan representation is supported in $\mathbb{R}_{>0}$. In Subsection III B, we will clarify what the support of $V^+(x,\cdot)$ is.

Remark III.4. We can also write

$$\mathbb{R} \ni k \mapsto v^+(k,x) = 1 + \pi \mathscr{F} B_x^+ \tag{42}$$

with $B_x^+ = \mathscr{R} V_x^+$,

$$\mathscr{R}\phi(x) = \phi(-x), \ x \in \mathbb{R},$$

for $\phi \in \mathscr{S}(\mathbb{R})$. Obviously, we have a similar writing for v^-.

B. Properties of the GLM kernel and the scattering data

Essentially, we can consider the quantities involved in the GLM equation only as distributions. However, the equivalence between Schrödinger and Helmholtz equations in 1D naturally lets us to "gain more regularity" for the scattering data and the kernel B_x^+. Our GLM equation would still hold, but it would be given in a more abstract form than its current compact writing (19). Following the classical method of associating the Helmholtz equation with the Schrödinger equation, we obtain that the associated Schrödinger scattering problem is

$$\left\{-\frac{d^2}{dz^2}+q(z)\right\}f(k,z)=k^2f(k,z), \tag{43}$$

$$f(k,z)=e^{-\imath kz}+f^s(k,z), \tag{44}$$

$$\lim_{z\to\pm\infty}\left\{\frac{d}{dz}f^s(k,z)\mp\imath kf^s(k,z)\right\}=0. \tag{45}$$

Remark III.5. Take a solution, say u, of the Helmholtz scattering problem. Define

$$\widetilde{f}=\theta u. \tag{46}$$

$\widetilde{f}$ now solves the Schrödinger equation as we have discussed, and we would like to see how $\widetilde{f}$ compares with the solution of the Schrödinger scattering problem. First, observe that

$$\widetilde{f}(k,z)=\theta(x)u(k,x)=\theta(x)\{u^+(-k,x)+R^+(k)u^+(k,x)\} \tag{47}$$

$$\theta(x)\left\{\frac{e^{\imath k(I_b-b)}}{\theta(x)}f^+(-k,z)+R^+(k)\frac{e^{-\imath k(I_b-b)}}{\theta(x)}f^+(k,z)\right\}, \tag{48}$$

$x\in\mathbb{R}$. This gives

$$\widetilde{f}(k,z)e^{-\imath k(I_b-b)}=f^+(-k,z)+R^+(k)e^{-2\imath k(I_b-b)}f^+(k,z). \tag{49}$$

Similarly,

$$\widetilde{f}(k,z)=T(k)u^-(k,x)\theta(x)=T(k)f^-(k,z). \tag{50}$$

Therefore,

$$\widetilde{f}(k,z)e^{-\imath k(I_b-b)}\sim e^{-\imath kz}+R^+(k)e^{-2\imath k(I_b-b)}e^{\imath kz},\quad z\to\infty,$$

and

$$\widetilde{f}(k,z)e^{-\imath k(I_b-b)}\sim e^{-\imath k(I_b-b)}T(k)e^{-\imath kz},\quad z\to-\infty.$$

Since $\widetilde{f}(k,z)e^{-\imath k(I_b-b)}-e^{-\imath kz},\quad z\in\mathbb{R}$, is radiating [i.e., satisfies (45)] and since solutions of the scattering problem are unique, we obtain that

$$\widetilde{R}^+(k)=R^+(k)e^{-2\imath k(I_b-b)}\quad\text{and}\quad\widetilde{T}(k)=T(k)e^{-\imath k(I_b-b)},$$

where $\widetilde{R}^+$ and $\widetilde{T}$ are the reflection and transmission coefficients of the Schrödinger scattering problem, respectively.

The previous remark leads to the following proposition.

Proposition III.2. The following relations hold true:

$$|R^+(k)|=|\widetilde{R}^+(k)|,\quad\forall k\in\mathbb{R}\backslash\{0\}, \tag{51}$$

$$|T(k)|=|\widetilde{T}(k)|,\quad\forall k\in\mathbb{R}\backslash\{0\}. \tag{52}$$

We also get the following.

Corollary III.2. $K^+=\mathscr{F}R^+$ is a well-defined $L^2(\mathbb{R})$-element.

Proof. For the Schrödinger equation case, the reflection coefficient is an element of $L^2(\mathbb{R})$; see Ref. 7. Since the absolute values of R^+ and $\widetilde{R}^+$ coincide, we get the result. □

Lemma III.3. Let $x \in \mathbb{R}$. Then, $V_x^+ \in \mathscr{S}'(\mathbb{R})$ is a distribution that consists of a singular part and an L^p-part. The same holds for B_x^+.

Proof. Let $x \in \mathbb{R}$. We get for $k \in \mathbb{R}$,

$$e^{-\imath kx}u^+(k,x) = 1 + \mathscr{F}^{-1}V_x^+(k) \tag{53}$$

and

$$e^{-\imath kz(x)}f^+(k,z(x)) = 1 + \mathscr{F}^{-1}\widetilde{V}_{z(x)}^+(k), \tag{54}$$

using the classical Paley–Wiener theory to the solution of the Jost solution f^+; see Ref. 7. Now, we use relation (24) and we get

$$e^{-\imath kz(x)}\theta(x)e^{\imath k(I_b-b)}u^+(k,x) = 1 + \mathscr{F}^{-1}\widetilde{V}_{z(x)}^+(k) \Rightarrow e^{-\imath kz(x)}\theta(x)e^{\imath k(I_b-b)}e^{\imath kx}\big(1 + \mathscr{F}^{-1}V_x^+(k)\big) = 1 + \mathscr{F}^{-1}\widetilde{V}_{z(x)}^+(k).$$

We define $\lambda = (\lambda(x) =)z(x) - x - I_b + b$. Thus,

$$\theta(x)e^{-\imath k\lambda} + \theta(x)e^{-\imath k\lambda}\mathscr{F}^{-1}V_x^+(k) = 1 + \mathscr{F}^{-1}\widetilde{V}_{z(x)}^+(k) \Rightarrow \tag{55}$$

$$\theta(x)e^{-\imath k\lambda}\mathscr{F}^{-1}V_x^+(k) = -\theta(x)e^{-\imath k\lambda} + 1 + \mathscr{F}^{-1}\widetilde{V}_{z(x)}^+(k) \Rightarrow \tag{56}$$

$$\mathscr{F}^{-1}V_x^+(k) = -1 + \frac{e^{\imath k\lambda}}{\theta(x)} + \frac{e^{\imath k\lambda}}{\theta(x)}\mathscr{F}^{-1}\widetilde{V}_{z(x)}^+(k). \tag{57}$$

Now, since $\widetilde{V}_{z(x)}^+ \in L^p(\mathbb{R})$ for every $p \in [1,\infty]$ and since the Fourier transforms of complex exponential functions are Dirac-delta distributions, we obtain the result. □

Remark III.6. We can identify the support of V_x^+ in view of relation (57). Moreover, (57) gives a full description of the singularities of the kernel V_x^+.

C. Proof of Theorem III.1

In this subsection, we combine our findings and give the proof of our main result.

Proof of Theorem III.1. Let $x \in \mathbb{R}$. The scattering identity that describes the solutions of the forward problem reads

$$y(k,x) = T(k)u^-(k,x) = u^+(-k,x) + R^+(k)u^+(k,x),$$

$k \in \mathbb{R}\backslash\{0\}$. As mentioned before, we set $v^\pm(k,x) = e^{\mp\imath kx}u^\pm(k,x)$ and we take

$$T(k)v^-(k,x) = v^+(-k,x) + R^+(k)e^{2\imath kx}v^+(k,x), \quad k \in \mathbb{R}\backslash\{0\}. \tag{58}$$

We can view relation (58) in $\mathscr{S}'(\mathbb{R})$ since $v^\pm(\cdot,x) \in \mathscr{S}'(\mathbb{R})$ (Corollary III.1) and $|T|, |R^+| < 1$. We get $\forall\psi \in \mathscr{S}(\mathbb{R})$,

$$\langle\mathscr{F}\{k \mapsto T(k)v^-(k,x)\},\psi\rangle = \langle\mathscr{F}\{k \mapsto v^+(-k,x)\},\psi\rangle + \langle\mathscr{F}\{k \mapsto R^+(k)e^{2\imath kx}v^+(k,x)\},\psi\rangle \tag{59}$$

(with the sense that we transform the distributions that are defined through integration). Using relation (32), we get

$$\mathbb{R} \ni k \mapsto v^+(-k,x) = 1 + (\mathscr{F}^{-1}V_x^+)(-k) = 1 + (\mathscr{R}\mathscr{F}^{-1}V_x^+)(k). \tag{60}$$

Therefore, we obtain

$$\langle\mathscr{F}\{k \mapsto T(k)v^-(k,x) - 1\},\psi\rangle = \langle\mathscr{F}\mathscr{R}\mathscr{F}^{-1}V_x^+,\psi\rangle + \langle\mathscr{F}\Big\{k \mapsto R^+(k)e^{2\imath kx},\psi\rangle + \langle\mathscr{F}\{k \mapsto R^+(k)e^{2\imath kx}(\mathscr{F}^{-1}V_x^+)(k)\},\psi\rangle. \tag{61}$$

Now, for $\phi \in \mathscr{S}(\mathbb{R})$,

$$\langle \mathscr{F}\mathscr{R}\mathscr{F}^{-1}V_x^+, \phi\rangle = \langle \mathscr{R}\mathscr{F}^{-1}V_x^+, \mathscr{F}\phi\rangle = \langle \mathscr{F}^{-1}V_x^+, \mathscr{R}\mathscr{F}\phi\rangle = \langle \mathscr{F}^{-1}V_x^+, \mathscr{F}\mathscr{R}\phi\rangle = \langle \mathscr{F}\mathscr{F}^{-1}V_x^+, \mathscr{R}\phi\rangle \tag{62}$$

$$= \langle \mathscr{R}V_x^+, \phi\rangle. \tag{63}$$

Therefore, we have for the first term of (61) that

$$\mathscr{F}\mathscr{R}\mathscr{F}^{-1}V_x^+ = \mathscr{R}V_x^+ \quad \text{in } \mathscr{S}'(\mathbb{R}). \tag{64}$$

Now, observe that $R^+ = \mathscr{F}^{-1}(K^+)$ and $\mathscr{F}^{-1}\delta_{-x} = k \mapsto e^{2\imath kx}$. We take for $\psi \in \mathscr{S}(\mathbb{R})$,

$$\langle \mathscr{F}\{k \mapsto R^+(k)e^{2\imath kx}\}, \psi\rangle = \langle \mathscr{F}\{\mathscr{F}^{-1}K^+\mathscr{F}^{-1}\delta_{-x}\}, \psi\rangle \tag{65}$$

$$\langle \mathscr{F}\mathscr{F}^{-1}(K^+ * \delta_{-x}), \psi\rangle = \langle K^+(x+\cdot), \psi\rangle. \tag{66}$$

Now, since $V_x^+ = V_x^{+,\mathscr{E}'} + V_x^{+,L^p}$, the convolution of V_x^+ and the shifted scattering data $K^+(x+\cdot) \in L^2(\mathbb{R})$ make sense and we obtain

$$k \mapsto R^+(k)e^{2\imath kx}(\mathscr{F}^{-1}V_x^+)(k) = \mathscr{F}^{-1}(K^+(x+\cdot))\mathscr{F}^{-1}(V_x^+) = \mathscr{F}^{-1}(K^+(x+\cdot) * V_x^+).$$

Combining the above, we get

$$\mathscr{F}\{k \mapsto R^+(k)e^{2\imath kx}(\mathscr{F}^{-1}V_x^+)(k)\} = K^+(x+\cdot) * V_x^+ = K^+(x+\cdot) \star \mathscr{R}V_x^+, \tag{67}$$

where for tempered distributions, we define $f \star g = f * \mathscr{R}g$. Now, putting all of our findings together, we get that

$$\mathscr{R}V_x^+ + K^+(x+\cdot) + K^+(x+\cdot) \star \mathscr{R}V_x^+ = \mathscr{F}\{k \mapsto T(k)v^-(k,x) - 1\} \tag{68}$$

or, equivalently,

$$B_x^+ + K^+(x+\cdot) + K^+(x+\cdot) \star B_x^+ = \mathscr{F}\{k \mapsto T(k)v^-(k,x) - 1\}. \tag{69}$$

□

IV. INVERSION

In this section, we use some of our theoretical findings to propose a practical method for solving the inverse medium problem. The right-hand side of the GLM equation [Eq. (19)] depends on unknown quantities, assuming that we only consider one-sided reflection measurements. To get around this obstacle, we will need to consider two sided data, namely, R^+, R^-, T, for all frequencies. Using them, we can form a coupled system for the GLM kernels B^+, B^- of the right and left going Jost solutions, respectively. After obtaining the Jost solutions, then either using Eqs. (5) and (6) or the equation error method,[15] we can obtain the coefficient of the Helmholtz operator.

First, observe that we can write $T(k) = 1 + \tau(k)$, $k \in \mathbb{R}\backslash\{0\}$, with $\tau(k) = \frac{k}{2\imath}\int_{\mathbb{R}} n(x)y(k,x)e^{\imath kx}dx$. τ is also well defined. Now, for fixed $x \in \mathbb{R}$, we have that

$$u^-(k,x)e^{\imath kx} = 1 + \mathscr{F}^{-1}V_x^-(k), \quad k \in \mathbb{R}. \tag{70}$$

Considering this, we can compute the right-hand side of the GLM equation as

$$\text{rhs} = \mathscr{F}\{(1+\tau)(1+\mathscr{F}^{-1}V_x^-) - 1\}. \tag{71}$$

The argument of the Fourier transform is $(1+\tau)(1+\mathscr{F}^{-1}V_x^-) - 1 = 1 + \tau + \mathscr{F}^{-1}V_x^- + \tau\mathscr{F}^{-1}V_x^- - 1 = \tau + \mathscr{F}^{-1}V_x^- + \tau\mathscr{F}^{-1}V_x^-$. Therefore,

$$\mathscr{F}\{\tau + \tau\mathscr{F}^{-1}V_x^- + \mathscr{F}^{-1}V_x^-\} = L + L * V_x^- + V_x^- = L + L \star B_x^- + \mathscr{R}B_x^- \tag{72}$$

with $B_x^- = \mathscr{R}V_x^-$ and $L = \mathscr{F}\tau$. We can set up an auxiliary scattering problem of the following form. We consider an incident wave traveling from $-\infty$ to $+\infty$ of the form $u^i(k,x) = e^{\imath kx}, x \in \mathbb{R}$. The scattering identity of this problem is

$$u^-(-k,x) + R^-(k)u^-(k,x) = T(k)u^+(k,x). \tag{73}$$

As mentioned before, we compute

$$u^-(-k,x)e^{-\imath kx} + R^-(k)u^-(k,x)e^{-\imath kx}e^{\imath kx}e^{-\imath kx} = T(k)u^+(k,x)e^{-\imath kx} \iff \tag{74}$$

$$v^-(-k,x) + R^-(k)v^-(k,x)e^{-2\imath kx} = T(k)v^+(k,x) \iff \tag{75}$$

$$\mathscr{F}^{-1}V_x^-(-k) + R^-(k)e^{-2\imath kx} + R^-(k)e^{-2\imath kx}\mathscr{F}^{-1}V_x^-(k) = T(k)v^+(k,x) - 1. \tag{76}$$

Now, we compute the Fourier transform to obtain

$$\mathscr{R}V_x^- + K^-(-x+\cdot) + K^-(-x+\cdot)\star V_x^- = \mathscr{F}\{T(k)v^+(k,x) - 1\}. \tag{77}$$

Therefore, we get the GLM equation,

$$B_x^- + K^-(-x+\cdot) + K^-(-x+\cdot)\star B_x^- = \mathscr{F}\{k \mapsto T(k)v^+(k,x) - 1\} \tag{78}$$

with $\mathscr{R}V_x^- = B_x^-$. Similarly as before, we obtain

$$\mathscr{F}\{k \mapsto T(k)v^+(k,x) - 1\} = L + L\star B_x^- + \mathscr{R}B_x^-. \tag{79}$$

Combining our findings, we are left with a system of two equations and two unknowns, B_x^-, B_x^+,

$$B_x^+ + K^+(x+\cdot) + K^+(x+\cdot)\star B_x^+ = L + L\star B_x^- + \mathscr{R}B_x^-, \tag{80}$$

$$B_x^- + K^-(-x+\cdot) + K^-(-x+\cdot)\star B_x^- = L + L\star B_x^+ + \mathscr{R}B_x^+. \tag{81}$$

Assuming the knowledge of discrete K^+, K^-, and L, we can solve for the GLM kernels B_x^+ and B_x^-. The solution of the system can be found using a conventional least-squares solver; see Ref. 8. Once we know B_x^+, we also know u^+ [relation (17)]. Thus, we can solve for n in (5). Alternatively, we can use the equation error method, see Ref. 15, to obtain n since u^+ also obeys the Helmholtz equation.

Remark IV.1. The coupled system of (80) and (81) yields the exact solutions $B_x^\pm$, assuming $K^\pm$ and L. Although one could consider only one sided reflection data and solve only (80) [or (81)] by approximating its right-hand side with zero, for example, for a discussion and comparison of the use of one sided vs two sided data for the computational solution of an inverse problem for estimating a diffusion potential from boundary measurements, we refer to Ref. 16.

V. DISCUSSION AND CONCLUSIONS

We have revisited the classical 1D Helmholtz scattering problem, and we have derived a generalized Gelfand–Levitan–Marchenko equation in the space of tempered distributions. In particular, we showed that the Jost solution of the Helmholtz equation minus a plane wave grows in a controlled way as the wave number grows. This allows us to consider a distributional framework where we derived a generalized version of the Gelfand–Levitan–Marchenko equation. We finally discussed a way to solve the inverse medium-problem using two-sided data, whereas one-sided should theoretically suffice. In the future, we will seek an explanation on why this is the case.

Recently, GLM-like methods have received renewed attention, especially in the area of seismic imaging, although the most significant limitation of GLM-like approaches for the Helmholtz equation is the difficulty in extending them in higher dimensional media. Without assuming symmetry to the medium (e.g., laterally stratified), we cannot transform the Helmholtz equation to the Schrödinger equation. With our new point of view, we believe that we have made a first step toward a possible extension of this particular GLM method to 2D and 3D Helmholtz scattering problems using the least amount of *a priori* assumptions.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Andreas Tataris: Conceptualization (lead); Formal analysis (lead); Investigation (lead); Methodology (lead); Writing – original draft (lead); Writing – review & editing (equal). **Tristan van Leeuwen**: Funding acquisition (lead); Supervision (lead); Visualization (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: CALCULATIONS WITH DISTRIBUTIONS AND SCHWARTZ FUNCTIONS

In this part, we recall certain properties of distributions that we used above,

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega t}d\omega = \delta(t) \iff \frac{1}{2\pi}\int_{\mathbb{R}} e^{2ikt}2dk = \frac{1}{\pi}\int_{\mathbb{R}} e^{2ikt}dk = \delta(t). \tag{A1}$$

According to our notation,

$$\mathscr{F}(1)(t) = \delta(t). \tag{A2}$$

Similarly,

$$\frac{1}{\pi}\int_{\mathbb{R}} e^{2ikt}e^{2ikg}dk = \frac{1}{\pi}\int_{\mathbb{R}} e^{2ik(t+g)}dk = \delta(t+g). \tag{A3}$$

Therefore,

$$\mathscr{F}(e^{2ikg}) = \delta(t+g). \tag{A4}$$

Similarly,

$$\frac{1}{\pi}\int_{\mathbb{R}} e^{2ikt}e^{ikg}dk = \frac{1}{\pi}\int_{\mathbb{R}} e^{2ik(t+\frac{g}{2})}dk = \frac{1}{\pi}\int_{\mathbb{R}} e^{ik(2t+g)}dk = \frac{2}{2\pi}\int_{\mathbb{R}} e^{ik(2t+g)}dk = 2\delta(2t+g). \tag{A5}$$

Therefore,

$$\mathscr{F}(e^{ikg}) = 2\delta(2t+g) = \delta\left(t+\frac{g}{2}\right). \tag{A6}$$

Another important property is

$$(\delta_a * f)(t) = f(t-a) \tag{A7}$$

for $f \in L^2(\mathbb{R})$ with $\delta_a(t) = \delta(t-a)$. In addition, previously, using the $\mathscr{R}$ (reflection) operator, we exchanged between the convolution and the correlation of distributions. Assuming a function $b \in \mathscr{S}(\mathbb{R})$, we get

$$(K(x+\cdot) * b)(t) = \int_{\mathbb{R}} K(x+z)b(t-z)dz. \tag{A8}$$

Now, we set $\zeta = -t+z$ and obtain $z = t+\zeta$. We get

$$(K(x+\cdot) * b)(t) = \int_{\mathbb{R}} K(x+t+\zeta)b(-\zeta)d\zeta = (K(x+\cdot) \star \mathscr{R}b)(t). \tag{A9}$$

For tempered distributions, we define

$$\langle \mathscr{R}f, \phi\rangle = \langle f, \mathscr{R}\phi\rangle.$$

See Ref. 17 (p. 334). In addition, $\mathscr{R}^2 = I$ since

$$\langle f, \phi\rangle = \langle f, \mathscr{R}^2\phi\rangle = \langle \mathscr{R}f, \mathscr{R}\phi\rangle = \langle \mathscr{R}^2 f, \phi\rangle. \tag{A10}$$

Finally, $\mathscr{F}$ and $\mathscr{R}$ commute since

$$\mathscr{R}\mathscr{F}\phi(t) = \frac{1}{\pi}\int_{\mathbb{R}}\phi(k)e^{-ikt}dk = \frac{1}{\pi}\int_{\mathbb{R}}\phi(-k)e^{ikt}dk = \mathscr{F}(\mathscr{R}\phi)(t), \tag{A11}$$

and this gives

$$\langle \mathscr{F}\mathscr{R}f,\phi\rangle = \langle \mathscr{R}f,\mathscr{F}\phi\rangle = \langle f,\mathscr{R}\mathscr{F}\phi\rangle = \langle f,\mathscr{F}\mathscr{R}\phi\rangle = \langle \mathscr{F}f,\mathscr{R}\phi\rangle = \langle \mathscr{R}\mathscr{F}f,\phi\rangle. \tag{A12}$$

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