Journal of Magnetism and Magnetic Materials 557 (2022) 169426

Contents lists available at ScienceDirect

Journal of Magnetism and Magnetic Materials

journal homepage: www.elsevier.com/locate/jmmm

Research articles

Theory of the dipole-exchange spin wave spectrum in ferromagnetic films with in-plane magnetization revisited

J.S. Harms [a,*], R.A. Duine [a,b]

[a] Institute for Theoretical Physics, Utrecht University, 3584CC Utrecht, The Netherlands
[b] Department of Applied Physics, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

ARTICLE INFO

Keywords:
Magnetism
Spin waves
Thin films
Magnonics

ABSTRACT

We present a refinement of the widely accepted spin-wave spectrum that Kalinikos and Slavin (Kalinikos, 1981; Kalinikos and Slavin, 1986) computed for magnetic films with an in-plane magnetization. The spin wave spectrum that follows from the diagonal approximation in this theory becomes inaccurate for relatively thick films, as has already been noted by Kreisel et al. (2009). Rather than solving an integrodifferential equation which follows from the magnetostatic Green's function, as done by Kalinikos and Slavin (Kalinikos, 1981; Kalinikos and Slavin, 1986), we impose the exchange and magnetostatic boundary conditions on bulk spin-wave solutions. This boundary problem has an accurate analytical solution which is quantitatively different from the commonly used diagonal theory (Kalinikos, 1981; Kalinikos and Slavin, 1986) for magnetic films.

1. Introduction

Dipole-exchange spin waves propagating in in-plane magnetized magnetic thin films have attracted lot of attention in recent years, due to their potential applications in magnonic devices [1]. Of special interest is the case in which spin waves travel perpendicular to the external magnetic field – in which case the spin wave velocity is the largest. As noted by Kreisel et al. [2] the spin wave spectrum that follows from the diagonal approximation in the commonly used theory [3,4] is inaccurate in this case for relatively thick films. The inaccuracy stems from the diagonal approximation, and disappears when solving the system numerically with interband interactions. This approach on the other hand is not feasible for analytic approximations.

In this article, we present an alternative analytic derivation of the dipole-exchange spin wave spectrum for this scenario. Rather than solving an integrodifferential equation following from the magnetostatic Green's functions as done by Kalinikos and Slavin [3,4], we use an approach resembling that of Wolfram and DeWames [5] which previously had no analytical solution. A similar approach has been used by Sonin [6] to derive the spectrum of spin waves propagating parallel to an in-plane magnetic field for sufficiently large wave numbers. Moreover, Arias [7] used this approach to find the spin-wave dispersion numerically and presented a uniform mode solution in the ultrathin-film limit similar to the uniform mode solution in the diagonal approximation by Kalinikos and Slavin [3,4]. Kalinikos and Slavin [3,4] approximately solved the integrodifferential equations by assuming a superposition of magnetization profiles which satisfy the exchange boundary conditions but do not satisfy the bulk equations of motion. Here, however we impose both the exchange and magnetostatic boundary conditions on bulk spin wave solutions. This boundary problem turns out to have an accurate analytical solution – compared with the numerical spectrum – and is quantitatively different from the commonly used diagonal spin wave theory [3,4] for relatively thick films.

2. Thin-film ferromagnet

2.1. Model and set-up

We consider the set-up in Fig. 1 of a ferromagnetic thin film of thickness L subject to an in-plane external magnetic field $\mathbf{H}_e$. We chose the $x-y$ axes to correspond to the in-plane directions, with the external magnetic field $\mathbf{H}_e = H_e\hat{y}$ pointing in the y direction. Furthermore, the z axis corresponds to the out of plane direction where the surfaces of the thin film are located at $z = \pm L/2$. For temperatures below the Curie temperature, amplitude fluctuations in the magnetization are negligible. Hence, the dynamics of the magnetization direction $\mathbf{n} = \mathbf{M}/M_s$ is described be the Landau–Lifshitz equation (LL), and the Maxwell equations in the magnetostatic limit – accounting for dipole–dipole interactions. The LL equation is given by

$$\partial_t \mathbf{n} = -\gamma \mathbf{n} \times \mathbf{H}_{\text{eff}}, \tag{1}$$

* Corresponding author.
E-mail addresses: J.S.Harms@uu.nl (J.S. Harms), R.A.Duine@uu.nl (R.A. Duine).

https://doi.org/10.1016/j.jmmm.2022.169426
Received 1 October 2021; Received in revised form 9 March 2022; Accepted 27 April 2022
Available online 6 May 2022

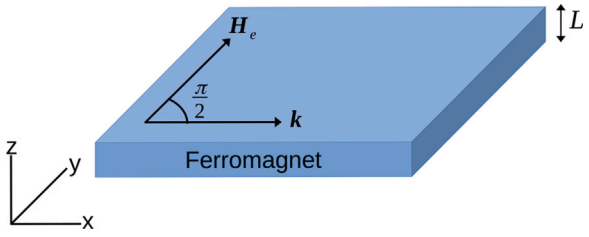

Fig. 1. Sketch of the set-up. We consider a ferromagnetic thin film of thickness L with the equilibrium magnetization pointing in the y direction.

which describes precession of the magnetization direction around the effective field $\mathbf{H}_{\text{eff}} = -\delta E/\delta(M_s\mathbf{n})$. Here, we consider the magnetic energy functional $E[\mathbf{n}]$ of the form

$$E[\mathbf{n}] = M_s \int dV \left[-\frac{1}{2} J\mathbf{n}\cdot\nabla^2\mathbf{n} - \mu_0\mathbf{H}\cdot\mathbf{n}\right]. \tag{2}$$

In the above J is the spin stiffness and $\mathbf{H} = \mathbf{H}_e + \mathbf{H}_D$ is the magnetic field strength, where $\mathbf{H}_D$ is the demagnetization field originating from dipole–dipole interactions. Furthermore, the magnetostatic Maxwell equations [8] – accounting for dipole–dipole interactions – are given by

$$\nabla\times\mathbf{H} = 0, \nabla\cdot\mathbf{B} = 0, \tag{3}$$

with $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$ the total magnetic field. The boundary conditions require the normal component of $\mathbf{B}$ and the tangential components of $\mathbf{H}$ to be continuous at the thin film surfaces.

In equilibrium the LL equation requires the equilibrium magnetization $\mathbf{M}_{\text{eq}}$ and the effective magnetic field strength $\mathbf{H}_{\text{eff}}$ to be parallel $\mathbf{M}_{\text{eq}} \parallel \mathbf{H}_{\text{eff}}$. In this case the internal magnetic field strength $\mathbf{H}_{\text{eq}} = \mathbf{H}_e + \mathbf{H}_D$ has a contribution from the external magnetic field $\mathbf{H}_e$ and the demagnetization field $\mathbf{H}_D = -\hat{z}M_z$, originating from the magnetostatic boundary conditions. For an external magnetic field pointing in the y direction, as discussed in this article, the uniform equilibrium magnetization $\mathbf{M}_{\text{eq}}$ should also point in the y direction.

Dipole-exchange spin-wave modes are generated by dynamical fluctuations of the magnetization and the demagnetizing field around the magnetostatic equilibrium

$$\mathbf{M} = \mathbf{M}_{\text{eq}} + \mathbf{m}(t),\ \mathbf{H} = \mathbf{H}_{\text{eq}} + \mathbf{h}_D(t), \tag{4}$$

where $\mathbf{m}$ is perpendicular to $\mathbf{M}_{\text{eq}}$ up to linear order in m_x and m_z, lying in the $x-z$ plane. The latter is a consequence of the magnitude of the magnetization being constant $|\mathbf{M}| = M_s$. Since the magnetostatic Maxwell Eqs. (3) are linear we require $\nabla\times\mathbf{h}_D = 0$, $\nabla\cdot\mathbf{b} = 0$, with $\mathbf{b} = \mu_0(\mathbf{h}_D + \mathbf{m})$. Using that the dynamic demagnetizing field is has vanishing curl, we express the dynamic demagnetization field in terms of a scalar potential $\mathbf{h}_D = \nabla\Phi_D$. Hence, the magnetostatic Maxwell equations become $\nabla^2\Phi_D = -\nabla\cdot\mathbf{m}$, where the magnetization outside the thin film vanishes.

The Landau–Lifschitz and magnetostatic Maxwell equations may be rewritten by means of $\mathbf{n} \simeq \hat{z}\sqrt{2}\text{Re}[\Psi] - \hat{x}\sqrt{2}\text{Im}[\Psi] + \hat{y}\left(1-|\Psi|^2\right)$, with $\Psi = (1/\sqrt{2})(\hat{z}-i\hat{x})\cdot\mathbf{n}$. Consequently the linearized LL and magnetostatic Maxwell equations become

$$\hat{\Omega}\Psi = -\left(\Omega_H - \Lambda^2\nabla^2\right)\Psi + \frac{(\partial_z - i\partial_x)}{\sqrt{2}M_s}\Phi_D, \tag{5a}$$

$$\frac{\nabla^2\Phi_D}{M_s^2} = -\frac{(\partial_z + i\partial_x)}{\sqrt{2}M_s}\Psi - \frac{(\partial_z - i\partial_x)}{\sqrt{2}M_s}\Psi^*. \tag{5b}$$

Additionally, the exchange boundary conditions for thin films [9,10] in the absence of surface anisotropy require

$$\pm\,\partial_z\Psi\Big|_{z=\pm L/2} = 0. \tag{6}$$

In the above, we defined the dimensionless magnetic field $\Omega_H = \mu_0 H_e/\mu_0 M_s$, exchange length $\Lambda = \sqrt{J/\mu_0 M_s}$ and the dimensionless frequency operator $\hat{\Omega} = i\partial_t/\gamma\mu_0 M_s$.

2.2. Bulk dipole-exchange spin-waves and its boundary conditions

Using the Bogoliubov ansatz, we write $\Psi(\mathbf{x},t) = u(\mathbf{x})e^{-i\omega t} + v^*(\mathbf{x})e^{i\omega t}$ and $\Phi_D(\mathbf{x},t) = w(\mathbf{x})e^{-i\omega t} + w^*(\mathbf{x})e^{i\omega t}$, where $\left(u(\mathbf{x}),\ v(\mathbf{x}),\ w(\mathbf{x})\right) \propto e^{i\mathbf{k}\cdot\mathbf{r}_\parallel}\left(u(\mathbf{k},z),\ v(\mathbf{k},z),\ w(\mathbf{k},z)\right)$, with $\mathbf{k} = \begin{pmatrix}k_x & k_y\end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix}x & y & z\end{pmatrix}$. In these coordinates the linearized LL and magnetostatic Maxwell equation (5) become

$$\mathcal{G}\cdot\begin{pmatrix}u(\mathbf{k},z) & v(\mathbf{k},z) & w(\mathbf{k},z)\end{pmatrix} = 0, \tag{7}$$

with

$$\mathcal{G} = \begin{pmatrix} -\sqrt{2}M_sF & 0 & (\partial_z + k_x) \\ 0 & -\sqrt{2}M_sF^* & (\partial_z - k_x) \\ \frac{M_s}{\sqrt{2}}(\partial_z - k_x) & \frac{M_s}{\sqrt{2}}(\partial_z + k_x) & (\partial_z^2 - k^2) \end{pmatrix}. \tag{8}$$

and $\hat{F} = \Omega + \Omega_h + \Lambda^2(k^2 - \partial_z^2)$, $\hat{F}^* = -\Omega + \Omega_h + \Lambda^2(k^2 - \partial_z^2)$ the dimensionless LL spin-wave operators and $\Omega = \omega/\gamma\mu_0 M_s$ the dimensionless frequency. Furthermore, $k^2 = k_x^2 + k_y^2$, which becomes $k^2 = k_x^2$ for spin waves travelling perpendicular to the external magnetic field. The above bulk equation of motion gives rise to a sixth order homogeneous differential equation in position space, which is cubic with respect to ∂_z^2. For spin waves travelling in the x direction, perpendicular to the external magnetic field, the general solution of Eq. (7) is given by the linear combination of plane waves

$$\begin{pmatrix}u(\mathbf{x})\\ v(\mathbf{x})\\ w(\mathbf{x})\end{pmatrix} = \sum_{l=1}^{6} C_{k_l}\begin{pmatrix}F_l^*(k_l + k_x)/\sqrt{2}M_s \\ F_l(k_l - k_x)/\sqrt{2}M_s \\ F_l^*F_l\end{pmatrix} e^{k_l z + ik_x x}, \tag{9}$$

where $F_l = \Omega + \Omega_h + \Lambda^2(k^2 - k_l^2)$ and $F_l^* = -\Omega + \Omega_h + \Lambda^2(k^2 - k_l^2)$. The wave numbers k_l satisfy the bulk equations of motion which follow from setting the determinant of Eq. (8) to zero

$$F_l^*F_l(k^2 - k_l^2) + (1/2)(F_l^* + F_l)(k_x^2 - k_l^2) = 0. \tag{10}$$

This is explicitly written as

$$(k^2 - k_l^2)\left\{\left[\Omega_h + 1/2 + \Lambda^2(k^2 - k_l^2)\right]^2 - 1/4 - \Omega^2\right\} = 0. \tag{11}$$

We find that the bulk equation of motion in Eq. (10) gives rise to one volume mode $k_l = \pm iq$ and two real surface modes $k_l = \pm k_{1,2}$, with $\{q, k_1, k_2\}$ real and positive. Furthermore, the bulk equation of motion Eq. (11) may be rewritten in the dispersive form

$$\Omega^2 = \left[\Omega_H + 1/2 + \Lambda^2k^2 + \Lambda^2q^2\right]^2 - 1/4, \tag{12}$$

where the precise form of the volume mode q follows from the boundary conditions on the system. From Eqs. (10) and (12) we find that the remaining two surface modes $k_{1,2}$ may be expressed as

$$k_1^2 = k_x^2, \tag{13a}$$

$$k_2^2 = k_x^2 + \Lambda^{-2}\left[2\Omega_H + 1 + \Lambda^2k^2 + \Lambda^2q^2\right]. \tag{13b}$$

The surface mode with wavelength k_1 is the Damon–Eshbach (DE) surface mode [11], which is confined to the surface with width $1/k_x$. The remaining surface mode $k_2 \gg k_1$ is a much more confined to the surface with a width of $1/k_2$. We will find this mode to be evanescent due to the boundary conditions of the thin film.

The exchange boundary conditions in Eq. (6) evaluated for the spin-wave modes in Eq. (9) give

$$\sum_{l=1}^{6} C_{k_l}k_lF_l^*(k_l + k_x)e^{\pm k_lL/2}\Big|_{z=\pm L/2} = 0, \tag{14a}$$

$$\sum_{l=1}^{6} C_{k_l}k_lF_l(k_l - k_x)e^{\pm k_lL/2}\Big|_{z=\pm L/2} = 0, \tag{14b}$$

with $k_l \in \{\pm k_1, \pm k_2, \pm iq\}$ as defined in Eqs. (10) and (13). The magnetostatic boundary conditions on the other hand require a bit more work. We start by noting that the magnetization vanishes $\Psi = 0$

outside the magnetic thin film ($z < -L/2$ and $L/2 < z$). Hence, the magnetostatic Maxwell Eqs. (3) outside the thin film give $\nabla^2 w(\mathbf{x}) = (-k^2+\partial_z^2)w(\mathbf{k},z)e^{i\mathbf{k}\cdot\mathbf{r}_\parallel} = 0$. The asymptotically bound solutions of outside the magnet are thus given by

$$w(\mathbf{k},z) \propto \begin{cases} e^{-kz}, & z > L/2, \\ e^{kz}, & z < -L/2. \end{cases} \tag{15}$$

Since the tangential components of $\mathbf{h}_\mathrm{D}$ are continuous across the thin film surfaces, the scalar field $w(\mathbf{k},z)$ should also be continuous across the thin film surface. Furthermore, continuity of the normal component of $\mathbf{b}$ in the Bogoliubov ansatz requires

$$\begin{aligned}\partial_z w(\mathbf{k},z^\pm) =& \partial_z w(\mathbf{k},z^\mp) \\ &+\frac{M_s}{\sqrt{2}}[u(\mathbf{k},z^\mp)+v(\mathbf{k},z^\mp)]|_{z=\pm L/2},\end{aligned} \tag{16}$$

with $z^\pm = z \pm 0^+$. This boundary condition (16) in combination with Eq. (15) gives the effective magnetostatic boundary condition

$$(\pm k + \partial_z)w(\mathbf{k},z) + \frac{M_s}{\sqrt{2}}[u(\mathbf{k},z)+v(\mathbf{k},z)]|_{z=\pm L/2} = 0. \tag{17}$$

Evaluated for the spin-wave modes in Eq. (9) the above effective magnetostatic boundary condition (17) gives

$$\sum_\pm C_{\pm k_1}\left[(2F_1^*F_1+F_1^*)\delta_\pm - F_1\delta_\mp\right]e^{\pm k_1 L/2} - \sum_l C_{k_l}F_l^* e^{k_l L/2}|_{z=L/2} = 0, \tag{18a}$$

$$\sum_\pm C_{\pm k_1}\left[(2F_1^*F_1+F_1)\delta_\mp - F_1^*\delta_\pm\right]e^{\mp k_1 L/2} - \sum_l C_{k_l}F_l e^{-k_l L/2}|_{z=-L/2} = 0, \tag{18b}$$

with $\delta_\pm \equiv \delta_{1,\pm 1}$ the Kronecker delta and $k_l \in \{\pm k_2, \pm iq\}$ as defined in Eqs. (10) and (13). Note here that we used the bulk equation of motion in Eq. (10) to simplify the above boundary conditions.

3. Dipole-exchange dispersion relation

3.1. General derivation

For notational simplicity we introduce the dimensionless wavenumbers $\Lambda k \to k$, $\Lambda q \to q$ and the dimensionless thickness $L/\Lambda \to L$.

3.1.1. Effective boundary conditions for spin waves

We start this section with by noting that Eq. (13) yields $k_2 \gg k \equiv k_x$, $e^{k_2 L/2} \gg e^{-k_2 L/2}$ and $|F_2^*| \gg |F_2|$. This allows us to approximate the exchange boundary conditions in Eq. (14) by

$$a_+F_k^*k^2e^{kL/2} + b_+ = cF_q^*\left(q^2\cos[(q+\delta)L/2] + kq\sin[(q+\delta)L/2]\right), \tag{19a}$$

$$a_+F_k^*k^2e^{-kL/2} + b_- = cF_q^*\left(q^2\cos[(q-\delta)L/2] - kq\sin[(q-\delta)L/2]\right), \tag{19b}$$

$$a_-F_kk^2e^{-kL/2} = cF_q\left(q^2\cos[(q+\delta)L/2] - kq\sin[(q+\delta)L/2]\right), \tag{19c}$$

$$a_-F_kk^2e^{kL/2} = cF_q\left(q^2\cos[(q-\delta)L/2] + kq\sin[(q-\delta)L/2]\right). \tag{19d}$$

where $a_\pm = C_{\pm k}$, $b_\pm \simeq C_{\pm k_2}F_2^*k_2^2e^{k_2L/2}/2$ and $e^{\pm i\delta L/2}c = C_{\pm q}$. Note that δ can in principle be a complex number. For future convenience we rewrite the above exchange boundary conditions to

$$\bar{b}_+ + a_-F_kk^2e^{-kL/2} = 2cF_qq^2\cos[(q+\delta)L/2], \tag{20a}$$

$$\bar{b}_+ - a_-F_kk^2e^{-kL/2} = 2cF_qkq\sin[(q+\delta)L/2], \tag{20b}$$

$$\bar{b}_- + a_-F_kk^2e^{kL/2} = 2cF_qq^2\cos[(q-\delta)L/2], \tag{20c}$$

$$-\bar{b}_- + a_-F_kk^2e^{kL/2} = 2cF_qkq\sin[(q-\delta)L/2]. \tag{20d}$$

where $\bar{b}_+ = (F_q/F_q^*)b_+ + a_+(F_q/F_q^*)F_k^*k^2e^{kL/2}$ and $\bar{b}_- = (F_q/F_q^*)b_- + a_+(F_q/F_q^*)F_k^*k^2e^{-kL/2}$ are free parameters since $k_2 \gg k$ implies that b_+ and b_- are – to good approximation – not restricted by the magnetostatic boundary conditions. From here we find that the contributions of q and δ can be separated by making use of the trigonometric identities

$$\begin{cases}\cos[(q\pm\delta)L/2] = \cos(qL/2)\cos(\delta L/2) \\ \qquad\qquad\qquad \mp \sin(qL/2)\sin(\delta L/2), \\ \sin[(q\pm\delta)L/2] = \sin(qL/2)\cos(\delta L/2) \\ \qquad\qquad\qquad \pm \cos(qL/2)\sin(\delta L/2).\end{cases}$$

The above trigonometric identities allow us to express $\bar{b}_+$ and $\bar{b}_-$ in Eq. (20) in terms of the variables a_-, k and q, which gives

$$\bar{b}_+ = \frac{a_-F_kk^2}{k^2+q^2}\Big[(q^2-k^2)e^{-kL/2} + 2kq\left(\csc(qL)e^{kL/2} - \cot(qL)e^{-kL/2}\right)\Big], \tag{21a}$$

$$\bar{b}_- = \frac{a_-F_kk^2}{k^2+q^2}\Big[(q^2-k^2)e^{kL/2} + 2kq\left(\cot(qL)e^{kL/2} - \csc(qL)e^{-kL/2}\right)\Big], \tag{21b}$$

where $\csc(qL) \equiv 1/\sin(qL)$ and $\cot(qL) \equiv \cos(qL)/\sin(qL)$. Hence, Eqs. (20) and (21) allow us to express $2cF_q^*F_qq^2\cos[(q\pm\delta)L/2]$ in terms of the variables a_-, k and q, which to leading order in exponential functions is given by

$$cF_q\cos[(q+\delta)L/2] = \frac{a_-F_kk^2e^{kL/2}}{k^2+q^2} \times\left(e^{-kL} + kL\csc(qL)/qL\right), \tag{22a}$$

$$cF_q^*\cos[(q-\delta)L/2] = \frac{a_-F_kk^2e^{kL/2}}{k^2+q^2}\frac{F_q^*}{F_q} \times(1+kL\cot(qL)/qL). \tag{22b}$$

So far we have used the exchange boundary conditions Eq. (19) to express $2cF_q^*F_qq^2\cos[(q\pm\delta)L/2]$ in terms of the variables a_-, k and q. From here, we impose the magnetostatic boundary conditions to find a closed expression for q satisfying all boundary conditions.

The remaining magnetostatic boundary conditions (18), for $k_2 \gg k_x$, $e^{k_2L/2} \gg e^{-k_2L/2}$ and $|F_2^*| \gg |F_2|$, are well approximated by

$$a_+(2F_kF_k^*+F_k^*)e^{kL/2} - a_-F_ke^{-kL/2} = 2cF_q\cos[(q+\delta)L/2], \tag{23a}$$

$$a_+F_k^*e^{-kL/2} - a_-(2F_kF_k^*+F_k)e^{kL/2} = -2cF_q^*\cos[(q-\delta)L/2]. \tag{23b}$$

3.1.2. Dipole-exchange spin-wave modes

The above magnetostatic boundary conditions together with the effective exchange boundary conditions in Eq. (22) give two linear homogeneous equations in a_+ and a_-. Hence, we have spin-wave solutions when the determinant of this square matrix vanishes. At leading order in exponential functions of the trigonometric contribution we find this to be the case when

$$\det\begin{bmatrix}(2F_k+1) & (3k^2+q^2)e^{-kL} \\ e^{-kL} & D(k,q)\end{bmatrix} = 0, \tag{24}$$

with

$$D(k,q) = (2F_k^*+1)\left[(3k^2+q^2) + 2k^3L\cot(qL)/qL\right] + 4k^2(k^2+q^2)(1+kL\cot(qL)/qL).$$

In the above we used the bulk equation of motion $2F_qF_q^* + F_q + F_q^* = 0$ to obtain $F_q^*/F_q = -(2F_q^*+1)$. Note that we neglected $2k^3L\csc(qL)/qL$ in Eqs. (24) and (22a) since it is exponentially suppressed in the

equation of motion and thus not of importance for the dispersion relation. The spin-wave modes hence satisfy

$$\begin{aligned}&\left[(F_k+1/2)(F_k^*+1/2)-e^{-2kL}/4\right]\\&\times(3k^2+q^2)\\&+(F_k+1/2)2k^2(k^2+q^2)\\&+(F_k+1/2)(F_k^*+1/2+k^2+q^2)\\&\times 2k^3L\cot(qL)/qL\\&=0.\end{aligned}\tag{25}$$

When interested in the nth spin-wave mode the above equation is well approximated by

$$\begin{aligned}&\left[(F_{k,n}+1/2)(F_k^*+1/2)-e^{-2kL}/4\right]\\&\times\left(3k^2+n^2\pi^2/L^2+\delta_n\pi^2/4L^2\right)\\&+(F_{k,n}+1/2)2k^2(k^2+n^2\pi^2/L^2)\\&+(F_{k,n}+1/2)(F_{k,n}^*+1/2+k^2+n^2\pi^2/L^2)\\&\times 2k^3L\cot(qL)/qL\\&=0,\end{aligned}\tag{26}$$

with $\delta_n \equiv \delta_{n,0}$ the Kronecker delta, $F_{k,n} \equiv F_k|_{q\to n\pi/L}$ and $F_{k,n}^* \equiv F_k^*|_{q\to n\pi/L}$. In order to proceed we use the identity

$$\pi\cot(\pi x)=\frac{1}{x}+2x\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}.\tag{27}$$

To make use of the above identity we consider $\pi x = qL$. Furthermore, we note that q for nth spin-wave is in the interval $n\pi/L < q < (n+1)\pi/L$. For the nth spin wave mode we obtain

$$\frac{\cot(qL)}{qL}\simeq\frac{2-\delta_n}{q^2L^2-n^2\pi^2}+\frac{2}{q^2L^2-(n+1)^2\pi^2}-\alpha_n,\tag{28}$$

with $\delta_n \equiv \delta_{n,0}$ the Kronecker delta and

$$\alpha_n=\frac{4}{3\pi^2}\frac{1}{(1+n)^2}.\tag{29}$$

We expand q^2L^2 around $n^2\pi^2$ for the nth spin-wave mode. From here it follows that Eq. (26) can be written explicitly as

$$a_{n,k}z^3+b_{n,k}z^2+c_{n,k}z+d_{n,k}=0,\tag{30}$$

with $z = q^2L^2 - n^2\pi^2$. Furthermore,

$$a_{n,k}\approx -1,\tag{31a}$$

$$b_{n,k}=B_{n,k}-\alpha_nC_{n,k}-a_{n,k}(2n+1)\pi^2,\tag{31b}$$

$$\begin{aligned}c_{n,k}=&(4-\delta_n)C_{n,k}\\&-(B_{n,k}-\alpha_nC_{n,k})(2n+1)\pi^2,\end{aligned}\tag{31c}$$

$$d_{n,k}=-C_{n,k}(2-\delta_n)(2n+1)\pi^2,\tag{31d}$$

and

$$\begin{aligned}B_{n,k}\simeq&\frac{1-e^{-2kL}}{4\gamma_{n,k}/L^2}-(k^2L^2+n^2\pi^2)\\&\times\frac{k^2L^2+n^2\pi^2+\delta_n\pi^2/4}{3k^2L^2+n^2\pi^2+\delta_n\pi^2/4},\end{aligned}\tag{32a}$$

$$2C_{n,k}\simeq\frac{k^3L^5\times\gamma_{n,k}^{-1}}{3k^2L^2+n^2\pi^2+\delta_n\pi^2/4},\tag{32b}$$

where $\gamma_{n,k} = 2(\Omega_H + 1/2 + k^2 + n^2\pi^2/L^2)$. For the nth spin-wave mode Eq. (30) gives the formal solution

$$\begin{aligned}z_{n,k}=&-\frac{1}{3}\frac{b_{n,k}}{a_{n,k}}+\sqrt{\frac{-4P_{n,k}}{3}}\\&\times\cos\left[\frac{1}{3}\arccos\left(\frac{3Q_{n,k}}{2P_{n,k}}\sqrt{\frac{-3}{P_{n,k}}}\right)-\frac{2\pi}{3}\right],\end{aligned}\tag{33a}$$

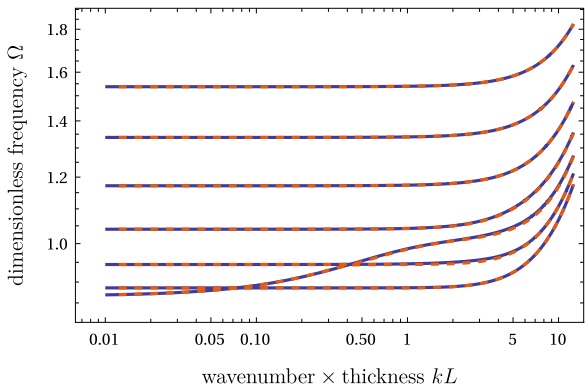

Fig. 2. The first seven modes of the dipole-exchange spin-wave dispersion relation for $\Omega_H = 1/2$ and $L = 24$ are shown. The dashed lines correspond to the analytic dispersion in Eq. (35), while the solid lines correspond to the full numeric solution of Eqs. (14) and (18).

with

$$Q_{n,k}=\frac{d_{n,k}}{a_{n,k}}-\frac{1}{3}\frac{b_{n,k}}{a_{n,k}}\frac{c_{n,k}}{a_{n,k}}+\frac{2}{27}\left(\frac{b_{n,k}}{a_{n,k}}\right)^3,\tag{34a}$$

$$P_{n,k}=\frac{c_{n,k}}{a_{n,k}}-\frac{1}{3}\left(\frac{b_{n,k}}{a_{n,k}}\right)^2.\tag{34b}$$

The dispersion relation of the nth spin-wave mode is accordingly given by Eq. (12)

$$\Omega_n^2=\left[\Omega_H+\frac{1}{2}+k^2+\frac{n^2\pi^2}{L^2}+\frac{z_n}{L^2}\right]^2-\frac{1}{4}.\tag{35}$$

Note that Eqs. (33) and (35) are expressed in terms of the dimensionless wavenumber $\Lambda k \to k$ and the dimensionless thickness $L/\Lambda \to L$. Furthermore, Eq. (35) gives the dispersion relation for the dimensionless frequency $\Omega = \omega/\gamma\mu_0M_s$. This is the main result of this paper. In Fig. 2 we compare the analytic dipole-exchange mode in Eq. (35) with the full numeric solution. We see that the analytic dispersion derived above is in good agreement with the full numeric solution of Eqs. (14) and (18). We like to stress that the analytic spin wave modes given in Eq. (35) do not experience level crossing. Hence, the nth mode does not cross the $(n-1)$ and $(n+1)$th energy modes. In the remaining Sections 3.2 and 3.3 we will simplify Eq. (35) for relative thin and thick films respectively.

3.2. Thick thin-film approximation for the lowest exchange mode

In relatively thick films, $L > \mathcal{O}(10\sqrt{J/\mu_0M_s})$, for sufficiently long wavelengths, the lowest energetic mode will be dominated by the first exchange mode. Away from the DE mode, we may approximate exchange mode solutions of Eq. (24) by $D(k,q) = 0$ with $F_k^* \to F_{k,n}^*$. This is equivalent to $a_{n,k} \to 0$ in Eq. (30) and $e^{-2kL} \to 0$ and $\delta_n\pi^2/4/L^4 \to 0$ in Eq. (32). The above results in a second order equation in z when the exchange mode has a small avoided crossing with the DE mode. For the lowest exchange mode $n \to 0$ this becomes

$$2kL\cot(qL)/qL=4k^2\gamma_k-3.\tag{36}$$

Hence, we obtain a quadratic equation in z

$$b_kz^2+c_kz+d_k=0,\tag{37}$$

where

$$b_k=B_k-\frac{4C_k}{3\pi^2},\tag{38a}$$

$$c_k=\frac{13}{3}C_k-\pi^2B_k,\tag{38b}$$

$$d_k=-\pi^2C_k,\tag{38c}$$

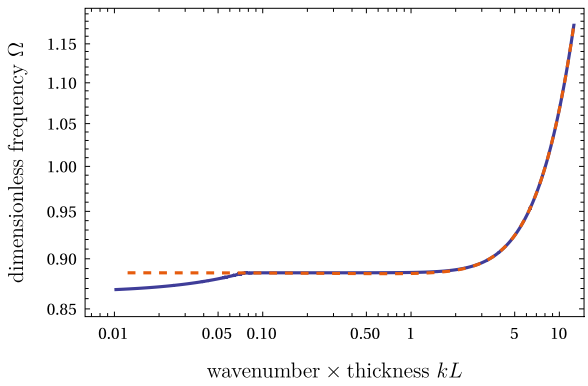

Fig. 3. Dipole-exchange dispersion relation of the lowest energy mode for $\Omega_H = 1/2$ and $L = 24$. The dashed line correspond to the analytically derived dispersion in Eq. (41), while the solid line gives the numeric solution to Eqs. (14) and (18) for the lowest energy mode.

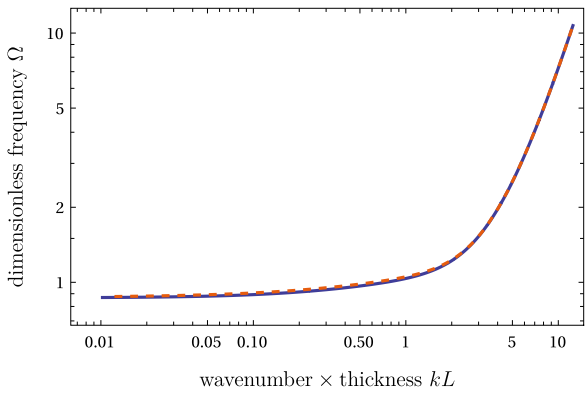

Fig. 4. Dipole-exchange dispersion of the lowest energy mode for $\Omega_H = 1/2$ and $L = 4$. The dashed line corresponds to the approximate dispersion relation in Eq. (47). The solid line corresponds to the numeric solution of Eqs. (14) and (18).

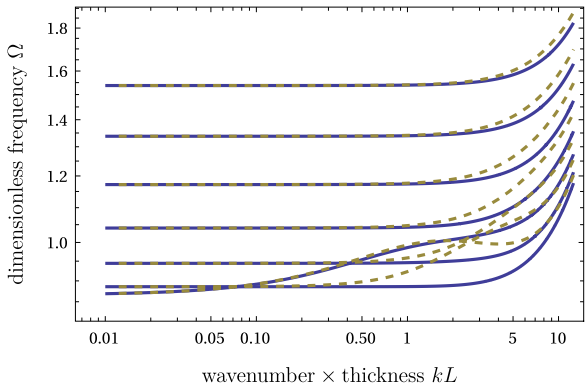

Fig. 5. The first seven modes of the dipole-exchange spin-wave dispersion relation for $\Omega_H = 1/2$ and $L = 24$ are shown. The dashed lines correspond to the analytic dispersion derived by Kalinikos and Slavin [3,4] in Eq. (48). The solid lines correspond to the full numerical solution of Eqs. (14) and (18). We did not plot the analytic results in Eq. (35) and Fig. 2, since it agrees very well with the numerical solution.

and

$$B_k \simeq 3 - 8(\Omega_H + 1/2 + k^2)k^2, \tag{39a}$$

$$C_k \simeq 2kL. \tag{39b}$$

We thus find

$$z = -\frac{c_k}{2b_k} + \mathrm{sgn}(b_k)\sqrt{\left(\frac{c_k}{2b_k}\right)^2 - \frac{d_k}{b_k}}. \tag{40}$$

The lowest exchange mode dispersion is thus

$$\Omega^2 = \left[\Omega_H + 1/2 + k^2 + z/L^2\right]^2 - 1/4. \tag{41}$$

In Fig. 3 we compare the above dispersion relation with the numeric solution of the lowest energy mode. We find good agreement between the approximated dispersion relation in Eq. (41) and the numerical lowest energy mode for wavelengths k larger than the level crossing point with the DE mode. Note that the above simplification is not restricted to the lowest energy exchange mode, but can be applied to the higher exchange modes as long as there is no large avoided crossing with the DE mode.

3.3. Thin film approximation for the lowest energy mode

For very thin films, $L \sim \mathcal{O}(\sqrt{J/\mu_0 M_s})$, only the DE wave is of importance for the lowest energy mode. For very thin films it is reasonable to assume $q^2L^2 < (3/4)^2\pi^2$, we may thus approximate

$$\frac{\cot(qL)}{qL} \simeq \frac{1}{(qL)^2} - \frac{1}{3}. \tag{42}$$

Thus the lowest energy mode in very thin films is described by a quadratic equation in z

$$a_k z^2 + b_k z + c_k = 0. \tag{43}$$

Where

$$a_k \approx -2(\Omega_H + 1/2 + k^2)/L^2, \tag{44a}$$

$$b_k \simeq B_k - c_k/3, \tag{44b}$$

$$c_k \simeq kL/6, \tag{44c}$$

and

$$B_k \simeq \frac{1 - e^{-2kL}}{4} - \frac{2}{3}(\Omega_H + 1/2 + k^2)k^2. \tag{45a}$$

The lowest energy mode in very thin films is thus given by

$$z = -\frac{b_k}{2a_k} + \sqrt{\left(\frac{b_k}{2a_k}\right)^2 - \frac{c_k}{a_k}}, \tag{46}$$

where the lowest energy dispersion relation is given by

$$\Omega^2 = \left[\Omega_H + 1/2 + k^2 + z/L^2\right]^2 - 1/4. \tag{47}$$

This is plotted in Fig. 4. We again find good agreement between the analytic result in Eq. (47) and the numerical solution of the full boundary conditions in Eqs. (14) and (18). Note that we took $\delta_n\pi^2/4L^2 \to 0$ of Eq. (32) in the very thin-film limit, since it simplifies the expressions for B_k and c_k and does not have a big impact on the dispersion relation for very thin-films.

3.4. Comparison with Kalinikos and Slavin [3,4]

In the paper by Kalinikos and Slavin [3,4] the dipole-exchange spin-wave spectrum that follows from the diagonal approximation for spin waves propagating perpendicular to a tangentially magnetized thin-film was given by

$$\Omega_n^2 = [\Omega_H + 1/2 + k^2 + n^2\pi^2/L^2]^2 - [P_n - 1/2]^2, \tag{48}$$

with

$$P_n = \frac{k^2L^2}{k^2L^2 + n^2\pi^2}\left[1 - \frac{k^2L^2}{k^2L^2 + n^2\pi^2} \times \frac{2}{1+\delta_n}\left(\frac{1 - (-1)^n e^{-kL}}{kL}\right)\right]. \tag{49}$$

In Fig. 5 we plotted the spin-wave dispersion by Kalinikos and Slavin [3,4] in Eq. (48) for relatively thick films. We see that the analytic

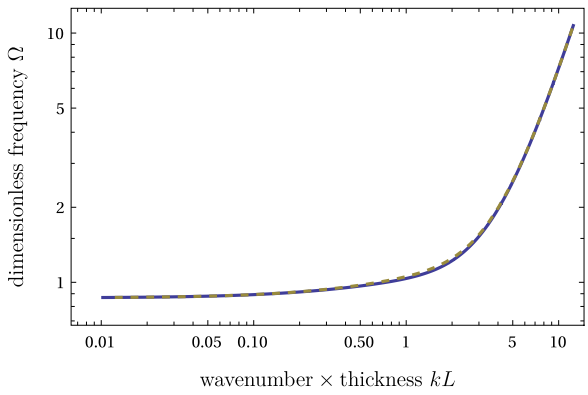

Fig. 6. Dipole-exchange dispersion of the lowest energy mode for $\Omega_H = 1/2$ and $L = 4$. The dashed line corresponds to the dispersion relation by Kalinikos and Slavin [3,4] in Eq. (48). The solid line on the other hand corresponds to the numeric solution of Eqs. (14) and (18). We did not plot the analytic results in Eq. (35) and Fig. 4, since it agrees very well with the numerical solution.

spin-wave dispersion by Kalinikos and Slavin [3,4] shows quantitative differences with the analytic spin-wave dispersion derived in this article Eqs. (35) and (41) and the numerical solution of the full problem. For very thin films on the other hand, we find good agreement between the theory of Kalinikos and Slavin [3,4], the analytic results derived in this article Eqs. (35) and (47) and the numeric solution of the full boundary conditions, see Fig. 6.

4. Discussion and conclusions

We considered the theory of spin waves in ferromagnetic films. More specifically, the theory of spin waves propagating perpendicular to an in-plane magnetic field. This case is of special interest since it is the most typical configuration used in spin wave experiments. The main result of this article is the spin wave spectrum in Eq. (35) which we derived by imposed the exchange and magnetostatic boundary conditions on bulk spin wave solutions. This derivation differs significantly from the derivation of Kalinikos and Slavin [3,4] where the magnetostatic Green's function was used to construct the spin wave spectrum. The boundary problem we obtained has an accurate analytical solution which agrees well with the numerical solution and shows quantitative differences with the commonly used theory in Refs. [3,4] in relative thick films. This inaccuracy of the spin wave spectrum that follows from the diagonal approximation in the theory by Kalinikos and Slavin [3,4] has already been observed by Kreisel et al. [2]. Furthermore, our results could be verified experimentally, for example using Brillouin light scattering and spin-wave spectroscopy [12].

Future research could generalize the method presented here to describe spin waves propagating in an arbitrary direction with respect to a generally oriented external magnetic field. This is relatively straightforward for in-plane magnetizations. Another way to generalize this model is to include the effects of both surface and boundary anisotropies. Lastly, the magnetization profile of spin wave modes could relatively straightforwardly be determined from the spin wave spectrum in Eqs. (35), (41) and (47).

CRediT authorship contribution statement

J.S. Harms: Conceptualization, Methodology, Validation, Formal analysis, Writing – original draft. **R.A. Duine:** Conceptualization, Writing – review & editing, Supervision, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

J.S.H. would like thank A. Rückriegel for helpful discussions and D. A. Bozhko for useful comments on the preprint. R.A.D. is member of the D-ITP consortium, a program of the Netherlands Organisation for Scientific Research (NWO) that is funded by the Dutch Ministry of Education, Culture and Science, Netherlands (OCW). This work is part of the Fluid Spintronics research programme with project number 182.069, which is financed by the Dutch Research Council, Netherlands (NWO).

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