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ABSTRACT

Shallow water inertio-gravity Poincaré waves in a rotating frame satisfy the Klein–Gordon equation, originally derived for relativistic, spinless quantum particles. Here, we compare these two superficially unrelated phenomena, suggesting a reason for them sharing the same equation. We discuss their energy conservation laws and the equivalency between the non-relativistic limit of the Klein–Gordon equation, yielding the Schrödinger equation, and the near-inertial wave limit in the shallow water system.

I. INTRODUCTION

The Klein–Gordon wave equation (hereafter KGE) was originally derived by Schrödinger in 1925[1] but was not published by him, as the equation failed to take into account the electron's spin. The equation was re-derived and published independently a year later by Klein[2] and by Gordon[3] in two separate papers. Generally, the equation can be written for a wave function Ψ as

$$\left(\frac{\partial^2}{\partial t^2} + f^2 - c^2\nabla^2\right)\Psi = 0, \tag{1}$$

where f and c are some constant frequency and speed, respectively, t denotes time, and the nabla denotes the gradient operator in 3D. Plane-wave solutions of the form $\Psi = \hat{\Psi}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ ($\mathbf{k}$ and ω are the wavenumber vector and frequency, respectively, and $\mathbf{x}$ denotes the position vector) yield the familiar dispersion relation

$$\omega^2 = f^2 + (kc)^2 \tag{2}$$

($k = |\mathbf{k}|$). For relativistic (spinless) quantum mechanics, the dispersion relation agrees with the Einstein relativistic energy–momentum relation

$$E^2 = E_0^2 + (pc)^2, \tag{3}$$

when $E_0 = mc^2 = \hbar f$ (where f is the reduced Compton frequency, so that $\lambda_{Compt} = c/f = \hbar/mc$ is the reduced Compton length) and the de Broglie postulates

$$E = \hbar\omega \quad \text{and} \quad \mathbf{p} = \hbar\mathbf{k} \tag{4}$$

are satisfied. Here, $E_0 = mc^2$ is the Einstein relativistic rest energy, m is the mass of the particle, and c is the speed of light. E and $\mathbf{p}$ are the energy and momentum of the quantum particle (where $p = |\mathbf{p}|$), respectively, and $\hbar$ is the reduced Planck constant.[4]

Consider, in turn, the momentum and continuity equations in the shallow water system, linearized about a basic state of rest, in a rotating frame with a rotation rate $f/2$, where f is the Coriolis frequency parameter[5]

$$\frac{\partial u}{\partial t} = fv - g\frac{\partial \eta}{\partial x}, \tag{5a}$$

$$\frac{\partial v}{\partial t} = -fu - g\frac{\partial \eta}{\partial y}, \tag{5b}$$

$$\frac{\partial \eta}{\partial t} = -H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{5c}$$

Here, g denotes gravity and (u, v) are the components of the velocity perturbations in the horizontal directions (x, y), H is the constant mean height of the shallow layer, and η is the height deviation from it. Writing the perturbation field as a plane wave normal mode $(u, v, \eta) = (\hat{u}, \hat{v}, \hat{\eta})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, one can bring (5) into the general form of KGE given in Eq. (1), where now the nabla indicates the horizontal gradient operator, with Ψ playing the role of any of the variables (u, v, η) to satisfy the Poincaré inertio-gravity wave dispersion relation given in (31). Now, f denotes the Coriolis parameter, and $c = \sqrt{gH}$ is the shallow water gravity wave's phase speed, so that $L_R = c/f = \sqrt{gH}/f$ is the Rossby deformation radius. The only difference is that, by construction, shallow water dynamics is restricted to the (x, y, t) space, whereas the general KGE is applied to the (x, y, z, t) space.

Although Poincaré is one of the major founders of both the theory of relativity[6] (along with Einstein) and the theory of tides[7] (along with Laplace), it is still hard to avoid asking why two such remote physical phenomena—the wave function dynamics of a quantum relativistic particle and the classical wave dynamics in a shallow water rotating system—obey the same equation. The analysis suggested here is aimed at providing some partial answers to this question. It is a follow-up of a previous paper in which we compared the propagation mechanism of inertio-gravity waves with the one of transverse electromagnetic waves in a cold collisionless plasma.[8]

In Sec. II, we consider the mechanical analog of the continuous limit of an infinite array of coupled harmonic oscillators, often invoked to provide a mental picture of a 1D quantum field,[9] to show how it could also be used to describe the Poincaré inertio-gravity waves in a rotating frame. Then in Sec. III, we compare the energy conservation laws for the quantum and fluid systems, where in Sec. IV, we compare their energy–momentum relations. In Sec. V, we consider the equivalency between the non-relativistic limit, in which the Schrödinger equation is extracted from the Klein–Gordon equation, and the near-inertial waves parabolic approximation range of (31). In Sec. VI, we conclude with a discussion about what we learned from the comparison between the two phenomena and suggestions for follow-up analyses.

II. MECHANICAL ANALOG

Consider an infinite 1D array of identical oscillators with the same natural frequency f and mass m, distributed evenly in the x direction with distance Δx between each other. The oscillators are coupled by springs of constant stiffness κ. Denote the small displacement from equilibrium of oscillator n by Ψ_n, it satisfies the momentum equation

$$\frac{\partial^2}{\partial t^2}\Psi_n = -f^2\Psi_n + \frac{\kappa}{m}\left[(\Psi_{n+1} - \Psi_n) - (\Psi_n - \Psi_{n-1})\right]. \tag{6}$$

The first term of the RHS represents the harmonic restoring force in isolation of oscillator n, while the second term denotes the coupling with the adjacent oscillators. [For instance, when the spring between oscillators n and $n+1$ is stretched, $(\Psi_{n+1} - \Psi_n) > 0$, and the spring between oscillators n and $n-1$ is compressed, $(\Psi_n - \Psi_{n-1}) < 0$, both interactions accelerate oscillator n in the positive direction of x.]

Equation (6) can be written equivalently as

$$\frac{\partial^2}{\partial t^2}\Psi(x,t) = -f^2\Psi(x,t) + \frac{\kappa}{m}[(\Psi(x+\Delta x, t) - \Psi(x,t)) \\ -(\Psi(x,t) - \Psi(x-\Delta x, t))]. \tag{7}$$

The 1D version of KGE is then recovered for the combined limit $\Delta x \to 0$ and $\sqrt{\frac{\kappa}{m}} \to \infty$, so that $\Delta x\sqrt{\frac{\kappa}{m}} = c$ (where c is finite). The generalization for a 3D array of oscillators is straightforward; however, for the sake of simplicity, we stay in 1D where x represents the direction of propagation.

Hence, without loss of generality, we assume that $\frac{\partial}{\partial y} = 0$ in (5). Define Ψ to be the small fluid particle displacement in the x direction, so that $u = \frac{\partial \Psi}{\partial t}$, then for modal dynamics (5b) and (5c) imply, respectively, that

$$v = -f\Psi, \tag{8a}$$

$$\eta = -H\frac{\partial \Psi}{\partial x}. \tag{8b}$$

Substituting back into (5a), we obtain the 1D version of KGE.

It seems, therefore, that the mechanical analog of coupled oscillators is the common denominator between the Poincaré propagation mechanism in rotating shallow water and the "mental model" of a scalar quantum field. The inertial oscillation frequency is analogous to the Compton frequency, and the action of the pressure gradient force is analogous to the coupling between the oscillators with the shallow water surface gravity wave's phase speed playing the role of the speed of light.

The analogy of the inertio-gravity waves to the coupled oscillators is illustrated in Fig. 1. This shows the vertical displacement of the free surface η (solid blue line) and associated vertically uniform horizontal velocities u, v (arrows and arrow-heads/tails in wave propagation and orthogonal directions, respectively) associated with a rightward propagating, rotationally modified surface gravity wave (see its location after a quarter cycle in red). The fluid element's horizontal displacement in the x-direction, ψ, is indicated by a green arrow. ψ can also refer to the small displacement of the coupled oscillator array described in (7), where the coupled oscillators are illustrated by simple pendulums whose natural frequency $f = \sqrt{\frac{g}{l}}$, where l is the length of the pendulum's string. The oscillating pendulums represent a metaphor for the inertial oscillations that each fluid element performs in the absence of free-surface displacements ($\eta = 0$, indicated by a blue dashed line), while the springs represent the pressure gradient forces associated with a sloping surface. For rightward propagation of the wave, u is in phase with the vertical displacement, η. Both Coriolis force (represented by the pendulum restoring force) as well as pressure gradient force (represented by coupling springs) displace the pendulum in

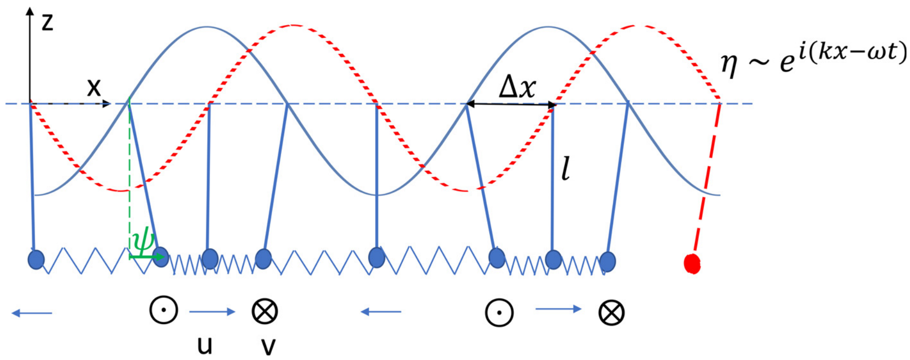

FIG. 1. Sketch of the mechanistic interpretation of the Klein–Gordon equation for long surface waves on a rotating plane. For further explanation, see the main text.

concert and together force its rightward propagation. For leftward propagation, u is in anti-phase with the vertical displacement so the blue arrows are now in the opposite direction but the arrow heads/tails (indicating the action of the Coriolis force) are left unchanged. Standing waves can be described as well by a superposition of leftward and rightward propagation; hence, u vanishes exactly under the crests and troughs of η. Note then that, apart from the horizontal arrows representing u, the snapshot in the figure of η, v and the pendulums' displacements fits equivalently for positive, negative, or zero (standing wave) propagation in the x direction.

III. ENERGY CONSERVATION

Defining $U \equiv \frac{\partial \Psi}{\partial t}$, the Hamiltonian of KGE can be obtained when multiplying (1) by U,

$$\frac{\partial}{\partial t}\mathcal{H} = \nabla \cdot (c^2 U \nabla \Psi), \tag{9a}$$

$$\mathcal{H} \equiv \frac{1}{2}\left[U^2 + (f\Psi)^2 + (c\nabla\Psi)^2\right], \tag{9b}$$

where $\mathcal{H}$ is the KGE Hamiltonian density.[10] Here, $\frac{1}{2}\Phi^2$ of a generic quantity $\Phi = Ae^{iS}$, possessing a complex amplitude A and real phase S, signifies its phase averaged value

$$\langle Re[\Phi]^2 \rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} Re[\Phi]^2 dS = \frac{1}{2}AA^*, \tag{10}$$

where an asterisk denotes the complex conjugate. For the conceptual model of continuous coupled oscillators, the first term of the Hamiltonian density can be regarded as the kinetic energy density of the oscillators, the second as the potential energy density stored in their intrinsic oscillation mechanism, and the third as the potential energy density resulting from the coupling between the oscillators. The term in the RHS of (9a) represents the energy flux convergence. For plane-waves satisfying dispersion relation (31), employing (10) it is straightforward to verify that

$$\frac{\partial}{\partial t}\mathcal{H} = \nabla \cdot (c^2 U \nabla \Psi) = -\nabla \cdot (\mathbf{c}_g \mathcal{H}), \tag{11a}$$

$$\mathbf{c}_g \equiv \nabla_{\mathbf{k}}\omega = \frac{c^2}{\omega}\mathbf{k}, \tag{11b}$$

where $\mathbf{c}_g$ denotes the group velocity. The energy flux convergence vanishes when integrated over the volume domain V, when assuming Dirichlet and/or Neumann boundary conditions on Ψ, or alternatively periodic boundary conditions. Then the KGE Hamiltonian becomes

$$H = \int \frac{1}{2}\left[U^2 + (f\Psi)^2 + (c\nabla\Psi)^2\right] dV. \tag{12}$$

For the continuous coupled oscillator system, plane wave solutions yield equipartition between the kinetic energy-like term of the Hamiltonian and the sum of the potential energy-like terms, associated with the self-oscillation and the coupling. When related to the Einstein energy–momentum relation (3), the first term is proportional to the square of the total energy, the second to the square of the rest energy, and the third to the square of the momentum term.

For the linearized rotating shallow water system (5), the energy E over area A can be written as the sum of the kinetic energy KE and the available (gravitational) potential energy PE,[5]

$$E = KE + PE = \int \rho \frac{H}{2}(u^2 + v^2) dA + \int \rho \frac{g}{2}\eta^2 dA, \tag{13}$$

where ρ is the constant density of the fluid. As is known in the literature, the presence of rotation alters the equipartition between kinetic and potential energy for surface gravity waves.[11] Although the Coriolis force does not perform work, the rotation affects the kinetic energy of the Poincaré inertio-gravity waves. For instance, denoting Ψ as the small fluid particle displacement in the x direction, in case we set $\frac{\partial}{\partial y} = 0$ and insert expressions (8) in (5), we find that, up to a common proportionality term, the energy partition reads

$$\underbrace{\omega^2}_{E} = \underbrace{\left[\frac{(kc)^2}{2} + f^2\right]}_{KE} + \underbrace{\frac{(kc)^2}{2}}_{PE}. \tag{14}$$

Here, evaluating (13) by employing (10), we use spectral fields, indicated by a hat, $(\hat{u}, \hat{v}) = \frac{1}{kH}(\omega, -if)\hat{\eta}$, obtained from (5c) and (8), respectively, so that in this normalization kinetic energy of motions in x and y directions contribute $\frac{1}{2}[(kc)^2 + f^2]$ and $\frac{1}{2}f^2$, respectively. Comparing (13) and (14) to (12) reveals that the Poincaré KE corresponds to the first two terms of the Hamiltonian $\int \frac{1}{2}[U^2 + (f\Psi)^2]dV$, where its available (gravitational) potential energy PE corresponds to the last term $\int \frac{1}{2}(c\nabla\Psi)^2 dV$. The analogy with the continuous coupled oscillators system suggests an explanation for this asymmetric energy partition. PE represents the energy signature of the pressure gradient force, $-g\nabla\eta$, which is analogous only to the coupling between the oscillators, whereas the rest of the oscillation energy, in the absence of coupling, is represented by *KE*. It is also straightforward to show that the energy flux $-c^2 U\nabla\Psi$, in the RHS of (9a), results from the work done by the wave pressure perturbation along the direction of propagation. For instance, in the case where we set $\frac{\partial}{\partial y}=0$ in (5) and having Ψ as the small fluid particle displacement in the x direction, $-c^2 U\frac{\partial\Psi}{\partial x}=u\,p'$, where here from (8b) the pressure perturbation is $p'=\rho g\eta=-c^2\frac{\partial\Psi}{\partial x}$ and $u=\frac{\partial\Psi}{\partial t}=U$.

It is interesting that KGE yields three different interpretations of the energy conservation law, when related to the continuous model of coupled oscillators (12), to the free relativistic, spinless quantum particle (3), and to the inertio-gravity waves in the rotating shallow water system (14).

IV. ENERGY–MOMENTUM RELATION

The de Broglie postulates (4) satisfy the fundamental wave energy–momentum relation

$$E=p\,c_p, \tag{15}$$

where $c_p=\frac{\omega}{k}$ is the wave phase speed. Applied to a quantum particle, for the non-relativistic case $E=\frac{p^2}{2m}$ (where v, $p=mv$, and $E=\frac{mv^2}{2}$ are the particle speed, momentum, and kinetic energy, respectively), Eqs. (4) and (15) imply that $v=c_g=2c_p$, where $c_g=\frac{d\omega}{dk}$ is the wave group speed.

For relativistic particles, it can be shown,[12] after some math, that relations (4), (31), and (15) hold when

$$E=\gamma mc^2,\quad p=\gamma mv,\quad v=c_g,\quad c^2=c_pc_g, \tag{16}$$

where

$$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{17}$$

represents the Lorentz contraction factor.

For classical waves, relation (15) holds in the average sense[13]

$$\langle E\rangle=\langle p\rangle c_p, \tag{18}$$

where here averaging, indicated by brackets, is over a wavelength, $\langle(\cdot)\rangle\equiv\frac{1}{\lambda}\int_0^\lambda(.)dx$. The total average energy of the Poincaré waves is[11]

$$\langle E\rangle=2\frac{\omega^2}{(kc)^2}\langle PE\rangle=2\frac{c_p^2}{c^2}\langle PE\rangle, \tag{19}$$

where $\langle PE\rangle=\frac{1}{4}\rho g\hat{\eta}^2$. The average wave-momentum, integrated over the total water depth from the bottom at $z=0$ to surface at $z=H+\eta$, is

$$\langle p\rangle=\left\langle\int_0^{H+\eta}\rho u\,dz\right\rangle=\left\langle\int_0^{\eta}\rho u\,dz'\right\rangle=\langle\rho\,u\,\eta\rangle, \tag{20}$$

where we use that density ρ and velocity u are z-independent and u is a plane wave. Hence, the integral to the mean, undisturbed surface, $z=H$, vanishes, leaving the net momentum flux to result from the surface displacement. For instance, for rightward propagation (Fig. 1), positive (negative) values of u correlate with crests (troughs) of η; thus, the vertical integration yields net positive eastward momentum in the direction of the wave propagation. Using again $\frac{\partial}{\partial y}=0$ in (5), from (8) we obtained $\hat{u}=\frac{\omega}{kH}\hat{\eta}$ that we now use to evaluate (20) as

$$\langle p\rangle=\langle\rho\,u\,\eta\rangle=\frac{\omega}{kc^2}\frac{1}{2}\rho g\hat{\eta}^2=2\frac{c_p}{c^2}\langle PE\rangle, \tag{21}$$

which indeed differs from the average energy (19) by a multiplication factor equal to the phase speed c_p.

A similar result can be derived for the chain of pendulums. For this, first associate the local density with the spatial distribution of the bobs attached to the pendulums (see Fig. 1). In the equilibrium rest position, the distance between every two adjacent pendulums is $x_{j+1}-x_j=\Delta x$, so the mean constant density of the bobs is $\rho=\frac{1}{\Delta x}$. When the pendulums are displaced, the in-between distance between two adjacent bobs becomes $\Delta x+(\psi_{j+1}-\psi_j)\approx\Delta x\left(1+\frac{\partial\psi}{\partial x}\right)$, assuming small displacements of the bobs. Thus, the local bob density becomes $\rho\left(1-\frac{\partial\psi}{\partial x}\right)$. As $-\frac{\partial\psi}{\partial x}=\eta$ in the continuum limit and $\langle u\rangle=0$, we obtain that $\langle p\rangle=\left\langle\rho\left(1-\frac{\partial\psi}{\partial x}\right)u\right\rangle=\langle\rho u\eta\rangle$. Using $u=\frac{\partial\Psi}{\partial t}$, this evaluates to $\langle p\rangle=\frac{1}{2}\rho\omega k|\hat{\Psi}|^2$, which indeed differs by a multiplication factor equal to the phase speed from the average energy obtained from (12) [as well as the constant density ρ which is assumed unity in (12)].

V. THE NON-RELATIVISTIC LIMIT AND NEAR-INERTIAL WAVES

The non-relativistic limit of the Klein–Gordon equation and the near-inertial wave limit are both central to their own respective fields of quantum mechanics and geophysical fluid dynamics. The non-relativistic limit yields the Schrödinger equation,[14] whereas near-inertial waves are the dominant mode of high-frequency variability in the ocean.[15] In this section, we show that the two limits are equivalent.

Beginning with the quantum field aspect, we write (31) in its non-dimensional form

$$\frac{\omega}{f}=\sqrt{1+(k\lambda_{Compt})^2}, \tag{22}$$

where the frequency ω is considered positive and recall that f is the reduced Compton frequency, satisfying $\hbar f=mc^2$. Now $k\lambda_{Compt}=\hbar k/mc=p/mc=\gamma\frac{v}{c}$, where we used the de Broglie postulate for the momentum (4) and its relativistic expression (16). For non-relativistic dynamics, $\frac{v}{c}\equiv\epsilon\ll 1$, thus using the definition of the Lorentz contraction factor $k\lambda_{Compt}=\frac{\epsilon}{\sqrt{1-\epsilon^2}}\to\epsilon$ in the non-relativistic limit. Expanding then (22) for small values of $k\lambda_{Compt}$, we obtain the parabolic approximation for the frequency

$$\frac{\omega}{f}=1+\frac{1}{2}(k\lambda_{Compt})^2+O\Big((k\lambda_{Compt})^4\Big). \tag{23}$$

Dropping the fourth order terms of $k\lambda_{Compt}$ and defining the non-relativistic frequency limit as the difference between the frequency and the reduced Compton frequency: $\omega_{NR}\equiv\omega-\frac{mc^2}{\hbar}$, we obtain, after multiplying by $\hbar$, the familiar de Broglie matter wave dispersion relation

$$E_{NR}\equiv\hbar\,\omega_{NR}=\frac{(\hbar k)^2}{2m}=\frac{p^2}{2m}, \tag{24}$$

which forms the basis for the Schrödinger equation, just as the Einstein relativistic energy–momentum relation (3) together with the dispersion relation of (31) form the basis for Klein–Gordon equation (1).

The formal way to obtain the Schrödinger equation from KGE is to note that the non-relativistic energy $E_{NR} = \hbar(\omega - f) = E - E_0$ is the small residual available energy in this limit; thus, we can regard the non-relativistic field Φ as a (complex) carrier's amplitude of the relativistic field Ψ, whose base frequency is f,

$$\Psi = \Phi e^{-ift}, \tag{25}$$

so that if $\Psi \sim e^{-i\omega t}$ then $\Phi \sim e^{-i\omega_{NR} t}$, where $\omega_{NR}/f \to \epsilon^2/2$ in the non-relativistic limit. Substituting Ψ back in (1), we obtain

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2\left(i\frac{2m}{\hbar}\frac{\partial \Phi}{\partial t} + \nabla^2\Phi\right). \tag{26}$$

Dividing (26) by f^2 on both sides, then it is straightforward to show that the LHS terms are of the order of ϵ^4, whereas the two terms on the RHS are each of the order of ϵ^2. Therefore, in the non-relativistic limit, the LHS term is negligible compared to the RHS; thus, the terms in brackets yield the Schrödinger equation

$$i\hbar\frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Phi. \tag{27}$$

Returning to the geophysical context [so that now the reduced Compton frequency and wavelength in (22) are replaced by the Coriolis frequency and the Rossby deformation radius, respectively], the two "textbook limits" that are usually considered[5] are the short wave limit of (non-rotational) surface gravity waves $(kL_R) \gg 1 \Rightarrow \omega \to \pm kc$ and the long wave limit of inertial oscillations $(kL_R) \ll 1 \Rightarrow \omega \to f$. Geometrically, the first limit is represented in Fig. 2 by the two linear dash-dotted lines having slopes of $\pm 45°$, whereas the second is represented by the horizontal dotted line, tangent to the minimum of ω at $k = 0$. The equivalent non-relativistic-like parabolic approximation in which time and space are scaled with ϵ^2 and ϵ, respectively, leading to $(kL_R) = \epsilon < 1 \Rightarrow \omega \to f[1 + \frac{1}{2}(kL_R)^2]$, can be, therefore, regarded as the limit corresponding to the dynamics of near-inertial waves (Fig. 2). Following the same logic for the geophysical context, we then obtain the equivalent Schrödinger equation

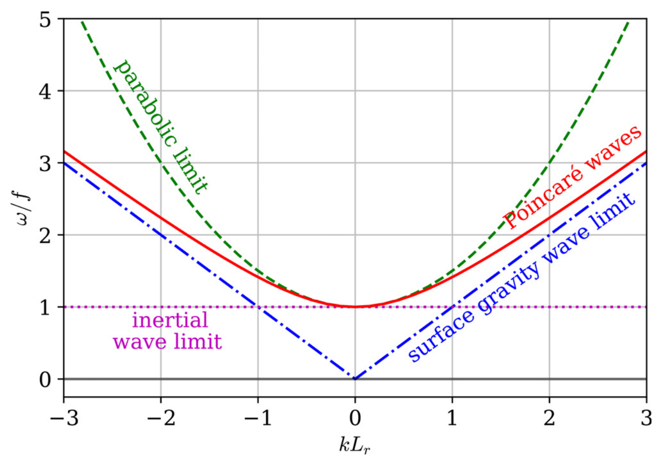

FIG. 2. Dispersion relation of the Poincaré waves $\omega/f = \sqrt{1 + (kL_R)^2}$, indicated by the solid (red) curve. The frequency ω is scaled by the Coriolis frequency f, and the wavenumber k is scaled by the Rossby deformation radius L_R. The other lines denote the limit of surface gravity waves $(kL_R) \gg 1 \Rightarrow \omega/f \to \pm kL_R$ $(\Rightarrow \omega \to \pm kc$, blue dashed-dotted curve), the long wave limit of inertial oscillations $(kL_R) \ll 1 \Rightarrow \omega/f \to 1$ (magenta dotted curve), and the parabolic limit $(kL_R) < 1 \Rightarrow \omega/f \to 1 + \frac{1}{2}(kL_R)^2$ (green dashed curve).

$$i\frac{\partial \Phi}{\partial t} = -\left(\frac{c^2}{2f}\right)\nabla^2\Phi. \tag{28}$$

Noting that in the quantum context, upon dividing (27) by $\hbar$ and using that $\hbar f = mc^2$, we retrieve this exact form except for the relevant change of meaning of c and f.

The near-inertial wave limit is illustrated in terms of coupled oscillators in Fig. 3. These are seen to be oscillating nearly in phase, owing to the weak coupling by pressure gradient forces. This is weak due to the large scale over which deflections of the surface appear, which also explains why the pendulums have to be considered nearly but not exactly in phase. Clearly, a better view of the inertial-gravity wave in this limit (as metaphorically expressed by the pendulums) is obtained by going to a co-oscillating frame in which the slow and slight deviations in inclination and spatial variations would become visible. In the physical plane, this would amount to a coordinate frame whose origin traverses a circle while the axes maintain their original orientation. This is what the demodulation (25) establishes. It suggests that the two restoring forces, viz., that due to the metaphor-gravity acting on the pendulums and the springs, start balancing each other due to phase delays involved. This quasi balance can be understood when substituting $\Phi \sim e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{NR} t)}$ in the right-hand side of (26) to obtain

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2\left(\frac{2m}{\hbar}\omega_{NR} - k^2\right)\Phi. \tag{29}$$

As ω_{NR} is defined positive, the two terms in the RHS act one against each other, and under the non-relativistic/near-inertial approximation, they are assumed to be in balance.

A different way to look at the Schrödinger equation as an approximation to the Klein–Gordon equation is to substitute back $\Phi = \Psi e^{ift}$ in the Schrödinger equation (27), take the time-derivative of that equation, but using this equation to replace first-order time derivatives of Ψ. This yields a modified Klein–Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} + f^2 - c^2\nabla^2\right)\Psi = -\left(\frac{c^2}{2f}\right)^2\nabla^4\Psi, \tag{30}$$

accounting indeed for the modified parabolic dispersion relation

$$\omega^2 = f^2 + (kc)^2 + \left[\frac{(kc)^2}{2f}\right]^2 \quad \Rightarrow \quad \omega = f\left[1 + \frac{1}{2}\left(\frac{kc}{f}\right)^2\right]. \tag{31}$$

Thus, the physical meaning of the Schrödinger-like approximation to the Klein–Gordon equation, both in the quantum and the geophysical fluid dynamics (GFD) setups, can be understood as adding a short scale (proportional to k^4) restoring force.

Based on multiple timescale separation, the same substitution of (25) has been suggested for Boussinesq flow,[16] and later for the shallow water system, to study the propagation of near-inertial waves in geostrophic currents.[17] Indeed, when neglecting the nonlinear advection terms, one obtains (28) with an additional potential, which is proportional to the vorticity of the steady geostrophic current.[17]

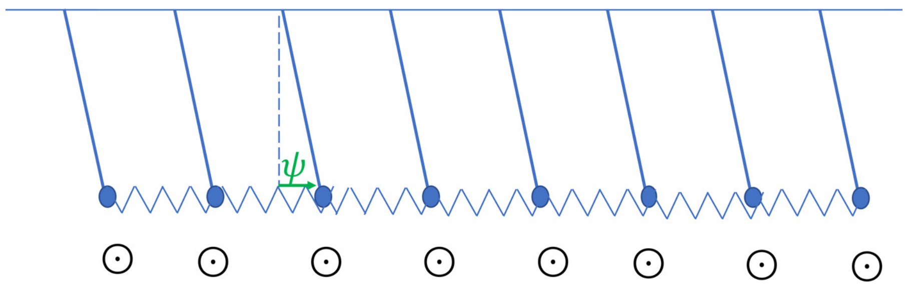

FIG. 3. Sketch of the mechanistic interpretation of the Klein–Gordon equation for long surface waves on a rotating plane when the pendulums turn synchronous for waves having infinite length scale and infinitesimal amplitudes to which the near-inertial limit $\epsilon \longrightarrow 0$ applies.

VI. DISCUSSION

When an equation describes seemingly two different phenomena in unrelated areas, it is intriguing to investigate whether a common underlying mechanism governs their dynamics.

Here, we suggest that the common denominator between the dynamics of relativistic quantum particles and the propagation of classical inertio-gravity waves is the mechanical analog paradigm of the continuous limit of a set of coupled oscillators. While this model (and its generalization to a 3D lattice) is invoked in the literature to provide a mental picture of quantum fields, its relevancy to the Poincaré waves, where the natural frequency of the oscillators is the Coriolis (inertial) frequency and the coupling is mediated by the pressure gradient force, seems to be new.

From the quantum mechanics aspect, the Madelung transformation[18] of the Schrödinger equation into the equations of a compressible fluid has been shown to provide a different point of view on quantum phenomena such as tunneling.[19] While the Madelung transformation can be extended to include relativistic dynamics,[20] we are not familiar with an extension in which the energy associated with the rotation of the fluid system becomes equivalent to the relativistic rest energy. Regarding the mechanical analog, it was shown by the authors in a previous paper[21] that the rotation of a shear flow of an ideal gas with zero absolute vorticity yields the solutions of the ground state harmonic oscillator, and the physical mechanism behind it is indeed the inertial oscillations. We do not know, however, whether these associations as well as of the surface gravity wave speed (or, equivalently, the speed of sound in compressible fluids) with the speed of light are merely analogies or that they contain some deeper meaning.

The universality of the Poincaré dispersion relation and its association with the Klein–Gordon equation describing both relativistic quantum particles as well as inertio-gravity waves in geophysical fluid dynamics (and also electromagnetic waves in cold plasmas[8]) is interesting by itself. The one-to-one correspondence between the non-relativistic limit and the near-inertial waves limit allows us both to define its "parabolic limit" as well as to understand why applying multi-timescale separation for near-inertial waves yields the Schrödinger equation, although the waves' dynamics is purely classical.

We envisage possible extensions of the KG and Schrödinger equations involving nonlinearities. For instance, in the shallow water wave context, this is possible by the following approach. Consider uni-directional ($\frac{\partial}{\partial y} = 0$) long surface gravity waves, $\eta(x,t)$, propagating on water of uniform depth H, so that total water depth $h = H + \eta$. Then, using the total derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$, the nonlinear shallow water equations read

$$\frac{Du}{Dt} = fv - g\frac{\partial h}{\partial x}, \tag{32a}$$

$$\frac{Dv}{Dt} = -fu, \tag{32b}$$

$$\frac{Dh}{Dt} = -h\frac{\partial u}{\partial x}. \tag{32c}$$

We combine these equations by taking the total time derivative of (32a), insert (32b), and in the near-inertial limit approximate total depth h by H, neglecting the weak surface elevation $\eta \ll H$, except when it is differentiated. However, we also neglect a nonlinear term containing the surface slope $\frac{\partial \eta}{\partial x}$ as this is small in this limit where the waves have large length scale. In contrast, since horizontal velocities are not small we do retain their nonlinear product terms so that these equations combine into

$$\left(\left[\frac{\partial^2}{\partial t^2} + f^2 - c^2\frac{\partial^2}{\partial x^2}\right] + 2u\frac{\partial^2}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial}{\partial t} + \left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2}{\partial x^2}\right)u = 0. \tag{33}$$

Here, in square brackets, we recognize the linear KG operator. The last two nonlinear terms in (33) appear to dynamically modify the Coriolis frequency, f, and long-wave speed, c, respectively. Quasi-linearization could handle this by replacing the u-dependent factors in the operator by their wavelength averages, leading to renormalized effective frequency and wave speed. Full nonlinear equation (33) can be seen as a nonlinear modification of the KGE (1) to be solved numerically in the subsequent work. This equation, describing bi-directional long surface waves, may offer an alternative for the well-studied uni-directional Ostrovsky equation (originally derived for internal wave modes),[22] extending the Korteweg–de Vries equation by adding the influence of Earth's rotation.[23] Of course, (33) also invites applying the demodulation (25) that, upon neglecting slow-time evolution of the carrier amplitude field, previously led to the Schrödinger equation, but as this requires some further phase-averaging this is left for future work.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Eyal Heifetz: Conceptualization (lead); Formal analysis (lead); Investigation (lead); Writing – original draft (lead); Writing – review & editing (equal). **Leo R. M. Maas:** Conceptualization (supporting); Formal analysis (supporting); Investigation (supporting); Visualization (supporting); Writing – original draft (supporting); Writing – review & editing (equal). **Julian Mak:** Funding acquisition (equal); Visualization (lead); Writing – review & editing (equal). **Ishay Pomerantz:** Conceptualization (supporting); Funding acquisition (equal); Writing – review & editing (supporting).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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