Developing Basic Principles of Calculus and Motion in Lower Secondary Education

Michiel Doorman, Rogier Bos, and Carolien Duijzer

1 Introduction

A common pitfall in the teaching and learning of calculus is to focus on algebraic manipulation—instead of conceptual understanding—as a result of the accessibility of the algebraic algorithms in contrast with the complexity of the involved concepts. Teachers' and students' efforts are mainly focused on students becoming *fluent* users of traditional calculus algorithms, while our current society also requires *critical* and *flexible* users. Meaning making and meaningful understanding are essential for students to be able to adapt algorithms—and other tools—to new situations.

A central representation in the mathematics of change is the graph of a quantity varying in relation to another quantity (e.g., velocity changing over time). Consequently, meaning making in this domain requires graph sense, which refers to the ability to interpret and construct graphs, to recognize the covariance of the represented variables, and to reason with slope and more general visual characteristics of the graph (Friel et al., 2001).

Situations where motion plays a role as a context (hereafter called motion contexts) have proven to be potentially rich in addressing the basic principles of calculus, for example, when modeling the relation between velocity and distance traveled. A didactical use of time–distance graphs, that represent distance changing over time, offers many opportunities for students to reason about these graphs, e.g., slope,

M. Doorman (✉) · R. Bos · C. Duijzer
Utrecht University, Utrecht, The Netherlands
e-mail: m.doorman@uu.nl

R. Bos
e-mail: r.d.bos@uu.nl

C. Duijzer
e-mail: c.duijzer@hsmarnix.nl

C. Michelsen et al. (eds.), *Mathematics and Its Connections to the Arts and Sciences (MACAS)*, Mathematics Education in the Digital Era 19,
https://doi.org/10.1007/978-3-031-10518-0_17

including reasoning about change and defining the relationship between the represented variables (Nemirovsky et al., 2013). Furthermore, the context of motion also offers ample opportunities to include direct motion experiences in learning environments (Nemirovsky et al., 1998), such as activities with analyzing changing speed of a toy car as developed in the ScienceMath project.[1] As a consequence, in the process of learning about the mathematics of change, the learning becomes rooted in experienced motion resulting in a better understanding of the relations between velocity, distance traveled, and acceleration. In sum, using rich contexts in mathematics education potentially stimulates context-related language, actions, and concepts.

Contexts in mathematics education can help to provide a sense of purpose to students and to root abstract concepts and procedures in solution strategies of meaningful problem situations. This use of meaningful contexts is advocated in Realistic Mathematics Education (RME). In this approach, these contexts are a didactical support for learning mathematics by evoking solution strategies that have the potential to introduce a new mathematical idea.

The contextual tasks provide information that can be organized mathematically and offer opportunities for students to use their knowledge and experiences. Moreover, a rich context allows for and elicits different approaches or solutions on different levels. A teacher using these approaches can focus on a more general process-related purpose, mathematically, than solving the problem (Freudenthal, 1991). When mathematical concepts and procedures emerge in a sequence of activities with a varying focus from situational solutions to generalized methods, the underlying process of teaching and learning is referred to as emergent modeling (Gravemeijer, 1999). Within the context of modeling motion, during carefully chosen activities, students are invited to create representations such as time–distance graphs, to discuss characteristics of the representations, to reason with them, and to generalize their use. In this process, more formal notations and language transform into conventional mathematical symbolizations and definitions (Doorman, 2019). In summary, we try to prevent that mathematics is experienced by students as adopting a set of pre-defined and isolated concepts and techniques. The use of contexts and emergent modeling in the RME approach aims at mathematics that is experienced by students as developing meaningful and relevant tools in activity.

When using motion contexts for learning calculus, students also learn about motion. Graphs are helpful tools that can connect the mathematics of change with kinematics (Michelsen, 1998). A context-rich learning trajectory around modeling motion activities needs to highlight differences between common mathematical language and ways of working, and those in physics (Boohan, 2016). For instance, in mathematics, the slope of a graph is usually approximated using a slope of a chord on that graph, while in physics lessons, the tangent to an s-t graph is often used to approximate speed at a certain point in time (Fig. 1).

[1] Toy car acceleration, retrieved from the ScienceMath-project https://sciencemath2030.wordpress.com/.

https://sciencemath2030.files.wordpress.com/2018/08/smallcar_en_material.pdf.

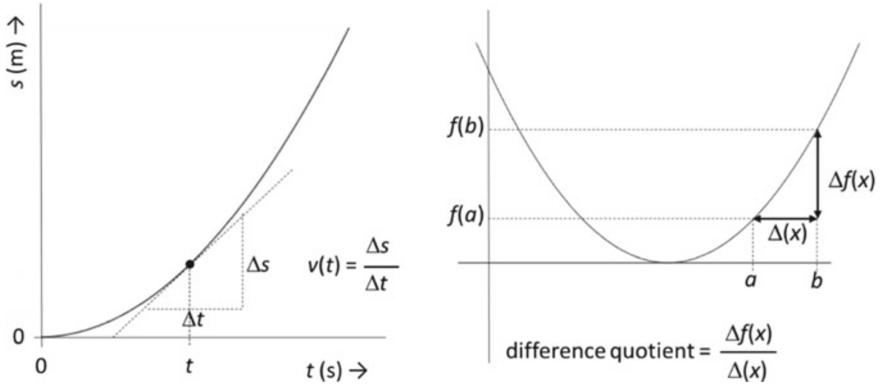

Fig. 1 Typical physics picture on the left and mathematics picture on the right

Table 1 Modeling change in a physics and a mathematics textbook

	Physics textbook	Mathematics textbook
Notation	$\Delta s/\Delta t$	$\Delta f(x)/\Delta(x)$
Language	Uniform motion, diagram	Difference quotient, chord in a graph
Way of working	Sketch tangent and select Δs and Δt at random and calculate quotient	Select a and then select b nearby, visualize the chord connecting (a,f(a)) and (b,f(b)) and calculate difference quotient

Although the two approaches of change in physics and mathematics textbooks are closely related, they vary in notations, language, and way of working (see Table 1). Moreover, in mathematics you would probably talk about a *t*-*s* graph instead of a *s*-*t* diagram, because the distinction between independent and dependent variables is important. In addition, both subjects have different conventions in the use of units at the axes and allowing negative values.

This example shows how these different conventions might limit possibilities for transfer, i.e., the recognition and awareness of similarities of affordances in both situations (Greeno et al., 1993; van Helden & Shvarts, n.d.).

In this paper, we discuss three cases that explore the potential of motion contexts for developing basic principles of calculus. The first example connects patterns in change with the relation between velocity and distance traveled in uniform and accelerated motion. In the second example, learning about tangent lines and slope is based on the design of a slide. The third and final example addresses the interpretation and construction of motion graphs while moving oneself in front of a motion sensor. With these three examples, we illustrate the value of motion contexts and the potential for involving students in the development of formal mathematical models with the aim to better understand the role of shared and specific representations, language, and ways of working in mathematics and science.

2 Example 1: Patterns in Change

This example is based on an approach to modeling motion for learning to understand patterns in change that was presented at MACAS2 (Doorman & Gravemeijer, 2008). The approach is inspired by the potential of students' constructions of graphs when supported with interactive video (Boyd & Rubin, 1996). The video showed the motion of a cat, and students were asked to describe this motion graphically. It appeared that they did not spontaneously construct a two-dimensional graph with a horizontal time axis and a vertical distance axis. This is surprising since their prior education included a lot of two-dimensional graphing activities. In earlier studies with more explicit questions in this direction, many students constructed graphs with a horizontal position axis and a vertical time axis. This choice was probably inspired by metaphoric resemblance between a horizontal distance that needs to be covered and a horizontal axis in the graph. The intervention that helped students to invent more conventional ways of graphing motion stimulated them to first create one-dimensional trace graphs to describe the varying velocity of a ball that was slowing down. The video of the ball focused attention on the change in the interval lengths in successive time intervals. These interval lengths appeared to be basic structuring elements of both the motion and its representation.

In line with this approach, we introduced weather forecasts and stroboscopic pictures of hurricanes and a falling ball (Fig. 2).

The main idea of this task is to invite students to focus on the change in the interval lengths of distances covered in equal time intervals. The prediction task is expected to create the need to get a better view of these changing lengths by positioning them next to each other in so-called interval graphs. The patterns in the interval graphs help to reason about changing velocity and to better predict distances covered when extrapolating the related graphs of distances traveled. In a 9th grade classroom that worked on the hurricane task, a student predicted the moment of reaching land by

The teacher discusses the change of position of a hurricane with students: when will it reach land?

In the picture you see traced and predicted successive positions of a hurricane (12h between positions)

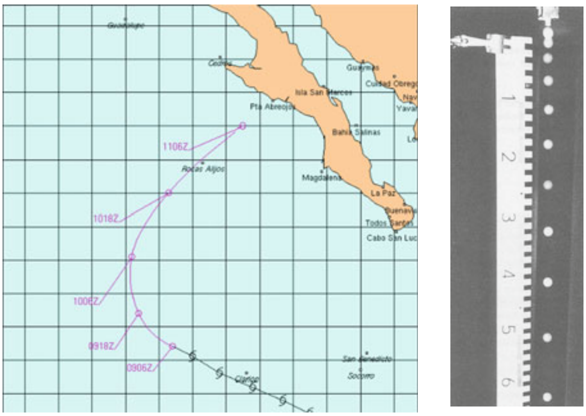

Fig. 2 Tracing a hurricane and a falling ball

extending the final displacement. The following discussion between the teacher and one of the students evolved:

> Teacher: What has she assumed?
>
> Tim: That the conditions are not going to change.
>
> Teacher: Exactly, she assumed that what happened during the last 12 hours will continue in exactly the same way. Is that probable?
>
> Tim: No.
>
> Teacher: Okay, who did not assume that the velocity would change? Suzanne?
>
> Suzanne: I looked every 12 hours, and then the intervals increased by half a centimetre.
>
> Teacher: Okay, you repeatedly add 0.5. And then?
>
> Suzanne: The last interval is about 3 cm [the teacher points to the 2.8] and if you continue this with 3.5, then it comes onto land at 1/3. And that's 10.00 o'clock.

In a follow-up task, students were asked to represent the motion of a falling ball and predict when it hits the floor. Working on that task students drew various types of graphs: graphs with either time or number of flashes on the horizontal axes and on the vertical axes either height, total distance traveled, or interval covered (Doorman & Gravemeijer, 2008). The variation in graphs was a productive starting point for discussing relations and their potential for predicting when the ball would hit the floor. As a result of the activities and these classroom discussions, the successive representations of motion created a chain of signification: Each representation derives its meaning from the activity with the previous representation (Fig. 3).

The idea of this approach is that it allows students to participate in the invention of continuous time graphs as conventional models of motion. The transition from discrete trace graphs and interval graphs to continuous time–distance and time–velocity graphs aligns with a development of reasoning about the relation between distance traveled and velocity in connection with characteristics of these graphs. In

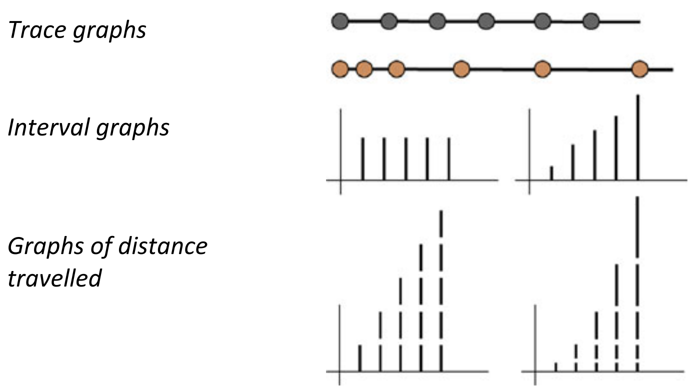

Fig. 3 Representations of uniform and accelerated motion

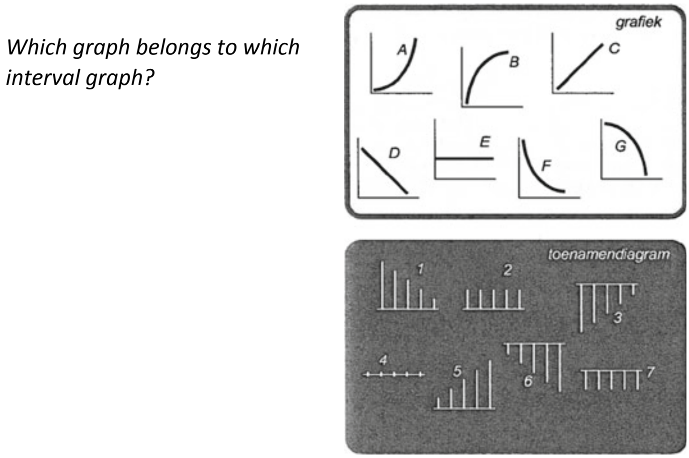

Fig. 4 Match graphs and their interval graphs (e.g., A-5 and C-2)

physics lessons, this understanding can be further developed to other types of motion and, for instance, investigating free fall, while in mathematics lessons various patterns of change can be investigated (Fig. 4).

With this example, we illustrate how contexts and (imaginary) motion experiments can support students during their modeling activities and contribute to identifying and relating variables (Zell & Beckmann, 2009). Furthermore, the learning trajectory is expected to create a meaningful introduction to two-dimensional motion graphs and reasoning about patterns in change by relating interval graphs and graphs of total distances traveled.

These patterns of change prepare students for the relationship between patterns in changing velocity and distances traveled, and patterns in graphs of functions and of their derivatives. In addition, a discrete version of the fundamental theorem of calculus can be introduced by showing that the difference between two distances traveled provides an interval, and the sum of all successive intervals results in the total distance traveled. This discrete theorem can also be used to prove that the difference between successive squares is an odd number, and the sum of successive odd numbers results in a square. However, this example does not address calculating or approaching instantaneous change or velocity, which needs further elaboration by averaging change with a difference quotient and the limit concept. The following example provides a context that has the potential to guide students toward this quotient.

3 Example 2: The Slide

This example is inspired by an inquiry-based approach to teaching the notion of slope and tangent to a curve (Bos et al., 2019, 2020). Introducing the notion of tangent and slope of a curve in a point is a didactical challenge. There is a risk that the students' focus is exclusively on differentiation techniques for calculating slope values, and not at all on the conceptual understanding. This might lead students to be lost in a situation where no explicit function description is available (Tall, 2013).

A guided (re)invention approach to slope is expected to support students in a meaningful conceptual understanding and in the ability to trace techniques and to adapt them to new situations. An attempt to support students in the (re)invention of the notion of slope of a curve in a point requires careful consideration of representations (e.g., Zandieh, 2000). Our approach refers to the daily life experience of a smooth motion along a slide. Students are asked to design such a slide consisting of a bended and a straight part joining without bumps (Fig. 5). The expected outcomes are concrete equations describing these parts and the notion of a difference quotient to quantify slope of a curve as a tool for this smooth joining.

The task invites students to discuss what it means for the line and curve to meet in a not-bumpy, i.e., smooth, way. We expect that they will invent methods that are essential to the notion of slope of a curve in a point. The context of this task is a non-kinematic one. Previous research exploited kinematic contexts, because dimensions (like meters/second) help support the meaning of slope and a connection with the physics curriculum (Doorman, 2005). However, the slide context offers opportunities that are not present in the kinematic approach. Students have tactile and visual experiences of smoothness of a surface (or curve) that provide an informal language to discuss the problem situation. Moreover, students are supposed to have a good prior knowledge of equations for lines and curves. This makes the task situation a realistic one, suited for grounding the new mathematical concept of slope of a curve. The slide task refers to a phenomenon that 'begs' to be mathematized by the tangent line and difference quotients.

Look at the picture of children's slides. Each slide has curved parts and straight parts. Use mathematics to design such a shape.
Focus on just one of the curved parts and a connected straight part. Remember, it is not nice to have a bumpy ride. Introduce a coordinate system and find equations for the curved part and the line; and provide the coordinates of the point where both parts meet.

Fig. 5 A smooth slide[2]

[2] Photo by Michal Jarmoluk via Pixabay.

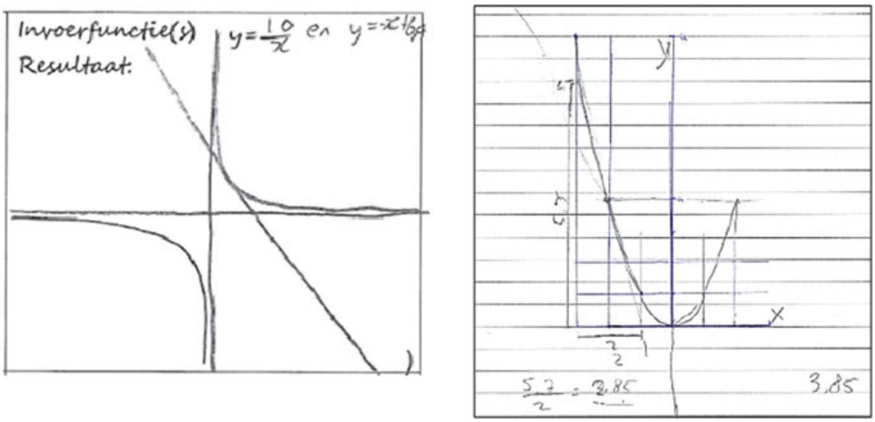

Fig. 6 Students attempt to model a slide (grade 8 students on the left and grade 10 students on the right)

A group of grade 8 students used GeoGebra to plot graphs and translated the line with fixed slope until it seemed to be a tangent to the hyperbola $y = 10/x$. The suggested line has equation $y = -x + 6.4$ (Fig. 6 left). This 'looks' acceptable. Clearly, these students have neither computed the intersection points (algebraic evaluation) nor zoomed in on the graphs (enhanced visual evaluation). Nevertheless, a classroom discussion could be used to address these issues.

A group of grade 10 students drew a tangent line to $y = x$^2 at the point (−2, 4) (Fig. 6 right). They somehow came up with a $\Delta y = 5.7$ for calculating the slope of their tangent line on the interval [−3, −1]. They were already familiar with slopes of straight lines and were able to adapt the method for this task. Their solution strategies provide potential starting point to have the difference quotient and the slope of a curve at a point emerge as models for quantifying instantaneous change.

A content-related connection with physics is less obvious for this task compared to the previous example, although the second group above showed a way of working that is similar to the one presented in physics textbooks (Fig. 1). What is further notable in this example is the inquiry-based way of working by the students. The task is really open and it is crucial for students to first explore the problem by making sketches in order to better understand how mathematics can help solving it. Next, they had to design a method and use it to find a solution, instead of applying a pre-designed procedure from a worked example. Finally, all drawings and findings had to be interpreted in the context of the original problem, evaluated, and communicated. These processes of inquiry have much in common with those from science. In science and mathematics education, these inquiry skills can be highlighted and related more explicitly, which requires specific teaching skills (Swan et al., 2013). The teacher needs to balance students' inquiry with classroom discussions and to balance a focus on inquiry skills with content-related issues. The Theory of Didactic Situations has proven to offer helpful tools for this balancing act by structuring lessons with teaching phases and phases during which students are working independently (Bos et al., 2019; Brousseau, 2002).

Finally, the slide task illustrates how bodily experiences can be used to support mathematical concepts. Embodied cognition theory posits that concepts are embedded in sensorimotor schemes and that gestural and other bodily activities are fundamental constituents of cognition (Radford, 2009a). This suggests that we need to reconsider the potential of contexts in mathematics from contexts providing a frame of reference that support the invention of solution procedures emerging from situational strategies, to contexts as situations that provide opportunities to act and to learn to recognize possibilities to interact mathematically. The third example illustrates the potential of such contexts for teaching grade 7 students to reason with graphs of motion.

4 Example 3: Interacting with Graphs of Motion

The third example in this paper was inspired by theories of embodied cognition. The potential of sensory-motor experiences is explored by designing activities that supported students' coordination and alignment of actions (vision, movement, and verbalization). The example below describes a starting activity in which students move in front of a motion sensor, while a distance–time graph of their own movements unfolded on the screen in front of them. This activity stems from a study in which students' development of reasoning about these motion graphs was investigated (Duijzer et al., 2019). The students were asked to 'walk a given graph' (see Fig. 7 left panel) and reason about their movements in relation to the graph emerging in the coordinate grid (see Fig. 7 right panel). As such, they had to reason in terms of distance from the sensor, where and when to start and stop, and about time, because each periodic shape takes a certain amount of time. We expected the students to slowly abandon iconic interpretations of graphs and reformulate their interpretations in terms of the motion and its graphical and dynamic representation as a mathematical graph (Radford, 2009b).

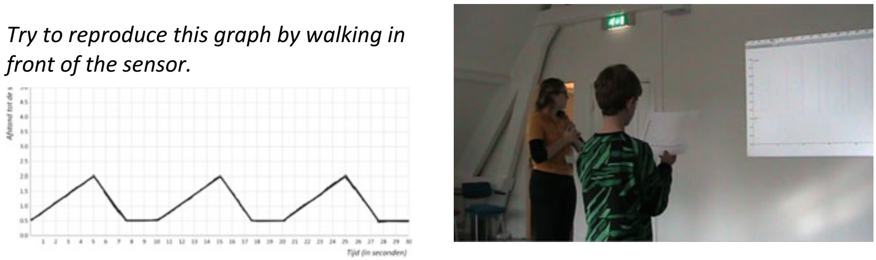

Fig. 7 Walk the graph task

In order to provide an immediate link between the dynamic situation of moving through space and its graphical representation, the motion sensor played a crucial role in directly representing the dynamic situation as a distance–time graph. The graphs created by the motion sensor offered students opportunities to conjecture, act, and reflect on these graphs. We present a vignette about Timon trying to walk the graph presented in Fig. 7.

[Timon holds the card firmly in front of his nose, and walks fast forwards and then a little faster backwards. After having made three such hills Timon stands still. The graph runs horizontally out of the projection screen. Timon finishes before the time runs out]

Timon: I am ready.

Teacher: You may continue... You have nothing to do anymore?

[Timon does nothing and seems to be satisfied with the result.]

Teacher: Oh.. there are no more?

[The teacher walks with Timon to the projection on the wall and holds the card next to the projected graph. The projected hills are not completely similar to the ones on the card.]

Teacher: What could you do differently?

Timon: [Timon points to the downward part of the second hill.] I could make it even more slanting.

Teacher: More slanting, okay. Let's try that.

Timon: Yes.

Teacher: Okay,

[Timon now walks slowly forwards, and quicker backwards. He repeats this again three times.]

This vignette illustrates the potential of a learning environment in which motion sensors are used. In the short excerpt, Timon seemed to be quite confident about his motion, yet overall his walking speed was too high to emulate the given graph correctly. Timon was able to verbalize how to improve graphical similarities between the projected graph as a result of his own movements in front of the sensor and the graph on his card by changing his speed. He shows that he is able to link the graph to movement in the walking space, even without having the direct feedback of the motion sensor. After this short intermezzo, Timon knows what he has to change in his walking speed in order to reproduce the given graph more accurately.

This example shows how connections between mathematics and motion provide a powerful learning environment to have students 'embody' graphical representations of faster and slower change in distance–time graphs. Also, some students were surprised that standing still did not stop the graphing tool, but resulted in a horizontal line (time runs and distance to the sensor does not change). Explicitly incorporating this embodied aspect of learning in the classroom can be an important addition to the emergent modeling practices that were neglected in the previous two examples.

However, taking the potential offered by the context requires from the teacher an understanding of both the mathematics of graphs and of the relation between velocity and distance traveled.

5 Discussion

In this paper, we discuss three examples related to the learning of basic principles of calculus. In all three examples, connections with science were expected to support the students' learning processes. The examples show the complexity and multiple faces of the involved concepts. Calculus involves graphing skills, patterns of change in graphs, tangents, difference quotients, and a limit concept. Moreover, models of motion require mathematics (e.g., understanding slope and tangents in graphs) and science (e.g., the relation between distance–time and velocity–time graphs and their dimensions).

The examples show that students can be involved in modeling practices that aim at the emergence of more formal and general notations, language, and ways of working. With the second and third examples, we suggest to extend the emergent modeling approach with ideas inspired by inquiry-based learning and embodied cognition. This implies a change of focus from contexts to illustrations that include a mathematical structure begging to be mathematized, toward contexts that can encourage students to develop inquiry processes and to embody and learn to recognize possibilities of mathematical or scientific actions. Furthermore, as part of the modeling motion practices students create and develop mathematical and scientific considerations in connection with each other. However, school practice with separated physics and mathematics lessons, and separated teacher education cultures, limits the potential of these connections between mathematics and science.

With the examples, we confirm the observation that despite the many overlaps between mathematics and science in school, it is still a significant challenge for subject teachers to find them and to use their potential (Wong & Dillon, 2019). Collaboration between subject teachers is difficult to organize in the current time-consuming teaching jobs. Recent studies in creating opportunities for teachers to work in teaching and learning networks showed a contribution to teachers' ownership of integrated teaching and learning approaches. With such networks, it appeared to be possible to enhance teachers' content knowledge, use of data logging technology and of inquiry-based pedagogies (Johnston et al., 2020). This is needed to provide teachers with the mechanism to cross the boundaries of subject disciplines. This boundary crossing, connecting mathematics and science, is fundamental to better prepare students for the interdisciplinary challenges in our quickly changing society (Maass et al., 2019). Consequently, awareness for and activity within (inter)national networks such as MACAS is still important for the required innovations of science and mathematics practices in secondary schools.

References

Boohan, R. (2016). *The Language of Mathematics in Science*. Retrieved from https://www.ase.org.uk/mathsinscience

Bos, R., Doorman, M., Cafuta, K., Praprotnik, S., Antoliš, S., & Bašić, M. (2019). Supporting the reinvention of the slope of a curve in a point. In U. T. Jankvist, M. van de Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education*. Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Bos, R., Doorman, M., & Piroi, M. (2020). Emergent models in a reinvention activity for learning the slope of a curve. *Journal of Mathematical Behavior, 59*. https://doi.org/10.1016/j.jmathb.2020.100773

Boyd, A., & Rubin, A. (1996). Interactive video: A bridge between motion and math. *International Journal of Computers for Mathematical Learning*, *1*(1). https://doi.org/10.1007/BF00191472

Brousseau, G. (2002). Theory of didactical situations in mathematics. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds.), *Mathematics Education Library* (19th ed.). https://doi.org/10.1007/0-306-47211-2

Doorman, L. M. (2005). Modelling motion: from trace graphs to instantaneous change. In *CD-β wetenschappelijke bibliotheek;nr. 51, 2005*. CD-ß.

Doorman, L. M. (2019). Design and research for developing local instruction theories. *Avances De Investigacion En Educacion Matematica, 15*, 29–42.

Doorman, L. M., & Gravemeijer, K. P. E. (2008). Learning Mathematics through Applications by Emergent modeling: The Case of Slope and Velocity. In B. Sriraman, C. Michelsen, A. Beckmann, & V. Freiman (Eds.), *Proceedings of MACAS2* (pp. 37–56). Odense: Centre for Science and Mathematics Education, University of Southern Denmark.

Duijzer, C., Van den Heuvel-Panhuizen, M., Veldhuis, M., & Doorman, M. (2019). Supporting primary school students' reasoning about motion graphs through physical experiences. *ZDM - Mathematics Education*, *51*(6). https://doi.org/10.1007/s11858-019-01072-6

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Kluwer Academic Publishers.

Friel, S. N., Curcio, F. R., & Bright, G. W. (2001). Making sense of graphs: Critical factors influencing comprehension and instructional implications. *Journal for Research in Mathematics Education, 32*(2), 124–158. https://doi.org/10.2307/749671

Gravemeijer, K. P. E. (1999). Mathematical thinking and learning how emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102

Greeno, J. G., Moore, J. L., & Smith, D. R. (1993). Transfer of situated learning. In D. K. Detterman & R. J. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 99–167). Erlbaum.

Johnston, J., Walshe, G., & Ríordáin, M. N. (2020). Supporting key aspects of practice in making mathematics explicit in science lessons. *International Journal of Science and Mathematics Education, 18*(7), 1399–1417. https://doi.org/10.1007/s10763-019-10016-1

Maass, K., Doorman, L. M., Jonker, V. H., & Wijers, M. M. (2019). Promoting active citizenship in mathematics teaching. *ZDM - International Journal on Mathematics Education, 51*(6), 991–1003. https://doi.org/10.1007/s11858-019-01048-6

Michelsen, C. (1998). Expanding context and domain: A cross-curricular activity in Mathematics and Physics. *ZDM, 30*(4), 100–106. https://doi.org/10.1007/BF02653149

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing Mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education*, *44*(2), 372–415. https://doi.org/10.5951/jresematheduc.44.2.0372

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119–172. https://doi.org/10.1207/s1532690xci1602_1

Radford, L. (2009a). "No! He starts walking backwards!": Interpreting motion graphs and the question of space, place and distance. *ZDM - Mathematics Education, 41*(4), 467–480. https://doi.org/10.1007/s11858-009-0173-9

Radford, L. (2009b). "No! He starts walking backwards!": Interpreting motion graphs and the question of space, place and distance. *ZDM Mathematics Education, 41*(4), 467–480. https://doi.org/10.1007/s11858-009-0173-9

Swan, M., Pead, D., Doorman, M., & Mooldijk, A. (2013). Designing and using professional development resources for inquiry-based learning. *ZDM - Mathematics Education*, *45*(7). https://doi.org/10.1007/s11858-013-0520-8

Tall, D. (2013). *How Humans Learn to Think Mathematically*. https://doi.org/10.1017/CBO9781139565202

van Helden, G., & Shvarts, A. (n.d.). Embodied transfer in mathematics learning: recognizing a unit circle as a sine-graph builder and anticipating a sine-graph-movement. *For the Learning of Mathematics.*

Wong, V., & Dillon, J. (2019). Crossing the boundaries: Collaborations between mathematics and science departments in English secondary (high) schools. *Research in Science & Technological Education, 1–21*. https://doi.org/10.1080/02635143.2019.1636024

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education, 8*, 103–127.

Zell, S., & Beckmann, A. (2009). Modelling activities while doing experiments to discover the concept of variable. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of CERME 6* (pp. 2216–2225). Lyon, France.