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Sequent Calculi for Intuitionistic Gödel–Löb Logic

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Abstract This paper provides a study of sequent calculi for intuitionistic Gödel–Löb logic (iGL), which is the intuitionistic version of the classical modal logic GL, known as Gödel–Löb logic. We present two different sequent calculi, one of which we prove to be the terminating version of the other. We study those systems from a proof-theoretic point of view. One of our main results is a syntactic proof for the cut-admissibility result for those systems. Finally, we apply this to prove Craig interpolation for intuitionistic Gödel–Löb logic.

1 Introduction

Intuitionistic Gödel–Löb logic (iGL) is an interesting logic for various reasons. It is the intuitionistic variant of one of the most well-known classical modal logics GL, which is the provability logic of Peano Arithmetic (PA). This logic is obtained by adding Gödel–Löb's axiom $\Box(\Box A \to A) \to \Box A$ to the standard Hilbert calculus for classical propositional normal modal logic K. In this logic, we read $\Box A$ as "A is provable in PA." Completeness of GL with respect to PA augmented with the provability predicate is shown in Solovay's famous completeness theorems in [16]. For more information about GL and PA, see Boolos [4].

Logic iGL is the intuitionistic variant of GL, meaning that it consists of the standard Hilbert calculus for intuitionistic normal modal logic K together with the Gödel–Löb axiom. Therefore, one might expect that iGL would be the provability logic for Heyting Arithmetic (HA), where $\Box A$ means "A is provable in HA." However, iGL is only sound with respect to the provability part of HA, but completeness fails. That is, there are principles of the provability logic of HA which are not provable in iGL (see Leivant [12]). What the provability logic for HA is, is a long-standing open question. Recently, Ardeshir and Mojtahedi [1] found the Σ_1-provability logic for HA. A key element in their study is logic iCGL, which is an extension of iGL by the so-called

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completeness axiom $A \to \Box A$. This logic is also known as *strong Löb logic* iSL and is complete with respect to an arithmetical theory based on what is called *slow provability* (see Visser and Zoethout [21]). It is certainly conceivable that iGL has a provability interpretation as well for some nonstandard provability predicate.

Logic iGL is also interesting because of its natural semantic framework. Ursini [19] is the first who devoted a paper to this logic (he denotes it as ID), in which he treats both a relational and an algebraic semantics. He shows the existence of a characteristic model and frame, the finite model property, the finite frame property, and decidability. The Kripke semantics for iGL is a natural combination of intuitionistic and modal logic where the modal relation has the classical GL properties: transitive and conversely well-founded. An overview of semantic results for iGL and other intuitionistic modal logics can be found in Litak [13]. Ursini's paper is marked by I°, and he announced that "two succeeding papers (II°, III°) with the same title could follow, devoted to a certain related double modal calculus and to the syntax of ID in the style of natural deduction." To the best of our knowledge, these papers have never appeared. The current paper continues Ursini's line of research, because it provides a proof theory for iGL based on the sequent calculus for GL developed by Avron [2].

In this article we give a proof-theoretic analysis of two different sequent calculi for iGL, one of which is terminating. We call the systems GL3i and GL4i. We study these calculi following the same line of research as in [10] for intuitionistic modal logics iK and iKD. The proof for termination of GL4i is interesting, because it uses a nonstandard induction based on Bílková [3]. Our aim here is to establish the cut-admissibility result for both systems.

We present a nontrivial syntactic proof for the cut-admissibility in GL3i based on results of Valentini [20] and Goré and Ramanayake [8]. They prove the cut-elimination theorem for sequent calculi for classical modal logic GL. The cut-elimination for sequent calculi for GL has an interesting history, nicely described in [8]. In short, Valentini presented a proof for sequents built from sets, a proof in which many steps remained implicit and therefore difficult to check. It is often assumed that set-based proofs for cut-elimination can easily be adopted to multiset-based calculi. For GL this is not obvious and led to the search of new cut-elimination proofs for GL. However, Goré and Ramanayake prove the cut-elimination theorem for multiset sequents, placing Valentini's argument in a formal setting.

The cut-admissibility theorem implies several results, such as the subformula property and consistency. It also implies that those sequent calculi, indeed, represent iGL. By the subformula property, we can conclude that intuitionistic Gödel–Löb logic is conservative over IPC, which means that no new propositional tautologies can be derived. In the last section we use our results to prove the Craig interpolation property for iGL.

The study of those systems is technically challenging, because it combines the difficulties of GL, with its highly nontrivial proofs of cut-elimination, with the already complicated framework of intuitionistic modal logics. In addition, termination proofs for sequent calculi for GL require rather complicated methods, because of the behavior of the so-called GLR rule. In this paper, we will adopt several methods used in the study of sequent calculi for GL and apply those in such a way that we can use them in our intuitionistic framework.

The paper is structured in the following way. Section 2 gives the preliminaries of both sequent systems for iGL. We present the syntactic cut-admissibility result for our nonterminating calculus GL3i in Section 3. In Section 4 we give the terminating result for GL4i. Section 5 states the equivalence between GL3i and GL4i which immediately implies the cut-admissibility for GL4i. The last two sections cover the Craig interpolation property for iGL and future work.

2 Intuitionistic Gödel–Löb Logic

We consider the modal language with *constant* $\bot$, *propositional variables* $p, q, \ldots$, *connectives* $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication), and the *modal operator* $\Box$. Note that we do not include $\Diamond$. In contrast to classical modal logic, $\Diamond$ is not interdefinable via $\Box$. Formulas are denoted by $A, B, C, \ldots$. If A is a formula, $\neg A$ is defined as $A \to \bot$. We call formulas of the form $\Box A$ *boxed formulas*.

Definition 2.1 *Intuitionistic Gödel-Löb logic* iGL is given by the Hilbert system containing:

1. intuitionistic propositional tautologies;
2. K-axiom: $\Box(A \to B) \to \Box A \to \Box B$;
3. Gödel–Löb's axiom: $\Box(\Box A \to A) \to \Box A$,
4. closed under *modus ponens*: if A and $A \to B$ in iGL, then also B in iGL;
5. closed under *necessitation*: if A is in iGL then also $\Box A$ in iGL.

We want to examine multiset sequent calculi for iGL. We use multisets of formulas that are denoted by Greek letters $\Gamma, \Delta, \ldots$. For two multisets Γ, Δ, we denote by $\Gamma \cup \Delta$ the multiset that contains only formulas A that belong to Γ and Δ, and the number of occurrences of A is the sum of the occurrences in Γ and Δ. Let Γ be a multiset. We define $\Box\Gamma$ to be the multiset $\{\Box A \mid A \in \Gamma\}$ and $\boxdot\Gamma$ to be the multiset $\Gamma \cup \Box\Gamma$.

We consider single-conclusion sequents, which are expressions of the form $\Gamma \Rightarrow C$, where Γ is a finite multiset of formulas and C is a formula. In a sequent notation, Γ, Δ denotes $\Gamma \cup \Delta$, Γ, A denotes $\Gamma \cup \{A\}$, and $\boxdot A$ denotes $\{A, \Box A\}$. So $\Gamma, \Delta, \boxdot A \Rightarrow C$ reads as $(\Gamma \cup \Delta \cup \{A, \Box A\}) \Rightarrow C$. In a sequent $\Gamma \Rightarrow C$, we call Γ the *antecedent* and C the *succedent*, which is standard terminology also for multi-conclusion sequents. We sometimes denote a sequent by S.

We study two calculi for intuitionistic Gödel–Löb logic. One is a terminating version of the other. The most important rule is the GLR rule, which is

$$\frac{\boxdot\Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}\ \text{GLR}$$

This rule has two flavors: in GL4i multiset Π cannot contain boxed formulas, while in GL3i there is no restriction on Π. Table 1 presents the rules for GL3i, which is the propositional intuitionistic calculus G3ip from Troelstra and Schwichtenberg [18] together with the modal rule GLR. To stress again, Π here is an arbitrary multiset of formulas, so it may contain boxed formulas. Table 2 presents the system GL4i, which contains the rules for the terminating calculus G4ip from Dyckhoff [6] together with the modal rule GLR and an additional left implication rule for box. So we have five left implication rules. Recall that, in contrast to GL3i, we put a restriction on multiset Π in the rules GLR and L$\Box\!\to$ in GL4i: Π does not contain boxed formulas. This is necessary to guarantee termination (see Section 4).

Both systems do not contain structural rules explicitly, but weakening and contraction are admissible. We use the nonterminating system GL3i to present a proof-theoretic proof for cut-admissibility based on the work of Valentini [20] and Goré

Table 1 Sequent calculus GL3i.

$\dfrac{}{\Gamma, p \Rightarrow p}$ At, p atomic	$\dfrac{}{\Gamma, \bot \Rightarrow C}$ L$\bot$
$\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$ R$\wedge$	$\dfrac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$ L$\wedge$
$\dfrac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}$ R$\vee_i$ $(i = 1, 2)$	$\dfrac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$ L$\vee$
$\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$ R$\to$	$\dfrac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}$ L$\to$
$\dfrac{\boxdot\Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}$ GLR	

Table 2 Sequent calculus GL4i.

$\dfrac{}{\Gamma, p \Rightarrow p}$ At, p atomic	$\dfrac{}{\Gamma, \bot \Rightarrow C}$ L$\bot$
$\dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$ R$\wedge$	$\dfrac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$ L$\wedge$
$\dfrac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}$ R$\vee_i$ $(i = 1, 2)$	$\dfrac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$ L$\vee$
$\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$ R$\to$	$\dfrac{\Gamma, p, A \Rightarrow C}{\Gamma, p, p \to A \Rightarrow C}$ L$p\to$, p atomic
$\dfrac{\Gamma, A \to (B \to C) \Rightarrow D}{\Gamma, A \wedge B \to C \Rightarrow D}$ L$\wedge\to$	$\dfrac{\Gamma, B \to C \Rightarrow A \to B \qquad \Gamma, C \Rightarrow D}{\Gamma, (A \to B) \to C \Rightarrow D}$ L$\to\to$
$\dfrac{\Gamma, A \to C, B \to C \Rightarrow D}{\Gamma, A \vee B \to C \Rightarrow D}$ L$\vee\to$	$\dfrac{\boxdot\Gamma, \Box A \Rightarrow A \qquad \Pi, \Box\Gamma, B \Rightarrow C}{\Pi, \Box\Gamma, \Box A \to B \Rightarrow C}$ L$\Box\to$
$\dfrac{\boxdot\Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}$ GLR	In GLR and L$\Box\to$, no boxed formulas in Π.

and Ramanayake [8], who proved cut-elimination for sequent calculi for classical GL. This approach is not applicable to GL4i, because of the restriction in the GLR rule. However, equivalence of GL3i and GL4i immediately implies the cut-admissibility result for GL4i. Furthermore, the cut-admissibility results for both GL3i and GL4i imply equivalences between the systems iGL, GL3i, and GL4i, which shows that all those systems express intuitionistic Gödel–Löb logic.

We fix some terminology. The sequents At and L$\bot$ are called *initial sequents* or *axioms*. The GLR rules are called *modal rules*, and all rules except for GLR are called *logical rules*. In the axioms and logical rules, the *principal* formula of an occurrence is defined as usual. In the GLR rule, all formulas in $\Box\Gamma$ as well as $\Box A$ are *principal*. Formula $\Box A$ on the right-hand side of the conclusion in the GLR rule is called the *diagonal formula*. We also call $\Box A$ in the principal formula of L$\Box\to$ the *diagonal formula*. A *derivation* in GL3i (resp., in GL4i) is a tree built up from the rules in GL3i (resp., GL4i) whose leaves are initial sequents. We use the same for GL4i. We use the letter Σ to denote derivations. The *height* of a derivation is the greatest number of successive applications of rules in it. We use standard notation $\vdash_{\mathsf{GL3i}} \Gamma \Rightarrow C$ for derivability of sequent $\Gamma \Rightarrow C$ in GL3i. We do the same for GL4i. The notation $\vdash_{\mathsf{GL3i},n} \Gamma \Rightarrow C$ means that sequent $\Gamma \Rightarrow C$ is derivable in GL3i with a height of derivation at most n.

In the following, and at later points, we will write $S_1 \cdots S_k / S$ as short notation for the rules

$$\frac{S_1}{S} \qquad \frac{S_1 \qquad S_2}{S}$$

for $k = 1, 2$, respectively. The following definition is useful in both systems.

Definition 2.2 A rule $S_1 \cdots S_k / S$ is invertible in GL3i (resp., GL4i) if whenever S is derivable in GL3i (resp., GL4i) we have that S_i is derivable in GL3i (resp., GL4i) for all i. That is, S / S_i is admissible in the calculus. A rule is *height-preserving invertible* if whenever $\vdash_n S$ we have $\vdash_n S_i$ for all i.

3 A Syntactic Proof for Cut-Admissibility

This section provides an analysis of the system GL3i. A substantial part is devoted to a syntactic proof of cut-admissibility. Immediate corollaries of the admissibility of cut are the subformula property, consistency, and conservativity over IPC. At the end of this section we will see the correspondence between GL3i and the Hilbert system for intuitionistic Gödel–Löb logic. This means that formula A is provable in iGL if and only if sequent ($\Rightarrow A$) is provable in GL3i. First we look at basic concepts. We define the *degree* $d(A)$ of a formula A inductively by $d(\bot) = 0$, $d(p) = 1$, and $d(A \wedge B) = d(A \vee B) = d(A \to B) = d(A) + d(B) + 1$. We start with a useful lemma.

Lemma 3.1 **(GL3i weakening, contraction, inversion)** *For each n, we have the following in* GL3i.

1. *Extended axiom rule*: $\vdash \Gamma, C \Rightarrow C$ *for every formula* C.
2. *Falsum rule*: $\vdash_n \Gamma \Rightarrow \bot$ *implies* $\vdash_n \Gamma \Rightarrow C$.
3. *Weakening*: $\vdash_n \Gamma \Rightarrow C$ *implies* $\vdash_n \Gamma, A \Rightarrow C$.
4. *Inversion*: *Rules* R$\wedge$, L$\wedge$, L$\vee$, *and* R $\to$ *are height-preserving invertible.*
5. *Inversion* L $\to$: *If* $\vdash_n \Gamma, A \to B \Rightarrow C$, *then* $\vdash_n \Gamma, B \Rightarrow C$.
6. *Contraction*: $\vdash_n \Gamma, D, D \Rightarrow C$ *implies* $\vdash_n \Gamma, D \Rightarrow C$.

Proof Statement (1) is proved by induction on the degree of formula C. All others are proved by induction on height n. Weakening is needed in the proofs for inversion, and inversion is used in the proof of contraction. □

Now we turn to the proof of cut-admissibility in GL3i. The proof is based on the works of Valentini [20] and Goré and Ramanayake [8]. Both prove cut-elimination for sequent calculi for classical GL in which—in contrast to our system—the structural rules are explicitly contained. Valentini considers sequents built from sets, whereas Goré and Ramanayake adapted his proof to multisets. The use of multisets instead of sets means that we have to take into account contraction. Goré and Ramanayake have formalized ideas of Valentini in order to give a robust proof for a multiset sequent calculus for GL. We will see that Valentini's proof idea can also be applied to GL3i, a system without explicit weakening and contraction, which are admissible in it. Since weakening and contraction are admissible, we do not need all the elements of the machinery used in [8], but we use these elements relevant for iGL.

A well-known method for establishing the cut-admissibility theorem is to transform topmost cuts of the form

$$\dfrac{\Gamma \Rightarrow D \qquad D, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ \text{cut}$$

into cut-free derivations with the same end-sequent. Standard is to use a double induction (d, h), where d is the degree of the cut-formula and h is the height of the cut, where the *cut-height* is defined as the height of its left premise derivation plus the height of its right premise derivation.

However, for provability logic, this is not sufficient. Therefore Valentini introduces a third induction parameter called "width," which is computed "globally." A third parameter is necessary when we encounter the following problem: consider a cut where both premises are derived from the GLR rule and the cut-formula is principal in both sides:

$$\dfrac{\text{GLR}\ \dfrac{\overset{\Sigma_l}{\boxdot\Gamma, \Box B \Rightarrow B}}{\Pi_l, \Box\Gamma \Rightarrow \Box B} \qquad \dfrac{\overset{\Sigma_r}{\Box B, B, \boxdot\Delta, \Box C \Rightarrow C}}{\Pi_r, \Box B, \Box\Delta \Rightarrow \Box C}\ \text{GLR}}{\Pi_l, \Pi_r, \Box\Gamma, \Box\Delta \Rightarrow \Box C}\ \text{cut}(\Box B)$$

A reasonable thing to do is the following, where we use the admissibility of contraction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\Sigma_l}{\boxdot\Gamma, \Box B \Rightarrow B}}{\Box\Gamma \Rightarrow \Box B} \qquad \overset{\Sigma_l}{\boxdot\Gamma, \Box B \Rightarrow B}}{\Box\Gamma, \boxdot\Gamma \Rightarrow B}\ \text{cut}_1}{\boxdot\Gamma \Rightarrow B}\ \text{contraction} \qquad \dfrac{\dfrac{\overset{\Sigma_l}{\boxdot\Gamma, \Box B \Rightarrow B}}{\Box\Gamma \Rightarrow \Box B} \qquad \overset{\Sigma_r}{\Box B, B, \boxdot\Delta, \Box C \Rightarrow C}}{B, \Box\Gamma, \boxdot\Delta, \Box C \Rightarrow C}\ \text{cut}_2}{\dfrac{\boxdot\Gamma, \Box\Gamma, \boxdot\Delta, \Box C \Rightarrow C}{\boxdot\Gamma, \boxdot\Delta, \Box C \Rightarrow C}\ \text{contraction}}\ \text{cut}_3}{\Pi_l, \Pi_r, \Box\Gamma, \Box\Delta \Rightarrow \Box C}\ \text{GLR}$$

But here, it is not possible to eliminate cut_1 when using the standard induction on (d, h). Although the cut-formula in cut_1 is the same as in $\text{cut}(\Box B)$, which means that the degree d remains the same, the cut-height h of cut_1 is not necessarily smaller than the cut-height of $\text{cut}(\Box B)$. The reason is that the cut-height of cut_1 is defined in terms of the height of Σ_l, but the cut-height of $\text{cut}(\Box B)$ also depends on Σ_r. So we cannot compare both cuts in terms of (d, h).

The width circumvents the problem, because it enables us to define a derivation of $\boxdot\Gamma \Rightarrow B$ in which each application of the cut rule is eliminable. Informally, the width is the number of GLR rules in the left premise of the cut in which the cut-formula is not introduced by weakening. We will now adopt Valentini's method to see that it works for GL3i.

Definition 3.2 Consider the rules in Table 1. We say that a formula A is *introduced by weakening* in rule ρ if ρ is an axiom rule and $A \in \Gamma$ or ρ is the GLR rule and $A \in \Pi$.

Note that formulas introduced by weakening are exactly the nonprincipal formulas of the corresponding rule. So one can ask why we redefine such formulas. We do this in order to be consistent with the terminology as in Valentini [20], where weakening is an explicit rule. It also gives an intuitive insight into the role of a particular formula in a GLR rule.

Definition 3.3 Let Σ be a derivation with end-sequent $\Gamma \Rightarrow C$. An instance ρ of the GLR rule appearing in Σ is *n-ary* (*over* $\Gamma \Rightarrow C$) if the segment between the conclusion of ρ and end-sequent $\Gamma \Rightarrow C$ contains exactly $n-1$ applications of the GLR rule.

For a cut-free derivation Σ with end-sequent $\Gamma \Rightarrow C$, let $\mathrm{GLR}(2, \Sigma)$ denote the number of GLR rules ρ in Σ satisfying the following:

1. ρ is 2-ary over $\Gamma \Rightarrow C$;
2. C is the diagonal formula of the 1-ary GLR rule in Σ below ρ;
3. C is not introduced by weakening in ρ.

Remark 3.4 The number $\mathrm{GLR}(2, \Sigma)$ may be different from 0, but only if C is a boxed formula, due to clause (2).

In the following we write $\Sigma / \Gamma \Rightarrow D$ to denote the derivation

$$\begin{array}{c} \Sigma \\ \Gamma \Rightarrow D \end{array}$$

Definition 3.5 (Width) Consider a topmost cut as shown below:

$$\dfrac{\begin{array}{c}\Sigma_l\\ \Gamma \Rightarrow D\end{array} \qquad \begin{array}{c}\Sigma_r\\ D, \Delta \Rightarrow C\end{array}}{\Gamma, \Delta \Rightarrow C}\ \mathrm{cut}_0$$

The *width* of cut_0 is defined as $w(\mathrm{cut}_0) = \mathrm{GLR}(2, \Sigma_l / \Gamma \Rightarrow D)$.

Remark 3.6 The width is defined on the basis of the left premise (and Σ_l) of the cut and is independent of the right premise (and Σ_r). The width has only been defined for topmost cuts as this restriction is sufficient for our purpose.

Example 3.7 Let us calculate the width $w(\mathrm{cut}(\Box\bot))$ in the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box\bot, \bot, \Box D \Rightarrow D}{\boxdot(\Box D \to \bot), \Box\bot \Rightarrow \Box D}\,\rho \qquad \Box\bot, \Box(\Box D \to \bot), \bot \Rightarrow \bot}{\boxdot(\Box D \to \bot), \Box\bot \Rightarrow \bot}\,\mathrm{L}{\to}}{\Box(\Box D \to \bot) \Rightarrow \Box\bot}\,\mathrm{GLR} \qquad \dfrac{\Box\bot, \bot, \Box D \Rightarrow D}{\Box\bot \Rightarrow \Box D}}{\Box(\Box D \to \bot) \Rightarrow \Box D}\,\mathrm{cut}(\Box\bot)$$

There is one 2-ary GLR rule over the left premise of the cut, which is rule ρ. Formula $\Box\bot$ is not introduced by weakening in ρ, so $w(\mathrm{cut}(\Box\bot)) = 1$.

The width is used as the induction parameter in the cut-elimination proof. In particular, the induction value for a topmost cut is (d, w, h), where d is the degree of the cut-formula, w is the width of the cut, and h is the cut-height. The idea of reducing the width is to transform a 2-ary GLR rule ρ over the left premise $\Gamma \Rightarrow D$ of the cut satisfying requirements (1)–(3) from Definition 3.3 into a GLR rule where D is introduced by weakening. What is important to note is that this only works because we allow boxed formulas in Π in the definition of the GLR rule in the rules of GL3i (see Table 1). This is not possible in the terminating system GL4i.

Before proving the cut-elimination theorem using the width, we introduce some derivation transformations that are useful in the proof. Some transformations are based on parts of a derivation. To this end, it is useful to look at *stub-derivations*, which are more general than derivations. Stub-derivations were introduced by Goré and Ramanayake in [8]. Informally, stub-derivations can be obtained by deleting a proper subderivation from a derivation, thereby obtaining a derivation with a "gap."

Definition 3.8 A *stub-derivation* in GL3i is defined recursively as follows:

1. initial sequents $p, \Gamma \Rightarrow p$ and $\bot, \Gamma \Rightarrow C$ are stub-derivations;
2. for any sequent S and stub-derivation Θ, each of
$$\text{(a)}\ \frac{\mathsf{stub}}{S} \qquad \text{(b)}\ \frac{\mathsf{stub} \qquad \Theta}{S} \qquad \text{(c)}\ \frac{\Theta \qquad \mathsf{stub}}{S}$$
is a stub-derivation;
3. an application of a logical or GLR rule to stub-derivation(s) that end with its premise(s) results in a stub-derivation;
4. an application of the cut rule to stub-derivations concluding its premises results in a stub-derivation.

If no cut rule is used in a stub-derivation Θ, we say that Θ is a *cut-free* stub-derivation. A stub-derivation with one occurrence of "stub" is called a *single stub-derivation*, also introduced in [8]. In the rest, we only consider those single stub-derivations. For a single stub-derivation Θ, we sometimes write $\Theta[\mathsf{stub}]$ to indicate the stub occurrence.

Definition 3.9 Let Σ be a derivation, and let $\Theta[\mathsf{stub}]$ be a single stub-derivation with an occurrence of one of the following:
$$\text{(a)}\ \frac{\mathsf{stub}}{S} \qquad \text{(b)}\ \frac{\mathsf{stub} \qquad \Sigma'}{S} \qquad \text{(c)}\ \frac{\Sigma' \qquad \mathsf{stub}}{S}$$
where Σ' is a derivation, and suppose that
$$\text{(a)}\ \frac{\Sigma}{S}\,\beta \qquad \text{(b)}\ \frac{\Sigma \qquad \Sigma'}{S}\,\beta \qquad \text{(c)}\ \frac{\Sigma' \qquad \Sigma}{S}\,\beta$$
are correct derivations in GL3i for some rule β. We say that a single stub-derivation Θ and a derivation Σ are *compatible* with *binding rule* β, and we obtain a correct derivation by replacing the "stub" in Θ by Σ, denoted by $\Theta[\Sigma]$.

Example 3.10 This example is taken from [8]. The left single stub-derivation and the derivation on the right are compatible with binding rule L$\vee$:
$$\Theta = \frac{\mathsf{stub} \qquad \begin{array}{c}\Sigma_1\\ A \to B \Rightarrow A \to B\end{array}}{B \vee (A \to B) \Rightarrow A \to B} \qquad\qquad \Sigma = \frac{\begin{array}{c}\Sigma_2\\ A, B \Rightarrow B\end{array}}{B \Rightarrow A \to B}\,\mathrm{R}{\to}$$
We have
$$\Theta[\Sigma] = \frac{\dfrac{\begin{array}{c}\Sigma_2\\ A, B \Rightarrow B\end{array}}{B \Rightarrow A \to B}\,\mathrm{R}{\to} \qquad \begin{array}{c}\Sigma_1\\ A \to B \Rightarrow A \to B\end{array}}{B \vee (A \to B) \Rightarrow A \to B}\,\mathrm{L}{\vee}$$

Definition 3.11 Let Θ be a cut-free stub-derivation. Let Δ be a (nonempty) multiset of formulas. We define Θ^{Δ} by recursion as follows. Intuitively, Θ^{Δ} is obtained from Θ by weakening with Δ.

1. axiom
   (a) atom: $(\Gamma, p \Rightarrow p)^{\Delta} = (\Gamma, p, \Delta \Rightarrow p)$,
   (b) $\bot$: $(\Gamma, \bot \Rightarrow C)^{\Delta} = (\Gamma, \bot, \Delta \Rightarrow C)$;
2. stub-instance
   (a) $(\mathsf{stub}/\,\Gamma \Rightarrow C)^{\Delta} = (\mathsf{stub}/\,\Gamma, \Delta \Rightarrow C)$,
   (b) $(\mathsf{stub}\ \ \Theta/\,\Gamma \Rightarrow C)^{\Delta} = (\mathsf{stub}\ \ \Theta^{\Delta}/\,\Gamma, \Delta \Rightarrow C)$,

   (c) $(\Theta\ \ \mathsf{stub}/\Gamma \Rightarrow C)^\Delta = (\Theta^\Delta\ \ \mathsf{stub}/\Gamma, \Delta \Rightarrow C)$;
3. unary connective rule: $(\Theta'/\Gamma \Rightarrow C)^\Delta = (\Theta'^\Delta/\Gamma, \Delta \Rightarrow C)$;
4. binary connective rule: $(\Theta_1\ \ \Theta_2/\Gamma \Rightarrow C)^\Delta = (\Theta_1^\Delta\ \ \Theta_2^\Delta/\Gamma, \Delta \Rightarrow C)$;
5. GLR rule: $(\Theta'/\Gamma \Rightarrow C)^\Delta = (\Theta'/\Gamma, \Delta \Rightarrow C)$.

Notice that the recursion terminates at an axiom rule, stub-instance (a) or a GLR rule. For a sequent $S = (\Gamma \Rightarrow C)$, we write $S^\Delta = (\Gamma, \Delta \Rightarrow C)$.

Lemma 3.12 *Let Θ be a cut-free stub-derivation, and let Δ be a multiset of formulas. Then Θ^Δ is also a cut-free stub-derivation.*

Proof This is by inspection of the recursion in Definition 3.11. □

Example 3.13 We continue with Example 3.10. If Δ is a (nonempty) formula multiset, then the single stub-derivation obtained from Θ by weakening with Δ is:

$$\Theta^\Delta = \frac{\mathsf{stub} \qquad \begin{array}{c}\Sigma_1^\Delta\\ A \to B, \Delta \Rightarrow A \to B\end{array}}{B \vee (A \to B), \Delta \Rightarrow A \to B}$$

We also see that the following is a correct derivation:

$$\Theta^\Delta[\Sigma^\Delta] = \dfrac{\dfrac{\begin{array}{c}\Sigma_2^\Delta\\ A, B, \Delta \Rightarrow B\end{array}}{B, \Delta \Rightarrow A \to B}\,\mathrm{R}{\to} \qquad \begin{array}{c}\Sigma_1^\Delta\\ A \to B, \Delta \Rightarrow A \to B\end{array}}{B \vee (A \to B), \Delta \Rightarrow A \to B}\,\mathrm{L}{\vee}$$

Lemma 3.14 *Let Θ be a cut-free single stub-derivation, and let Σ be a cut-free derivation with end-sequent $\Gamma \Rightarrow C$ such that Θ and Σ are compatible with binding rule β yielding a correct derivation $\Theta[\Sigma]$. For any multiset Δ, we have the following.*

1. *If there is an application of the* GLR *rule in the segment from β to the end-sequent in $\Theta[\Sigma]$, then $\Theta^\Delta[\Sigma']$ is a correct derivation for any cut-free derivation Σ' with end-sequent $\Gamma \Rightarrow C$. In particular, $\Theta^\Delta[\Sigma]$ is a correct derivation.*
2. *If there is no application of the* GLR *rule in the segment from β to the end-sequent in $\Theta[\Sigma]$, then $\Theta^\Delta[\Sigma']$ is a correct derivation for any cut-free derivation Σ' with end-sequent $\Gamma, \Delta \Rightarrow C$. In particular, $\Theta^\Delta[\Sigma^\Delta]$ is a correct derivation.*

Proof From Lemma 3.12, we know that Θ^Δ is a correct cut-free single stub-derivation. So for both (1) and (2) we only have to show that the end-sequent of Σ' is a correct premise regarding binding rule β and Θ^Δ.

The single stub-derivation Θ has one of the following stub-instances (which are sub-stub-derivations of Θ^Δ):

$$\text{(a)}\ \frac{\mathsf{stub}}{S} \qquad \text{(b)}\ \frac{\mathsf{stub} \qquad \Theta'}{S} \qquad \text{(c)}\ \frac{\Theta' \qquad \mathsf{stub}}{S}$$

By assumption, Θ and Σ are compatible, so the following are correct instances of binding rule β, where $\Gamma \Rightarrow C$ is the end-sequent of Σ and $S_{\mathrm{end}(\Theta')}$ is the end-sequent in Θ':

$$\text{(a)}\ \frac{\Gamma \Rightarrow C}{S}\,\beta \qquad \text{(b)}\ \frac{\Gamma \Rightarrow C \qquad S_{\mathrm{end}(\Theta')}}{S}\,\beta \qquad \text{(c)}\ \frac{S_{\mathrm{end}(\Theta')} \qquad \Gamma \Rightarrow C}{S}\,\beta$$

For case (1) we distinguish two easy cases. If some rule below β is a GLR rule, we know, by inspection of Definition 3.11, that sequent S remains the same in Θ^{Δ}, so the stub-instances in Θ^{Δ} remain the same as in Θ. This immediately implies the desired result.

If no rule below β is a GLR rule, but β itself is, we know, by inspection of Definition 3.11, that the antecedent of sequent S becomes enlarged with Δ in Θ^{Δ}, so the following is the stub-instance in Θ^{Δ} in which the right presents a correct instance of GLR rule β:

$$\frac{\mathsf{stub}}{S^{\Delta}} \qquad \frac{\Gamma \Rightarrow C}{S^{\Delta}}\,\beta$$

So every cut-free derivation Σ' with end-sequent $\Gamma \Rightarrow C$ is compatible with Θ^{Δ}.

For case (2), in which rule β and all rules below β are not a GLR rule, we know, by inspection of Definition 3.11, that the following is a correct stub-instance of Θ^{Δ}:

$$\text{(a)}\ \frac{\mathsf{stub}}{S^{\Delta}} \qquad \text{(b)}\ \frac{\mathsf{stub} \qquad \Theta'^{\Delta}}{S^{\Delta}} \qquad \text{(c)}\ \frac{\Theta'^{\Delta} \qquad \mathsf{stub}}{S^{\Delta}}$$

Since β is not a GLR rule, we have a correct instance of β, where $S_{\mathrm{end}(\Theta'^{\Delta})}$ is the end-sequent in Θ'^{Δ}:

$$\text{(a)}\ \frac{\Gamma, \Delta \Rightarrow C}{S^{\Delta}}\,\beta \qquad \text{(b)}\ \frac{\Gamma, \Delta \Rightarrow C \quad S_{\mathrm{end}(\Theta'^{\Delta})}}{S^{\Delta}}\,\beta \qquad \text{(c)}\ \frac{S_{\mathrm{end}(\Theta'^{\Delta})} \quad \Gamma, \Delta \Rightarrow C}{S^{\Delta}}\,\beta$$

So every cut-free derivation Σ' with end-sequent $\Gamma, \Delta \Rightarrow C$ is compatible with Θ^{Δ}. □

We examine one more transformation of derivations, also used in [20] and [8].

Definition 3.15 Let Σ be a cut-free derivation with end-sequent of the form $\Gamma, \Box B \Rightarrow C$, where an occurrence of $\Box B$ is introduced by weakening in every 1-ary GLR rule over $\Gamma, \Box B \Rightarrow C$. Let Δ be a multiset of formulas. We define $\Sigma^{\Delta \rightsquigarrow \Box B}$ by recursion as follows, by replacing occurrence $\Box B$ by Δ.

1. axiom
   (a) atom: $(\Gamma, p, \Box B \Rightarrow p)^{\Delta \rightsquigarrow \Box B} = (\Gamma, p, \Delta \Rightarrow p)$,
   (b) $\bot$: $(\Gamma, \bot, \Box B \Rightarrow C)^{\Delta \rightsquigarrow \Box B} = (\Gamma, \bot, \Delta \Rightarrow C)$;
2. unary connective rule: $(\Sigma' / \Gamma, \Box B \Rightarrow C)^{\Delta \rightsquigarrow \Box B} = (\Sigma'^{\Delta \rightsquigarrow \Box B} / \Gamma, \Delta \Rightarrow C)$;
3. binary connective rule:
   $(\Sigma_1 \ \ \Sigma_2 / \Gamma, \Box B \Rightarrow C)^{\Delta \rightsquigarrow \Box B} = (\Sigma_1^{\Delta \rightsquigarrow \Box B} \ \ \Sigma_2^{\Delta \rightsquigarrow \Box B} / \Gamma, \Delta \Rightarrow C)$;
4. GLR rule: $(\Sigma' / \Gamma, \Box B \Rightarrow C)^{\Delta \rightsquigarrow \Box B} = (\Sigma' / \Gamma, \Delta \Rightarrow C)$.

Lemma 3.16 *Let Σ be a cut-free derivation with end-sequent of the form $\Gamma, \Box B \Rightarrow C$, where an occurrence of $\Box B$ is introduced by weakening in every 1-ary* GLR *rule over $\Gamma, \Box B \Rightarrow C$. Let Δ be a multiset of formulas. Then $\Sigma^{\Delta \rightsquigarrow \Box B}$ is a well-defined cut-free derivation.*

Proof We use induction on the height n of derivation Σ. For $n = 0$, Σ is an initial sequent of the form $(\Gamma, p, \Box B \Rightarrow p)$ or $(\Gamma, \bot, \Box B \Rightarrow C)$. Then also $(\Gamma, p, \Delta \Rightarrow p)$ and $(\Gamma, \bot, \Delta \Rightarrow C)$ are initial sequents, since $\Box B$ is not an atom or $\bot$.

Suppose that $\vdash_n \ \Gamma, \Box B \Rightarrow C$ and that the last rule applied is a logical rule. Formula $\Box B$ cannot be a principal formula; therefore the premise(s) have the form

$\Gamma', \Box B \Rightarrow C'$ for some Γ' and C'. By the induction hypothesis, $\Gamma', \Delta \Rightarrow C'$ are derivable. Applying the logical rule again, we obtain $\vdash_n \Gamma, \Delta \Rightarrow C$.

Now suppose that the last rule applied is the GLR rule. We use the fact that $\Box B$ is introduced by weakening in any 1-ary GLR rule; therefore, introducing Δ with weakening also gives a correct application of the GLR rule. This gives us a correct derivation.

Note that cut-freeness is guaranteed by the fact that Σ is cut-free. □

Theorem 3.17 (Cut-admissibility) *Let*

$$\dfrac{\begin{array}{c}\Sigma_l\\ \Gamma \Rightarrow D\end{array} \qquad \begin{array}{c}\Sigma_r\\ D, \Delta \Rightarrow C\end{array}}{\Gamma, \Delta \Rightarrow C}\ \mathrm{cut}(D)$$

be a topmost cut. This can be transformed into a cut-free derivation with the same end-sequent.

Proof Let (d, w, h) be the induction value of $\mathrm{cut}(D)$, where d is the degree of the cut-formula D, w is the width of $\mathrm{cut}(D)$, and h is the cut-height. We have four cases:

(a) at least one of the premises is an axiom;
(b) both premises are not axioms and the cut formula D is not principal in the left premise;
(c) both premises are not axioms and D is principal in the left premise only;
(d) D is principal in both the left and right premise.

Case (a)
For case (a), we refer to Negri, von Plato, and Ranta [15].

Case (b)
In case (b), we assume that D is not principal in the left premise of the cut. This means that an L-rule is applied to the left premise (so no GLR rule). Here we look at the L∧-rule. The derivation

$$\dfrac{\dfrac{\begin{array}{c}\Sigma\\ A, B, \Gamma' \Rightarrow D\end{array}}{A \wedge B, \Gamma' \Rightarrow D}\ \mathrm{L}\wedge \qquad \begin{array}{c}\Sigma_r\\ D, \Delta \Rightarrow C\end{array}}{A \wedge B, \Gamma', \Delta \Rightarrow C}\ \mathrm{cut}(D)$$

with $\Gamma = A \wedge B, \Gamma'$ is transformed into the following derivation with one cut with degree d:

$$\dfrac{\dfrac{\begin{array}{c}\Sigma\\ A, B, \Gamma' \Rightarrow D\end{array} \qquad \begin{array}{c}\Sigma_r\\ D, \Delta \Rightarrow C\end{array}}{A, B, \Gamma', \Delta \Rightarrow C}\ \mathrm{cut}_1}{A \wedge B, \Gamma', \Delta \Rightarrow C}\ \mathrm{L}\wedge$$

We have $w(\mathrm{cut}(D)) = w(\mathrm{cut}_1)$, because the 2-ary GLR rules in the left premise of cut_1 remain the same as those in the left premise of $\mathrm{cut}(D)$. And we see that the cut-height of cut_1 is $h - 1$. So by induction we can remove cut_1.

For L∨ and L→, we refer to [15]. Note that the presence of the new parameter w does not affect the correctness of the proofs in a similar way as for L∧.

Case (c)

For case (c), we assume that D is not principal in the right premise of the cut and principal in the left. In this case we only have to focus on the last rule applied in the right premise. For proofs of nonmodal R- and L-rules, see [15]. Here we provide the details for case L $\to$ to see that the introduction of the width does not affect the proof. In the following, we omit subderivations Σ_l, and so on, for readability. Derivation

$$\dfrac{\Gamma \Rightarrow D \qquad \dfrac{D, A \to B, \Delta' \Rightarrow A \qquad D, B, \Delta' \Rightarrow C}{D, A \to B, \Delta' \Rightarrow C}\,\mathrm{L}{\to}}{\Gamma, A \to B, \Delta' \Rightarrow C}\,\mathrm{cut}(D)$$

with $\Delta = A \to B, \Delta'$ is transformed into the following derivation with two cuts of degree d:

$$\dfrac{\dfrac{\Gamma \Rightarrow D \qquad D, A \to B, \Delta' \Rightarrow A}{\Gamma, A \to B, \Delta' \Rightarrow A}\,\mathrm{cut}_1 \qquad \dfrac{\Gamma \Rightarrow D \qquad D, B, \Delta' \Rightarrow C}{\Gamma, B, \Delta' \Rightarrow C}\,\mathrm{cut}_2}{\Gamma, A \to B, \Delta' \Rightarrow C}\,\mathrm{L}{\to}$$

We have $w(\mathrm{cut}_1) = w(\mathrm{cut}_2) = w(\mathrm{cut}(D))$, since the width is defined solely on the basis of the left premise of the cut which is not changed in the transformation. The cut-height of both cuts is reduced to a height of at most $h - 1$.

For case (c), we are left with one more possibility, which is the case where the right premise ends with a GLR rule. Note that we are in the case where D is not principal. We can get rid of cut(D) as follows:

$$\dfrac{\Gamma \Rightarrow D \qquad \dfrac{\boxdot\Delta', \Box A \Rightarrow A}{D, \Pi, \Box\Delta' \Rightarrow \Box A}\,\mathrm{GLR}}{\Gamma, \Pi, \Box\Delta' \Rightarrow \Box A}\,\mathrm{cut}(D) \quad \rightsquigarrow \quad \dfrac{\boxdot\Delta', \Box A \Rightarrow A}{\Gamma, \Pi, \Box\Delta' \Rightarrow \Box A}\,\mathrm{GLR}$$

Case (d)

Now we turn to case (d), where formula D is principal in both the left and right premise. If D is not boxed, then again we refer to [15]. We work out the case in which $D = A \to B$ to see that the width does not change the proof. Derivation

$$\dfrac{\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\mathrm{R}{\to} \qquad \dfrac{A \to B, \Delta \Rightarrow A \qquad B, \Delta \Rightarrow C}{A \to B, \Delta \Rightarrow C}\,\mathrm{L}{\to}}{\Gamma, \Delta \Rightarrow C}\,\mathrm{cut}(A \to B)$$

can be transformed into the following derivation with three cuts:

$$\dfrac{\dfrac{\dfrac{\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\mathrm{R}{\to} \qquad A \to B, \Delta \Rightarrow A}{\Gamma, \Delta \Rightarrow A}\,\mathrm{cut}_1 \qquad \dfrac{A, \Gamma \Rightarrow B \qquad B, \Delta \Rightarrow C}{A, \Gamma, \Delta \Rightarrow C}\,\mathrm{cut}_2}{\Gamma, \Delta, \Gamma, \Delta \Rightarrow C}\,\mathrm{cut}_3}{\Gamma, \Delta \Rightarrow C}\,\text{contraction}$$

In cut_1 the degree and width remain d and w, but its cut-height reduces to at most $h - 1$. Both in cut_2 and cut_3 the degree of the cut-formula is lower than d. So in these cases it does not matter what happens with the width or height. So we have a cut-free derivation for $\Gamma, \Delta, \Gamma, \Delta \Rightarrow C$. Contraction gives us a cut-free derivation for $\Gamma, \Delta \Rightarrow C$.

Now we look at the most interesting case, where D is boxed, say, $D = \Box B$. Cut-formula $\Box B$ is principal in both the left and right premise, so the cut is as follows:

$$\dfrac{\mathrm{GLR}\ \dfrac{\begin{matrix}\Sigma_l\\ \boxdot\Gamma,\Box B \Rightarrow B\end{matrix}}{\Pi_l,\Box\Gamma\Rightarrow\Box B} \qquad \dfrac{\begin{matrix}\Sigma_r\\ \Box B, B,\boxdot\Delta,\Box C\Rightarrow C\end{matrix}}{\Pi_r,\Box B,\Box\Delta\Rightarrow\Box C}\ \mathrm{GLR}}{\Pi_l,\Pi_r,\Box\Gamma,\Box\Delta\Rightarrow\Box C}\ \mathrm{cut}(\Box B)$$

The reduction is divided into two cases:

(i) $w = w(\mathrm{cut}(\Box B)) = 0$. This means that in any 1-ary GLR rule ρ over $\boxdot\Gamma, \Box B \Rightarrow B$, we have that $\Box B$ is introduced by weakening in ρ. This means that we can apply Definition 3.15 and Lemma 3.16 to obtain a cut-free derivation $\Sigma_l^{\Box\Gamma\rightsquigarrow\Box B}$. (Strictly speaking, if $\boxdot\Gamma, \Box B \Rightarrow B$ is the conclusion of a GLR rule, we have that $\boxdot\Gamma, \Box\Gamma \Rightarrow B$ can be derived from Σ_l.) So the reduction is the following:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Sigma_l^{\Box\Gamma\rightsquigarrow\Box B}\\ \boxdot\Gamma,\Box\Gamma\Rightarrow B\end{matrix}}{\boxdot\Gamma\Rightarrow B}\ \text{contraction} \qquad \dfrac{\dfrac{\begin{matrix}\Sigma_l\\ \boxdot\Gamma,\Box B\Rightarrow B\end{matrix}}{\Box\Gamma\Rightarrow\Box B} \qquad \begin{matrix}\Sigma_r\\ \Box B,B,\boxdot\Delta,\Box C\Rightarrow C\end{matrix}}{B,\Box\Gamma,\boxdot\Delta,\Box C\Rightarrow C}\ \mathrm{cut}_1}{\dfrac{\boxdot\Gamma,\Box\Gamma,\boxdot\Delta,\Box C\Rightarrow C}{\dfrac{\boxdot\Gamma,\boxdot\Delta,\Box C\Rightarrow C}{\Pi_l,\Pi_r,\Box\Gamma,\Box\Delta\Rightarrow\Box C}\ \mathrm{GLR}}\ \text{contraction}}\ \mathrm{cut}_2$$

Since $\Sigma_l^{\Box\Gamma\rightsquigarrow\Box B}$ is a well-defined cut-free derivation of $\boxdot\Gamma, \Box\Gamma \Rightarrow B$, we are allowed to apply contraction to get a cut-free derivation of $\boxdot\Gamma \Rightarrow B$. We can eliminate cut_1 since its degree and width are d and w and its height is $h-1$. We can eliminate cut_2 because its degree is less than d.

(ii) For $w = w(\mathrm{cut}(\Box B)) > 0$, first note that, by inspection of the rules, in the backward direction of the proof tree, boxed formulas do not disappear in the antecedent of the sequents. We have that Σ_l is of the following form, where Θ_l is a single stub-derivation compatible with sequent $\Pi, \Box\Pi', \Box\Gamma', \Box B, \Box\Lambda \Rightarrow \Box A$:

$$\Sigma_l = \begin{matrix}\dfrac{\begin{matrix}\Sigma_l'\\ \boxdot\Gamma',\Box B,B,\boxdot\Lambda,\Box A\Rightarrow A\end{matrix}}{\Pi,\Box\Pi',\Box\Gamma',\Box B,\Box\Lambda\Rightarrow\Box A}\ \rho\\ \Theta_l\end{matrix}$$

So the topmost cut is

$$\dfrac{\mathrm{GLR}\ \dfrac{\begin{matrix}\dfrac{\begin{matrix}\Sigma_l'\\ \boxdot\Gamma',\Box B,B,\boxdot\Lambda,\Box A\Rightarrow A\end{matrix}}{\Pi,\Box\Pi',\Box\Gamma',\Box B,\Box\Lambda\Rightarrow\Box A}\ \rho\\ \Theta_l\\ \boxdot\Gamma,\Box B \Rightarrow B\end{matrix}}{\Pi_l,\Box\Gamma\Rightarrow\Box B} \qquad \dfrac{\begin{matrix}\Sigma_r\\ \Box B, B,\boxdot\Delta,\Box C\Rightarrow C\end{matrix}}{\Pi_r,\Box B,\Box\Delta\Rightarrow\Box C}\ \mathrm{GLR}}{\Pi_l,\Pi_r,\Box\Gamma,\Box\Delta\Rightarrow\Box C}\ \mathrm{cut}(\Box B)$$

where ρ is a 2-ary GLR rule over the left premise $\Pi_l, \Box\Gamma \Rightarrow \Box B$ and $\Box B$ is not introduced by weakening in ρ (so $\Box B \notin \Pi, \Box\Pi'$). We write $\Box\Gamma = \Box\Pi', \Box\Gamma'$ in the conclusion of ρ. The goal is to eliminate $\mathrm{cut}(\Box B)$.

Rule ρ is a 1-ary GLR rule over $\boxdot\Gamma, \Box B \Rightarrow B$, so there is no GLR rule in the segment between ρ and GLR. We can apply Lemma 3.14(2) to conclude that the following is a correct derivation of the sequent $\Box A, \Box\Gamma \Rightarrow \Box B$, which we call Φ:

$$\Phi = \begin{array}{c} \dfrac{\dfrac{\text{cut-free}}{\Box A, A, \Box A \Rightarrow A}}{\Box A, A, \Pi, \Box\Gamma, \Box B, \Box\Lambda \Rightarrow \Box A}\, \rho' \\ \Theta_l^{\Box A, A} \\ \dfrac{\Box A, A, \boxdot\Gamma, \Box B \Rightarrow B}{\Box A, \Box\Gamma \Rightarrow \Box B}\, \text{GLR} \end{array}$$

Now consider the following derivation with three cuts, where cut_1 and cut_2 cut on formula $\Box B$:

$$\dfrac{\dfrac{\dfrac{\Phi \qquad \begin{array}{c}\Sigma_l \\ \boxdot\Gamma, \Box B \Rightarrow B\end{array}}{\Box A, \Box\Gamma, \boxdot\Gamma \Rightarrow B}\,\text{cut}_1 \qquad \dfrac{\Phi \qquad \begin{array}{c}\Sigma_l' \\ \Box B, B, \boxdot\Lambda, \Box A \Rightarrow A\end{array}}{\Box A, \Box\Gamma, B, \boxdot\Lambda, \Box A \Rightarrow A}\,\text{cut}_2}{\Box A, \Box A, \Box A, \Box\Gamma, \Box\Gamma, \boxdot\Gamma, \boxdot\Lambda \Rightarrow A}\,\text{cut}_3}{\Box A, \boxdot\Gamma, \boxdot\Lambda \Rightarrow A}\,\text{contraction}$$

We first look at cut_1 and cut_2. We have $d(\text{cut}_1) = d(\text{cut}_2) = d$. When comparing the width of cut_1 and cut_2 to the width of $\text{cut}(\Box B)$, we see that the 2-ary rule ρ over the left premise in $\text{cut}(\Box B)$ is replaced by the 2-ary GLR rule ρ' in the left derivation Φ of cut_1 and cut_2. In the GLR applications ρ', $\Box B$ is derived by weakening, so ρ' does not contribute to the width in cut_1 and cut_2. Therefore, $w(\text{cut}_1) = w(\text{cut}_2) \leq w - 1$. In particular, the width becomes $w - 1$ or 0, where $w(\text{cut}_1) = w(\text{cut}_2) = 0$ in case $A = \Box B$ or $\Box A = \Box B$, because then $\Box B$ is also introduced by weakening in all other 2-ary GLR rules in $\Theta^{\Box A, A}$. Therefore, we can eliminate cut_1 and cut_2. Also, cut_3 is eliminable, because the degree of the cut-formula is $d - 1$. So we can apply contraction to get a cut-free proof Σ for $\Box A, \boxdot\Gamma, \boxdot\Lambda \Rightarrow A$.

Now consider

$$\dfrac{\dfrac{\begin{array}{c} \dfrac{\begin{array}{c}\Sigma \\ \Box A, \boxdot\Gamma, \boxdot\Lambda \Rightarrow A\end{array}}{\Pi, \Box B, \Box\Gamma, \Box\Lambda \Rightarrow \Box A}\,\rho'' \\ \Theta_l \\ \dfrac{\boxdot\Gamma, \Box B \Rightarrow B}{\Box\Gamma \Rightarrow \Box B}\,\text{GLR} \end{array} \qquad \begin{array}{c}\Sigma_l \\ \boxdot\Gamma, \Box B \Rightarrow B\end{array}}{\Box\Gamma, \boxdot\Gamma \Rightarrow B}\,\text{cut}_4}{\boxdot\Gamma \Rightarrow B}\,\text{contraction}$$

We can get rid of cut_4, because $w(\text{cut}_4) = w - 1$, since $\Box B$ is introduced by weakening in GLR rule ρ''. So there is a cut-free proof with end-sequent $\Box\Gamma, \boxdot\Gamma \Rightarrow B$. Applying contraction gives us a cut-free proof Σ' with conclusion $\boxdot\Gamma \Rightarrow B$. Now we can conclude the proof with the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Sigma' \\ \boxdot\Gamma \Rightarrow B\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\Sigma_l \\ \boxdot\Gamma, \Box B \Rightarrow B\end{array}}{\Box\Gamma \Rightarrow \Box B} \qquad \begin{array}{c}\Sigma_r \\ \Box B, B, \boxdot\Delta, \Box C \Rightarrow C\end{array}}{B, \Box\Gamma, \boxdot\Delta, \Box C \Rightarrow C}\,\text{cut}_5}{\boxdot\Gamma, \Box\Gamma, \boxdot\Delta, \Box C \Rightarrow C}\,\text{cut}_6}{\dfrac{\boxdot\Gamma, \boxdot\Delta, \Box C \Rightarrow C}{\Pi_l, \Pi_r, \Box\Gamma, \Box\Delta \Rightarrow \Box C}\,\text{GLR}}\,\text{contraction}$$

In this reduction, cut_5 and cut_6 are eliminable as in case (i) above yielding a cut-free proof of $\boxdot\Gamma, \Box\Gamma, \boxdot\Delta, \Box C \Rightarrow C$. Applying contraction gives us a cut-free proof. $\Box$

The elimination of cut_4 deserves more attention. We can remove the cut because the width reduces. Note that the cut-free derivation Σ does not have any effect on the calculation of the width of cut_4. This means that the elimination of $\text{cut}_1, \text{cut}_2$, and cut_3 does not affect the width of cuts lower in the tree. In [8], it is said that cut_4 is "shielded" by the GLR instance ρ''. This shielding is crucial.

The reduction in the proof of the theorem is quite complex and blows up the proof complexity enormously. See Overview 3.20 to see the cut-elimination of case d(ii) in one proof tree.

Sometimes it is easier to find cut-free proofs in a direct way. We illustrate this in the following example.

Example 3.18 We look at an example for $w(\text{cut}(\Box B)) > 0$. Take $B = \bot$. We consider the following derivation from Example 3.7, where $\Box\bot$ is principal in the 2-ary GLR rule ρ over $\Box(\Box D \to \bot) \Rightarrow \Box\bot$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\Box\bot, \bot, \Box D \Rightarrow D}{\boxdot(\Box D \to \bot), \Box\bot \Rightarrow \Box D}\,\rho
\qquad
\Box\bot, \Box(\Box D \to \bot), \bot \Rightarrow \bot
}{\boxdot(\Box D \to \bot), \Box\bot \Rightarrow \bot}\,\text{L}{\to}
}{\Box(\Box D \to \bot) \Rightarrow \Box\bot}\,\text{GLR}
\qquad
\dfrac{\Box\bot, \bot, \Box D \Rightarrow D}{\Box\bot \Rightarrow \Box D}
}{\Box(\Box D \to \bot) \Rightarrow \Box D}\,\text{cut}(\Box\bot)
$$

It is possible to reduce this derivation to a cut-free derivation using the reductions in Theorem 3.17. But the following gives a shorter cut-free derivation of the sequent $\Box(\Box D \to \bot) \Rightarrow \Box D$:

$$
\dfrac{
\dfrac{\boxdot(\Box D \to \bot), \Box D \Rightarrow \Box D \qquad \Box(\Box D \to \bot), \Box D, \bot \Rightarrow D}{\boxdot(\Box D \to \bot), \Box D \Rightarrow D}\,\text{L}{\to}
}{\Box(\Box D \to \bot) \Rightarrow \Box D}\,\text{GLR}
$$

The admissibility of cut implies the subformula property, consistency, and conservativity over IPC. The following corollary states the equivalence between the Hilbert calculus and our sequent calculus GL3i.

Corollary 3.19 *Formula $\bigwedge\Gamma \to C$ is provable in* iGL *if and only if sequent $\Gamma \Rightarrow C$ is derivable in* GL3i.

Proof For the proof from left to right, we prove that $\vdash_{\mathsf{iGL}} A$ implies $\vdash_{\mathsf{GL3i}} (\Rightarrow A)$. We show that the axioms and rules in the Hilbert system are derivable in GL3i. The intuitionistic tautologies are evident. For the K-axiom we have

$$
\dfrac{
\dfrac{
\dfrac{\boxdot(A \to B), \Box A, A, \Box B \Rightarrow A \qquad \Box(A \to B), \boxdot A, \Box B, B \Rightarrow B}{\boxdot(A \to B), \boxdot A, \Box B \Rightarrow B}\,\text{L}\to
}{\Box(A \to B), \Box A \Rightarrow \Box B}\,\text{GLR}
}{\Rightarrow \Box(A \to B) \to \Box A \to \Box B}\,\text{R}\to
$$

The other axioms are left for the reader. The cut rule is used in the proof for *modus ponens*.

The direction from right to left is done by induction on the height of the derivation of $\Gamma \Rightarrow C$. $\Box$

Overview 3.20 This is an overview of the cut-elimination reduction of the proof of Theorem 3.17 for a cut with cut-formula $\Box B$ with $w(\mathrm{cut}(\Box B)) > 0$. Double lines indicate contraction.

$$\dfrac{\stackrel{\Sigma_l'}{\boxdot\Gamma', \Box B, B, \boxdot\Lambda, \Box A \Rightarrow A}}{\Pi, \Box\Gamma, \Box B, \Box\Lambda \Rightarrow \Box A}\,\rho$$

$$\dfrac{\text{GLR}\,\dfrac{\stackrel{\Theta_l}{\boxdot\Gamma, \Box B \Rightarrow B}}{\Pi_l, \Box\Gamma \Rightarrow \Box B} \qquad \dfrac{\stackrel{\Sigma_r}{\Box B, B, \boxdot\Delta, \Box C \Rightarrow C}}{\Pi_r, \Box B, \Box\Delta \Rightarrow \Box C}\,\text{GLR}}{\Pi_l, \Pi_r, \Box\Gamma, \Box\Delta \Rightarrow \Box C}\,\mathrm{cut}(\Box B)$$

reduces to cut-free derivation

$$\dfrac{\dfrac{}{\Box A, A, \Box A \Rightarrow A}\,\text{cut-free}}{\Box A, A, \Pi, \Box\Gamma, \Box B, \Box\Lambda \Rightarrow \Box A}\,\text{GLR'} \qquad \dfrac{\dfrac{}{\Box A, A, \Box A \Rightarrow A}\,\text{cut-free}}{\Box A, A, \Pi, \Box\Gamma, \Box B, \Box\Lambda \Rightarrow \Box A}\,\text{GLR'}$$

$$\dfrac{\dfrac{\dfrac{\stackrel{\Theta_l^{\Box A, A}}{\Box A, A, \boxdot\Gamma, \Box B \Rightarrow B}}{\Box A, \Box\Gamma \Rightarrow \Box B}\,\text{GLR} \qquad \stackrel{\Sigma_l}{\boxdot\Gamma, \Box B \Rightarrow B}}{\Box A, \Box\Gamma, \boxdot\Gamma \Rightarrow B}\,\mathrm{cut}_1 \qquad \dfrac{\dfrac{\stackrel{\Theta_l^{\Box A, A}}{\Box A, A, \boxdot\Gamma, \Box B \Rightarrow B}}{\Box A, \Box\Gamma \Rightarrow \Box B}\,\text{GLR} \qquad \stackrel{\Sigma_l'}{\Box B, B, \boxdot\Lambda, \Box A \Rightarrow A}}{\Box A, \Box\Gamma, B, \boxdot\Lambda, \Box A \Rightarrow A}\,\mathrm{cut}_2}{\Box A, \Box A, \Box A, \Box\Gamma, \Box\Gamma, \boxdot\Gamma, \boxdot\Lambda \Rightarrow A}\,\mathrm{cut}_3$$

$$\dfrac{\Box A, \Box A, \Box A, \Box\Gamma, \Box\Gamma, \boxdot\Gamma, \boxdot\Lambda \Rightarrow A}{\dfrac{\Box A, \boxdot\Gamma, \boxdot\Lambda \Rightarrow A}{\Pi, \Box B, \Box\Gamma, \Box\Lambda \Rightarrow \Box A}\,\text{GLR}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\stackrel{\Theta_l}{\boxdot\Gamma, \Box B \Rightarrow B}}{\Box\Gamma \Rightarrow \Box B}\,\text{GLR} \qquad \stackrel{\Sigma_l}{\boxdot\Gamma, \Box B \Rightarrow B}}{\Box\Gamma, \boxdot\Gamma \Rightarrow B}\,\mathrm{cut}_4}{\boxdot\Gamma \Rightarrow B} \qquad \dfrac{\dfrac{\stackrel{\Sigma_l}{\boxdot\Gamma, \Box B \Rightarrow B}}{\Box\Gamma \Rightarrow \Box B} \qquad \stackrel{\Sigma_r}{\Box B, B, \boxdot\Delta, \Box C \Rightarrow C}}{B, \Box\Gamma, \boxdot\Delta, \Box C \Rightarrow C}\,\mathrm{cut}_5}{\boxdot\Gamma, \Box\Gamma, \boxdot\Delta, \Box C \Rightarrow C}\,\mathrm{cut}_6$$

$$\dfrac{\boxdot\Gamma, \Box\Gamma, \boxdot\Delta, \Box C \Rightarrow C}{\dfrac{\boxdot\Gamma, \boxdot\Delta, \Box C \Rightarrow C}{\Pi_l, \Pi_r, \Box\Gamma, \Box\Delta \Rightarrow \Box C}\,\text{GLR}}$$

4 Termination

In this section we turn to the terminating sequent calculus GL4i. There are various concepts of termination. We are interested in *strong termination*: there is a well-founded ordering on sequents such that, for all the rules in the calculus, the premises are smaller in this ordering than the conclusion. Strictly speaking, GL4i is not strongly terminating, but it has a property very close to it, which we call *termination modulo extended axioms*. We will see that all sequents of the form $\Gamma, C \Rightarrow C$ are provable in GL4i. Including these in the system yields strong termination. Termination modulo extended axioms is a key ingredient in the syntactic proof of equivalence between GL3i and GL4i (see Section 5). We prove termination modulo extended axioms for GL4i based on a loop-preventing proof search for a sequent calculus for GL from [3].

Calculus GL4i is an extension of the terminating calculus G4ip discovered independently by Dyckhoff [6] and Hudelmaier [9]. The extension is analogous to the extensions of G4ip to calculi for iK and iKD as developed in [10]. The propositional system G3ip is not strongly terminating, because in the standard ordering on sequents the left premise of the left implication rule L $\rightarrow$ does not decrease in complexity. However, it is *weakly terminating*; that is, there is a terminating process of deciding the derivability of a sequent involving a global check in the proof search (see [18]). Dyckhoff and Hudelmaier replaced L $\rightarrow$ by four left implication rules, corresponding to the outermost connective in A for principal formula $A \rightarrow B$. They defined an ordering on sequents to show that G4ip strongly terminates. To do so, define the *degree* d of formulas as $d(\bot) = d(p) = 1$, $d(A \vee B) = d(A \rightarrow B) = d(A) + d(B) + 1$, and $d(A \wedge B) = d(A) + d(B) + 2$. In addition, we have to deal with the modality and define $d(\Box A) = d(A) + 1$. This is slightly different from the standard degree which we use for GL3i. Dyckhoff's ordering extends to multisets in the following way, as in Dershowitz and Manna [5] : $\Gamma_0 \ll \Gamma_1$ if and only if Γ_0 is the result of replacing one or more formulas in Γ_1 by zero or more formulas of lower degree. Extend this to sequents: $(\Gamma_1 \Rightarrow C_1) \ll (\Gamma_2 \Rightarrow C_2)$ if $\Gamma_1, C_1 \ll \Gamma_2, C_2$.

Terminating calculi for iK and iKD from [10] strongly terminate in this ordering. This is not the case for GL4i, because the premise of the GLR rule is not lower than the conclusion with respect to $\ll$. Recall the rule

$$\frac{\boxdot\Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}\ \text{GLR}$$

where Π does not contain any boxed formulas. The premise of the rule is not necessarily lower than the conclusion with respect to the ordering $\ll$. Intuitively, the size of the sequent in the premise is "doubled" compared to the sequent in its conclusion, because Γ is "duplicated." The same problem arises in the left premise of L$\Box \rightarrow$. This means that the degree alone does not suffice to act as the induction parameter on sequents to ensure termination. We define the appropriate ordering in the proof of Theorem 4.2. Similar problems are discussed for termination of tableau systems for GL in Goré and Kelly [7].

We need the following lemma for GL4i. Compare this lemma to Lemma 3.1.

Lemma 4.1 (GL4i **weakening, contraction, inversion)** *For all n, we have the following in* GL4i.

1. *Falsum rule*: $\vdash_n \Gamma \Rightarrow \bot$ *implies* $\vdash_n \Gamma \Rightarrow C$.
2. *Weakening*: $\vdash_n \Gamma \Rightarrow C$ *implies* $\vdash_n \Gamma, A \Rightarrow C$.
3. *Inversion*: R$\wedge$, L$\wedge$, L$\vee$, R $\to$, L$p\to$, L$\wedge\to$, *and* L$\vee\to$ *are height-preserving invertible.*
4. *Inversion* L $\to\to$: *If* $\vdash_n \Gamma, (A \to B) \to C \Rightarrow D$, *then* $\vdash_n \Gamma, C \Rightarrow D$.
5. *Inversion* L$\Box\to$: *If* $\vdash_n \Pi, \Box\Gamma, (\Box A \to B) \Rightarrow C$, *then* $\vdash_n \Pi, \Box\Gamma, B \Rightarrow C$.
6. *Extended axiom*: $\vdash \Gamma, C \Rightarrow C$ *for every formula* C.
7. *Contraction*: $\vdash_n \Gamma, D, D \Rightarrow C$ *implies* $\vdash_n \Gamma, D \Rightarrow C$.

Proof Statements (1)–(5) and (7) are proved by induction on height n. Weakening is used in proofs for inversion. Inversion is used in the proof for contraction. Statement (6) is proved by induction on Dyckhoff's degree of formula C using inversion of R $\to$. □

The reason that we need extended axioms is due to the form of the GLR rule and L$\Box\to$. Infinite branches can occur by repeated GLR and L$\Box\to$ inferences. In the following tree, we have a loop where we can apply L$\Box\to$ with diagonal formula $\Box A$ infinitely many times, indicated by the vertical dots:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\boxdot(\Box A \to \bot), \boxdot C, \Box A, \Box A, A \Rightarrow A}\,\mathrm{L}\Box\to \qquad \dfrac{}{\Gamma', \bot \Rightarrow A}}{\boxdot(\Box A \to \bot), \boxdot C, \Box A \Rightarrow A}\,\mathrm{L}\Box\to \qquad \dfrac{}{\Gamma, \bot \Rightarrow C}}{\boxdot(\Box A \to \bot), \Box C \Rightarrow C}\,\mathrm{L}\Box\to}{\Box(\Box A \to \bot) \Rightarrow \Box C}\,\mathrm{GLR}$$

However, we see that we create an infinite branch for the provable sequent

$$\boxdot(\Box A \to \bot), \boxdot C, \Box A, \Box A, A \Rightarrow A,$$

where formula A occurs in both the antecedent and the conclusion of the sequent. In fact, we will see in the proof of termination that infinite branches always contain sequents of the form $\Gamma, C \Rightarrow C$. The reason is that there is a finite number of boxed subformulas in the end-sequent. For the GLR rule, we observe that each boxed formula may appear at most once in a conclusion of a sequent in a single branch, because it moves into the antecedent and stays there. For a second application of the GLR rule with the same boxed formula $\Box A$ we obtain an extended axiom $\Gamma, \Box A \Rightarrow \Box A$, which is provable. In the proof we will also take into account the left premise of L$\Box\to$, which may end in a provable sequent of the form $\Gamma, A \Rightarrow A$, as in the example above. This makes it possible to stop the proof search at that point in the tree, cutting off the infinite branch.

Theorem 4.2 *Proof search in* GL4i *is terminating modulo extended axioms.*

Proof Consider a proof search for sequent $\Gamma \Rightarrow C$. Let c be the number of all boxed subformulas in $\Gamma \Rightarrow C$ counted as a set. We have at most c different boxed formulas in an antecedent of a sequent in the proof search counted as a set. We use c to define the induction parameter. For a sequent $(\Delta \Rightarrow D)$ in the proof search, define $b(\Delta \Rightarrow D)$ to be the number of boxed formulas in Δ counted as a set. We have $c - b(\Delta \Rightarrow D) > 0$. We prove the theorem by induction on $(c - b(\Delta \Rightarrow D), \ll)$ ordered in a lexicographical way. Note that the first entry ranges over natural numbers with the standard ordering, and the second entry ranges over sequents with ordering $\ll$.

If a sequent is of the form $\Gamma, C \Rightarrow C$, then we are done by definition of termination modulo extended axioms. So suppose that it is not an extended axiom.

If we backwards apply a logical rule different from $L\Box\to$, then $c - b$ decreases or $c - b$ stays the same and the premises are lower with respect to $\ll$. Therefore, we can apply the induction hypothesis to conclude that the proof search terminates.

If we backwards apply $L\Box\to$ to a sequent of the form $\Pi, \Box\Gamma, \Box A \to B \Rightarrow C$, then for the right premise we have the same reasoning as above. For the left premise, we have two cases. If $\Box A$ is not contained in $\Box\Gamma$, then $c - b$ decreases, because Π contains no boxed formulas. If $\Box A$ is in $\Box\Gamma$, say, $\Box\Gamma = \Box\Gamma', \Box A$, then the left premise is of the form $\boxdot\Gamma', \Box A, \Box A, A \Rightarrow A$ and we close the branch, because it is an extended axiom. So we can apply the induction hypothesis to conclude that it terminates.

If we backwards apply the GLR rule, say, to a sequent of the form $\Pi, \Box\Gamma \Rightarrow \Box A$, then $c - b$ decreases since $\Box A$ is assumed not to be in $\Box\Gamma$, and Π in the GLR is assumed to not contain boxed formulas. Again we apply the induction hypothesis to conclude termination modulo extended axioms. $\square$

Note that the induction pair depends on the end-sequent. So this ordering cannot easily be used in general for GL4i for all sequents. In the proof it only works for sequents in the particular proof search tree. This makes it difficult to compare sequents from different proofs. However, for every c, this ordering easily extends to an ordering on sequents that have less than c boxed subformulas counted as a set.

Remark 4.3 In the proof we distinguish two cases when dealing with $L\Box\to$. Another way to deal with these cases is to replace $L\Box\to$ by the following rules in the system GL4i:

$$\frac{\Pi, \Box\Gamma, B \Rightarrow C}{\Pi, \Box\Gamma, \Box A, \Box A \to B \Rightarrow C}\ L\Box\!\to_1 \qquad \frac{\boxdot\Gamma, \Box A \Rightarrow A \qquad \Pi, \Box\Gamma, B \Rightarrow C}{\Pi, \Box\Gamma, \Box A \to B \Rightarrow C}\ L\Box\!\to_2$$

where $\Box A$ is not contained in $\Box\Gamma$ in rule $L\Box\to_2$. In this new system, termination can be proved using the same induction parameter.

5 Equivalence of GL3i and GL4i

We use the method from [6] and [10] to prove the equivalence between GL3i and GL4i in the sense that both derive the same sequents. This equivalence and the cut-admissibility result for GL3i immediately implies the cut-admissibility in GL4i.

Definition 5.1 A multiset is *irreducible* if it has no element that is a disjunction, conjunction, or $\bot$, and there is no atom p and formula A such that p and $p \to A$ are both contained in it. A sequent $\Gamma \Rightarrow C$ is *irreducible* if its antecedent Γ is.

Definition 5.2 A proof in GL3i is *sensible* if the following holds: if the last inference is $L\to$, then its principal formula is not of the form $p \to A$ for some atom p and formula A.

Definition 5.3 A proof in GL3i is *strict* if the following holds: if the last inference is $L\to$ with principal formula of the form $\Box A \to B$, then the left premise is an axiom or the conclusion of the GLR rule.

Lemma 5.4 *Every irreducible sequent provable in* GL3i *has a sensible strict proof.*

Proof For a contradiction, assume there are irreducible provable sequents that have no sensible strict proof. Consider, among all proofs of all such sequents, the proof Σ with the shortest leftmost branch. Since Σ is not sensible or not strict, Σ is of the form

$$\dfrac{\begin{array}{c}\Sigma_l\\ \Gamma, A \to B \Rightarrow A\end{array} \qquad \begin{array}{c}\Sigma_r\\ \Gamma, B \Rightarrow C\end{array}}{\Gamma, A \to B \Rightarrow C}\ \mathrm{L}\to$$

with principal formula $A \to B$, where A is atomic or A is boxed. Since the end-sequent is irreducible, we know that $\bot \notin \Gamma$ and if A is atomic, then $A \notin \Gamma$. Therefore, the left premise cannot be an axiom, but is derived from a rule, say, ρ. Formula A is atomic or boxed, so ρ is a left rule or the GLR rule. However, ρ cannot be the GLR rule, because the proof would then be strict and sensible.

So suppose that ρ is a left rule. Sequent $(\Gamma, A \to B \Rightarrow A)$ is irreducible and has a shorter leftmost branch in its proof than Σ. By the assumption, it has a strict and sensible proof Σ_0. Note that Σ_0 is not necessarily the same as Σ_l. In addition, since sequent $(\Gamma, A \to B \Rightarrow A)$ is irreducible, ρ is an instance of $\mathrm{L}\to$. Let $A' \to B'$ be its principal formula. Since Σ_0 is sensible, A' is not atomic. So we have a proof of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\Sigma_0'\\ \Gamma', A \to B, A' \to B' \Rightarrow A'\end{array} \qquad \begin{array}{c}\Sigma_0''\\ \Gamma', A \to B, B' \Rightarrow A\end{array}}{\Gamma', A \to B, A' \to B' \Rightarrow A}\ \rho \qquad \begin{array}{c}\Sigma_r\\ \Gamma', A' \to B', B \Rightarrow C\end{array}}{\Gamma', A \to B, A' \to B' \Rightarrow C}\ \mathrm{L}\to$$

Now consider the following proof with the same end-sequent:

$$\dfrac{\begin{array}{c}\Sigma_0'\\ \Gamma', A \to B, A' \to B' \Rightarrow A'\end{array} \qquad \dfrac{\begin{array}{c}\Sigma_0''\\ \Gamma', A \to B, B' \Rightarrow A\end{array} \qquad \begin{array}{c}\Sigma_1\\ \Gamma', B, B' \Rightarrow C\end{array}}{\Gamma', A \to B, B' \Rightarrow C}}{\Gamma', A \to B, A' \to B' \Rightarrow C}$$

where the existence of Σ_1 is justified by applying inversion from Lemma 3.1 to Σ_r. This proof is sensible, because A' is not atomic. In case A' is not boxed, we immediately see that the proof is also strict. If A' is boxed, then the proof is strict since Σ_0 is strict. $\square$

Lemma 5.5 *In* GL3i *we have* $\vdash \Gamma, (A \to B) \to C \Rightarrow A \to B$ *if and only if* $\vdash \Gamma, B \to C \Rightarrow A \to B$.

Proof Both $(A \to B) \to C \Rightarrow B \to C$ and $B \to C, A \Rightarrow (A \to B) \to C$ are provable sequents in GL3i. The admissibility of cut, contraction, and inversion of $\mathrm{R}\to$ implies the desired result. $\square$

Theorem 5.6 *The calculi* GL3i *and* GL4i *are equivalent, that is,* $\vdash_{\mathsf{GL3i}} S$ *if and only if* $\vdash_{\mathsf{GL4i}} S$.

Proof The proof from right to left is straightforward using the admissibility of cut, weakening, and contraction in GL3i. The proof is done by induction on the height of the proof of sequent S in GL4i. We write down the two interesting cases where the last inferences are $\mathrm{L} \to\!\!\to$ and $\mathrm{L}\Box\!\to$.

If $\mathrm{L} \to\!\!\to$ is the last rule, then S is of the form $\Gamma, (A \to B) \to C \Rightarrow D$ derived from premises $\Gamma, B \to C \Rightarrow A \to B$ and $\Gamma, C \Rightarrow D$. By the induction hypothesis, we know that those premises are also derivable in GL3i. By Lemma 5.5, we have $\vdash_{\mathsf{GL3i}} \Gamma, (A \to B) \to C \Rightarrow A \to B$. Applying L $\to$ to this sequent and $\Gamma, C \Rightarrow D$ we conclude that $\vdash_{\mathsf{GL3i}} \Gamma, (A \to B) \to C \Rightarrow D$.

If L$\Box \to$ is the last rule, then S is of the form $\Pi, \Box\Gamma, \Box A \to B \Rightarrow C$ derived from premises $\boxdot\Gamma, \Box A \Rightarrow A$ and $\Pi, \Box\Gamma, B \Rightarrow C$, where Π does not contain a boxed formula. By the induction hypothesis, we have that those premises are also derivable in GL3i. Applying the GLR rule from GL3i to $\boxdot\Gamma, \Box A \Rightarrow A$ we have $\vdash_{\mathsf{GL3i}} \Pi, \Box\Gamma \Rightarrow \Box A$. Weakening gives us a GL3i proof of $\Pi, \Box\Gamma, \Box A \to B \Rightarrow \Box A$. An application of L $\to$ to this sequent and $\Pi, \Box\Gamma, B \Rightarrow C$ shows that S is provable in GL3i.

The other direction is done by induction on the ordering of sequents similarly defined as in the terminating proof for GL4i as follows. Let S be provable in GL3i. Let c be the number of different boxed subformulas in S. So c is a constant. We fix the ordering on sequents containing c or less than c different boxed subformulas. For such a sequent S', let $b(S')$ be the number of different boxed formulas in its antecedent counted as a set. We know that $c - b(S') \geq 0$. We perform induction on the pair $(c - b(S'), \ll)$.

The case where S is an axiom is trivial, since both calculi have the same axioms. Also, for S of the form $\Gamma, A \Rightarrow A$ we are done. So assume that this is not the case. We distinguish between S being irreducible or not. If S is not irreducible, its antecedent contains a disjunction, conjunction, or both p and $p \to A$. For those cases we can backwards apply the rules L$\vee$, L$\wedge$, and L $\to$, respectively. The premises of those rules decrease in the ordering $(c - b, \ll)$, where for L $\to$ we only need to consider the right premise, since the left premise is of the form $\Gamma, p \to A, p \Rightarrow p$. By inversion in GL3i (see Lemma 3.1), those premises are derivable in GL3i. So we can apply the induction hypothesis to see that those premises are derivable in GL4i. Using the rules in GL4i gives the desired result. See [10] for details.

Now suppose that S is irreducible. By Lemma 5.4, we may assume that the proof of S is sensible and strict. The last rule ρ applied is a right rule, GLR or L $\to$. If ρ is a right rule, we inductively have a proof in GL4i, since the premises of those rules are lower in the ordering $(c - b, \ll)$ and the right rule ρ belongs to both calculi. When ρ is a GLR application, we have in GL3i

$$\frac{\boxdot\Gamma, \Box A \Rightarrow A}{S = (\Box\Pi, \Pi', \Box\Gamma \Rightarrow \Box A)}\ \text{GLR}$$

where Π' does not contain boxed formulas. We have $\vdash_{\mathsf{GL3i}} \boxdot\Pi, \boxdot\Gamma, \Box A \Rightarrow A$ by weakening. This sequent has c or less than c different boxed subformulas, so we can compare it to S in the ordering. Our assumption is that $\Box A$ does not appear in $\Box\Pi$ or $\Box\Gamma$, therefore making it smaller than S in the ordering. So we apply the induction hypothesis to conclude that $\vdash_{\mathsf{GL4i}} \boxdot\Pi, \boxdot\Gamma, \Box A \Rightarrow A$. Using GLR in GL4i gives us $\vdash_{\mathsf{GL4i}} S$.

For L $\to$, suppose that $S = (\Gamma, A \to B \Rightarrow C)$ with principal formula $A \to B$. Since the proof of S is sensible, A is not atomic. We continue with the different forms of A.

If $A = A_1 \to A_2$, then $S = (\Gamma, (A_1 \to A_2) \to B \Rightarrow C)$ is derivable in GL3i with premises $\Gamma, (A_1 \to A_2) \to B \Rightarrow A_1 \to A_2$ and $\Gamma, B \Rightarrow C$. By Lemma 5.5, we have that $\Gamma, A_2 \to B \Rightarrow A_1 \to A_2$ is derivable in GL3i. This sequent has c or less than c boxed subformulas, and we see that it is smaller than S in our ordering. The same holds for $\Gamma, B \Rightarrow C$. Hence we can apply the induction hypothesis to conclude that those are derivable in GL4i. Applying L $\to\!\to\!\to$ gives $\vdash_{\mathsf{GL4i}} S$.

The cases where A is a conjunction, disjunction, or $\bot$ are treated in a similar way. If $A = \Box A_1$, then the fact that the proof is strict implies that the left premise in L $\to$ is the conclusion of GLR. So we have in GL3i:

$$\dfrac{\dfrac{\boxdot\Gamma, \Box A_1 \Rightarrow A_1}{\Box\Pi, \Pi', \Box\Gamma, \Box A_1 \to B \Rightarrow \Box A_1}\,\text{GLR} \qquad \Box\Pi, \Pi', \Box\Gamma, B \Rightarrow C}{S = (\Box\Pi, \Pi', \Box\Gamma, \Box A_1 \to B \Rightarrow C)}\,\text{L}\to$$

with Π' not containing boxed formulas. By weakening, $\vdash_{\mathsf{GL3i}} \boxdot\Pi, \boxdot\Gamma, \Box A_1 \Rightarrow A_1$. This sequent and the right premise are smaller than S. The induction hypothesis and an application of L$\Box\!\to$ results in $\vdash_{\mathsf{GL4i}} S$. $\Box$

From the previous theorem and the cut-admissibility of GL3i in Theorem 3.17, we obtain the following.

Corollary 5.7 *The cut rule is admissible in* GL4i.

6 An Application: Craig Interpolation

We can use the admissibility of cut in GL3i in order to prove the Craig interpolation property for intuitionistic Gödel–Löb logic. The Craig interpolation property for a logic L is the statement that if $\vdash_L A \to B$, then there exists a formula I having only propositional variables shared by A and B such that $\vdash_L A \to I$ and $\vdash_L I \to B$. Such a formula I is called the *interpolant* of A and B. There are several generalizations of Craig interpolation for sequent calculi (see Mints [14]). Here we use the following characterization. We write $\mathrm{Var}(\Gamma)$ to mean all the atoms occurring in formulas from Γ.

Lemma 6.1 *Let $\Gamma_1, \Gamma_2 \Rightarrow C$ be provable in* GL3i *(Γ_1 and Γ_2 may be empty). Then there exists a formula I (*interpolant*) such that*

1. $\Gamma_1 \Rightarrow I$ *and* $\Gamma_2, I \Rightarrow C$,
2. $\mathrm{Var}(I) \subseteq \mathrm{Var}(\Gamma_1) \cap \mathrm{Var}(\Gamma_2, C)$.

Before we prove the lemma, we show how this implies the Craig interpolation property for intuitionistic Gödel–Löb logic.

Theorem 6.2 (Craig interpolation) *If $\vdash_{\mathsf{iGL}} A \to B$, then there exists a formula I (*interpolant*) such that*

1. $\vdash_{\mathsf{iGL}} A \to I$ *and* $\vdash_{\mathsf{iGL}} I \to B$,
2. $\mathrm{Var}(I) \subseteq \mathrm{Var}(A) \cap \mathrm{Var}(B)$.

Proof Suppose that $\vdash_{\mathsf{iGL}} A \to B$. By the interpretation of formulas in the sequent calculus GL3i, we have $\vdash_{\mathsf{GL3i}} A \Rightarrow B$. By Lemma 6.1, we can find a formula I such that $\vdash_{\mathsf{GL3i}} A \Rightarrow I$ and $\vdash_{\mathsf{GL3i}} I \Rightarrow B$ and $\mathrm{Var}(I) \subseteq \mathrm{Var}(A) \cap \mathrm{Var}(B)$. Again by the interpretation, we conclude that $\vdash_{\mathsf{iGL}} A \to I$ and $\vdash_{\mathsf{iGL}} I \to B$. Note that we immediately have the second requirement. $\Box$

Lemma 6.1 is proved by induction on the proof-height of $\Gamma_1, \Gamma_2 \Rightarrow C$. This is a well-known strategy when proving Craig interpolation (see, e.g., Takeuti [17]).

Proof of Lemma 6.1 We proceed by induction on the proof-height k of the derivation of $\Gamma_1, \Gamma_2 \Rightarrow C$. At each stage there are several cases to consider. We deal with some examples only.

1. $k = 0$ and $\Gamma_1, \Gamma_2 \Rightarrow C$ is derived by the At-rule; that is, $\Gamma_1, \Gamma_2 \Rightarrow C$ has the form $\Gamma, p \Rightarrow p$ for some p. There are two cases, $p \in \Gamma_1$ or $p \in \Gamma_2$. Take $I = p$ and $I = \bot \to \bot$, respectively.
2. $k = 0$ and $\Gamma_1, \Gamma_2 \Rightarrow C$ is derived by L$\bot$. There are two cases, $\bot \in \Gamma_1$ or $\bot \in \Gamma_2$. Take $I = C$ and $I = \bot \to \bot$, respectively.
3. $k > 0$ and the last rule applied is R$\wedge$:
$$\frac{\Gamma_1, \Gamma_2 \Rightarrow A \qquad \Gamma_1, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \wedge B}\,\mathrm{R}\wedge$$
Applying the induction hypothesis to both premises, we have that
   - there exists an interpolant I_1 such that $\Gamma_1 \Rightarrow I_1$ and $\Gamma_2, I_1 \Rightarrow A$ with $\mathrm{Var}(I_1) \subseteq \mathrm{Var}(\Gamma_1) \cap \mathrm{Var}(\Gamma_2, A)$;
   - there exists an interpolant I_2 such that $\Gamma_1 \Rightarrow I_2$ and $\Gamma_2, I_2 \Rightarrow B$ with $\mathrm{Var}(I_2) \subseteq \mathrm{Var}(\Gamma_1) \cap \mathrm{Var}(\Gamma_2, B)$.

   Take $I = I_1 \wedge I_2$ as the required interpolant. The cases for L$\wedge$, R$\vee$, and L$\vee$ are proved in a similar way.
4. $k > 0$ and the last rule applied is L $\to$. We have two cases, $A \to B \in \Gamma_1$ or $A \to B \in \Gamma_2$. We look at the first case. This case is somehow distinct from the other steps, in the sense that we apply the induction hypotheses to sequents where Γ_1 and Γ_2 are "reversed." Write $\Gamma_1 = \Gamma_1', A \to B$. We have
$$\frac{\Gamma_1', A \to B, \Gamma_2 \Rightarrow A \qquad \Gamma_1', B, \Gamma_2 \Rightarrow C}{\Gamma_1', A \to B, \Gamma_2 \Rightarrow C}\,\mathrm{L}\to$$
We now apply the induction hypothesis on the left premise in the following way:
   - there is an interpolant I_1 such that $\Gamma_2 \Rightarrow I_1$ and $\Gamma_1', A \to B, I_1 \Rightarrow A$ with $\mathrm{Var}(I_1) \subseteq \mathrm{Var}(\Gamma_2) \cap \mathrm{Var}(\Gamma_1', A \to B, A)$.

   For the second premise, we obtain that
   - there exists an interpolant I_2 for which $\Gamma_1', B \Rightarrow I_2$ and $\Gamma_2, I_2 \Rightarrow C$ with $\mathrm{Var}(I_2) \subseteq \mathrm{Var}(\Gamma_1', B) \cap \mathrm{Var}(\Gamma_2, C)$.

   Take $I = I_1 \to I_2$. It is easily shown that the second requirement of the lemma is fulfilled. For the first we have to show that $\Gamma_1', A \to B \Rightarrow I$ and $\Gamma_2, I \Rightarrow C$ are derivable. This is shown in the following derivation trees using the observations made before. Double lines indicate weakening:
$$\frac{\dfrac{\Gamma_1', A \to B, I_1 \Rightarrow A \qquad \dfrac{\Gamma_1', B \Rightarrow I_2}{\Gamma_1', B, I_1 \Rightarrow I_2}}{\Gamma_1', A \to B, I_1 \Rightarrow I_2}\,\mathrm{L}\to}{\Gamma_1', A \to B \Rightarrow I_1 \to I_2}\,\mathrm{R}\to \qquad \frac{\dfrac{\Gamma_2 \Rightarrow I_1}{\Gamma_2, I_1 \to I_2 \Rightarrow I_1} \qquad \Gamma_2, I_2 \Rightarrow C}{\Gamma_2, I_1 \to I_2 \Rightarrow C}\,\mathrm{L}\to$$
5. $k > 0$ and the last rule applied is GLR:
$$\frac{\boxdot\Gamma_1, \boxdot\Gamma_2, \Box B \Rightarrow B}{\Pi_1, \Box\Gamma_1, \Pi_2, \Box\Gamma_2 \Rightarrow \Box B}\,\mathrm{GLR}$$
We apply the induction hypothesis to the premise to obtain an interpolant I' such that $\boxdot\Gamma_1 \Rightarrow I'$ and $\boxdot\Gamma_2, \Box B, I' \Rightarrow B$ with $\mathrm{Var}(I') \subseteq \mathrm{Var}(\boxdot\Gamma_1) \cap \mathrm{Var}(\boxdot\Gamma_2, B)$. Take $I = \Box I'$. Weakening of both sequents with $\Box I'$ results in sequents $\boxdot\Gamma_1, \Box I' \Rightarrow I'$ and $\boxdot\Gamma_2, \Box B, I', \Box I' \Rightarrow B$. Now apply the

GLR rule to both to obtain the desired result, that is, $\Pi_1, \Box\Gamma_1 \Rightarrow I$ and $\Pi_2, \Box\Gamma_2, I \Rightarrow \Box B$ with $\mathrm{Var}(I) \subseteq \mathrm{Var}(\Pi_1, \Box\Gamma_1) \cap \mathrm{Var}(\Pi_2, \Box\Gamma_2, \Box B)$.

$\Box$

7 Conclusion and Future Research

This article consists of a proof-theoretic study of intuitionistic Gödel–Löb logic, with a focus on the single-conclusion sequent calculi GL3i and GL4i. The main results are the syntactic cut-admissibility proof and the termination proof for GL4i. What is especially interesting is that our cut-admissibility proof for GL3i highly depends on the structure of the calculus. The small difference in the definition of the GLR rule in GL4i compared to GL3i makes this proof strategy fail for GL4i. This example is one among others that asks for general syntactic treatments of cut-admissibility.

At the end of the paper, we proved Craig interpolation for iGL using the cut-free system GL3i. We conjecture that iGL also admits uniform interpolation. In [11], Iemhoff provides a uniform modular method to prove uniform interpolation for several intuitionistic modal logics using terminating calculi. It would be interesting to know whether this method can be extended to GL4i to prove uniform interpolation for iGL.

As mentioned in the introduction, there is a variety of semantic frameworks for iGL. We chose not to include a semantic study of iGL in this paper. However, we expect that the completeness result for the terminating calculus GL4i can also be proved by a counter-model construction similar to the one Avron [2] provides for the completeness result for GL.

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