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The tadpole conjecture in asymptotic limits

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Abstract: The tadpole conjecture suggests that the complete stabilization of complex structure deformations in Type IIB and F-theory flux compactifications is severely obstructed by the tadpole bound on the fluxes. More precisely, it states that the stabilization of a large number of moduli requires a flux background with a tadpole that scales linearly in the number of stabilized fields. Restricting to the asymptotic regions of the complex structure moduli space, we give the first conceptual argument that explains this linear scaling setting and clarifies why it sets in only for a large number of stabilized moduli. Our approach relies on the use of asymptotic Hodge theory. In particular, we use the fact that in each asymptotic regime an orthogonal sl(2)-block structure emerges that allows us to group fluxes into sl(2)-representations and decouple complex structure directions. We show that the number of stabilized moduli scales with the number of sl(2)-representations supported by fluxes, and that each representation fixes a single modulus. Furthermore, we find that for Calabi-Yau four-folds all but one representation can be identified with representations occurring on two-folds. This allows us to discuss moduli stabilization explicitly and establish the relevant scaling constraints for the tadpole.

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Contents

1 Introduction

String theory compactifications typically give rise to a large number of massless scalar fields in the lower-dimensional effective theory. These (moduli) fields correspond to deformations of the compactification background and are ruled-out by experiments. A very well-studied setting is that of Calabi-Yau orientifold compactifications of type II string theories, in particular, type IIB or its non-perturbative completion F-theory. The reason why this

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corner of the landscape is special is because one can turn on three-form fluxes (or four-form fluxes in the F/M-theory realisation) such that their back-reaction still allows for a Calabi-Yau geometry, as long as they are self-dual [1–3]. Furthermore, these fluxes generically lift the moduli corresponding to complex structure deformations of the Calabi-Yau manifold [2, 4], and have therefore been used at length in string phenomenology.

The in general complicated structure of the moduli space of complex structure deformations suggests that turning on flux quanta in a few cycles is enough to generate a potential capable of stabilizing a large number of moduli. However, this reasoning has been challenged by the "Tadpole Conjecture" [5, 6], which states that the charge Q induced by the fluxes needed to stabilize a large number, n, of moduli grows linearly with n. This growth yields an obstruction on complete moduli stabilization if its slope is larger than the slope for the linear growth observed for the tadpole. The refined version of the tadpole conjecture makes precisely this claim. If this refined version of the tadpole conjecture is true, then the only phenomenologically-relevant Calabi-Yau manifolds are those with a small number of complex structure moduli, since large numbers are not amenable to flux-moduli stabilisation. While the tadpole conjecture suggests a mathematically precise statement, its proof is extremely challenging and currently wide open. Even in specific situations, such as e.g. K3$\times$K3 compactifications, any direct proof soon faces technical challenges.

In this work we initiate the first systematic and general approach to establish the tadpole conjecture. We collect strong evidence that this conjecture is satisfied for all (strict) asymptotic limits in moduli space. Moreover, we give the first conceptual argument that explains the linear scaling of the tadpole with the number of stabilized moduli and the requirement that one has to consider a large number of moduli. Our approach uses the powerful machinery of asymptotic Hodge structures [7, 8] and does neither rely on explicit examples nor on a specific choice of asymptotic limit. Rather, we exploit the fact that asymptotic Hodge theory provides us with an explicit expression for the Hodge-star operator acting on four-forms in terms of n moduli that are taken to be in the asymptotic region. Requiring the self-duality condition for the fluxes then becomes an explicit, polynomial equation for the moduli that has been explored before in [9, 10]. The degrees of these polynomials are fixed by the weights of the flux components under n commuting sl(2)-algebras characterizing the asymptotic region. We show that there is a one-to-one correspondence between the number of moduli fixed and the number of sl(2)-representations supported by flux. Each representation yields an independent positive-definite term in the tadpole cancellation condition such that a scaling of the flux tadpole with the number of moduli appears to be immediate if the individual terms are lower-bounded.

The crucial task of this work is to establish this scaling of the flux charge and argue for a lower bound on individual terms arising from fluxes in an sl(2)-representation, thus showing that the refined Tadpole Conjecture applies. This requires us to understand in detail which sl(2)-representations can arise in an asymptotic Calabi-Yau four-fold compactification. We find that the richest structure hereby comes from the presence of the single (4,0)-form, which accounts for sl(2)-representations with weights reaching from -4 to 4. The remaining other sl(2)-representations have a maximal weight 2 and match with sl(2)-representations found for Calabi-Yau two-folds (K3 surfaces). Since only the number of these K3-type representations is increasing with the number of moduli, we are lead to study the much simpler and more constrained problem of examining the tadpole contribution of such shorter representations. We find that fluxes in these representations have either all n sl(2)-weights greater or equal to zero, or smaller or equal to zero. The self-duality condition furthermore forbids purely zero-weight fluxes and relates fluxes with positive and negative weights. Any asymptotic stabilization thus has to use these K3-type flux pairs which individually fix a single modulus and contribute to the tadpole with positive powers of fractions of moduli that become large in the asymptotic regime. Remarkably, this tadpole can be lower-bounded by a moduli-independent sum of positive terms depending on the number sl(2)-fluxes times a large parameter ensuring that the moduli are evaluated in the asymptotic region.[1] We complete our argument by providing evidence that the flux terms do not scale inversely with the number of moduli and most of them are bounded from below by 1/4 in all examples we worked out.

The paper is organized as follows: in section 2 we introduce the basic features of F-theory flux compactifications and the Tadpole Conjecture. In section 3 we discuss asymptotic Hodge theory and show that there are only very few weights that appear in a generic four-fold. In section 4 we show how moduli stabilization by self-dual fluxes works in general, while in section 5 we work out the explicit solution for the representations appearing in a four-fold. In section 6 we analyze the tadpole, and show how the tadpole conjecture is verified. We conclude in section 7. In appendix A we provide more details about asymptotic Hodge structures, and in appendix B we give explicit bases for the sl(2)-representations and the action of the Hodge star on them. In appendix C we illustrate the concepts introduced with a two-moduli example, while in appendix D we work out examples that illustrate flux quantization in the sl(2) basis with a large number of moduli.

2 F-theory fluxes and the tadpole conjecture

In this section we give a brief review of flux compactifications in F-theory. We focus on the aspects needed for our study of the tadpole conjecture [5, 6], and refer for a more comprehensive discussion to [11].

2.1 F-theory flux compactifications

We are interested in compactifications of F-theory on Calabi-Yau four-folds in the presence of four-form fluxes. Such manifolds have to admit a two-torus fibration, which from the Type IIB perspective encodes the variation of the dilaton-axion field. To study the resulting four-dimensional effective action we first consider the dual M-theory setting by compactifying eleven-dimensional supergravity on a resolved four-fold Y_4 to obtain a three-dimensional effective theory. To connect with F-theory one then takes the limit of shrinking

[1] This parameter, called γ, does not need to be extremely large. It was shown that already for $\gamma \gtrsim 4$ moduli stabilization using asymptotic Hodge theory gives a very good approximation of the situation, where all corrections are taken into account [10].

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the torus-fiber [11, 12].[2] Investigating the impact of corrections in the resulting F-theory effective actions has been the focus of [13–20].

In the three-dimensional theory and in the absence of fluxes the deformations of the four-fold give rise to massless scalar fields, which correspond to the $h^{3,1}$ complex-structure moduli and $h^{1,1}$ Kähler moduli of Y_4.[3] However, when considering a non-trivial four-form flux $G_4 \in H^4(Y_4, \mathbb{Z}/2)$, a potential is induced which can be brought into the form [11, 21]

$$V = \frac{1}{\mathcal{V}^3}\left(\|G_4\|^2 - \langle G_4, G_4\rangle\right). \tag{2.1}$$

Here $\mathcal{V}$ denotes the volume of the four-fold Y_4 which depends on the Kähler moduli, and we have defined the norm and inner product of a real four-form $v \in H^4(Y_4, \mathbb{R})$ as

$$\|v\|^2 = \int_{Y_4} v \wedge \star v\,, \qquad\qquad \langle v, v\rangle = \int_{Y_4} v \wedge v\,, \tag{2.2}$$

where $\star$ denotes the Hodge-star operator of Y_4. The dependence of the potential (2.1) on the complex-structure moduli is encoded in the Hodge-star operator in the first term, while the second term is an on-shell contribution obtained using the Bianchi identity for G_4.

We are interested in Minkowski minima of the potential (2.1). These are obtained when G_4 is self-dual and primitive [1, 2], which reads in formulas

$$G_4 = \star G_4\,, \qquad\qquad J \wedge G_4 = 0\,. \tag{2.3}$$

Let us emphasize that in the following we consider $G_4 = \star G_4$ as a condition in cohomology and not as a local condition in target space. We furthermore note that the primitive four-form cohomology decomposes as

$$H^4_{\text{prim}} = H^{4,0} \oplus H^{3,1} \oplus H^{2,2}_{\text{prim}} \oplus H^{1,3} \oplus H^{0,4}\,, \tag{2.4}$$

and that the Hodge-star operator $\star$ acts on the $H^{p,q}$ cohomologies by multiplication with $(-1)^{(p-q)/2}$. The self-duality condition in (2.3) then implies that the $(3,1)$-components of the four-form flux have to vanish, which can be written as $h^{3,1}$ equations for the $h^{3,1}$ complex structure moduli. Generically these equations fix the moduli at some value — but the tadpole conjecture, to which we turn now, challenges this naïve expectation.

2.2 Tadpole conjecture

The four-form flux G_4 induces a D3-brane charge Q (or M2-brane charge in the M-theory dual picture) that has to be cancelled globally. Defining

$$Q = \frac{1}{2}\langle G_4, G_4\rangle = \frac{1}{2}\int_{Y_4} G_4 \wedge G_4\,, \tag{2.5}$$

[2] Note that we consider the general case in which the F-theory setting can admit non-trivial 7-brane configurations with non-Abelian gauge groups. In this case Y_4 is the smooth Calabi-Yau four-fold obtained by resolving the gauge theory singularities. The volumes of the resolution cycles are parameterized by Kähler moduli and are shrunk in the F-theory limit.

[3] All the $h^{3,1}$ complex structure moduli lift directly to complex scalars in the four-dimensional F-theory lift, whereas only $h^{1,1} - 1$ Kähler moduli, corresponding to the $(1,1)$-cohomology on the base, give rise to Kähler moduli in four dimensions.

this leads to the tadpole cancelation condition and inequality

$$Q + N_{\rm D3} = \frac{\chi(Y_4)}{24} \qquad \Rightarrow \qquad Q \leq \frac{\chi(Y_4)}{24}\,, \tag{2.6}$$

where $\chi(Y_4)$ is the Euler number of Y_4 and $N_{\rm D3} \geq 0$ denotes the number of space-time filling D3-(or M2-)branes. For solutions to the equations of motion with self-dual G_4 flux the charge Q is always positive. The fact that Q is also bounded from above may suggest that the number of self-dual flux configurations on a given Calabi-Yau manifold is finite. However, as we will discuss in detail below, the Hodge star can degenerate near the boundaries of the moduli space and vacua could accumulate in such asymptotic regimes [22]. Remarkably, it turns out that one can prove the absence of such accumulation points and a general finiteness theorem for self-dual flux vacua using asymptotic Hodge theory and tame geometry [23].[4] Although the number of flux configurations is finite, it is expected to be extremely large. For instance, for the Calabi-Yau four-fold with the largest-known Euler number the number of vacua has been estimated to be of order 10^{272000} [27]. We note, though, that this number was obtained by counting lattice sites and does not take into account any of the intricate structure of the complex-structure moduli space and of the Hodge-star operator. The tadpole conjecture scrutinizes these estimates even further as it challenges the idea that full moduli stabilization can be achieved for manifolds with large $h^{3,1}$.

The tadpole conjecture [5] postulates that for a large number of moduli there are always remaining flat directions. To be more precise, the conjecture states that when stabilizing a large number of complex moduli $n_{\rm stab} \gg 1$, the charge induced by the flux grows linearly with the moduli in the form

$$Q > \alpha\, n_{\rm stab}\,. \tag{2.7}$$

The refined version of the conjecture then gives a precise lower bound for the slope:

$$\alpha > \frac{1}{3}\,. \tag{2.8}$$

If the tadpole conjecture is true, one cannot stabilize a large number of complex structure moduli within the tadpole bound (2.6), since for a large $h^{3,1}$ the Euler number behaves as $\chi(Y_4) \sim h^{3,1}/4$ and one would have

$$\frac{1}{3} n_{\rm stab} < Q \leq \frac{\chi(Y_4)}{24} \sim \frac{1}{4} h^{3,1} \qquad \Rightarrow \qquad n_{\rm stab} < \frac{3}{4} h^{3,1}\,. \tag{2.9}$$

We note that the tadpole conjecture refers to stabilization of the real and imaginary part of the complex moduli. If the conjecture is true there is always some left-over moduli space, and it is a very interesting question (beyond the scope of this paper) to understand its structure, in particular, if it is compact or not.

[4] This result is a generalization of a famous theorem [24] about the finiteness of Hodge loci, and captures finiteness along one-dimensional limits first shown in [25, 26].

3 Aspects of asymptotic Hodge theory

In this work we are interested in the behavior of the flux-induced charge Q when stabilizing moduli near the boundary of complex-structure moduli space. A suitable framework for discussing this question is asymptotic Hodge theory, which we briefly review in the following [9]. Let us stress that a boundary in complex-structure moduli space corresponds not only to the familiar large complex-structure limit, but it includes also the conifold point and more general degenerations [28]. Our analysis is valid for all of such boundaries. In appendix C we present a two-moduli example that illustrates the concepts we introduce in this section, where the asymptotic region we explore is that close to the conifold point and the weak coupling limit.

3.1 Strict asymptotic regimes and sl(2)-decomposition

The complex-structure moduli space of Calabi-Yau four-folds is parametrized by $h^{3,1}$ complex scalar fields. Let us consider an asymptotic region in this moduli space and separate these fields into two groups in the following way

$$\{t^i, \zeta^\alpha\}\,, \qquad\qquad i = 1, \ldots, n\,, \qquad \alpha = n+1, \ldots, h^{3,1}\,, \tag{3.1}$$

where $0 < n \leq h^{3,1}$. The ζ^α are called *spectator* fields and will not play any role in our subsequent discussion, so we will mostly ignore them. The scalars t^i correspond to the coordinates (on the covering space) of the moduli space that parametrize how far away we are from one of its boundaries. The real and imaginary parts of t^i will loosely be called axions and saxions, respectively,[5]

$$t^i = \phi^i + i\, s^i\,, \qquad s^i > 0\,, \qquad\qquad i = 1, \ldots, n\,, \tag{3.2}$$

and the boundary of the moduli space corresponds to the limit $s^i \to \infty$.

Asymptotic regimes. In the near-boundary region we can consider the following two regimes:

1. The *asymptotic regime* is characterized by the following condition for the saxions s^i with $i = 1, \ldots, n$

$$s^i \gg 1\,, \tag{3.3}$$

which corresponds to dropping corrections $\mathcal{O}(e^{2\pi i t^i})$ in, for instance, the Hodge-star operator.

2. In the *strict asymptotic regime* the s^i are ordered according to the hierarchy

$$\frac{s^1}{s^2} > \gamma\,, \quad \frac{s^2}{s^3} > \gamma\,, \quad \ldots\,, \quad \frac{s^{n-1}}{s^n} > \gamma\,, \quad s^n > \gamma\,, \qquad |\phi^i| < \delta\,, \tag{3.4}$$

[5] Note that the real parts are not necessarily axions, i.e. they might not have a continuous shift symmetries even in the leading moduli space metric, such as the intersection of two conifold loci [28].

where $\gamma \gg 1$ and $\delta > 0$. Dropping polynomial corrections of order $\mathcal{O}(\gamma^{-1})$ corresponds to the sl(2)-approximation [8], which is the setting where our work will take place. Note that these inequalities specify a certain hierarchy of field values and one cannot simply permute indices. In general, using the same coordinates with a different ordering, and hence a different hierarchy, implies that one probes another strict asymptotic regime with different properties.

Let us emphasize that the spectator fields ζ^α are not send to the boundary, however, quantities such as the Hodge-star operator can still depend on them.

Sl(2)-decomposition. We now turn to the sl(2)-decomposition of the fourth cohomology of Y_4. A pedagogical introduction to this subject can be found for instance in section 3.1 of [10], and more detailed discussions can be found in [7, 8, 29, 30]. We summarize the main aspects that we will use in the rest of the paper as follows:

- To each boundary of complex-structure moduli space one can associate monodromy transformations acting on the fourth cohomology of the Calabi-Yau four-fold Y_4. If n saxions s^i are sent to the boundary, this action can be realized by n commuting matrices T_i acting on $H^4_{\text{prim}}(Y_4, \mathbb{R})$. These matrices can always be made unipotent, that is $(T_i - \mathbb{1})^{m+1} = 0$ for some $m \geq 0$, and to each T_i we can associate a so-called log-monodromy matrix $N_i = \log T_i$. Since the T_i are unipotent it follows that the N_i are nilpotent (when acting on the fourth cohomology), and we note that the N_i commute among each other.

- Each nilpotent N_i can be completed into an sl(2)-triple, in which it acts as lowering operator. However, the choice of weight operator is not unique. Although the N_i commute with each other, in general the other generators of the sl(2)-triples do not. It is a non-trivial result of the sl(2)-orbit theorem [8] that for each N_i one can construct n sets of commuting sl(2)-triples as[6]

$$\text{commuting sl(2)-triples:} \qquad \{N_i^-, N_i^+, N_i^0\}\,, \tag{3.5}$$

  with the standard commutation relations

$$\left[N_i^0, N_j^\pm\right] = \pm 2\, N_i^\pm\, \delta_{ij}\,, \qquad \left[N_i^+, N_j^-\right] = N_i^0\, \delta_{ij}\,. \tag{3.6}$$

- The triples shown in (3.5) can be used to split the vector space $H^4_{\text{prim}}(Y_4, \mathbb{R})$ in the following way (see appendix A for technical details)

$$H^4_{\text{prim}}(Y_4, \mathbb{R}) = \bigoplus_{\ell \in \mathcal{E}} V_\ell\,, \qquad \ell = (\ell_1, \ldots, \ell_n)\,, \tag{3.7}$$

  where V_ℓ are the eigenspaces of N_i^0 characterized by $N_i^0 v_\ell = (\ell_i - \ell_{i-1}) v_\ell$, which can be rewritten as

$$(N_1^0 + \ldots + N_i^0)\, v_\ell = \ell_i\, v_\ell\,, \qquad v_\ell \in V_\ell\,. \tag{3.8}$$

  The indices ℓ_i are integers that for Calabi-Yau four-folds are in the range $\ell_i \in \{-4, \ldots, +4\}$, and $\mathcal{E}$ denotes the set of all possible vectors ℓ.

[6] Here the lowering operator N_i^- is closely related to N_i, but only $N_1^- = N_1$ holds generally.

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- It is instructive to consider the periods of the (up to rescaling) unique, holomorphic $(4,0)$-form Ω of Y_4. Its period vector Π admits an expansion in the strict asymptotic regime (3.4) as

$$\Pi_{\mathrm{sl}(2)} = e^{t^i N_i^-}\,\tilde{a}_0\,, \tag{3.9}$$

where we did not display subleading polynomial terms of order $1/\gamma$ and exponentially suppressed corrections. We note that for derivatives of the periods both kinds of corrections can be essential, as it happens for example for the Kähler metric near a conifold point (see appendix C). These essential corrections are taken into account when computing e.g. the asymptotic form of the Hodge star operator. The leading polynomial term $\tilde{a}_0$ has a precise location in the sl(2)-decomposition

$$\mathrm{Re}\,\tilde{a}_0\,,\ \mathrm{Im}\,\tilde{a}_0 \in V_d\,, \tag{3.10}$$

with d a vector of the form $d = (d_1, \ldots, d_n)$. The d_i are the largest integers such that the condition $(N_1^- + \ldots + N_k^-)^{d_k}\tilde{a}_0 \neq 0$ is satisfied, i.e. $\tilde{a}_0$ is a highest-weight state. We also note that $0 \leq d_1 \leq \ldots \leq d_n \leq 4$.

- We can make the above sl(2)-decomposition (3.7) more refined using sl(2) highest-weight states and their descendants. Let us introduce the subspaces

$$P_\ell = V_\ell \cap \ker\big[(N_1^-)^{\ell_1-\ell_0+1}\big] \cap \ldots \cap \ker\big[(N_n^-)^{\ell_n-\ell_{n-1}+1}\big]\,, \tag{3.11}$$

with $\ell_0 \equiv 0$. The decomposition (3.7) can then be rewritten as a weight-space decomposition in the following way

$$H^4_{\mathrm{prim}}(Y_4,\mathbb{R}) = \bigoplus_{\ell\in\mathcal{E}_{\mathrm{hw}}} \bigoplus_{k_1=0}^{\ell_1-\ell_0} \cdots \bigoplus_{k_n=0}^{\ell_n-\ell_{n-1}} (N_1^-)^{k_1}\cdots(N_n^-)^{k_n} P_\ell\,, \tag{3.12}$$

where we defined the index set for highest-weight states as

$$\mathcal{E}_{\mathrm{hw}} = \{\ell = (\ell_1,\ldots,\ell_n) \mid 0 \leq \ell_1 \leq \ldots \leq \ell_n \leq 4\}\,. \tag{3.13}$$

To summarize, the elements of $H^4_{\mathrm{prim}}(Y_4,\mathbb{R})$ are arranged in irreducible representation of the boundary sl(2)-algebras. This means that all their information is encoded in the highest-weight subspaces P_ℓ (and in the lowering matrices), so that by successively applying N_i^- the full primitive four-form cohomology can be obtained.

We already note now that by introducing a boundary Hodge decomposition, we will be able to break down the index set (3.13) into smaller components (cf. equation (3.21)), thereby further reducing the set of allowed sl(2)-eigenspaces.

3.2 Boundary Hodge decomposition

In addition to the sl(2)-decomposition introduced above, there is another algebraic structure associated to the boundary: the boundary Hodge decomposition given by

$$H^4_{\mathrm{prim}}(Y_4,\mathbb{C}) = H^{4,0}_\infty \oplus H^{3,1}_\infty \oplus H^{2,2}_\infty \oplus H^{1,3}_\infty \oplus H^{0,4}_\infty\,, \tag{3.14}$$

where $\overline{H_\infty^{p,q}} = H_\infty^{q,p}$. This decomposition is independent of the limiting coordinates t^i. From a more physical perspective one can interpret (3.14) as a charge decomposition, with the index p of $H_\infty^{p,q}$ as charge, and its counterpart q fixed by $p+q=4$.[7] We now discuss the following aspects:

- We first introduce a Weil operator $\star_\infty$ that acts on an element in one of the subspaces $w_{p,q} \in H_\infty^{p,q}$ as
$$\star_\infty \, w_{p,q} = i^{p-q} w_{p,q} \,. \tag{3.15}$$
The Weil operator thus squares to the identity, that is $\star_\infty^2 = \mathbb{1}$, and it is independent of the moduli t^i. It also maps the V_ℓ appearing in the sl(2)-decomposition (3.7) as
$$\star_\infty : V_\ell \to V_{-\ell} \,. \tag{3.16}$$
Here we have used that dim $V_{+\ell}$ = dim $V_{-\ell}$, which can be shown using the action of the sl(2)-triples.

- It is now instructive to split the highest-weight subspaces P_ℓ introduced in (3.11) as
$$P_\ell = \bigoplus_{p+q=\ell_n+4} P_\ell^{p,q} \,, \tag{3.17}$$
based on the Deligne splittings associated to the strict asymptotic regime (see appendix A for more details). For our purposes here it is sufficient to note that this splitting is correlated with the boundary Hodge decomposition given in (3.14), as becomes apparent by noting that
$$e^{iN^-_{(n)}} P_\ell^{p,q} \subseteq H_\infty^{p,4-p} \,, \qquad p \geq q \,, \qquad N^-_{(n)} \equiv \sum_{i=1}^n N_i^- \,. \tag{3.18}$$

- Combining (3.18) with (3.16), expanding the exponentials and identifying sl(2)-eigenspaces, we can then show that
$$\star_\infty \left(\prod_{i=1}^n \frac{(iN_i^-)^{k_i}}{k_i!} v_\ell^{p,q} \right) = i^{2p-4} \prod_{i=1}^n \frac{(iN_i^-)^{\ell_i-\ell_{i-1}-k_i}}{(\ell_i-\ell_{i-1}-k_i)!} v_\ell^{p,q} \,, \tag{3.19}$$
for $v_\ell^{p,q} \in P_\ell^{p,q}$ (with $p \geq q$), and where k_i are any set of integers that give a non-vanishing contribution on the left-hand side.[8] Let us note the similarity of this relation with the Kähler-form identity $\star J^k/k! = J^{d-k}/(d-k)!$ on a d-dimensional Kähler manifold. This identity will prove to be very useful in the study of moduli stabilization, where we need to identify self-dual fluxes under the Hodge star.

[7] This operator-based approach has been explored in more detail in [26, 31, 32]: for (3.14) one can introduce a corresponding charge operator with $H_\infty^{p,q}$ as eigenspaces; for the sl(2)-triples one can rotate to a complex basis such that the generators commute with the charge operator. This thereby allows for a simultaneous decomposition into eigenstates, which — although we do not use these operators explicitly in this work — is described in this subsection.

[8] This identity has also appeared before in [33] for the leading term a_0 of the periods, where it played an important role in determining charge-to-mass ratios of BPS states.

- It is also helpful to correlate the highest-weights given in (3.13) with the charge decomposition described by (3.17). This refined splitting of indices is given by

$$\mathcal{E}_{\text{hw,charge}} = \{(p,q,\ell)\,|\, 0 \le p,q \le 4\,,\ \ell \in \mathcal{E}_{\text{prim}}\,,\ p+q = \ell_n + 4\}\,. \tag{3.20}$$

For $p = 4$ or $q = 4$ there is one state — $\tilde{a}_0$ obtained from (3.9) or its conjugate — spanning the corresponding highest-weight subspace, which can be attributed to the Calabi-Yau condition $h^{4,0} = h^{0,4} = 1$. The remaining highest-weight states have $1 \le p,q \le 3$ and weights similarly bounded as $0 \le \ell_1 \le \ldots \le \ell_n \le 2$. A more detailed explanation of this decomposition is given in appendix A. Here we state its result for (3.20) as

$$\mathcal{E}_{\text{hw,charge}} = \{(4,d_n,d),(d_n,4,d)\} \cup \mathcal{E}_{\text{K3}}\,, \tag{3.21}$$

where we defined

$$\mathcal{E}_{\text{K3}} = \{(p,q,\ell)\,|\, 1 \le p,q \le 3\,,\ 0 \le \ell_1 \le \ldots \le \ell_n \le 2\,,\ p+q = \ell_n + 4\}. \tag{3.22}$$

The remainder $\mathcal{E}_{\text{K3}}$ of highest-weight states are related to $(3,1)$-, $(2,2)$-, and $(1,3)$-forms in the boundary Hodge decomposition, as can be seen by using (3.18). Relabeling these (p,q)-forms by shifting both degrees down by one, we recover a decomposition reminiscent of the middle cohomology of a Calabi-Yau two-fold, namely a K3 surface. Likewise, the weights ℓ_i are bounded between zero and two. However, note that $(3,1)$-forms are not unique for four-folds, so this situation is analogous to having multiple copies of K3-like blocks.

- The decomposition (3.21) is one of the key insights for our work on moduli stabilization later in this paper. It tells us that for only one or two sl(2)-representations the weights lie within the range $-4 \le \ell_i \le 4$ that characterizes four-folds, while the remainder of states have weights restricted as $-2 \le \ell_i \le 2$. More concretely, we can decompose the primitive middle cohomology as

$$H^4_{\text{prim}}(Y_4,\mathbb{R}) = H_\Omega \oplus \bigoplus_{\ell \in \mathcal{E}_{\text{K3}}} H_{\text{K3},\ell}\,, \tag{3.23}$$

where we defined the subspaces

$$\begin{aligned} H_\Omega &= \bigoplus_{k_1=0}^{d_1-d_0} \cdots \bigoplus_{k_n=0}^{d_n-d_{n-1}} (N_1^-)^{k_1}\cdots(N_n^-)^{k_n}\,(P_d^{4,d_n} \oplus P_d^{d_n,4})\,,\\ H_{\text{K3},\ell} &= \bigoplus_{\substack{p,q\\ (p,q,\ell)\in\mathcal{E}_{\text{K3}}}} \bigoplus_{k_1=0}^{\ell_1-\ell_0} \cdots \bigoplus_{k_n=0}^{\ell_n-\ell_{n-1}} (N_1^-)^{k_1}\cdots(N_n^-)^{k_n}\,P_\ell^{p,q}\,. \end{aligned} \tag{3.24}$$

For the first equation in (3.24) we take just the subspace $P_d^{4,4}$ once in the case that $d_n = 4$. For the second equation the first sum means that, for a given set of highest weights ℓ, we take all possible values p,q such that $(p,q,\ell) \in \mathcal{E}_{\text{K3}}$. In (3.23) we then sum over all blocks of K3 subspaces $H_{\mathrm{K3},\ell}$ to recover the primitive cohomology $H^4_{\mathrm{prim}}(Y_4,\mathbb{R})$, where we suppressed the p,q indices in the summation subscript.

For the purposes of moduli stabilization the majority of fluxes thus comes from the $H_{\mathrm{K3},\ell}$: we will see that fluxes in H_Ω can only stabilize few moduli, so at large $h^{3,1}$ the essence of the problem is captured by the K3 subblocks. We stress that this splitting of the sl(2)-representations is a general result from asymptotic Hodge theory about the sl(2)-decomposition of $H^4_{\mathrm{prim}}(Y_4,\mathbb{R})$; the appearance of these blocks is not a simplifying assumption we make in this work, but a consequence of the existence of the boundary structure.

3.3 The strict-asymptotic form of the Hodge star

In this subsection we discuss the action of the Hodge-star operator $\star$ in the strict asymptotic regime. For rigorous derivations of the relevant formulas we refer to [8, 30], while here we only state that schematically we have

$$\star \xrightarrow{\text{strict asymptotic regime}} \star_{\mathrm{sl}(2)}\,, \tag{3.25}$$

where $\star_{\mathrm{sl}(2)}$ denotes the Hodge-star operator in the strict asymptotic regime. This operator can be written as the following matrix

$$\star_{\mathrm{sl}(2)} = e^{+\phi^i N_i^-}\left[e^{-\frac{1}{2}\log(s^i)N_i^0}\,\star_\infty\, e^{+\frac{1}{2}\log(s^i)N_i^0}\right]e^{-\phi^i N_i^-}\,, \tag{3.26}$$

where N_i^- and N_i^0 are elements of the sl(2)-triplets introduced in (3.5). For later reference it is worthwhile to look more closely at the sl(2)-approximated Hodge decomposition:

- In analogy to (3.14) we can write down a Hodge decomposition in the strict asymptotic regime as

$$H^4_{\mathrm{prim}}(Y_4,\mathbb{C}) = H^{4,0}_{\mathrm{sl}(2)} \oplus H^{3,1}_{\mathrm{sl}(2)} \oplus H^{2,2}_{\mathrm{sl}(2)} \oplus H^{1,3}_{\mathrm{sl}(2)} \oplus H^{0,4}_{\mathrm{sl}(2)}\,, \tag{3.27}$$

  with $\overline{H^{p,q}_{\mathrm{sl}(2)}} = H^{q,p}_{\mathrm{sl}(2)}$. As an example, the sl(2)-approximated period vector $\Pi_{\mathrm{sl}(2)}$ of the holomorphic $(4,0)$-form given in (3.9) spans the subspace $H^{4,0}_{\mathrm{sl}(2)}$.

- There is also a straightforward way to pass between the boundary Hodge structure $H^{p,q}_\infty$ and the sl(2)-approximated $H^{p,q}_{\mathrm{sl}(2)}$. We interpolate by applying the saxion- and axion-dependent factors in (3.26) as

$$H^{p,q}_{\mathrm{sl}(2)} = e^{\phi^i N_i^-}\,e^{-\frac{1}{2}\log(s^i)N_i^0}\,H^{p,q}_\infty\,. \tag{3.28}$$

  This identity proves to be useful when lifting a boundary (p,q)-form to a (p,q)-form in the strict asymptotic regime. To be more explicit, by using (3.18) for a highest-weight element $v_\ell^{p,q}\in P_\ell^{p,q}$ (with $p\geq q$) we can show with Baker-Campbell-Hausdorff that[9]

$$e^{i t^i N_i^-}\,v_\ell^{p,q} \in H^{p,4-p}_{\mathrm{sl}(2)}\,. \tag{3.29}$$

[9]In particular, one can prove that $e^{-\frac{1}{2}\log(s^i)N_i^0}\,e^{iN^-_{(n)}}\,e^{+\frac{1}{2}\log(s^i)N_i^0} = e^{is^iN_i^-}$ in the strict asymptotic regime.

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Note that the $(4,0)$-form period vector in (3.9) is a special case of this identity with $v_d^{4,d_n} = \tilde{a}_0$.

Our goal for the rest of this section is to determine the strict asymptotic limit of the norm defined in (2.2). We first note that for each of the subspaces V_ℓ appearing in the decomposition (3.7), one can introduce a basis $\{(v_\ell)_1, \dots (v_\ell)_{\dim V_\ell}\} \in V_\ell$. To find it, one can first introduce such a basis for the corresponding highest-weight subspaces, P_ℓ and then find the remaining basis elements by successive application of the N_i^-, in analogy with eq. (3.12). This basis can then be normalized such that

$$\langle (v_\ell)_i, (v_{\ell'})_j \rangle = \pm\, \delta_{i,j}\, \delta_{\ell,-\ell'} \,, \tag{3.30}$$

where $\langle \cdot,\cdot \rangle$ is the pairing defined in (2.2), and depending on the particular vector within each V_ℓ the sign can be positive or negative. We then define a norm $\|\cdot\|_\infty$ using the Weil operator $\star_\infty$ as follows

$$\|v_\ell\|^2_\infty = \langle \star_\infty v_\ell, v_\ell \rangle \,. \tag{3.31}$$

The pairing and norms for the basis elements associated with the K3-like blocks are explicitly shown in appendix B. Next, we observe that the inner product of two vectors v and v' satisfies

$$\langle e^{\phi^i N_i^-} v, e^{\phi^j N_j^-} v' \rangle = \langle v, v' \rangle \,, \tag{3.32}$$

where ϕ^i are the axionic parts of the complex-structure moduli. Using the explicit form of $\star_{\mathrm{sl}(2)}$ shown in (3.26) one can derive the following expression

$$\begin{aligned} \left\langle v_\ell, \left[e^{-\phi^i N_i^-} \star_{\mathrm{sl}(2)} e^{+\phi^i N_i^-} \right] v_\ell \right\rangle &= \left(s^1\right)^{\ell_1} \left(s^2\right)^{\ell_2-\ell_1} \dots (s^n)^{\ell_n-\ell_{n-1}} \|v_\ell\|^2_\infty \\ &= \left(\frac{s^1}{s^2}\right)^{\ell_1} \dots \left(\frac{s^{n-1}}{s^n}\right)^{\ell_{n-1}} (s^n)^{\ell_n} \|v_\ell\|^2_\infty \\ &\equiv \kappa_\ell \, \|v_\ell\|^2_\infty \,. \end{aligned} \tag{3.33}$$

We emphasize that $\|v_\ell\|^2_\infty$ is independent of the moduli and, explicitly, κ_ℓ is given by

$$\kappa_\ell = \left(\frac{s^1}{s^2}\right)^{\ell_1} \dots \left(\frac{s^{n-1}}{s^n}\right)^{\ell_{n-1}} (s^n)^{\ell_n} \,. \tag{3.34}$$

Note that from this definition it is obvious that $\kappa_{-\ell} = (\kappa_\ell)^{-1}$ where $-\ell$ stands for the vector with all entries opposite as those of ℓ.

V_{heavy}, V_{light} and V_{rest}. Before closing this subsection, we want to define three subspaces of $H^4_{\text{prim}}(Y_4,\mathbb{R})$ which are distinguished by the behavior of their norm in the strict asymptotic regime. We define V_{heavy} and V_{light} by

$$\begin{aligned} V_{\text{heavy}} &= \bigoplus_\ell V_\ell \,, \qquad \text{with } \ell_1, \ell_2, \dots, \ell_n \geq 0 \text{ and at least one } \ell_i > 0 \,, \\ V_{\text{light}} &= \bigoplus_\ell V_\ell \,, \qquad \text{with } \ell_1, \ell_2, \dots, \ell_n \leq 0 \text{ and at least one } \ell_i < 0 \,, \end{aligned} \tag{3.35}$$

and V_{rest} includes all the elements not belonging to V_{heavy} nor V_{light}. The primitive four-form cohomology splits as $H^4_{\text{prim}}(Y_4, \mathbb{R}) = V_{\text{heavy}} \oplus V_{\text{light}} \oplus V_{\text{rest}}$, and we come back to this splitting below.

3.4 Explicit sl(2) subspaces on Calabi-Yau four-folds

The expansion of $H^4_{\text{prim}}(Y_4, \mathbb{R})$ into sl(2)-eigenspaces shown in (3.7) contains in general a large number of terms. However, in this section we show that only a small number of sl(2)-subspaces are populated, which makes possible a very explicit analysis of moduli stabilization.

Sl(2)-representations I. In a first step we do not consider the sl(2)-representation coming from the holomorphic $(4,0)$-form, i.e. we focus on the subspaces corresponding to the second part in the set $\mathcal{E}$ shown in (3.21). (These correspond to the inner part of the Deligne diamond (A.11).) According to (3.21) the possible highest weight states for this part are of the form

$$P_0\,, \qquad P_{0\,1_i}\,, \qquad P_{0\,2_i}\,, \qquad P_{0\,1_i\,2_j}\,, \tag{3.36}$$

where we employ the notation

$$P_{0\,1_i\,2_j} = P_{0,\ldots,0,\underset{i}{1},\ldots,1,\underset{j}{2},\ldots,2}\,, \tag{3.37}$$

with i and j denoting the positions at which the first time a 1 or 2 appears. Note that this includes the possibility $i = 1$ for which a 0 is absent, i.e. the spaces P do not need to start with a 0. All the sl(2)-representations can then be obtained by successively applying the lowering operators N_i^-, so that the full primitive four-form cohomology (up to the sl(2) state corresponding to the holomorphic $(4,0)$-form) can be decomposed into the following V_ℓ (repeated indices are not summed over)

$$\begin{aligned} V_{\text{heavy}} &= \begin{cases} V_{02_i} &= P_{02_i} \\ V_{01_i} &= P_{01_i} \\ V_{01_i 2_j} &= P_{01_i 2_j} \\ V_{01_i 0_j} &= N_j^-\, P_{01_i 2_j} \end{cases} \\ V_{\text{light}} &= \begin{cases} V_{0-2_i} &= (N_i^-)^2\, P_{02_i} \\ V_{0-1_i} &= N_i^-\, P_{01_i} \\ V_{0-1_i-2_j} &= N_i^-\, N_j^-\, P_{01_i 2_j} \\ V_{0-1_i 0_j} &= N_i^-\, P_{01_i 2_j} \end{cases} \end{aligned} \qquad V_{\text{rest}} = V_0 = P_0 \oplus N_i^-\, P_{02_i}\,. \tag{3.38}$$

We mention that a crucial point for what follows is the absence of $V_\ell \in V_{\text{rest}}$ with some positive and some negative ℓ_i, that is, V_{rest} is entirely composed by V_0.

Sl(2)-representations II. Let us now present the highest-weight states corresponding to the sl(2)-representation coming from the holomorphic (4,0)-form. As will become clear in section 5, these will not play an essential role in the stabilization of many moduli, but we display here for completeness. The possible highest-weight spaces are the following

$$\begin{array}{llllll} P_0\,, \\ P_{01_i}\,, & P_{02_i}\,, & P_{03_i}\,, & P_{04_i}\,, \\ P_{01_i2_j}\,, & P_{01_i3_j}\,, & P_{01_i4_j}\,, & P_{02_i3_j}\,, & P_{02_i4_j}\,, & P_{03_i4_j}\,, \\ P_{0\,1_i\,2_j3_k}\,, & P_{0\,1_i\,2_j4_k}\,, & P_{0\,1_i\,3_j4_k}\,, & P_{0\,2_i\,3_j4_k}\,, \\ P_{0\,1_i\,2_j3_k4_l}\,. \end{array} \tag{3.39}$$

Let us remark that, as opposed to the highest-weight states in equation (3.36), for a given Calabi-Yau four-fold and for a given strict asymptotic regime (cf. eq. (3.4)) only one of these (4,0)-form highest-weight states will be present.[10] This corresponds to the outer part of the Deligne diamond (A.11), and it can be directly related to the enhancement chain of the singularity types associated to the corresponding strict asymptotic regime.

By direct comparison of equations (3.36) and (3.39), we see that the contribution from the (4,0)-form to the subspaces V_{light}, V_{heavy} and V_{rest} can be much richer than the one presented in (3.38). We will not display all the possibilities here since, as mentioned, their detailed expression will not play a relevant role for our later discussion. They can however be obtained by applying all possible lowering operators to the corresponding highest-weight states.

4 Moduli stabilization — general considerations

In this section we discuss complex-structure moduli-stabilization in the strict asymptotic limit using the framework of asymptotic Hodge theory. We give an overview of the general structure, but provide a more detailed picture in section 5.

4.1 Self-duality condition

Minkowski minima of the scalar potential (2.1) are obtained when solving the self-duality condition of the four-form flux G_4 shown in equation (2.3). We now bring this condition into the framework of asymptotic Hodge theory. According to (3.7) we can expand G_4 as $G_4 = \sum_{\ell\in\mathcal{E}} G_\ell$ with $G_\ell \in V_\ell$. However, given the form of (3.33), it turns out to be convenient to define an axion-dependent four-form flux and perform its expansion as [9]

$$\hat{G}_4 \equiv e^{-\phi^i N_i^-} G_4\,, \qquad\qquad \hat{G}_4 = \sum_{\ell\in\mathcal{E}} \hat{G}_\ell\,, \tag{4.1}$$

where $\hat{G}_\ell \in V_\ell$. Similar redefinitions using the log-monodromy matrices have been used in [9, 10, 34–38]. Let us stress that in our setting the lowering operators N_i^- appear, which in general are valued over the rationals instead of the integers. Note that an integral shift of the flux quanta can therefore require the axions to wind around multiple times

[10] Note that by exploring different asymptotic limits for a given CY generally only a subset of the spaces shown in eq. (3.39) (one for each asymptotic limit) are realized. That is, not necessarily all of them can be realized in a given CY.

instead of just once. The self-duality condition (2.3) in the strict asymptotic regime reads $G_4 = \star_{\mathrm{sl}(2)} G_4$, which can be rewritten as

$$\hat{G}_4 = \left[e^{-\phi^i N_i^-} \star_{\mathrm{sl}(2)} e^{+\phi^i N_i^-} \right] \hat{G}_4 \,. \tag{4.2}$$

Comparing now with (3.26) and recalling that the N_i^0 act as $N_i^0 : V_\ell \to V_\ell$ while $\star_\infty : V_\ell \to V_{-\ell}$, we note that the expression in parenthesis in (4.2) maps V_ℓ to $V_{-\ell}$. Taking the inner product with $\hat{G}_\ell$ and using the orthogonality condition (3.30) as well as (3.33) we obtain

$$\langle \hat{G}_{+\ell}, \hat{G}_{-\ell} \rangle = \left\langle \hat{G}_\ell, \left[e^{-\phi^i N_i^-} \star_{\mathrm{sl}(2)} e^{+\phi^i N_i^-} \right] \hat{G}_\ell \right\rangle = \kappa_\ell \| \hat{G}_\ell \|_\infty^2 \,. \tag{4.3}$$

Let us emphasize that the axions appear only in $\hat{G}_\ell$ and the saxions appear only in κ_ℓ. We furthermore note that the norm $\| \cdot \|_\infty$ is positive-definite and that in our conventions the saxions satisfy $s^i > 0$. Hence, (4.3) implies the two relations

$$\langle \hat{G}_{+\ell}, \hat{G}_{-\ell} \rangle \geq 0 \,, \tag{4.4}$$

$$\hat{G}_{+\ell} \neq 0 \quad \Rightarrow \quad \hat{G}_{-\ell} \neq 0 \,. \tag{4.5}$$

We also observe that the pairing $\langle \hat{G}_{+\ell}, \hat{G}_{-\ell} \rangle$ is invariant under $\ell \to -\ell$, and so we obtain from (4.3) the relation $\langle \hat{G}_{+\ell}, \hat{G}_{-\ell} \rangle = \kappa_{-\ell} \| \hat{G}_{-\ell} \|_\infty^2$. Multiplying this expression with (4.3) leads to

$$\langle \hat{G}_{+\ell}, \hat{G}_{-\ell} \rangle^2 = \| \hat{G}_{+\ell} \|_\infty^2 \, \| \hat{G}_{-\ell} \|_\infty^2 \,, \tag{4.6}$$

where we used that $\kappa_\ell \kappa_{-\ell} = 1$. This relation corresponds to the equality in a Cauchy-Schwarz inequality for the pairing $\langle \star_\infty \cdot , \cdot \rangle$, which can only be satisfied for

$$\hat{G}_{-\ell} = \kappa_\ell \star_\infty \hat{G}_{+\ell} \,. \tag{4.7}$$

The tadpole. Let us now turn to the tadpole contribution of the flux G_4 given in (2.5). In terms of sl(2)-representations this is

$$Q = \frac{1}{2} \langle G_4, G_4 \rangle = \frac{1}{2} \langle \hat{G}_4, \hat{G}_4 \rangle = \frac{1}{2} \sum_\ell \langle \hat{G}_{+\ell}, \hat{G}_{-\ell} \rangle \,, \tag{4.8}$$

where we have used (4.6). We can use the self-duality condition, and the properties of the different sl(2)-weights to give a more explicit expression, as we will see in section 6.

4.2 Saxion stabilization

Let us now discuss the stabilization of the saxions s^i through the conditions (4.7). (We address the stabilization of the axions in section 4.3 below.) Taking the ratio between equation (4.3) and its version with $\ell \to -\ell$, and using $\langle \hat{G}_{+\ell}, \hat{G}_{-\ell} \rangle = \langle \hat{G}_{-\ell}, \hat{G}_{+\ell} \rangle$ and $\kappa_{-\ell} = \kappa_{+\ell}^{-1}$, we arrive at

$$\kappa_\ell = \frac{\| \hat{G}_{-\ell} \|_\infty}{\| \hat{G}_{+\ell} \|_\infty} \qquad \text{with} \qquad \kappa_\ell = \left(\frac{s^1}{s^2} \right)^{\ell_1} \cdots \left(\frac{s^{n-1}}{s^n} \right)^{\ell_{n-1}} (s^n)^{\ell_n} \,. \tag{4.9}$$

As we have seen in (4.5), if $\hat{G}_{+\ell} \neq 0$ then also $\hat{G}_{-\ell} \neq 0$. Let us therefore label all non-vanishing $\hat{G}_\ell$ by an index α and define

$$y^i = \log \frac{s^i}{s^{i+1}}\,, \qquad\qquad \mathcal{B}_\alpha = \log \frac{\|\hat{G}_{-\alpha}\|_\infty}{\|\hat{G}_{+\alpha}\|_\infty}\,, \tag{4.10}$$

together with $s^{n+1} \equiv 1$. Note that in the strict asymptotic limit we are considering (cf. equation (3.4)), all the y^i are positive. Taking then the logarithm of equation (4.9) leads to

$$\ell_{(\alpha)1}\, y^1 + \ell_{(\alpha)2}\, y^2 + \ldots + \ell_{(\alpha)n}\, y^n = \mathcal{B}_\alpha\,, \tag{4.11}$$

which in matrix notation is expressed as

$$\mathcal{A}_{\alpha i}\, y^i = \mathcal{B}_\alpha\,. \tag{4.12}$$

The number of saxions s^i stabilized by this condition is equal to the rank of $\mathcal{A}$, and in order to stabilize all saxions we have to require $\mathcal{A}$ to be of maximal rank, that is $\operatorname{rank}\mathcal{A} = h^{3,1}$. This implies in particular, that we need to have at least $h^{3,1}$ non-vanishing pairs $(\hat{G}_{+\ell}, \hat{G}_{-\ell})$. The values of the stabilized saxions s^i are then determined via the relation

$$y^i = \left[\mathcal{A}^+\right]^{i\beta} \mathcal{B}_\beta\,, \tag{4.13}$$

where $\mathcal{A}^+$ denotes the pseudo-inverse of $\mathcal{A}$ which can be computed for instance through a singular value decomposition. Note that when $\operatorname{rank}\mathcal{A} = h^{3,1}$ and $\mathcal{B} = 0$ we have $y^i = 0$, which is not compatible with the growth requirement (3.4). We therefore require at least one $\kappa_\alpha \neq 1$.

4.3 Axion stabilization

Let us now briefly consider the stabilization of the axions ϕ^i from a general perspective, while a more detailed discussion of this question will be given in section 5 below. We first recall that the axions ϕ^i appear in the self-duality condition (4.7) through the relation (4.1), that is

$$\hat{G}_4 = e^{-\phi^i N_i^-}\, G_4\,, \tag{4.14}$$

where N_i^- are the lowering operators in the commuting sl(2)-triples (3.5). The flux G_4 can be expanded into the sl(2)-eigenspaces V_ℓ appearing in (3.7) as

$$G_4 = \sum_{\ell\in\mathcal{E}} G_\ell\,. \tag{4.15}$$

Now, if G_4 has only components G_ℓ which are all annihilated by the action of a particular N_i^-, then the corresponding axion ϕ^i does not appear in (4.14) and will not be stabilized. Generalizing this argument, we therefore have at most

$$\dim \operatorname{span}\{N_i^- G_4\} \tag{4.16}$$

linearly independent combinations of axions ϕ^i appearing in $\hat{G}_4$, which is therefore the maximal number of axions that can be stabilized through the self-duality condition (4.7).

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5 Moduli stabilization — explicit analysis

After having discussed moduli stabilization from a general point of view in section 4, we now turn to a more detailed treatment. We are going to make use of the decomposition of $H^4_{\text{prim}}(Y_4, \mathbb{R})$ into V_{heavy} , V_{light} and V_{rest} shown in section 3.4. Let us note that we will mostly ignore the sl(2)-eigenspaces coming from the irreducible representation(s) of the $(4,0)$-form and focus our attention on those coming from the middle part. As explained below, the former do not play an important role for large numbers of moduli.

Decomposition using K3-like blocks. Recall from section 3.4 that the highest-weight subspaces in the sl(2)-decomposition (3.12) can be divided into a one- or two-dimensional part corresponding to the $(4,0)$-form (and its conjugate) with weights $0 \le d_1 \le \ldots \le d_n \le 4$. For all others these are bounded by $0 \le \ell_1 \le \ldots \le \ell_n \le 2$. To be more explicit, for completeness we restate (3.23) as

$$H^4_{\text{prim}}(Y_4, \mathbb{R}) = H_\Omega \oplus \bigoplus_{\ell \in \mathcal{E}_{\text{K3}}} H_{\text{K3},\ell}\,, \tag{5.1}$$

where the subspaces on the right-hand side were defined in (3.24). The first piece H_Ω denotes the sl(2)-eigenspaces that descend from the highest-weight state of the $(4,0)$-form and its complex conjugate. The remainder $H_{\text{K3},\ell}$ (with $\ell \in \mathcal{E}_{\text{K3}}$) is comprised of descendants of highest-weight states corresponding to $(3,1)$-, $(2,2)$- and $(1,3)$-forms. For the purposes of moduli stabilization we can stabilize at most four moduli (eight real scalars) via fluxes in H_Ω, since this is the maximum number of different V_ℓ that one can populate in this sl(2)-representation. For any strict asymptotic regime with large $h^{3,1}$ the majority of moduli thus has to be stabilized through fluxes in subspaces $H_{\text{K3},\ell}$. In the following we will therefore restrict our attention to fluxes in $H_{\text{K3},\ell}$, and comment afterwards on the inclusion of fluxes in H_Ω.

Hodge decomposition of K3-like blocks. For our study of moduli stabilization it is useful to write out a Hodge decomposition for $H_{\text{K3},\ell}$ in the strict asymptotic regime as

$$H_{\text{K3},\ell} = (H_{\text{K3},\ell})^{3,1}_{\text{sl}(2)} \oplus (H_{\text{K3},\ell})^{2,2}_{\text{sl}(2)} \oplus (H_{\text{K3},\ell})^{1,3}_{\text{sl}(2)}\,. \tag{5.2}$$

For the self-duality condition on the fluxes (2.3) we can only allow for $G_4 \in (H_{\text{K3},\ell})^{2,2}_{\text{sl}(2)}$. In particular, we cannot have $(4,0)$- or $(0,4)$-fluxes coming from these subspaces, so the vacuum loci will be supersymmetric by construction, i.e. as many axions and saxions will be stabilized, both with the same mass.[11] As a complementary perspective, it is convenient to rewrite the self-duality condition restricted to $H_{\text{K3},\ell}$ into an orthogonality condition with $(H_{\text{K3},\ell})^{3,1}_{\text{sl}(2)}$. Recalling (3.29), we can write down a basis for these vector spaces in terms of the highest-weight states (3.17) as

$$\chi_\ell(t^i) = e^{it^i N_i^-}\, v_\ell^{3,1+\ell_n} \in (H_{\text{K3},\ell})^{3,1}_{\text{sl}(2)}\,, \qquad v_\ell^{3,1+\ell_n} \in P_\ell^{3,1+\ell_n}\,. \tag{5.3}$$

[11] One can also include supersymmetry-breaking pieces coming from fluxes in H_Ω, as discussed at the end of section 5.2. These on one hand do not alter the analysis of the vacuum loci, and on the other they can only fix a handful of moduli, so we do not consider them here.

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The self-duality condition restricted to $H_{\mathrm{K3},\ell}$ is then implemented by

$$\langle e^{it^i N_i^-} v_\ell^{3,1+\ell_n}, G_4 \rangle = 0 \,. \tag{5.4}$$

This gives us a simple set of algebraic equations, holomorphic in the moduli, that we need to solve to determine the vacuum loci for a given flux $G_4 \in H_{\mathrm{K3},\ell}$.

5.1 No moduli stabilization with fluxes in $V_{\mathrm{rest}} \subset H_{K3}$

We start by considering fluxes in V_{rest}. As can be seen from equation (3.38), V_{rest} splits into P_0 and $N_i^- P_{02_i}$ which we discuss in turn. First, P_0 is the sl(2)-singlet that is annihilated by all N_i^-, which implies for $G_4 \in P_0$ that

$$G_4 \in P_0 \qquad \Rightarrow \qquad G_4 = \hat{G}_4 \,. \tag{5.5}$$

The self-duality condition (4.7) then reads

$$G_4 = \star_\infty G_4 \,, \tag{5.6}$$

in particular, it is independent of the axions and saxions. Hence, through such fluxes no moduli are stabilized. Moreover, the only fluxes in P_0 that are self-dual, namely the ones in $P_0^{(2,2)}$, do not contribute to the potential (see eq. (B.5)). Second, fluxes in V_{rest} that are not highest-weight, namely $G_4 \in N_i^- P_{02_i}$, are anti-self dual (see equation (B.11)) and therefore do not satisfy the self-duality condition (4.7). To summarize, fluxes in V_{rest} do not stabilize moduli.

Let us note that the situation for the fluxes in V_{rest} coming from the (4,0)-form can be slightly more involved, and such fluxes can indeed be used to fix moduli. However, as emphasized at the beginning of this section, this will not play an important role when trying to fix many of moduli, since at most four (complex) moduli can be fixed by turning on these fluxes.

5.2 Vacua with fluxes in V_{heavy} and V_{light}

Let us consider the moduli stabilization including fluxes in V_{heavy}, given by the spaces in (3.38). The self-duality condition will relate these fluxes to their counterparts in V_{light}, so we will also introduce these. We will consider the situation where there is flux in each of these individual subspaces, and then argue that combining them leads to the same conclusions.

Flux along $\mathbf{V_{01_i}}$. We begin by considering a highest-weight four-form flux $G^R_{01_i} v^R_{01_i} + G^I_{01_i} v^I_{01_i} \in V_{01_i} = P_{01_i}$, where we denote by $G^{R,I}_{01_i}$ the two possible flux component along the V_{01_i} space in the basis introduced in eq. (B.6). Note that each V_{01_i} has (real) dimension 2, so that we need to allow for both kinds of fluxes. The flux-axion polynomial then reads

$$\hat{G}_4 = G^R_{01_i} v^R_{01_i} + G^I_{01_i} v^I_{01_i} + \left(G^R_{0-1_i} - \phi^i G^R_{01_i}\right) v^R_{0-1_i} + \left(G^I_{0-1_i} - \phi^i G^I_{01_i}\right) v^I_{0-1_i} \,, \tag{5.7}$$

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where no sum over i is implied and where we have explicitly substituted the basis elements shown in eq. (B.6). With this flux configuration, the tadpole (4.8) is $Q = G^R_{01_i} G^I_{0-1_i} - G^I_{01_i} G^R_{0-1_i}$, and the self-duality conditions read

$$
\begin{aligned}
s^i G^R_{01_i} &= G^I_{0-1_i} - \phi^i G^I_{01_i} \,, \\
-s^i G^I_{01_i} &= G^R_{0-1_i} - \phi^i G^R_{01_i} \,.
\end{aligned}
\tag{5.8}
$$

These equations have solutions whenever at least one of the two pairs of fluxes $(G^R_{01_i}, G^I_{0-1_i})$ or $(G^I_{01_i}, G^R_{0-1_i})$ are non-zero. For concreteness, if one considers for instance the former, the moduli are fixed as[12]

$$
s^i = \frac{G^I_{0-1_i}}{G^R_{01_i}} \,, \qquad \phi^i = 0 \,, \tag{5.9}
$$

The sign of s^i is the same as the sign of the tadpole, so that a physical solution (i.e. $s^i > 0$) requires a positive contribution to the tadpole. If both pairs of fluxes are turned on there is also a solution provided that the tadpole is positive, that is

$$
s^i = \frac{G^R_{01_i} G^I_{0-1_i} - G^I_{01_i} G^R_{0-1_i}}{(G^I_{01_i})^2 (G^R_{01_i})^2} \,, \qquad \phi^i = \frac{G^I_{01_i} G^I_{0-1_i} + G^R_{01_i} G^R_{0-1_i}}{(G^I_{01_i})^2 (G^R_{01_i})^2} \,, \tag{5.10}
$$

and the tadpole appears in the numerator of the saxion vev.

Flux along V_{02_i}. We consider now the four-form highest-weight flux in $V_{02_i} = P_{02_i}$ and its descendants, and use the basis presented in eq. (B.9). The flux-axion polynomial is now

$$
\hat{G}_4 = G_{02_i} v_{02_i} + \left(G_0 - \phi^i G_{02_i}\right) v_0 + \left(G_{0-2_i} - \phi^i G_0 + \frac{1}{2}(\phi^i)^2 G_{02_i}\right) v_{0-2_i} \,, \tag{5.11}
$$

and the corresponding tadpole reads $Q = G_{02_i} G_{0-2_i} - \frac{1}{2} G_0^2$. The self-duality condition gives the following equations

$$
\begin{aligned}
(G_0 - \phi^i G_{02_i}) = -(G_0 - \phi^i G_{02_i}) \quad &\Longrightarrow \quad G_0 - \phi^i G_{02_i} = 0 \,, \\
\frac{G_{02_i}}{2} (s^i)^2 = G_{0-2_i} - \phi^i G_0 &+ \frac{1}{2}(\phi^i)^2 G_{02_i} \,.
\end{aligned}
\tag{5.12}
$$

Note that moduli stabilization requires $G_{02_i} \neq 0$. Without loss of generality, we can use the axionic shift symmetry to set $G_0 = 0$ and we get the solution

$$
s^i = \sqrt{\frac{2 G_{0-2_i}}{G_{02_i}}} \,, \qquad \phi^i = 0 \,, \tag{5.13}
$$

where again a real positive saxion implies a positive tadpole. Note that the general solution can be obtained from (5.13) by shifting in the axionic field $\phi^i \to \phi^i + \phi^i_b$, together with $G_4 \to e^{\phi^i_b N_i^-} G_4$. This yields the general solution

$$
s^i = \frac{\sqrt{2\, G_{0-2_i} G_{02_i} - G_0^2}}{G_{02_i}} \,, \qquad \phi^i = \frac{G_0}{G_{0-2_i}} \,. \tag{5.14}
$$

[12] If the other pair is considered, a similar solution is obtained upon exchanging $(G^R_{01_i}, G^I_{0-1_i}) \to (-G^I_{01_i}, G^R_{0-1_i})$. If any of such pairs is turned on and also the component of the other pair along V_{light}, the result is just a shift in the vev of the axion $\phi^i \to \phi^i + G^I_{0-1_i}/G^R_{01_i}$ in the solution in (5.9).

Flux along $V_{01_i2_j}$. Let us finally consider the highest weight flux in $V_{01_i2_j}$, together with its descendants. Using the basis in eq. (B.12), the corresponding flux-axion polynomial reads

$$\begin{aligned}\hat{G}_4 = G_{01_i2_j} v_{01_i2_j} &+ \left(G_{01_i0_j} - \phi^j G_{01_i2_j}\right) v_{01_i0_j} + \left(G_{0-1_i0_j} - \phi^i G_{01_i2_j}\right) v_{0-1_i0_j} \\ &+ \left(G_{0-1_i-2_j} - \phi^i G_{01_i0_j} - \phi^j G_{0-1_i0_j} + \phi^i\phi^j G_{01_i0_j}\right) v_{0-1_i-2_j} \,.\end{aligned} \tag{5.15}$$

The tadpole contribution of these fluxes is $Q = G_{01_i2_j} G_{0-1_i-2_j} - G_{01_i0_j} G_{0-1_i0_j}$. Using the action of $\star_\infty$ on the different basis vectors, displayed in (B.14), the self-duality condition yields the following two equations.

$$\begin{aligned} s^i\, s^j\, G_{01_i2_j} &= \left(G_{0-1_i-2_j} - \phi^i G_{01_i0_j} - \phi^j G_{0-1_i0_j} + \phi^i\phi^j G_{01_i0_j}\right), \\ -\frac{s^i}{s_j}\left(G_{01_i0_j} - \phi^j G_{01_i2_j}\right) &= \left(G_{0-1_i0_j} - \phi^i G_{01_i2_j}\right) .\end{aligned} \tag{5.16}$$

Instead of solving these equations directly, it is more practical to switch to the orthogonality condition (5.4) with the sl(2)-approximated (3, 1)-form. We find that the (3, 1)-form is given by

$$\begin{aligned}\chi_{01_i2_j} &= (1 + t^i N_i^-)(1 + t^j N_j^-) v_{01_i2_j} \\ &= v_{01_i2_j} + t^i v_{0-1_i0_j} + t^j v_{01_i0_j} + t^i t^j v_{0-1_i-2_j} \,.\end{aligned} \tag{5.17}$$

When $G_{01_i2_j} = 0$, the equations simplify considerably and we can use the shift symmetry of the axions to set G_{01_i2j} to zero without loss of generality. The solution then reads

$$s^j = -\frac{G_{01_i0_j}}{G_{0-1_i0_j}} s^i \,, \qquad \phi^j = -\frac{G_{01_i0_j}}{G_{0-1_i0_j}} \phi^i \,, \tag{5.18}$$

which has physical solutions with positive saxions only when the tadpole is positive.

When $G_{01_i2_j} \neq 0$, requiring orthogonality of G_4 with $\chi_{01_i2_j}$ then yields[13]

$$(G_{01_i2_j} t^i - G_{0-1_i0_j})(G_{01_i2_j} t^j - G_{01_i0_j}) = G_{01_i0_j} G_{0-1_i0_j} - G_{01_i2_j} G_{0-1_i-2_j} \,, \tag{5.19}$$

which can easily be solved for $t^i = \phi^i + is^i$ or $t^j = \phi^j + is^j$ by moving the other factor to the right-hand side. For concreteness, let us write the explicit solution for the case $G_{01_i0_j} = G_{0-1_i0_j} = 0$ (from which the general case above can also be obtained by exploiting the shift-symmetries of the axions):

$$s^j = \frac{G_{0-1_i-2_j}}{G_{01_i2_j}} \frac{s^i}{(s^i)^2 + (\phi^i)^2} \,, \qquad \phi^j = -\frac{G_{0-1_i-2_j}}{G_{01_i2_j}} \frac{\phi^i}{(s^i)^2 + (\phi^i)^2} \,. \tag{5.20}$$

As before, this is a physical solution with positive saxions only when the signs of the relevant fluxes are such that the tadpole is positive. Thus, one positive contribution to the tadpole is able to fix one linear combination of the saxions and of the axions.

[13] Up to proportionality factors, the real part of this equation corresponds to the first equation in (5.16), and the imaginary part to the second.

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Flux along all K3-like subspaces. Let us discuss the situation with fluxes along an arbitrary combination of subspaces in V_{heavy} and V_{light}. Since self-duality conditions for each subspace decouple, each of the individual equations will have to be satisfied. This means that there will generically be no solutions, unless the fluxes satisfy certain properties. For instance, for fluxes in V_{01_i} and V_{02_i} at the same time, there is a solution only if (5.9) and (5.13) are satisfied, which implies a relation between the ratios of fluxes appearing in these equations. On the other hand, these pairs contribute to the tadpole, so tadpole-wise the most economic way of fixing the modulus t^i moduli is having a single pair of fluxes either of the type V_{01_i} or of the type V_{02_i}. In order to fix two moduli t^i and t^j, one can either use V_{01_i} or V_{02_i} together with V_{01_j} or V_{02_j}, or alternatively any of these four together with $V_{01_i 2_j}$. Any of these possibilities will fix both moduli in the most economic way tadpole-wise. We come back to this point in section 6.

Adding fluxes along H_Ω. Finally, let us comment on the effect of including the fluxes in H_Ω coming from the sl(2)-representation whose highest-weight state corresponds to the (4,0)-form (and its complex conjugate). We do not perform a systematic study of moduli stabilization with all possible flux choices in H_Ω here, but we explain instead why it is not relevant for determining the scaling of the tadpole charge with a large number moduli stabilized in the strict asymptotic region.

Using the expression (3.26) for the Hodge star it can be seen that at most $d_n \leq 4$ moduli can appear in the self-duality condition for such fluxes. Thus, a large number of moduli cannot be stabilized by means of such fluxes, as opposed to the ones in $\bigoplus_{\ell\in\mathcal{E}_{\text{K3}}} H_{\text{K3},\ell}$. Moreover, the fact that the self-duality condition for each subspace decouples implies that if these fluxes yield equations for some moduli that also appear in the conditions coming from the K3-like fluxes, they will only be satisfied for compatible flux configurations, but they will never lower the tadpole. As before, tadpole-wise the most economic way to fix moduli is by choosing fluxes in such a way that the moduli that appear in the conditions for the fluxes in H_Ω do not appear in the rest. This will still lead to (at least) a linear scaling of the tadpole with the (large) number of stabilized moduli, coming from the K3-like fluxes.

It is also interesting to consider the effect of supersymmetry breaking introduced by these fluxes. As mentioned at the beginning of the section, the K3-like fluxes do not induce any supersymmetry breaking effect, so that fixing moduli using only them always yields supersymmetric vacua, where both the saxions and their corresponding axions are fixed and have equal mass. However, this does not mean that our analysis cannot capture non-supersymmetric vacua. This is the case due to the decoupling of the self-duality conditions for the different subspaces, which crucially ensures that the inclusion of fluxes in H_Ω does not modify the scaling of the tadpole with the number of fixed moduli. Note, however, that these supersymmetry breaking fluxes can alter the masses of the moduli, as they can introduce some mixing at the level of the scalar potential, even though they cannot do it at the level of the vacuum equations. To sum up, adding fluxes in H_Ω does not change the discussion with respect to the vacuum loci or the scaling of the tadpole with the number of fixed moduli, but it can change the values of the masses of the moduli, as expected from the fact that they can break supersymmetry.

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6 The tadpole contribution

In this section we analyze the scaling of the flux-induced tadpole charge Q with the number of fixed moduli. We compare this with the behavior predicted by the tadpole conjecture [5], which was introduced in section 2.2.

Linear scaling with $h^{3,1}$. Let us start by recalling the tadpole contribution of the flux G_4 from equation (2.5) as $Q = \frac{1}{2}\int G_4 \wedge G_4 = \frac{1}{2}\langle G_4, G_4\rangle$. Using the self-duality condition (4.7), the property (3.32) and the condition (4.6) we compute

$$\langle G_4, G_4\rangle = \langle \hat{G}_4, \hat{G}_4\rangle = \sum_\ell \langle \hat{G}_{+\ell}, \hat{G}_{-\ell}\rangle = \sum_\ell \|\hat{G}_\ell\|_\infty \, \|\hat{G}_{-\ell}\|_\infty \, . \tag{6.1}$$

As argued below equation (4.12), if we want to stabilize n_{stab} saxions we need to have n_{stab} non-vanishing pairs $(\hat{G}_{+\ell}, \hat{G}_{-\ell})$ and hence the sum (6.1) contains at least $2n_{\text{stab}}$ non-vanishing positive terms. As shown explicitly in the previous section, this would also stabilize the corresponding n_{stab} axions. Using then for instance (4.9), we can rewrite (6.1) in the following way

$$\begin{aligned}
\langle G_4, G_4\rangle &= \sum_{\ell} \kappa_\ell \|\hat{G}_\ell\|^2_\infty \\
&= \sum_{\ell\in V_{\text{heavy}}} \kappa_\ell \|\hat{G}_\ell\|^2_\infty + \sum_{\ell\in V_{\text{rest}}} \kappa_\ell \|\hat{G}_\ell\|^2_\infty + \sum_{\ell\in V_{\text{light}}} \kappa_\ell \|\hat{G}_\ell\|^2_\infty \\
&= 2\sum_{\ell\in V_{\text{heavy}}} \kappa_\ell \|\hat{G}_\ell\|^2_\infty + \sum_{\ell\in V_{\text{rest}}} \kappa_\ell \|\hat{G}_\ell\|^2_\infty \, ,
\end{aligned} \tag{6.2}$$

where in the second line we split the sum into contributions coming from the different subspaces defined in eq. (3.35). In the third line we used $\kappa_{-\ell} = (\kappa_\ell)^{-1}$ and $\kappa_{-\ell}\|\hat{G}_{-\ell}\|^2_\infty = \kappa_\ell\|\hat{G}_\ell\|^2_\infty$ (cf. eq. (4.9)) to pair the contributions from V_{heavy} and V_{light} and make the summation explicitly in terms of elements of V_{heavy}. As we saw in section 5.1, fluxes in V_{rest} are either anti-self dual, or those in V_0 do not fix any moduli. Note that they do contribute to the tadpole, though. We thus get

$$Q = \frac{1}{2}\langle G_4, G_4\rangle \geq \sum_{\ell>0} \gamma^{\sum \ell_i} \|\hat{G}_\ell\|^2_\infty \, . \tag{6.3}$$

Since, again, there should at least be one $\hat{G}_\ell$ per moduli stabilized, this sum has at least n_{mod} terms, confirming the tadpole conjecture. Each term is weighted by a positive power of γ, which makes moduli stabilization in the asymptotic regime more difficult to achieve within the tadpole bound, even for a relatively small number of moduli (a similar observation was made in [39]). Note however that, as mentioned in the introduction, γ need not be very large: $\gamma \gtrsim 4$ is enough for the sl(2) approximation to reproduce the actual vacua with high accuracy [10].

Quantization. The norms $\|\hat{G}_\ell\|^2_\infty$ appearing in (6.3) are in general not integer quantized and in principle could depend on the spectator moduli. It is therefore not obvious that the tadpole conjecture (reviewed in section 2.2) is satisfied since — in an extreme situation —

the norms may scale as $\|\hat{G}_\ell\|^2_\infty \sim 1/n_{\rm stab}$ and thereby violate the conjecture. However, we see no indications of such a scaling:

- In table 1 of appendix D we analyzed four examples with $n_{\rm stab}$ or order 20. For each of these examples the spectator moduli decouple and the majority (more than 70%) of norms of the subspaces satisfies $\|G_\ell\|^2_\infty \geq \frac{1}{4}$. Hence, there is no inverse scaling with the number of stabilized moduli $n_{\rm stab}$.

- Let us now clarify why our bounds on the norm of G_ℓ, as opposed to the norm of the axion dependent $\hat{G}_\ell$ (which are the ones that appear in the definition of the tadpole, cf. eq. (6.3)), are meaningful. First, note that the pairing that appears in the definition of the tadpole charge fulfills $\langle G_4, G_4\rangle = \langle \hat{G}_4, \hat{G}_4\rangle$, whereas this is not the case for the boundary norm, for which $\|G_4\|_\infty \neq \|\hat{G}_4\|_\infty$ in general (the equality holds if e.g. all the axion vevs vanish).

  Nevertheless, the sum over ℓ's in equation (6.3) contains at least $n_{\rm stab}$ terms for which $\hat{G}_\ell = G_\ell$ (a single sl(2)-representation fixes a single modulus, see section 5.2) and the rest, if non-zero, give extra positive contributions.[14] Therefore, we can bound eq. (6.3) as

$$Q \geq \sum_{\ell>0} \gamma^{\sum \ell_i} \|\hat{G}_\ell\|^2_\infty \geq \sum_{\ell'>0} \gamma^{\sum \ell'_i} \|G_{\ell'}\|^2_\infty \,, \tag{6.4}$$

  where the summation over ℓ' indicates that only the fluxes in $V_{\rm heavy}$ that correspond to the highest-weight within each sl(2)-representation are included, for which $G_{\ell'} = \hat{G}_{\ell'}$.

- The parameter $\gamma \gg 1$ parametrizes the strict asymptotic regime, and in [10] it was found that $\gamma \gtrsim 4$ provides already a good approximation of the moduli space structure for Calabi-Yau three-folds. This parameter appears in (6.3) as $\gamma^{\sum \ell_i}$, where the sum is over positive integers taking values one or two. A very conservative estimate therefore is

$$\gamma^{\sum \ell_i} \geq \gamma \gtrsim 4 \,. \tag{6.5}$$

  Furthermore, in our analysis in appendix D we found that 70% of the norms $\|G_\ell\|^2_\infty$ are larger than 1/4. Using then that for stabilizing $n_{\rm stab}$ moduli there needs to be $n_{\rm stab}$ terms, we can estimate

$$\sum_{\ell>0} \|\hat{G}_\ell\|^2_\infty > \frac{1}{4} \cdot 0.7 \cdot n_{\rm stab} \,. \tag{6.6}$$

  However, our analysis provides only a lower bound on $\|G_\ell\|^2$ since we determined the lowest non-zero values for each $\|G_\ell\|^2_\infty$ individually but did not account for the requirement that the corresponding fluxes have to stabilize $n_{\rm stab}$ moduli. (See the appendix for details on our analysis, in particular the discussion after (D.10)). The values $\|G_\ell\|^2_\infty$ are therefore expected to be much larger.

[14]Note also that, if an axion dependence is generated for a lower ℓ that happens to be populated also by the highest weight state in a different sl(2)-representation, there will be no mixing between the two because different sl(2)-representations are orthogonal.

- Combining now the individual results above, we can give the following very conservative estimate for the tadpole charge

$$Q > 0.7 n_{\rm stab} \,, \tag{6.7}$$

which is in agreement with the refined version of the Tadpole Conjecture, eq. (2.8) (i.e. $\alpha > 1/3$).

7 Conclusions

In this work we have studied the tadpole conjecture in asymptotic regions of complex structure moduli space by using the powerful tools of asymptotic Hodge theory. Our analysis was carried out in F-theory compactifications on Calabi-Yau four-folds in which the complex structure moduli are stabilized by a self-dual four-form flux G_4. We were able to give general evidence that the scaling of the tadpole with the number of stabilized directions is eminent if the moduli are stabilized in the strict asymptotic regions. These regions are close to the boundaries of the moduli space at which the Calabi-Yau space degenerates and one can additionally establish a hierarchy among the moduli values. Let us stress that apart from this hierarchy constraint our analysis was neither restricted to specific examples nor to specific asymptotic limits, such as the large complex structure limit. This generality puts much weight on the collected evidence and one can hope that the presented arguments are a first step to prove the conjecture in full generality.

Our approach to the tadpole conjecture relied on the remarkable fact that in the strict asymptotic regimes the (p,q)-decomposition of the middle cohomology splits into representations of an $\mathrm{sl}(2)^n$ algebra with commuting factors. Here the n refers to the number of fields pushed to the asymptotic regime, which by assumption are the fields that we aim to stabilize in a vacuum. Switching on fluxes in the sl(2) eigenspaces we found that the commutativity of the sl(2)s reduces the generally complicated and coupled system of vacuum equations into a set of constraints that can be analyzed systematically. In particular, we showed that the Calabi-Yau condition ensures that there is maximally a single sl(2)-representation with weights reaching from -4 to 4. All other sl(2)-representations have a maximal weight 2 and can formally be identified with sl(2)-representations found for Calabi-Yau two-folds (K3 surfaces). For a large number n of fields stabilized in the asymptotic regime we thus found that merely these K3-type representations are relevant for stabilization. This has led to additional constraints that ensure, for example, that when stabilizing $n_{\rm stab}$ scalars the tadpole always admits order $n_{\rm stab}$ terms that grow in the vacuum expectation values of these fields. These findings give an explanation of the requirement in the tadpole conjecture to consider a large number of stabilized moduli. In fact, when considering only few moduli, the larger sl(2)-representations stemming from the existence of a (4,0)-form, and not being related to K3-representations, can be used to stabilize moduli and escape some of the stringent constraints. While at first counter-intuitive, we now see that as soon as we consider a large number of fields, we encounter more structure and constraints. We believe that this feature persists when including corrections.

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Before turning to a brief discussion of the corrections to the strict sl(2)-splitting, let us stress an important subtle aspect that needs further clarification within our approach. It is known that the sl(2)-splitting of the flux cannot be generally performed over the integers but requires to use rational numbers. We have shown that in concrete examples the denominators do not scale with the number of moduli, but note that an abstract analysis of the largest occurring denominator would require a more sophisticated mathematical argument which goes beyond the scope of this work. It would be desirable to address this question together with an in-detail analysis of moduli stabilization of axions ϕ^i. The relation of these two issues arises from the fact that we have implemented the sl(2)-approximation also in the axion sector and worked with the axion-dependent flux $\hat{G}_4$. The latter links axion monodromies and fluxes and we expect additional constraints to arise from the integral quantization of fluxes. This highlights an important direction for future work.[15]

The analysis of this paper concerns the moduli that are lifted in the strict asymptotic regime, for which we showed that whenever n_{stab} moduli are stabilized by the leading contributions (i.e. ignoring the corrections), the tadpole grows linearly in the number of stablized moduli. An interesting challenge for future work is to extend our analysis to include corrections breaking the sl(2)-splitting and leave the strict asymptotic regime. In particular, one may want to keep the saxions s^i large but allow for a stabilization without a hierarchy among the s^i. In this case polynomial corrections in the ratios of the s^i will no longer be suppressed and play a major role in moduli stabilization. We expect that as one gets closer to the boundaries, the masses of moduli which are not stabilized in the strict asymptotic regime are asymptotically vanishing compared to those of moduli which are stabilized in the strict asymptotic regime. For families of vacua close to the boundaries, the tadpole should then grow (at least) linearly with the number of moduli that remain stabilized at a hierarchically higher mass than the rest. Asymptotic Hodge theory provides powerful tools to systematically include such corrections (see, e.g. [26]). In the multi-moduli case, however, we expect that such an analysis will quickly get very involved. Nevertheless, such an extension will be essential to give a definite answer about the validity of the tadpole conjecture in asymptotic regimes. This extended analysis would then also cover the linear scenario presented in [41, 42], which might pose a challenge to this conjecture.[16] An important feature of this stabilization scenario is that in the associated sl(2)-approximation a flat direction remains that then gets lifted after including corrections.

Let us also comment on the prospects of studying moduli stabilization and the tadpole conjecture in full generality. At first, one would expect that moduli stabilization is generic in this case, since even switching on a single flux quantum results in a highly non-trivial self-duality condition with polynomial and exponentially suppressed terms. This complexity is, however, deceiving when incorporating the results of the famous theorem about Hodge loci by Cattani, Deligne, and Kaplan [24]. Concretely, one can use this mathematical result to conclude that the locus in complex structure moduli space at which the flux

[15] In particular, let us stress that this quantization issue has appeared before in [40] in the analysis of the distance conjecture and weak gravity conjecture.

[16] Note that it has recently been argued in [10, 43] (see also [44]) that there is no clear structural reason to expect a counter example from the linear scenario.

G_4 is of type $(2,2)$ is actually given by algebraic equations. In other words, upon choosing appropriate coordinates, the moduli stabilization conditions are simply vanishing conditions on polynomials. While it is not known how the dimension of these spaces grows with the tadpole, it is conceivable from our asymptotic analysis that there is in fact a scaling with the number of moduli and it would be very interesting to prove such a scaling. Also allowing for a $(4,0)+(0,4)$ piece will, in general, destroy the algebraicity of the vacuum locus [23]. However, note that this generalization for a Calabi-Yau fourfold results in only a single complex equation independent of the number of moduli. This might indicate why in cases with few moduli the tadpole scaling can be violated while eventually it will be generally present when studying the stabilization of a large number of fields.

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A Asymptotic Hodge structures

In this appendix we provide some background on the algebraic structures that underly strict asymptotic regimes: the sl(2)-decomposition and the boundary Hodge decomposition. In particular, we argue for the decomposition of the highest-weight states as described by (3.21) given in the main text.

Pure Hodge structure. Let us first recall the pure Hodge structure that lives in the strict asymptotic regime given by (3.27). It is described by a Hodge decomposition of the primitive cohomology $H^4_{\text{prim}}(Y_4)$ into (p,q)-form pieces as

$$H^4_{\text{prim}}(Y_4) = \bigoplus_{p+q=4} H^{p,q}_{\text{sl}(2)}\,, \tag{A.1}$$

where $\bar{H}^{p,q}_{\text{sl}(2)} = H^{q,p}_{\text{sl}(2)}$. In order to make the underlying boundary structures more precise, it is helpful to recast this splitting in terms of a so-called Hodge filtration F^p. These vector spaces collect the (p,q)-eigenspaces of the Hodge decomposition as

$$F^p_{\text{sl}(2)} = \sum_{q\geq p} H^{q,4-q}_{\text{sl}(2)}\,. \tag{A.2}$$

This filtration varies holomorphically in the complex structure moduli t^i. By the sl(2)-approximation it can be described as

$$F^p_{\text{sl}(2)}(t) = e^{t^i N_i^-} F^p_{(n)}\,, \tag{A.3}$$

where the limiting filtration $F^p_{(n)}$ only depends on moduli not taken to the boundary. For later reference, let us note that we can generate n other limiting filtrations $F^p_{(k)}$ through the recursion relation

$$F^p_{(k)} = e^{iN^-_{k+1}} F^p_{(k+1)} . \tag{A.4}$$

Mixed Hodge structure. Let us point out that the filtration $F^p_{(0)}$ obtained in this way is precisely the Hodge decomposition associated to the boundary described by (3.14). For the other limiting filtrations there generically is no notion of a pure Hodge structure; instead these give rise to mixed Hodge structures, which can be made precise through Deligne splittings. These first require us to introduce monodromy weight filtrations W_ℓ, which are vector spaces constructed out of the kernels and images of the lowering operators as

$$W_\ell(N) = \sum_{j \geq \max(-1,\ell-4)} \ker N^{j+1} \cap \operatorname{img} N^{j-\ell+4} , \qquad N = N^-_{(k)} = N^-_1 + \dots N^-_k , \tag{A.5}$$

with $N^-_{(0)} = 0$. The Deligne splitting describing the mixed Hodge structure at step k is then given by

$$I^{p,q}_{(k)} = F^p_{(k)} \cap \overline{F}^q_{(k)} \cap W_{p+q}(N^-_{(k)}) . \tag{A.6}$$

We stress that we already specialized to the sl(2)-splitting from the beginning in this paper, and in general other pieces have to be taken into account in this intersection of vector spaces altering the conjugation property $\overline{I}^{p,q} = I^{q,p}$. The weight operators of the sl(2)-triples are then understood as multiplying an element by its row $p+q$ as

$$v_{p,q} \in I^{p,q}_{(k)} : \quad N^0_{(k)} v_{p,q} = (p+q-4) v_{p,q} . \tag{A.7}$$

Of special importance to us are the highest-weight components under the lowering operators $N^-_{(k)}$, which can be defined as

$$P^{p,q}_{(k)} = I^{p,q}_{(k)} \cap \ker[(N^-_{(k)})^{p+q-3}] . \tag{A.8}$$

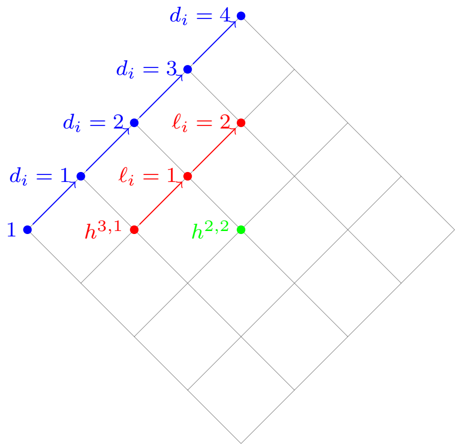
The Deligne splitting can then be recovered from the highest-weight subspaces and their descendants as

$$I^{p,q}_{(k)} = \bigoplus_r (N^-_{(k)})^k P^{p+r,q+r}_{(k)} . \tag{A.9}$$

Allowed weights. Having introduced the Deligne splittings and their highest-weight decompositions, we are finally in the position to look more closely at the allowed weights for highest-weight states, i.e. the splitting into the $(4,0)$-form part and K3 Hodge structures given in (3.23). To this end, the crucial relation between the highest-weight subspaces is given by

$$e^{iN^-_{k+1}} P^{p,q}_{(k+1)} \subseteq P^{p,q-\ell_{k+1}+\ell_k}_{(k)} , \tag{A.10}$$

assuming $p \geq q$. Pictorially this relation can be interpreted as follows: a highest-weight state starting at position (p,q) in the Deligne diamond can only end up in another position (p,q') with $q' \geq q$ and $q' \leq p$. In other words, only diagonal displacements to right-above are allowed, and never across the middle. This rule is rather abstract, so let us draw it explicitly for the Deligne diamond of a Calabi-Yau fourfold



(A.11)

We indicated the (up to rescaling) unique state starting in $I^{4,0}_{(0)}$ — corresponding to the leading term $\tilde{a}_0$ of the $(4,0)$-form periods — in blue, while the remainder of the diagram has been highlighted in red and green. Note also that, in principle, a highest weight state is allowed to skip in-between steps, e.g. it can move directly from $d_1 = 1$ to $d_2 = 4$. From this decomposition we see that: (1) there is one state corresponding to the outer blue part of the diamond, (2) there are $h^{3,1}$ states that can move up at most two steps in red, and (3) the remainder of the highest-weight states in green is fixed at $I^{2,2}$. This matches precisely with the allowed indices given in (3.21).

B Bases of sl(2)-representations and $\star_\infty$ action

We include here some useful explicit expressions and properties the basis vectors for the subspaces $V_{\boldsymbol{\ell}}$, including the explicit action of $\star_\infty$ and the lowering operators. Recall that the pairing $\langle \cdot , \cdot \rangle$ fulfills

$$\langle \, \cdot \, , N_i^- \, \cdot \, \rangle = -\langle N_i^- \, \cdot \, , \, \cdot \, \rangle \, . \tag{B.1}$$

The space P_0. The space P_0 is formed by $P_0^{3,1}$, $P_0^{1,3}$, and $P_0^{2,2}$, according to their decomposition in the last Hodge-Deligne diamond. All of them are annihilated by all the N_i^-. The former two form a (real) two-dimensional space, $P_0^{3,1} \oplus P_0^{1,3}$. A real basis can be obtained from the real and imaginary parts of the complex basis vector in $P_0^{3,1}$, which we denote

$$v_0^R \, , \qquad v_0^I , \tag{B.2}$$

with the non-vanishing pairings

$$\langle v_0^R, v_0^R \rangle = -1 \, , \qquad \langle v_0^I, v_0^I \rangle = -1 \, . \tag{B.3}$$

The action of $\star_\infty$ is

$$\star_\infty \, v_0^R = -v_0^R \, , \qquad \star_\infty v_0^I = -v_0^I \, . \tag{B.4}$$

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The space $P_0^{2,2}$ is one dimensional, with the pairing and the action of the boundary Hodge star on the basis vector given by

$$\langle v_0^{(2,2)}, v_0^{(2,2)} \rangle = 1\,, \qquad \star_\infty v_0^{(2,2)} = v_0^{(2,2)}\,. \tag{B.5}$$

The spaces generated by P_{01_i}. Each of these spaces is a complex space of dimension two, since it includes both the $P_{01_i}^{3,2}$ and $P_{01_i}^{2,3}$ in the last Hodge-Deligne diagram associated to the real splitting. It consists of V_{01_i} and V_{0-1_i}, each of them of real dimension two. We can define a real basis by taking the real and imaginary parts of the complex vector that sits in $P_{01_i}^{3,2}$ and applying N_i^- to each of them separately.

$$v_{01_i}^R\,, \qquad v_{01_i}^I\,, \qquad v_{0-1_i}^R = N_i^- v_{01_i}^R \qquad v_{0-1_i}^I = N_i^- v_{0-1_i}^I\,, \tag{B.6}$$

with the following non-vanishing pairings

$$\langle v_{01_i}^R, v_{0-1_i}^I \rangle = 1\,, \qquad \langle v_{01_i}^I, v_{0-1_i}^R \rangle = -1\,. \tag{B.7}$$

The action of $\star_\infty$ on the basis vectors is

$$\begin{aligned} \star_\infty v_{01_i}^R &= +v_{0-1_i}^I\,, \quad \star_\infty v_{0-1_i}^I = +v_{01_i}^R\,, \\ \star_\infty v_{01_i}^I &= -v_{0-1_i}^R\,, \quad \star_\infty v_{0-1_i}^R = -v_{01_i}^I\,. \end{aligned} \tag{B.8}$$

The spaces generated by P_{02_i}. Each of these spaces has (real) dimension three, and it consists on $V_{02_i} \oplus N_i^- P_{02_i} \oplus V_{0-2_i}$ as defined in eq. (3.38), where we recall that $N_i^- P_{02_i}$ includes the part of V_0 that is not highest-weight. We introduce the following basis vector for each of these $V_{\boldsymbol{\ell}}$, respectively

$$v_{02_i}\,, \qquad v_0 = N_i^- v_{02_i}\,, \qquad v_{0-2_i} = (N_i^-)^2 v_{02_i}\,. \tag{B.9}$$

Their non-zero inner products are

$$\langle v_{02_i}, v_{0-2_i} \rangle = 1\,, \qquad \langle v_0, v_0 \rangle = -1\,, \tag{B.10}$$

and the action of $\star_\infty$ on each of them takes the form

$$\star_\infty\, v_{02_i} = \frac{1}{2} v_{0-2_i}\,, \qquad \star_\infty v_0 = -v_0\,, \qquad \star_\infty v_{0-2_i} = 2\, v_{02_i}\,. \tag{B.11}$$

The spaces generated by $P_{01_i 2_j}$. In this case, each of these subspaces has (real) dimension four, and it consists of $V_{01_i 2_j} \oplus V_{01_i 0_j} \oplus V_{0-1_i 0_j} \oplus V_{0-1_i -2j}$ (cf. eq. (3.38)). Introducing the basis vectors

$$v_{01_i 2_j}\,, \quad v_{01_i 0_j} = N_j^- v_{01_i 2_j}\,, \quad v_{0-1_i 0_j} = N_i^- v_{01_i 2_j}\,, \quad v_{0-1_i -2j} = N_i^-\, N_j^- v_{01_i 2_j}\,, \tag{B.12}$$

we obtain the following non-vanishing pairings

$$\langle v_{01_i 2_j}, v_{0-1_i -2_j} \rangle = 1\,, \qquad \langle v_{01_i 0_j}, v_{0-1_i 0_j} \rangle = -1\,. \tag{B.13}$$

Finally, the $\star_\infty$ operator acts as

$$\begin{aligned} \star_\infty v_{01_i 2_j} &= +v_{0-1_i -2_j}\,, \quad \star_\infty v_{0-1_i -2_j} = +v_{01_i 2_j}\,, \\ \star_\infty v_{01_i 0_j} &= -v_{0-1_i 0_j}\,, \quad \star_\infty v_{0-1_i 0_j} = -v_{01_i 0_j}\,. \end{aligned} \tag{B.14}$$

C Weak coupling-conifold example

In this appendix we consider a simple two-moduli example, where one complex structure modulus is sent to weak coupling and the other towards a conifold point. The first corresponds to Sen's limit [45] and this modulus can be understood as the axio-dilaton of Type IIB; the second is the conifold modulus of the Calabi-Yau threefold in this setup. This example highlights that asymptotic Hodge theory can be applied near any boundary in complex structure moduli space, allowing us to probe regions away from large complex structure. More concretely, it illustrates how the sl(2)-decomposition splits into representations associated to the (4,0)-form and a K3 subblock. This example demonstrates the interplay between fluxes in different representations in the stabilization of complex structure moduli.

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Period vector data. We begin by describing the period vector of the (4,0)-form near such a weak coupling-conifold point. Let us point out that we do not refer to any explicit geometrical example here, but merely use the typical form of the periods in such a regime. The leading part of the period vector takes the form

$$\begin{aligned}\Pi = \bigg(&1+\frac{a^2z^2}{8\pi},\ az,\ i-\frac{ia^2z^2}{8\pi},\ \frac{iaz(\log(z)-1)}{2\pi},\\ &\tau\Big(1+\frac{a^2z^2}{8\pi}\Big),\ a\tau z,\ \tau\Big(i-\frac{ia^2z^2}{8\pi}\Big),\ \frac{ia\tau z(\log(z)-1)}{2\pi}\bigg)\,,\end{aligned} \tag{C.1}$$

where $0\neq a\in\mathbb{R}$, with the conventions for the conifold periods used in [28]. The axio-dilaton is denoted by $\tau = c+is$, and the conifold modulus by $z=e^{2\pi i t}=e^{2\pi i(x+iy)}$. The period vector can be brought to the standard form of the nilpotent orbit approximation as[17]

$$\Pi = e^{\tau N_1 + tN_2}(a_{00}+e^{2\pi i t}a_{01}+e^{4\pi i t}a_{02})\,, \tag{C.2}$$

where the log-monodromy matrices N_i are given by

$$N_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0 \end{pmatrix}, \tag{C.3}$$

and the terms a_{ij} in the expansion of the period vector read

$$\begin{aligned} &a_{00} = (1,\ 0,\ i,\ 0,\ 0,\ 0,\ 0,\ 0)\,, \quad a_{01} = a\Big(0,\ 1,\ 0,\ -\frac{i}{2\pi},\ 0,\ 0,\ 0,\ 0\Big)\,,\\ &a_{02} = \frac{a^2}{4\pi}(1,\ 0,\ -i,\ 0,\ 0,\ 0,\ 0,\ 0)\,. \end{aligned} \tag{C.4}$$

[17] The terms proportional to a_{01} and a_{02} are the exponential corrections to the leading order term, that we did not display in the main text (see equation (3.9)). These are essential for computing a non-degenerate Hodge star.

For later reference, let us write down the expression for the bilinear pairing defined in (2.2)

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{C.5}$$

Given this data one has to compute the sl(2) approximation (see details in [9, 10]). The lowering operators $N_1^- = N_1$ and $N_2^- = N_2$ in (C.3) are completed into sl(2)-triples by weight operators

$$N_1^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad N_2^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{C.6}$$

which indeed fulfill the standard commutation relations (3.5). For our purposes we will not need the raising operators, but in principle one could obtain these by solving the remaining commutation relations.

Highest-weight states and Hodge decompositions. Having set up the period vector and sl(2)-algebras, we next study the decomposition of the middle cohomology $H^4_{\mathrm{prim}}(Y_4)$ in the strict asymptotic regime under the sl(2)-approximation. We begin with the highest-weight states (3.11) of the sl(2)-representations. Two highest-weight states are obtained from a_0 and its conjugate, which have weights $\ell = (1,1)$ under N_1^0 and $N_1^0 + N_2^0$. The remaining highest-weight state is given by $a_{01} + \frac{i}{2\pi} N_2 a_{01}$, which has weights $\ell = (1,2)$. Altogether we find

$$\begin{aligned} P^{4,1}_{11}: &\quad (1,\ 0,\ i,\ 0,\ 0,\ 0,\ 0,\ 0)\,, \\ P^{3,3}_{12}: &\quad (0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\,, \\ P^{1,4}_{11}: &\quad (1,\ 0,\ -i,\ 0,\ 0,\ 0,\ 0,\ 0)\,. \end{aligned} \tag{C.7}$$

These sl(2)-representations are completed by considering descendants under the action of N_1, N_2, which are given by

$$\begin{aligned} N_1 P^{4,1}_{11} \subset V_{-1-1}: &\quad (0,\ 0,\ 0,\ 0,\ 1,\ 0,\ +i,\ 0)\,, \\ N_1 P^{1,4}_{11} \subset V_{-1-1}: &\quad (0,\ 0,\ 0,\ 0,\ 1,\ 0,\ -i,\ 0)\,, \\ N_1 P^{3,3}_{12} \subset V_{-10}: &\quad (0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0)\,, \\ N_2 P^{3,3}_{12} \subset V_{10}: &\quad (0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)\,, \\ N_1 N_2 P^{3,3}_{12} \subset V_{-1-2}: &\quad (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1)\,. \end{aligned} \tag{C.8}$$

Recalling the splitting (3.23) of the sl(2)-representations we then decompose the middle cohomology into two parts as

$$H^4_{\mathrm{prim}}(Y_4,\mathbb{C}) = H_\Omega \oplus H_{\mathrm{K3}}\,, \tag{C.9}$$

where we defined the terms

$$\begin{aligned} H_\Omega &= P^{4,1}_{11} \oplus N_1 P^{4,1}_{11} \oplus P^{1,4}_{11} \oplus N_1 P^{1,4}_{11}\,, \\ H_{\mathrm{K3}} &= P^{3,3}_{12} \oplus N_1 P^{3,3}_{12} \oplus N_2 P^{3,3}_{12} \oplus N_1 N_2 P^{3,3}_{12}\,. \end{aligned} \tag{C.10}$$

Note that H_Ω is spanned by vectors with non-vanishing entries in odd positions, while H_{K3} is spanned by vectors with non-vanishing entries in even positions.

We next consider the Hodge decomposition of these individual terms H_Ω and H_{K3} in the strict asymptotic regime. As Hodge structure on the $(4,0)$-form representations we find

$$H_\Omega = H^{4,0}_\Omega \oplus H^{3,1}_\Omega \oplus H^{1,3}_\Omega \oplus H^{0,4}_\Omega\,, \tag{C.11}$$

with subspaces spanned by

$$\begin{aligned} H^{4,0}_\Omega:&\quad (1,\ 0,\ i,\ 0,\ c+is,\ 0,\ i(c+is),\ 0)\,, \\ H^{3,1}_\Omega:&\quad (1,\ 0,\ i,\ 0,\ c-is,\ 0,\ i(c-is),\ 0)\,. \end{aligned} \tag{C.12}$$

and the others determined by complex conjugation. The $(4,0)$-form subspace is straightforwardly determined as $H^{4,0}_\Omega = e^{t^i N_i} P^{4,1}_{11}$ from the highest-weight subspace according to (3.29); the $(3,1)$-form subspace is spanned by the linear combination of $\Pi_{\mathrm{sl}(2)}$ and $\partial_\tau \Pi_{\mathrm{sl}(2)}$ that is orthogonal to $\overline{\Pi}_{\mathrm{sl}(2)}$ under the bilinear pairing (which is precisely the one picked by the Kähler covariant derivative). The other part of the cohomology corresponds to the K3 block: its Hodge decomposition takes the form

$$H_{\mathrm{K3}} = (H_{\mathrm{K3}})^{3,1} \oplus (H_{\mathrm{K3}})^{2,2} \oplus (H_{\mathrm{K3}})^{1,3}\,, \tag{C.13}$$

with subspaces spanned by

$$\begin{aligned} (H_{\mathrm{K3}})^{3,1}:&\quad (0,\ 1,\ 0,\ -(x+iy),\ 0,\ c+is,\ 0,\ -(x+iy)(c+is))\,, \\ (H_{\mathrm{K3}})^{2,2}:&\quad (0,\ y,\ 0,\ 0,\ 0,\ cy+sx,\ 0,\ -s(x^2+y^2))\,, \\ &\quad (0,\ 0,\ 0,\ y,\ 0,\ s,\ 0,\ cy-sx)\,, \end{aligned} \tag{C.14}$$

and the $(1,3)$-form subspace fixed by complex conjugation. Now the $(3,1)$-form subspace is straightforwardly determined as $(H_{\mathrm{K3}})^{3,1} = e^{t^i N_i} P^{3,3}_{12}$ from the highest-weight subspace according to (3.29); the $(2,2)$-subspace is determined as the part of H_{K3} which is orthogonal to the $(3,1)$- and $(1,3)$-subspaces under the bilinear pairing. Note that in this example where the number of moduli is not large, the subspaces H_Ω and H_{K3} have roughly the same dimensions, while for large moduli the dimension of H_Ω is $\mathcal{O}(1)$ while the dimension of H_{K3} is $\mathcal{O}(n)$.

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Putting the above two Hodge decompositions (C.12) and (C.14) together, we can write the sl(2)-approximated Hodge star operator of the middle cohomology $H^4_{\text{prim}}(Y_4, \mathbb{C})$ as

$$\star_{\text{sl}(2)} = \begin{pmatrix} 0 & 0 & \frac{c}{s} & 0 & 0 & 0 & -\frac{1}{s} & 0 \\ 0 & \frac{cx}{sy} & 0 & \frac{c}{sy} & 0 & -\frac{x}{sy} & 0 & -\frac{1}{sy} \\ -\frac{c}{s} & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 \\ 0 & -\frac{c(x^2+y^2)}{sy} & 0 & -\frac{cx}{sy} & 0 & \frac{x^2+y^2}{sy} & 0 & \frac{x}{sy} \\ 0 & 0 & \frac{c^2}{s}+s & 0 & 0 & 0 & -\frac{c}{s} & 0 \\ 0 & \frac{x(c^2+s^2)}{sy} & 0 & \frac{c^2+s^2}{sy} & 0 & -\frac{cx}{sy} & 0 & -\frac{c}{sy} \\ -\frac{c^2+s^2}{s} & 0 & 0 & 0 & \frac{c}{s} & 0 & 0 & 0 \\ 0 & -\frac{(c^2+s^2)(x^2+y^2)}{sy} & 0 & -\frac{x(c^2+s^2)}{sy} & 0 & \frac{c(x^2+y^2)}{sy} & 0 & \frac{cx}{sy} \end{pmatrix} . \tag{C.15}$$

One can verify straightforwardly that $\star_{\text{sl}(2)}$ acts as $(-1)^{(p-q)/2}$ on elements of the subspaces $(H_{\text{K3}})^{p,q}$ and $H^{p,q}_{\Omega}$.

Self-duality conditions and flux vacua. Having constructed the sl(2)-approximation and corresponding Hodge star operator, we next study the self-duality condition for the fluxes in the strict asymptotic regime. We treat H_Ω of the $(4,0)$-form and the K3 block H_{K3} individually. Let us denote the four-form flux by a vector as $G_4 = (h_1, h_2, h_3, h_4, f_1, f_2, f_3, f_4)$. The self-duality condition on the subspace H_Ω then reads

$$-f_3 + ch_3 - h_1 s = 0 \,, \qquad f_1 - ch_1 - h_3 s = 0 \,. \tag{C.16}$$

Note that these constraints can equivalently be obtained by demanding orthogonality under the bilinear pairing with the $(3,1)$-form subspace given in (C.12), since the $(4,0)$-form subspace and its conjugate are self-dual. It is solved by

$$c = \frac{f_1 h_1 + f_3 h_3}{h_1^2 + h_3^2} \,, \qquad s = \frac{-f_3 h_1 + f_1 h_3}{h_1^2 + h_3^2} \,. \tag{C.17}$$

We next turn to the self-duality condition of the K3 subblock H_{K3}. Recall from (5.4) that it is most conveniently imposed by demanding orthogonality with the $(3,1)$-form subspace. Taking the $(3,1)$-form given in (C.14) we obtain as condition

$$(h_2 \tau - f_2)(h_2 t + h_4) = h_2 f_4 - f_2 h_4 \,. \tag{C.18}$$

The flux quanta here are identified with those in (5.15) as

$$G_{12} = h_2, \quad G_{-10} = f_2, \quad G_{10} = -h_4, \quad G_{-1-2} = -f_4 \,. \tag{C.19}$$

The solution to (C.18) is described by some simple complex function $t(\tau)$. In (5.20) this function parametrized the flat direction of the scalar potential; here these are lifted by the inclusion of fluxes in H_Ω, which fixes τ by (C.17), and thus also t.

D Tadpole contribution and flux quantization

In this appendix we investigate how the quantization of fluxes affects the tadpole for a few examples. To be precise, we consider Calabi-Yau three-fold geometries near the large

	$h^{1,1}$	n_{gen}	n_{stab}	heavy V_ℓ	$\|G_\ell\|^2_\infty \geq \frac{1}{4}$
example 1	70	53	20	21	15
example 2	75	65	20	21	19
example 3	100	47	24	25	21
example 4	100	67	22	23	18

Table 1. In this table we summarize the statistics of our study of geometries with large $h^{1,1}$: the number n_{gen} indicates the number of generators we managed to compute for the Kähler cone; the number n_{stab} gives the number of moduli in which the sl(2)-approximated Hodge star $\star_{\text{sl}(2)}$ varies in the strict asymptotic regime we chose; the next column specifies the number of sl(2)-eigenspaces V_ℓ which are heavy asymptotically; the last column indicates how many out of these heavy V_ℓ have a boundary Hodge norm bounded from below by at least 1/4.

complex structure regime with large $h^{2,1}$.[18] By using mirror symmetry we compute the relevant data with CYTools [46].

Quantization of sl(2)-eigenspaces. Let us first recall the form of the tadpole in strict asymptotic regimes as given in (6.3). We want to identify the relevant quantity to check for flux quantization, and make sure that there are no pieces that can scale inversely with the number of moduli. We decomposed the tadpole contribution of the fluxes as

$$\langle G_4, G_4 \rangle \geq 2 \sum_{\ell > 0} \gamma^{\sum \ell_i} \|\hat{G}_\ell\|^2_\infty \,. \tag{D.1}$$

This sum runs over at least as many terms as the number of stabilized moduli n_{stab}. Therefore, in order for the tadpole to grow linearly with n_{stab}, we have to require that each of these terms is order one. The strict asymptotic regime already requires $\gamma \gg 1$, so the problem at hand reduces to checking whether the boundary norm $\|\hat{G}_\ell\|^2_\infty$ defined in (3.31) is of order one. We explicitly derive lower bounds for these coefficients for each of the heavy sl(2)-eigenspaces V_ℓ in a few three-fold examples, given in table 1.

Details of the computation. We now provide some details of how the calculation above was implemented.

- We compute the relevant topological data of the examples using CYTools. We take four examples at various large values of $h^{1,1}$ in the Kreuzer-Skarke dataset [47]. The intersection numbers of a given geometry are efficiently computed using this program, as well as the generators of the Mori cone.[19] However, for our purposes we need to

[18] For simplicity we use Calabi-Yau three-folds but we do not expect the quantization issues to be very different in four-folds, in particular the absence of inverse scaling with the number of moduli, as we will show.

[19] To be more precise, let us note that it is much easier in practice to compute the Mori and Kähler cone for the ambient space rather than the Calabi-Yau manifold itself. The Kähler cone of the ambient space is in general contained in the Kähler cone of the Calabi-Yau manifold, so we thereby restrict our attention to a smaller portion of the moduli space. We refer to [48–50] for more details on constructions of Calabi-Yau hypersurfaces in toric ambient spaces.

know generators ω_i of the Kähler cone: in this basis we can expand the Kähler form as $J = t^i\omega_i$, where sending the saxion $\mathrm{Im}\, t^i = s^i$ to infinity corresponds to the large field limit to the boundary. The computational cost of dualizing a generic Mori cone to the Kähler cone scales exponentially with $h^{1,1}$, and for $h^{1,1} \gtrsim 20$ it becomes essentially impossible to compute all of these generators.

- Let us first briefly review some key aspects of dualizing a Mori cone $\mathcal{M}$ to the Kähler cone $\mathcal{K}$, and in particular highlight the complications that arise at large $h^{1,1}$. We write the generators of the Mori cone as $M_a = (M_{ai}) \in \mathcal{M}$, where a labels the generators and $i = 1, \ldots, h^{1,1}$ the components of this vector. A vector $K = (K_i)$ then lies in the dual Kähler cone $\mathcal{K}$ if it satisfies the conditions

$$M_a \cdot K = \sum_{i=1}^{h^{1,1}} M_{ai} K_i \geq 0\,, \tag{D.2}$$

  for all Mori cone generators M_a. Computing the dual of a simplicial cone (the number of generators M_a is equal to the dimension $h^{1,1}$ of the cone) can be performed very quickly, even at large dimensions. The complexity of the problem arises when the Mori cone is non-simplicial, i.e. the number of generators is larger than $h^{1,1}$. In practice, geometries in the Kreuzer-Skarke database at large $h^{1,1}$ have considerably larger numbers of generators, for instance at $h^{1,1} = 100$ there are about $150 - 200$ generators. One way the exponential scaling of the computational cost now becomes apparent is by scanning over all simplicial subcones of $\mathcal{M}$: each of these can be dualized efficiently, however, we have to consider roughly $\binom{150}{100}$ such subcones at $h^{1,1} = 100$.

- In this paper we therefore take a more pragmatic approach, and content ourselves with determining a subset of the Kähler cone generators. We consider only linearly independent generators, and denote the number of generators we find for our examples in the end by n_{gen}. Our method works as follows. We consider a fixed number of simplicial subcones $\mathcal{M}_{\text{sim}}$ of the Mori cone $\mathcal{M}$, of order 10^4, and dualize each of these individually to a simplicial cone $\mathcal{K}_{\text{sim}}$. Since the subcones we start from are smaller than the Mori cone $\mathcal{M}_{\text{sim}} \subset \mathcal{M}$, the resulting dual cones are larger than the Kähler cone $\mathcal{K} \subset \mathcal{K}_{\text{sim}}$. In other words, most of the generators of $\mathcal{K}_{\text{sim}}$ do not satisfy all conditions in (D.2). However, typically we do encounter some generators that satisfy (D.2), and we use precisely these rays as generators of the Kähler cone. In principle one could recover all generators of the Kähler cone $\mathcal{K}$ with this approach, however around $h^{1,1} \gtrsim 15$ the number of subcones $\mathcal{M}_{\text{sim}}$ is already too large. Nevertheless, by taking only a small subset of Mori subcones we have been able to determine a sizeable number of Kähler cone generators for a handful of examples.

- With the relevant geometric data of the Calabi-Yau hypersurface in hand — the intersection numbers and the Kähler cone generators — we proceed and study strict asymptotic regimes in the large volume limit. These regions are defined by the ordering of the saxions s^i that specify how far we move along the Kähler cone generators. For the procedure to construct the sl(2)-approximation in this strict asymptotic regime we refer to [10] for a pedagogical introduction. One of the main messages we want to convey here is that the resulting boundary Hodge star operator $\star_{\mathrm{sl}(2)}$ need not depend on all available moduli — there can be some trivial sl(2)-triples $(N_i^{\pm}, N_i^0) = 0$. By carefully selecting appropriate orderings of the saxions we were able to find sl(2)-approximations depending on about $n_{\text{stab}} \sim 20$ moduli in these moduli spaces with $h^{1,1} = 70 - 100$. Note that there are some technical limitations at play here: even for a simplicial Kähler cone one would have to consider $(h^{1,1}!)$ different orderings; moreover, we were not able to identify all Kähler cone generators but only a subset $n_{\text{gen}} < h^{1,1}$. We expect that a complete description of the full non-simplicial Kähler cones would enable us to identify asymptotic regimes with n_{stab} much closer to the actual number of moduli $h^{1,1}$.

- Given the sl(2)-approximation for a strict asymptotic regime, we proceed and derive lower bounds for the norms $\|G_\ell\|^2_\infty$. We consider real, quantized three-form fluxes $G \in H^3(Y_3, \mathbb{Z}) \cap V_{\text{heavy}}$ valued in the heavy sl(2)-eigenspaces (see (3.35)). We subsequently project G onto one of the sl(2)-eigenspaces as $G_\ell \in V_\ell \subset V_{\text{heavy}}$. Let us note that the sl(2)-splitting is generically only realized over the rationals, so for these individual components of the three-form flux G we have $G_\ell \in H^3(Y_3, \mathbb{Q})$. In order to obtain a lower bound on $\|G_\ell\|^2_\infty$, we rewrite it as a sum over squares of integer flux quanta: the smallest coefficient in front of these squares then gives a lower bound on the norm.

- Let us elaborate on this rewriting of $\|G_\ell\|^2_\infty$ for a moment. We take a normalized basis $v_{\ell,a} \in H^3(Y_3, \mathbb{R}) \cap V_\ell$ with $a = 1, \ldots, \dim V_\ell$. We represent the $2(h^{2,1}+2) \times \dim(V_\ell)$ matrix of basis vectors by $B_\ell = (v_{\ell,a})$, satisfying $B_\ell^T B_\ell = \mathbb{1}_{\dim(V_\ell)}$. Subsequently we can represent G_ℓ by a $\dim(V_\ell)$-component vector as $B_\ell^T \cdot G_\ell = (G_{\ell,a})$ such that $G_\ell = G_{\ell,a} v_{\ell,a}$. In this basis the boundary norm can be represented by a $\dim(V_\ell) \times \dim(V_\ell)$ matrix as
$$(M_\ell)_{ab} = \langle v_{\ell,a}, \star_\infty v_{\ell,b} \rangle \,. \tag{D.3}$$
The problem at hand then reduces to decomposing this matrix as $M_\ell = Q_\ell^T Q_\ell$ by a Cholesky decomposition, with Q_ℓ a $\dim(V_\ell) \times \dim(V_\ell)$ matrix: the boundary norm $\|G_\ell\|^2_\infty$ simplifies to the Euclidean norm of a vector as
$$\|G_\ell\|^2_\infty = |Q \cdot B_\ell^T \cdot G_\ell|^2 \,. \tag{D.4}$$
Expanding G in an integral basis as $G = (a_I, 0)$, where $I = 0, \ldots, h^{2,1}$ with a_I integral coefficients, we can then write out this Euclidean norm as
$$\|G_\ell\|^2_\infty = \sum_a b_a \Big(\sum_I n_{aI} a_I\Big)^2 \,, \tag{D.5}$$
with $n_{aI} \in \mathbb{Z}$ such that $\gcd_I(n_{aI}) = 1$ for each $a = 1, \ldots, \dim(V_\ell)$, and $b_a \in \mathbb{Q}_{>0}$. The lower bound is then given by
$$\|G_\ell\|^2_\infty \geq \min(b_a) \,. \tag{D.6}$$

- Let us demonstrate the above procedure on an example. We take the sl(2)-eigenspace $V_{01_{13}}$ of example 4 in table 1, which can be spanned by the unit vectors e_{86} and e_{87} with a 1 in the corresponding positions. The flux $G_{01_{13}}$ along this basis can be expanded as

$$G_{01_{13}} = \frac{1}{3}(-a_{36} - 2a_{84} + 3a_{86})e_{86} + \frac{1}{3}(-2a_{36} - a_{84} + 3a_{87})e_{87} \,. \tag{D.7}$$

In the basis e_{86}, e_{87} the Hodge norm reduces to

$$(M_{01_2 1_{10}})_{ab} = \begin{pmatrix} 14 & -7 \\ -7 & 14 \end{pmatrix}, \qquad Q = \begin{pmatrix} \sqrt{14} & -\frac{7}{2} \\ 0 & \frac{\sqrt{21}}{2} \end{pmatrix} \,. \tag{D.8}$$

The norm then simplifies to

$$\|G_{01_{13}}\|^2_\infty = \frac{7}{2}(a_{84} - 2a_{86} + a_{87})^2 + \frac{7}{6}(2a_{36} + a_{84} - 3a_{87})^2 \,, \tag{D.9}$$

which has as lower bound

$$\|G_{01_{13}}\|^2_\infty \geq \frac{7}{6} \,. \tag{D.10}$$

Note, however, that this particular bound is not easily saturated. For instance, by putting only $a_{84} = 1$ and all others to zero we also turn on the first square, in which case $\|G_{01_{13}}\|^2_\infty = 14/3$. This interplay between different squares arises for other sl(2)-eigenspaces as well. It even shows up between different boundary norms $\|G_\ell\|^2_\infty$, where saturating the lower bound for one coefficient results in another satisfying its bound marginally.

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