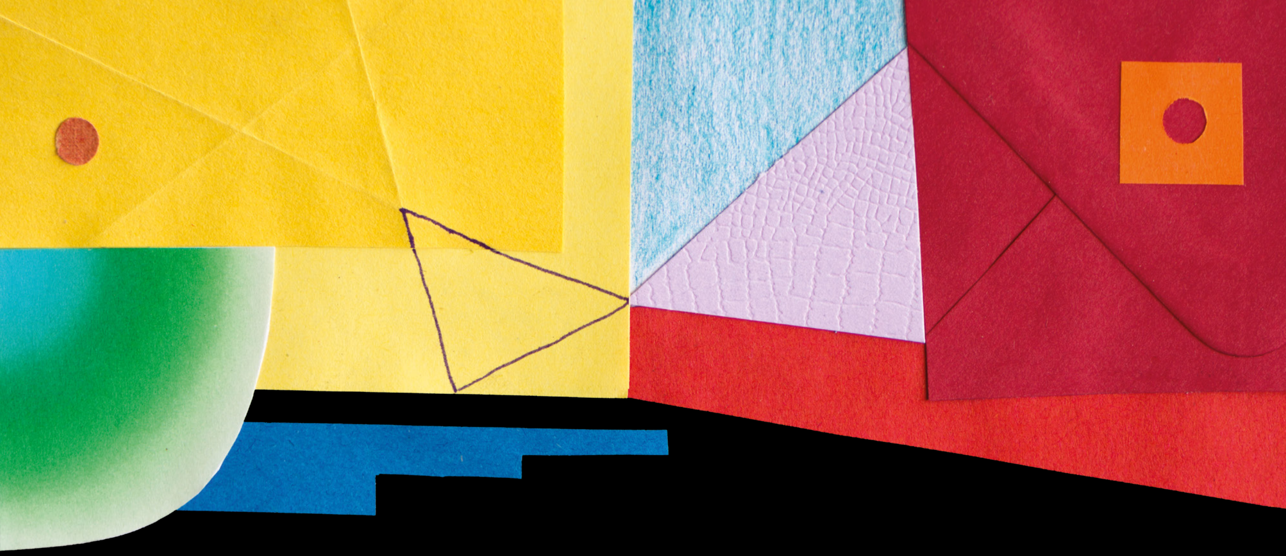

# Uniform Interpolation and Admissible Rules

## Proof-theoretic investigations into (intuitionistic) modal logics

Iris van der Giessen

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# Uniform Interpolation and Admissible Rules

Proof-theoretic investigations into (intuitionistic) modal logics

## *Uniforme Interpolatie en Toelaatbare Regels*

*Bewijstheoretisch onderzoek naar (intuïtionistische) modale logica's*

(met een samenvatting in het Nederlands)

**Proefschrift**

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door

**Iris van der Giessen**

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# Contents

# Introduction

This thesis investigates classical and intuitionistic modal logics via proof-theoretic methods for two important and widely applied topics in logic: uniform interpolation and admissible rules. Both topics are treated in separate parts of the thesis. In what follows we briefly set the scene of the thesis and refer to the respective chapters for more extensive introductions with historical overviews.

Classical modal logics and intuitionistic modal logics form rich classes of logics that find applications in, for example, epistemic, temporal, and provability logics. Classical modal logics have been thoroughly studied and the interest in intuitionistic modal logics has grown more recently. For a good overview of applications of intuitionistic modal logics see (Stewart *et al.*, 2015). A well-known distinction between the two classes is that the modal operators $\Box$ and $\Diamond$ are interdefinable for classical modal logics, but are independent in intuitionistic modal logics. This leads to a variety of intuitionistic modal logics and makes the study of logics that only contain $\Box$ a natural first step in investigations into intuitionistic modal logics, as we will do in this thesis. In particular, we focus on the so-called coreflection principle $A \to \Box A$ that cannot have a meaningful classical interpretation but has interesting intuitionistic modal applications in epistemic logic, provability logic, and lax logics. We further refer to Chapter 1 for a brief history on classical and intuitionistic modal logics.

Proof theory is a broad field in mathematical logic that studies proofs as mathematical entities and analyses their properties and relations between them. Its applications range over various areas in mathematics (e.g., ordinal analysis, proof mining, and reverse mathematics), computer science (e.g., automated deduction, type theory, and proof complexity), and beyond. This thesis belongs to the discipline of structural proof theory that is concerned with the design of proof systems that describe formal systems and the analysis of their structural properties (for textbook accounts see Troelstra and Schwichtenberg, 2000; Negri *et al.*, 2001). Proof systems contain inference rules that allow us to determine the valid principles of a given logic. In this thesis we use proof systems that originate from the sequent calculus by Gentzen (1935a,b). It is well known that the sequent framework is not applicable to each logic. Therefore various generalizations of ordinary sequents have emerged, such as nested sequents (Bull, 1992; Brünnler, 2009; Poggiolesi, 2009a), hypersequents (Avron, 1996; Pottinger, 1983), and labelled sequents (Gabbay, 1996; Negri, 2005). The analysis of proof systems can reveal many important properties of the logic at hand such as consistency and decidability, and also uniform interpolation.

Uniform interpolation is a property stronger than the well-known Craig interpolation property. A logic has Craig interpolation if for each valid implication between formulas $A \to B$ one can find a formula (the interpolant) $C$ with shared variables from $A$ and $B$ such that $A \to C$ and $C \to B$ are also valid. Uniform interpolation is stronger in the sense that $C$ only depends on either $A$ or $B$. The property finds applications in computer science and is studied in different fields of logic such as algebraic logic and model theory. This thesis takes a proof-theoretic point of view which originates from Pitts (1992). Proof-theoretic research of uniform interpolation has several advantages. On one hand, proof systems of a particular form can provide constructive proofs of uniform interpolation. On the other hand, lack of uniform interpolation may exclude the existence of certain sequent calculi for the logic at hand. This general perspective on the interaction between uniform interpolation and concrete proof systems is initiated by Iemhoff (2019a,b). We refer to Chapter 2 for a more extensive introduction.

The admissible rules of a logic are those rules that can be added to the logic without changing its valid formulas. Interestingly, these rules are not bound to a concrete proof system, but reflect the interaction between valid formulas of the logic. Admissible rules can describe many properties of logics, such as the well-known disjunction property stating that whenever $A \vee B$ is valid, then $A$ is valid or $B$ is valid. In this thesis we study different problems that concern the set of *all* admissible rules of a logic. An early one originates from Friedman (1975): given a logic $\mathsf{L}$, can we decide whether a rule is admissible in $\mathsf{L}$ or not? This question has been resolved for many logics, establishing, among other things, the decidability of the admissibility problem for $\mathsf{IPC}$ and many classical modal logics, see, e.g., (Rybakov, 1997). Other problems are concerned with a solid description of all admissible rules in a logic. In this thesis we use proof theory to describe admissible rules of intuitionistic modal logics based on (Iemhoff and Metcalfe, 2009b). See Chapter 5 for a historical and technical introduction to the topic.

These topics come together in this thesis. In Part I, called ‘Uniform Interpolation in Proof Theory’, we investigate different sequent-like proof systems for classical and intuitionistic modal logics in relation to uniform interpolation. In Part II, called ‘Admissible Rules and Proof Theory’, we study proof systems to describe admissible rules for classical and intuitionistic modal logics.

## Main results

Our investigation in Part I is inspired by the work of Iemhoff (2019a,b) who characterizes sufficient conditions for sequent calculi for proving uniform interpolation in the setting of classical and intuitionistic modal logics. An important concept is termination of the proof search. She also characterizes so-called negative results: logics without uniform interpolation cannot have certain terminating sequent calculi. Our aim is to extend this line of research in two ways: investigate other intuitionistic modal logics and widen the scope to other proof formalisms.

Concerning the former, we provide sequent calculi for two intuitionistic modal logics, $\mathsf{iGL}$ and $\mathsf{iSL}$ (both have connections to provability). We give syntactic cut-elimination proofs based on a non-trivial cut-elimination strategy for classical $\mathsf{GL}$ (Valentini, 1983). See the beginning of Chapter 3 for a historical note on cut-elimination proofs for $\mathsf{GL}$. Some of the calculi that we develop are terminating. While we do not use these to show uniform interpolation, we establish the Craig interpolation property for both logics. We use the termination to develop a countermodel construction for $\mathsf{iSL}$.

In addition, we show that intuitionistic modal logics $\mathsf{iK4}$ and $\mathsf{iS4}$ do not have the uniform interpolation property. In light of the negative results from Iemhoff (2019a,b), we obtain that these logics cannot be described by certain terminating sequent calculi.

Concerning the latter, we study uniform interpolation for classical modal logics via nested sequents and hypersequents. We develop a method to reprove uniform interpolation for logics $\mathsf{K}$, $\mathsf{T}$, $\mathsf{D}$, and $\mathsf{S5}$. We construct uniform interpolants via terminating nested sequent calculi and hypersequent calculi. To the best of our knowledge, this provides a first constructive definition of uniform interpolants for $\mathsf{S5}$. Although the interpolants are defined constructively, our proof incorporates semantic reasoning based on so-called bisimulation quantifiers.

Part II provides an investigation of admissible rules in classical and intuitionistic modal logics. We are interested in a characterization of all admissible rules of a given logic. A characterization can be given by a basis, which is a set of admissible rules from which all other admissible rules can be derived. Admissible rules have been extensively studied for classical modal logics (e.g., Rybakov, 1997).

This thesis provides a first study of admissible rules for intuitionistic modal logics. We are able to describe bases for the admissible rules in six intuitionistic modal logics with coreflection: $\mathsf{iCK4}, \mathsf{iCS4} \equiv \mathsf{IPC}, \mathsf{iSL}, \mathsf{KM}, \mathsf{mHC}$, and $\mathsf{PLL}$. In addition, we show decidability of admissibility for these logics. Our technique relies on a proof theory for admissibility based on (Iemhoff and Metcalfe, 2009b). This proof theory is special because it does not reason on the level of formulas, but it contains rules that reason about rules.

The proof also relies on a semantic approach developed by Ghilardi (1999, 2000) about the interaction between so-called projective formulas and the extension property. We analyse their importance in the field of admissible rules. We explore the method in (Ghilardi, 2000) for classical modal logic which is based on a bisimulation argument. Our aim is to highlight the results by identifying key aspects of the proof. In turn, we propose a minor simplification. In addition, we show that the same interaction holds for the logics $\mathsf{iCK4}$, $\mathsf{iCS4} \equiv \mathsf{IPC}$, $\mathsf{iSL}$, $\mathsf{KM}$, $\mathsf{mHC}$, and $\mathsf{PLL}$, by using the simpler method from (Ghilardi, 1999).

## Structure of the thesis

The thesis consists of two parts with a preceding chapter introducing the logics that we work with throughout the thesis. We would like to stress that the Parts I and II can be read independently. Each part consists of three chapters. The first chapter of each part provides comprehensive overviews of history and known results. We sometimes give proofs for folklore results. The other two chapters in each part contain new results. In detail, the thesis is structured as follows.

Chapter 1 reviews syntax and Kripke semantics for classical and intuitionistic modal logics, including their history. We discuss intuitionistic modal logics with coreflection and their background in epistemic, provability, and lax logics.

Part I investigates uniform interpolation in proof theory. Chapter 2 introduces basic concepts and it gives a historical overview of the subject. Chapter 3 is a study of uniform interpolation in intuitionistic modal logics. It contains the analysis of sequent calculi for $\mathsf{iGL}$ and $\mathsf{iSL}$ and the proof that $\mathsf{iK4}$ and $\mathsf{iS4}$ lack the uniform interpolation property. Chapter 4 provides the method to prove uniform interpolation for $\mathsf{K}$, $\mathsf{T}$, and $\mathsf{D}$ via nested sequents and for $\mathsf{S5}$ via hypersequents.

Part II forms a study of admissible rules. Chapter 5 introduces important concepts and contains a historical overview of the subject. Chapter 6 explores the importance of projectivity in the field of admissible rules starting with an exposition of known results. It provides our analysis of (Ghilardi, 2000) about the extension property in classical modal logics and studies the extension property for intuitionistic modal logics with coreflection. Chapter 7 investigates the proof theory and bases for the admissible rules in the six intuitionistic modal logics $\mathsf{iCK4}, \mathsf{iCS4} \equiv \mathsf{IPC}, \mathsf{iSL}, \mathsf{KM}, \mathsf{mHC}$, and $\mathsf{PLL}$.

We conclude the thesis with Conclusions and Future Work exploring future research and the connection between uniform interpolation and admissible rules.

## Publications

Chapters 3, 4, 6, 7 contain new contributions. The majority of the work originally appeared in articles as indicated below.

- Chapter 3: the work on iGL and iSL is based on joint work with Rosalie Iemhoff and merges the two papers that treat the logics separately, respectively (van der Giessen and Iemhoff, 2021, 2022).
- Chapter 4 is based on joint work with Raheleh Jalali and Roman Kuznets. The method for K,T, and D can be found in (van der Giessen *et al.*, 2021a) and the one for S5 in an extended version (van der Giessen *et al.*, 2022).
- Chapter 6: the analysis of (Ghilardi, 2000) is published in (van der Giessen, 2021b). The study of the extension property for intuitionistic modal logics can be found in (van der Giessen, 2021a).
- Chapter 7 is based on (van der Giessen, 2021a).

# 1

# Classical and Intuitionistic Modal Logic

The research on classical modal logic is often traced back to Lewis (1918), who aimed to axiomatize strict implications as intentional counterparts of material implication. In classical logic this binary implication can be replaced by a unary box operator together with the material implication which became standard practice in the study of modal logics.[1] The contemporary representation of classical modal logic has been formed by many in terms of, for example, axiomatization, algebra, topology, and possible world semantics. See (Blackburn *et al.*, 2001, §1.7) for a historical overview. Languages of modal logics contain special operators, called *modalities*, expressing some quality of the truth of a statement. Well-known examples are *alethic* modalities modelling necessity and possibility, *epistemic* modalities modelling knowledge and beliefs, *temporal* modalities modelling truth over time, and *provability* modalities expressing provability in arithmetical theories. These modalities are added to a classical propositional logic and, in turn, adhere a classical meaning. This results in dual modalities. For example, the alethic modality for possibility (usually denoted by $\Diamond$) is the dual of necessity (usually denoted by $\Box$) in a similar way that $\exists$ is the dual of $\forall$ in classical quantified logic.

The research on intuitionistic modal logic focuses on modalities with an intuitionistic reading. Early publications on intuitionistic modal logic are by Fitch (1948)

[1] The strict implication is also known as *Lewis arrow* and is often denoted by $\prec\!\!\!\!\!\!\;3$. Although it did not became an object of study in the classical setting, it is more expressive than the box operator $\Box$ intuitionistically. See also footnote 16 of this thesis with its relation to provability logic. We refer to (Litak and Visser, 2018) who plead for a revive of Lewis arrow in intuitionistic modal logic. So, although the research on intuitionistic modal logic had to wait, it was hidden at the beginning after all.

and Prior (1957). In the intuitionistic setting, $\Box$ and $\Diamond$ are not dual to each other, so one must define the behavior of $\Box$ and $\Diamond$ independently, which can be done in different ways. In the literature often intuitionistic versions of classical modal logics are considered. One can distinguish two mainstream approaches to find intuitionistic modal counterparts: *constructive* and *intuitionistic* modal logic.

*Constructive* modal logics appear in computer science in which the intuitionistic account on modalities is justified from a computational point of view. This work goes back to Prawitz (1965) who investigated proof theory for intuitionistic modalities. Some works that can be gathered under this approach are communication systems by Stirling (1987); the dynamic systems by Wijesekera (1990); research on an extended Curry-Howard correspondence by Bierman and de Paiva (2000) with applications in staged computation by Davies and Pfenning (2001); and nested sequent calculi developed by Arisaka *et al.* (2015). In general, the addition of the law of excluded middle $p \vee \neg p$, a typical classical axiom, to constructive modal logics do not result in classical modal logics.

This in contrast to the *intuitionistic* approach of intuitionistic modal logic in which the addition of the excluded middle yields a classical modal logic. Examples of this approach are the studies of the intuitionistic modal logic $\mathsf{MIPC}$ corresponding to the monadic fragment of intuitionistic predicate logic mimicking behavior of $\Box$ and $\Diamond$ as $\forall$ and $\exists$, respectively, in Prior (1957), Bull (1965, 1966), Ono (1977), Bezhanishvili (1998); translations of intuitionistic modal logic into classical bimodal logics, based on Gödel's translation of intuitionistic logic into the classical modal logic $\mathsf{S4}$, by Fischer-Servi (1977), Wolter and Zakharyaschev (1999b); analyses of algebraic and Kripke semantics by Ono (1977), Sotirov (1984), Božić and Došen (1984), Došen (1985), Bezhanishvili (1998), Wolter and Zakharyaschev (1997, 1999a); and proof theory by Simpson (1994), Amati and Pirri (1994), Galmiche and Salhi (2015), Marin and Straßburger (2017). We refer to (Simpson, 1994) for a good survey on intuitionistic modal logic.

The difference between *constructive* and *intuitionistic* modal logic lies in the interpretation of $\Diamond$, but they agree on the interpretation of $\Box$. So in general, the $\Box$-fragments of both approaches coincide. Intuitionistic modal logics only dealing with $\Box$ are also studied in the literature and form the objects of our study. We will see that these pop up in different fields. We will also see that we can study 'real' intuitionistic modal axioms that do not have a meaningful classical interpretation. We zoom in on the coreflection principle $p \to \Box p$.

This chapter introduces basic definitions and concepts of the classical and intuitionistic modal logics that we will be working with throughout this thesis. The chapter is structured as follows. Section 1.1 introduces syntax and classical and intuitionistic modal logics with $\Box$ (we do not deal with intuitionistic $\Diamond$). Section 1.2 introduces relational semantics for classical and intuitionistic modal logic, separately. All the logics that we consider have the finite model property and are decidable. For the classical case we focus on different kinds of bisimulation. In addition, we discuss negative translations from classical modal logic into intuitionistic modal logic by Litak *et al.* (2017). Finally, Section 1.3 presents a survey on intuitionistic modal logics with the coreflection principle $p \to \Box p$ where we discuss applications in epistemic logic, provability logic, and lax logic.

## 1.1 Syntax

We study classical and intuitionistic modal logics over a language only containing $\Box$ and without $\Diamond$. For surveys on classical modal logic see (Chagrov and Zakharyaschev, 1997) and (Blackburn *et al.*, 2001). For the intuitionistic setting see (Božić and Došen, 1984), (Došen, 1985), and (Wolter and Zakharyaschev, 1999a).

Let $\mathcal{L}$ denote the *modal language* consisting of countably many *(propositional) variables* $p_1, p_2, \ldots$, *constant* $\bot$ (falsum), *propositional connectives* $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication), and *modal operator* $\Box$. *Modal formulas* are defined as usual by the following grammar, where we use auxiliary parentheses:

$$A ::= p \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (\Box A).$$

We denote by $\mathsf{Prop}$ the countable set of propositional variables and denote by $\mathsf{Form}$ the set of all well-formed formulas in language $\mathcal{L}$. For $P \subseteq \mathsf{Prop}$, let $\mathcal{L}(P)$ denote the language $\mathcal{L}$ restricted to propositional variables in $P$ and, similarly, let $\mathsf{Form}(P)$ be the set of formulas only consisting of propositional variables from $P$. We use letters $p, q, r, \ldots$ to range over propositional variables, $\overline{p}, \overline{q}, \overline{r}, \ldots$ to present finite lists of propositional variables, capital letters $A, B, C, \ldots$ to range over formulas, and Greek letters $\Gamma, \Delta, \ldots$ to range over finite (multi)sets of formulas. Given formulas $A$ and $B$, we use the following standard abbreviations:

$$\begin{aligned} \neg A &:= A \to \bot & \qquad A \leftrightarrow B &:= (A \to B) \wedge (B \to A) \\ \top &:= \neg\bot & \boxdot A &:= A \wedge \Box A \end{aligned}$$

As usual, we drop some parentheses in the representation of formulas so that $\Box$, $\boxdot$, and $\neg$ bind stronger than $\wedge$ and $\vee$, which in turn bind stronger than $\to$ and $\leftrightarrow$. Given a finite (multi)set $\Gamma$, we write $\bigwedge \Gamma$ and $\bigvee \Gamma$ for the iterated conjunction and disjunction of $\Gamma$, where $\bigwedge \emptyset := \top$ and $\bigvee \emptyset := \bot$, by definition. We define

$$\begin{aligned} \Box\Gamma &:= \{\Box A \mid A \in \Gamma\}, \\ \boxdot\Gamma &:= \{\boxdot A \mid A \in \Gamma\}, \\ \Box\Gamma \to \Gamma &:= \{\Box A \to A \mid A \in \Gamma\}. \end{aligned}$$

In many contexts, $\Gamma, \Delta$ is short for $\Gamma \cup \Delta$ and $\Gamma, A$ is short for $\Gamma \cup \{A\}$. Finally, for given formula $A$, $\mathsf{Var}(A)$ is the set of propositional variables occurring in $A$ and $\mathsf{Sub}(A)$ is the set of all subformulas of $A$ (including $A$ itself).

**1.1.1 Definition** (Substitution)
A *substitution* is a function $\sigma : \mathsf{Form}(P) \to \mathsf{Form}(Q)$ that commutes with the connectives and modality, that is, for all formulas $A, B \in \mathsf{Form}(P)$,

$$\begin{aligned} \sigma(\bot) &= \bot; \\ \sigma(A \cdot B) &= \sigma(A) \cdot \sigma(B), \text{ for } \cdot = \wedge, \vee, \to; \\ \sigma(\Box A) &= \Box \sigma(A). \end{aligned}$$

Classical and intuitionistic modal logics are defined on the basis of classical propositional logic CPC and intuitionistic propositional logic IPC, respectively.[2]

**1.1.2 Definition** (Normal modal logic; classical and intuitionistic)
A *classical (intuitionistic) normal modal logic* is a set of formulas $\mathsf{L} \subseteq \mathsf{Form}$ that contains

- classical (intuitionistic) propositional calculus CPC (IPC);
- *Normality axiom* (k)*:* $\Box(p \to q) \to (\Box p \to \Box q)$;

and is closed under the inference rules:

- *Modus ponens* (MP): from $A$ and $A \to B$ infer $B$;
- *Substitution* (Subst): for any substitution $\sigma$, from $A$ infer $\sigma(A)$;
- *Necessitation* (N): from $A$ infer $\Box A$.

So the difference of classical and intuitionistic normal modal logic lies in the propositional base. Since we only work with normal modal logics, we drop the term 'normal' and just speak about classical modal logic and intuitionistic modal logic. We denote K and iK for the smallest classical modal logic and smallest intuitionistic modal logic, respectively.

**1.1.3 Convention**
All results in this thesis concern classical and intuitionistic modal logics only. Therefore, when we say 'let L be a logic' we always mean 'let L be a classical or intuitionistic modal logic.'

[2]We assume that the reader is familiar with the logics CPC and IPC. Discussions can be found in many books, see, e.g., (Chagrov and Zakharyaschev, 1997, Chapter 1 and 2) and (Troelstra and van Dalen, 1988). In particular, IPC and CPC are defined in the language without $\Box$, and so we treat definitions such as that of Prop and Form accordingly when concerned with these logics.

| | | |
|---|---|---|
| (k) | $\Box(p \to q) \to (\Box p \to \Box q)$ | *normality axiom* |
| (t) | $\Box p \to p$ | *reflection axiom* |
| (4) | $\Box p \to \Box\Box p$ | *transitivity axiom* |
| (d) | $\Box p \to \neg\Box\neg p$ | *seriality axiom* |
| (5) | $\neg\Box p \to \Box\neg\Box p$ | *Euclidean axiom* |
| (wlöb) | $\Box(\Box p \to p) \to \Box p$ | *weak Löb axiom* |

**Figure 1.1.** Modal axioms[3]

| | |
|---|---|
| $\mathsf{K} := \mathsf{CPC} + (\mathsf{k})$ | $\mathsf{iK} := \mathsf{IPC} + (\mathsf{k})$ |
| $\mathsf{D} := \mathsf{K} + (\mathsf{d})$ | $\mathsf{iD} := \mathsf{iK} + (\mathsf{d})$ |
| $\mathsf{T} := \mathsf{K} + (\mathsf{t})$ | $\mathsf{iT} := \mathsf{iK} + (\mathsf{t})$ |
| $\mathsf{K4} := \mathsf{K} + (4)$ | $\mathsf{iK4} := \mathsf{iK} + (4)$ |
| $\mathsf{S4} := \mathsf{K} + (\mathsf{t}) + (4)$ | $\mathsf{iS4} := \mathsf{iK} + (\mathsf{t}) + (4)$ |
| $\mathsf{S5} := \mathsf{K} + (\mathsf{t}) + (5)$ | |
| $\mathsf{GL} := \mathsf{K} + (\mathsf{wl\ddot{o}b})$ | $\mathsf{iGL} := \mathsf{iK} + (\mathsf{wl\ddot{o}b})$ |

**Figure 1.2.** Classical and intuitionistic modal logics

If $\mathsf{L}$ is a logic and $\Gamma$ is a set of formulas, $\Gamma \vdash_{\mathsf{L}} A$ means that $A$ belongs to the smallest set of formulas containing $\mathsf{L} \cup \Gamma$ and that is closed under the rules (MP) and (N). This is the *global* consequence relation.[4] In particular, we write $\vdash_{\mathsf{L}} A$ for $\emptyset \vdash_{\mathsf{L}} A$, which means $A \in \mathsf{L}$. When $\Gamma \vdash_{\mathsf{L}} A$, one can think of it as a derivation from $\Gamma$ to $A$, which is a sequence of formulas $A_1, \dots, A_n$ such that $A_n = A$ and for every $1 \leq i \leq n$, $A_i$ is either an axiom from $\mathsf{L}$, or in $\Gamma$, or is obtained from some of the preceding formulas by (MP), (Subst), (N), where (Subst) is only applied to axioms from $\mathsf{L}$.

A set of formulas $\Gamma$ is an *axiomatization* for a classical (intuitionistic) modal logic $\mathsf{L}$ if $\mathsf{L}$ is the smallest classical (intuitionistic) modal logic containing $\Gamma$. A logic is called *finitely axiomatizable* if there is a finite axiomatization for it. In addition, for given classical (intuitionistic) logic $\mathsf{L}$ and set of formulas $\Gamma$, let $\mathsf{L} + \Gamma$ denote the smallest classical (intuitionistic) modal logic containing $\mathsf{L} \cup \Gamma$, and we say that $\mathsf{L} + \Gamma$ *extends* $\mathsf{L}$ and that $\Gamma$ is its *axiomatization over* $\mathsf{L}$. We write $\mathsf{L} + A_1 + \dots + A_n$

[3]The reflection axiom is usually called the reflexivity axiom, but we choose to use terminology in line with the coreflection axiom (see Figure 1.3). In addition, weak Löb axiom is known as Löb's axiom, but we would like to stress its relation to a stronger variant used to define strong Löb logic iSL (see Figure 1.4)

[4]Notice that closure under (Subst) is not required. For our purposes in Part I, this definition suffices. For the characterization of the admissible rules in Part II, we carefully introduce the concepts of consequence relations and formally define the local and global one (Section 5.2).

to mean $\mathsf{L} + \{A_1, \ldots, A_n\}$. Different axiomatizations can result in the same modal logic (considered as a set of formulas) and write $\mathsf{L}_1 \equiv \mathsf{L}_2$ if that is the case for logics $\mathsf{L}_1$ and $\mathsf{L}_2$.

We recall well-known modal logics in its classical and intuitionistic setting in Figure 1.2 using modal axioms from Figure 1.1. Note that all these logics are finitely axiomatizable. In Figure 1.4 in Section 1.3 we will introduce more intuitionistic modal logics, all containing the coreflection principle $p \to \Box p$.

It is well known that for classical modal logic we can define $\to$ in terms of $\neg$ and $\vee$ and we can define the modality $\Diamond$ as the dual of $\Box$ as follows $\Diamond A := \neg\Box\neg A$. It is not in general the case for intuitionistic modal logics. For each intuitionistic modal logic from Figure 1.2 it results in its classical counterpart by adding the law of excluded middle $p \vee \neg p$. We omit a definition of $\mathsf{iS5}$ on purpose, because there is no standard intuitionistic counterpart of $\mathsf{S5}$ also when restricted to $\Box$-fragments. Došen (1985) has distinguished different intuitionistic counterparts of axiom (5) that are classically equivalent. In Section 1.2.3 we discuss the relation between the classical and intuitionistic modal logics from Figure 1.2 in terms of negative translations studied by Litak *et al.* (2017).

We close this section with a fundamental property of the global consequence relation $\vdash_{\mathsf{L}}$ for logics extending $\mathsf{K4}$ or $\mathsf{iK4}$, i.e., each (classical or intuitionistic) modal logic that contains the transitivity axiom $\Box p \to \Box\Box p$. See (Hakli and Negri, 2012) for a discussion on the deduction theorem in modal logic.

**1.1.4 Theorem** (Deduction theorem)
Let $\mathsf{L}$ be a classical modal logic extending $\mathsf{K4}$ or an intuitionistic modal logic extending $\mathsf{iK4}$. For all sets of formulas $\Gamma$ and formulas $A, B$ it holds that,

$$\Gamma, A \vdash_{\mathsf{L}} B \text{ if and only if } \Gamma \vdash_{\mathsf{L}} \boxdot A \to B.$$

Proof. It follows by induction on the derivations and applications of $\Box A \to \Box\Box A$, cf. (Chagrov and Zakharyaschev, 1997, Theorem 3.51). ■

## 1.2 Semantics

In this section we recall Kripke semantics for classical and intuitionistic modal logics in two separate subsections. For the classical modal logics, we especially focus on different variants of bisimulations. We discuss the semantics for intuitionistic modal logics with the coreflection principle $p \to \Box p$ in Section 1.3. We end with a small subsection about translations between the classical and intuitionistic modal logics from Figure 1.2.

### 1.2.1 Classical relational semantics

For general overviews on classical modal logics, we refer to (Chagrov and Zakharyaschev, 1997) and (Blackburn *et al.*, 2001). We introduce relational semantics. For relation $R \subseteq W \times W$, we write $wRv$ to mean $(w, v) \in R$.

**1.2.1 Definition**
A *classical modal Kripke frame* is a pair $(W, R)$, where $W$ is a nonempty set equipped with a binary relation $R \subseteq W \times W$. A *classical modal Kripke model* is a triple $(W, R, V)$, where $(W, R)$ is a classical modal Kripke frame and $V$ is a map $V : W \to \mathcal{P}(\mathsf{Prop})$ called a *valuation*. We say that $(W, R, V)$ *is a model over* $P \subseteq \mathsf{Prop}$ if the codomain of $V$ is restricted to $\mathcal{P}(P)$.

We only consider modal Kripke frames and modal Kripke models. So, we will simply call them classical frames and classical models or even just frames and models when there can be no confusion with the intuitionistic setting. We use letters $K, M, N$ to denote Kripke models and we call elements in $W$ *worlds*.

We often write $w \in K$ to mean $w \in W$ when $K = (W, R, V)$, and write $K(w)$ to mean $V(w)$. And for worlds $w, v \in K = (W, R, V)$, if $wRv$ we call $w$ a *predecessor* of $v$ and $v$ a *successor* of $w$, or say that $w$ is *below* $v$ and $v$ is *above* $w$.

**1.2.2 Definition**
Let $K = (W, R, V)$ be a model with world $w \in W$. For formula $A$, we inductively define a forcing relation as usual:

$$
\begin{array}{lll}
K, w \Vdash p & \text{iff} & p \in V(w); \\
K, w \nVdash \bot; & & \\
K, w \Vdash A \wedge B & \text{iff} & K, w \Vdash A \text{ and } K, w \Vdash B; \\
K, w \Vdash A \vee B & \text{iff} & K, w \Vdash A \text{ or } K, w \Vdash B; \\
K, w \Vdash A \to B & \text{iff} & \text{if } K, w \Vdash A, \text{ then } K, w \Vdash B; \\
K, w \Vdash \Box A & \text{iff} & \text{for all } v \text{ such that } wRv \text{ we have } K, v \Vdash A.
\end{array}
$$

If $K, w \Vdash A$, we say that $A$ *is true at* $w$, $w$ *forces* $A$, or $w$ *satisfies* $A$. We write $w \Vdash A$ if model $K$ is clear from the context. We write $K \models A$ to mean $K, w \Vdash A$ for every $w \in K$ and say that $K$ *satisfies* $A$, or $A$ *is true* in $K$. We say that $w$ *refutes* $A$ and $K$ refutes $A$, if $w \nVdash A$ and $K \not\models A$, respectively. For frame $F = (W, R)$, we say that $F$ *satisfies* $A$, and write $F \models A$, if for every valuation $V$ on $F$ we have $K \models A$ where $K = (W, R, V)$.

We recall standard properties of binary relations, frames, and models. We use first order notation in the following definition.

### 1.2.3 Definition

In the following, let $w, v, u$ range over elements in $W$. Binary relation $R \subseteq W \times W$ is called

- *reflexive* if for all $w$, $wRw$;
- *irreflexive* if for all $w$, not $wRw$;
- *transitive* if for all $w, v, u$, if $wRv$ and $vRu$, then $wRu$;
- *intransitive* if for all mutually distinct $w, v, u$, $wRv$ and $vRu$ yield not $wRu$;
- *serial* if for all $w$, there exists $v$ such that $wRv$;
- *Euclidean* if for all $w, v, u$, if $wRv$ and $wRu$, then $vRu$;
- *symmetric* if for all $w, v$, if $wRv$, then $vRw$;
- *antisymmetric* if for all $w, v$, if $wRv$, then not $vRw$;
- an *equivalence relation* if it is reflexive, transitive, and symmetric;
- *conversely well-founded* if there is no infinite ascending sequence $wRvRuR\ldots$ of not necessarily distinct elements from $W$;
- *dense* if for all $w, v$ such that $wRv$ there exists $u$ such that $wRu$ and $uRv$;
- *total* if $R = W \times W$.

For a binary relation $R$, we denote by

- $R^+$ the reflexive closure of $R$;
- $R^*$ the reflexive and transitive closure of $R$;
- $wR^>v$ to mean $wRv$ and not $vRw$.

Set $W$ is called

- *rooted* (w.r.t. $R$) if there exists $w$ such that for every $v$, $wR^*v$. Such a $w$ does not have to be unique, but sometimes we pinpoint one such $w$ and call it the *root* and we usually denote it by $\rho$;
- *treelike* (w.r.t. $R$) if it is rooted w.r.t. $R^*$ with a unique root, and for every $w$ that is not the root, the set $\{v \in W \mid vR^*w\}$ is finite, and there exists a unique $v$ distinct from $w$ such that $vRw$ and there is no other $u$ with $vRu$ and $uRw$. We call such $v$ the *parent* of $w$ and $w$ a *child* of $v$. World $w$ is called a *leaf* if $wRv$ implies $v = w$.

### 1.2.4 Remark

Note that we adopt a nonstandard definition of intransitivity in which we require the $w, v, u$ to be mutually distinct. This allows us to speak about intransitive models with reflexive worlds. In addition, when $W$ is rooted w.r.t. a transitive relation $R$ there exists a $w$ such that for each distinct $v$ we have $wRv$.

### 1.2.5 Definition

Let $F = (W, R)$ be a frame. We define the following properties, which are defined similarly for a model $K = (W, R, V)$.

- $F$ is called *finite* if $W$ is finite;
- For any property P of $R$ from Definition 1.2.3, $F$ is said to have property P if $R$ satisfies property P. For example, $F$ is called reflexive if $R$ is.
- For any property P from Definition 1.2.3, $F$ is said to have property P if $W$ has property P (w.r.t. $R$). For example, $F$ is rooted if $W$ is.

A logic $\mathsf{L}$ is *sound* with respect to a class of frames (models) $\mathcal{K}$ if for all formulas $A$, $\vdash_{\mathsf{L}} A$ implies $K \models A$ for all $K \in \mathcal{K}$. A logic $\mathsf{L}$ is *complete* with respect to a class of models $\mathcal{K}$ if the implication holds in the other direction, that is, if for any formula $A$, if $K \models A$ for all $K \in \mathcal{K}$, then $\vdash_{\mathsf{L}} A$. This is also known as *weak* completeness. It is common to say 'complete' when meaning 'sound and complete.'

**1.2.6 Definition** (Finite model property)
A logic is said to satisfy the *finite model property* if it is complete with respect to a finite class of models (or frames).

We recall the following completeness results, see, e.g., (Blackburn *et al.*, 2001, §4.2 and §4.3) and (Chagrov and Zakharyaschev, 1997, §5.2).

**1.2.7 Theorem** (Completeness finite frames and models)
The following statements also hold for models instead of frames.

1. $\mathsf{K}$ is complete with respect to the class of finite frames.
2. $\mathsf{T}$ is complete with respect to the class of finite reflexive frames.
3. $\mathsf{D}$ is complete with respect to the class of finite serial frames.
4. $\mathsf{K4}$ is complete with respect to the class of finite transitive frames.
5. $\mathsf{S4}$ is complete with respect to the class of finite transitive reflexive frames.
6. $\mathsf{S5}$ is complete with respect to the class of finite total frames.
7. $\mathsf{GL}$ is complete with respect to the class of finite transitive irreflexive frames.

In Part II we use the slightly stronger result for rooted frames. This follows from the previous completeness theorem and properties of generated subframes which will be defined in Definition 1.2.17, see, e.g., (Chagrov and Zakharyaschev, 1997, §3.3 and §5.2).

**1.2.8 Theorem**
The completeness results from Theorem 1.2.7 also holds when the mentioned classes of frames are replaced by the class of their respective rooted members.

**1.2.9 Definition**
A classical modal logic $\mathsf{L}$ is called *transitive* if $\mathsf{L} \supseteq \mathsf{K4}$. In turn, a transitive logic $\mathsf{L}$ is called *reflexive* if $\mathsf{L} \supseteq \mathsf{S4}$ and *irreflexive* if $\mathsf{L} \supseteq \mathsf{GL}$.

**1.2.10 Remark**
For transitive logics, we have for each rooted model $K$ with root $\rho$,

$$K \models A \text{ iff } K, \rho \Vdash \boxdot A.$$

**1.2.11 Definition**
Let $F = (W, R)$ be a transitive frame. The *cluster* $cl(w)$ of a world $w$ is the equivalence class of $w$ under the equivalence relation $\sim_R$ defined as

$$w \sim_R v \text{ iff } wR^+v \text{ and } vR^+w.$$

Note that if $cl(w)$ is not a singleton, all $v \in cl(w)$ are reflexive because of transitivity of the frame.

**1.2.12 Definition**
Let $K$ be a rooted transitive model with root $\rho$. We say that $K$ *almost satisfies* formula $A$ if $K, w \Vdash A$ for all $w \notin cl(\rho)$.

In Part I we are interested in completeness with respect to more restricted classes of frames and models, see, e.g., (Goré, 1999, p. 360).[5]

**1.2.13 Theorem** (Completeness finite treelike frames and models)
The following statements also hold for models instead of frames.

1. K is complete with respect to the class of finite intransitive irreflexive treelike frames.
2. T is complete with respect to the class of finite intransitive reflexive treelike frames.
3. D is complete with respect to the class of finite intransitive treelike frames with irreflexive worlds except the leaves that are reflexive.
4. GL is complete with respect to the class of finite transitive irreflexive treelike frames.

**1.2.14 Remark**
The completeness Theorems 1.2.7, 1.2.8, and 1.2.13 are with respect to different classes of frames. As mentioned before we use different classes in different contexts. If logic L is complete with respect to a class of frames $\mathcal{F}_{\mathsf{L}}$, we say that $F$ is an L-*frame* if $F \in \mathcal{F}_{\mathsf{L}}$ and $K$ is an L-*model* if $K$ is based on an L-frame. Each time we use this terminology the class of frames in question is fixed at the particular chapter or section.

[5] The result for D in (Goré, 1999) does not require the non-leaves to be irreflexive, but this can be obtained by unraveling. We do not go into further details.

**1.2.15 Remark**
For transitive logics K4 and S4, we do not have completeness with respect to finite treelike models. Instead, these are complete with respect to classes of finite trees of clusters defined in the following definition. K4 is complete with respect to the class of finite trees of finite clusters and S4 with respect to finite trees of finite clusters only containing reflexive worlds. Again see, e.g., Goré (1999, p. 360).

Another fundamental property of logics next to the finite model property is decidability.

**1.2.16 Definition** (Decidability)
A logic L is said to be *decidable* if for any formula $A$ there exists an algorithm that decides whether $\vdash_{\mathsf{L}} A$ or not.

It is well known that finitely axiomatizable logics with the finite model property are decidable. This fact is due to Harrop (1958), see, e.g., (Blackburn *et al.*, 2001, Theorem 6.15). So all the classical modal logics discussed above are decidable.

Now we turn to bisimulations. These are truth-preserving relations between models. Standard examples are the common operations on Kripke models: generated submodels, bounded morphisms, and disjoint unions. For our purposes, we recall the definitions of generated submodels and different variants of bisimulations. For this material we refer to (Blackburn *et al.*, 2001, Chapter 2) and (Chagrov and Zakharyaschev, 1997, Chapter 3).

**1.2.17 Definition** (Generated submodel)
Let $K = (W, R, V)$ be a model. A model $K' = (W', R', V')$ is a *submodel* of $K$ if $W' \subseteq W$, $R' = R \cap (W' \times W')$, and $V'$ is the restriction of $V$ to $W'$. A *generated submodel* of $K$, is a submodel $K' = (W', R', V')$ that is upward closed, i.e., for all $w' \in W'$, if $w'Rv$, then $v \in W'$. A *submodel generated by* $w \in W$, denoted $K_w = (W_w, R_w, V_w)$, is the smallest generated submodel containing $w$. A *generated subframe* (*by a world* $w$) is defined analogously.

Note that $K_w$ is a rooted model with root $w$.

**1.2.18 Definition** (Bisimulation)
Let $K = (W, R, V)$ and $K' = (W', R', V')$ be two models. A non-empty binary relation $Z \subseteq W \times W'$ is called a *bisimulation* between $K$ and $K'$ if the following conditions hold for all $w \in W$ and $w' \in W'$ with $wZw'$:

- (atoms): $p \in V(w)$ if and only if $p \in V'(w')$ for all $p \in \mathsf{Prop}$;
- (forth): if $wRv$, then there exists $v' \in W'$ such that $w'R'v'$ and $vZv'$;
- (back): if $w'R'v'$, then there exists $v \in W$ such that $wRv$ and $vZv'$.

We say that $w$ and $w'$ are *bisimilar*, and we write $(K, w) \sim (K', w')$ when $wZw'$ for some bisimulation $Z$. If the models are clear from the context we just write $w \sim w'$. For rooted models $K$ and $K'$ we write $K \sim K'$ to mean that their roots are bisimilar.

Note that $\sim$ forms an equivalence relation. The following variant of bisimulation is used in the literature on quantifier bisimulation on which we will expand more in Part I (see Section 2.4).

**1.2.19 Definition** (Bisimulation modulo $p$)
Let $K = (W, R, V)$ and $K' = (W', R', V')$ be two models. A non-empty binary relation $Z \subseteq W \times W'$ is called a *bisimulation modulo* $p$ between $K$ and $K'$ if conditions (forth) and (back) hold from Definition 1.2.18 and condition (atoms) is replaced by the following restricted condition:

- (atoms$^p$): $q \in V(w)$ if and only if $q \in V'(w')$ for all $q \in \mathsf{Prop} \setminus \{p\}$.

Relation $\sim^p$ is similarly defined as $\sim$ from Definition 1.2.18 and we say that the objects are $p$-*bisimilar*.

Clearly, bisimulation implies bisimulation modulo $p$. And again, $\sim^p$ forms an equivalence relation. The following shows that generated submodels can be seen as special cases of bisimulations. For a proof we refer to (Blackburn *et al.*, 2001, Theorem 2.19). Recall that we write $w \in K$ to mean $w \in W$ when $K = (W, R, V)$.

**1.2.20 Theorem**
Let $K$ be a model and let $K'$ be a generated submodel of $K$. Then for every $w' \in K'$ we have $(K, w') \sim (K', w')$.

It is well known that bisimulations preserve modal equivalence. In a similar way, bisimulations modulo $p$ preserve modal equivalence between formulas in the language without propositional variable $p$. The following is proved by induction on formulas, see, e.g., (Blackburn *et al.*, 2001, Theorem 2.20).

**1.2.21 Theorem** (Modal equivalence)
Let $K$ and $K'$ be two models. For every $w \in K$ and $w' \in K'$,

1. if $(K, w) \sim (K, w')$, then $K, w \Vdash A$ iff $K', w' \Vdash A$ for every formula $A$;
2. if $(K, w) \sim^p (K, w')$, then $K, w \Vdash A$ iff $K', w' \Vdash A$ for every formula $A$ with $p \notin \mathsf{Var}(A)$.

So together with Theorem 1.2.20 we have the following corollary.

**1.2.22 Corollary**
Let $K$ be a model and let $K'$ be a generated submodel of $K$. Then for every formula $A$ and every $w' \in K'$, we have $K, w' \Vdash A$ iff $K', w' \Vdash A$.

It is also well known that the converse of Theorem 1.2.21 is not true in general (see e.g. Blackburn *et al.*, 2001, Example 2.23).[6] A way to obtain both directions is to look at $n$-bisimulations.

**1.2.23 Definition** ($n$-Bisimulation)
Let $K = (W, R, V)$ and $K' = (W', R', V')$ be two models. A sequence of binary relations $Z_n \subseteq \cdots \subseteq Z_0$ is an *$n$-bisimulation* between $K$ and $K'$ if the following conditions hold for all $w \in W$ and $w' \in W'$ and $0 < i \leq n$:

- ($\mathsf{atoms}_n$): if $wZ_0w'$, then $p \in V(w)$ if and only if $p \in V'(w')$ for all $p \in \mathsf{Prop}$;
- ($\mathsf{forth}_n$): if $wZ_iw'$ and $wRv$, then there exists $v' \in W'$ such that $w'R'v'$ and $vZ_{i-1}v'$;
- ($\mathsf{back}_n$): if $wZ_iw'$ and $w'R'v'$, then there exists $v \in W$ such that $wRv$ and $vZ_{i-1}v'$.

We say that $w$ and $w'$ are *$n$-bisimilar*, and we write $(K, w) \sim_n (K', w')$ when $wZ_nw'$ for some $n$-bisimulation $Z_n \subseteq \cdots \subseteq Z_0$. If the models are clear from the context we simply write $w \sim_n w'$. For rooted models $K$ and $K'$ we write $K \sim_n K'$ to mean that their roots are bisimilar.

Note that for every $k < n$, if $(K, w) \sim_n (K', w')$, then $(K, w) \sim_k (K', w')$. Again, $\sim_n$ forms an equivalence relation on rooted models. For rooted model $K$, we denote $[K]_n$ for the $n$-bisimulation equivalence class of $K$. We recall the following well-known result, which can be proved by induction on $n$.

**1.2.24 Theorem**
Let $\overline{p}$ be a finite list of propositional variables. For each $n$, there are finitely many $n$-bisimulation equivalence classes of models over $\overline{p}$.

Bisimulation implies $n$-bisimulation for all $n$, but the converse is not true. Recall the following theorem stating that $n$-bisimulation exactly yields modal equivalence for formulas of modal degree $n$ when restricting to a finite number of propositional variables.

[6] Nonetheless, it is true for image-finite models known as the Hennessy-Milner Theorem (see e.g. Blackburn *et al.*, 2001, Theorem 2.24). In particular, modal equivalence between finite models implies bisimilarity.

**1.2.25 Definition** (Modal degree)
The *modal degree* $md(A)$ of a formula $A$ is defined recursively as follows:

$$\begin{aligned} md(p) &= 0, \text{ for } p \in \mathsf{Prop}; \\ md(\bot) &= 0; \\ md(A \cdot B) &= \max(md(A), md(B)), \text{ for } \cdot = \wedge, \vee, \to; \\ md(\Box A) &= md(A) + 1. \end{aligned}$$

It is well known that, for each $n$ and given finite set of propositional variables $\overline{p}$, there are finitely many non-equivalent formulas in $\mathsf{Form}(\overline{p})$ of modal degree less or equal to $n$, see, e.g., (Blackburn *et al.*, 2001, Proposition 2.29). For a proof of the following theorem we refer to (Blackburn *et al.*, 2001, Proposition 2.31).

**1.2.26 Theorem**
Let $\overline{p}$ be a finite list of propositional variables and let $K$ and $K'$ be models over $\overline{p}$. Then, for every $w \in K$ and $w' \in K'$, we have $(K, w) \sim_n (K', w')$ if and only if for all formulas $A \in \mathsf{Form}(\overline{p})$ with $md(A) \leq n$ we have $K, w \Vdash A$ iff $K', w' \Vdash A$.

Sometimes $n$-bisimulation is defined as in the next lemma. It is a folklore result that this is equivalent to the definition used here. We would like to provide a proof, because we could not easily find it in the literature.

**1.2.27 Lemma**
Let $K = (W, R, V)$ and $K' = (W', R', V')$ be two models with $w \in W$ and $w' \in W'$. We have $(K, w) \sim_n (K', w')$ if and only if the following conditions hold, where conditions (2) and (3) are not required for $n = 0$:

1. $(K, w) \sim_0 (K', w')$;
2. for all $v$ with $wRv$ there exists a $v'$ such that $w'R'v'$ and $(K, v) \sim_{n-1} (K', v')$;
3. for all $v'$ with $w'R'v'$ there exists a $v$ such that $wRv$ and $(K, v) \sim_{n-1} (K', v')$.

Proof. The case for $n = 0$ is straightforward. So suppose $n > 0$. The direction from left to right is easy by noticing that if $Z_n \subseteq \cdots \subseteq Z_0$ is an $n$-bisimulation, then $Z_{n-1} \subseteq \cdots \subseteq Z_0$ is an $(n-1)$-bisimulation. For the other direction we have to define an $n$-bisimulation $Z_n \subseteq \cdots \subseteq Z_0$ such that $wZ_nw'$. By condition (2), for every $u$ such that $wRu$, there exists $u'$ such that $w'R'u'$ and $(K, u) \sim_{n-1} (K', u')$. For each such $u$, let $Z^u_{n-1} \subseteq \cdots \subseteq Z^u_0$ be the $(n-1)$-bisimulation between $(K, u)$ and $(K', u')$. Similarly, by condition (3), for each $v'$ such that $w'R'v'$, there exists $v$ such that $wRv$ with an $(n-1)$-bisimulation $Z^{v'}_{n-1} \subseteq \cdots \subseteq Z^{v'}_0$ between $(K, v)$ and $(K', v')$. Note that each of $u, v$ can be $w$ and $u', v'$ can equal $w'$. We define $Z_n \subseteq \cdots \subseteq Z_0$ as follows for $i < n$:

$$Z_n := \{(w, w')\} \quad \text{and} \quad Z_i := \{(w, w')\} \cup \bigcup_{wRu} Z^u_i \cup \bigcup_{w'R'v'} Z^{v'}_i.$$

We have to check that this is indeed an $n$-bisimulation. Condition ($\mathsf{atoms}_n$) follows immediately from condition (1). For ($\mathsf{forth}_n$) suppose $xZ_ix'$ and $xRy$. For $i = n$, we know $x = w$ and $x' = w'$ and by construction $yZ^y_{n-1}y'$ for some $y'$ with $x'R'y'$, hence $yZ_{n-1}y'$. Now suppose $i < n$. For $x = w$ and $x' = w'$ we again have $yZ^x_{n-1}y'$ for some $x'R'y'$. Since $Z^y_{n-1} \subseteq Z^y_{i-1}$ we have $yZ^y_{i-1}y'$, hence $yZ_{i-1}y'$. In the other cases that $xZ^u_ix'$ or $xZ^{v'}_ix'$ for some $u$ such that $wRu$ or some $v'$ such that $w'R'v'$ the result follows immediately by the fact that $Z^u_i$ and $Z^{v'}_i$ are $(n-1)$-bisimulations. Condition ($\mathsf{back}_n$) is shown in a similar way. ■

Bisimulation modulo $p$ is used in the study of uniform interpolation in Part I and $n$-bisimulation is used in the study of admissible rules in Part II. We conclude with the remark that the combination of the two, i.e., $n$-bisimulation modulo $p$, can similarly be defined as studied by Visser (1996).

### 1.2.2 Intuitionistic birelational semantics

Intuitionistic modal Kripke semantics combines intuitionistic propositional and classical modal Kripke semantics. We use so-called birelational models, where a partial order $\leq$ acts as an intuitionistic relation and a binary relation $R$ acts as the modal relation. In order to provide a meaningful intuitionistic interpretation of the modality, different choices can be made. For full discussions see, e.g., (Simpson, 1994) and (Litak, 2014). The main difficulty is to provide a precise definition of the valuation clause of $\Box A$ under which the monotonicity lemma holds. The monotonicity lemma is an essential feature of intuitionistic models stating that valuation of a formula is upward closed in any model (see Lemma 1.2.31). One option is to use both relations in the valuation clause of the modality, that is,

$$K, w \Vdash \Box A \text{ iff for all } x \geq w \text{ and for all } v \text{ such that } xRv \text{ we have } K, v \Vdash A,$$

and not imposing any condition on the interaction between the two relations. Here we choose to use a classical reading of the valuation clause of the modality, and to impose the frame condition $\leq; R \subseteq R$, where ; denotes relation composition. Other frame conditions also suffice like $\leq; R \subseteq R; \leq$ (see, e.g. Božić and Došen (1984) and Simpson (1994)), and $\leq; R; \leq = R$ (see, e.g., Sotirov (1984) and Wolter and Zakharyaschev (1997, 1999a)). Models satisfying $\leq; R \subseteq R$ are present in (Goldblatt, 1981) and (Božić and Došen, 1984), and are called *condensed* models in the latter.

This section introduces the condensed models for intuitionistic modal logics $\mathsf{iK}$, $\mathsf{iD}$, $\mathsf{iT}$, $\mathsf{iK4}$, and $\mathsf{iS4}$ from Božić and Došen (1984) and Došen (1985). We treat $\mathsf{iGL}$ later in Subsection 1.3.2 where we discuss its provability interpretation in relation to other provability logics with the coreflection principle $p \to \Box p$.

**1.2.28 Definition**
A *partial order* is a reflexive, transitive, antisymmetric relation. We denote it by $\leq$ as usual and write $<$ to mean: $w < x$ if $w \leq x$ and $w \neq x$.

**1.2.29 Definition**
An *intuitionistic modal Kripke frame* is a triple $(W, \leq, R)$, where $W$ is a nonempty set equipped with a partial order $\leq$ and binary relation $R$. We require that $R$ is closed under prefixing with $\leq$, that is,

- $(\mathsf{R}_\Box)$: $\quad \leq; R \subseteq R.$, i.e., if $w < x$ and $xRv$, then $wRv$.

An *intuitionistic modal Kripke model* is a structure $(W, \leq, R, V)$, where $(W, \leq, R)$ is an intuitionistic modal Kripke frame and $V$ is a *valuation* $V : W \to \mathcal{P}(\mathsf{Prop})$ that is *monotonic* in $\leq$, i.e., $w \leq x$ implies $V(w) \subseteq V(x)$. We say that $(W, \leq, R, V)$ *is a model over* $P \subseteq \mathsf{Prop}$ if the codomain of $V$ is $\mathcal{P}(P)$.

Analogously to the classical setting, we use letters $K, M, N$ to indicate intuitionistic modal Kripke models and we often write $w \in K$ to mean $w \in W$ when $K = (W, \leq, R, V)$. In case there can be no confusion with the classical setting, we simply say frame and model.

**1.2.30 Definition**
Let $K = (W, \leq, R, V)$ be a model with world $w \in W$. For formula $A$, we inductively define a forcing relation as follows:

$$
\begin{array}{lll}
K, w \Vdash p & \text{iff} & p \in V(w); \\
K, w \nVdash \bot; & & \\
K, w \Vdash A \wedge B & \text{iff} & K, w \Vdash A \text{ and } K, w \Vdash B; \\
K, w \Vdash A \vee B & \text{iff} & K, w \Vdash A \text{ or } K, w \Vdash B; \\
K, w \Vdash A \to B & \text{iff} & \text{for all } x \geq w, \text{ if } K, x \Vdash A \text{ then } K, x \Vdash B; \\
K, w \Vdash \Box A & \text{iff} & \text{for all } v \text{ such that } wRv \text{ we have } K, v \Vdash A.
\end{array}
$$

The definitions of truth and validity are similarly defined as for classical modal Kripke models in Definition 1.2.2.

**1.2.31 Lemma** (Monotonicity lemma)
Let $K$ be an intuitionistic modal Kripke model. For any formula $A$ and worlds $w, x \in K$, if $K, w \Vdash A$ and $w \leq x$, then $K, x \Vdash A$.

Proof. Induction on the structure of $A$. ■

Recall the frame properties from Definition 1.2.3. The following is the intuitionistic analogue of Definition 1.2.5. The only difference lies in the third item where it combines the two relations $\leq$ and $R$.

**1.2.32 Definition**
Let $F = (W, \leq, R)$ be a frame. We define the following properties, which is defined similarly for a model $K = (W, \leq, R, V)$.

- $F$ is called *finite* if $W$ is finite;
- For any property P of $R$ from Definition 1.2.3, $F$ is said to have property P if $R$ satisfies property P. For example, $F$ is called reflexive if $R$ is.
- For any property P of $W$ from Definition 1.2.3, $F$ is said to have property P if $W$ has property P w.r.t. $\leq \cup R$. For example, $F$ is rooted if $W$ is rooted w.r.t. $\leq \cup R$.

Recall the following completeness results for intuitionistic counterparts of classical modal logics (we treat $\mathsf{iGL}$ in Section 1.3.2). Božić and Došen (1984) show completeness for $\mathsf{iK}$ via a canonical model construction and Došen (1985) for extensions of $\mathsf{iK}$.

**1.2.33 Theorem** (Completeness)
The following statements also hold for intuitionistic modal models instead of intuitionistic modal frames.

1. $\mathsf{iK}$ is complete with respect to the class of all frames.
2. $\mathsf{iD}$ is complete with respect to the class of serial frames.
3. $\mathsf{iT}$ is complete with respect to the class of reflexive frames.
4. $\mathsf{iK4}$ is complete with respect to the class of transitive frames.
5. $\mathsf{iS4}$ is complete with respect to the class of reflexive transitive frames.

**1.2.34 Definition**
An intuitionistic modal logic $\mathsf{iL}$ is called *conservative* over $\mathsf{IPC}$ if for every box-free formula $A$, if $\vdash_{\mathsf{iL}} A$, then $\vdash_{\mathsf{IPC}} A$.

Since the propositional connectives are only evaluated by the intuitionistic relation $\leq$ in the forcing relation of intuitionistic modal logics, we have the following easy corollary.

**1.2.35 Corollary**
Logics $\mathsf{iK}$, $\mathsf{iD}$, $\mathsf{iT}$, $\mathsf{iK4}$, and $\mathsf{iS4}$ are conservative over $\mathsf{IPC}$.

The finite model property can be proved by filtration methods, see, e.g., (Fairtlough and Mendler, 1997).[7] This can be rather complicated when dealing with both $\Box$ and $\Diamond$ simultaneously. For instance, Simpson (1994, §8.2) corrected a

[7] For filtration in classical modal logics, see, e.g., (Chagrov and Zakharyaschev, 1997, §5.3) or (Blackburn *et al.*, 2001, §2.3).

proof given by Ewald (1986) in the setting of tense logics. Here we present the proof for the logics considered in this section.

**1.2.36 Theorem** (Finite model property)
Each logic mentioned in Theorem 1.2.33 is complete with respect to the class of its finite models.

Proof. We use a filtration method and first prove it for iK. Let $K = (W, \leq, R, V)$ be a (possibly infinite) model refuting $A$. Define the following set where $\mathsf{Sub}(A)$ is the set of subformulas of $A$:

$$T_w := \{B \in \mathsf{Sub}(A) \mid K, w \Vdash B\}.$$

We define equivalence relation $\equiv$ on $W$ as follows:

$$w \equiv v \text{ if and only if } T_w = T_v.$$

Since $\mathsf{Sub}(A)$ is finite, there are only finitely many equivalence classes. We now define the filtration model

$$K_\equiv := (W_\equiv, \leq_\equiv, R_\equiv, V_\equiv),$$

where

- $W_\equiv$ is the finite set of equivalence classes $[w] = \{v \in W \mid w \equiv v\}$;
- $[w] \leq_\equiv [v]$ if and only if $T_w \subseteq T_v$;
- $[w]R_\equiv[v]$ if and only if for all $\Box A \in T_w$ we have $A \in T_v$;
- $p \in V_\equiv([w])$ if and only if $p \in \mathsf{Sub}(A)$ and $p \in V(w)$.

This is a well-defined model. Especially, $(\mathsf{R}_\Box)$ is satisfied for relations $\leq_\equiv$ and $R_\equiv$. Now we show that for each $B \in \mathsf{Sub}(A)$ and each world $w$ that $K, w \Vdash B$ if and only if $K_\equiv, [w] \Vdash B$. This is done by an easy induction on the structure of $B$. Here we show the details for $\Box B$. First suppose $K, w \Vdash \Box B$ and let $[w]R_\equiv[v]$. It follows that $B \in T_v$ and so $K, v \Vdash B$. The induction hypothesis implies $K_\equiv, [v] \Vdash B$. Now suppose $K_\equiv, [w] \Vdash \Box B$ and $wRv$. By definition of $R_\equiv$ it follows that $[w]R_\equiv[v]$, and so $K_\equiv, [v] \Vdash B$. By the induction hypothesis we conclude $K, v \Vdash B$. Hence, $K_\equiv$ is a finite model that refutes $A$.

For logics iT and iD, the same filtration suffices as reflexivity (seriality) of model $K$ implies reflexivity (seriality) of model $K_\equiv$. For logics iK4 and iS4 we adjust the filtration in the following standard way, only changing the definition of $R_\equiv$: $[w]R_\equiv[v]$ if and only if for all $\Box A \in T_w$ we have $A \in T_v$ and $\Box A \in T_v$. ■

The intuitionistic analogue of Theorem 1.2.8 holds and follows from constructions of submodels as defined below (cf. Chagrov and Zakharyaschev, 1997, §3.3, §5.2).

**1.2.37 Theorem**
The logics mentioned in Theorem 1.2.33 are complete with respect to the class of its finite rooted models.

**1.2.38 Definition** (Generated submodel intuitionistic)
Let $K = (W, \leq, R, V)$ be a model. Model $K' = (W', \leq', R', V')$ is a *submodel* of $K$ if $W' \subseteq W$, $\leq' = \leq \cap (W' \times W')$, $R' = R \cap (W' \times W')$, and $V'$ is the restriction of $V$ to $W'$. A *generated submodel* of $K$, is a submodel $K' = (W', \leq', R', V')$ that is upward closed, i.e., for all $w' \in W'$, if $w' \leq x$, then $x \in W'$, and if $w'Rv$, then $v \in W'$. A *submodel generated by* $w \in W$, denoted $K_w = (W_w, \leq_w, R_w, V_w)$, is the smallest generated submodel containing $w$.

Note that $K_w$ is a rooted model with root $w$.

The following is the analogue of Corollary 1.2.22. In the classical setting it is an immediate consequence of the properties of bisimulations. Bisimulation for intuitionistic modal logics falls outside the scope of this thesis. However, the following could be easily shown directly from the definitions.

**1.2.39 Theorem**
Let $K$ be an intuitionistic modal model and let $K'$ be a generated submodel of $K$. Then for every formula $A$ and every $w' \in K'$, we have $K, w' \Vdash A$ iff $K', w' \Vdash A$.

The finite model property and finite axiomatizations of the logics imply decidability of the logics.

**1.2.40 Theorem** (Decidability)
Logics $\mathsf{iK}, \mathsf{iT}, \mathsf{iD}, \mathsf{iK4}$, and $\mathsf{iS4}$ are decidable.

### 1.2.3 Negative translations

In the literature, there exist different translations between classical and intuitionistic modal logic providing justifications for certain intuitionistic counterparts (also with $\Diamond$) of classical modal logics. The first translations were focused on classical bimodal logics provided by Fischer-Servi (1977) and Wolter and Zakharyaschev (1999b). The translations are based on Gödel's translation of $\mathsf{IPC}$ into classical $\mathsf{S4}$. One can think of translating the intuitionistic part into $\mathsf{S4}$ and the modal part into a corresponding classical modal logic resulting in a classical bimodal logic. For instance, Fischer-Servi (1977) provides a translation of Prior's $\mathsf{MIPC}$ into classical bimodal logic $(\mathsf{S4}, \mathsf{S5})$ justifying that $\mathsf{MIPC}$ is the 'true' intuitionistic counterpart for $\mathsf{S5}$. Later, translations into single classical modal logic are considered. For instance, Bezhanishvili (2001) provides Glivenko type theorems between $\mathsf{MIPC}$ and $\mathsf{S5}$.

Here we briefly discuss double negation translations[8] between the classical and intuitionistic modal logics presented in Figure 1.2 following results from Litak *et al.* (2017). They study the question for what type of double negation translations $t$ and for what logics $\mathsf{L}$ and its intuitionistic version $\mathsf{iL}$ we have

$$\vdash_{\mathsf{L}} A \text{ iff } \vdash_{\mathsf{iL}} t(A).$$

Surprisingly, not every $\mathsf{iL}$ defined by standard axioms over $\mathsf{IPC}$ can be described by a double negation translation for its classical $\mathsf{L}$. They provide the example for $\mathsf{iK}$ extended by axiom $\Box\Box p \to \Box p$. But for logics $\mathsf{iK}, \mathsf{iT}, \mathsf{iK4}$ and $\mathsf{iS4}$ we have the following result. We expect the same result for $\mathsf{iGL}$, but we leave that for future work.

**1.2.41 Definition** (Kuroda's translation)
Define translation $A^{\mathsf{kur}} := \neg\neg A_{\mathsf{kur}}$, where $(\cdot)_{\mathsf{kur}}$ is defined as

$$\begin{aligned} p_{\mathsf{kur}} &:= p, \text{ for } p \in \mathsf{Prop}; \\ \bot_{\mathsf{kur}} &:= \bot; \\ (A \cdot B)_{\mathsf{kur}} &:= A_{\mathsf{kur}} \cdot B_{\mathsf{kur}}, \text{ for } \cdot = \wedge, \vee, \to; \\ (\Box A)_{\mathsf{kur}} &:= \Box\neg\neg A_{\mathsf{kur}}. \end{aligned}$$

**1.2.42 Theorem** (Litak *et al.*, 2017)
Let $\mathsf{L}$ denote $\mathsf{K}, \mathsf{T}, \mathsf{K4}$, or $\mathsf{S4}$. For all formulas $A$, it holds that

$$\vdash_{\mathsf{L}} A \leftrightarrow A^{\mathsf{kur}} \quad \text{and} \quad \vdash_{\mathsf{L}} A \text{ iff } \vdash_{\mathsf{iL}} A^{\mathsf{kur}}.$$

Theorem 1.2.42 does not work for Glivenko's translation defined as $A^{\mathsf{glv}} := \neg\neg A$. However, it suffices for logics with the coreflection principle $p \to \Box p$. We do not discuss this further, but we refer to (Litak *et al.*, 2017).

## 1.3 Intuitionistic modal logic with coreflection

Intuitionistic modal logic is not only interesting in its connection to classical modal logic. It also opens the door to study 'real' intuitionistic modal axioms that do not make sense in the eyes of a classical modal logician. It turns out that intuitionistic

[8] Well-known double negation translations are Glivenko's translation, Gödel-Gentzen translation, and Kolmogorov's translation, see, e.g., (Troelstra and van Dalen, 1988).

modal logics reveal constructive behaviors that meet interesting applications. One axiom that reveals an intuitionistic modal character is the *coreflection principle*

$$(\mathsf{c}) \quad p \to \Box p,$$

which can be considered as the dual of the reflection axiom $\Box p \to p$. The axiom pops up in different areas under different names. In subsequent subsections, we discuss its application in intuitionistic epistemic logic (where the axiom is known as the *constructivity of truth*), provability logic (under the name *completeness principle*), and lax logics.

The coreflection principle is a 'real' intuitionistic modal axiom, meaning that it cannot have a meaningful reading in the classical setting. Classically, adding the principle results in logics where the modality collapses as shown in the following lemma. In terms of Kripke semantics, one ends up with one-world classical modal models that are either reflexive or irreflexive by definition. In particular, adding it to classical Gödel-Löb logic GL yields $\Box A \leftrightarrow \top$ and adding it to reflexive logics such as T yields $\Box A \leftrightarrow A$.

**1.3.1 Lemma**
Formula $(\Box A \leftrightarrow A) \vee (\Box A \leftrightarrow \top)$ is derivable from the coreflection principle in any classical normal modal logic.

Proof. The result follows from derivations of $(\Box A \to A) \vee (\Box(\Box A \to A) \to \Box A)$, $(\Box A \to A) \to (\Box A \leftrightarrow A)$, and $(\Box(\Box A \to A) \to \Box A) \to (\Box A \leftrightarrow \top)$. We only show a Hilbert style derivation for the first, the others are left to the reader.

1. $(\Box A \to A) \vee \neg(\Box A \to A)$ (classical truth)
2. $\neg(\Box A \to A)$ (assumption)
3. $\bot \to A$ (classical truth)
4. $(\Box A \to A) \to A$ ((2) + (3))
5. $\Box(\Box A \to A) \to \Box A$ ((3) + (c) + (k))
6. $\neg(\Box A \to A) \to (\Box(\Box A \to A) \to \Box A)$ ((2) + (6))
7. $(\Box A \to A) \vee (\Box(\Box A \to A) \to \Box A)$ ((1) + (6))

■

Figure 1.4 gives the definitions of the logics with the coreflection principle that we consider, referring to axioms from Figure 1.1 and 1.3. Figure 1.5 forms an overview of finite semantics that we will discuss in subsequent material. See (Litak, 2014) for a schema of intuitionistic modal logics including those presented here.

Before going into the different interpretations of the coreflection principle, let us introduce the minimal coreflection logic and its relational semantics. We call it iCK4 for reasons explained below, but it is also known as logic R after Fairtlough

| | | |
|---|---|---|
| (c) | $p \to \Box p$ | *coreflection axiom* |
| (it) | $\Box p \to \neg\neg p$ | *intuitionistic reflection axiom* |
| (slöb) | $(\Box p \to p) \to p$ | *strong Löb axiom* |
| (cb) | $\Box p \to ((q \to p) \vee q)$ | *Cantor-Bendixson axiom* |
| (bind) | $\Box\Box p \to \Box p$ | *bind axiom* |

**Figure 1.3.** Modal axioms for logics with coreflection

| | |
|---|---|
| *Minimal coreflection logic* | iCK4 := iK + (c) |
| *Logic of intuitionistic belief* | $\mathsf{IEL}^-$ := iCK4 |
| *Logic of intuitionistic knowledge* | IEL := iCK4 + (it) |
| *Intuitionistic propositional logic* | iCS4 := iCK4 + (t) $\equiv$ IPC |
| *Intuitionistic strong Löb logic* | iSL := iCK4 + (wlöb) $\equiv$ iK + (slöb) |
| *Modalized Heyting calculus* | mHC := iCK4 + (cb) |
| *Kuznetsov-Muravitsky logic* | KM := iCK4 + (wlöb) + (cb) |
| *Propositional lax logic* | PLL := iCK4 + (bind) |

**Figure 1.4.** Logics with coreflection

| | | |
|---|---|---|
| iCK4 | finite strong models | Theorem 1.3.6 |
| IEL | finite strong serial models | Theorem 1.3.12 |
| iCS4 | finite strong reflexive models, i.e., $R = \leq$ | Theorem 1.3.11 |
| iSL | finite strong irreflexive models | Theorem 1.3.15 |
| mHC | finite strong models with $< \subseteq R$ | Theorem 1.3.15 |
| KM | finite strong irreflexive models with $< \subseteq R$, i.e., $R = <$ | Theorem 1.3.15 |
| PLL | finite FM-models | Theorem 1.3.20 |

**Figure 1.5.** Finite model property for logics with coreflection

and Mendler (1997). As defined in Figure 1.4 the *minimal coreflection logic* is defined as

$$\mathsf{iCK4} := \mathsf{iK} + p \to \Box p.$$

The i stands for 'intuitionistic' and the C refers to the coreflection principle (c). Note that the coreflection principle imposes a very strong condition on logics. It immediately implies $\boxdot A \leftrightarrow A$. It also implies transitivity of its corresponding Kripke frames as noted below, which is the reason why we include 4 in the name of the base logic iCK4.

**1.3.2 Remark**
It is worthwhile to note that + includes closure under the necessitation rule (N) by definition. However for logics with (c) this is not necessary anymore, since (c) can be seen as the internalized version of rule (N).

A stronger deduction theorem holds for logics with the coreflection principle, so it holds for all logics presented in Figure 1.4.

**1.3.3 Theorem** (Deduction theorem)
Let L be a logic extending iCK4. For all sets of formulas $\Gamma$ and formulas $A, B$ it holds that,

$$\Gamma, A \vdash_{\mathsf{L}} B \text{ if and only if } \Gamma \vdash_{\mathsf{L}} A \to B.$$

Proof. Similarly proved as the modal deduction theorem in Theorem 1.1.4. ■

**1.3.4 Definition** (Strong Kripke frames and models)
A *strong Kripke frame* $(W, \leq, R)$ is an intuitionistic modal frame such that the following condition holds:

- *strong* (S): $R \subseteq \leq$, i.e., if $wRv$, then $w \leq v$.

An *intuitionistic strong Kripke model* is an intuitionistic modal model based on a strong Kripke frame.

We adopt the terminology from Litak and Visser (2018), but the (S) is also known under the name *realistic* (Visser and Zoethout, 2019). Transitivity of the modal relation $R$ follows from condition (S) and the frame condition ($\mathsf{R}_\Box$) on intuitionistic modal Kripke models from Definition 1.2.29. Note that we have the monotonicity lemma for both relations.

**1.3.5 Lemma** (Strong monotonicity lemma)
Let $K = (W, \leq, R, V)$ be a strong Kripke model. For any formula $A$ and worlds $w, v, x \in W$,

1. if $K, w \Vdash A$ and $w \leq x$, then $K, x \Vdash A$;
2. if $K, w \Vdash A$ and $wRv$, then $K, v \Vdash A$.

Proof. Condition (1) follows from Lemma 1.2.31. Fact (2) follows immediately from (1) by (S). ■

It is easily seen that there is a correspondence between the coreflection principle (c) and the strong condition (S), see, e.g., Visser and Zoethout (2019). Completeness of iCK4 with respect to these models is shown via a standard Henkin construction of the canonical model by Artemov and Protopopescu (2016). Using a filtration method, we can prove the finite model property.

**1.3.6 Theorem** (Finite model property iCK4)
Logic iCK4 is complete with respect to the class of finite strong models.

Proof. We use a similar filtration as defined in the proof of Theorem 1.2.36. We only change the definition of $R_{\equiv}$ to: $[w]R_{\equiv}[v]$ if and only if $T_w \subseteq T_v$ and for all $\Box A \in T_w$ we have $A \in T_v$. This guarantees the strong condition $R_{\equiv} \subseteq \leq_{\equiv}$. ■

Strong models are rooted if they are rooted with respect to $\leq$. We have the following analogues of Theorem 1.2.37 and Corollary 1.2.35.

**1.3.7 Theorem**
Logic iCK4 is complete with respect to the class of finite rooted strong models.

**1.3.8 Corollary**
Logic iCK4 is conservative over IPC.

Recall the definition of a cluster for the classical modal setting in Definition 1.2.11. For strong models, clusters always contain one world, so we do not have to take these into account in the following intuitionistic version of Definition 1.2.12.

**1.3.9 Definition**
Let $K$ be a rooted strong model with root $\rho$. We say that $K$ *almost satisfies* formula $A$ if $K, w \Vdash A$ for all $w \neq \rho$.

The finite model property entails the decidability of iCK4.

**1.3.10 Theorem** (Decidability iCK4)
Logic iCK4 is decidable.

We conclude with some remarks on logic iCS4 which is defined as iCK4 plus the reflection axiom

$$(\mathsf{t}) \quad \Box p \to p.$$

It is clear that together with the reflection principle (c) yields $A \leftrightarrow \Box A$ for all formulas $A$, meaning that all modal formulas are equivalent to the formula obtained from it by dropping all the boxes. Although the language of IPC does not contain $\Box$, we abuse notation and write iCS4 $\equiv$ IPC. This fact implies the completeness result where the modal relation $R$ is equal to $\leq$ so that the models for iCS4 are essentially intuitionistic Kripke models for IPC. We state the theorem without proof.[9]

[9] We did not formally introduce the Kripke semantics for IPC. In our notation, it is a structure

**1.3.11 Theorem**
Logic iCS4 is complete with respect to finite strong reflexive models.

### 1.3.1 Intuitionistic epistemic logic

Among the interpretations of the coreflection principle that we discuss, we start with the most recent one by Artemov and Protopopescu (2016) who discuss the coreflection principle in the setting of intuitionistic epistemic logic. The reason that we start with epistemic logic, is that it provides a surprising interpretation for the minimal coreflection logic iCK4.

The coreflection principle has been a central point of discussion in the setting of epistemic logic in which $\Box A$ is interpreted as 'it is known that $A$.' In this setting, the principle is known as the *principle of omniscience* saying that *all truths are known.* The principle plays a big role in the famous Church-Fitch paradox of knowability, showing that *all truths are knowable* only if *all truths are known.*[10] The first is known as the *knowability principle.* The paradox concerns any theory committed to this principle such as Dummett's *anti-realism* and *verificationalism* which is the view that any truth is verifiable. The paradox lies in the fact that the knowability principle is considered to be valid, while the principle of omniscience is rejected.[11]

Many ways to tackle the paradox have been proposed. In (Maffezioli *et al.*, 2013), they are divided into three categories, each pointing to relevant literature there.

1. Restriction on the possible instances of the knowability principle;
2. Reformulation of the knowability principle;
3. Revision of the logical framework in which the Church-Fitch derivation is made.

All these solutions aim to avoid a derived conclusion of omniscience from a reasonable verificationist formalization of the knowability principle.

In contrast to these three categories, we can distinguish a fourth category in which the coreflection principle is not considered to be the principle of omniscience, but is given another interpretation. In this way, the coreflection principle is embraced

$(W, \leq, V)$ in which the valuation function $V$ is monotone in $\leq$. For more information, see, e.g., Troelstra and van Dalen (1988, Chapter 2) and Chagrov and Zakharyaschev (1997, Chapter 2).

[10] See Theorem 5 in (Fitch, 1963) and (Church, 2009).

[11] The Church-Fitch derivation as originally presented by Fitch (1963) was not supposed to outline a paradox. However, the result was rediscovered in (Hart and McGinn, 1976) and (Hart, 1979) in which Fitch's argument is put forward as a threat of the anti-realists position of knowledge.

concluding that there is no paradox after all.

4. Acceptance of the principle $p \to \Box p$.

This is the road that Artemov and Protopopescu (2016) take and that we follow here. This is not a mainstream standpoint in the discussion on Church-Fitch paradox, but is defended in (Martino and Usberti, 1994; Usberti, 2016; Khlentzos, 2004; Artemov and Protopopescu, 2016). Also in the work of Hart (1979) and Williamson (1988) similar arguments are lined out, but are still questioned. See (Murzi, 2010) and (Artemov and Protopopescu, 2016) for expositions of arguments for and against this standpoint.

The general outline of the argument commits to the common idea that intuitionistic meaning of truth is identified by the existence of a proof. Knowledge is considered to be a result of verification. Now from a proof of $A$ we immediately have a verification for $A$ because proofs are considered to be particular kinds of verification, justifying $A \to \Box A$. Using this argument, we do not read $p \to \Box p$ as a principle of omniscience, but rather as the *constructivity of truth*. Artemov and Protopopescu (2016) formalize these ideas by extending the well-known Brouwer-Heyting-Kolmogorov (BHK) interpretation[12] for intuitionism with an extra clause for the knowledge operator $\Box$, denoted by $\mathbf{K}$ in their work. Accordingly, a proof of $A \to \Box A$ is a construction that converts a proof of $A$ into a proof of $\Box A$, which in turn should be conclusive evidence of verification that $A$ has a proof. This construction is a proof checking procedure checking the proof of $A$.

In classical epistemic logic, the reflection axiom $\Box p \to p$ is taken to be primitive for knowledge, but here it is too strict and instead we adopt *intuitionistic reflection*

$$(\mathsf{it}) \quad \Box p \to \neg\neg p,$$

meaning that *known principles cannot be false*. Coreflection and intuitionistic reflection give

$$A \to \Box A \quad \text{and} \quad \Box A \to \neg\neg A,$$

for all formulas $A$. Using the well-known double negation translation from classical to intuitionistic logic[13], intuitionistic knowledge can be considered to lie strictly between intuitionistic truth and classical truth.

Artemov and Protopopescu (2016) present two logics, one for belief and one for knowledge. So far, we did not focus on belief. In contrast to knowledge, false beliefs are not *a priori* ruled out. We recall the definitions of the logics from

[12] The concept that intuitionistic truth is adhered from proofs, cf. (Troelstra and van Dalen, 1988). See the PhD thesis of Akbar Tabatabai (2018) for a new broad perspective on the BHK-interpretation.

[13] cf. Glivenko (1929).

Figure 1.4. The *basic intuitionistic logic of belief* $\mathsf{IEL}^-$ is the least intuitionistic modal logic extending $\mathsf{iK}$ and satisfying

$$(\mathsf{c}) \quad p \to \Box p,$$

and the *the logic of intuitionistic knowledge* $\mathsf{IEL}$ contains in addition to that the axiom

$$(\mathsf{it}) \quad \Box p \to \neg\neg p.$$

So $\mathsf{IEL}^-$ equals $\mathsf{iCK4}$ and provides an epistemic interpretation for the minimal coreflection logic. As shown in Theorem 1.3.6, $\mathsf{IEL}^-$ is complete with respect to finite strong models. Logic $\mathsf{IEL}$ is complete with respect to strong serial models, i.e., models in which for all worlds $w$ there is a $v$ such that $wRv$ (Artemov and Protopopescu, 2016). Note that for both semantics we have the strong monotonicity lemma (Lemma 1.3.5). We also have the finite model property.

**1.3.12 Theorem** (Finite model property $\mathsf{IEL}^-$ and $\mathsf{IEL}$)
Logic $\mathsf{IEL}^-$ is complete with respect to the class of finite strong models. Logic $\mathsf{IEL}$ is complete with respect to the class of finite strong serial models.

Proof. The result for $\mathsf{IEL}^- \equiv \mathsf{iCK4}$ is already shown in Theorem 1.3.6. For $\mathsf{IEL}$ a similar filtration argument is used. ∎

**1.3.13 Corollary**
Logics $\mathsf{IEL}^-$ and $\mathsf{IEL}$ are conservative over $\mathsf{IPC}$.

Similarly to Theorem 1.3.7, $\mathsf{IEL}$ is complete with respect to finite strong rooted models. From this semantics it can be easily shown that $\vdash_{\mathsf{IEL}} \neg\Box\bot$. We have the following analogy of Theorem 1.3.10.

**1.3.14 Theorem** (Decidability $\mathsf{IEL}^-$ and $\mathsf{IEL}$)
Logics $\mathsf{IEL}^-$ and $\mathsf{IEL}$ are decidable.

## 1.3.2 Provability logic

This section is devoted to the logics $\mathsf{iGL}$, $\mathsf{iSL}$, $\mathsf{mHC}$ and $\mathsf{KM}$ and their provability interpretation. The latter three contain the coreflection principle which is known as the *completeness principle* in this area. Before we go to intuitionistic provability, let us quickly go to the classical story.

The research on provability logic goes back to Gödel (1931), who showed that arithmetical theories can encode properties about themselves enabling formal reasoning about the theory within that theory. In particular, one can define a provability predicate $Pr_T(\ulcorner\varphi\urcorner)$ in $T$, meaning '$\varphi$ is provable in $T$,' where $\ulcorner\varphi\urcorner$ represents the Gödel number of arithmetical sentence $\varphi$. Provability logic is a modal logic that reflects the behavior of formal provability of some arithmetical system $T$, in which, informally, $\Box A$ reads as '$A$ is provable in $T$.' Solovay (1976) has shown in his famous completeness theorem that $\mathsf{GL}$ is the provability logic of Peano Arithmetic $\mathsf{PA}$. We remind the reader of Figure 1.2 where $\mathsf{GL}$ is defined as the least classical modal logic extending $\mathsf{K}$ containing the *weak Löb axiom*

$$(\mathsf{wl\ddot{o}b}) \quad \Box(\Box p \to p) \to \Box p,$$

also known as the *Löb* axiom or *Gödel-Löb* axiom. The completeness result involves *realizations* $(\cdot)^*$ of modal formulas that are functions from propositional variables $p$ to arithmetical sentences such that it commutes with the connectives and $(\Box A)^* = Pr_{\mathsf{PA}}(\ulcorner A^*\urcorner)$. Solovay's completeness theorem states that

$$\vdash_{\mathsf{GL}} A \text{ if and only if for all realizations } (\cdot)^*, \mathsf{PA} \vdash A^*.$$

Interestingly, axiom $(\mathsf{wl\ddot{o}b})$ of $\mathsf{GL}$ reflects a formalized version of Gödel's second incompleteness theorem stating that $\mathsf{PA}$, if consistent, cannot prove its own consistency, i.e.,

$$\vdash_{\mathsf{GL}} \neg\Box\bot \to \neg\Box(\neg\Box\bot).$$

Logic $\mathsf{GL}$ is also the provability logic for other theories. In addition, there are other provability logics in the classical setting, such as Grzegorczyk's logics $\mathsf{K4.Grz}$ and $\mathsf{S4.Grz}$.[14] The reader is referred to (Boolos, 1979) and (Artemov and Beklemishev, 2005) for more information.

Intuitionistic provability logic focuses on Heyting Artithmetic $\mathsf{HA}$, the intuitionistic counterpart of $\mathsf{PA}$. An intriguing open problem is the characterization of the provability logic of $\mathsf{HA}$. Here we consider a standard provability predicate for $\mathsf{HA}$, i.e., $Pr_{\mathsf{HA}}(\ulcorner\varphi\urcorner)$ and the question is what (intuitionistic) modal logic characterizes $\mathsf{HA}$ such as $\mathsf{GL}$ characterizes $\mathsf{PA}$. At first sight, a natural candidate seems *intuitionistic Gödel-Löb logic* $\mathsf{iGL}$[15] defined in Figure 1.2, i.e., $\mathsf{iGL}$ is the least intuitionistic modal logic extending $\mathsf{iK}$ and satisfying the *weak Löb* axiom

$$(\mathsf{wl\ddot{o}b}) \quad \Box(\Box p \to p) \to \Box p.$$

Although $\mathsf{iGL}$ is sound with respect to formal provability in $\mathsf{HA}$, it is not complete. Different principles of provability in $\mathsf{HA}$ have been discovered throughout the past

[14] $\mathsf{K4.Grz}$ is the logic that results by adding axiom $(\mathsf{grz})$ $\Box(\Box(A \to \Box A) \to A) \to \Box A$ to $\mathsf{K4}$. Similarly, $\mathsf{S4.Grz}$ is obtained by adding $(\mathsf{grz})$ to $\mathsf{S4}$ (originally by $\Box(\Box(A \to \Box A) \to A) \to A$). The provability interpretation for $\mathsf{S4.Grz}$ is based on the translation $t$ from $\mathsf{S4.Grz}$ into $\mathsf{GL}$ with $t(\Box A) = \Box A \wedge A$, see, e.g., (Maksimova, 2007). For $\mathsf{K4.Grz}$, see (Esakia, 2006).

[15] First studies on $\mathsf{iGL}$ focus on algebras and fixed points in (Sambin, 1976) and (Ursini, 1979), where $\mathsf{iGL}$ is called **ID**.

fifty years, as discussed in, e.g., (Iemhoff, 2001b). In order to get closer to the solution, research on preservativity logic[16] seems promising because it has lead to a conjecture of the axiomatization of the provability logic of HA by Iemhoff (2001b).

Logics with coreflection also open the door to a better understanding of provability in the intuitionistic context. In this area, it is natural to call axiom $p \to \Box p$ the *completeness principle*, since it reads as $p^* \to Pr(\ulcorner p^* \urcorner)$. In words, if $p^*$ is true then there exists a proof for $p^*$. For classical theories, Gödel's first incompleteness theorem implies that the presence of the completeness principle results in an inconsistent theory. Surprisingly, the completeness principle survives in the intuitionistic setting with, in words of Visser (1982), an 'intrinsic interest' and 'technical applications.'

Here we elaborate on the logics iSL, mHC, and KM that all contain the completeness principle. Recall Figure 1.4 for the definitions, i.e., *intuitionistic strong Löb logic* iSL extends iK with *strong Löb axiom*

$$(\mathsf{sl\ddot{o}b}) \quad (\Box p \to p) \to p,$$

or equivalently extends iGL with the *completeness principle*

$$(\mathsf{c}) \quad p \to \Box p.$$

Therefore iSL is also known as iGLC. The *modalized Heyting calculus* mHC was introduced by Esakia (2006) and is the least intuitionistic modal logic extending iK that satisfies the axioms

$$\begin{aligned} (\mathsf{c}) &\quad p \to \Box p, \\ (\mathsf{cb}) &\quad \Box p \to ((q \to p) \vee q). \end{aligned}$$

The name (cb) stands for *Cantor-Bendixson* for its connections with the Cantor-Bendixson derivative in topology, and is also known as (derv), see (Litak, 2014). The *Kuznetsov-Muravitsky logic* KM[17] is introduced by Kuznetsov and further studied together with Muravitsky in different papers, see (Muravitsky, 2014) for an excellent overview. Logic KM is the least intuitionistic modal logic extending iGL with previous axioms (c) and (cb).

Logic iSL is very important in the work of Ardeshir and Mojtahedi (2018), who characterized the $\Sigma_1$-provability logic of HA. More precisely, let a $\Sigma_1$-realization

[16] Preservativity logic is an extension of provability logic, see, e.g., (Visser, 1994; Iemhoff, 2003a; Iemhoff *et al.*, 2005). For an arithmetical theory $T$ and sentences $\varphi$ and $\psi$, $\varphi$ *$\Sigma_1$-preserves $\psi$ w.r.t. $T$*, if for all $\Sigma_1$-sentences $\theta$, if $T \vdash \theta \to \varphi$ then $T \vdash \theta \to \psi$. Provability of $\varphi$ can be defined as $\top$ $\Sigma_1$-preserves $\varphi$, and indeed the provability predicate is a $\Sigma_1$-sentence. Formal behaviour of preservativity logics can be studied in terms of the Lewis arrow $\prec\!\!\!\!3$, in which $\Box A \leftrightarrow \top \prec\!\!\!\!3\, A$, see (Litak and Visser, 2018).

[17] In early papers the logic was called $I^{\Delta}$.

be a realization $(\cdot)^*$ where $(p)^*$ is a $\Sigma_1$-sentence for propositional variable $p$. They provide an axiomatization for the logic called $\mathsf{iH}_\sigma$ and show that

$$\vdash_{\mathsf{iH}_\sigma} A \text{ if and only if for all } \Sigma_1\text{-realizations } (\cdot)^*, \mathsf{HA} \vdash A^*.$$

In addition, Visser and Zoethout (2019) show arithmetical completeness results for iSL in various theories and for various provability interpretations for $\Box$. For example, it is the provability logic of an extension of HA with the completeness principle with respect to what is called 'slow provability.' Logics mHC and KM mostly lend their interest in provability to embeddings into classical provability logics. The lattice of the normal extensions of KM and those of GL are isomorphic (Kuznetsov and Muravitsky, 1986). The lattice of the normal extensions of mHC is isomorphic to the lattice of normal extensions if K4.Grz, which was announced by Esakia (2006) and proved by Muravitsky (2017). In addition, KM is the provability logic of a theory closely related to PA (Visser, 1982).

In connection to Part I, we are interested in the proof theory of these logics. For KM there is a sequent calculus developed by Darjania (1984). Ursini (1979) discusses different axiomatizations for iGL. In Chapter 3 we develop (terminating) sequent calculi for iGL and iSL and show the Craig interpolation property.

In connection to Part II, we are interested in admissibility in provability logics and HA. Regarding HA, it is known for a long time that the disjunction property holds in HA, i.e., if $\mathsf{HA} \vdash \varphi \vee \psi$, then $\mathsf{HA} \vdash \varphi$ or $\mathsf{HA} \vdash \psi$ for arithmetical sentences $\varphi$ and $\psi$, but that its formalized version does not belong the the provability logic of HA (Leivant, 1975). Visser (1999) has shown that the propositional rules of HA are the same as those for IPC. And Iemhoff (2001b) has shown, based on facts by Visser (1994), that HA recognizes its propositional admissible rules, i.e., that for any propositional admissible rule $A/B$ in HA, formula $\Box A \to \Box B$ belongs to the provability logic of HA. In Chapter 7 we characterize the admissible rules for iSL, mHC, and KM.

A lot more can be said about intuitionistic provability. For instance, the existence and uniqueness of fixed points (Sambin, 1976; Muravitsky, 2014; Litak, 2014), preservativity logic and logics with Lewis arrow (see footnote 16), algebraic and topological semantics (see Litak (2014) for an overview).

In the rest of the section we recall relational semantics for the four logics iGL, iSL, mHC, and KM. Recall Definition 1.2.32 for the terminology about intuitionistic modal frames. Also recall the definition of strong frames from Definition 1.3.4. The four logics admit the finite model property as listed in the following theorem. To stress, by frame and model we mean intuitionistic modal frame and intuitionistic modal model.

**1.3.15 Theorem** (Finite model property)
The following statements also hold for models instead of frames.

1. (Ursini, 1979) iGL is complete with respect to the class of finite irreflexive frames.
2. (Visser and Zoethout, 2019) iSL is complete with respect to the class of finite strong irreflexive frames.
3. (Muravitsky, 1981) KM is complete with respect to the class of finite strong frames with $<\subseteq R$.
4. (Litak, 2014) mHC is complete with respect to the class of finite strong irreflexive frames with $<\subseteq R$.

We would like to make a few comments. The proof of the finite model property for mHC is published in (Litak, 2014) and uses the completeness results from Esakia (2006) with respect to strong conversely well-founded intuitionistic modal frames with $<\subseteq R$. Also stronger results have been published with respect to finite treelike models, such as for iSL by Ardeshir and Mojtahedi (2018) and KM by Muravitsky (see the overview by Muravitsky (2014)). In Section 3.3, we obtain similar results for iSL via a countermodel construction.

Theorem 1.3.15 also holds when restricted to rooted frames (cf. Theorem 1.3.7). We have the analogue of Corollary 1.2.35.

**1.3.16 Corollary**
Logics iGL, iSL, KM, and mHC are conservative over IPC.

Since the finite model property of the four logics, we can easily derive the known decidability properties of the logics.

**1.3.17 Theorem** (Decidability)
Logics iGL, iSL, mHC, and KM are decidable.

### 1.3.3 Lax logic

Lax logics are fascinating intuitionistic modal logics. They feature a non-standard modality that combines some properties of a classical $\Box$ and some properties of a classical $\Diamond$. Despite the non-standard modality they appear in interesting applications in different fields of mathematics and computer science. In this section we recall definitions and semantics of propositional lax logic PLL and give an overview of its applications.

To stress the non-standard reading of the modality it is common to denote it by a new symbol $\bigcirc$. So when we are concerned with lax logics we understand $\mathcal{L}$ to be the language as defined in Section 1.1 with $\Box$ replaced by $\bigcirc$.

Recall the definition of *propositional lax logic* $\mathsf{PLL}$ from Figure 1.4, i.e., $\mathsf{PLL}$ is defined as the least intuitionistic modal logic extending $\mathsf{iK}$ and satisfying the axioms

$$\begin{array}{rl} (\mathsf{c}) & p \to \bigcirc p, \\ (\mathsf{bind}) & \bigcirc\bigcirc p \to \bigcirc p. \end{array}$$

Any intuitionistic modal extension of $\mathsf{PLL}$ is called a *lax logic* and $\bigcirc$ is called the *lax modality*.

The first axiom is usually denoted by label $(\mathsf{R})$ in this field, see (Fairtlough and Mendler, 1997). For the second axiom, we use the terminology from Litak (2014) referring to its connections with monads in functional programming. The lax modality shows features of $\Box$ and $\Diamond$ from classical modal logic. Indeed, the normality axiom $(\mathsf{k})$ is reserved for $\Box$ and not for $\Diamond$. Also $(\mathsf{bind})$ can be taken as an axiom for $\Box$ reflecting dense Kripke models. On the other hand, axioms $(\mathsf{c})$ and $(\mathsf{bind})$ are taken to be axioms for $\Diamond$ in classical $\mathsf{S4}$. But the combination of these axioms do not have a meaningful interpretation for a classical logician (Lemma 1.3.1).

Now we give a brief historical overview of lax logic in which we highlight the applications of our interest. We refer to (Fairtlough and Mendler, 1997) and (Melzer, 2020, Section 3.1) for more elaborated overviews.

The first appearance of a lax-like modality can be traced back to a proof-theoretic study by Curry (1957), but Melzer (2020) points out that the first steps towards lax modalities may already have been made by Dedekind (1888) in connection to Galois theory. Goldblatt (1981) provides algebraic semantics in terms of *local algebras* and relational semantics. In addition he studies topological properties. In type theory and category theory $\bigcirc$ appears as a unary type constructor in the computational lambda calculus of Moggi (1991), where it features as a *strong monad*. A Curry-Howard isomorphism between $\mathsf{PLL}$ and Moggi's computational lambda calculus is provided by Benton *et al.* (1998).

The terminology *lax logic* stems from the noteworthy work of Mendler (1993) and his work together with Fairthlough in (Fairtlough and Mendler, 1994, 1997). They study lax modalities in the context of behavioral constraints analysis in hardware verification. The reading of $\bigcirc A$ is '$A$ is true under some constraint.' The term 'lax' is chosen to indicate the looseness of the notion of correctness up to constraints. As a specific application, they study timing analysis of combinational circuits. Of our interest is their birelational Kripke models for $\mathsf{PLL}$ which they call *constraint models*. We study them under the name FM-models after Fairtlough and Mendler.

Lax logic naturally arises from the algebraic study of nuclei and subframe logics. *Nuclear algebras* are the same as the local algebras of Goldblatt, which are Heyting algebras equipped with a so-called *nucleus*. Bezhanishvili and Ghilardi (2007) have shown that nuclear algebras are in one-to-one correspondence with subframes of Esakia spaces. Furthermore, Bezhanishvili *et al.* (2019) study translations of intermediate logics into lax logics providing a new characterization of subframe logics in terms of lax logics. In addition, Melzer (2020) develops canonical formulas for lax logics based on the algebraic study in (Bezhanishvili *et al.*, 2021), and uses these to prove some preservation results for the translations of lax logics into intermediate logics.

Finally, the proof-theoretic study of PLL is interesting in light of Part I. Sequent-style calculi are developed by Fairtlough and Mendler (1997) and Howe (2001). Moreover, Iemhoff (2021) provides a terminating sequent calculus and proves the uniform interpolation property for PLL. In light of Part II, we mention the work by Ghilardi and Lenzi (2022) who show finite unification for PLL.

In the rest of this section we recall definitions of FM-models. These are not considered to form a standard Kripke semantics for PLL. Standard is the semantics provided by Goldblatt (1981).

**1.3.18 Theorem** (Goldblatt, 1981)
Logic PLL is complete with respect to the class of dense strong intuitionistic modal Kripke models.

**1.3.19 Definition** (FM-frames and FM-models)
An *FM-frame* is a structure $(W, \leq, R, F)$, where $W$ is a non-empty set with partial order $\leq$, binary relation $R$, and $F \subseteq W$ the set of *fallible* worlds, and we require the following conditions:

- $R$ is reflexive and transitive;
- *strong* (S): $R \subseteq \leq$;
- $F$ is upwards closed, i.e., if $w \leq x$ and $w \in F$, then $x \in F$.

An *FM-model* is a structure $K = (W, \leq, R, F, V)$, where $(W, \leq, R, F)$ is an FM-frame and $V$ is the *valuation* map $V : W \to \mathcal{P}(\mathsf{Prop})$ which is *monotonic* in $\leq$, i.e., $w \leq x$ implies $V(w) \subseteq V(x)$ and, in addition,

- $V$ is *full* on $F$, i.e., if $w \in F$, then $V(w) = \mathsf{Prop}$.

Similarly to Definition 1.2.29 we can define an FM-model *over a set of variables* $P \subseteq \mathsf{Prop}$. In that case we require that if $w \in F$, then $V(w) = P$.

The forcing relation $\Vdash$ on FM-models is defined as in Definition 1.2.30 modulo the following changes:

$$
\begin{array}{lll}
K, w \Vdash \bot & \text{iff} & w \in F; \\
K, w \Vdash \bigcirc A & \text{iff} & \text{for all } x \geq w \text{ there exists a } v \text{ such that } xRv \text{ and } K, v \Vdash A.
\end{array}
$$

The definitions of truth and validity remain the same as for intuitionistic modal Kripke models in Definition 1.2.30.

By induction on $A$ we have $K, w \Vdash A$ for all $w \in F$. Observe that the monotonicity lemma holds with respect to both relations (Fairtlough and Mendler, 1997), similarly shown as Lemma 1.3.5.

The finite model property with respect to FM-models is shown by a filtration argument in (Fairtlough and Mendler, 1997, Theorem 4.6).

**1.3.20 Theorem** (Finite model property)
Logic PLL is complete with respect to the class of finite FM-models.

A generated submodel of an FM-model $(W, \leq, R, F, V)$ is similarly defined as in Definition 1.2.38 while taking into account the restrictions on $F$. An FM-model is rooted if it is rooted with respect to $\leq$. Analogously to Theorem 1.2.8, we have completeness with respect to the classes of finite rooted FM-frames.

We have the following analogue of Corollary 1.2.35. See also (Fairtlough and Mendler, 1997, Theorem 2.4).

**1.3.21 Corollary**
Logic PLL is conservative over IPC.

As a consequence of the finite model property and finite axiomatization of PLL we know that PLL is decidable. The theorem was first shown by Goldblatt (1981).

**1.3.22 Theorem** (Decidability)
Logic PLL is decidable.

This concludes this chapter in which we recalled all the necessary definitions and results for classical modal logics and intuitionistic modal logics (with coreflection). The definitions of the logics that we will work with can be found in Figures 1.2 and 1.4.

# Part I

## Uniform Interpolation in Proof Theory

# 2

# Basics of Uniform Interpolation

Uniform interpolation is a property of logics stronger than the well-known Craig interpolation property. It plays an essential role in proof theory. On one hand, proof theory provides a constructive approach to uniform interpolation. On the other hand, uniform interpolation plays a role in the existence of calculi. Part I of the thesis studies different proof-theoretic methods of Craig interpolation and uniform interpolation in the realm of classical and intuitionistic modal logic.

This chapter forms the introduction to Part I where we introduce the necessary basics. We start with a historical overview, where we pay special attention to proof-theoretic research in the area. We also discuss several applications of uniform interpolation in for instance data bases and computer science. Section 2.2 presents the formal concepts on Craig interpolation and uniform interpolation in classical and intuitionistic modal logic. Section 2.3 discusses the role of sequent calculi, where termination is a key concept for uniform interpolation. Section 2.4 studies the link to propositional quantification in syntax and semantics, the latter known as bisimulation quantification.

## 2.1 History

This historical overview focuses on results on Craig interpolation and uniform interpolation obtained for classical and intuitionistic modal logics with a special focus on its relation to proof theory in Section 2.1.3.

### 2.1.1 Craig interpolation

The study of interpolation in logic finds its roots in Craig's famous interpolation lemma for first order logic (Craig, 1957b). A logic has Craig interpolation if for any provable implication $A \to B$ there is an interpolant $C$ with non-logical symbols that occur in both $A$ and $B$ such that the implications $A \to C$ and $C \to B$ are provable. One could say that the purpose of the interpolant is to state the reason why $B$ is implied by $A$ by using the common language of the two. What this language is, depends on the logic in question. For propositional (modal) logics it amounts to the common propositional variables occurring in $A$ and $B$.

It is well known that Craig interpolation has close connections to other important logical properties. It implies the definability theorem of Beth (1953) imposing a good balance between syntax and semantics and it plays a role to show functional completeness (Henkin, 1963). There are different ways to prove the Craig interpolation property. In model theory, it is implied by Robinson's joint consistency theorem (Robinson, 1956). The algebraic counterpart of Craig interpolation is (super)amalgamation of a corresponding variety of algebras. And in proof theory, cut-free proof systems are used to construct explicit interpolants.

All these concepts are widely studied for many logics. For introductions we refer to the PhD thesis (Hoogland, 2001), the textbook (Gabbay and Maksimova, 2005), and the chapters, (Takeuti, 1987, §1.6) and (Chagrov and Zakharyaschev, 1997, Chapter 14).

Although Craig interpolation is considered to be a desirable property, it is quite rare. Most propositional logics lack the property. A landmark result is that only seven consistent intermediate logics enjoy the Craig interpolation property (Maksimova, 1977). In the setting of classical modal logic, at most 37 logics extending $\mathsf{S4}$ enjoy Craig interpolation (Maksimova, 1979). And although there is a continuum of extensions of $\mathsf{GL}$ with interpolation, there is also a continuum of extensions of $\mathsf{GL}$ without it (Chagrov and Zakharyaschev, 1997, §14.4).

Surprisingly, Craig interpolation is not much studied yet in intuitionistic modal logics. Luppi (1996) establishes the property for $\mathsf{S4}$- and $\mathsf{S5}$-type of intuitionistic modal logics, one of which is $\mathsf{iK4}$ as defined in Figure 1.2. Wijesekera (1990) studies Craig interpolation in constructive modal logics. Muravitsky (2014) proves Craig interpolation for logic $\mathsf{KM}$. In addition, Simonova (1990) shows that a continuum of extensions of $\mathsf{KM}$ have Craig interpolation, but she also constructs such a logic that has not. Other intuitionistic modal logics that are known to enjoy Craig interpolation follow from the study on the more difficult uniform interpolation property. We discuss these in the next section.

There are several versions of Craig interpolation. The one that we consider is known as local interpolation. Instead of considering $\to$, one can also interpolate a consequence relation $\vdash$ known as deductive or global interpolation. This is in general weaker than local interpolation. Besides uniform interpolation, there are also other stronger versions of Craig interpolation. Among these concepts are that of the well-known Lyndon interpolation (Lyndon, 1959) and feasible interpolation used in proof complexity (Krajíček, 2019), but these will not be studied in this thesis.

### 2.1.2 Uniform interpolation

A first occurrence of uniform interpolation, although not called by that name, that we could find, is by Henkin (1963) who has shown that classical propositional logic has uniform interpolation, but that first order logic does not.

Uniform interpolation states that for any formula $A$ and non-logical symbol $p$, there exist a post-interpolant, $\dot{\exists} pA$, and a pre-interpolant, $\dot{\forall} pA$, which are formulas that do not contain $p$ such that for every formula $B$ without symbol $p$ we have

$$\begin{aligned} A \to B &\text{ is provable iff } \dot{\exists} pA \to B \text{ is provable,} \\ B \to A &\text{ is provable iff } B \to \dot{\forall} pA \text{ is provable.} \end{aligned}$$

It is well known that this property is stronger than Craig interpolation as the interpolant of $A \to B$ only depends on either $A$ or $B$. Indeed, a pre-interpolant defined on $A$ uniformly serves as an interpolant for fixed $A$ and all $B$ with a given common language.

Uniform interpolants can simulate propositional quantifiers. This is the reason for the suggestive notation above. The quantifiers are not part of the language, but uniform interpolants mimick quantification over a propositional variable $p$.

The quantifier simulation was first observed by Pitts (1992) who provided a first proof-theoretic proof for uniform interpolation, showing the property for $\mathsf{IPC}$. His work is pioneering work for further proof-theoretic studies, including this thesis. Section 2.1.3 is devoted to the history on uniform interpolation in proof theory. Here we touch upon semantic, algebraic, and model theoretic approaches.

Semantically, quantifier simulation is studied via bisimulation quantification. This was already indirectly present in the algebraic proof of uniform interpolation for $\mathsf{GL}$ by Shavrukov (1993). In the same line of research, Ghilardi (1995) shows the property for $\mathsf{K}$. Independently, Visser (1996) uses bounded $p$-bisimulation on Kripke models to show uniform interpolation for $\mathsf{IPC}$, $\mathsf{K}$, $\mathsf{GL}$, and $\mathsf{S4.Grz}$. This method is adapted by Kurahashi (2020) to prove Lyndon uniform interpolation for a wide range of modal logics. In addition, D'Agostino and Hollenberg (2000) prove uniform interpolation for the modal $\mu$-calculus. See (D'Agostino, 2007) and (French, 2006) for overviews on bisimulation quantification.

In model theory, the bisimulation quantifiers are closely related to model completions. Ghilardi and Zawadowski (2002) investigate in their book properties of varieties of algebras for intermediate and modal logics so that uniform interpolation implies the existence of a model completion for the first order theory of the variety. In combination with the result from (Maksimova, 1977) that there are seven consistent intermediate logics with Craig interpolation, they show that these are also the only intermediate logics that enjoy uniform interpolation (Ghilardi and Zawadowski, 1997). In addition, uniform interpolation fails for the well-behaved logic S4 (Ghilardi and Zawadowski, 1995).

Model completeness and uniform interpolation are also studied for equational consequences in algebra, see, e.g., (van Gool *et al.*, 2017) and (Kowalski and Metcalfe, 2019, 2018). They investigate deductive uniform interpolation, i.e., uniform interpolation with respect to the global consequence relation. Although logics K and T have (local) uniform interpolation, Kowalski and Metcalfe (2019, 2018) show the failure of uniform deductive interpolation for many modal logics, including K, T, K4, and S4.

Finally, translations play a role to (dis)prove uniform interpolation. Based on a translation from S4 to K4, Bílková (2006) shows that uniform interpolation also fails for K4. Using a similar technique, we show that it fails for iK4 and iS4 in Section 3.4. And Visser (2005) reproves uniform interpolation for the modal $\mu$-calculus based on the result for GL.

### 2.1.3 Proof theory

The proof-theoretic research on uniform interpolation, and other variants of interpolation in general, has two advantages. The first is concerned with so-called positive results for logics that enjoy (uniform) interpolation. The second is concerned with so-called negative results for logics that lack it.

Considering the former, a proof-theoretic proof for uniform interpolation enables one to find interpolants constructively rather than merely prove their existence. Whereas analytic sequent calculi can be used to prove Craig interpolation constructively, *terminating* cut-free sequent calculi play a similar role for uniform interpolation. Pitts (1992) has provided a first syntactic proof of this kind, establishing the uniform interpolation property for IPC. The method is successfully adjusted to classical modal logics to (re)prove the uniform interpolation property for logics including K, T, GL, and S4.Grz by Bílková (2006) and for K and D by

Iemhoff (2019a). Kracht (2007) uses a tableau calculus for K. Akbar Tabatabai *et al.* (2021) focus on uniform Lyndon interpolation for non-normal modal logics.

For intuitionistic modal logics little is known about uniform interpolation and the prominent proof methods employed in the literature thus far are proof-theoretic. Iemhoff (2019b) proves among others uniform interpolation for iK and iD. Iemhoff (2021) also shows the property for PLL. And Akbar Tabatabai *et al.* (2022) continue their study on uniform Lyndon interpolation for non-normal intuitionistic modal logics. In Chapter 3 we define terminating calculi for iGL and iSL, but we did not yet succeed to use it to prove uniform interpolation. For iGL the problem is open, but iSL is believed to have uniform interpolation as shown by semantic means in the unpublished manuscript by Litak and Visser (2020).

The methods discussed so far use standard Gentzen-style sequents. As mentioned in the Introduction, the types of generalizations of standard sequents has grown tremendously throughout recent years. Such variants of sequent calculi have been used to prove Craig interpolation. For classical modal logics, Fitting and Kuznets (2015) use nested sequents and Kuznets (2016) uses hypersequents to show Craig interpolation. A modular proof-theoretic framework encompassing these and also labelled sequents is provided by Kuznets (2018). The same ideas were extended to intermediate logics by Kuznets and Lellmann (2018). Lyon *et al.* (2020) explore Craig interpolation via nested sequents for bi-intuitionistic logic and tense logics.

In Chapter 4 we extend the same line of research, but now for uniform interpolation. We explore terminating nested sequent and hypersequent calculi in classical modal logics. Bílková (2011) has also provided a syntactic proof for the uniform interpolation property for K based on nested sequents. Another example in this line of research is the syntactic treatment of uniform interpolation for the modal $\mu$-calculus via cyclic proofs (Afshari *et al.*, 2021). Not only sequent-like systems are used, but also resolution-based methods in for instance (multi-)modal logics by Herzig and Mengin (2008) and Alassaf *et al.* (2021).

Let us now discuss the second advantage of (uniform) interpolation in proof theory. It serves as a powerful tool in the study of the existence of proof systems. We mention that this was already observed by Barwise (1985) who states the following (in our discussion it does not matter what logic $\mathcal{L}(Q_1)$ is):

> 'One can use the interpolation property as a yardstick for measuring whether there is a good proof theory. In case of $\mathcal{L}(Q_1)$ knowing that interpolation fails shows that one is not going to have a good Gentzen style proof system for $\mathcal{L}(Q_1)$.'

Here we specifically think of modal logics, the uniform interpolation property, and termination as a good property of a proof system.

The non-existence of calculi based on the failure of uniform interpolation is termed a negative result. Negative results are obtained by Iemhoff (2019a,b) who has provided a modular method to prove uniform interpolation for (intuitionistic) modal logics and intermediate logics with sequent calculi consisting of so-called focused rules.[18] It shows that all intermediate logics, except the seven that enjoy uniform interpolation, cannot have a proof system of that kind. The same holds for $\mathsf{K4}$ and $\mathsf{S4}$. Akbar Tabatabai and Jalali (2018a,b) obtain negative results for modal and substructural logics using Craig interpolation and uniform interpolation.

However, one has to be careful, because it means that logics that fail to have uniform interpolation do not have a terminating calculi of a particular type, but it does not exclude any other terminating proof system.

### 2.1.4 Applications

Some forms of uniform interpolation have a great practical value. A substantial amount of work has been conducted on Second-order Quantifier Elimination (SOQE), which is another name for uniform interpolation and is also known as forgetting, projection, and variable elimination. The two conferences devoted to SOQE witness the fruitful development in the area. We mention a few applications that appear in the conference proceedings of SOQE (Koopmann *et al.*, 2017) and (Schmidt *et al.*, 2021), and the textbook (Gabbay *et al.*, 2008).

SOQE is used in modal correspondence theory, which is concerned with semantic characterizations of Kripke frames reflecting modal axioms. Modal axioms can be translated into second-order formulas and the idea is to use uniform interpolation to eliminate the second-order quantifiers and reach a first-order formula. Note that this uniform interpolation is in a second-order theory and is different from the uniform interpolation for modal logics studied in this thesis.

SOQE is important in data bases that can logically be described by description logics via so-called ontologies. Koopmann (2015) lists a lot of applications. The general idea is that uniform interpolants extract implicit information from an ontology, by restricting to a part of the language, but keeping all the logical consequences for that restricted part. Examples are knowledge forgetting where agents are able to forget information while still knowing all the logical consequences for a restricted set of variable, information hiding of sensitive data that should not

[18] This is independent from what is known as focused proof systems that arise from a focusing technique in linear logic programming, see, e.g., for such a system in intuitionistic modal logic (Chaudhuri *et al.*, 2016).

be shared in public, and abduction in which uniform interpolants can be used to express a plausible relevant hypothesis.

Practical applications ask for practical algorithms to compute interpolants. For a lot of theories, uniform interpolants do not always exist but can still be computed for a restricted class of formulas. Several algorithms for SOQE have been implemented. For some online references see (Gabbay *et al.*, 2008, §1.5).

Finally, uniform interpolation is used in computer science in program slicing. Program slicing is used to easily indicate the parts of a program where bugs occur. A program slice is a restricted part of the program where one hopes that it performs the same way as the original program on the restricted inputs.

## 2.2 Craig and uniform interpolation

We present the basics of Craig interpolation and uniform interpolation in classical and intuitionistic modal logics. We only introduce definitions and results that we need for subsequent Chapters 3 and 4. Craig interpolation in classical modal logics is discussed in (Chagrov and Zakharyaschev, 1997, Chapter 14). For a short general overview of interpolation in non-classical logics see (D'Agostino, 2008).

All definitions apply for both classical and intuitionistic modal logics. Recall from Section 1.1 that $\mathsf{Var}(A)$ denotes the set of propositional variables occurring in formula $A$.

**2.2.1 Definition** (Craig interpolation)
Logic $\mathsf{L}$ has the *Craig interpolation property*, or simply *Craig interpolation*, if for every formulas $A, B \in \mathsf{Form}$ such that $\vdash_{\mathsf{L}} A \to B$, there exists a formula $C \in \mathsf{Form}$ such that

(i) $\mathsf{Var}(C) \subseteq \mathsf{Var}(A) \cap \mathsf{Var}(B)$, and
(ii) $\vdash_{\mathsf{L}} A \to C$ and $\vdash_{\mathsf{L}} C \to B$.

Such a formula $C$ is called the *interpolant* of $A$ and $B$.

Interpolation can also be defined with respect to a consequence relation instead of a provable implication. The version with the *local* consequence relation coincides with the implication variant and is called *local* interpolation. The version with the *global* consequence relation is known as *global*, *deductive*, or *weak* interpolation and is implied by local interpolation with the presence of a reasonable deduction theorem. See page 11 for an informal definition of the local and global consequence relations and see Example 5.2.12 for a formal definition.

**2.2.2 Example**
Logics CPC and IPC and all classical modal logics from Figure 1.2, that is, K, D, T, K4, S4, S5, and GL have Craig interpolation, see, e.g., (Chagrov and Zakharyaschev, 1997, Chapter 14). In addition, intuitionistic modal logics iK, iD, KM, and PLL are known to have Craig interpolation (for references see Section 2.1). In Section 3.2 we show Craig interpolation for logics iGL and iSL. We conjecture that Craig interpolation also holds for iT, iK4, and iS4.

We now define the uniform interpolation property, where we use the suggestive notation with $\dot{\forall}$ and $\dot{\exists}$, which are not present in the object language $\mathcal{L}$ (Section 1.1), in order to already stress the connection to bisimulation quantification explained in Section 2.4. In the literature, e.g. in (Visser, 1996; Iemhoff, 2019b), the dot is usually omitted, and one simply writes $\forall$ and $\exists$, but we prefer to use the dot to stress that these quantifiers are not part of the object language, and so $\dot{\forall}pA$ and $\dot{\exists}pA$ should be understood as formulas from $\mathsf{Form}$.

**2.2.3 Definition** (Uniform interpolation)
Logic $\mathsf{L}$ has the *uniform interpolation property*, or simply *uniform interpolation*, if for every formula $A \in \mathsf{Form}$ and propositional variable $p$, there exist formulas $\dot{\forall}pA, \dot{\exists}pA \in \mathsf{Form}$ such that

(i) $\mathsf{Var}(\dot{\exists}pA) \subseteq \mathsf{Var}(A) \setminus \{p\}$ and $\mathsf{Var}(\dot{\forall}pA) \subseteq \mathsf{Var}(A) \setminus \{p\}$,
(ii) $\vdash_{\mathsf{L}} A \to \dot{\exists}pA$ and $\vdash_{\mathsf{L}} \dot{\forall}pA \to A$, and
(iii) for each formula $B$ with $p \notin \mathsf{Var}(B)$:

$$\begin{aligned} \vdash_{\mathsf{L}} A \to B \quad &\text{implies} \quad \vdash_{\mathsf{L}} \dot{\exists}pA \to B, \\ \vdash_{\mathsf{L}} B \to A \quad &\text{implies} \quad \vdash_{\mathsf{L}} B \to \dot{\forall}pA. \end{aligned}$$

We call $\dot{\exists}pA$ the *post-interpolant* and $\dot{\forall}pA$ the *pre-interpolant*, or simply *uniform interpolants*, of $A$ with respect to $p$.

In the literature, $\dot{\exists}pA$ is also known as the *right* interpolant and $\dot{\forall}pA$ as the *left* interpolant. Indeed, in corresponding algebraic theories, uniform interpolants can be seen as left and right adjoints of particular morphisms, see, e.g., (Pitts, 1992). In Definition 2.2.3, points (ii) and (iii) are equivalent to the following: for all formulas $B$ with $p \notin \mathsf{Var}(B)$,

$$\begin{aligned} \vdash_{\mathsf{L}} A \to B \quad &\text{iff} \quad \vdash_{\mathsf{L}} \dot{\exists}pA \to B, \\ \vdash_{\mathsf{L}} B \to A \quad &\text{iff} \quad \vdash_{\mathsf{L}} B \to \dot{\forall}pA. \end{aligned}$$

The notation $\dot{\forall}pA$ and $\dot{\exists}pA$ suggests that these are uniquely defined.

**2.2.4 Lemma**
Pre- and post-interpolants are unique up to provable equivalence.

Proof. Let $C_1$ and $C_2$ both be post-interpolants of $A$ with respect to $p$. In particular, by property (ii), $\vdash_{\mathsf{L}} A \to C_1$ and $\vdash_{\mathsf{L}} A \to C_2$. Since $p \notin C_1$ and $p \notin C_2$ by property (i), we can apply property (iii) to $\vdash_{\mathsf{L}} A \to C_1$ with respect to $C_2$ in order to obtain $\vdash_{\mathsf{L}} C_2 \to C_1$. Similarly, (iii) applied to $\vdash_{\mathsf{L}} A \to C_2$ with respect to $C_1$ yields $\vdash_{\mathsf{L}} C_1 \to C_2$. Therefore $\vdash_{\mathsf{L}} C_1 \leftrightarrow C_2$. Similar argument applies to pre-interpolants. ■

It is well known that uniform interpolants define Craig interpolants for provable implications $A \to B$ such that they only depend on either $A$ or $B$. For propositional variables $\overline{p} = \{p_1, \ldots, p_m\}$ we write $\dot{\exists}\overline{p}A$ meaning $\dot{\exists}p_1 \ldots \dot{\exists}p_m A$, similarly for $\dot{\forall}$.

### 2.2.5 Lemma

Let $\mathsf{L}$ be a logic. If $\mathsf{L}$ has uniform interpolation, then it has Craig interpolation. Moreover, for $A \in \mathsf{Form}(\overline{p}, \overline{q})$ and $B \in \mathsf{Form}(\overline{q}, \overline{r})$ such that $\vdash_{\mathsf{L}} A \to B$, $\dot{\exists}\overline{p}A$ and $\dot{\forall}\overline{r}B$ are respectively the *least* and *greatest* interpolants of $A$ and $B$, that is, for any interpolant $C$ of $A$ and $B$ it holds that $\vdash_{\mathsf{L}} \dot{\exists}\overline{p}A \to C$ and $\vdash_{\mathsf{L}} C \to \dot{\forall}\overline{r}B$.

Proof. Suppose $\vdash_{\mathsf{L}} A \to B$ with $A \in \mathsf{Form}(\overline{p}, \overline{q})$ and $B \in \mathsf{Form}(\overline{q}, \overline{r})$. By computing uniform interpolants consecutively, one could say that $\dot{\exists}\overline{p}A$ is a uniform interpolant of $A$ with respect to $\overline{p}$. What we mean by that is that $\dot{\exists}\overline{p}A$ only contains variables shared by $A$ and $B$, $\vdash_{\mathsf{L}} A \to \dot{\exists}\overline{p}A$, and $\vdash_{\mathsf{L}} \dot{\exists}\overline{p}A \to B$. Moreover, $\vdash_{\mathsf{L}} \dot{\exists}\overline{p}A \to C$ for any interpolant $C$ by (iii) from Definition 2.2.3. A similar argument works for $\dot{\forall}\overline{r}B$. ■

Note that in this terminology we have $\vdash_{\mathsf{L}} \dot{\exists}\overline{p}A \to \dot{\forall}\overline{r}B$.

### 2.2.6 Example

Logics $\mathsf{CPC}$ and $\mathsf{IPC}$ and classical modal logics $\mathsf{K}$, $\mathsf{D}$, $\mathsf{T}$, $\mathsf{S5}$, and $\mathsf{GL}$ enjoy uniform interpolation, but $\mathsf{K4}$ and $\mathsf{S4}$ lack it. Also intuitionistic modal logics $\mathsf{iK}$, $\mathsf{iD}$, $\mathsf{PLL}$, and (almost certainly) $\mathsf{iSL}$ are known to have uniform interpolation. See Section 2.1.2 for references. In Section 3.4 we show that $\mathsf{iK4}$ and $\mathsf{iS4}$ do not have uniform interpolation. We conjecture that $\mathsf{iT}$ and $\mathsf{iGL}$ have uniform interpolation, but leave these as open problems.

### 2.2.7 Remark

For classical logics, the pre- and post-interpolants are interdefinable. Indeed, one can define

$$\dot{\exists}pA := \neg\dot{\forall}\neg A \quad \text{or} \quad \dot{\forall}pA := \neg\dot{\exists}\neg A.$$

In contrast, pre-interpolants cannot be defined in terms of post-interpolants in $\mathsf{IPC}$, but as noted in (Pitts, 1992), post-interpolants can be defined in terms of pre-interpolants as follows:

$$\dot{\exists}pA := \dot{\forall}q(\dot{\forall}p(A \to q) \to q),$$

where $q \notin \mathsf{Var}(A)$. This result is folklore, but let us spell it out for a modal logic $\mathsf{L}$. Property (i) is trivial. For (ii), we apply (ii) to see that $\vdash_{\mathsf{L}} \dot{\forall} p(A \to q) \to (A \to q)$. Equivalently, $\vdash_{\mathsf{L}} A \to (\dot{\forall} p(A \to q) \to q)$. Since $q \notin \mathsf{Var}(A)$ we can use (iii) to conclude $\vdash_{\mathsf{L}} A \to \dot{\forall} q(\dot{\forall} p(A \to q) \to q)$. Finally, for property (iii), let $B$ be a formula with $p \notin \mathsf{Var}(B)$ such that $\vdash_{\mathsf{L}} A \to B$. So $\vdash_{\mathsf{L}} \top \to (A \to B)$ and (iii) applied to this yields $\vdash_{\mathsf{L}} \top \to \dot{\forall} p(A \to B)$, and thus

$$\vdash_{\mathsf{L}} \dot{\forall} p(A \to B).$$

By (ii) we have $\vdash_{\mathsf{L}} \dot{\forall} q(\dot{\forall} p(A \to q) \to q) \to (\dot{\forall} p(A \to q) \to q)$. The trick is now to substitute $B$ for $q$, which is possible since $q \notin \mathsf{Var}(A)$ and $p \notin \mathsf{Var}(B)$. So we have

$$\vdash_{\mathsf{L}} \dot{\forall} q(\dot{\forall} p(A \to q) \to q) \to (\dot{\forall} p(A \to B) \to B).$$

Therefore, we conclude $\vdash_{\mathsf{L}} \dot{\forall} q(\dot{\forall} p(A \to q) \to q) \to B$.

It is well known that locally tabular logics with Craig interpolation are easily shown to have uniform interpolation.

**2.2.8 Definition**
A logic $\mathsf{L}$ is *locally tabular* if there are up to equivalence only finitely many formulas in a given finite set of propositional variables.

Examples of locally tabular logics are CPC and S5, see, e.g., (Chagrov and Zakharyaschev, 1997).

**2.2.9 Lemma**
Let $\mathsf{L}$ be a locally tabular logic with the Craig interpolation property. Then $\mathsf{L}$ has also the uniform interpolation property.

Proof. Let $A \in \mathsf{Form}(p, \overline{q})$. Consider formulas $B \in \mathsf{Form}(\overline{q})$ such that $\vdash_{\mathsf{L}} A \to B$. Since $\mathsf{L}$ is locally tabular, there are finitely many non-equivalent of such $B$'s, say $B_1, \ldots, B_n$. Define $\dot{\exists} pA := B_1 \wedge \cdots \wedge B_n$. It is easy to see that properties (i) and (ii) from Definition 2.2.3 are satisfied. For (iii), let $B \in \mathsf{Form}(\overline{q}, \overline{r})$ be a formula such that $p \notin \mathsf{Var}(B)$ and suppose $\vdash_{\mathsf{L}} A \to B$. Since $\mathsf{L}$ has Craig interpolation, there exists an interpolant $C$ of $A$ and $B$, i.e., $C \in \mathsf{Form}(\overline{q})$, $\vdash_{\mathsf{L}} A \to C$, and $\vdash_{\mathsf{L}} C \to B$. Since $\mathsf{L}$ is locally tabular, there is an equivalent $B_i \in \mathsf{Form}(\overline{q})$ to $C$. So $\vdash_{\mathsf{L}} A \to B_i$ and $\vdash_{\mathsf{L}} B_i \to B$. Therefore $\vdash_{\mathsf{L}} \dot{\exists} pA \to B_i$ by definition of $\dot{\exists} pA$ and so $\vdash_{\mathsf{L}} \dot{\exists} pA \to B$ as desired. Analogously, $\dot{\forall} pA$ can be defined as a finite disjunction of formulas $B \in \mathsf{Form}(\overline{q})$ that satisfy $\vdash_{\mathsf{L}} B \to A$. ■

## 2.3 Interpolation via sequents

This section briefly reviews the concepts for a proof-theoretic research of uniform interpolation with respect to sequent calculi.

#### 2.3.1 Definition

A *sequent* is a pair of finite multisets of formulas $\Gamma$ and $\Delta$, written $\Gamma \Rightarrow \Delta$. An *intuitionistic sequent* is a sequent $\Gamma \Rightarrow \Delta$ with $|\Delta| = 1$. The *formula interpretation* $I$ of a sequent is defined as $I(\Gamma \Rightarrow \Delta) := \bigwedge \Gamma \to \bigvee \Delta$.

#### 2.3.2 Remark

A sequent can also be defined on sets instead of multisets $\Gamma, \Delta$. Such sequents will be used in Chapter 7. The formula interpretation $I$ is defined similarly as when working with multisets.

In a sequent $\Gamma \Rightarrow \Delta$, we call $\Gamma$ the *antecedent* and $\Delta$ the *succedent*. In a sequent notation, $\Gamma, \Pi$ denotes $\Gamma \cup \Pi$ and $\Gamma, A$ denotes $\Gamma \cup \{A\}$. Recall from Section 1.1 that $\boxdot\Gamma$ is defined as $\{\boxdot A \mid A \in \Gamma\}$. However, if $\boxdot\Gamma$ occurs in the antecedent of a sequent it is defined as

$$\boxdot\Gamma = \Gamma \cup \Box\Gamma.$$

Similarly, if the expression $\boxdot A$ occurs in the antecedent of a sequent it stands for $A, \Box A$. So $(\Gamma, \boxdot A \Rightarrow \Delta, \Sigma)$ should be read as $(\Gamma \cup \{A\} \cup \{\Box A\} \Rightarrow \Delta \cup \Sigma)$. We sometimes denote a sequent by $S$.

#### 2.3.3 Definition

A sequent calculus consists of rules of the form

$$\frac{S_1 \qquad \dots \qquad S_n}{S}$$

where the sequents $S_i$ are called the *premises* of the rule and $S$ the *conclusion*. A rule with no premises is called an *axiom*. A calculus is *intuitionistic* if the rules are defined for intuitionistic sequents. A sequent is *provable* or *derivable* in a sequent calculus $\mathsf{SC}$ if there is a tree of rules from $\mathsf{SC}$ with the sequent in the root ending in leaves that are rules with empty premises. Such a tree is called a *proof* or *derivation* of $S$ and we write $\vdash_{\mathsf{SC}} S$. We say that a logic is *complete* with respect to a sequent calculus $\mathsf{SC}$ if $S$ is provable in $\mathsf{SC}$ if and only if $\vdash_{\mathsf{L}} I(S)$.

Cut-free analytic sequent calculi are suitable to prove Craig interpolation. These are calculi without the cut rule:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ \text{cut}$$

that amounts to calculi with the subformula property, meaning that formulas occurring in the premises are subformulas from formulas in the conclusion. There are several generalizations of Craig interpolation for sequent calculi, see (Mints, 2001). A well-known method is the split method by Maehara (1960) which we define as follows.

**2.3.4 Definition**
A sequent calculus $\mathsf{SC}$ has *split Craig interpolation*, if for every provable sequent $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ in $\mathsf{SC}$, there exists a formula $C$ such that

(i) $\mathsf{Var}(C) \subseteq \mathsf{Var}(\Gamma_1, \Delta_1) \cap \mathsf{Var}(\Gamma_2, \Delta_2)$, and
(ii) $\vdash_{\mathsf{SC}} \Gamma_1 \Rightarrow \Delta_1, C$ and $\vdash_{\mathsf{SC}} C, \Gamma_2 \Rightarrow \Delta_2$.

For intuitionistic calculi we require $|\Delta_2| = 1$ and $\Delta_1 = \emptyset$.

**2.3.5 Theorem**
Let $\mathsf{L}$ be a logic complete with respect to a sequent calculus $\mathsf{SC}$. Suppose $\mathsf{SC}$ has split Craig interpolation. Then $\mathsf{L}$ has Craig interpolation.

Proof. This is easily shown by taking $\Gamma_1 = \{A\}$, $\Gamma_2 = \emptyset$, $\Delta_1 = \emptyset$, and $\Delta_2 = \{B\}$ in Definition 2.3.4. ■

The sequent-based definitions of uniform interpolation that one finds in the literature are different for classical and intuitionistic modal logics. The sequent-based definition originates from Pitts (1992) for $\mathsf{IPC}$ and is adopted by Iemhoff (2019b) for intuitionistic modal logics. Bílková (2006) defines it for classical modal logics.

Although we will not use these definitions, we would like to present them in this chapter on the basics of uniform interpolation, especially, because the classical one forms the basis for our proof-theoretic research for nested sequents and hypersequents in Chapter 4.

**2.3.6 Definition**
A sequent calculus $\mathsf{SC}$ for a classical modal logic has *sequent uniform interpolation* if for each sequent $\Gamma \Rightarrow \Delta$ and $p \in \mathsf{Prop}$ there exists a formula $\mathcal{A}_p(\Gamma; \Delta) \in \mathsf{Form}$ such that:

(i) $\mathsf{Var}(\mathcal{A}_p(\Gamma; \Delta)) \subseteq \mathsf{Var}(\Gamma, \Delta) \setminus \{p\}$,
(ii) $\vdash_{\mathsf{SC}} \Gamma, \mathcal{A}_p(\Gamma; \Delta) \Rightarrow \Delta$, and
(iii) for every finite multisets $\Pi, \Sigma$ of formulas not containing $p$, if it holds that $\vdash_{\mathsf{SC}} \Pi, \Gamma \Rightarrow \Delta, \Sigma$, then:

$$\vdash_{\mathsf{SC}} \Pi \Rightarrow \mathcal{A}_p(\Gamma; \Delta), \Sigma.$$

**2.3.7 Definition**
A sequent calculus $\mathsf{SC}$ for an intuitionistic modal logic has *sequent uniform interpolation* if for each sequent $\Gamma \Rightarrow C$ and $p \in \mathsf{Prop}$ there exist formulas $\mathcal{A}_p(\Gamma; C) \in \mathsf{Form}$ and $\mathcal{E}_p(\Gamma) \in \mathsf{Form}$ such that:

(i) $\mathsf{Var}(\mathcal{A}_p(\Gamma; C)) \subseteq \mathsf{Var}(\Gamma, C) \setminus \{p\}$ and $\mathsf{Var}(\mathcal{E}_p(\Gamma)) \subseteq \mathsf{Var}(\Gamma) \setminus \{p\}$,
(ii) $\vdash_{\mathsf{SC}} \Gamma, \mathcal{A}_p(\Gamma; C) \Rightarrow C$ and $\vdash_{\mathsf{SC}} \Gamma \Rightarrow \mathcal{E}_p(\Gamma)$, and
(iii) for every finite multiset $\Pi$ of formulas not containing $p$ with $\vdash_{\mathsf{SC}} \Pi, \Gamma \Rightarrow C$ it holds that:

$$\begin{aligned} &\vdash_{\mathsf{SC}} \Pi, \mathcal{E}_p(\Gamma) \Rightarrow \mathcal{A}_p(\Gamma; C), \text{ and} \\ &\vdash_{\mathsf{SC}} \Pi, \mathcal{E}_p(\Gamma) \Rightarrow C \text{ if } p \notin \mathsf{Var}(C). \end{aligned}$$

**2.3.8 Theorem**
Let $\mathsf{L}$ be a logic complete with respect to a sequent calculus $\mathsf{SC}$. Suppose $\mathsf{SC}$ has sequent uniform interpolation. Then $\mathsf{L}$ has the uniform interpolation property.

Proof. For classical and intuitionistic modal logic we define $\dot\forall pA = \mathcal{A}_p(\emptyset; A)$. In classical modal logic we define $\dot\exists pA$ as its dual explained in Remark 2.2.7. For intuitionistic modal logic we define $\dot\exists pA = \mathcal{E}_p(A)$. It is straightforward to show that conditions (i), (ii), and (iii) from Definition 2.2.3 hold, where in the intuitionistic case one uses the fact that $\mathcal{E}_p(\emptyset) = \top$. ■

Classically, one only has to deal with the pre-interpolant by Remark 2.2.7. Although for intuitionistic modal logic, post-interpolants can also be defined in terms of pre-interpolants, the sequent definition intertwines both due to technical reasons (Pitts, 1992; Iemhoff, 2019b). One sees that $\mathcal{E}_p(\Gamma)$ only depends on $\Gamma$. This is in line with Remark 2.2.7 stating that post-interpolants cannot be expressed in terms of pre-interpolants.

Whereas cut-free sequent calculi are suitable to prove split Craig interpolation, for uniform interpolation we need stronger conditions on the calculi. Craig interpolants are inductively defined along a proof. The idea is that uniform interpolants are inductively defined on the basis of a proof search. For the uniform interpolants to be well defined, one needs a *finite* proof search. Iemhoff (2019b) provides sufficient conditions on sequent calculi for classical and intuitionistic modal logics in order to show uniform interpolation.

Termination will play a role in Chapters 3 and 4 so let us introduce some concepts.

**2.3.9 Definition**
A sequent calculus is said to *strongly terminate* if for each sequent, any order of bottom-up applications of the rules is finite, i.e., it stops with leaves that are either axioms or sequents to which no rule can be applied.

There are also other concepts of termination, but not all of them are sufficient to prove uniform interpolation. For instance, *weak termination*, the fact that one can find *at least one* finite process of bottom-up termination of the rules determining the (un)derivability of the sequent, is not sufficient. In general, weak termination is based on a global check on the proof search. Instead, one would like to have that each proof search terminates in certain saturated sequents for which termination is guaranteed by a local check on these sequents. These are simple sequents on which the uniform interpolants are easily definable. Strongly terminating calculi form examples of this.

Let us briefly look ahead to Chapter 3 in which we develop terminating sequent calculi for intuitionistic modal logics iGL and iSL. Developing terminating sequent calculi for intuitionistic modal logics is in general harder than for their classical counterparts, as the standard calculi for IPC are already non-terminating (Iemhoff, 2018, 2019b).

Consider the two sequent calculi G3ip and G4ip presented in Figures 3.1 and 3.2 in Chapter 3. Calculus G3ip from (Troelstra and Schwichtenberg, 2000) is known to be weakly terminating by a global check on sequents, but it is not strongly terminating because of the rule

$$\frac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}\ \mathrm{L}\!\to$$

in which the principal formula $A \to B$ is copied into the left premise.

However, a strongly terminating sequent calculus G4ip has been developed by Dyckhoff (1992) and used by Pitts (1992) to show sequent uniform interpolation for IPC. Independently, Hudelmaier (1988, 1992, 1993) and Vorob'ev (1952, 1970) introduced similar calculi. Calculus G4ip is obtained from G3ip in which rule L $\to$ is replaced by four implication rules.

A standard measure to show strong termination is based on the degree of formulas. The degree is the formalization of doing induction on the structure of the formula. Do not confuse it with the modal degree defined in Definition 1.2.25.

**2.3.10 Definition** (Degree)
The *degree* $d(A)$ of a formula $A$ is defined recursively as follows:

$$\begin{aligned} d(p) &= 1, \text{ for } p \in \mathsf{Prop}; \\ d(\bot) &= 0; \\ d(A \cdot B) &= \max(d(A), d(B)) + 1, \text{ for } \cdot = \wedge, \vee, \to; \\ d(\Box A) &= d(A) + 1. \end{aligned}$$

The degree $d(S)$ of a sequent $S$ is the sum of the degrees of its formulas.

This measure will not work for $\mathsf{G4ip}$, but Dyckhoff defines a new measure as follows, where we already extend it to the modal language.

**2.3.11 Definition** (Dyckhoff degree)
The *Dyckhoff degree* $d_{\mathrm{D}}(A)$ of a formula $A$ is defined recursively as follows:

$$\begin{aligned}
d_{\mathrm{D}}(p) &= 1, \text{ for } p \in \mathsf{Prop};\\
d_{\mathrm{D}}(\bot) &= 1;\\
d_{\mathrm{D}}(A \cdot B) &= \max(d_{\mathrm{D}}(A), d_{\mathrm{D}}(B)) + 1, \text{ for } \cdot = \vee, \to;\\
d_{\mathrm{D}}(A \wedge B) &= \max(d_{\mathrm{D}}(A), d_{\mathrm{D}}(B)) + 2;\\
d_{\mathrm{D}}(\Box A) &= d_{\mathrm{D}}(A) + 1.
\end{aligned}$$

For sequents $S_1$ and $S_2$ we define the ordering $\ll$ as follows:

$$S_1 \ll S_2 \quad \text{iff} \quad \begin{array}{l} S_1 \text{ is the result of replacing one or more formulas in } S_2 \\ \text{by zero or more formulas of lower Dyckhoff degree.} \end{array}$$

The formalization of $\ll$ is due to (Dershowitz and Manna, 1979).

**2.3.12 Theorem** (Dyckhoff, 1992)
Calculus $\mathsf{G4ip}$ is strongly terminating in the ordering $\ll$.

## 2.4 Propositional quantification

Uniform interpolation has close connections to second order quantification (see Section 2.1.4). For the uniform interpolation property that we study for modal logics it connects to propositional quantification. In particular, once we know that propositional quantification can be simulated in the propositional modal logic, we know that uniform interpolants $\dot\forall pA$ and $\dot\exists pA$ exist. Quantifier simulation can be considered both from a syntactic and semantic point of view.

The syntactic view was first observed by Pitts (1992) for $\mathsf{IPC}$. We informally explain the idea, because we do not use it explicitly in this thesis. The idea is to translate the propositional (modal) logic into a second order propositional extension where $\forall$ is explicitly present in the language and formulas are formed by the following grammar (recall Remark 2.2.7 that $\exists$ can be defined in terms of $\forall$):

$$A ::= p \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (\Box A) \mid (\forall p A).$$

Second order propositional logics can be defined by sequent calculi by adding rules for $\forall$. See for instance (Pitts, 1992) for second order propositional intuitionistic logic $\mathsf{IPC}^2$ or (Bílková, 2006) for second order propositional classical modal logics. The interpretation of the second order propositional (modal) logic into the propositional (modal) logic is a function $(\cdot)^*$ satisfying:

$$\begin{aligned} p^* &= p, \text{ for } p \in \mathsf{Prop}; \\ \bot^* &= \bot; \\ (A \cdot B)^* &= A^* \cdot B^*, \text{ for } \cdot = \wedge, \vee, \to; \\ (\Box A)^* &= \Box A^*; \\ (\forall p A)^* &= \dot{\forall} p A^*, \end{aligned}$$

where $\dot{\forall} p A^*$ is a propositional (modal) formula and such that the function interprets the second order logic into the propositional modal logic that restricts to the identity on propositional modal formulas. This means that whenever a second order formula $A$ is provable, then its translation $A^*$ is provable in the propositional modal logic. Moreover, if $A$ is quantifier-free, then $A^* = A$.

Second order propositional logics have uniform interpolation which immediately follows from quantification over propositions. And with such a translation in place, uniform interpolation for the propositional logic follows directly.

Note that the translation cannot be conservative. This follows from the fact that the quantified logic is undecidable, while the logics that we have in mind are decidable. For instance for classical modal logic $\mathsf{K}$, Bílková (2006, page 47) gives formula

$$(\Diamond \forall p A) \leftrightarrow (\forall p \Diamond A) \tag{2.1}$$

as an example that is not true in second order logic $\mathsf{K}^2$, but its translation is true in $\mathsf{K}$.

From a semantic point of view, propositional quantifiers can be described by bisimulation quantifiers, which are also used to provide semantic proofs of uniform interpolation in, e.g., (Visser, 1996). For a good overview of its connection to uniform interpolation see (D'Agostino, 2007).

Since these play an essential role in Chapter 4 we choose to formally define these in our classical modal setting. Recall Definition 1.2.19 for bisimulation modulo $p$ with the notation $\sim^p$. We have not defined bisimulations for intuitionistic modal logics and consider that for future research.

**2.4.1 Definition**
Let $\mathsf{L}$ be a classical modal logic complete with respect to a class of models $\mathcal{K}_{\mathsf{L}}$. We

say that *bisimulation quantifiers* $\overline{\exists}$ and $\overline{\forall}$ are definable over class $\mathcal{K}_\mathsf{L}$ if for every formula $A$ and propositional variable $p$ there exist formulas $\overline{\forall}pA, \overline{\exists}pA \in \mathsf{Form}$ such that

(i) $\mathsf{Var}(\overline{\exists}pA) \subseteq \mathsf{Var}(A) \setminus \{p\}$ and $\mathsf{Var}(\overline{\forall}pA) \subseteq \mathsf{Var}(A) \setminus \{p\}$,
(ii) for all models $K \in \mathcal{K}_\mathsf{L}$ and $w \in K$:

$$\begin{aligned} K, w \Vdash \overline{\exists}pA \text{ iff there exists } M \in \mathcal{K}_\mathsf{L} \text{ and } v \in M \text{ such that} \\ (M, v) \sim^p (K, w) \text{ and } M, v \Vdash A, \\ K, w \Vdash \overline{\forall}pA \text{ iff for all } M \in \mathcal{K}_\mathsf{L} \text{ and } v \in M, \\ \text{if } (M, v) \sim^p (K, w), \text{ then } M, v \Vdash A. \end{aligned}$$

#### 2.4.2 Lemma

Let $\mathsf{L}$ be a classical modal logic complete with respect to a class of models $\mathcal{K}_\mathsf{L}$. Suppose that the bisimulation quantifiers are definable over class $\mathcal{K}_\mathsf{L}$. Then $\mathsf{L}$ has the uniform interpolation property.

Proof. Let $\dot{\forall}pA := \overline{\forall}pA$ and $\dot{\exists}pA := \overline{\exists}pA$. We establish the uniform interpolation properties (i), (ii) and (iii) from Definition 2.2.3. We only show it for $\dot{\forall}pA := \overline{\forall}pA$, the proof for $\dot{\exists}pA := \overline{\exists}pA$ proceeds similarly.

Property (i) follows immediately by property (i) from Definition 2.4.1.

For condition (ii), assume towards a contradiction that $\nvdash_\mathsf{L} \overline{\forall}pA \to A$. By completeness $K, w \nVdash \overline{\forall}pA \to A$ for some $K \in \mathcal{K}_\mathsf{L}$ and $w \in K$. So $K, w \Vdash \overline{\forall}pA$ but $K, w \nVdash A$. This contradicts to property (ii), since $(K, w) \sim^p (K, w)$.

For (iii), let $p \notin \mathsf{Var}(B)$ and suppose $\nvdash_\mathsf{L} B \to \overline{\forall}pA$. So, $K, w \nVdash B \to \overline{\forall}pA$ for some $K \in \mathcal{K}_\mathsf{L}$ and $w \in K$. It follows that $K, w \Vdash B$ and $K, w \nVdash \overline{\forall}pA$. By (ii) of bisimulation quantification, there exists $M \in \mathcal{K}_\mathsf{L}$ and $v \in M$ such that $(M, v) \sim^p (K, w)$ and $M, v \nVdash A$. Since $p \notin \mathsf{Var}(B)$, we use Theorem 1.2.21 to conclude $M, v \Vdash B$. Therefore it holds that $M, v \nVdash B \to A$. Thus, by soundness of $\mathsf{L}$, we have $\nvdash_\mathsf{L} B \to A$. ■

Visser (1996) shows that bisimulation quantifiers are definable in transitive treelike models for $\mathsf{K}$. We show it for intransitive models in Corollaries 4.2.16 and 4.2.21 for logics $\mathsf{K}$, $\mathsf{T}$, and $\mathsf{D}$.

Finally, note that a standard interpretation of semantic quantification cannot work. Indeed, the formula in (2.1) with $A = \Diamond p \wedge \Diamond \neg p \to p$ in logic $\mathsf{K}$ is not bisimulation invariant which should be the case for propositional modal formulas by Theorem 1.2.21. To see this, note that the formula is valid in a one-world reflexive model, but the right-to-left direction is not valid in the bisimilar two-world model with a total relation. However, we can relax the semantic quantification to bisimulation modulo $p$.

# 3

# Towards Uniform Interpolation in Intuitionistic Modal Logic

The title of this chapter might be misleading, because we do not provide any proof demonstrating that some intuitionistic modal logic has uniform interpolation. To the contrary, we show that logics iK4 and iS4 do not enjoy the uniform interpolation property. This is a short proof, so why can we still write an entire chapter with this title?

The reason is that this chapter is motivated by the goal expressed in the title: establishing the uniform interpolation property for intuitionistic modal logics. As described in the history (Section 2.1.3), only a few intuitionistic modal logics are known to have the property, such as iK, iD, and PLL (Iemhoff, 2019b, 2021). Our ultimate goal is to provide a proof-theoretic method to show uniform interpolation for logics iGL and iSL.

The interest of this chapter lies in the development of multiset-based sequent calculi for iGL and iSL. We define four main systems called G3iGL, G3iSL, G4iGL, and G4iSL. All systems are based on a sequent system for IPC extended by modal rules for GL. The former two are based on a standard system for IPC. The latter two are based on a terminating system for IPC developed by Dyckhoff (1992) and are shown to be terminating. These calculi are developed in order to get closer to our goal.

We present a nontrivial syntactic proof for the cut-admissibility in G3iGL and G3iSL. The proof is based on existing methods for classical modal logic GL. The problem of cut-elimination for GL has an interesting history. A first proof was claimed by Leivant (1981) for set-based sequents, but had a flaw and a correction was proposed by Valentini (1983). Other methods are proposed for various systems: set-based sequents (Borga, 1983; Sasaki, 2001; Brighton, 2016), multiset-based sequents (Mints, 2005; Goré and Ramanayake, 2012b; Goré *et al.*, 2021), labeled systems (Negri, 2005), and hypersequents (Poggiolesi, 2009b). Some of these are later shown to be incorrect or incomplete. In particular, Valentini's method for sets is considered to be informal and difficult to check. Moen (2001) states that Valentini's proof is incorrect, but this argument has later been contested by Goré and Ramanayake (2012b). In contrast, they formalize Valentini's method and confirm that the proof is correct, and even generalize it to multisets. In addition, the method by Brighton did not work well for the set-based framework, but is shown to be successful for multisets (Goré *et al.*, 2021). This demonstrates the difficult character of cut-elimination for provability logics.

The calculi G4iGL and G4iSL are obtained similarly to other terminating calculi for intuitionistic modal logics (Iemhoff, 2018, 2019b, 2020). To show termination, we use a well-order based on an order by Bílková (2006) used in her syntactic proof of uniform interpolation for GL. Termination of the proof systems enables us to show equivalence with their corresponding systems G3iGL and G3iSL. In turn, it implies the cut-elimination theorem for G4iGL and G4iSL.

We establish a few properties. We use G3iGL and G3iSL to show Craig interpolation for iGL and iSL. And we use a terminating calculus related to G4iSL to provide a countermodel construction for iSL. This provides a semantic treatment of the cut-elimination theorem.

This chapter is a combination of the results for iGL from (van der Giessen and Iemhoff, 2021) and iSL from (van der Giessen and Iemhoff, 2022). The chapter is structured as follows. Section 3.1 provides the sequent calculi for iGL and iSL and presents the proofs for syntactic cut-elimination and termination. Sections 3.2 and 3.3 provide the Craig interpolation theorem and countermodel construction respectively. Section 3.4 presents the proof that iK4 and iS4 do not have uniform interpolation and we end with a conclusion in Section 3.5.

## 3.1 Sequent calculi for iGL and iSL

Recall that at the beginning of Section 2.3 we introduced the notion of a sequent. We use intuitionistic sequents based on multisets as defined in Definition 2.3.1, i.e., a sequent $\Gamma \Rightarrow A$ is a pair of finite multiset of formulas $\Gamma$ and formula $A$, and its formula interpretation is defined as $I(\Gamma \Rightarrow A) := \bigwedge \Gamma \to A$. We call $\Gamma$ the *antecedent* and $C$ the *succedent*. The notation of $(\Gamma, \boxdot\Sigma, \boxdot A \Rightarrow B)$ reads as $(\Gamma \cup \Box\Sigma \cup \Sigma \cup \{\Box A\} \cup \{A\} \Rightarrow B)$. We use letters $S$ to range over sequents.

One of the standard calculi without structural rules for IPC is G3ip, given in Figure 3.1, which is the propositional part of the calculus G3i from (Troelstra and Schwichtenberg, 2000). The calculus G4ip in Figure 3.2 is the terminating sequent calculus for IPC by Dyckhoff (1992), where we have renamed some of the rules.

As one can see, G4ip is the result of replacing the single left implication rule L $\to$ in G3ip by the four left implication rules of G4ip, each corresponding to the outermost connective of the antecedent of the principal implication. Since in this thesis the propositional language is extended by the modal operator $\Box$, we use five instead of four replacing implication rules, one extra for the case that the antecedent of the principal implication is boxed.

**3.1.1 Definition**
We define the following calculi, where G3im and G4im stand for the logic G3ip and G4ip, respectively, but then formulated for the modal language $\mathcal{L}$ of this thesis defined at the beginning of Section 1.1. For the rules see Figures 3.1, 3.2, 3.3.

$$\mathsf{G3iGL} := \mathsf{G3im} + \mathcal{R}_{\mathrm{GL}}, \qquad \mathsf{G4iGL} := \mathsf{G4im} + \to_{\mathrm{GL}} + \mathcal{R}^4_{\mathrm{GL}},$$
$$\mathsf{G3iSL} := \mathsf{G3im} + \mathcal{R}_{\mathrm{SL}}, \qquad \mathsf{G4iSL} := \mathsf{G4im} + \to_{\mathrm{SL}} + \mathcal{R}^4_{\mathrm{SL}}.$$

Let us give some terminology that we use throughout this chapter. First recall from Definition 2.3.3 the standard notions of *premise* and *conclusion*. We call sequents obtained from rule Ax or L$\bot$ *axioms*.

In the rules of G3iX, the *principal* formula(s) of an inference are defined as usual for the connectives, see, e.g., (Buss, 1998). For $\mathrm{X} \in \{\mathrm{GL}, \mathrm{SL}\}$, the principal formulas are $\Box A$ and all formulas in $\Box\Gamma$ for $\mathcal{R}_{\mathrm{X}}$, and for L $\to_{\mathrm{X}}$ the principal formulas are $\Box A \to B$ and the formulas in $\Box\Gamma$. The *diagonal* formula of $\mathcal{R}_{\mathrm{X}}$ is $\Box A$. In an application of $\mathcal{R}_{\mathrm{GL}}$, the formulas in $\Sigma$ are said to be *introduced* in that inference step. Similarly for the formulas $\Box\Sigma$ in an application of $\mathcal{R}_{\mathrm{SL}}$. Note that $\Pi$ ranges over multisets that do not contain boxed formulas. In case of iK and iKD in (Iemhoff, 2018) there is no requirement on $\Pi$ in the modal rules of the calculi G3iK and G3iKD. The reason for the restriction on $\Pi$ and the presence of $\Box\Sigma$ in G3iSL (of $\Sigma$ in G3iGL) lies in the proof of the cut-elimination theorem.

The notion *ancestor* is defined as usual (Buss, 1998). We use this explicitly in the cut-elimination proof for the systems G3iGL and G3iSL, so let us spell out the definition here for these systems. We first define the *immediate ancestors* of a formula occurring in the conclusion of a rule. For the rules in G3im, if the formula occurs in $\Gamma$ or is the right formula in a left rule, its immediate ancestor is its corresponding formula (in $\Gamma$) in the premise of the rule. Principal formulas have multiple immediate ancestors, namely their corresponding subformulas in the premise. In the modal rule $\mathcal{R}_{\mathrm{SL}}$, the immediate ancestor of formulas in $\Pi$ are

$$\frac{}{\Gamma, p \Rightarrow p}\ \mathrm{Ax} \qquad \frac{}{\Gamma, \bot \Rightarrow C}\ \mathrm{L}\bot$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \mathrm{R}\wedge \qquad \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\ \mathrm{L}\wedge$$

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \mathrm{R}\vee_i\ (i = 1, 2) \qquad \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}\ \mathrm{L}\vee$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \mathrm{R}\to \qquad \frac{\Gamma, A \to B \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \to B \Rightarrow C}\ \mathrm{L}\to$$

**Figure 3.1.** Sequent calculus G3ip

$$\frac{}{\Gamma, p \Rightarrow p}\ \mathrm{Ax} \qquad \frac{}{\Gamma, \bot \Rightarrow C}\ \mathrm{L}\bot$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \mathrm{R}\wedge \qquad \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\ \mathrm{L}\wedge$$

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \mathrm{R}\vee_i\ (i = 1, 2) \qquad \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}\ \mathrm{L}\vee$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \mathrm{R}\to \qquad \frac{\Gamma, p, A \Rightarrow C}{\Gamma, p, p \to A \Rightarrow C}\ \mathrm{L}p\!\to$$

$$\frac{\Gamma, A \to (B \to C) \Rightarrow D}{\Gamma, A \wedge B \to C \Rightarrow D}\ \mathrm{L}\wedge\!\to \qquad \frac{\Gamma, A \to C, B \to C \Rightarrow D}{\Gamma, A \vee B \to C \Rightarrow D}\ \mathrm{L}\vee\!\to$$

$$\frac{\Gamma, B \to C \Rightarrow A \to B \qquad \Gamma, C \Rightarrow D}{\Gamma, (A \to B) \to C \Rightarrow D}\ \mathrm{L}\to\!\to$$

**Figure 3.2.** Sequent calculus G4ip

their corresponding formulas from $\Pi$ in the premise, the immediate ancestors of formula $\Box B \in \Gamma$ are the corresponding formulas $B, \Box B$ coming from $\boxdot\Gamma$, and the immediate ancestors of $\Box A$ are $\Box A$ and $A$ displayed in the premise. Formulas in $\Box\Sigma$ do not have an ancestor, but are introduced in the rule, as defined above. Similarly for rule $\mathcal{R}_{\mathrm{GL}}$. The *ancestor* relation between two formulas is defined as the reflexive, transitive closure of the immediate ancestor relation. Finally, a *strict ancestor* of a formula $A$ is defined to be an ancestor of $A$ that as a formula is equal to $A$. In the rest of the chapter we often omit these words. When in a sequent $S$ on a branch there occurs a formula $A$, then if we speak of an occurrence of $A$ in a

$$\frac{\boxdot\Gamma, \Box A \Rightarrow A}{\Sigma, \Box\Gamma \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{GL}} \qquad \frac{\Pi, \boxdot\Gamma, \Box A \Rightarrow A}{\Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}}$$

$$\frac{\boxdot\Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}\,\mathcal{R}^4_{\mathrm{GL}} \qquad \frac{\boxdot\Gamma, \Box A \Rightarrow A \qquad \Pi, \Box\Gamma, B \Rightarrow C}{\Pi, \Box\Gamma, \Box A \to B \Rightarrow C}\,{\to}_{\mathrm{GL}}$$

$$\frac{\Pi, \boxdot\Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}\,\mathcal{R}^4_{\mathrm{SL}} \qquad \frac{\Pi, \boxdot\Gamma, \Box A, \Box A \to B \Rightarrow A \qquad \Pi, \Box\Gamma, B \Rightarrow C}{\Pi, \Box\Gamma, \Box A \to B \Rightarrow C}\,{\to}_{\mathrm{SL}}$$

**Figure 3.3.** Modal rules. $\Pi$ ranges over multisets that do not contain boxed formulas.

sequent higher than $S$ along that branch, then we mean a strict ancestor of that occurrence of $A$ in $S$.

The notion of a derivation in a sequent calculus is defined in Definition 2.3.3. So we write $\vdash_{\mathsf{G3iGL}}$ for derivability in G3iGL, etc. In addition, we define the *height* of a derivation as the length of its longest branch, where branches consisting of one node are considered to have height zero. The height of a sequent in a derivation is the height of its subderivation. If $\vdash$ stands for derivability in a given calculus, then we write $\vdash^l S$ if sequent $S$ has a derivation of height at most $l$ in that calculus.

### 3.1.1 Structural rules

We have the usual lemma on the height-preserving admissibility of weakening, contraction and inversion, proved in Lemma 3.1.3.

**3.1.2 Definition**

Let SC be a sequent calculus. A rule

$$\frac{S_1 \qquad \dots \qquad S_n}{S}$$

is called

- *admissible in* SC if whenever $S_i$ is derivable in SC for all $i$, $S$ is derivable in SC. A rule with one premise $S_1$ is called *height-preserving admissible* if whenever $\vdash^l_{\mathsf{SC}} S_1$ we have $\vdash^l_{\mathsf{SC}} S$,
- *(height-preserving) invertible in* SC if all rules with premise $S$ and conclusion $S_i$ are (height-preserving) admissible in SC.

Consider the standard rules for weakening and contraction. Note that we only need the left versions, since we work with intuitionistic sequents in which the right hand side is fixed to be one formula.

$$\frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C}\text{ weakening} \qquad \frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}\text{ contraction}$$

**3.1.3 Lemma** (Weakening, contraction, and inversion)
For $\vdash$ denoting $\vdash_{\mathsf{G3iGL}}$, $\vdash_{\mathsf{G4iGL}}$, $\vdash_{\mathsf{G3iSL}}$, or $\vdash_{\mathsf{G4iSL}}$, the following statements hold, where $A, B, C, D$ range over formulas, $\Gamma$, $\Sigma$ range over finite sets of formulas, and $\Pi$ ranges over finite sets of formulas that do not contain boxed formulas.

| | | |
|---|---|---|
| 1. | *Extended axiom:* | $\vdash \Gamma, C \Rightarrow C$. |
| 2. | *Falsum rule:* | $\vdash^l \Gamma \Rightarrow \bot$ implies $\vdash^l \Gamma \Rightarrow C$. |
| 3. | *Weakening:* | weakening is height-preserving admissible. |
| 4. | *Inversion:* | rules R$\wedge$, L$\wedge$, L$\vee$, R $\to$, L$p\to$, L$\wedge\to$, L$\vee\to$ are height-preserving invertible. |
| 5. | *Inversion* L $\to$*:* | if $\vdash^l \Gamma, A \to B \Rightarrow C$, then $\vdash^l \Gamma, B \Rightarrow C$. |
| 7. | *Inversion* L $\to\to$*:* | if $\vdash^l \Gamma, (A \to B) \to C \Rightarrow D$, then $\vdash^l \Gamma, C \Rightarrow D$. |
| 8. | *Inversion* $\to_{\mathrm{GL}}$*:* | if $\vdash^l \Pi, \Box\Gamma, \Box A \to B \Rightarrow C$, then $\vdash^l \Pi, \Box\Gamma, B \Rightarrow C$. |
| 9. | *Inversion* $\to_{\mathrm{SL}}$*:* | if $\vdash^l \Pi, \Box\Gamma, \Box A \to B \Rightarrow C$, then $\vdash^l \Pi, \Box\Gamma, B \Rightarrow C$. |
| 10. | *Contraction* G3iX*:* | In G3iGL and G3iSL, contraction is height-preserving admissible. |

Proof. The first statement is proved by induction, on the structure of $A$ in the case of G3iGL and G3iSL, and on the Dyckhoff degree (Definition 2.3.11) of $A$ in the case of G4iGL and G4iSL. The other statements are proved by induction on height $l$. Weakening is used in the proofs of inversion, which are used in the proof of contraction. The proofs of these properties are standard. For details, see page 66–67 in (Troelstra and Schwichtenberg, 2000). Let us only present two proofs.

For weakening in G3iSL, suppose $\vdash^l S$. Let us only treat the case for $l > 0$ and the last inference is $\mathcal{R}_{\mathrm{SL}}$. Thus $S = (\Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box B)$ and the premise is $(\Pi, \boxdot\Gamma, \Box B \Rightarrow B)$, where $\Pi$ does not contain boxed formulas. We have to show that $\vdash^l \Box\Sigma, \Pi, \Box\Gamma, A \Rightarrow \Box B$. If $A$ is a boxed, this follows from an application of $\mathcal{R}_{\mathrm{SL}}$ to the premise in which $A$ is introduced in the conclusion. Otherwise, by the induction hypothesis $\vdash^{l-1} \Pi, \boxdot\Gamma, A, \Box B \Rightarrow B$. Apply $\mathcal{R}_{\mathrm{SL}}$ to obtain the result.

For inversion of $\to_{\mathrm{SL}}$, we use weakening in G4iSL. We treat the case where the last inference is $\to_{\mathrm{SL}}$ for some principal formula $\Box A_1 \to B_1$. So let us suppose $\vdash^l \Pi', \Box A_1 \to B_1, \Box\Gamma, \Box A \to B \Rightarrow C$ with premises $(\Pi', B_1, \Box\Gamma, \Box A \to B \Rightarrow C)$ and $(\Pi', \Box A_1, \Box A_1 \to B_1, \boxdot\Gamma, \Box A \to B \Rightarrow A_1)$. By induction hypothesis we have $\vdash^{l-1} \Pi', \Box A_1, \Box A_1 \to B_1, \boxdot\Gamma, B \Rightarrow A_1$ and $\vdash^{l-1} \Pi', B_1, \Box\Gamma, B \Rightarrow C$. If $B$ is not boxed, apply $\to_{\mathrm{SL}}$ for the desired result. If $B$ has the form $\Box B'$, weakening implies $\vdash^{l-1} \Pi', \Box A_1, \Box A_1 \to B_1, \boxdot\Gamma, \Box B', B' \Rightarrow A_1$ and again $\to_{\mathrm{SL}}$ suffices. ∎

**3.1.4 Remark**
We are only certain about the height-preserving admissibility of contraction in G3iGL and G3iSL. For G4iGL and G4iSL a direct proof might be difficult and it is the question whether it would be height-preserving. In (van der Giessen and Iemhoff, 2021) we claimed this result for G4iGL in Lemma 4.1, but this statement might be wrong. However, all other results remain true because these are independent from this. Finally, note that the admissibility of contraction in G4iGL and G4iSL follows once we have shown equivalence to G3iGL and G3iSL (Corollary 3.1.24).

In the syntactic proof of cut-elimination for G3iGL and G3iSL, we need a strengthening of the closure under weakening (Lemma 3.1.10), for which we have to introduce two transformations on derivations (Definition 3.1.9).

**3.1.5 Definition**
Let $\mathcal{D}$ be a derivation in G3iGL (in G3iSL) with endsequent $\Gamma \Rightarrow \Box A$. A *$\Box A$-critical inference over* $\Gamma \Rightarrow \Box A$ is an $\mathcal{R}_{\mathrm{GL}}$-inference ($\mathcal{R}_{\mathrm{SL}}$-inference) $R$ in $\mathcal{D}$ such that

(i) $\Box A$ is principal in $R$;
(ii) between $\Gamma \Rightarrow \Box A$ and the conclusion of $R$ there is exactly one $\mathcal{R}_{\mathrm{GL}}$-inference ($\mathcal{R}_{\mathrm{SL}}$-inference) in which $\Box A$ is principal, and it is the diagonal formula of that inference.

The segment from the premise of $R$ till $\Gamma \Rightarrow \Box A$ is a *$\Box A$-critical segment.*

It is important to recall our convention from page 64 about strict ancestors; in the definition of $\Box A$-critical inference, the $\Box A$ mentioned in (i) is required to be a strict ancestor of the $\Box A$ mentioned in (ii).

**3.1.6 Remark**
Note that above $\Gamma \Rightarrow \Box A$ there may be more sequents than the ones in a $\Box A$-critical segment, as in the following example on the left, where $A = (A' \to \Box D)$ and $A'$ is not boxed, and the leftmost branch is a $\Box A$-critical segment:

$$
\dfrac{\dfrac{\dfrac{\dfrac{B_1, \boxdot A, A', \Box D \Rightarrow D}{B_1, \Box A, A' \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{SL}}}{B_1, \Box A \Rightarrow A}\,\mathrm{R}\!\to \qquad \begin{array}{c}\mathcal{D}' \\ B_2, \Box A \Rightarrow A\end{array}}{B_1 \vee B_2, \Box A \Rightarrow A}\,\mathrm{L}\vee}{B_1 \vee B_2 \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}}
\qquad\qquad
\dfrac{\begin{array}{c}\dfrac{B_1, \boxdot A, A', \Box D \Rightarrow D}{B_1, \Box A, A' \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{SL}} \\ \mathcal{B} \\ B_1 \vee B_2, \Box A \Rightarrow A\end{array}}{B_1 \vee B_2 \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}}
$$

Most of the time we focus on the $\Box A$-critical sequent and would like to picture this schematically as presented on the right, where $\mathcal{B}$ should be understood as covering the hidden part of the proof. In (van der Giessen and Iemhoff, 2021), we have made this explicit for G3iGL by introducing so-called *stub-derivations* (in this case $\mathcal{B}$), based on (Goré and Ramanayake, 2012b). Informally, stub-derivations can be obtained by deleting a proper subderivation from a derivation, thereby obtaining a derivation with a ‘gap.’ Here we take the slightly more informal approach that we took for G3iSL in (van der Giessen and Iemhoff, 2022).

**3.1.7 Remark**

We are mostly interested in the case that $\Gamma \Rightarrow \Box A$ is the conclusion of an $\mathcal{R}_{\mathrm{SL}}$-inference (or $\mathcal{R}_{\mathrm{GL}}$-inference). In G3iSL, the $\Box A$-critical segment looks as follows, with $\Gamma = \Box\Sigma_1, \Pi_1, \Box\Gamma_1$ and where $\Pi_1$ and $\Pi_2$ do not contain boxed formulas, and where $\Gamma_1' \subseteq \Gamma_1$:

$$\begin{array}{c} \dfrac{\Pi_2, \boxdot\Gamma_1', \boxdot\Gamma_2, \boxdot A, \Box D \Rightarrow D}{\Box\Sigma_2, \Pi_2, \Box\Gamma_1, \Box\Gamma_2, \Box A \Rightarrow \Box D}\ R \text{ (application of } \mathcal{R}_{\mathrm{SL}}) \\ \mathcal{B} \\ \dfrac{\Pi_1, \boxdot\Gamma_1, \Box A \Rightarrow A}{\Box\Sigma_1, \Pi_1, \Box\Gamma_1 \Rightarrow \Box A}\ \mathcal{R}_{\mathrm{SL}} \end{array}$$

It easily follows that there can be no applications of $\mathcal{R}_{\mathrm{SL}}$ in the showed segment in $\mathcal{B}$, for at such an inference $\Box A$ would have disappeared in the sequents above it, as it is not principal in that inference by definition. This is where we use the restriction that all formulas in the $\Pi$ of an application of $\mathcal{R}_{\mathrm{SL}}$ are not boxed. As mentioned above, by definition the $\Box A$ in the top sequent is required to be a strict ancestor of the $\Box A$ in the bottom sequent. So only rules from G3im can be applied in the segment of $\mathcal{B}$ indicated above, and therefore boxed formulas in the antecedents of the sequents do not disappear in the backward direction of the proof tree. Therefore $\Box\Gamma_1$ is still present in the sequent pictured above $\mathcal{B}$.

In the proof of cut-elimination for G3iGL and G3iSL, we need to weaken $\Box A$-critical segments in a specific way. Namely, the situation may occur that we wish to weaken the sequents in such a segment in such a way that all inferences remain valid and the top sequent is weakened by formula $D$ and the bottom sequent by $\Box D$. As an example, consider a $\Box\Box A$-critical segment of minimal length in G3iSL:

$$\dfrac{\dfrac{\boxdot\Box A, \Box A \Rightarrow A}{\Box\Box A \Rightarrow \Box A}\ \mathcal{R}_{\mathrm{SL}}}{\Rightarrow \Box\Box A}\ \mathcal{R}_{\mathrm{SL}}$$

The following are two ways to achieve this, depending on whether $D$ is boxed (left) or not boxed (right).

$$\dfrac{\dfrac{\Box D, D, \boxdot\Box A, \Box A \Rightarrow A}{\Box D, D, \Box\Box A \Rightarrow \Box A}\ \mathcal{R}_{\mathrm{SL}}}{\Box D \Rightarrow \Box\Box A}\ \mathcal{R}_{\mathrm{SL}} \qquad \dfrac{\dfrac{\Box D, D, D, \boxdot\Box A, \Box A \Rightarrow A}{\Box D, D, \Box\Box A \Rightarrow \Box A}\ \mathcal{R}_{\mathrm{SL}}}{\Box D \Rightarrow \Box\Box A}\ \mathcal{R}_{\mathrm{SL}}$$

In general, if for a $\Box A$-critical segment we need to weaken its top sequent with at least $D$ and its bottom sequent $S$ with $\Box D$, then every sequent in $\mathcal{D}$ has to be weakened at the left by $\Box D, D^n$, where $n$ is the number of applications of $\mathcal{R}_{\mathrm{SL}}$ below that sequent, except in the case that $D$ is boxed, as in that case it can be introduced at every application of $\mathcal{R}_{\mathrm{SL}}$. Of course, in order to remain a valid derivation also sequents not in that segment but above $\Gamma \Rightarrow \Box A$ have to be considered. In what follows the details behind this idea are spelled out.

**3.1.8 Definition**
Let $\mathcal{D}$ be a derivation in G3iGL (in G3iSL) with endsequent $S$. The $\mathcal{R}_{\mathrm{GL}}$-*grade* ($\mathcal{R}_{\mathrm{SL}}$-*grade*) $g_{\mathcal{D}}(S)$ of $S$ is defined as

$$g_{\mathcal{D}}(S) := \text{the number of applications of } \mathcal{R}_{\mathrm{GL}} \ (\mathcal{R}_{\mathrm{SL}}) \text{ below } S \text{ in } \mathcal{D}.$$

We write $g(S)$ if $\mathcal{D}$ is clear from the context. In this measure, if $S$ is the premise of an application of $\mathcal{R}_{\mathrm{SL}}$, then that application counts in $g(S)$. For example, on the following two branch segments,

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \dfrac{S_6}{S_5}\,\mathcal{R}_{\mathrm{SL}}\\ \vdots\\ S_3\end{array} \qquad \begin{array}{c}\vdots\\ S_4\end{array}}{S_2}\,\text{nonmodal rule}}{\begin{array}{c}S_1\\ \vdots\end{array}}\,\mathcal{R}_{\mathrm{SL}}$$

if there is no application of $\mathcal{R}_{\mathrm{SL}}$ below $S_1$, then $g(S_1) = 0$, $g(S_6) = 2$, and $g(S_i) = 1$ for all other $i$.

**3.1.9 Definition**
Given a sequent $S = (\Gamma \Rightarrow C)$ in a derivation $\mathcal{D}$ and given a formula $D$, translations $(\cdot)_D$ and $(\cdot)^D$ on sequents in $\mathcal{D}$ are defined differently for G3iGL and G3iSL, as follows, where the second is only defined for G3iSL. We suppress the dependence on $\mathcal{D}$ in the notation, as it will always be clear from the context which derivation is meant.

In G3iGL:

$$S_D = (\Gamma \Rightarrow C)_D := \begin{cases} \Gamma, \Box D \Rightarrow C & \text{if } g(S) = 0, \\ \Gamma, \Box D, D \Rightarrow C & \text{if } g(S) > 0. \end{cases}$$

In G3iSL:

$$S_D = (\Gamma \Rightarrow C)_D := \begin{cases} \Gamma, \Box D \Rightarrow C & \text{if } D \text{ is boxed and } g(S) = 0, \\ \Gamma, \Box D, D \Rightarrow C & \text{if } D \text{ is boxed and } g(S) > 0, \\ \Gamma, \Box D, D^{g(S)} \Rightarrow C & \text{if } D \text{ is not boxed.} \end{cases}$$

$$S^D = (\Gamma \Rightarrow C)^D := \begin{cases} \Gamma \Rightarrow C & \text{if } D \text{ is boxed and } g(S) > 0, \\ \Gamma, D \Rightarrow C & \text{if } D \text{ is not boxed or } g(S) = 0. \end{cases}$$

$\mathcal{D}_D$ is the result of replacing each sequent $S$ in $\mathcal{D}$ by $S_D$, likewise for $\mathcal{D}^D$. Given a multiset of formulas $\{D_1, \ldots, D_n\}$, define

$$\mathcal{D}^{\{D_1,\ldots,D_n\}} := (\ldots((\mathcal{D}^{D_1})^{D_2})\ldots)^{D_n},$$

and similarly for $\mathcal{D}_{\{D_1,\ldots,D_n\}}$.

**3.1.10 Lemma** (Strong weakening)
For any sequent $S = (\Gamma \Rightarrow C)$ and multiset of formulas $\Theta$ we have:

1. If $\mathcal{D}$ is a cutfree proof of $S$ in G3iSL, then $\mathcal{D}^\Theta$ is a cutfree proof of $(\Gamma, \Theta \Rightarrow C)$ in G3iSL of the same height as $\mathcal{D}$.
2. If $\mathcal{D}$ is a cutfree proof of $S$ in G3iSL, then $\mathcal{D}_\Theta$ is a cutfree proof in G3iSL of the sequent $(\Gamma, \Box\Theta \Rightarrow C)$ of the same height as $\mathcal{D}$. Moreover, given the following derivation $\mathcal{D}$ that ends with $\mathcal{R}_{\mathrm{SL}}$ and has a $\Box A$-critical inference above its endsequent $S$:

$$\dfrac{(S') \quad \Pi_2, \boxdot\Gamma_2, \boxdot A, \Box D \Rightarrow D}{\Box\Sigma_2, \Pi_2, \Box\Gamma_2, \Box A \Rightarrow \Box D}\mathcal{R}_{\mathrm{SL}}$$
$$\mathcal{B}$$
$$\dfrac{\Pi_1, \boxdot\Gamma_1, \Box A \Rightarrow A}{(S) \quad \Box\Sigma_1, \Pi_1, \Box\Gamma_1 \Rightarrow \Box A}\mathcal{R}_{\mathrm{SL}}$$

Then in $\mathcal{D}_D$ this part of the proof becomes:

$$\dfrac{\Pi_2, \boxdot\Gamma_2, \boxdot A, \Box D, D^k, \Box D \Rightarrow D}{\Box\Sigma_2, \Pi_2, \Box\Gamma_2, \Box D, D, \Box A \Rightarrow \Box D}\mathcal{R}_{\mathrm{SL}}$$
$$\mathcal{B}'$$
$$\dfrac{\Pi_1, \boxdot\Gamma_1, \Box D, D, \Box A \Rightarrow A}{\Box\Sigma_1, \Pi_1, \Box\Gamma_1, \Box D \Rightarrow \Box A}\mathcal{R}_{\mathrm{SL}}$$

where $k = 1$ if $D$ is a boxed formula and $k = g(S') = 2$ otherwise. $\mathcal{B}'$ is that part of the branch segment in $\mathcal{D}_D$ that corresponds to $\mathcal{B}$ in $\mathcal{D}$.
3. Similar result to 2 holds for G3iGL in which the $\Pi's$ and boxes before each $\Sigma$ are dropped, and $k = 1$.

Proof. The proof of 1 is straightforward. Without lose of generality, we show 2 for $\Theta = \{D\}$ and we leave 3 to the reader. It is clear that for every axiom $S$ in $\mathcal{D}$, $S_D$ is an axiom too. So all leafs of $\mathcal{D}_D$ are axioms. Therefore it suffices to show that all inference steps remain valid under translation $(\cdot)_D$. Consider an application of a two-premise rule:

$$\dfrac{S_2 \qquad S_3}{S_1} R$$

If $R$ is an instance of a nonmodal rule $\mathcal{R}$, then $g(S_1) = g(S_2) = g(S_3)$. Thus sequents $S_i = (\Gamma_i \Rightarrow C_i)$ are in $(S_i)_D$ all weakened with the same formula(s) on the left or remain all three unchanged. Therefore the inference becomes, for $\Pi$ being either $\varnothing$, $\{\Box D\}$, $\{\Box D, D\}$, $\{\Box D, D^{g(S_1)}\}$:

$$\dfrac{\Pi, \Gamma_2 \Rightarrow C_2 \qquad \Pi, \Gamma_3 \Rightarrow C_3}{\Pi, \Gamma_1 \Rightarrow C_1}\, R$$

This clearly is an instance of $\mathcal{R}$ as well. Single premise rules are treated in exactly the same way. For the case that $R$ is an instance of a modal rule, suppose it is of the form

$$\dfrac{(S_2)\ \Pi, \boxdot\Gamma, \Box A \Rightarrow A}{(S_1)\ \Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box A}\, \mathcal{R}_{\mathrm{SL}}$$

Here $\Pi$ does not contain boxed formulas. We distinguish the case that $D$ is a boxed formula and that it is not. In the first case, the inference becomes one of

$$\dfrac{\Pi, \boxdot\Gamma, \Box D, D, \Box A \Rightarrow A}{\Box\Sigma, \Pi, \Box\Gamma, \Box D, D \Rightarrow \Box A}\, \mathcal{R}_{\mathrm{SL}} \qquad \dfrac{\Pi, \boxdot\Gamma, \Box D, D, \Box A \Rightarrow A}{\Box\Sigma, \Pi, \Box\Gamma, \Box D \Rightarrow \Box A}\, \mathcal{R}_{\mathrm{SL}}$$

depending on whether $g(S_1) > 0$ or $g(S_1) = 0$. These are instances of $\mathcal{R}_{\mathrm{SL}}$, as the $D$ in the conclusion of the leftmost case can be introduced because it is a boxed formula. If $D$ is not a boxed formula, then because $g(S_2) = g(S_1) + 1$, the inference becomes

$$\dfrac{\Pi, \boxdot\Gamma, \Box D, D, D^{g(S_1)}, \Box A \Rightarrow A}{\Box\Sigma, \Pi, \Box\Gamma, \Box D, D^{g(S_1)} \Rightarrow \Box A}\, \mathcal{R}_{\mathrm{SL}}$$

Since $D$ is not boxed, the formulas $D^{g(S_1)}$ in the premise remain in the conclusion, and the inference is indeed valid.

It is not hard to see that $\mathcal{D}_D$ is cutfree and has the same height as $\mathcal{D}$. Finally, since for the endsequent $S$ we have $g(S) = 0$, the endsequent of $\mathcal{D}_D$ is $(\Gamma, \Box D \Rightarrow C)$. ■

### 3.1.2 Cut-elimination

In this section we provide a syntactic proof for the admissibility of the cut rule

$$\dfrac{\Gamma_1 \Rightarrow A \qquad A, \Gamma_2 \Rightarrow C}{\Gamma_1, \Gamma_2 \Rightarrow C}\, \mathrm{cut}$$

in systems G3iGL and G3iSL. The proof of cut-admissibility is not straightforward, which is based on the method to prove cut-elimination for GL from (Valentini, 1983; Goré and Ramanayake, 2012b). The key idea behind these proofs is the use of a new measure on cuts.

A well-known method for establishing the cut-admissibility theorem is to transform topmost cuts of the form

$$\dfrac{\stackrel{\mathcal{D}_1}{\Gamma_1 \Rightarrow A} \qquad \stackrel{\mathcal{D}_2}{A, \Gamma_2 \Rightarrow C}}{\Gamma_1, \Gamma_2 \Rightarrow C}\,\text{cut} \tag{3.1}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are cutfree, into cutfree derivations with the same end-sequent. Standard is to use a double induction $(d, h)$ defined as follows. Recall the degree $d$ of a formula from Definition 2.3.10.

#### 3.1.11 Definition

Consider a topmost cut as in (3.1).

- The *degree* $d$ of the cut is the degree of the cut-formula $A$.
- The *level* $h$ of a cut is the sum of the heights of the derivations of its premises.[19]

However, for G3iGL and G3iSL, this is not sufficient. We use a third induction parameter called 'width' which is computed 'globally.' A third parameter is necessary when we encounter the following problem: consider a cut where both premises are derived from rule $\mathcal{R}_{\mathrm{GL}}$ and the cut-formula is principal in both sides:

$$\dfrac{\dfrac{\stackrel{\mathcal{D}_1}{\boxdot\Gamma_1, \Box B \Rightarrow B}}{\Sigma_1, \Box\Gamma_1 \Rightarrow \Box B}\,\mathcal{R}_{\mathrm{GL}} \qquad \dfrac{\stackrel{\mathcal{D}_2}{\Box B, B, \boxdot\Gamma_2, \Box C \Rightarrow C}}{\Sigma_1, \Box B, \Box\Gamma_2 \Rightarrow \Box C}\,\mathcal{R}_{\mathrm{GL}}}{\Sigma_1, \Sigma_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box C}\,\text{cut}_1$$

A reasonable thing to do is the following, where we use the admissibility of contraction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\stackrel{\mathcal{D}_1}{\boxdot\Gamma_1, \Box B \Rightarrow B}}{\Box\Gamma_1 \Rightarrow \Box B} \qquad \stackrel{\mathcal{D}_1}{\boxdot\Gamma_1, \Box B \Rightarrow B}}{\Box\Gamma_1, \boxdot\Gamma \Rightarrow B}\,\text{cut}_2}{\boxdot\Gamma_1 \Rightarrow B}\,\text{contraction} \qquad \dfrac{\dfrac{\stackrel{\mathcal{D}_1}{\boxdot\Gamma_1, \Box B \Rightarrow B}}{\Box\Gamma_1 \Rightarrow \Box B} \qquad \stackrel{\mathcal{D}_2}{\Box B, B, \boxdot\Gamma_2, \Box C \Rightarrow C}}{B, \Box\Gamma_1, \boxdot\Gamma_2, \Box C \Rightarrow C}\,\text{cut}_3}{\dfrac{\boxdot\Gamma_1, \Box\Gamma, \boxdot\Gamma_2, \Box C \Rightarrow C}{\boxdot\Gamma_1, \boxdot\Gamma_2, \Box C \Rightarrow C}\,\text{contraction}}\,\text{cut}_4}{\Sigma_1, \Sigma_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box C}\,\mathcal{R}_{\mathrm{GL}}$$

But here, it is not possible to eliminate $\text{cut}_2$, when using the standard induction on $(d, h)$. Although the cut-formula in $\text{cut}_2$ is the same as in $\text{cut}_1$, which means

[19] In (van der Giessen and Iemhoff, 2021) the level of an inference was called its *height*, following (Goré and Ramanayake, 2012b). Here we use *level* instead, the terminology from (Troelstra and Schwichtenberg, 2000)

that the degree $d$ remains the same, the level $h$ of $\mathrm{cut}_2$ is not necessarily smaller than the level of $\mathrm{cut}_1$. The reason is that the level of $\mathrm{cut}_2$ is defined in terms of the height of $\mathcal{D}_1$, but the level of $\mathrm{cut}_1$ also depends on $\mathcal{D}_2$. So we cannot compare both cuts in terms of $(d, h)$.

The width circumvents the problem, because it enables us to define a derivation of $\boxdot\Gamma \Rightarrow B$ in which each application of the cut rule is eliminable. Recall Definition 3.1.5 of $\Box A$-critical inferences.

**3.1.12 Definition** (Width)
Consider a topmost cut as depicted in (3.1). The *width* of the cut equals 0 if $A$ is not boxed. In case $A$ is boxed, the *width* is the number of $A$-critical inferences over $\Gamma_1 \Rightarrow A$.

Note that the width is defined on the basis of the left premise (and $\mathcal{D}_1$) of the cut and is independent of the right premise (and $\mathcal{D}_2$). The width has only been defined for topmost cuts as this restriction is sufficient for our purpose.

We define the *dwh-order* on topmost cuts to be the lexicographic ordering on the triples $(d, w, h)$.

**3.1.13 Remark**
The definition of the width originates from Valentini (1983), who defines it for a set-based sequent system for GL including explicit weakening rules. Goré and Ramanayake (2012b) point out that the definition is a bit imprecise, especially to check the cut-elimination proof in combination with the explicit weakening rules. They formalize the width more precisely and show that the argument holds for a multiset-based sequent system for GL containing explicit weakening and contraction rules. Here, we use the original definition from Valentini, because in our setting it is the correct formalization, as we do not deal with explicit weakening and contraction rules, but are admissible in the systems.

**3.1.14 Lemma**
In G3iSL, if a cut of the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Pi_1, \boxdot\Gamma_1, \Box A \Rightarrow A\end{matrix}}{\Box\Sigma_1, \Pi_1, \Box\Gamma_1 \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}} \qquad \begin{matrix}\mathcal{D}'\\ \Box A, \Pi_2, \Box\Gamma_2 \Rightarrow C\end{matrix}}{\Box\Sigma_1, \Pi_1, \Box\Gamma_1, \Pi_2, \Box\Gamma_2 \Rightarrow C}\,\mathrm{cut} \tag{3.2}$$

has width zero and $\mathcal{D}$ is cutfree, then there exists a cutfree derivation of the sequent $(\Pi_1, \boxdot\Gamma_1 \Rightarrow A)$. Similar result holds for G3iGL, in which all the $\Pi$'s and boxes before all $\Sigma$'s are dropped.

Proof. Let us only show the proof for G3iSL, and we let the case for G3iGL to the reader. As the cut has width zero, in any inference in the left premise, any strict ancestor of the indicated $\Box A$ in the antecedent of the conclusion of $\mathcal{D}$ is not principal. Therefore in any inference in $\mathcal{D}$ (strict ancestors of) $\Box A$ occur either in both conclusion and premises or in none, or the inference has the following form

$$\frac{\Pi, \boxdot\Gamma, \Box B \Rightarrow B}{\Box A, \Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box B}\ \mathcal{R}_{\mathrm{SL}}$$

All these inferences remain valid if the strict ancestors of the occurrence of $\Box A$ that we are considering are removed in the antecedents, if present. The same applies to axioms. Thus removing in the antecedents of the sequents in $\mathcal{D}$ bottom-up the strict ancestors of $\Box A$ as long as they are present, results in a cutfree proof of $(\Pi_1, \boxdot\Gamma_1 \Rightarrow A)$. ■

**3.1.15 Theorem** (Cut-admissibility)
The cut rule is admissible in G3iGL and G3iSL.

Proof. We prove the statement for both G3iGL and G3iSL. Many steps apply to both, and these will be presented uniformly. We deviate between the two when necessary. Following the proof of the cut elimination proof for G3ip in (Troelstra and Schwichtenberg, 2000), we successively eliminate cuts from the proof, always considering those cuts that have no cuts above them, the *topmost* cuts. For this it suffices to show that for cutfree proofs $\mathcal{D}_1$ and $\mathcal{D}_2$, the following proof $\mathcal{D}$ can be transformed into a cutfree proof $\mathcal{D}'$ of the same endsequent:

$$\frac{\begin{matrix}\mathcal{D}_1 \\ \Gamma_1 \Rightarrow A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ A, \Gamma_2 \Rightarrow C\end{matrix}}{\Gamma_1, \Gamma_2 \Rightarrow C}\ \text{cut}$$

This is proved by induction on the *dwh*-order. We use the fact that G3iGL and G3iSL are closed under weakening and contraction implicitly at various places.

There are three possibilities:

1. at least one of the premises is an axiom;
2. both premises are not axioms and the cutformula is not principal in at least one of the premises;
3. the cutformula is principal in both premises, which are not axioms.

**Case 1:**
As in (Troelstra and Schwichtenberg, 2000), straightforward, by checking all possible cases: If $\mathcal{D}_1$ is axiom $L\bot$, then we let $\mathcal{D}'$ be the instance $S$ of that axiom. If $\mathcal{D}_2$ is axiom $L\bot$ and $\bot$ is not the cutformula, then we let $\mathcal{D}'$ be the instance $S$ of that axiom. If $\bot$ is the cutformula, then $(\Gamma_1 \Rightarrow \bot)$ has a cutfree derivation. Thus so has $(\Gamma_1, \Gamma_2 \Rightarrow C)$ by Lemma 3.1.3, where we use the additional fact that in this proof no new cuts are introduced with respect to $\mathcal{D}_1$. Therefore we let $\mathcal{D}'$

be this proof.

Assume both premises are not instances of $L\bot$. If $\mathcal{D}_1$ is axiom Ax, then $A$ is a propositional variable. If $\mathcal{D}_2$ also is an instance of Ax, then $S$ is an instance of Ax, so we let $\mathcal{D}'$ be that instance. If $\mathcal{D}_2$ is not an axiom, then $A$ cannot be principal in its last inference, because it is a propositional variable. We obtain $\mathcal{D}'$ by *cutting at a lower level*, by which we mean the following. Suppose the last inference of $\mathcal{D}_2$ is R $\to$:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow A\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2'\\ \Gamma_2, A, B_1 \Rightarrow B_2\end{array}}{\Gamma_2, A \Rightarrow B_1 \to B_2}\,\mathrm{R}\to}{\Gamma_1, \Gamma_2, B_1 \Rightarrow B_2}\,\mathrm{cut}$$

Consider the following:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma_1 \Rightarrow A\end{array} \qquad \begin{array}{c}\mathcal{D}_2'\\ \Gamma_2, A, B_1 \Rightarrow B_2\end{array}}{\Gamma_1, \Gamma_2, B_1 \Rightarrow B_2}\,\mathrm{cut}}{\Gamma_1, \Gamma_2 \Rightarrow B_1 \to B_2}\,\mathrm{R}\to$$

Since the degree and width remain the same, but the level lowers, we can apply the induction hypothesis to obtain a cutfree proof of $(\Gamma_1, \Gamma_2 \Rightarrow B_1 \to B_2)$. The cases where the last inference of $\mathcal{D}_2$ is another rule than $R \to$ are treated in a similar way.

The case that $\mathcal{D}_1$ is not an axiom and $\mathcal{D}_2$ is an instance of rule Ax remains. Here we also cut at a lower level, the proof is completely analogous to the case just treated.

**Case 2:**
First, the case that $A$ is not principal in the last inference of $\mathcal{D}_1$. Thus the last inference in $\mathcal{D}_1$ is one of the rules $\mathcal{R}$ presented in Figure 3.1. In case $\mathcal{R}$ is L $\to$, the last part of the proof looks as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma, B \to B' \Rightarrow B\end{array} \qquad \begin{array}{c}\mathcal{D}_1''\\ \Gamma, B' \Rightarrow A\end{array}}{\Gamma, B \to B' \Rightarrow A}\,\mathrm{L}\to \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_2, A \Rightarrow C\end{array}}{\Gamma, \Gamma_2, B \to B' \Rightarrow C}\,\mathrm{cut}$$

where $\Gamma_1 = \Gamma, B \to B'$. The cut can be pushed upwards in the following way (double lines suppress applications of weakening):

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma, B \to B' \Rightarrow B\end{array}}{\Gamma, \Gamma_2, B \to B' \Rightarrow B} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1''\\ \Gamma, B' \Rightarrow A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma_2, A \Rightarrow C\end{array}}{\Gamma, \Gamma_2, B' \Rightarrow C}\,\mathrm{cut}}{\Gamma, \Gamma_2, B \to B' \Rightarrow C}\,\mathrm{L}\to$$

Thus we obtain a proof of $(\Gamma_1, \Gamma_2, B \to B' \Rightarrow C)$ with a cut of the same degree and at most the same width as the cut in $\mathcal{D}$ but of lower level, and the induction hypothesis can be applied. Other nonmodal rules can be treated in a similar way.

Second, the case that $A$ is not principal in the last inference of $\mathcal{D}_2$. The nonmodal rules are handled as in the previous cases. We treat the remaining case $\mathcal{R}_{\mathrm{GL}}$ and $\mathcal{R}_{\mathrm{SL}}$. Here we only treat the different cases for $\mathcal{R}_{\mathrm{SL}}$ as the proof for $\mathcal{R}_{\mathrm{GL}}$ is analogous to the second case. For $\mathcal{R}_{\mathrm{GL}}$, $\mathcal{D}$ takes one of the following forms:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma_1 \Rightarrow A\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ A, \Pi, \boxdot\Gamma, \Box B \Rightarrow B\end{matrix}}{A, \Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box B}\,\mathcal{R}_{\mathrm{SL}}}{\Gamma_1, \Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box B}\,\text{cut} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma_1 \Rightarrow A\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ \Pi, \boxdot\Gamma, \Box B \Rightarrow B\end{matrix}}{A, \Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box B}\,\mathcal{R}_{\mathrm{SL}}}{\Gamma_1, \Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box B}\,\text{cut}$$

depending on whether $A$ is not a boxed formula (the leftmost case) or is a boxed formula (the rightmost case). Here $\Gamma_2 = \Box\Sigma, \Pi, \Box\Gamma$ and $\Pi$ contains no boxed formulas. In the rightmost case, consider the following cutfree proof:

$$\dfrac{\begin{matrix}\mathcal{D}_2'\\ \Pi, \boxdot\Gamma, \Box B \Rightarrow B\end{matrix}}{\Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box B}\,\mathcal{R}_{\mathrm{SL}}$$

By the closure under weakening there is a cutfree proof of $(\Gamma_1, \Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box B)$. In the leftmost case, let $\Gamma', \Pi'$ be such that $\Gamma_1 = \Box\Gamma' \cup \Pi'$ and $\Pi'$ does not contain boxed formulas. The following is a proof of the same endsequent of the same degree and width but of lower level (double lines suppress applications of weakening):

$$\dfrac{\genfrac{}{}{2pt}{0}{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma_1 \Rightarrow A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2'\\ A, \Pi, \boxdot\Gamma, \Box B \Rightarrow B\end{matrix}}{\Gamma_1, \Pi, \boxdot\Gamma, \Box B \Rightarrow B}\,\text{cut}}{\boxdot\Gamma', \Pi', \Pi, \boxdot\Gamma, \Box B \Rightarrow B}\,\text{weakening}}{\Gamma_1, \Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box B}\,\mathcal{R}_{\mathrm{SL}}$$

**Case 3:**
The cutformula is principal in both premises, which are not axioms. We distinguish by cases according to the form of the cutformula, and treat implications and boxed formulas.

If the cutformula is an implication, the last inference has the following form:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1'\\ \Gamma_1, A \Rightarrow B\end{matrix}}{\Gamma_1 \Rightarrow A \to B}\,\mathrm{R}\to \qquad \dfrac{\begin{matrix}\mathcal{D}_2'\\ \Gamma_2, A \to B \Rightarrow A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2''\\ \Gamma_2, B \Rightarrow C\end{matrix}}{\Gamma_2, A \to B \Rightarrow C}\,\mathrm{L}\to}{\Gamma_1, \Gamma_2 \Rightarrow C}\,\text{cut}$$

This is replaced by the following proof in which each cut either is of lower degree or of the same degree and width but of lower level as the cut in $\mathcal{D}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1'}{\Gamma_1, A \Rightarrow B}}{\Gamma_1 \Rightarrow A \to B} \qquad \overset{\mathcal{D}_2'}{\Gamma_2, A \to B \Rightarrow A}}{\Gamma_1, \Gamma_2 \Rightarrow A}\,\text{cut} \qquad \overset{\mathcal{D}_1'}{\Gamma_1, A \Rightarrow B}}{\Gamma_1, \Gamma_1, \Gamma_2 \Rightarrow B}\,\text{cut} \qquad \overset{\mathcal{D}_2''}{\Gamma_2, B \Rightarrow C}}{\Gamma_1, \Gamma_1, \Gamma_2, \Gamma_2 \Rightarrow C}\,\text{cut}$$

If the cutformula is a boxed formula, then the proofs of the premises end with an application of $\mathcal{R}_{\mathrm{GL}}$ or $\mathcal{R}_{\mathrm{SL}}$, depending on the logic in question:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1'}{\boxdot\Gamma_1, \Box A \Rightarrow A}}{\Sigma_1, \Box\Gamma_1 \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{GL}} \qquad \dfrac{\overset{\mathcal{D}_2'}{\boxdot A, \boxdot\Gamma_2, \Box B \Rightarrow B}}{\Sigma_2, \Box\Gamma_2, \Box A \Rightarrow \Box B}\,\mathcal{R}_{\mathrm{GL}}}{\Sigma_1, \Sigma_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B}\,\text{cut}_1$$

$$\dfrac{\dfrac{\overset{\mathcal{D}_1'}{\Pi_1, \boxdot\Gamma_1, \Box A \Rightarrow A}}{\Box\Sigma_1, \Pi_1, \Box\Gamma_1 \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}} \qquad \dfrac{\overset{\mathcal{D}_2'}{\Pi_2, \boxdot A, \boxdot\Gamma_2, \Box B \Rightarrow B}}{\Box\Sigma_2, \Pi_2, \Box\Gamma_2, \Box A \Rightarrow \Box B}\,\mathcal{R}_{\mathrm{SL}}}{\Box\Sigma_1, \Box\Sigma_2, \Pi_1, \Pi_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B}\,\text{cut}_1$$

where the $\Pi_i$ do not contain boxed formulas. We distinguish the cases that the width of the cut is zero and that it is bigger than zero. In the first case, by Lemma 3.1.14, there is a cutfree proof $\mathcal{D}_3$ of $(\boxdot\Gamma_1 \Rightarrow A)$ in case of G3iGL and there is a cutfree proof $\mathcal{D}_3$ of $(\Pi_1, \boxdot\Gamma_1 \Rightarrow A)$ in case of G3iSL. Therefore in the following derivation for G3iSL all cuts are either of lower degree or of the same degree and width but of lower level than $\text{cut}_1$. A similar derivation works for G3iGL in which all $\Pi$'s and boxes in front of $\Sigma$'s are dropped.

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_3}{\Pi_1, \boxdot\Gamma_1 \Rightarrow A} \qquad \dfrac{\dfrac{\overset{\mathcal{D}_1'}{\Pi_1, \boxdot\Gamma_1, \Box A \Rightarrow A}}{\Pi_1, \Box\Gamma_1 \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}} \qquad \overset{\mathcal{D}_2'}{\boxdot A, \Pi_2, \boxdot\Gamma_2, \Box B \Rightarrow B}}{A, \Pi_1, \Pi_2, \Box\Gamma_1, \boxdot\Gamma_2, \Box B \Rightarrow B}\,\text{cut}_3}{\Pi_1, \Pi_1, \Pi_2, \Box\Gamma_1, \boxdot\Gamma_1, \boxdot\Gamma_2, \Box B \Rightarrow B}\,\text{cut}_2}{\Pi_1, \Pi_2, \boxdot\Gamma_1, \boxdot\Gamma_2, \Box B \Rightarrow B}\,\text{contraction}}{\Box\Sigma_1, \Box\Sigma_2, \Pi_1, \Pi_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B}\,\mathcal{R}_{\mathrm{SL}}$$

Thus the induction hypothesis can be applied, which in combination with the closure under contraction and an application of $\mathcal{R}_{\mathrm{SL}}$ yields a desired cutfree proof.

Finally, we treat the case in which the width of $\text{cut}_1$ is greater than zero. The proofs for G3iGL and G3iSL are analogous, but differ in some steps. Therefore we present the proofs in parallel on the subsequent four pages; left pages for G3iGL and right pages for G3iSL. We use the transformations from Definition 3.1.9. The case of G3iSL is more involved, with extra weakening transformations. Both proofs presented here form a simplification from those in (van der Giessen and Iemhoff, 2021, 2022). Figure 3.4 might help to understand all steps that we will take.

For G3iGL, if the width of $\mathrm{cut}_1$ is bigger than zero, the derivation of the left premise contains the following $\Box A$-critical branch segment, where the showed segment in $\mathcal{B}$ does not contain applications of $\mathcal{R}_{\mathrm{GL}}$ and $\Gamma_1' \subseteq \Gamma_1$ by Remark 3.1.7:

$$\dfrac{\begin{array}{c} \mathcal{D}_3 \\ \dfrac{\boxdot\Gamma_1', \boxdot\Gamma_3, \boxdot A, \Box D \Rightarrow D}{\Sigma_3, \Box\Gamma_1, \Box\Gamma_3, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{GL}} \\ \mathcal{B} \\ \dfrac{\boxdot\Gamma_1, \Box A \Rightarrow A}{\Sigma_1, \Box\Gamma_1 \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{GL}} \end{array} \qquad \begin{array}{c} \mathcal{D}_2 \\ \Sigma_2, \Box\Gamma_2, \Box A \Rightarrow \Box B \end{array}}{\Sigma_1, \Sigma_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B}\,\mathrm{cut}_1$$

Recall that $\mathcal{D}_1$ is the derivation in $\mathcal{D}$ of $(\Sigma_1, \Box\Gamma_1 \Rightarrow \Box A)$ and that $\mathcal{D}_1'$ is the derivation in $\mathcal{D}$ of $(\boxdot\Gamma_1, \Box A \Rightarrow A)$. Let $\overline{\mathcal{D}}_1$ be equal to $\mathcal{D}_1$ except for the last inference, which is still an application of $\mathcal{R}_{\mathrm{SL}}$, but one in which $\Sigma_1$ is not introduced. Let $S_3$ be the sequent at the top of the segment, $S_3 = (\boxdot\Gamma_1', \boxdot\Gamma_3, \boxdot A, \Box D \Rightarrow D)$. Since there is no $\mathcal{R}_{\mathrm{GL}}$-inference in $\mathcal{B}$, this means that $g_{\overline{\mathcal{D}}_1}(S_3) = 2$, all sequents in $\mathcal{B}$ have $\mathcal{R}_{\mathrm{GL}}$-grade 1, and the endsequent of $\overline{\mathcal{D}}_1$ has $\mathcal{R}_{\mathrm{GL}}$-grade 0. Therefore by Lemma 3.1.10, $(\overline{\mathcal{D}}_1)_D$ is a derivation in which the $\Box A$-critical branch segment becomes the following, where $\mathcal{B}'$ does not contain applications of $\mathcal{R}_{\mathrm{GL}}$:

$$(\overline{\mathcal{D}}_1)_D \quad = \quad \begin{array}{c} \mathcal{D}_3' \\ \dfrac{\boxdot\Gamma_1', \boxdot\Gamma_3, \boxdot A, \Box D, D, \Box D \Rightarrow D}{\Sigma_3, \Box\Gamma_1, \Box\Gamma_3, \Box D, D, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{GL}} \\ \mathcal{B}' \\ \dfrac{\boxdot\Gamma_1, \Box D, D, \Box A \Rightarrow A}{\Box\Gamma_1, \Box D \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{GL}} \end{array}$$

Sequent $S_4 = (\boxdot\Gamma_1', \boxdot\Gamma_3, \Box D, D, \Box D \Rightarrow D)$ that one obtains by removing $\boxdot A$ from the antecedent of $(S_3)_D$ is derivable, because $D$ is in its antecedent, and clearly has a cutfree derivation. We call the new obtained derivation $\mathcal{D}_1^\circ$:

$$\mathcal{D}_1^\circ \quad = \quad \begin{array}{c} \dfrac{\dfrac{\text{cutfree}}{\boxdot\Gamma_1', \boxdot\Gamma_3, \Box D, D, \Box D \Rightarrow D}}{\Sigma_3, \Box\Gamma_1, \Box\Gamma_3, \Box D, D, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{GL}} \\ \mathcal{B}' \\ \dfrac{\boxdot\Gamma_1, \Box D, D, \Box A \Rightarrow A}{\Box\Gamma_1, \Box D \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{GL}} \end{array}$$

Note that $\mathcal{D}_1^\circ$ is a cutfree derivation with endsequent $S_1 = (\Box\Gamma_1, \Box D \Rightarrow \Box A)$, and that it contains one $\Box A$-critical inference less than $\mathcal{D}_1$. *(Proof continues on page 80)*

For G3iSL, if the width of $\mathrm{cut}_1$ is bigger than zero, the derivation of the left premise contains the following $\Box A$-critical branch segment, , where the showed segment in $\mathcal{B}$ does not contain applications of $\mathcal{R}_{\mathrm{SL}}$ and $\Gamma_1' \subseteq \Gamma_1$ by Remark 3.1.7:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_3}{\Pi_3, \boxdot\Gamma_1', \boxdot\Gamma_3, \boxdot A, \Box D \Rightarrow D}}{\Box\Sigma_3, \Pi_3, \Box\Gamma_1, \Box\Gamma_3, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{SL}}}{\dfrac{\mathcal{B}}{\Pi_1, \boxdot\Gamma_1, \Box A \Rightarrow A}}}{\Box\Sigma_1, \Pi_1, \Box\Gamma_1 \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}} \qquad \dfrac{\mathcal{D}_2}{\Box\Sigma_2, \Pi_2, \Box\Gamma_2, \Box A \Rightarrow \Box B}$$

$$\overline{\Box\Sigma_1, \Box\Sigma_2, \Pi_1, \Pi_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B}\ \mathrm{cut}_1$$

Recall that $\mathcal{D}_1$ is the derivation in $\mathcal{D}$ of $(\Box\Sigma_1, \Pi_1, \Box\Gamma_1 \Rightarrow \Box A)$ and that $\mathcal{D}_1'$ is the derivation in $\mathcal{D}$ of $(\Pi_1, \boxdot\Gamma_1, \Box A \Rightarrow A)$. Let $\overline{\mathcal{D}}_1$ be equal to $\mathcal{D}_1$ except for the last inference, which is still an application of $\mathcal{R}_{\mathrm{SL}}$, but one in which $\Box\Sigma_1$ is not introduced. Let $S_3$ be the sequent at the top of the segment, $S_3 = (\Pi_3, \boxdot\Gamma_1', \boxdot\Gamma_3, \boxdot A, \Box D \Rightarrow D)$. Since there is no $\mathcal{R}_{\mathrm{SL}}$-inference in $\mathcal{B}$, this means that $g_{\overline{\mathcal{D}}_1}(S_3) = 2$, all sequents in $\mathcal{B}$ have $\mathcal{R}_{\mathrm{SL}}$-grade 1, and the endsequent of $\overline{\mathcal{D}}_1$ has $\mathcal{R}_{\mathrm{SL}}$-grade 0. Therefore by Lemma 3.1.10, $(\overline{\mathcal{D}}_1)_D$ is a derivation in which the $\Box A$-critical branch segment becomes the following, where $\mathcal{B}'$ does not contain applications of $\mathcal{R}_{\mathrm{SL}}$ and $k$ equals 1 or 2 depending on whether $D$ is boxed or not:

$$(\overline{\mathcal{D}}_1)_D \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_3'}{\Pi_3, \boxdot\Gamma_1', \boxdot\Gamma_3, \boxdot A, \Box D, D^k, \Box D \Rightarrow D}}{\Box\Sigma_3, \Pi_3, \Box\Gamma_1, \Box\Gamma_3, \Box D, D, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{SL}}}{\dfrac{\mathcal{B}'}{\Pi_1, \boxdot\Gamma_1, \Box D, D, \Box A \Rightarrow A}}}{\Pi_1, \Box\Gamma_1, \Box D \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}}$$

Sequent $S_4 = (\Pi_3, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D, D^k, \Box D \Rightarrow D)$ that one obtains by removing $\boxdot A$ from the antecedent of $(S_3)_D$ is derivable, because $D$ is in its antecedent, and clearly has a cutfree derivation. We call the new obtained derivation $\mathcal{D}_1^\circ$:

$$\mathcal{D}_1^\circ \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\text{cutfree}}{\Pi_3, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D, D^k, \Box D \Rightarrow D}}{\Box\Sigma_3, \Pi_3, \Box\Gamma_1, \Box\Gamma_3, \Box D, D, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{SL}}}{\dfrac{\mathcal{B}'}{\Pi_1, \boxdot\Gamma_1, \Box D, D, \Box A \Rightarrow A}}}{\Pi_1, \Box\Gamma_1, \Box D \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}}$$

Note that $\mathcal{D}_1^\circ$ is a cutfree derivation with endsequent $S_1 = (\Pi_1, \Box\Gamma_1, \Box D \Rightarrow \Box A)$, and that it contains one $\Box A$-critical inference less than $\mathcal{D}_1$. *(Proof continues on page 81)*

Consider the following proof, where $\text{cut}_3$ and $\text{cut}_4$ cut on formula $\Box A$:

$$\dfrac{\dfrac{\mathcal{D}_1^\circ \qquad \overset{\mathcal{D}_1'}{\boxdot\Gamma_1, \Box A \Rightarrow A}}{\Box\Gamma_1, \boxdot\Gamma_1, \Box D \Rightarrow A}\text{cut}_3 \qquad \dfrac{\mathcal{D}_1^\circ \qquad \overset{\mathcal{D}_3}{\boxdot\Gamma', \boxdot\Gamma_3, \boxdot A, \Box D \Rightarrow D}}{A, \Box\Gamma_1, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D^2 \Rightarrow D}\text{cut}_4}{\Box\Gamma_1^2, \boxdot\Gamma_1, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D^3 \Rightarrow D}\text{cut}_2$$

Given that $\mathcal{D}_1', \mathcal{D}_1^\circ, \mathcal{D}_3$ are cutfree, the only cuts in the proof are $\text{cut}_2$, $\text{cut}_3$, and $\text{cut}_4$. By the observation above, the widths of $\text{cut}_3$ and $\text{cut}_4$ are lower than that of $\text{cut}_1$. Thus there exist cutfree proofs of the two premises of $\text{cut}_2$. Since $\text{cut}_2$ has lower degree than $\text{cut}_1$ the induction hypothesis applies. Together with the closure under contraction, and the fact that $\Gamma_1' \subseteq \Gamma_1$, this proves that there exists a cutfree proof, say $\mathcal{D}_4$, of $S_5 = (\boxdot\Gamma_1, \boxdot\Gamma_3, \Box D \Rightarrow D)$.

Recall $\mathcal{D}_1$, in which the segment is the following, where $\mathcal{B}$ does not contain applications of $\mathcal{R}_{\text{GL}}$:

$$\dfrac{\boxdot\Gamma_1', \boxdot\Gamma_3, \boxdot A, \Box D \Rightarrow D}{\Sigma_3, \Box\Gamma_1, \Box\Gamma_3, \Box A \Rightarrow \Box D}\mathcal{R}_{\text{GL}}$$
$$\mathcal{B}$$
$$\dfrac{\boxdot\Gamma_1, \Box A \Rightarrow A}{\Sigma_1, \Box\Gamma_1 \Rightarrow \Box A}\mathcal{R}_{\text{GL}}$$

Let $\mathcal{D}_1^\triangledown$ denote the result of replacing, in $\mathcal{D}_1$, the derivation of the sequent at the top of the segment by derivation $\mathcal{D}_4$ of $S_5$. And let $\mathcal{D}'$ be the result of replacing $\mathcal{D}_1$ in $\mathcal{D}$ by $\mathcal{D}_1^\triangledown$. Thus in $\mathcal{D}'$ the segment, the part above the segments and the last inference become

$$\dfrac{\overset{\mathcal{D}_4}{\boxdot\Gamma_1, \boxdot\Gamma_3, \Box D \Rightarrow D}}{\Sigma_3, \Box\Gamma_1, \Box\Gamma_3, \Box A \Rightarrow \Box D}\mathcal{R}_{\text{GL}}$$
$$\mathcal{B}$$
$$\dfrac{\dfrac{\boxdot\Gamma_1, \Box A \Rightarrow A}{\Sigma_1, \Box\Gamma_1 \Rightarrow \Box A}\mathcal{R}_{\text{GL}} \qquad \overset{\mathcal{D}_2}{\Sigma_2, \Box\Gamma_2, \Box A \Rightarrow \Box B}}{\Sigma_1, \Sigma_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B}\text{cut}_5$$

Note that $\mathcal{D}'$ still consists of valid inferences. The $\Box A$ at the top of the segment is now introduced in the first inference of the segment, and therefore no longer principal in that inference. Thus the width of $\text{cut}_5$ is lower than that of $\text{cut}_1$.

Since their degrees are the same, the induction hypothesis can be applied to obtain a cutfree proof of the endsequent $(\Sigma_1, \Sigma_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B)$, which is what we had to show. This concludes the proof for G3iGL.

Consider the following proof, where $\text{cut}_3$ and $\text{cut}_4$ cut on formula $\Box A$:

$$\dfrac{\dfrac{\mathcal{D}_1^\circ \qquad \begin{array}{c}\mathcal{D}_1' \\ \Pi_1, \boxdot\Gamma_1, \Box A \Rightarrow A\end{array}}{\Pi_1^2, \Box\Gamma_1, \boxdot\Gamma_1, \Box D \Rightarrow A}\,\text{cut}_3 \qquad \dfrac{\mathcal{D}_1^\circ \qquad \begin{array}{c}\mathcal{D}_3 \\ \Pi_3, \boxdot\Gamma', \boxdot\Gamma_3, \boxdot A, \Box D \Rightarrow D\end{array}}{A, \Pi_1, \Pi_3, \Box\Gamma_1, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D^2 \Rightarrow D}\,\text{cut}_4}{\Pi_1^3, \Pi_3, \Box\Gamma_1^2, \boxdot\Gamma_1, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D^3 \Rightarrow D}\,\text{cut}_2$$

Given that $\mathcal{D}_1', \mathcal{D}_1^\circ, \mathcal{D}_3$ are cutfree, the only cuts in the proof are $\text{cut}_2$, $\text{cut}_3$, and $\text{cut}_4$. By the observation above, the widths of $\text{cut}_3$ and $\text{cut}_4$ are lower than that of $\text{cut}_1$. Thus there exist cutfree proofs of the two premises of $\text{cut}_2$. Since $\text{cut}_2$ has lower degree than $\text{cut}_1$ the induction hypothesis applies. Together with the closure under contraction, and the fact that $\Gamma_1' \subseteq \Gamma_1$, this proves that there exists a cutfree proof, say $\mathcal{D}_4$, of $S_5 = (\Pi_1, \Pi_3, \boxdot\Gamma_1, \boxdot\Gamma_3, \Box D \Rightarrow D)$.

Observe that in $(\mathcal{D}_1)^{\Pi_1}$, the segment becomes the following, where $\mathcal{B}''$ does not contain applications of $\mathcal{R}_{\text{SL}}$:

$$\begin{array}{c}\dfrac{\Pi_1, \Pi_3, \boxdot\Gamma_1', \boxdot\Gamma_3, \boxdot A, \Box D \Rightarrow D}{\Pi_1, \Box\Sigma_3, \Pi_3, \Box\Gamma_1, \Box\Gamma_3, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\text{SL}} \\ \mathcal{B}'' \\ \dfrac{\Pi_1^2, \boxdot\Gamma_1, \Box A \Rightarrow A}{\Box\Sigma_1, \Pi_1^2, \Box\Gamma_1 \Rightarrow \Box A}\,\mathcal{R}_{\text{SL}}\end{array}$$

Let $\mathcal{D}_1^\triangledown$ denote the result of replacing, in $(\mathcal{D}_1)^{\Pi_1}$, the derivation of the sequent at the top of the segment by derivation $\mathcal{D}_4$ of $S_5$. And let $\mathcal{D}'$ be the result of replacing $\mathcal{D}_1$ in $\mathcal{D}$ by $\mathcal{D}_1^\triangledown$. Thus in $\mathcal{D}'$ the segment, the part above the segments and the last inference become

$$\begin{array}{c}\mathcal{D}_4 \\ \dfrac{\Pi_1, \Pi_3, \boxdot\Gamma_1, \boxdot\Gamma_3, \Box D \Rightarrow D}{\Pi_1, \Box\Sigma_3, \Pi_3, \Box\Gamma_1, \Box\Gamma_3, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\text{SL}} \\ \mathcal{B}'' \\ \dfrac{\dfrac{\Pi_1^2, \boxdot\Gamma_1, \Box A \Rightarrow A}{\Box\Sigma_1, \Pi_1^2, \Box\Gamma_1 \Rightarrow \Box A}\,\mathcal{R}_{\text{SL}} \qquad \begin{array}{c}\mathcal{D}_2 \\ \Box\Sigma_2, \Pi_2, \Box\Gamma_2, \Box A \Rightarrow \Box B\end{array}}{\Box\Sigma_1, \Box\Sigma_2, \Pi_1^2, \Pi_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B}\,\text{cut}_5\end{array}$$

Note that $\mathcal{D}'$ still consists of valid inferences. The $\Box A$ at the top of the segment is now introduced in the first inference of the segment, and therefore no longer principal in that inference. Thus the width of $\text{cut}_5$ is lower than that of $\text{cut}_1$.

Since their degrees are the same, the induction hypothesis can be applied to obtain a cutfree proof of the endsequent. Closure under contraction implies that $(\Box\Sigma_1, \Box\Sigma_2, \Pi_1, \Pi_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B)$ has a cutfree proof as well, which is what we had to show. This concludes to proof for G3iSL. ■

$$\dfrac{\dfrac{\text{cutfree}}{\Pi_3, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D, D^k, \Box D \Rightarrow D}}{\Box\Sigma_3, \Pi_3, \Box\Gamma_1, \Box\Gamma_3, \Box D, D, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{SL}} \qquad \dfrac{\dfrac{\text{cutfree}}{\Pi_3, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D, D^k, \Box D \Rightarrow D}}{\Box\Sigma_3, \Pi_3, \Box\Gamma_1, \Box\Gamma_3, \Box D, D, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{SL}}$$

$$\mathcal{B}' \qquad\qquad\qquad\qquad \mathcal{B}'$$

$$\dfrac{\dfrac{\Pi_1, \boxdot\Gamma_1, \Box D, D, \Box A \Rightarrow A}{\Pi_1, \Box\Gamma_1, \Box D \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}} \quad \begin{matrix}\mathcal{D}_1' \\ \Pi_1, \boxdot\Gamma_1, \Box A \Rightarrow A\end{matrix}}{\Pi_1^2, \Box\Gamma_1, \boxdot\Gamma_1, \Box D \Rightarrow A}\,\text{cut}_3 \qquad \dfrac{\dfrac{\Pi_1, \boxdot\Gamma_1, \Box D, D, \Box A \Rightarrow A}{\Pi_1, \Box\Gamma_1, \Box D \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}} \quad \begin{matrix}\mathcal{D}_3 \\ \Pi_3, \boxdot\Gamma_1', \boxdot\Gamma_3, \boxdot A, \Box D \Rightarrow D\end{matrix}}{A, \Pi_1, \Pi_3, \Box\Gamma_1, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D^2 \Rightarrow D}\,\text{cut}_4$$

$$\dfrac{\dfrac{\dfrac{\Pi_1^2, \Box\Gamma_1, \boxdot\Gamma_1, \Box D \Rightarrow A \qquad A, \Pi_1, \Pi_3, \Box\Gamma_1, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D^2 \Rightarrow D}{\Pi_1^3, \Pi_3, \Box\Gamma_1^2, \boxdot\Gamma_1, \boxdot\Gamma_1', \boxdot\Gamma_3, \Box D^3 \Rightarrow D}\,\text{cut}_2}{\Pi_1, \Pi_3, \boxdot\Gamma_1, \boxdot\Gamma_3, \Box D \Rightarrow D}\,\text{contraction}}{\Pi_1, \Box\Sigma_3, \Pi_3, \Box\Gamma_1, \Box\Gamma_3, \Box A \Rightarrow \Box D}\,\mathcal{R}_{\mathrm{SL}}$$

$$\mathcal{B}''$$

$$\dfrac{\dfrac{\dfrac{\Pi_1^2, \boxdot\Gamma_1, \Box A \Rightarrow A}{\Box\Sigma_1, \Pi_1^2, \Box\Gamma_1 \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}} \qquad \begin{matrix}\mathcal{D}_2 \\ \Box\Sigma_2, \Pi_2, \Box\Gamma_2, \Box A \Rightarrow \Box B\end{matrix}}{\Box\Sigma_1, \Box\Sigma_2, \Pi_1^2, \Pi_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B}\,\text{cut}_5}{\Box\Sigma_1, \Box\Sigma_2, \Pi_1, \Pi_2, \Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box B}\,\text{contraction}$$

**Figure 3.4.** Illustration of the main steps in the proof of Theorem 3.1.15 for $\mathcal{R}_{\mathrm{SL}}$. It does not contain $\mathcal{D}_4$, but the derivation that $\mathcal{D}_4$ replaces. Strictly speaking, it is not a derivation, since it contains contraction that is not part of the calculus. For $\mathcal{R}_{\mathrm{GL}}$, one can consider a similar picture where all $\Pi$'s and boxes before $\Sigma$'s are dropped, $k = 1$, and in which $\mathcal{B}''$ is $\mathcal{B}$.

**3.1.16 Remark**
The elimination of $\mathrm{cut}_5$ deserves more attention. We can remove the cut because the width reduces. Note that the cut-free derivation $\mathcal{D}_4$ does not have any effect on the calculation of the width of $\mathrm{cut}_5$. This means that the elimination of $\mathrm{cut}_2$, $\mathrm{cut}_3$, and $\mathrm{cut}_4$ does not affect the width of $\mathrm{cut}_5$ lower in the tree. Goré and Ramanayake (2012b) say that $\mathrm{cut}_5$ is 'shielded' by the $\mathcal{R}_{\mathrm{GL}}$ (or $\mathcal{R}_{\mathrm{SL}}$) pictured above $\mathcal{B}''$. This shielding is crucial.

The admissibility of cut implies standard properties such as the subformula property, consistency, and conservativity over IPC. In particular we have the completeness with respect to the logics.

**3.1.17 Corollary**
Sequent $S$ is provable in G3iGL (in G3iSL) if and only if its formula interpretation $I(S)$ is derivable in iGL (in iSL).

### 3.1.3 Termination

Iemhoff (2020) provides a method to turn a G3-like system into a terminating G4-like system for a large class of intuitionistic modal logics. This stems from (Dyckhoff, 1992) for IPC, showing strong termination of G4ip in the ordering $\ll$ from Definition 2.3.11 (Theorem 2.3.12). In turn, he shows the equivalence to G3ip. This is not the case for G4iGL and G4iSL, because, consider rule $\mathcal{R}^4_{\mathrm{GL}}$:

$$\frac{\boxdot\Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}\ \mathcal{R}^4_{\mathrm{GL}}$$

where $\Pi$ does not contain any boxed formulas. The premise of the rule is not necessarily lower than the conclusion with respect to the ordering $\ll$. Intuitively, the size of the sequent in the premise is 'doubled' compared to the sequent in its conclusion, because $\Gamma$ is 'duplicated.'

We show that systems G4iGL and G4iSL have a property close to strong termination. Using this, we can establish the equivalence between G3iGL (G3iSL) and G4iGL (G4iSL) as well. Recall Lemma 3.1.3, in which sequents of the form $\Gamma, A \Rightarrow A$ are called *extended axioms* and are proved to be derivable in G4iGL and G4iSL.

**3.1.18 Definition**
A sequent calculus is said to *terminate modulo extended axioms* if for each sequent, any process of bottom-up applications of the rules terminate in sequents that are either of the form $\Gamma, A \Rightarrow A$ or $\Gamma, \bot \Rightarrow A$, or to which no rule can be applied.

The reason that we need extended axioms is due to the form of the modal rules. For example, infinite branches can occur by repeated $\mathcal{R}^4_{\mathrm{GL}}$ and $\rightarrow_{\mathrm{GL}}$ inferences. In the following tree, we have a loop where we can apply $\rightarrow_{\mathrm{GL}}$ with diagonal formula $\Box A$ infinitely many times, indicated by the vertical dots.

$$
\dfrac{\dfrac{\dfrac{\vdots}{\boxdot(\Box A \to \bot), \boxdot C, \Box A, \Box A, A \Rightarrow A}\rightarrow_{\mathrm{GL}} \quad \dfrac{}{\Gamma', \bot \Rightarrow A}}{\boxdot(\Box A \to \bot), \boxdot C, \Box A \Rightarrow A}\rightarrow_{\mathrm{GL}} \quad \dfrac{}{\Gamma, \bot \Rightarrow C}}{\dfrac{\boxdot(\Box A \to \bot), \Box C \Rightarrow C}{\Box(\Box A \to \bot) \Rightarrow \Box C}\mathcal{R}^4_{\mathrm{GL}}}\rightarrow_{\mathrm{GL}}
$$

However, we see that we have created an infinite branch for the provable sequent $(\boxdot(\Box A \to \bot), \boxdot C, \Box A, \Box A, A \Rightarrow A)$, where formula $A$ occurs in both the antecedent and the conclusion of the sequent.

We define the appropriate ordering in the proof of the following theorem based on (Bílková, 2006). Similar problems are discussed for termination of tableau systems for GL by Goré and Kelly (2007).

**3.1.19 Definition**

For a sequent $S = (\Gamma \Rightarrow A)$, let $b(S)$ be the number of different boxed formulas in $\Gamma$, considered as a set. Given a number $c$, define the ordering $\sqsubset^c$ on sequents as follows, where $\ll$ is the ordering from Definition 2.3.11:

$$
S_1 \sqsubset^c S_2 \quad \text{iff} \quad \begin{array}{l} c - b(S_1) < c - b(S_2), \text{ or} \\ b(S_1) = b(S_2) \text{ and } S_1 \ll S_2. \end{array}
$$

As we will see, this order is used only in cases where $c \geq b(S)$.

**3.1.20 Theorem** (Termination)

Sequent calculi G4iGL and G4iSL terminate modulo extended axioms.

Proof. Let $S$ be the sequent for which we perform the proof search. Let $c$ be the number of all boxes occurring in $S$. Note that there are at most $c$ different boxed formulas in an antecedent of a sequent in the proof search counted as a set. For any rule application in the proof search all premises come before the conclusion in the order $\sqsubset^c$ as proved as follows.

If a sequent is of the form $\Gamma, C \Rightarrow C$ we are done by definition of termination modulo extended axioms. So suppose it is not an extended axiom.

If we apply backwards a rule from Figure 3.2, then $c - b$ decreases, or $c - b$ stays the same and the premises are lower with respect to $\ll$. Therefore, the premise is lower in $\sqsubset^c$ than the conclusion. If we apply backwards $\to_{\mathrm{GL}}$, say, to a sequent of the form $(\Pi, \Box\Gamma, \Box A \to B \Rightarrow C)$, then for the right premise we have the same reasoning as above. For the left premise, we have two cases. If $\Box A$ is not contained in $\Box\Gamma$, then $c-b$ decreases, because $\Pi$ contains no boxed formulas. If $\Box A$ is in $\Box\Gamma$, say, $\Box\Gamma = \Box\Gamma', \Box A$, then the left premise is of the form $(\boxdot\Gamma', \Box A, \Box A, A \Rightarrow A)$ and we close the branch, because it is an extended axiom. Therefore the premises are lower in $\sqsubset^c$ than the conclusion. Similar reasoning applies to $\to_{\mathrm{SL}}$.

If we apply backwards rule $\mathcal{R}^4_{\mathrm{GL}}$, say, to a sequent of the form $(\Pi, \Box\Gamma \Rightarrow \Box A)$, then $c-b$ decreases since $\Box A$ is assumed not to be in $\Box\Gamma$, and $\Pi$ is assumed to not contain boxed formulas. Therefore the premise is lower in $\sqsubset^c$ than the conclusion. Similarly so for rule $\mathcal{R}^4_{\mathrm{SL}}$. ■

In rule $\mathcal{R}^4_{\mathrm{SL}}$ we assume that $\Pi$ does not contain boxed formulas, but we might drop this condition while keeping termination modulo extended axioms as we discuss in Lemma 7 from (van der Giessen and Iemhoff, 2022). Important to note is that the condition on the $\Pi$'s in rule $\to_{\mathrm{SL}}$ is necessary for proving termination.

Now we use the termination of G4iGL and G4iSL to show their equivalence to G3iGL and G3iSL, respectively, meaning that the calculi derive the same sequents. This implies that cut is also admissible in G4iGL and G4iSL and that these systems are indeed proof systems for the logics iGL and iSL. First, we prove a normal form theorem for proofs in the calculi G3iGL and G3iSL.

**3.1.21 Definition**
We define the following notions.

- A multiset is *irreducible* if it has no element that is a disjunction or a conjunction or $\bot$ and for no propositional variable $p$ does it contain both $p \to B$ and $p$.
- A sequent $\Gamma \Rightarrow A$ is *irreducible* if $\Gamma$ is irreducible.
- A proof in G3iGL or G3iSL is *sensible* if the following holds: if the last inference is L $\to$, then its principal formula is not of the form $p \to A$ for some propositional variable $p$ and formula $A$.
- A proof in G3iGL (in G3iSL) is *strict* if the following holds: if the last inference is L $\to$ with principal formula of the form $\Box A \to B$, then the left premise is an axiom or the conclusion of an application of rule $\mathcal{R}_{\mathrm{GL}}$ (rule $\mathcal{R}_{\mathrm{SL}}$).

Note that for strict proofs in case the left premise is an instance of an axiom, it can only be an instance of $L\bot$, because the formula in the succedent of the left premise is $\Box A$. This implies that if $S$ is irreducible, the left premise cannot be an instance of an axiom and thus is required to be the conclusion of an application of a modal rule.

**3.1.22 Lemma**

In G3iGL and G3iSL, every irreducible sequent that is provable has a sensible strict proof.

Proof. This is proved in the same way as the corresponding lemma (Lemma 1) in (Dyckhoff, 1992). Arguing by contradiction, assume that among all provable irreducible sequents that have no sensible strict proofs, $S$ is such a sequent with the shortest proof, $\mathcal{D}$, where the *length* of a proof is the length of its leftmost branch. Thus the last inference in the proof is an application

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A \to B \Rightarrow A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma, B \Rightarrow C\end{array}}{\Gamma, A \to B \Rightarrow C}\ \mathrm{L} \to$$

of $\mathrm{L} \to$, where $A$ is a propositional variable or a boxed formula. Since $\Gamma, A \to B$ is irreducible, $\bot \notin \Gamma$ and if $A$ is a variable, $A \notin \Gamma$. Therefore the left premise cannot be an axiom and hence is the conclusion of a rule, say $\mathcal{R}$. Since the succedent of the conclusion of $\mathcal{R}$ consists of a variable or a boxed formula, $\mathcal{R}$ is a left rule or it is $\mathcal{R}_{\mathrm{GL}}$ or $\mathcal{R}_{\mathrm{SL}}$. The latter cases cannot occur, since the proof then would be strict and sensible. Thus $\mathcal{R}$ is a left rule.

Sequent $(\Gamma, A \to B \Rightarrow A)$ is irreducible and has a shorter proof than $S$. Thus its subproof $\mathcal{D}_1$ is strict and sensible. Since the sequent is irreducible and $A$ is a variable or a boxed formula, the last inference of $\mathcal{D}_1$ is $\mathrm{L} \to$ with a principal formula $A' \to B'$ such that $A'$ is not a variable. Let $\mathcal{D}'$ be the proof of the left premise $(\Gamma, A \to B \Rightarrow A')$. Thus the last part of $\mathcal{D}$ looks as follows, where $\Pi, A' \to B' = \Gamma$.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Pi, A \to B, A' \to B' \Rightarrow A'\end{array} \qquad \begin{array}{c}\mathcal{D}''\\ \Pi, A \to B, B' \Rightarrow A\end{array}}{\Pi, A \to B, A' \to B' \Rightarrow A} \qquad \begin{array}{c}\mathcal{D}_2\\ \Pi, B, A' \to B' \Rightarrow C\end{array}}{\Pi, A \to B, A' \to B' \Rightarrow C}$$

Consider the following proof of $S$.

$$\dfrac{\begin{array}{c}\mathcal{D}'\\ \Pi, A \to B, A' \to B' \Rightarrow A'\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}''\\ \Pi, A \to B, B' \Rightarrow A\end{array} \qquad \begin{array}{c}\mathcal{D}'''\\ \Pi, B, B' \Rightarrow C\end{array}}{\Pi, A \to B, B' \Rightarrow C}}{\Pi, A \to B, A' \to B' \Rightarrow C}$$

The existence of $\mathcal{D}'''$ follows from Lemma 3.1.3 (5) (inversion) and the existence of $\mathcal{D}_2$. The obtained proof is strict and sensible: In case $A'$ is not a boxed formula, this is straightforward. In case $A'$ is a boxed formula, it follows from the fact, observed above, that $\mathcal{D}_1$ is strict and sensible. ■

**3.1.23 Theorem** (Equivalence)
For all sequents $S$ we have $\vdash_{\mathsf{G3iGL}} S$ if and only if $\vdash_{\mathsf{G4iGL}} S$. Similarly so for G3iSL and G4iSL.

Proof. The proof is an adaptation of the proof of Theorem 1 in (Dyckhoff, 1992). The direction from right to left is straightforward because G3iGL and G3iSL are closed under the structural rules and the cut rule, but let us fill in some of the details. Let us prove it simultaneously for iGL and iSL by writing G3iX and G4iX. We use induction to the height of the proof of a sequent in G4iX. Suppose $\vdash_{\mathsf{G4iX}}$. If $S$ is an instance of an axiom, then clearly $\vdash_{\mathsf{G3iX}} S$ as well. Suppose $S$ is not an instance of an axiom and consider the last inference of the proof of $S$. We distinguish according to the rule of which the last inference is an instance. We only treat two cases, the remaining cases are left to the reader.

If the rule is $\mathrm{L}p \to$, then $S$ is of the form $\Gamma, p, p \to A \Rightarrow C$. The premise is $\Gamma, p, A \Rightarrow C$, which, by the induction hypothesis, is derivable in G3iX. It is not hard to show that $\Gamma, p, p \to A \Rightarrow A$ is also derivable in G3iX. Applications of cut and contraction show that so is $S$.

If the rule is $\mathrm{L} \to\to$, then $S$ is of the form $\Gamma, (A \to B) \to C \Rightarrow D$ and the premises are $\Gamma, C \Rightarrow D$ and $\Gamma, B \to C \Rightarrow A \to B$. The premises are derivable in G3iX by the induction hypothesis. It is not difficult to show that then $\Gamma, (A \to B) \to C, A \Rightarrow B$ is derivable in G3iX as well. Hence so is $\Gamma, (A \to B) \to C \Rightarrow A \to B$. An application of $\mathrm{L} \to$ proves that $S$ is derivable in G3iX.

If the rule is $\to_{\mathrm{SL}}$, then $S$ is of the form $(\Pi, \Box\Gamma, \Box A \to B \Rightarrow C)$ and the premises are $(\Pi, \boxdot\Gamma, \Box A, \Box A \to B \Rightarrow A)$ and $(\Pi, \Box\Gamma, B \Rightarrow C)$. The premises are derivable in G3iSL by the induction hypothesis. The following derivation shows that $S$ is derivable in G3iSL:

$$\dfrac{\dfrac{\Pi, \boxdot\Gamma, \Box A, \Box A \to B \Rightarrow A}{\Pi, \Box\Gamma, \Box A \to B \Rightarrow \Box A}\,\mathcal{R}_{\mathrm{SL}} \qquad \Pi, \Box\Gamma, B \Rightarrow C}{(S)\ \Pi, \Box\Gamma, \Box A \to B \Rightarrow C}\,\mathrm{L} \to$$

The remaining cases are left to the reader.

For the other direction we have to show that every sequent $S$ that is provable in G3iX is provable in G4iX. This is proved by induction on the well-ordering $\sqsubset^c$, where $c$ is the number of boxed formulas that occur in formulas in $S$ as defined in Definition 3.1.19. We consider sequents $S'$ that only contain subformulas from $S$, meaning that $c - b(S') \geq 0$ for all such sequents $S'$. From Theorem 3.1.20 we know that the proof search on $S$ in G4iX terminates modulo extended axioms in this ordering. Let $\sqsubset$ denote $\sqsubset^c$ in the rest of the proof and suppose $\vdash_{\mathsf{G3iX}} S$.

Sequents lowest in the ordering $\sqsubset$ do not contain connectives and no boxed formula in the succedent. Thus in this case $S$ has to be an instance of an axiom, and since G3iX and G4iX have the same axioms, $S$ is provable in G4iX.

We turn to the case that $S$ is not the lowest in the ordering $\sqsubset$. If the antecedent of $S$ contains a conjunction, $S = (\Gamma, A_1 \wedge A_2 \Rightarrow C)$, then $S' = (\Gamma, A_1, A_2 \Rightarrow C)$ is provable in G3iX too by inversion (Lemma 3.1.3). As $S' \sqsubset S$, $S'$ is provable in G4iX by the induction hypothesis. Thus so is $(\Gamma, A_1 \wedge A_2 \Rightarrow C)$. A disjunction in the antecedent as well as the case that both $p$ and $p \to A$ belong to the antecedent, can be treated in the same way.

Thus only the case that $S$ is irreducible remains, and by Lemma 3.1.22 we may assume its proof to be sensible and strict. Thus its last inference is an application of a rule, $\mathcal{R}$, that is either a nonmodal right rule, $\mathcal{R}_{\mathrm{GL}}$, $\mathcal{R}_{\mathrm{SL}}$ or L $\to$. In the first case, $\mathcal{R}$ belongs to both calculi and the premise of $\mathcal{R}$ is lower in the $\sqsubset$ ordering than $S$ and thus the induction hypothesis applies.

In the case of $\mathcal{R}_{\mathrm{SL}}$, let $S = (\Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box A)$ and let $(\Pi, \boxdot\Gamma, \Box A \Rightarrow A)$ be the premise of the inference. There are two cases: either $\Box A$ does occur in $\Box\Sigma$ or $\Box\Gamma$, or it does not. In the first case, $S$ is an extended axiom which is derivable in G4iSL. In the latter case, we consider sequent $S' = (\boxdot\Sigma, \Pi, \boxdot\Gamma, \Box A \Rightarrow A)$ which is derivable in G3iSL by weakening (Lemma 3.1.3). We have $S' \sqsubset S$, so $S'$ is by the induction hypothesis derivable in G4iSL. An application of $\mathcal{R}^4_{\mathrm{SL}}$ shows that $S$ is derivable in G4iSL. Similar argument applies to $\mathcal{R}_{\mathrm{GL}}$.

Finally, we turn to the implication rule L $\to$. Suppose that the principal formula of the last inference is $A \to B$ and $S = (\Gamma, A \to B \Rightarrow C)$. Since the proof is sensible, $C$ is not a propositional variable. We distinguish according to the main connective of $A$.

- If $A = \bot$, then $(\Gamma \Rightarrow C)$ is derivable in G3iX (follows easily from admissibility of cut), and therefore in G4iX. As G4iX is closed under weakening (Lemma 3.1.3), $S$ is derivable in G4iX too.
- If $A = A_1 \wedge A_2$, then $(\Gamma, A_1 \to (A_2 \to B) \Rightarrow C)$ is derivable in G3iX (using cut). Thus the sequent is derivable in G4iX by the induction hypothesis. Hence so is $(\Gamma, A_1 \wedge A_2 \to B \Rightarrow C)$.
- The case that $A = A_1 \vee A_2$ is analogous.
- If $A = A_1 \to A_2$, then because $A \to B$ is the principal formula, both sequents $(\Gamma, A \to B \Rightarrow A)$ and $(\Gamma, B \Rightarrow C)$ are derivable in G3iX. Thus so is sequent $(\Gamma, A_2 \to B \Rightarrow A_1 \to A_2)$ (using cut and inversion of the R $\to$ rule (Lemma 3.1.3)). Since this sequent and $(\Gamma, B \Rightarrow C)$ are lower in the ordering $\sqsubset$ than $S$, they are derivable in G4iX by the induction hypothesis. Hence so is $S$.
- If $A = \Box A_1$, we distinguish between G3iGL and G3iSL. Let us only treat G3iSL as the case for G3iGL is treated similarly. The fact that the proof is strict implies that the left premise is the conclusion of an application of $\mathcal{R}_{\mathrm{SL}}$, so

the derivation in G3iSL looks as follows, where $\Gamma = \Box\Sigma \cup \Pi \cup \Box\Gamma'$ for some $\Gamma', \Sigma, \Pi$ such that $\Pi$ does not contain boxed formulas:

$$\dfrac{\dfrac{\Pi, \boxdot\Gamma', \Box A_1 \to B, \Box A_1 \Rightarrow A_1}{\Box\Sigma, \Pi, \Box\Gamma', \Box A_1 \to B \Rightarrow \Box A_1}\,\mathcal{R}_{\mathrm{SL}} \qquad \Box\Sigma, \Pi, \Box\Gamma', B \Rightarrow C}{(S)\ \Box\Sigma, \Pi, \Box\Gamma', \Box A_1 \to B \Rightarrow C}\,\mathrm{L}\to$$

The right premise is smaller in $\sqsubset$ than $S$, so by induction hypothesis it is also provable in G4iSL. There are two cases: either $\Box A_1$ does occur in $\Box\Sigma$ or in $\Box\Gamma$, or it does not. We show that in both cases the sequent $S' = (\boxdot\Sigma, \Pi, \boxdot\Gamma', \Box A_1 \to B, \Box A_1 \Rightarrow A_1)$ is provable in G4iSL, and an application of rule $\to_{\mathrm{SL}}$ shows that $S$ is also provable in G4iSL. In the first case, $S'$ is an extended axiom and so it is derivable in G4iSL. In the latter case, note that $S' \sqsubset S$ and $S'$ is provable in G3iSL by weakening (Lemma 3.1.3), so by induction hypothesis we have that $S'$ is provable in G4iSL.

This concludes the proof showing that $\vdash_{\mathsf{G3iX}} S$ if and only if $\vdash_{\mathsf{G4iX}} S$. ■

The previous theorem and Theorem 3.1.15 imply the following results.

**3.1.24 Corollary**
The cut rule and the contraction rule are admissible in G4iGL and G4iSL.

**3.1.25 Corollary**
Sequent $S$ is provable in G4iGL (in G4iSL) if and only if its formula interpretation $I(S)$ is derivable in iGL (in iSL).

## 3.2 Craig interpolation

We can use the admissibility of cut in order to prove the Craig interpolation property for the logics iGL and iSL. Recall the definition of Craig interpolation from Definition 2.2.1. We will show that the calculi G3iGL and G3iSL have split Craig interpolation by the Maehara method as defined in Definition 2.3.4 which immediately yields the Craig interpolation property for the logics by Theorem 2.3.5.

**3.2.1 Theorem**
Sequent systems G3iGL and G3iSL have split Craig interpolation. That is, when $\vdash$ denotes $\vdash_{\mathsf{G3iGL}}$ or $\vdash_{\mathsf{G3iSL}}$, for every finite sets of formulas $\Gamma_1$ and $\Gamma_2$, and formula $D$, if $\vdash \Gamma_1, \Gamma_2 \Rightarrow D$, then there exists a formula $C$ such that

(i) $\mathsf{Var}(C) \subseteq \mathsf{Var}(\Gamma_1) \cap \mathsf{Var}(\Gamma_2, D)$, and
(ii) $\vdash \Gamma_1 \Rightarrow C$ and $\vdash C, \Gamma_2 \Rightarrow D$.

Proof. We proceed by induction on the derivation height $l$ of $\Gamma_1, \Gamma_2 \Rightarrow D$. This is a well-known strategy when proving Craig interpolation, see, i.e., (Takeuti, 1987).

First suppose $l = 0$. If $(\Gamma_1, \Gamma_2 \Rightarrow D)$ is derived by the Ax-rule, that is, it has the form $(\Gamma, p \Rightarrow p)$ for some $p$. There are two cases, $p \in \Gamma_1$ or $p \in \Gamma_2$. Take $C = p$ and $C = \top$, respectively. If $(\Gamma_1, \Gamma_2 \Rightarrow D)$ is derived by L$\bot$. There are two cases, $\bot \in \Gamma_1$ or $\bot \in \Gamma_2$. Take $C = D$ and $C = \top$, respectively.

Now suppose $l > 0$. We distinguish according to the last rule applied. Suppose that the last rule applied is R$\wedge$, where $(\Gamma_1, \Gamma_2 \Rightarrow D)$ is of the form $(\Gamma_1, \Gamma_2 \Rightarrow A \wedge B)$ with premises $(\Gamma_1, \Gamma_2 \Rightarrow A)$ and $(\Gamma_1, \Gamma_2 \Rightarrow B)$. Applying the induction hypothesis to both premises we have that

- there exists an interpolant $C_1$ such that $\vdash \Gamma_1 \Rightarrow C_1$ and $\vdash \Gamma_2, C_1 \Rightarrow A$ with $\mathsf{Var}(C_1) \subseteq \mathsf{Var}(\Gamma_1) \cap \mathsf{Var}(\Gamma_2, A)$,
- there exists an interpolant $C_2$ such that $\vdash \Gamma_1 \Rightarrow C_2$ and $\vdash \Gamma_2, C_2 \Rightarrow B$ with $\mathsf{Var}(C_2) \subseteq \mathsf{Var}(\Gamma_1) \cap \mathsf{Var}(\Gamma_2, B)$.

Take $C = C_1 \wedge C_2$ as the required interpolant. The cases for L$\wedge$, R$\vee$, and L$\vee$ are proved in a similar way.

If the last rule applied is L $\rightarrow$, we have two cases; the principal formula $A \rightarrow B$ of the rule is in $\Gamma_1$ or in $\Gamma_2$. We look at the first case. This case is somehow distinct from the other steps, in the sense that we apply the induction hypotheses to sequents where $\Gamma_1$ and $\Gamma_2$ are 'reversed.' Write $\Gamma_1 = \Gamma_1', A \rightarrow B$. We have

$$\dfrac{\Gamma_1', A \rightarrow B, \Gamma_2 \Rightarrow A \qquad \Gamma_1', B, \Gamma_2 \Rightarrow D}{\Gamma_1', A \rightarrow B, \Gamma_2 \Rightarrow D}\ \mathrm{L}\rightarrow$$

We now apply the induction hypothesis on the left premise in the following way:

- there is an interpolant $C_1$ such that $\vdash \Gamma_2 \Rightarrow C_1$ and $\vdash \Gamma_1', A \rightarrow B, C_1 \Rightarrow A$ with $\mathsf{Var}(C_1) \subseteq \mathsf{Var}(\Gamma_2) \cap \mathsf{Var}(\Gamma_1', A \rightarrow B, A)$.

For the second premise we get that

- there exists an interpolant $C_2$ for which $\vdash \Gamma_1', B \Rightarrow C_2$ and $\vdash \Gamma_2, C_2 \Rightarrow D$ with $\mathsf{Var}(C_2) \subseteq \mathsf{Var}(\Gamma_1', B) \cap \mathsf{Var}(\Gamma_2, D)$.

Take $C = C_1 \rightarrow C_2$. It is easily shown that the first requirement is fulfilled. For the second we have to show that $\Gamma_1', A \rightarrow B \Rightarrow C$ and $\Gamma_2, C \Rightarrow D$ are derivable. This is shown in the following derivation trees using the observations made before. Double lines indicate weakening:

$$\dfrac{\dfrac{\Gamma_2 \Rightarrow C_1}{\Gamma_2, C_1 \rightarrow C_2 \Rightarrow C_1} \qquad \Gamma_2, C_2 \Rightarrow D}{\Gamma_2, C_1 \rightarrow C_2 \Rightarrow D}\ \mathrm{L}\rightarrow$$

$$
\dfrac{\Gamma_1', A \to B, C_1 \Rightarrow A \qquad \dfrac{\Gamma_1', B \Rightarrow C_2}{\Gamma_1', B, C_1 \Rightarrow C_2}}{\dfrac{\Gamma_1', A \to B, C_1 \Rightarrow C_2}{\Gamma_1', A \to B \Rightarrow C_1 \to C_2}\, \mathrm{R}\to}\, \mathrm{L}\to
$$

Finally, we consider the rules $\mathcal{R}_{\mathrm{GL}}$. The proof for $\mathcal{R}_{\mathrm{SL}}$ is completely analogous. For $\mathcal{R}_{\mathrm{GL}}$ the inference looks as follows:

$$
\dfrac{\boxdot\Gamma_1, \boxdot\Gamma_2, \Box B \Rightarrow B}{\Sigma_1, \Box\Gamma_1, \Sigma_2, \Box\Gamma_2 \Rightarrow \Box B}\, \mathcal{R}_{\mathrm{GL}}
$$

We apply the induction hypothesis to the premise to obtain an interpolant $C'$ such that $\mathsf{Var}(C') \subseteq \mathsf{Var}(\boxdot\Gamma_1) \cap \mathsf{Var}(\boxdot\Gamma_2, B)$, and such that $\vdash_{\mathsf{G3iGL}} \boxdot\Gamma_1 \Rightarrow C'$ and $\vdash_{\mathsf{G3iGL}} \boxdot\Gamma_2, \Box B, C' \Rightarrow B$. Take $C = \Box C'$. Weakening of both sequents with $\Box C'$ results in $\vdash_{\mathsf{G3iGL}} \boxdot\Gamma_1, \Box C' \Rightarrow C'$ and $\vdash_{\mathsf{G3iGL}} \boxdot\Gamma_2, \Box B, C', \Box C' \Rightarrow B$. Now apply rule $\mathcal{R}_{\mathrm{GL}}$ to both to obtain the desired result, that is, $\vdash_{\mathsf{G3iGL}} \Sigma_1, \Box\Gamma_1 \Rightarrow C$ and $\vdash_{\mathsf{G3iGL}} \Sigma_2, \Box\Gamma_2, C \Rightarrow \Box B$ with $\mathsf{Var}(C) \subseteq \mathsf{Var}(\Sigma_1, \Box\Gamma_1) \cap \mathsf{Var}(\Sigma_2, \Box\Gamma_2, \Box B)$. ■

**3.2.2 Corollary**
Logics iGL and iSL have the Craig interpolation property.

## 3.3 Countermodel construction

We return to the Kripke semantics of iSL discussed earlier in Section 1.3.2. Different semantics for iSL are provided by Visser and Zoethout (2019), Ardeshir and Mojtahedi (2018), Litak (2014), and Litak and Visser (2018). Most of these results are based on a Henkin construction.

In this section we provide a countermodel construction using a terminating calculus for iSL, providing a new proof for folklore results. We show that iSL is complete with respect to the class of *strong conversely well-founded treelike models* (that may be infinite) and the class of *finite strong irreflexive treelike models.* Closely related are the works by Švejdar (2006) and Avron (1984), who provide countermodel constructions for G4ip for IPC, and a standard sequent calculus for GL, respectively.

To be more precise, recall Definition 1.3.4 for intuitionistic strong models, which are models of the form $(W, \leq, R, V)$ satisfying:

$$
(\mathsf{R}_\Box) \quad \leq; R \subseteq R, \qquad \text{and} \qquad (\mathsf{S}) \quad R \subseteq \leq .
$$

In addition we assume $W$ to be treelike with respect to $\leq$. For the infinite case we assume that $R$ is conversely well-founded. For the finite case, we assume $W$ to be finite and $R$ to be irreflexive.

The countermodel construction is based on a terminating proof system closely related to G4iSL that we call G4iSL′.

**3.3.1 Definition**
We define calculus G4iSL′ to be G4iSL, but where L $\to\!\!\to$ is replaced by the following rule:

$$\frac{\Gamma, B \to C, A \Rightarrow B \qquad \Gamma, C \Rightarrow D}{\Gamma, (A \to B) \to C \Rightarrow D} \; \mathrm{L} \to\!\!\to'$$

The only difference between G4iSL and G4iSL′ is another representation of rule L $\to\!\!\to$ that enforces an immediate backward application of R $\to$ to the left premise. This change is necessary in the proof of completeness. It is easy to see that the properties from Lemma 3.1.3 that hold for G4iSL also hold for G4iSL′. In addition, it is terminating modulo extended axioms similarly proved as Theorem 3.1.20.

For the countermodel construction we define the termination procedure modulo extended axioms in a more rigorous way imposing an order on rule applications following (Bílková, 2006) where we first apply invertible rules and after that the non-invertible rules. Note that all rules in G4iSL′ are invertible except for L $\to\!\!\to'$, R$\vee$, $\to_{\mathrm{SL}}$, and $\mathcal{R}^4_{\mathrm{SL}}$. To make this explicit, we introduce the following concept.

**3.3.2 Definition**
We call a sequent *saturated* in G4iSL′ if it is not an extended axiom and it cannot be the conclusion of an invertible rule from G4iSL′.

**3.3.3 Lemma**
A sequent of the form $\Pi, \Box\Gamma \Rightarrow C$ is saturated if it satisfies the following conditions:

(i) $C$ is of the form $p$, $\bot$, $A_1 \vee A_2$ or $\Box A$,
(ii) all formulas in $\Pi$ have the form $p, p \to A, (A \to B) \to C$ or $\Box A \to B$,
(iii) not both $p \in \Pi$ and $p \to A \in \Pi$,
(iv) not both $A \in \Pi \cup \Box\Gamma$ and $C = A$.

Proof. This is an easy observation by the form of the rules. ■

Given a sequent $S = (\Gamma \Rightarrow C)$, the *proof search tree* of $S$ is defined as follows. We create a tree whose nodes are labeled by sequents. The root is labeled by $(\Gamma \Rightarrow C)$ and we apply the rules backwards. By backwards applying a rule to a sequent that is a label of a node in the tree, we create predecessor node(s) and label them by the premise(s) of the rule. We first apply all invertible rules, in arbitrary order. We continue, until no invertible rule can be applied. If such a sequent is an extended axiom, then stop the search for that node. Otherwise, it is a saturated sequent and apply each possible non-invertible rule and create for each rule predecessor nodes of the node marked by the saturated sequent, and again label them by the premise(s) of the corresponding rule. Repeat the procedure for those nodes.

In addition we mark the sequents in the proof search tree as *positive* or *negative*. If a leave is labeled by an extended axiom or an instance of L$\bot$, mark it as *positive*. If not, mark it as *negative*. For other nodes, we move down the proof search tree marking nodes in the following way. A saturated sequent is marked as *positive* if for at least one backwards applied rule all its corresponding predecessors have been marked as positive. Non-saturated sequents are marked as *positive* if all its predecessors have been marked as positive.

Similarly to Theorem 3.1.20 we can prove the following.

**3.3.4 Theorem**
Sequent calculus $\mathsf{G4iSL}'$ terminates modulo extended axioms. In addition, according to the proof search described above, a sequent is provable in $\mathsf{G4iSL}'$ if it is marked as positive in the proof search tree.

Let us first prove soundness and then completeness. We show it for strong conversely well-founded treelike models. The exact same proof applies to finite strong irreflexive treelike models. Recall notation on models from Definition 1.2.30. In addition, for a multiset $\Gamma$ we write $K, w \Vdash \Gamma$ if $K, w \Vdash A$ for every $A \in \Gamma$. We say that sequent $\Gamma \Rightarrow C$ is *valid in model* $K$, denoted $K \models \Gamma \Rightarrow C$, if for all $w$, $K, w \Vdash \Gamma$ implies $K, w \Vdash C$. We say that $\Gamma \Rightarrow C$ is *refuted in* $K$ if it is not valid in $K$. And we say that sequent $\Gamma \Rightarrow C$ is *valid*, denoted $\models \Gamma \Rightarrow C$, if it is valid in all strong conversely well-founded treelike models. And we say that sequent $\Gamma \Rightarrow C$ is *refuted* if it is not valid.

**3.3.5 Theorem** (Soundness)
Calculus $\mathsf{G4iSL}'$ is sound with respect to the class of strong conversely well-founded treelike models. That is, for all sequents $S$, if $\vdash_{\mathsf{G4iSL}'} S$, then $\models S$.

Proof. As usual, we use induction on the height $d$ of the derivation of $S$. We consider a few steps. First consider $d = 0$. If $S$ has the form $(p, \Gamma \Rightarrow p)$ for some $p$, suppose $K, w \Vdash A$ for each $A \in \Gamma \cup \{p\}$. Then clearly $K, w \Vdash p$, thus $K \models p, \Gamma \Rightarrow p$. If $S$ has the form $(\bot, \Gamma \Rightarrow C)$, note that for each model $K$ we have $K, w \nVdash \bot$, hence $K \models \bot, \Gamma \Rightarrow C$.

Now consider $d > 0$. We treat four cases. First suppose the last rule applied is R$\wedge$, where $S$ has the form $(\Gamma \Rightarrow A \wedge B)$ with premises $(\Gamma \Rightarrow A)$ and $(\Gamma \Rightarrow B)$. Let $K$ be a model and $w \in K$ such that $K, w \Vdash \Gamma$. By the induction hypothesis we know $K, w \Vdash A$ and $K, w \Vdash B$, hence $K, w \Vdash A \wedge B$. Thus $K \models \Gamma \Rightarrow A \wedge B$.

Now assume the last inference is an instance of L $\rightarrow\!\!\!\!\rightarrow'$. So let $S$ be of the form

$(\Gamma, (A \to B) \to C \Rightarrow D)$, with premises $(\Gamma, B \to C, A \Rightarrow B)$ and $(\Gamma, C \Rightarrow D)$. Let $K$ be a model and $w \in K$ such that $K, w \Vdash \Gamma, (A \to B) \to C$. It follows from $K, w \Vdash (A \to B) \to C$ that $K, w \Vdash B \to C$. The induction hypothesis of the first premise gives us $\models \Gamma, B \to C, A \Rightarrow B$ and so $K, w \Vdash A \to B$ by monotonicity (Lemma 1.3.5). Since $K, w \Vdash (A \to B) \to C$ we have $K, w \Vdash C$. Hence, by the induction hypothesis applied to the second premise we have $K, w \Vdash D$. Therefore $\models \Gamma, (A \to B) \to C \Rightarrow D$.

For $\to_{\mathrm{SL}}$, write $S$ as $(\Pi, \Box\Gamma, \Box A \to B \Rightarrow C)$, with the premises of the rule $(\Pi, \boxdot\Gamma, \Box A, \Box A \to B \Rightarrow A)$ and $(\Pi, \Box\Gamma, B \Rightarrow C)$, where $\Pi$ does not contain boxed formulas. Suppose $K, w \Vdash \Pi, \Box\Gamma, \Box A \to B$. We show that this implies $K, w \Vdash B$ and therefore $K, w \Vdash C$ by the induction hypothesis of the second premise. Suppose for a contradiction that $K, w \nVdash B$. By $K, w \Vdash \Box A \to B$ we have $K, w \nVdash \Box A$. This implies the following reasoning:

- there exists $x_1$ such that $wRx_1$ and $K, x_1 \nVdash A$. Because $K, w \Vdash \Box\Gamma$, we have $K, x_1 \Vdash \Gamma$. By condition (S) on the models, we also have $w \leq x_1$ and therefore by monotonicity $K, x_1 \Vdash \Pi, \Box\Gamma, \Box A \to B$. So together we have $K, x_1 \Vdash \Pi, \boxdot\Gamma, \Box A \to B$ and $K, x_1 \nVdash A$. By induction hypothesis of the first premise, $K, x_1 \nVdash \Box A$.
- So, there exists $x_2$ such that $x_1 R x_2$ and $K, x_2 \nVdash A$. By the above reasoning: $K, x_2 \Vdash \Pi, \boxdot\Gamma, \Box A \to B$ and $K, x_2 \nVdash A$. So by the induction hypothesis we have $K, x_2 \nVdash \Box A$.
- So, there exists $x_3 \dots$

Thus we constructed an infinite sequence $wRx_1Rx_2Rx_3R\dots$. This is in contradiction with the fact that the we deal with conversely well-founded relation $R$. Hence we proved $K, w \Vdash B$, and hence $K, w \Vdash C$. So $\models \Pi, \Box\Gamma, \Box A \to B \Rightarrow C$.

The last rule we treat is $\mathcal{R}^4_{\mathrm{SL}}$. So we have $S = (\Pi, \Box\Gamma' \Rightarrow \Box A)$ with premise $(\Pi, \boxdot\Gamma', \Box A \Rightarrow A)$, where $\Pi$ does not contain boxed formulas. Suppose that $K, w \Vdash \Pi, \Box\Gamma'$. We would like to prove $K, w \Vdash \Box A$. Suppose for a contradiction that $K, w \nVdash \Box A$. Then by a similar reasoning as for rule $\to_{\mathrm{SL}}$ we obtain an infinite sequence $wRx_1Rx_2Rx_3\dots$. This is a contradiction, so $K, w \Vdash \Box A$. ■

In the proof of completeness we use the following lemma. Recall that L $\to\to'$, R$\vee$, $\to_{\mathrm{SL}}$, and $\mathcal{R}^4_{\mathrm{SL}}$ are the non-invertible rules for G4iSL$'$. All other rules are invertible.

### 3.3.6 Lemma

Let $K$ be a strong conversely well-founded treelike model with world $w$. For any invertible rule in G4iSL$'$ we have that whenever one of its premises is refuted by $w$ in $K$, the conclusion of the rule is also refuted by $w$ in $K$.

Proof. This is an easy calculation for each invertible rule. ■

**3.3.7 Theorem** (Completeness)
Calculus $\mathsf{G4iSL}'$ is complete with respect to the class of strong conversely well-founded treelike models. That is, for all sequents $S$, if $\models S$, then $\vdash_{\mathsf{G4iSL}'} S$.

Proof. We prove by contradiction that whenever there is no cutfree proof for $S$, then there is a strong conversely well-founded treelike model that refutes $S$. Recall that the modal relation $R$ should be transitive and irreflexive. We assume that $\nvdash_{\mathsf{G4iSL}'} S$, which means that $S$ is marked as negative in the proof search tree. We use an induction on the height $d$ of the finite proof search tree for $S$. We start with the leaves for which $d = 0$. For $d > 0$ we distinguish between saturated and non-saturated negative sequents. Recall that for each non-saturated sequent in the proof search tree marked as negative, there is *at least* one predecessor marked as negative. And for each saturated sequent marked as negative, *for all* backwards applied rules, *at least* one predecessor is marked as negative.

If $d = 0$, then $S = (\Gamma \Rightarrow C)$ is a saturated sequent with no possible backwards rule. It cannot be an extended axiom because $S$ is underivable. Together with Lemma 3.3.3, we see that formulas in $\Gamma$ have the form: $p$ with $p \neq C$, $p \to A$ with $p \notin \Gamma$, or $\Box A$. And $C = q$ with $q \notin \Gamma$ or $C = \bot$. We consider a one-world model $K = (W, \leq, R, V)$ with $W = \{w\}$, $\leq = \{(w, w)\}$, $R = \emptyset$, and $p \in V(w)$ iff $p \in \Gamma$. This is clearly a model of the right shape with $K, w \Vdash \Gamma$, but $K, w \nVdash C$.

Now suppose $d > 0$ and $S = (\Gamma \Rightarrow C)$ is not saturated. This means that the last rule applied is invertible. Sequent $S$ is marked as negative, so there is a premise $\Gamma' \Rightarrow C'$ marked as negative. This means that $\Gamma' \Rightarrow C'$ is underivable. By induction hypothesis, there is a countermodel $K$ such that $K \not\models \Gamma' \Rightarrow C'$. By Lemma 3.3.6, we know that the rule is also semantic invertible, so $K \not\models \Gamma \Rightarrow C$.

Now let $d > 0$ and suppose $S = (\Gamma \Rightarrow C)$ is saturated. This means that last rules applied in the proof search are non-invertible rules. We have different predecessor nodes of $S$, depending on the possible rules applied to $S$. For each applied rule, at least one premise is marked as negative. For each possible application of a non-invertible rule we have the following:

(i) (L $\to\to'$): $S$ has the form $(\Gamma'_i, (A_i \to B_i) \to C_i \Rightarrow D)$ with one of the sequents $(\Gamma'_i, B_i \to C_i, A_i \Rightarrow B_i)$ or $(\Gamma'_i, C_i \Rightarrow D)$ as negative predecessor in the proof search tree.
(ii) ($\to_{\mathrm{SL}}$): $S$ has the form $(\Pi_j, \Box\Gamma'_j, \Box A_j \to B_j \Rightarrow C_j)$ with one of the sequents $(\Pi_j, \boxdot\Gamma'_j, \Box A_j, \Box A_j \to B_j \Rightarrow A_j)$ or $(\Pi_j, \Box\Gamma'_j, B_j \Rightarrow C)$ as a negative predecessor.
(iii) ($\mathcal{R}^4_{\mathrm{SL}}$): $S$ has the form $(\Pi, \Box\Gamma' \Rightarrow \Box A)$ where premise $(\Pi, \boxdot\Gamma', \Box A \Rightarrow A)$ is negative,
(iv) (R$\vee$): $S$ has the form $(\Gamma \Rightarrow A \vee B)$ and both premises $(\Gamma \Rightarrow A)$ and $(\Gamma \Rightarrow B)$ are negative predecessors.

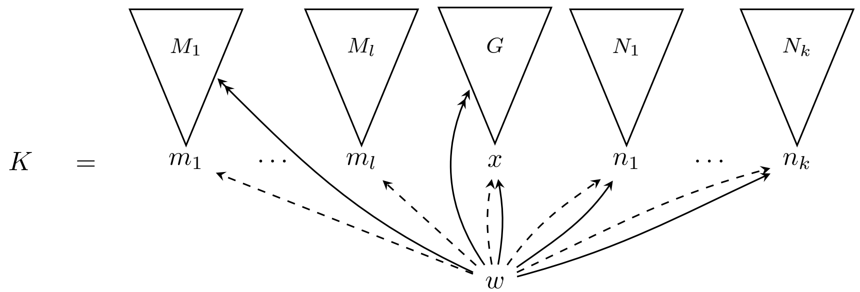
Note that case (iii) and (iv) cannot occur both at the same time. In the following we just say predecessor to mean negative predecessor, since those are the only of interest. We continue by analyzing two different possibilities.

First assume that at least one of the following occurs: either (i) with predecessor $(\Gamma'_i, C_i \Rightarrow D)$ for some $i$ or (ii) with predecessor $(\Pi_j, \Box\Gamma'_j, B_j \Rightarrow C)$ for some $j$. Those cases are treated in a similar way as the semantic invertible rules. In each case, we find a countermodel for the corresponding predecessor using the induction hypothesis. In both cases the found countermodel is also a countermodel for $S$.

Now assume that none of the previous cases occur. We apply the induction hypothesis to each of the applied rules of the concerned saturated sequent:

- In (i): for each $i$ we have predecessor $S_i = (\Gamma'_i, B_i \to C_i, A_i \Rightarrow B_i)$. So by induction hypothesis there exists for each $i$ a model $M'_i$ and world $m_i$ such that $M'_i, m_i \nVdash S_i$. Let $M_i$ be the submodel with root $m_i$, then $M_i, m_i \nVdash S_i$.
- In (ii): for each $j$, the predecessor is $S_j = (\Pi_j, \boxdot\Gamma'_j, \Box A_j, \Box A_j \to B_j \Rightarrow A_j)$. So by induction hypothesis there exists for each $j$ a model $N'_j$ and world $n_j$ such that $N'_j, n_j \nVdash S_j$. Let $N_j$ be the submodel with root $n_j$. So $N_j, n_j \nVdash S_j$.
- In (iii): we have predecessor $S' = (\Pi, \boxdot\Gamma', \Box A \Rightarrow A)$ and apply the induction hypothesis to this. So we have a model $G'$ and world $x$ such that $G', x \nVdash S'$. Let $G$ be the submodel with root $x$, then $G, x \nVdash S'$.
- In (iv): apply the induction hypothesis to both predecessors $S_1 = (\Gamma \Rightarrow A)$ and $S_2 = (\Gamma \Rightarrow B)$ to get two models $H_1, H_2$ with roots $h_1, h_2$ such that $H_1, h_1 \nVdash S_1$ and $H_2, h_2 \nVdash S_2$.

Sequent $S = (\Gamma \Rightarrow C)$ is saturated, so $C$ is a disjunction, boxed formula, a propositional variable, or $\bot$. The latter two are equivalent. When $C$ is a boxed formula, say $C = \Box A$, case (iii) occurs, but case (iv) does not. Then we construct the following model $K$. In this picture, relation $R$ should be understood as the transitive closure and whenever $v \leq yRz$ for some $v, y, z$ then $vRz$. Dashed arrows represent relation $\leq$ and the other arrows represent relation $R$. Double headed arrows are examples of $R$ relations defined by the closure conditions.



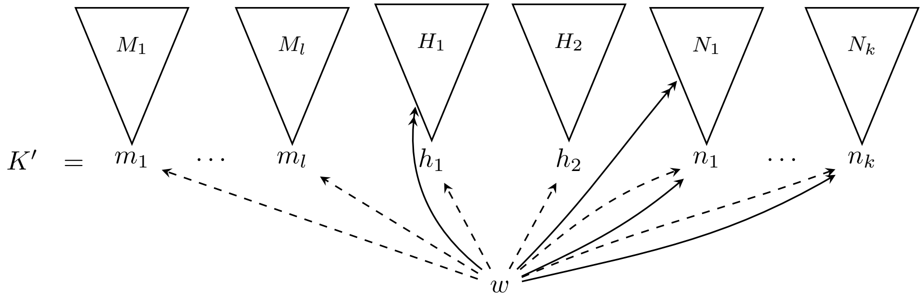

$w \Vdash p$ if $p \in \Gamma$

For each $y, v, z$ we have $y \leq vRz$ implies $yRv$, and $yRv$ implies $w \leq v$ by construction, so it satisfies the frame conditions ($\mathsf{R}_\Box$) and ($\mathsf{S}$). Relation $R$ is conversely well-founded by construction. Further note that the valuation in $K$ respects monotonicity, since $K, m_i \Vdash p$, $K, x \Vdash p$ and $K, n_j \Vdash p$ for each $p \in \Gamma$. This can be seen by inspection of the rules. So we can conclude that $K$ is a strong conversely well-founded treelike model.

We claim that $K$ refutes $S = (\Gamma \Rightarrow C)$, where $C = \Box A$. We show that $K, w \Vdash B$ for each $B \in \Gamma$ and $K, w \nVdash \Box A$. The latter follows directly by the induction assumption in (iii) implying $K, x \nVdash A$. For the first, recall that sequent $(\Gamma \Rightarrow \Box A)$ is a saturated sequent. We treat each $B \in \Gamma$ depending on its form (Lemma 3.3.3).

- For $\Box B \in \Gamma$, let $y$ such that $wRy$. We have to show that $K, y \Vdash B$. If $y \in M_i$, then $K, y \Vdash B$ since $m_i R y$ and $K, m_i \models \Box B$. Now consider $y \in G$. By construction of $R$, we find a finite sequence $y = z_1, z_2, \ldots, z_{n-1}, z_n = x$ such that $xRz_{n-1}R \ldots Rz_2Ry$. So $xRy$, because $R$ is transitive. By induction assumption (iii) we have $K, x \Vdash \Box B, B$. Hence $K, y \Vdash B$. For $y \in N_j$ we proceed by a similar argument.
- For $p \in \Gamma$, then $K, w \Vdash p$ by definition,
- For $p \to B \in \Gamma$, then $p \notin \Gamma$ because $S$ is saturated. Let $v \geq w$ and $K, v \Vdash p$. Since $p \notin \Gamma$ we have $v > w$ and therefore $v \geq m_i$ for some $i$, or $v \geq x$, or $v \geq n_j$ for some $j$. For each case we have that $p \to B$ is also present in the antecedent of the predecessor. So $K, m_i \Vdash p \to B$ for each $i$, $K, x \Vdash p \to B$, and $K, n_j \Vdash p \to B$ for each $j$. So indeed $K, v \Vdash B$.
- For $(A_{i'} \to B_{i'}) \to C_{i'} \in \Gamma$ for some $i'$. Let $v \geq w$ with $K, v \Vdash A_{i'} \to B_{i'}$. We treat different cases for $v$. For $v = w$, we use induction in (i) saying that $K, m_{i'} \nVdash B_{i'}$ and $K, m_{i'} \Vdash A_{i'}$. This implies $K, m_{i'} \nVdash A_{i'} \to B_{i'}$. By monotonicity, $K, w \nVdash A_{i'} \to B_{i'}$ which contradicts our assumption. So $v > w$ and again we have several possibilities: $v \geq m_{i'}$, $v \geq m_i$ for some $i \neq i'$, $v \geq x$, or $v \geq n_j$ for some $j$. For $v \geq m_{i'}$ we use $K, m_{i'} \Vdash B_{i'} \to C_{i'}$, $K, m_{i'} \Vdash A_{i'}$ and $K, m_{i'} \nVdash B_{i'}$. By monotonicity, $K, v \Vdash B_{i'} \to C_{i'}$ and $K, v \Vdash A_{i'}$. Using assumption $K, v \Vdash A_{i'} \to B_{i'}$ gives $K, v \Vdash C_{i'}$. For all other cases of $v$ we can use the fact that $(A_{i'} \to B_{i'}) \to C_{i'}$ stays present in the predecessor of the rule. So we can conclude $K, m_i \Vdash (A_{i'} \to B_{i'}) \to C_{i'}$ for each $i \neq i'$, $K, x \Vdash (A_{i'} \to B_{i'}) \to C_{i'}$, and $K, n_j \Vdash (A_{i'} \to B_{i'}) \to C_{i'}$ for each $j$. This implies $K, v \Vdash C_{i'}$.
- For $\Box A_{j'} \to B_{j'} \in \Gamma$ for some $j'$, let $v \geq w$ and $K, v \Vdash \Box A_{j'}$. We see that $v \neq w$, because suppose $v = w$, we have by induction in (ii) $K, n_{j'} \nVdash A_{j'}$, hence $K, w \nVdash \Box A_{j'}$ which contradicts our assumption. So $v \geq m_i$ for some $i$, $v \geq x$ or $v \geq n_j$ for some $j$. For all those cases we have $\Box A_{j'} \to B_{j'}$. This is also true for $v \geq n_{j'}$ by keeping $\Box A_{j'} \to B_{j'}$ in the premise of rule $\to_{\mathrm{SL},1}$.

To conclude, we found a strong conversely well-founded treelike model $K$ such that $K \not\models S$, where $S = (\Gamma \Rightarrow C)$ with $C = \Box A$. When $C$ is a disjunction $A \vee B$, case (iv) occurs, but case (iii) does not. In this case we construct the following countermodel $K'$, defined in a similar way as before.




$w \Vdash p$ if $p \in \Gamma$

The proof is similar as before. When $C$ is a propositional variable $p$ or $\bot$ we only create the model using $M_i$ and $N_j$, because both case (iii) and (iv) do not occur. ■

**3.3.8 Corollary**
The contraction rule and the cut rule are admissible in G4iSL′.

**3.3.9 Corollary**
We have the following completeness results:

1. Logic iSL is complete with respect to the class of strong conversely well-founded treelike models.
2. Logic iSL is complete with respect to the class of finite strong irreflexive treelike models.

## 3.4 Logics iK4 and iS4

Ghilardi and Zawadowski (1995) prove that classical modal logic S4 does not have uniform interpolation by providing an explicit counterexample. Using a translation from S4 to K4 and the fact that K4 is a subsystem of S4, Bílková (2006) concludes that K4 does not enjoy the uniform interpolation property either. Here we use similar translations, but now from K4 into iK4 and S4 into iS4.

Recall Section 1.2.3 about the negative translations from classical modal logics into intuitionistic modal logics studied in (Litak *et al.*, 2017). Definition 1.2.41 gives the Kuroda's translation $(\cdot)^{\mathsf{kur}}$ of formulas. Recall from Theorem 1.2.42 that we have

$$\vdash_{\mathsf{K4}} A \leftrightarrow A^{\mathsf{kur}} \quad \text{and} \quad \vdash_{\mathsf{S4}} A \leftrightarrow A^{\mathsf{kur}}, \tag{3.3}$$

$$\vdash_{\mathsf{K4}} A \ \text{ iff } \vdash_{\mathsf{iK4}} A^{\mathsf{kur}} \quad \text{and} \quad \vdash_{\mathsf{S4}} A \ \text{ iff } \vdash_{\mathsf{iS4}} A^{\mathsf{kur}}. \tag{3.4}$$

**3.4.1 Lemma** (Ghilardi and Zawadowski, 1995)
There is a formula $B$ with $\mathsf{Var}(B) = \{p_1, p_2, q\}$ which does not have a uniform interpolant $\dot{\exists} p_1 \dot{\exists} p_2 B$ in $\mathsf{S4}$. That is, there is no formula $\dot{\exists} p_1 \dot{\exists} p_2 B$ such that

(i) $\mathsf{Var}(\dot{\exists} p_1 \dot{\exists} p_2 B) \subseteq \mathsf{Var}(B) \setminus \{p_1, p_2\}$,
(ii) $\vdash_{\mathsf{S4}} B \to \dot{\exists} p_1 \dot{\exists} p_2 B$,
(iii) for each formula $C$ with $p_1, p_2 \notin \mathsf{Var}(C)$:

$$\vdash_{\mathsf{S4}} B \to C \text{ implies } \vdash_{\mathsf{S4}} \dot{\exists} p_1 \dot{\exists} p_2 B \to C.$$

This means that $\exists p_1 \exists p_2 B$ cannot be simulated in $\mathsf{S4}$ for that particular $B$.

**3.4.2 Corollary** (Bílková, 2006)
There is a formula $B$ with $\mathsf{Var}(B) = \{p_1, p_2, q\}$ which does not have a uniform interpolant $\dot{\exists} p_1 \dot{\exists} p_2 B$ in $\mathsf{K4}$.

**3.4.3 Corollary**
There is a formula $B$ with $\mathsf{Var}(B) = \{p_1, p_2, q\}$ which does not have a uniform interpolant $\dot{\exists} p_1 \dot{\exists} p_2 B$ in $\mathsf{iS4}$.

Proof. Consider formula $B$ from Lemma 3.4.1 and suppose for a contradiction that $\mathsf{iS4}$ has uniform interpolation. So for $B^{\mathsf{kur}}$ there exists $\dot{\exists} p_1 \dot{\exists} p_2 B^{\mathsf{kur}}$ such that the three properties from Lemma 3.4.1 do hold for $\mathsf{iS4}$. We will show that it would then also be a uniform interpolant of $B$ in $\mathsf{S4}$, a contradiction.

Of course, (i) is satisfied. Since $\mathsf{iS4} \subseteq \mathsf{S4}$, it follows that $\vdash_{\mathsf{S4}} B^{\mathsf{kur}} \to \dot{\exists} p_1 \dot{\exists} p_2 B^{\mathsf{kur}}$. By property (3.3) applied to $B$ we have $\vdash_{\mathsf{S4}} B \to \dot{\exists} p_1 \dot{\exists} p_2 B^{\mathsf{kur}}$ and so (ii) is true. For (iii), let $C$ be a formula with $p_1, p_2 \notin \mathsf{Var}(C)$ and suppose $\vdash_{\mathsf{S4}} B \to C$. By (3.4) we have $\vdash_{\mathsf{iS4}} (B \to C)^{\mathsf{kur}}$ and by definition $\vdash_{\mathsf{iS4}} B^{\mathsf{kur}} \to C^{\mathsf{kur}}$. Formula $C^{\mathsf{kur}}$ does also not contain $p_1, p_2$, so since we assumed $\mathsf{iS4}$ to have uniform interpolation, it holds that $\vdash_{\mathsf{iS4}} \dot{\exists} p_1 \dot{\exists} p_2 B^{\mathsf{kur}} \to C^{\mathsf{kur}}$. So $\vdash_{\mathsf{S4}} \dot{\exists} p_1 \dot{\exists} p_2 B^{\mathsf{kur}} \to C^{\mathsf{kur}}$, because $\mathsf{iS4} \subseteq \mathsf{S4}$. Again using (3.3) we obtain $\vdash_{\mathsf{S4}} \dot{\exists} p_1 \dot{\exists} p_2 B^{\mathsf{kur}} \to C$ showing property (iii). This is in contradiction with Lemma 3.4.1. ∎

By the exact same proof one can show that uniform interpolation also fails for $\mathsf{iK4}$.

**3.4.4 Corollary**
There is a formula $B$ with $\mathsf{Var}(B) = \{p_1, p_2, q\}$ which does not have a uniform interpolant $\dot{\exists} p_1 \dot{\exists} p_2 B$ in $\mathsf{iK4}$.

## 3.5 Conclusion

The admissibility of the cut rule is a central theme and we provide different proofs for different systems for iGL and iSL. We introduce several sequent calculi, the four main ones are G3iGL, G3iSL, G4iGL and G4iSL, and one other, G4iSL′, a variant of G4iSL. One of the main results is the syntactic cut-admissibility proof for G3iGL and G3iSL. What is especially interesting is that our cut-admissibility proof for G3iGL and G3iSL highly depends on the structure of the calculus and makes this proof strategy fail for G4iGL and G4iSL.

The interest of systems G4iGL and G4iSL lies in their termination proved in this chapter. A constructive proof of their equivalence to G3iGL and G3iSL, respectively, is provided, showing that cut is admissible in G4iGL and G4iSL.

We defined G4iSL′, a variant of G4iSL, and use its termination to provide a countermodel construction providing a semantic proof of cut-admissibility. We expect that a similar construction would work for iGL. Goré *et al.* (2021) use a terminating calculus for GL to provide a syntactic proof (checked in Coq) of cut-elimination closely related to semantic completeness proofs. They specifically put forward the idea of exploring a similar technique for iGL, for which our systems could be used.

The main reason for the development of the terminating calculi is to prove uniform interpolation for the logics. Note that for iSL, it is almost certain that the uniform interpolation property holds, because a semantic proof is proposed in the unpublished manuscript by Litak and Visser (2020). For iGL the problem is open. We leave this question for further research. Here we established the Craig interpolation theorems for iGL and iSL.

Like iGL and iSL, logics mHC and KM have connections to provability (see Section 1.3.2). It would be good to see whether similar sequent calculi can be developed for these logics. The only calculus that we are aware of is for KM by Darjania (1984).

On a technical note, one might ask whether the following rules are sound and complete for iSL, where we drop each $\boxdot$, since $\boxdot C \leftrightarrow C$ for each formula $C$ by coreflection.

$$\frac{\Pi, \Gamma, \Box A \Rightarrow A}{\Box\Sigma, \Pi, \Box\Gamma \Rightarrow \Box A}\,\mathcal{R}'_{\mathrm{SL}} \qquad \frac{\Pi, \Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}\,\mathcal{R}^{4'}_{\mathrm{SL}}$$

$$\frac{\Pi, \Gamma, \Box A \Rightarrow A \qquad \Pi, \Box\Gamma, B \Rightarrow C}{\Pi, \Box\Gamma, \Box A \to B \Rightarrow C}\,\mathrm{L} \to'_{\mathrm{SL}}$$

The current order to show termination with rule $\mathcal{R}^4_{\mathrm{SL}}$ relies on the fact that boxed formulas do not disappear in the antecedents by bottom-up applications of the rules. This crucial fact is not true for $\mathcal{R}^{4'}_{\mathrm{SL}}$. As formula $A$ is 'duplicated' in the premise, the question is with what order the system with this rule would terminate.

Finally, we proved that iK4 and iS4 do not have uniform interpolation. This is in line with the fact that the method of turning G3 systems into G4 systems in (Iemhoff, 2020) is tight to the modal rules, and cannot for instance work for logics iK4 and iS4.[20] In fact, the lack of uniform interpolation, implies the negative result that these logics do not have a terminating sequent calculus satisfying conditions from (Iemhoff, 2019b).

[20] In (Iemhoff, 2020), there is an error in the presentation of rule $\mathcal{R}_T$ on page 3, in which $\Box\varphi$ should also be present in the premise. This shows that the general method does not work for iT, but needs additional treatment analogously to the terminating sequent calculus for T from (Bílková, 2006).

# 4

# Uniform Interpolation via Multicomponent Sequents

As we explained in Section 2.1.3, proof-theoretic methods establishing the uniform interpolation for a logic can be very successful. In particular, cut-free terminating sequent calculi are suitable to explicitly construct uniform interpolants.

The aim of this chapter is to widen the scope to other proof formalisms that generalize the ordinary sequent calculus. As mentioned in the Introduction, diverse proof systems are invented to incorporate extra structure on ordinary sequents leading to systems with for example *hypersequents*, *nested sequents*, and *labelled sequents*. The first two are examples of what Kuznets (2018) calls *multicomponent* sequents in which individual ordinary sequents are combined in a specific way to define the new structures.[21]

These proof formalisms have recently been used to show the Craig interpolation property for certain logics as discussed in Section 2.1.3. In this chapter we rely on (Kuznets, 2018) that presents a modular proof-theoretic framework to show Craig interpolation for classical modal logics via multicomponent sequents and labelled calculi based on earlier publications (Fitting and Kuznets, 2015) and (Kuznets, 2016). The method combines syntactic and semantic reasoning. Generalized Craig interpolants are constructed using the calculus in a purely syntactic manner, but the method's correctness uses semantic notions of Kripke models of the underlying logic.

[21] Hypersequent and nested sequent systems are also examples of *internal* systems whose expressions correspond to formulas. This in contrast to the labelled systems that form an example of *external* systems that truly enrich the language.

We focus on multicomponent sequent calculi in classical modal logic. To be specific, we prove uniform interpolation for $\mathsf{K}$, $\mathsf{T}$, and $\mathsf{D}$ via nested sequents, and for $\mathsf{S5}$ via hypersequents. It has been known that these logics have uniform interpolation, but we provide a new method that can hopefully be extended to other logics. Similar to (Kuznets, 2018), we combine syntactic and semantic reasoning. We use terminating calculi to define the uniform interpolants and then provide model modifications and use bisimulations to prove the correctness of these interpolants.

Although, the method that we present is not purely proof-theoretic, it reveals the close interplay between the constructive definitions of the uniform interpolants and the semantic bisimulation quantifiers. This is in contrast to the proof-theoretic proofs using ordinary sequent calculi for $\mathsf{K}$, $\mathsf{T}$ (Bílková, 2006) and for $\mathsf{D}$ (Iemhoff, 2019a). In addition, although proving uniform interpolation for $\mathsf{S5}$ is simple (see Lemma 2.2.9), we provide a first direct construction of the uniform interpolants in $\mathsf{S5}$.

The work in this chapter is joint work with Raheleh Jalali and Roman Kuznets. The method for $\mathsf{K}$, $\mathsf{T}$, and $\mathsf{D}$ is published in (van der Giessen *et al.*, 2021a). The work on $\mathsf{S5}$ is provided in the follow-up (van der Giessen *et al.*, 2022). Section 4.1 introduces the nested sequents and hypersequents that we use. Section 4.2 presents the method for $\mathsf{K}$, $\mathsf{T}$, and $\mathsf{D}$ and Section 4.3 for $\mathsf{S5}$. We end with a conclusion in Section 4.4.

## 4.1 Multicomponent sequents

In this chapter we introduce nested sequent calculi for $\mathsf{K}$, $\mathsf{D}$, and $\mathsf{T}$, and a hypersequent system for $\mathsf{S5}$. Following (Brünnler, 2009), we use a different language for modal logics than the one we use in the rest of this thesis, which is denoted $\mathcal{L}$ and defined in Section 1.1.

Let $\mathcal{L}_c$ denote the *classical modal language* consisting of countably many *(propositional) variables* $p_1, p_2, \ldots,$ *constants* $\bot$ *and* $\top$, *connectives* $\wedge$, $\vee$, and *modal operators* $\Box$ and $\Diamond$. The difference between language $\mathcal{L}_c$ and $\mathcal{L}$ is that $\top$ and $\Diamond$ are primitively defined, whereas $\to$ is not part of the language anymore. We again denote by $\mathsf{Prop}$ the countable set of propositional variables. We consider modal formulas in *negation normal form* defined by the following grammar:

$$A ::= p \mid \neg p \mid \bot \mid \top \mid (A \wedge A) \mid (A \vee A) \mid (\Box A) \mid (\Diamond A),$$

where $\neg p$ is the negation of $p$ for each $p \in \mathsf{Prop}$. The set $\mathsf{Lit}$ of *literals* consists of all propositional variables and their negations, with $\ell$ used to denote its elements.

Given formulas $A$ and $B$, we define $\neg A$ recursively as usual using De Morgan's laws to push the negation inwards and define the implication as

$$A \to B := \neg A \lor B.$$

For a multiset $\Gamma$, in addition to $\Box\Gamma$ which is defined at the beginning of Section 1.1, we define

$$\Diamond\Gamma := \{\Diamond A \mid A \in \Gamma\}.$$

In the previous chapter, we worked with ordinary two-sided sequents of the form $\Gamma \Rightarrow \Delta$ with formula interpretation $\bigwedge \Gamma \to \bigvee \Delta$. Using classical reasoning, this is equivalent to $\bigvee_{A\in\Gamma} \neg A \lor \bigvee \Delta$. This amounts to one-sided sequents.

**4.1.1 Definition**
An *ordinary one-sided sequents* is a multisets of formulas, say $\Gamma$, and its formula interpretation is $\bigvee \Gamma$.

These are the sequents on which we base the nested sequents and hypersequents in the next sections. In the following we let Greek letters $\Gamma, \Delta, \ldots$ range over one-sided sequents and simply say sequent. Be careful that we also use $\Gamma, \Delta, \ldots$ to range over nested sequents in the following section, but it will not lead to any confusion as nested sequents generalize sequents.

### 4.1.1 Nested sequents

We introduce the nested sequent calculus from (Brünnler, 2009). In this section we let $\mathsf{L}$ range over $\mathsf{K}$, $\mathsf{D}$, and $\mathsf{T}$.

**4.1.2 Definition** (Nested sequent)
A *nested sequents* $\Gamma$ are recursively defined in the following form:

$$A_1, \ldots, A_n, [\Gamma_1], \ldots, [\Gamma_m]$$

is a nested sequent where $A_1, \ldots, A_n$ are modal formulas for $n \geq 0$ and $\Gamma_1, \ldots, \Gamma_m$ are nested sequents for $m \geq 0$. We call brackets [ ] a *structural box*. The *formula interpretation* $I$ of a nested sequent is defined recursively by

$$I(A_1, \ldots, A_n, [\Gamma_1], \ldots, [\Gamma_m]) := A_1 \lor \cdots \lor A_n \lor \Box I(\Gamma_1) \lor \cdots \lor \Box I(\Gamma_m).$$

As for formulas, we let $\mathsf{Var}(\Gamma) \subseteq \mathsf{Prop}$ denote the propositional variables occurring in the nested sequent $\Gamma$.

One way of looking at a nested sequent is to consider a tree of ordinary one-sided sequents. Each structural box in the nested sequent creates a child in the tree. In order to be able to reason about formulas in a particular tree node, we introduce labels.

### 4.1.3 Definition

A *label* is a finite sequence of natural numbers. We denote labels by $\overline{s}, \overline{t}, \dots$, and a label $\overline{s} * n$ denotes the label $\overline{s}$ extended by the natural number $n$.

We usually write $\overline{s}n$ instead of $\overline{s} * n$, unless it is ambiguous, as, e.g., for $1 * 2 * 3$, which is different from $1 * 23$. The labeled nested sequents defined in the following definition are closely related to labelled sequents from (Negri and von Plato, 2011) but retain the nested notation (see Goré and Ramanayake, 2012a).

### 4.1.4 Definition (Labeling)

For a nested sequent $\Gamma$ and label $\overline{s}$ we define a *labeling function* $l_{\overline{s}}$ to recursively label structural boxes in nested sequents as follows:

$$l_{\overline{s}}(A_1, \dots, A_n, [\Gamma_1], \dots, [\Gamma_m]) := A_1, \dots, A_n, [l_{\overline{s}1}(\Gamma_1)]_{\overline{s}1}, \dots, [l_{\overline{s}m}(\Gamma_m)]_{\overline{s}m}.$$

Let $\mathfrak{L}_{\overline{s}}(\Gamma)$ be the set of labels occurring in $l_{\overline{s}}(\Gamma)$ plus label $\overline{s}$ (for formulas outside all structural boxes). Define the labeled nested sequent $l(\Gamma) := l_1(\Gamma)$, and let $\mathfrak{L}(\Gamma) := \mathfrak{L}_1(\Gamma)$. Formulas in a nested sequent $\Gamma$ are *labeled* according to the labeling of the structural boxes containing them. We write $1 : A \in \Gamma$ if the formula $A$ occurs in $\Gamma$ outside all structural boxes. Otherwise, $\overline{s} : A \in \Gamma$ whenever $A$ occurs in $l(\Gamma)$ within a structural box labeled $\overline{s}$.

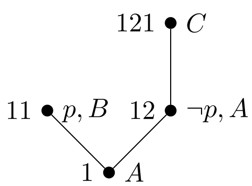
The set $\mathfrak{L}(\Gamma)$ can be considered as the set of nodes of the corresponding tree of $\Gamma$, with 1 being the root of this tree. Often, we do not distinguish between a nested sequent $\Gamma$ and its labeled sequent $l(\Gamma)$.

### 4.1.5 Example

Consider a nested sequent $\Gamma = A, [p, B], \big[\neg p, A, [C]\big]$. The corresponding labeled nested sequent is written below, where we write 11, 12, and 121 for $1 * 1$, $1 * 2$, and $1 * 2 * 1$, respectively:

$$l(\Gamma) = A, [p, B]_{11}, \big[\neg p, A, [C]_{121}\big]_{12}.$$

So $\mathfrak{L}(\Gamma) = \{1, 11, 12, 121\}$. The corresponding tree is pictured as follows, where each node is labeled on the left and marked by its formulas on the right (in particular, here $1 : A \in \Gamma$ and $121 : C \in \Gamma$, but $12 : C \notin \Gamma$):



$$\frac{}{\Gamma\{p, \neg p\}}\ \mathsf{id_P} \qquad \frac{}{\Gamma\{\top\}}\ \mathsf{id_\top} \qquad \frac{\Gamma\{A \vee B, A, B\}}{\Gamma\{A \vee B\}}\ \vee$$

$$\frac{\Gamma\{A \wedge B, A\} \qquad \Gamma\{A \wedge B, B\}}{\Gamma\{A \wedge B\}}\ \wedge \qquad \frac{\Gamma\{\Box A, [A]\}}{\Gamma\{\Box A\}}\ \Box$$

$$\frac{\Gamma\{\Diamond A, [\Delta, A]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ \mathsf{k} \qquad \frac{\Gamma\{\Diamond A, [A]\}}{\Gamma\{\Diamond A\}}\ \mathsf{d} \qquad \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}}\ \mathsf{t}$$

**Figure 4.1.** Terminating nested rules: the principal formula is not saturated.

Following (Brünnler, 2009), we will work with contexts in rules to signify that the rules can be applied in an arbitrary node of the nested sequent. We will work with *unary contexts* which are nested sequents with exactly one *hole*, denoted by the symbol $\{\ \}$. Such contexts are denoted by $\Gamma\{\ \}$. The insertion $\Gamma\{\Delta\}$ of a nested sequent $\Delta$ into a context $\Gamma\{\ \}$ is obtained by replacing the occurrence $\{\ \}$ with $\Delta$. The hole $\{\ \}$ can be labeled the same way as formulas. We write $\Gamma\{\ \}_{\overline{s}}$ to denote the label of the hole.

**4.1.6 Example**
Consider $\Gamma'\{\ \} = A, [p, B], [\neg p, \{\ \}]$. This is a context and its labeled context is

$$\Gamma'\{\ \}_{12} = A, [p, B]_{11}, [\neg p, \{\ \}]_{12}.$$

Let $\Delta = A, [C]$. Then $\Gamma'\{\Delta\}$ equals $\Gamma$ from Example 4.1.5.

We define the nested calculi which are extensions of the multiset-based version from (Brünnler, 2009) to the language with constants $\bot$ and $\top$, necessitating an addition of the rule $\mathsf{id_\top}$ for handling these (cf. the treatment of constants in Fitting and Kuznets (2015)).

**4.1.7 Definition**
The nested sequent calculus NK consists of all rules in the first two rows in Figure 4.1 plus the rule k. The nested calculus ND is obtained by adding to NK the rule d, and likewise, nested calculus NT is obtained by adding to NK the rule t.

In the rules, the *principal* formula is defined as usual. The nested sequent calculi NK, ND, and NT are sound and complete for modal logics K, D, and T, respectively, as stated in the following theorem.

**4.1.8 Theorem** (Brünnler, 2009)
Nested sequent $\Gamma$ is provable in $\mathsf{NL}$ if and only if its formula interpretation $I(\Gamma)$ is derivable in $\mathsf{L}$.

The rules in Figure 4.1 with embedded contraction are sometimes called Kleene'd rules. Following (Brünnler, 2009), in order to ensure termination, we only apply a rule when the principal formula in the conclusion is not saturated as defined below. In case of rule $\mathsf{k}$ it means it is not saturated w.r.t. to the label of the structural box containing $\Delta$.

**4.1.9 Definition** (Saturation)
Consider a nested sequent $\Gamma = \Gamma'\{C\}_{\overline{s}}$, i.e., $\overline{s} : C \in \Gamma$. The formula $C$ is called $\mathsf{K}$-*saturated in* $\Gamma$ if the following conditions hold depending on the form of $C$:

- $C$ is $\bot$, $\top$, or $p$ or $\neg p$ for some $p \in \mathsf{Prop}$;
- if $C = A \vee B$, then both $\overline{s} : A \in \Gamma$ and $\overline{s} : B \in \Gamma$;
- if $C = A \wedge B$, then either $\overline{s} : A \in \Gamma$ or $\overline{s} : B \in \Gamma$;
- if $C = \Box A$, then there is a label $\overline{s}n \in \mathfrak{L}(\Gamma)$ such that $\overline{s}n : A \in \Gamma$.

The formula $C$ of the form $\Diamond A$ is

- $\mathsf{K}$-*saturated in* $\Gamma$ *w.r.t.* $\overline{s}n \in \mathfrak{L}(\Gamma)$ if $\overline{s}n : A \in \Gamma$;
- $\mathsf{D}$-*saturated in* $\Gamma$ if there is a label $\overline{s}n \in \mathfrak{L}(\Gamma)$;
- $\mathsf{T}$-*saturated in* $\Gamma$ if $\overline{s} : A \in \Gamma$.

A nested sequent $\Gamma$ is called $\mathsf{K}$-*saturated* if

(i) it is neither of the form $\Gamma'\{p, \neg p\}$ for some $p \in \mathsf{Prop}$ nor of the form $\Gamma'\{\top\}$;
(ii) all its formulas $\overline{s} : \Diamond A$ are $\mathsf{K}$-saturated w.r.t. every child of $\overline{s}$; and
(iii) all its other formulas are $\mathsf{K}$-saturated in $\Gamma$.

A nested sequent $\Gamma$ is $\mathsf{D}$-*saturated* ($\mathsf{T}$-saturated) if it is $\mathsf{K}$-saturated and all its labeled formulas $\overline{s} : \Diamond A$ are $\mathsf{D}$-saturated ($\mathsf{T}$-saturated) in $\Gamma$.

**4.1.10 Example**
The nested sequent $\Gamma = [\Diamond A]_{1*1}$ is $\mathsf{K}$-saturated but it is neither $\mathsf{D}$-saturated nor $\mathsf{T}$-saturated. Indeed, for the logic $\mathsf{D}$ we would need $1 * 1 * n : A$ to be present for some $n$ and for $\mathsf{T}$ we would need to have $1{*}1 : A$ in order to saturate $1{*}1 : \Diamond A \in \Gamma$.

We formally prove strong termination (see Definition 2.3.9) in the next theorem. Such a proof is not provided in (van der Giessen *et al.*, 2021a) on which this chapter is based. Let us first define some terminology. Recall Definition 1.2.25 for the modal degree of a formula. In the new language $\mathcal{L}_c$, we change the definition as follows.

**4.1.11 Definition** (Modal degree)
The *modal degree* $md(A)$ of a formula $A$ in negation normal form is defined recursively as follows:

$$\begin{aligned} md(p) = md(\neg p) &= 0, \text{ for } p \in \mathsf{Prop}; \\ md(\bot) = md(\top) &= 0; \\ md(A \wedge B) = md(A \vee B) &= \max(md(A), md(B)); \\ md(\Box A) = md(\Diamond A) &= md(A) + 1. \end{aligned}$$

**4.1.12 Definition**
For label $\overline{s}$ we denote by $|\overline{s}|$ the *length* of $\overline{s}$. Let $\Gamma$ be a nested sequent. The *depth* of $\Gamma$, denoted $dp(\Gamma)$, is the maximum length among its labels. Define the *maximal depth* of $\Gamma$ as

$$h(\Gamma) = \max_{\overline{s}:A \in \Gamma} \{md(A) + |\overline{s}|\}.$$

For each $\overline{s} \in \mathfrak{L}(\Gamma)$ let $c_{\overline{s}}(\Gamma)$ denote the number of labels of the form $\overline{s}n \in \mathfrak{L}(\Gamma)$. The *size* of $\Gamma$, denoted $s(\Gamma)$ is the number of nodes of its corresponding tree. The *weight* of a node $\overline{s}$ in $\Gamma$, denoted $w(\overline{s})$, is the number of formulas in $\overline{s}$ counted as a set. Define the *weight* of $\Gamma$ as

$$w(\Gamma) = \sum_{\overline{s} \in \Gamma} w(\overline{s}).$$

Let $sf(\Gamma)$ be the number of all subformulas occurring in $\Gamma$ counted as a set.

In words, $|\overline{s}|$ presents the depth of $\overline{s}$ in the nested sequent tree, $c_{\overline{s}}(\Gamma)$ presents the number of children of $\overline{s}$ in $\Gamma$, and $h(\Gamma)$ presents the maximum depth of formulas occurring in $\Gamma$ in terms of the modal degree plus the depth of structural boxes.

**4.1.13 Theorem** (Brünnler, 2009)
The nested sequent calculi $\mathsf{NK}, \mathsf{ND}$, and $\mathsf{NT}$ are strongly terminating.

Proof. Consider a proof search for nested sequent $\Gamma$. We show that the proof search terminates by increasing weight. First observe that, bottom-up, the weight indeed strictly increases for each rule, because we only apply the rules if the principal formula is not saturated. To ensure termination, we show that the weights of the nested sequents $\Delta$ occurring in the proof search are bounded. Once we know that the size of all such $\Delta$'s is bounded by say $r$, then the weight is bounded by $r \cdot sf(\Gamma)$.

Let $\Delta$ be a nested sequent in the proof search. Let $h = h(\Gamma)$. The depth $dp(\Delta)$ is bounded by $h$. This follows from the fact that each premise in any of the rules has the same maximal depth as its conclusion. So $h(\Delta) = h$, hence $dp(\Delta) \leq h$. Now we show that branching in the tree is also bounded. For each $\overline{s} \in \mathfrak{L}(\Delta)$ we show

$$c_{\overline{s}}(\Delta) \leq \begin{cases} sf(\Gamma) + c_{\overline{s}}(\Gamma), & \text{if } \overline{s} \in \mathfrak{L}(\Gamma), \\ sf(\Gamma), & \text{if } \overline{s} \notin \mathfrak{L}(\Gamma). \end{cases}$$

A child $\overline{s}n$ can only be added to $\Delta$ by the $\Box$-rule or rule $\mathsf{d}$. And since rules can only be applied if the principal formula is not saturated, $\overline{s}n$ contains a formula $A$ that is not present in any other $\overline{s}m \in \mathfrak{L}(\Delta)$. For any rule applied bottom-up, no formula can disappear in any node of the nested sequent. So for $\overline{s}$ we can only apply the $\Box$-rule and rule $\mathsf{d}$ at most once for any formula $\Box A$ or $\Diamond A$. Since the number of possible formulas in the proof search tree is bounded by $sf(\Gamma)$ one can only add at most $sf(\Gamma)$ new children of $\overline{s}$ (to the possibly children already present in $\Gamma$). Summarizing, the depth $dp(\Delta)$ is bounded and the number of children $c_{\overline{s}}(\Delta)$ for each $\overline{s} \in \mathfrak{L}(\Delta)$ is bounded, hence the size $s(\Delta)$ is bounded. ■

#### 4.1.14 Remark

Our proof of termination is different from the one provided by Brünnler (2009). The main difference is that we do not rely on what we call a 'check on tree nodes' where the proof search can be stopped for nested sequents that contain two nodes representing the same ordinary sequent. Our proof search ends with leaves that are either non-derivable and saturated, or derivable and of the form $\Gamma\{\top\}$ or $\Gamma\{p, \neg p\}$ for some $p \in \mathsf{Prop}$. It is important to mention that this argument does *not* hold for the nested sequent calculi for logics like $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{S5}$ from (Brünnler, 2009).

Intuitively, nested sequents capture the tree structure of Kripke models.

#### 4.1.15 Convention

In this chapter we rely on the completeness theorem with respect to *intransitive treelike* models (Theorem 1.2.13). By Remark 1.2.4, there can possibly be reflexive worlds. Taking into account Remark 1.2.14, in this chapter we call such models $\mathsf{K}$-models, $\mathsf{D}$-models, and $\mathsf{T}$-models accordingly.

Following (Kuznets, 2018), we extend the definition of validity to nested sequents and see that we have completeness for validity of nested sequents.

#### 4.1.16 Definition

Let $K = (W, R, V)$ be an $\mathsf{L}$-model. A (*treelike*) *multiworld interpretation of a nested sequent* $\Gamma$ *into* $K$ is a function $\mathcal{I} : \mathfrak{L}(\Gamma) \to W$ such that $\mathcal{I}(\overline{s}) R \mathcal{I}(\overline{s}n)$ whenever $\{\overline{s}, \overline{s}n\} \subseteq \mathfrak{L}(\Gamma)$. Then we define

$$K, \mathcal{I} \models \Gamma \quad \text{iff} \quad K, \mathcal{I}(\overline{s}) \Vdash A \text{ for some } \overline{s} : A \in \Gamma.$$

We say that $\Gamma$ is *valid in* $K$ or $K$ *satisfies* $\Gamma$, denoted $K \models \Gamma$, if $K, \mathcal{I} \models \Gamma$ for all multiworld interpretations $\mathcal{I}$ of $\Gamma$ into $K$.

**4.1.17 Lemma**
For a nested sequent $\Gamma$ and an $\mathsf{L}$-model $K$, we have $K \models \Gamma$ iff $K \models I(\Gamma)$.

Proof. Both directions are shown by induction on the structure of $\Gamma$. Let $\Gamma$ be of the form $A_1, \ldots, A_n, [\Gamma_1], \ldots, [\Gamma_m]$. First suppose $K \not\models \Gamma$, i.e., $K, \mathcal{I} \not\models \Gamma$ for some $\mathcal{I}$. Then for all $\overline{s} : B \in \Gamma$ we have $K, \mathcal{I}(\overline{s}) \nVdash B$, in particular, $K, \mathcal{I}(1) \nVdash A_i$ for all $1 \leq i \leq n$. In addition, we show that $K, \mathcal{I}(1) \nVdash \Box I(\Gamma_j)$ for all $1 \leq j \leq m$. To prove this, we define $\mathcal{I}_j$ as follows: $\mathcal{I}_j(1 * \overline{s}') := \mathcal{I}(1 * j * \overline{s}')$ for each $1 * \overline{s}' \in \mathfrak{L}(\Gamma_j)$; in particular, $\mathcal{I}_j(1) := \mathcal{I}(1 * j)$. It is easy to see that $\mathcal{I}_j$ is a multiworld interpretation of $\Gamma_j$ into $K$ and that $K, \mathcal{I}_j \not\models \Gamma_j$. Thus, by induction hypothesis, $K, \mathcal{I}_j(1) \nVdash I(\Gamma_j)$, i.e., $K, \mathcal{I}(1 * j) \nVdash I(\Gamma_j)$. Since $\mathcal{I}(1) R \mathcal{I}(1 * j)$, it follows that $K, \mathcal{I}(1) \nVdash \Box I(\Gamma_j)$. We conclude that $K, \mathcal{I}(1) \nVdash I(\Gamma)$.

Now suppose $K, w \nVdash I(\Gamma)$. For each $1 \leq j \leq m$, there is a world $v_j$ such that $wRv_j$ and $K, v_j \nVdash I(\Gamma_j)$. By induction hypothesis, there exists a multiworld interpretation $\mathcal{I}_j$ of $\Gamma_j$ into $K$ such that $\mathcal{I}_j(1) = v_j$ and $K, \mathcal{I}_j \not\models \Gamma_j$. Define $\mathcal{I}$ as follows: $\mathcal{I}(1) := w$ and $\mathcal{I}(1 * j * \overline{s}) := \mathcal{I}_j(1 * \overline{s})$. So $K, \mathcal{I} \not\models \Gamma$, and hence $K \not\models \Gamma$. ■

### 4.1.2 Hypersequents

Remark 4.1.14 tells us that the nested sequent calculus for $\mathsf{S5}$ in (Brünnler, 2009) does not terminate in saturated sequents. Therefore we use a hypersequent calculus. Cut-free hypersequent calculi for $\mathsf{S5}$ were first, independently, introduced by Mints (1968), Pottinger (1983), and Avron (1996). Among the many existing hypersequent calculi, we use the one closest to tableaus.

**4.1.18 Definition** (Hypersequent)
A *hypersequent* $\mathcal{G}$ is a multiset of ordinary (one-sided) sequents $\Gamma_i$, written

$$\mathcal{G} = \Gamma_1 \mid \cdots \mid \Gamma_n,$$

and its *formula interpretation* $I$ is defined recursively by

$$I(\mathcal{G}) := \Box\left(\bigvee \Gamma_1\right) \vee \cdots \vee \Box\left(\bigvee \Gamma_n\right).$$

Each $\Gamma_i$ is called a *sequent component*. We call $n$ the *size* of $\mathcal{G}$ denoted by $|\mathcal{G}|$.

In this section and Section 4.3, we use letters $\Gamma$ and $\Delta$ for ordinary sequents and sequent components, and letters $\mathcal{G}$ and $\mathcal{H}$ to denote hypersequents. We denote $\mathsf{Var}(\mathcal{G}) \subseteq \mathsf{Prop}$ for the propositional variables occurring in the hypersequent $\mathcal{G}$.

**4.1.19 Definition**
The hypersequent calculus $\mathsf{HS5}$ is presented in Figure 4.2.

$$\frac{}{\mathcal{G} \mid \Gamma, p, \neg p}\ \mathsf{id_P} \qquad \frac{}{\mathcal{G} \mid \Gamma, \top}\ \mathsf{id_\top}$$

$$\frac{\mathcal{G} \mid \Gamma, A \vee B, A, B}{\mathcal{G} \mid \Gamma, A \vee B}\ \vee \qquad \frac{\mathcal{G} \mid \Gamma, A \wedge B, A \qquad \mathcal{G} \mid \Gamma, A \wedge B, B}{\mathcal{G} \mid \Gamma, A \wedge B}\ \wedge$$

$$\frac{\mathcal{G} \mid \Gamma, \Box A \mid A}{\mathcal{G} \mid \Gamma, \Box A}\ \Box \qquad \frac{\mathcal{G} \mid \Gamma, \Diamond A \mid \Delta, A}{\mathcal{G} \mid \Gamma, \Diamond A \mid \Delta}\ \mathsf{k} \qquad \frac{\mathcal{G} \mid \Gamma, \Diamond A, A}{\mathcal{G} \mid \Gamma, \Diamond A}\ \mathsf{t}$$

**Figure 4.2.** Terminating hypersequent rules for S5

These rules are obtained by Kleene'ing the S5 hypersequent calculus from (Restall *et al.*, 2007) as explained in (Kuznets and Lellmann, 2016, Section 5). Strictly speaking, rules in the latter are grafted hypersequent rules for logic K5, which is defined as K plus the axiom (5) $\neg\Box p \to \Box\neg\Box p$, but the crown rules for these grafted hypersequents are exactly the hypersequent rules for S5. Another difference is that we are using one-sided sequents and negation normal form. HS5 can also be viewed as a sequent-style equivalent of what Fitting (2007) calls the 'Simple S5 Tableau System.'

**4.1.20 Theorem**
Hypersequent $\mathcal{G}$ is provable in HS5 if and only if its formula interpretation $I(\mathcal{G})$ is derivable in S5.

Proof. From an examination of the cut-elimination theorem from (Kuznets and Lellmann, 2016) for logic K5, we conclude that the cut-elimination theorem proceeds in the same way for HS5 from which completeness of the system follows. ■

These rules form a terminating calculus for S5 under the proviso that their principal formulas are not saturated (with respect to the active component in rules k and t), as defined presently.

**4.1.21 Definition** (Saturation in hypersequents)
A formula $C$ is *saturated* in a hypersequent $\mathcal{G}$ if it satisfies the following conditions according to the form of $C$:

- $C$ is $\bot$, $\top$, or $p$ or $\neg p$ for some $p \in \mathsf{Prop}$;
- if $C = A \vee B$, then both $A$ and $B$ are in the same sequent component as $C$;
- if $C = A \wedge B$, then $A$ or $B$ belongs to the same sequent component as $C$;
- if $C = \Box A$, then $A$ occurs in the hypersequent $\mathcal{G}$.

In addition we define the following.

- The formula $C = \Diamond A$ is *saturated with respect to a sequent component of* $\mathcal{G}$ if $A$ belongs to this sequent component.
- A hypersequent $\mathcal{G}$ is *propositionally saturated* if all disjunctions and conjunctions in it are saturated, and, additionally, $\mathcal{G}$ is neither of the form $\mathcal{H} \mid \Gamma, \top$ nor of the form $\mathcal{H} \mid \Gamma, p, \neg p$ for any $p \in \mathsf{Prop}$.
- A hypersequent $\mathcal{G}$ is *saturated* if it is propositionally saturated, all boxed formulas are saturated, and all diamond formulas in it are saturated w.r.t. each sequent component of $\mathcal{G}$.

The definition of saturation in nested sequents explicitly refers to labels as sequences of natural numbers. In the hypersequent setting we use natural numbers.

**4.1.22 Definition** (Labeling hypersequents)
For a hypersequent $\mathcal{G} = \Gamma_1 \mid \cdots \mid \Gamma_m$ we use the set of *labels* $\mathfrak{L}(\mathcal{G}) = \{1, \ldots, m\}$. We write $n : A \in \mathcal{G}$ if $A \in \Gamma_n$.

Strictly speaking, these labels impose an ordering on the sequent components turning it into a sequence of sequents rather than a multiset of sequents. However, we continue with the multiset representation, stating labels explicitly if necessary.

**4.1.23 Definition**
The *weight* of a label $n$ in a hypersequent $\mathcal{G}$, denoted $w(n)$, is the number of formulas in $n$ counted as a set. Define the *weight* of $\mathcal{G}$ as

$$w(\mathcal{G}) = \sum_{n \in \mathfrak{L}(\mathcal{G})} w(n).$$

Let $sf(\mathcal{G})$ be the number of all subformulas occurring in $\mathcal{G}$ counted as a set.

**4.1.24 Theorem**
The hypersequent calculus $\mathsf{HS5}$ is strongly terminating.

Proof. We show that the proof search terminates by increasing weight. Indeed, bottom-up, the weight strictly increases for each rule, because rules are only applied when the principal formula is not saturated. To ensure termination, we show that the weights of all hypersequents in the proof search are bounded. Similarly to Theorem 4.1.13, we do so by showing that the size is bounded.

Consider a proof search for hypersequent $\mathcal{G}$. We show that the size of each hypersequent in the proof search is bounded by $sf(\mathcal{G}) + |\mathcal{G}|$. The size of the premises in the rules grow or stay the same compared to the size of the conclusion. In addition, the size can only grow by the $\Box$-rule. By saturation, $\Box$-rule can be applied at most once for each different boxed subformula, so there are at most $sf(\mathcal{G})$ applications of this rule along a branch. So the size is bounded by $sf(\mathcal{G}) + |\mathcal{G}|$. ■

In our construction of uniform interpolants in Section 4.3, we additionally use the following rule, which shows similarities with the nested sequent rule d from the calculus ND for logic D:

$$\frac{\mathcal{G} \mid \Gamma_1, \Diamond\Delta_1 \mid \cdots \mid \Gamma_k, \Diamond\Delta_k \mid \Delta_1, \ldots, \Delta_k}{\mathcal{G} \mid \Gamma_1, \Diamond\Delta_1 \mid \cdots \mid \Gamma_k, \Diamond\Delta_k} \;\mathsf{d} \tag{4.1}$$

Its admissibility is easy to show by showing the admissibility of the rule

$$\frac{\mathcal{G} \mid}{\mathcal{G}}$$

that creates an empty new component and then applying multiple instances of the rule k. Creating a new empty component is admissible. This can be seen via Theorem 4.1.20 and the fact that $\Box\bot$ is false. Note that the rule d does not play a role in the saturation in Definition 4.1.21. In our interpolation construction, the rule d is applied exclusively to saturated hypersequents.

**4.1.25 Convention**

In this chapter we use total models for S5 from the completeness theorem in Theorem 1.2.7. From now on we call these S5-models in line with Remark 1.2.14.

The following definitions are given by analogy with nested sequents, but now using natural numbers as labels.

**4.1.26 Definition**

A *(cluster-like) multiworld interpretation of a hypersequent* $\mathcal{G} = \Gamma_1 \mid \cdots \mid \Gamma_n$ *into an* S5*-model* $K = (W, W \times W, V)$ is a function $\mathcal{I} : \{1, \ldots, n\} \to W$.

In the hypersequent setting, by multiworld interpretation we always mean cluster-like multiworld interpretation. Note that there is no restriction on the image of $\mathcal{I}$, because we work with models with a total modal relation, meaning that all worlds are related to each other. For a fixed multiworld interpretation $\mathcal{I}$, we usually write $w_i$ instead of $\mathcal{I}(i)$ and represent the whole $\mathcal{I}$ by $w_1, \ldots, w_n$. A multiworld interpretation $w_1, \ldots, w_n$ is injective if the worlds $w_i$ are pairwise disjoint.

**4.1.27 Definition**

Let $K$ be an S5-model with (not necessarily distinct) worlds $w_1, \ldots, w_n$ and let $\mathcal{G} = \Gamma_1 \mid \cdots \mid \Gamma_n$ be a hypersequent. We define

$$K, w_1, \ldots, w_n \models \mathcal{G} \quad \text{iff} \quad K, w_i \Vdash A \text{ for some } i \text{ and } A \in \Gamma_i.$$

We say that $\mathcal{G}$ is *valid in* $K$ or $K$ *satisfies* $\mathcal{G}$, denoted $K \models \mathcal{G}$, if $K, w_1, \ldots, w_n \models \mathcal{G}$ for all multiworld interpretations $w_1, \ldots, w_n$ of $\mathcal{G}$ into $K$.

**4.1.28 Theorem**
For a hypersequent $\mathcal{G}$ and an $\mathsf{S5}$-model $K$, we have $K \models \mathcal{G}$ iff $K \models I(\mathcal{G})$.

Proof. Follows easily by working out the definitions of the formula interpretation and Definition 4.1.27. ■

### 4.1.3 Multiformulas

We import some notation from Kuznets (2018) in order to formulate the uniform interpolation property for nested sequents and hypersequents in subsequent sections. We present the definitions for the nested sequent setting where $\mathsf{L}$ denotes $\mathsf{K}$, $\mathsf{D}$, or $\mathsf{T}$ and we work with labels $\overline{s}$. All definitions and results do also apply to hypersequents for $\mathsf{L} = \mathsf{S5}$ where one uses natural numbers $n$ as labels.

**4.1.29 Definition**
*Multiformulas* are defined by the grammar

$$\mho ::= \overline{s} : A \mid (\mho \owedge \mho) \mid (\mho \ovee \mho),$$

where $\overline{s}$ is a label and $A$ a formula. $\mathfrak{L}(\mho)$ denotes the set of labels occurring in $\mho$.

Fun fact: the symbol $\mho$ is pronounced 'mho', which is the reverse of 'ohm' the same way as $\mho$ is the reverse of $\Omega$, the symbol for ohm in physics.

**4.1.30 Definition** (Suitability)
A multiworld interpretation $\mathcal{I}$ of a nested sequent $\Gamma$ into $\mathsf{L}$-model $K$ is *suitable for a multiformula* $\mho$ if $\mathfrak{L}(\mho) \subseteq \mathfrak{L}(\Gamma)$, in which case we call it a *multiworld interpretation of* $\mho$ *into* $K$.

**4.1.31 Definition** (Truth for multiformulas)
Let $\mathcal{I}$ be a multiworld interpretation of a multiformula $\mho$ into an $\mathsf{L}$-model $K$. We define $K, \mathcal{I} \models \mho$ recursively as follows:

$$\begin{aligned}
K, \mathcal{I} &\models \overline{s} : A && \text{iff } K, \mathcal{I}(\overline{s}) \Vdash A; \\
K, \mathcal{I} &\models \mho_1 \owedge \mho_2 && \text{iff } K, \mathcal{I} \models \mho_i \text{ for both } i = 1, 2; \\
K, \mathcal{I} &\models \mho_1 \ovee \mho_2 && \text{iff } K, \mathcal{I} \models \mho_i \text{ for at least one } i = 1, 2.
\end{aligned}$$

Note that $\mathfrak{L}(\mho_i) \subseteq \mathfrak{L}(\mho)$, meaning that $\mathcal{I}$ is also a multiworld interpretation of each $\mho_i$ into $K$.

We define the label-erasing function $\mathsf{form}$ from multiformulas to formulas, as well as multiformula equivalence, and state some of the latter's easily provable properties without proof.

**4.1.32 Definition**
The *function* form from multiformulas to formulas is defined as follows:

$$\begin{aligned}\mathsf{form}(\overline{s} : A) &:= A,\\ \mathsf{form}(\mho_1 \owedge \mho_2) &:= \mathsf{form}(\mho_1) \wedge \mathsf{form}(\mho_2),\\ \mathsf{form}(\mho_1 \ovee \mho_2) &:= \mathsf{form}(\mho_1) \vee \mathsf{form}(\mho_2).\end{aligned}$$

**4.1.33 Definition** (Multiformula equivalence)
Multiformulas $\mho_1$ and $\mho_2$ are *equivalent*, denoted $\mho_1 \equiv_{\mathsf{L}} \mho_2$, or simply $\mho_1 \equiv \mho_2$, if $\mathfrak{L}(\mho_1) = \mathfrak{L}(\mho_2)$ and $K, \mathcal{I} \vDash \mho_1$ iff $K, \mathcal{I} \vDash \mho_2$ for any multiworld interpretation $\mathcal{I}$ of $\mho_1$ into an $\mathsf{L}$-model $K$.

**4.1.34 Lemma** (Equivalence property)
For any multiformula $\mho$, label $\overline{s}$, and formulas $A$ and $B$,

1. $\mho \owedge \mho \equiv \mho \ovee \mho \equiv \mho$,
2. $\overline{s} : A \owedge \overline{s} : B \equiv \overline{s} : (A \wedge B)$, and
3. $\overline{s} : A \ovee \overline{s} : B \equiv \overline{s} : (A \vee B)$.

**4.1.35 Definition** (Normal forms)
We say that a multiformula $\mho$ is in

- *special disjunctive normal form* (*SDNF*) if $\mho$ is a $\ovee$-disjunction of $\owedge$-conjunctions of labeled formulas $\overline{s} : A$ such that each disjunct contains exactly one occurrence of each label $\overline{s} \in \mathfrak{L}(\mho)$.
- *special conjunctive normal form* (*SCNF*) if $\mho$ is a $\owedge$-conjunction of $\ovee$-disjunctions of labeled formulas $\overline{s} : A$ such that each conjunct contains exactly one occurrence of each label $\overline{s} \in \mathfrak{L}(\mho)$.

**4.1.36 Lemma**
For each multiformula $\mho$, there exists an equivalent multiformula in SDNF and an equivalent multiformula in SCNF.

Proof. Since $\ovee$ and $\owedge$ behave classically, one can employ the standard transformation into the DNF/CNF. In order to ensure one label per disjunct/conjunct rule, multiple labels can be combined using Lemma 4.1.34, whereas missing labels can be added in the form of $\overline{s} : \bot$ in case of SDNF and $\overline{s} : \top$ in case of SCNF. ■

Recall the definition of bisimulation modulo $p$ from Definition 1.2.19. We extend it to include multiworld interpretations.

**4.1.37 Definition**
Let $K = (W, R, V)$ and $K' = (W', R', V')$ be $\mathsf{L}$-models and let $\mathcal{I} : \mathfrak{L} \to W$ and $\mathcal{I}' : \mathfrak{L} \to W'$ be functions with a common domain $\mathfrak{L}$. We write $(K, \mathcal{I}) \sim^p (K', \mathcal{I}')$ if

there is a bisimulation $Z$ modulo $p$ between $K$ and $K'$ with $\mathcal{I}(\bar{s})Z\mathcal{I}'(\bar{s})$ for all $\bar{s} \in \mathfrak{L}$.

**4.1.38 Lemma**
Let $\Gamma$ be a nested sequent not containing $p$ and let $\mathcal{I}$ and $\mathcal{I}'$ be multiworld interpretations of $\Gamma$ into $K$ and $K'$ respectively such that $(K,\mathcal{I}) \sim^p (K',\mathcal{I}')$. Then $K,\mathcal{I} \models \Gamma$ iff $K',\mathcal{I}' \models \Gamma$. Similar result holds when $\Gamma$ is replaced by a multiformula $\mho$.

Proof. If $(K,\mathcal{I}) \sim^p (K',\mathcal{I}')$, then $(K,\mathcal{I}(\bar{s})) \sim^p (K,\mathcal{I}'(\bar{s}))$ for all $\bar{s} \in \mathfrak{L}(\Gamma)$. By Theorem 1.2.21 we have $K,\mathcal{I}(\bar{s}) \Vdash A$ iff $K',\mathcal{I}'(\bar{s}) \Vdash A$ for all $\bar{s} : A \in \Gamma$. The statement follows from Definition 4.1.16. A similar proof applies to multiformulas using Definition 4.1.31. ■

## 4.2 Uniform interpolation via nested sequents

We define a new notion of uniform interpolation for nested sequents in Section 4.2.1 that involves bisimulation modulo $p$ in Kripke semantics. We use this to prove uniform interpolation for $\mathsf{K}$ in Section 4.2.2, and for $\mathsf{D}$ and $\mathsf{T}$ in Section 4.2.3.

### 4.2.1 Bisimulation

Recall from Remark 2.2.7 that for classical modal logics, the existence of post-interpolants ensures the existence of pre-interpolants, and vice versa. Thus, from now on, we focus on $\dot{\forall} pA$.

**4.2.1 Definition** (NUIP)
Let a nested sequent calculus $\mathsf{NL}$ be sound and complete w.r.t. a logic $\mathsf{L}$. We say that $\mathsf{NL}$ has the *nested sequent uniform interpolation property*, or *NUIP*, if for each nested sequent $\Gamma$ and $p \in \mathsf{Prop}$ there exists a multiformula $\mathcal{A}_p(\Gamma)$, called a *nested uniform interpolant*, such that

(i) $\mathsf{Var}\big(\mathcal{A}_p(\Gamma)\big) \subseteq \mathsf{Var}(\Gamma) \setminus \{p\}$ and $\mathfrak{L}\big(\mathcal{A}_p(\Gamma)\big) \subseteq \mathfrak{L}(\Gamma)$;

(ii) for each multiworld interpretation $\mathcal{I}$ of $\Gamma$ into an $\mathsf{L}$-model $K$

$$K,\mathcal{I} \models \mathcal{A}_p(\Gamma) \quad \text{implies} \quad K,\mathcal{I} \models \Gamma;$$

(iii) for each nested sequent $\Sigma$ with $p \notin \mathsf{Var}(\Sigma)$ and $\mathfrak{L}(\Sigma) = \mathfrak{L}(\Gamma)$ and for each multiworld interpretation $\mathcal{I}$ of $\Gamma$ into an $\mathsf{L}$-model $K$,

$$K,\mathcal{I} \not\models \mathcal{A}_p(\Gamma) \text{ and } K,\mathcal{I} \not\models \Sigma \quad \text{imply} \quad K',\mathcal{I}' \not\models \Gamma \text{ and } K',\mathcal{I}' \not\models \Sigma$$

for some multiworld interpretation $\mathcal{I}'$ of $\Gamma$ into some $\mathsf{L}$-model $K'$.

The condition on labels (i) ensures that interpretations of $\Gamma$ are suitable for $\mathcal{A}_p(\Gamma)$.

Compare the definition to the sequent style uniform interpolation (Definition 2.3.6). Now we immediately incorporate semantic notions. In order to stay close to the form of the nested sequents, the uniform interpolants are defined as multiformulas instead of formulas.

**4.2.2 Remark**
Bílková (2011) also defines uniform interpolation for nested sequents which differs in several ways. Apart from a minor difference in condition (iii), our definition involves semantic notions and uses multiformula interpolants instead of formulas.

**4.2.3 Lemma**
If a nested calculus $\mathsf{NL}$ has the NUIP, then its logic $\mathsf{L}$ has uniform interpolation.

Proof. To show the existence of $\dot{\forall} pA$, consider a nested uniform interpolant $\mathcal{A}_p(A)$ of the nested sequent $A$, with $\mathfrak{L}(A) = \{1\}$. By Lemma 4.1.36, w.l.o.g. we may assume that $\mathcal{A}_p(A) = 1 : C$. Let $\dot{\forall} pA := C$. We prove the uniform interpolation properties (i), (ii), and (iii) from Definition 2.2.3 based on the corresponding NUIP properties. By NUIP(i), $\mathsf{Var}(\dot{\forall} pA) = \mathsf{Var}(1 : C) \subseteq \mathsf{Var}(A) \setminus \{p\}$ which proves (i).

For condition (ii), assume towards a contradiction that $\nvdash_{\mathsf{L}} C \to A$. By completeness $K, w \nVdash C \to A$ for some $\mathsf{L}$-model $K$ and $w \in K$. Consider a multiworld interpretation $\mathcal{I}$ of $A$ into $K$ such that $\mathcal{I}(1) := w$. Then $K, \mathcal{I} \models 1 : C$ but $K, \mathcal{I} \not\models A$, in contradiction to NUIP(ii). Hence, $\vdash_{\mathsf{L}} \dot{\forall} pA \to A$.

For (iii), let $p \notin \mathsf{Var}(B)$ and suppose $\nvdash_{\mathsf{L}} B \to C$. So, $K, w \nVdash B \to C$ for some $\mathsf{L}$-model $K$ and $w \in K$. Note $\mathfrak{L}(\neg B) = \mathfrak{L}(A) = \{1\}$. Consider a multiworld interpretation $\mathcal{I}$ of $A$ into $K$ with $\mathcal{I}(1) := w$. We have that $K, \mathcal{I} \not\models 1 : C$ and $K, \mathcal{I} \not\models \neg B$. By NUIP(iii), there is an $\mathsf{L}$-model $K'$ and a multiworld interpretation $\mathcal{I}'$ of the nested sequent $A$ into $K'$ such that $K', \mathcal{I}' \not\models A$ and $K', \mathcal{I}' \not\models \neg B$. So, $K', \mathcal{I}'(1) \nVdash A$ and $K', \mathcal{I}'(1) \Vdash B$. Thus, by soundness of $\mathsf{L}$, we have $\nvdash_{\mathsf{L}} B \to A$. ■

With our knowledge on bisimulation quantifiers, we replace the third condition in NUIP with a (possibly) stronger condition (iii)′ as follows.

**4.2.4 Definition** (BNUIP)
Let a nested sequent calculus $\mathsf{NL}$ be sound and complete w.r.t. a logic $\mathsf{L}$. Calculus $\mathsf{NL}$ has the *bisimulation nested sequent uniform interpolation property*, or simply *BNUIP*, if, in addition to conditions NUIP(i)–(ii) from Definition 4.2.1,

(iii)′ for each $\mathsf{L}$-model $K$ and multiworld interpretation $\mathcal{I}$ of $\Gamma$ into $K$, we have

$$K, \mathcal{I} \not\models \mathcal{A}_p(\Gamma) \quad \text{implies} \quad (K', \mathcal{I}') \sim^p (K, \mathcal{I}) \text{ and } K', \mathcal{I}' \not\models \Gamma,$$

for some multiworld interpretation $\mathcal{I}'$ of $\Gamma$ into some $\mathsf{L}$-model $K'$.

**4.2.5 Lemma**
If $\Gamma$ and $\mathcal{A}_p(\Gamma)$ satisfy condition (iii)$'$ of Definition 4.2.4, then they satisfy condition (iii) of Definition 4.2.1.

Proof. Let $\Sigma$ be a nested sequent with $p \notin \mathsf{Var}(\Sigma)$ and $\mathfrak{L}(\Sigma) = \mathfrak{L}(\Gamma)$. Let $K$ be an $\mathsf{L}$-model such that $K, \mathcal{I} \not\models \mathcal{A}_p(\Gamma)$ and $K, \mathcal{I} \not\models \Sigma$. By BNUIP(iii)$'$ we find an $\mathsf{L}$-model $K'$ and $\mathcal{I}'$ from $\Gamma$ into $K'$ such that $(K', \mathcal{I}') \sim^p (K, \mathcal{I})$ and $K', \mathcal{I}' \not\models \Gamma$. By Lemma 4.1.38, we also conclude $K', \mathcal{I}' \not\models \Sigma$. ■

**4.2.6 Corollary**
If a nested calculus $\mathsf{NL}$ has the BNUIP, then its logic $\mathsf{L}$ has uniform interpolation.

Recall Definition 2.4.1 of bisimulation quantification in a particular class of models. The previous corollary also follows from the following stronger observation.

**4.2.7 Lemma**
Let $\mathsf{L}$ be complete with respect to a class of models $\mathcal{K}_\mathsf{L}$. If its nested calculus $\mathsf{NL}$ has the BNUIP, then propositional quantifiers are definable over $\mathcal{K}_\mathsf{L}$.

Proof. To show the existence of $\dot{\forall} pA$, consider an interpolant $\mathcal{A}_p(A)$ of the nested sequent $A$, with $\mathfrak{L}(A) = \{1\}$, that satisfies all properties of BNUIP. Again by Lemma 4.1.36, we may assume that $\mathcal{A}_p(A) = 1 : C$. Define $\overline{\forall} pA := C$. It is easy to check that this is indeed a bisimulation quantifier by the properties of BNUIP. ■

We are interested in manipulations of treelike models that preserve bisimulation modulo $p$.

**4.2.8 Definition** (Model transformations)
Let $K = (W, R, V)$ be an intransitive tree, $K_w = (W_w, R_w, V_w)$ be its generated subtree with root $w \in W$, and $M = (W_M, R_M, V_M)$ be another tree with root $\rho_M$. A model $K' = (W', R', V')$ is the result of *replacing the subtree $K_w$ with $M$ in $K$* if

$$
\begin{aligned}
W' &:= (W \setminus W_w) \sqcup W_M, \\
R' &:= \big(R \cap (W \setminus W_w)^2\big) \sqcup R_M \sqcup \big\{(v, \rho_M) \mid vRw\big\}, \\
V'(v) &:= \begin{cases} V(v) & \text{if } v \in W \setminus W_w, \\ V_M(v) & \text{if } v \in W_M. \end{cases}
\end{aligned}
$$

A model $K'' := (W'', R'', V'')$ is the result of *duplicating* $K_w$ in $K$ if another copy[22] $K_w^c := (W_w^c, R_w^c, V_w^c)$ of $K_w$ is inserted alongside $K_w$, i.e., if

$$\begin{aligned} W'' &:= W \sqcup W_w^c, \\ R'' &:= R \sqcup R_w^c \sqcup \{(v, w^c) \mid vRw\}, \\ V''(v) &:= \begin{cases} V(v) & \text{if } v \in W, \\ V_w(v) & \text{if } v \in W_w^c. \end{cases} \end{aligned}$$

Similarly, $K''$ is the result of *cloning* $K_w$ in $K$ if $K_w^c$ is inserted as a subtree of $K_w$, where the definition of $R''$ reads now as follows:

$$R'' := R \sqcup R_w^c \sqcup \{(w, w^c)\}$$

**4.2.9 Lemma**
In the setup from Definition 4.2.8, let $Z \subseteq W_M \times W_w$ be a bisimulation demonstrating that $(M, \rho_M) \sim^p (K_w, w)$. Let $K'$ obtained by replacing $K_w$ with $M$ in $K$ and let $K''$ obtained by duplicating $K_w$ in $K$. Then

1. $(K', v) \sim^p (K, v)$ for all $v \in W \setminus W_w$ and $(K', u_M) \sim^p (K, u)$ whenever $u_M Z u$. Moreover, if both $K$ and $M$ are K-models (D-models, T-models), then so is $K'$.
2. $(K'', v) \sim^p (K, v)$ for all $v \in W$ and, in addition, $(K'', u^c) \sim^p (K, u)$ for all $u \in W_w$. If $K$ is a K-model (D-model, T-model) not rooted at $w$, so is $K''$.
3. The same holds when $K''$ is obtained by cloning if $wRw$ except that cloning does not preserve D-models.

Proof. It is easy to see that one bisimulation witnesses all of the stated bisimilarities in each case. For replacing use $Z'$ and for duplicating and cloning use $Z''$ as follows:

$$\begin{aligned} Z' &:= \{(v, v) \mid v \in W \setminus W_w\} \sqcup Z. \\ Z'' &:= \{(v, v) \mid v \in W\} \sqcup \{(u^c, u) \mid u \in W_w\}. \end{aligned}$$

Both the tree structure and the reflexivity of the worlds are preserved by all of the operations. Leaves are preserved by replacement and duplication, whereas cloning turns a leaf $w$ into a non-leaf without removing its reflexivity as required in D-models. ■

[22] Formally, we define $v^c := (v, c)$, $W_w^c := \{v^c \mid v \in W_w\}$, $R_w^c := \{(v^c, u^c) \mid (v, u) \in R_w\}$, and $V_w^c(q) := \{v^c \mid v \in V_w(q)\}$.

### 4.2.2 Uniform interpolation for K

In this section, we present our method of constructing nested uniform interpolants satisfying BNUIP for the calculus NK.

Interpolants $\mathcal{A}_p(\Gamma)$ are defined recursively on the basis of the terminating calculus from Figure 4.1. If $\Gamma$ is not K-saturated, $\mathcal{A}_p(\Gamma)$ is defined recursively in Table 4.1 based on the form of $\Gamma$. For rows 3–5, we assume that the formula displayed in the left column is not K-saturated in $\Gamma$, whereas for $\Diamond A$ in the last row we assume it not to be K-saturated w.r.t. $\overline{s}n$ in $\Gamma$. Each row in the table corresponds to a rule in the proof search, where the left column in the table corresponds to the conclusion of a rule and the right column uses the premise(s) of the rule. Strictly speaking, this is a non-deterministic algorithm, since the order does not affect our results, we do not specify it. However, it is more efficient to apply rows 1–2 of Table 4.1 first and row 5 last.

For K-saturated $\Gamma$, we define $\mathcal{A}_p(\Gamma)$ recursively as follows:

$$\mathcal{A}_p(\Gamma) := \bigovee_{\substack{\overline{s}:\ell\in\Gamma \\ \ell\in\mathsf{Lit}\setminus\{p,\neg p\}}} \overline{s} : \ell \qquad \ovee \qquad \bigovee_{\substack{\overline{t}\in\mathfrak{L}(\Gamma) \\ (\exists B)\overline{t}:\Diamond B\in\Gamma}} \overline{t} : \Diamond\mathcal{A}_p^{\mathsf{form}}\Big(\bigvee\nolimits_{\overline{t}:\Diamond B\in\Gamma} B\Big), \tag{4.2}$$

where

$$\mathcal{A}_p^{\mathsf{form}}(\Gamma) := \mathsf{form}\big(\mathcal{A}_p(\Gamma)\big).$$

| $\Gamma$ matches | $\mathcal{A}_p(\Gamma)$ equals |
|---|---|
| $\Gamma'\{\top\}_{\overline{s}}$ | $\overline{s} : \top$ |
| $\Gamma'\{p, \neg p\}_{\overline{s}}$ | $\overline{s} : \top$ |
| $\Gamma'\{A \vee B\}$ | $\mathcal{A}_p\big(\Gamma'\{A \vee B, A, B\}\big)$ |
| $\Gamma'\{A \wedge B\}$ | $\mathcal{A}_p\big(\Gamma'\{A \wedge B, A\}\big) \owedge \mathcal{A}_p\big(\Gamma'\{A \wedge B, B\}\big)$ |
| $\Gamma'\{\Box A\}_{\overline{s}}$ | $\bigowedge_{i=1}^{m}\Big(\overline{s} : \Box D_i \ovee \bigovee_{\overline{t}\neq\overline{s}n} \overline{t} : C_{i,\overline{t}}\Big)$ where $n$ is the smallest integer such that $\overline{s}n \notin \mathfrak{L}(\Gamma)$ and the SCNF of $\mathcal{A}_p\big(\Gamma'\{\Box A, [A]_{\overline{s}n}\}\big)$ is $\bigowedge_{i=1}^{m}\Big(\overline{s}n : D_i \ovee \bigovee_{\overline{t}\neq\overline{s}n} \overline{t} : C_{i,\overline{t}}\Big)$ |
| $\Gamma'\{\Diamond A, [\Delta]_{\overline{s}n}\}$ | $\mathcal{A}_p\big(\Gamma'\{\Diamond A, [\Delta, A]\}\big)$ |

**Table 4.1.** Construction of $\mathcal{A}_p(\Gamma)$ for NK for $\Gamma$ that are not K-saturated.

Recall that form is the label forgetting function defined in Definition 4.1.32. Since here we apply form to the multiformula $\mathcal{A}_p(\Gamma)$ with 1 being its only label, we have $K, \mathcal{I} \models \mho$ iff $K, \mathcal{I}(1) \Vdash \mathsf{form}(\mho)$ for such multiformulas $\mho$. As usual, we define the empty disjunction to be false, which in this format means $\bigovee \emptyset := 1 : \bot$.

The construction of $\mathcal{A}_p(\Gamma)$ is well-defined (modulo a chosen order) as shown below.

### 4.2.10 Definition

For a nested sequent $\Gamma$, let $b(\Gamma)$ be the number of its distinct diamond subformulas.

### 4.2.11 Theorem

Let $\Gamma$ be a nested sequent. Then $\mathcal{A}_p(\Gamma)$ is well defined, that is, the calculation of $\mathcal{A}_p(\Gamma)$ terminates.

Proof. In Theorem 4.1.13 we proved that the rules of NK terminate by increasing weight $w(\Gamma)$, which is shown to be bounded. Consider the lexicographical ordering based on the pairs $(b(\Gamma), w(\Gamma))$. For each row in Table 4.1, $d$ stays the same for the recursive calls for premise(s), but $w$ increases. The recursive call in (4.2) for a K-saturated $\Gamma$ decreases $d$ because the set of diamond subformulas of $\bigvee_{\bar{t}:\Diamond B \in \Gamma} B$ is strictly smaller than that of $\Gamma$. If $b(\Gamma) = 0$ for a K-saturated $\Gamma$, the second disjunct of (4.2) is empty, thus no such new recursive calls are generated. ∎

In the following examples we use Lemmas 4.1.34 and 4.1.36 as necessary.

### 4.2.12 Example

The algorithm for $\mathcal{A}_p(\Box p, \Box\neg p)$ calls the calculation of $\mathcal{A}_p\left(\Box p, \Box\neg p, [p]_{11}\right)$, which in turn calls $\mathcal{A}_p\left(\Box p, \Box\neg p, [p]_{11}, [\neg p]_{12}\right)$. The latter nested sequent is K-saturated, and the algorithm returns $1 : \bot \ovee 1 : \bot$, the first disjunct corresponding to the empty disjunction of literals other than $p$ and $\neg p$ and the second one representing the absent diamond formulas. Computing its SCNF we get

$$\mathcal{A}_p\left(\Box p, \Box\neg p, [p]_{11}, [\neg p]_{12}\right) \equiv 1 : \bot \ovee 11 : \bot \ovee 12 : \bot.$$

Applying the transformation from the penultimate row of Table 4.1, we first get

$$\mathcal{A}_p\left(\Box p, \Box\neg p, [p]_{11}\right) = 1 : \bot \ovee 11 : \bot \ovee 1 : \Box\bot \equiv 1 : \Box\bot \ovee 11 : \bot,$$

and finally

$$\mathcal{A}_p\left(\Box p, \Box\neg p\right) = 1 : \Box\bot \ovee 1 : \Box\bot \equiv 1 : \Box\bot.$$

It is easy to check that $1 : \Box\bot$ is a bisimulation nested uniform interpolant of the nested sequent $\Box p, \Box\neg p$ w.r.t. $p$, and, accordingly, $\Box\bot$ is a uniform interpolant of the formula $\Box p \vee \Box\neg p$ w.r.t. $p$.

### 4.2.13 Example

Consider the nested sequent $\Gamma = \neg p, \Diamond q \wedge \Diamond p, [q]$. In the absence of boxes, the algorithm amounts to processing the K-saturated nested sequents in the leaves of the proof search tree

$$\dfrac{\neg p, \Diamond q \wedge \Diamond p, \Diamond q, [q]_{11} \qquad \dfrac{\neg p, \Diamond q \wedge \Diamond p, \Diamond p, [q,p]_{11}}{\neg p, \Diamond q \wedge \Diamond p, \Diamond p, [q]_{11}}}{\neg p, \Diamond q \wedge \Diamond p, [q]_{11}}$$

We have

$$\mathcal{A}_p(\neg p, \Diamond q \wedge \Diamond p, \Diamond q, [q]_{11}) = 11 : q \oslash 1 : \Diamond \mathcal{A}_p^{\mathsf{form}}(q),$$
$$\mathcal{A}_p(\neg p, \Diamond q \wedge \Diamond p, \Diamond p, [q,p]_{11}) = 11 : q \oslash 1 : \Diamond \mathcal{A}_p^{\mathsf{form}}(p).$$

Since $A_p^{\mathsf{form}}(q)$ and $A_p^{\mathsf{form}}(p)$ can be simplified to $q$ and $\bot$ respectively, we obtain

$$\mathcal{A}_p(\Gamma) \equiv (11 : q \oslash 1 : \Diamond q) \owedge (11 : q \oslash 1 : \Diamond \bot) \equiv 11 : q,$$

where the latter equivalence holds since $\Diamond\bot$ can never be true. It is easy to verify that $11 : q$ is a bisimulation nested uniform interpolant of $\neg p, \Diamond q \wedge \Diamond p, [q]_{11}$ w.r.t. $p$.

**4.2.14 Theorem**

The nested calculus NK has the BNUIP.

Proof. It is easy to see that BNUIP(i) is satisfied. In order to prove BNUIP(ii), let $\Gamma$ be a nested sequent and $\mathcal{I}$ be a multiworld interpretation of $\Gamma$ into a K-model $K = (W, R, V)$ such that $K, \mathcal{I} \models \mathcal{A}_p(\Gamma)$ (by BNUIP(i), $\mathcal{I}$ is suitable for $\mathcal{A}_p(\Gamma)$). We show $K, \mathcal{I} \models \Gamma$ by induction on the nested sequent ordering $(b(\Gamma), w(\Gamma))$. Considering the construction of $\mathcal{A}_p(\Gamma)$, we treat the cases of Table 4.1 first.

Cases in rows 1–2 of Table 4.1 are trivial. Those in rows 3, 4, and 6 are similar. Here we only discuss row 6. So suppose that $\Gamma = \Gamma'\{\Diamond A, [\Delta]_{\overline{s}n}\}$ and suppose $K, \mathcal{I} \models \mathcal{A}_p(\Gamma'\{\Diamond A, [\Delta, A]_{\overline{s}n}\})$. By induction, $K, \mathcal{I} \models \Gamma'\{\Diamond A, [\Delta, A]_{\overline{s}n}\}$. Since $K, \mathcal{I}(\overline{s}n) \Vdash A$ implies $K, \mathcal{I}(\overline{s}) \Vdash \Diamond A$, it follows that $K, \mathcal{I} \models \Gamma'\{\Diamond A, [\Delta]_{\overline{s}n}\}$.

For row 5, let $\Gamma = \Gamma'\{\Box A\}_{\overline{s}}$, and

$$\mathcal{A}_p\big(\Gamma'\{\Box A, [A]_{\overline{s}n}\}\big) \equiv \bigowedge_{i=1}^{m} \left( \overline{s}n : D_i \oslash \bigovee_{\overline{t} \neq \overline{s}n} \overline{t} : C_{i,\overline{t}} \right),$$

for some $\overline{s}n \notin \mathfrak{L}(\Gamma)$, and

$$K, \mathcal{I} \models \bigowedge_{i=1}^{m} \left( \overline{s} : \Box D_i \oslash \bigovee_{\overline{t} \neq \overline{s}n} \overline{t} : C_{i,\overline{t}} \right). \tag{4.3}$$

For any $v$ with $\mathcal{I}(\overline{s})Rv$, define multiworld interpretation $\mathcal{I}_v := \mathcal{I} \sqcup \{(\overline{s}n, v)\}$ of $\Gamma'\{\Box A, [A]_{\overline{s}n}\}$ into $K$. It follows from (4.3) that, for each $i$, either $K, \mathcal{I}_v(\overline{t}) \Vdash C_{i,\overline{t}}$

for some $\overline{t} \in \mathfrak{L}(\Gamma)$ or $K, \mathcal{I}_v(\overline{s}n) \Vdash D_i$, meaning that $K, \mathcal{I}_v \models \mathcal{A}_p(\Gamma'\{\Box A, [A]_{\overline{s}n}\})$. By induction hypothesis, we have $K, \mathcal{I}_v \models \Gamma'\{\Box A, [A]_{\overline{s}n}\}$ when $\mathcal{I}(\overline{s})Rv$. Clearly, $K, \mathcal{I} \models \Gamma$ if $K, \mathcal{I}(\overline{s}) \Vdash \Box A$. Otherwise, there exists a $v$ such that $\mathcal{I}(\overline{s})Rv$ and $K, v \nVdash A$. For this world $K, \mathcal{I}_v \models \Gamma'\{\Box A, [A]_{\overline{s}n}\}$ implies $K, \mathcal{I}_v \models \Gamma'\{\Box A\}_{\overline{s}}$, which yields $K, \mathcal{I} \models \Gamma$ because $\mathcal{I}_v$ agrees with $\mathcal{I}$ on all labels from $\Gamma$.

Finally, let $\Gamma$ be $\mathsf{K}$-saturated and $K, \mathcal{I} \models \mathcal{A}_p(\Gamma)$ with $\mathcal{A}_p(\Gamma)$ as defined in (4.2). Clearly, $K, \mathcal{I} \models \Gamma$ if we have $K, \mathcal{I}(\overline{s}) \Vdash \ell$ for some $\overline{s} : \ell \in \Gamma$. Thus, it only remains to consider the case when $K, \mathcal{I}(\overline{t}) \Vdash \Diamond\mathcal{A}_p^{\mathsf{form}}\left(\bigvee_{\overline{t}:\Diamond B\in\Gamma} B\right)$ for some $\overline{t} \in \mathfrak{L}(\Gamma)$. Then $K, v \Vdash \mathcal{A}_p^{\mathsf{form}}\left(\bigvee_{\overline{t}:\Diamond B\in\Gamma} B\right)$ for some $v$ such that $\mathcal{I}(\overline{t})Rv$ and, accordingly, $K, \mathcal{J} \models \mathcal{A}_p\left(\bigvee_{\overline{t}:\Diamond B\in\Gamma} B\right)$ for $\mathcal{J} := \{(1, v)\}$. By induction hypothesis (for smaller $d$), $K, \mathcal{J} \models \bigvee_{\overline{t}:\Diamond B\in\Gamma} B$, and, hence, $K, v \Vdash B$ for some $\overline{t} : \Diamond B \in \Gamma$. Now $K, \mathcal{I} \models \Gamma$ follows from $\mathcal{I}(\overline{t})Rv$. This case concludes the proof for BNUIP(ii).

It remains to prove BNUIP(iii)$'$. Let $\mathcal{I}$ be a multiworld interpretation of $\Gamma$ into a $\mathsf{K}$-model $K$ such that $K, \mathcal{I} \not\models \mathcal{A}_p(\Gamma)$. We must find another multiworld interpretation $\mathcal{I}'$ into some $\mathsf{K}$-model $K'$ such that $(K', \mathcal{I}') \sim^p (K, \mathcal{I})$ and $K', \mathcal{I}' \not\models \Gamma$. We construct these $K'$ and $\mathcal{I}'$ while simultaneously proving BNUIP(iii)$'$ by induction on the lexicographic order $(b(\Gamma), w(\Gamma))$. Recall that $\mathsf{K}$-models (and their submodels) are irreflexive intransitive trees.

Let us start with the difficult case for $\mathsf{K}$-saturated $\Gamma$. So suppose $K, \mathcal{I} \not\models \mathcal{A}_p(\Gamma)$ for $\mathcal{A}_p(\Gamma)$ from (4.2). We first briefly sketch the construction and the proof. The labeled literals $\overline{s} : \ell$ from (4.2) can determine the truth values of variables other than $p$ in the worlds in the range of $\mathcal{I}$. Saturation takes care of the appropriate truth values for formulas except for diamond formulas. By contrast, truth values of $p$ cannot be specified in $\mathcal{A}_p(\Gamma)$. To refute $\Gamma$, they must be adjusted on a world-by-world basis, which prompts the additional requirement that $\mathcal{I}'$ be injective, that is, $\mathcal{I}'(\overline{s}) = \mathcal{I}'(\overline{t})$ implies $\overline{s} = \overline{t}$. This avoids incompatible requirements on the truth value of $p$ in a world $\mathcal{I}(\overline{s}) = \mathcal{I}(\overline{t})$ that originates from distinct $\overline{s}$ and $\overline{t}$. Finally, for $\Diamond A$ to be false at a world $w$ in the range of $\mathcal{I}$, one must falsify $A$ at all children of $w$, including those outside the range of $\mathcal{I}$. This is achieved by replacing subtrees with bisimilar models obtained by the induction hypothesis from the right disjunct of (4.2), as schematically depicted in Figure 4.3. We now describe it in detail.

(1) First, we make the interpretation injective. It is easy to see (though tedious to describe in detail) that by a breadth-first recursion on nodes $\overline{s}$ in $\Gamma$, one can duplicate $K_{\mathcal{I}(\overline{s}n)}$ according to Definition 4.2.8 whenever $\mathcal{I}(\overline{s}n) = \mathcal{I}(\overline{s}m)$ for some $m < n$ to obtain a model $M$ and an injective multiworld interpretation $\mathcal{J}$ of $\Gamma$ into it such that $(M, \mathcal{J}) \sim^p (K, \mathcal{I})$. Thus, $\mathcal{J}(\overline{s}) \neq \mathcal{J}(\overline{t})$ whenever $\overline{s} \neq \overline{t}$ and $M, \mathcal{J} \not\models \mathcal{A}_p(\Gamma)$ by Lemma 4.1.38.

(2) Then we deal with out-of-range children. A model $M'$ is constructed from $M$ by applying the following $\Diamond$-processing step for each node $\overline{t} \in \mathfrak{L}(\Gamma)$ that

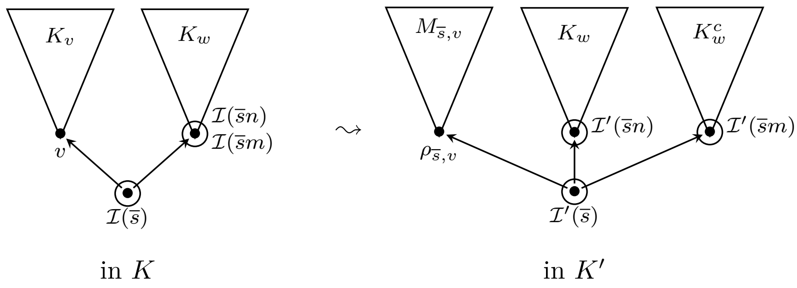



**Figure 4.3.** Main transformations for constructing model $K'$: circles represent worlds in the range of $\mathcal{I}$.

contains at least one formula of the form $\Diamond A$ (nodes can be chosen in any order). Start by setting $M^0 := M$ and $j := 0$:

*Step:* Since $M^j, \mathcal{J} \not\models \mathcal{A}_p(\Gamma)$, it follows from the second disjunct in (4.2) that

$$M^j, \mathcal{J}(\bar{t}) \nVdash \Diamond\mathcal{A}_p^{\mathsf{form}}\left(\bigvee_{\bar{t}:\Diamond B\in\Gamma} B\right).$$

Thus, $M^j, v \nVdash \mathcal{A}_p^{\mathsf{form}}\left(\bigvee_{\bar{t}:\Diamond B\in\Gamma} B\right)$ for any child $v$ of $\mathcal{J}(\bar{t})$ in $M^j$, so

$$M_v^j, \mathcal{I}_v \not\models \mathcal{A}_p\left(\bigvee_{\bar{t}:\Diamond B\in\Gamma} B\right)$$

for the multiworld interpretation $\mathcal{I}_v := \{(1, v)\}$ of $\bigvee_{\bar{t}:\Diamond B\in\Gamma} B$ into the subtree $M_v^j$ of $M^j$ with root $v$. By the induction hypothesis for smaller $d$, there exists a $\mathsf{K}$-model $M_{\bar{t},v}$ with root $\rho_{\bar{t},v}$ such that $(M_v^j, v) \sim^p (M_{\bar{t},v}, \rho_{\bar{t},v})$ and

$$M_{\bar{t},v}, \rho_{\bar{t},v} \nVdash \bigvee_{\bar{t}:\Diamond B\in\Gamma} B.$$

Let $M^{j+1}$ be the result of replacing each subtree $M_v^j$ for children $v$ of $\mathcal{J}(\bar{t})$ not in the range of $\mathcal{J}$ with $M_{\bar{t},v}$ in $M^j$ according to Definition 4.2.8. Note that all these subtrees are disjoint because the models are intransitive trees and, hence, these replacements do not interfere with one another. Note also that since the range of $\mathcal{J}$ is downward closed and the roots of the replaced subtrees are outside, no world from the range is modified. Thus, $\mathcal{J}$ remains an injective interpretation into $M^{j+1}$. Finally, it follows from Lemma 4.2.9 that $(M^j, \mathcal{J}) \sim^p (M^{j+1}, \mathcal{J})$. Hence, $M^{j+1}, \mathcal{J} \not\models \mathcal{A}_p(\Gamma)$.

Let $M' = (W', R', V')$ be the model obtained after replacements for all $\bar{t}$'s are completed (again they do not interfere with each other). Then we have

$(M, \mathcal{J}) \sim^p (M', \mathcal{J})$ and, for each out-of-range child $v$ of $\mathcal{J}(\bar{t})$ in $M$, the world $\rho_{\bar{t},v}$ is a child of $\mathcal{J}(\bar{t})$ in $M'$ and

$$M', \rho_{\bar{t},v} \not\Vdash \bigvee_{\bar{t}:\Diamond B \in \Gamma} B.$$

This accounts for all children of $\mathcal{J}(\bar{t})$ in $M'$.

(3) It remains to adjust the truth values of $p$. We define $K' := (W', R', V_p')$ by modifying the valuation $V'$ of $M'$ as follows:

$$V_p'(w) := \begin{cases} V'(w) \cup \{p\} & \text{if there is an } \bar{s} \text{ with } w = \mathcal{J}(\bar{s}) \text{ and } \bar{s} : \neg p \in \Gamma; \\ V'(w) \setminus \{p\} & \text{if there is an } \bar{s} \text{ with } w = \mathcal{J}(\bar{s}) \text{ and } \bar{s} : p \in \Gamma; \\ V'(w) & \text{otherwise.} \end{cases}$$

This is well defined, since $\mathcal{J}$ is injective and not both $p$ and $\neg p$ occur in node $\bar{s}$ since $\Gamma$ is $\mathsf{K}$-saturated. For $\mathcal{I}' := \mathcal{J}$, it follows that

$$K', \mathcal{I}'(\bar{s}) \not\Vdash \neg p \text{ whenever } \bar{s} : \neg p \in \Gamma; \tag{4.4}$$

$$K', \mathcal{I}'(\bar{s}) \not\Vdash p \text{ whenever } \bar{s} : p \in \Gamma. \tag{4.5}$$

Moreover, since subtrees $K'_{\rho_{\bar{t},v}}$ are disjoint from worlds in the range of $\mathcal{I}'$,

$$K', \rho_{\bar{t},v} \not\Vdash B \text{ whenever } \bar{t} : \Diamond B \in \Gamma. \tag{4.6}$$

After these three steps, we have a model $(K', \mathcal{I}') \sim^p (M', \mathcal{J}) \sim^p (M, \mathcal{J}) \sim^p (K, \mathcal{I})$ that satisfies (4.4), (4.5), and (4.6). It remains to prove that $K', \mathcal{I}' \not\models \Gamma$ by showing that $K', \mathcal{I}'(\bar{s}) \not\Vdash A$ for all $\bar{s} : A \in \Gamma$, which is done by induction on $A$.

- $A = \bot$ is trivial, while $\top$ cannot occur in a $\mathsf{K}$-saturated nested sequent.
- For $A \in \{p, \neg p\}$, this follows from (4.4) and (4.5).
- Let $A = \ell \in \mathsf{Lit} \setminus \{p, \neg p\}$. According to (4.2), we have $K, \mathcal{I}(\bar{s}) \not\Vdash A$ because $K, \mathcal{I} \not\models \mathcal{A}_p(\Gamma)$, which transfers to $K'$ and $\mathcal{I}'$ by bisimilarity modulo $p$.
- The cases $B \vee B'$ and $B \wedge B'$ follow from saturation and are left to the reader.
- Let $A = \Box B$. We get $\bar{s}n : B \in \Gamma$ for some label $\bar{s}n$ by $\mathsf{K}$-saturation. By induction hypothesis, $K', \mathcal{I}'(\bar{s}n) \not\Vdash B$. Since $\mathcal{I}'(\bar{s}) R' \mathcal{I}'(\bar{s}n)$, we conclude $K', \mathcal{I}'(\bar{s}) \not\Vdash \Box B$ as required.
- Finally, let $A = \Diamond B$. To falsify $\Diamond B$ at $\mathcal{I}'(\bar{s})$, we need to show that $K', u \not\Vdash B$ whenever $\mathcal{I}'(\bar{s}) R' u$. If $u = \mathcal{I}'(\bar{s}n)$ for some $\bar{s}n \in \mathfrak{L}(\Gamma)$, saturation ensures that $\bar{s}n : B \in \Gamma$, hence, $K', u \not\Vdash B$ by induction hypothesis. The only other children of $\mathcal{I}'(\bar{s})$ are $u = \rho_{\bar{s},v}$, for which $K', u \not\Vdash B$ follows from (4.6). This completes the proof of BNUIP(iii)$'$ for $\mathsf{K}$-saturated nested sequents.

Now we prove BNUIP(iii)′ for all nested sequents that are not $\mathsf{K}$-saturated based on Table 4.1. $\mathcal{A}_p(\Gamma'\{\top\}_{\overline{s}}) = \mathcal{A}_p(\Gamma'\{p, \neg p\}_{\overline{s}}) = \overline{s} : \top$, which cannot be false, thus, BNUIP(iii)′ for them is vacuously true. For $\Gamma'\{A \vee B\}$, $\Gamma'\{A \wedge B\}$, and $\Gamma'\{\Diamond A, [\Delta]\}$, the requisite statement easily follows by induction hypothesis. For instance, for the last of the three, one obtains $(K', \mathcal{I}') \sim^p (K, \mathcal{I})$ such that $K', \mathcal{I}' \not\models \Gamma'\{\Diamond A, [\Delta, A]\}$. Since $\Gamma'\{\Diamond A, [\Delta]\}$ consists of some of these formulas in the same nodes, clearly it is also falsified by $K', \mathcal{I}'$.

For the remaining case, assume $K, \mathcal{I} \not\models \mathcal{A}_p(\Gamma'\{\Box A\}_{\overline{s}})$, i.e.,

$$K, \mathcal{I} \not\models \bigotimes_{i=1}^{m} \left( \overline{s} : \Box D_i \otimes \bigotimes_{\overline{t} \neq \overline{s}n} \overline{t} : C_{i,\overline{t}} \right) \tag{4.7}$$

where

$$\mathcal{A}_p\big(\Gamma'\{\Box A, [A]_{\overline{s}n}\}\big) \equiv \bigotimes_{i=1}^{m} \left( \overline{s}n : D_i \otimes \bigotimes_{\overline{t} \neq \overline{s}n} \overline{t} : C_{i,\overline{t}} \right). \tag{4.8}$$

By (4.7), for some $i$, we have $K, \mathcal{I}(\overline{s}) \not\Vdash \Box D_i$ and $K, \mathcal{I}(\overline{t}) \not\Vdash C_{i,\overline{t}}$ for all $\overline{t} \neq \overline{s}n$. The former means that $K, v \not\Vdash D_i$ for some $v$ such that $\mathcal{I}(\overline{s}) R v$. Therefore, a multiworld interpretation $\mathcal{J} := \mathcal{I} \sqcup \{(\overline{s}n, v)\}$ of $\Gamma'\{\Box A, [A]_{\overline{s}n}\}$ into $K$ falsifies (4.8), and, by induction hypothesis, there is a multiworld interpretation $\mathcal{J}'$ into a $\mathsf{K}$-model $K'$ such that $(K', \mathcal{J}') \sim^p (K, \mathcal{J})$ and $K', \mathcal{J}' \not\models \Gamma'\{\Box A, [A]_{\overline{s}n}\}$. Define $\mathcal{I}'$ to be $\mathcal{J}'$ restricted to the domain of $\mathcal{I}$. Since all formulas from $\Gamma'\{\Box A\}_{\overline{s}}$ are present in $\Gamma'\{\Box A, [A]_{\overline{s}n}\}$, we have $(K, \mathcal{I}) \sim^p (K', \mathcal{I}')$ and $K', \mathcal{I}' \not\models \Gamma'\{\Box A\}_{\overline{s}}$.

This concludes the proof of BNUIP(iii)′, as well as of BNUIP. ■

Theorem 4.2.14 together with Corollary 4.2.6 gives us the following corollary.

#### 4.2.15 Corollary

Logic $\mathsf{K}$ has the uniform interpolation property.

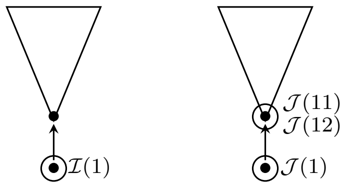
In addition, together with Lemma 4.2.7 we conclude the following. This result is new compared to (van der Giessen *et al.*, 2021a) on which this chapter is based.

#### 4.2.16 Corollary

Bisimulation quantifiers are definable over the class of $\mathsf{K}$-models, i.e., finite irreflexive intransitive trees.

Note that the structure of models as irreflexive intransitive trees was substantially used to ensure that the replacements applied to the original model do not interfere with each other.

**4.2.17 Example**
In Example 4.2.12 we saw that $\mathcal{A}_p(\Box p, \Box\neg p) \equiv 1 : \Box\bot$. We now use this example to demonstrate the importance of injectivity in BNUIP(iii)$'$. Indeed, suppose $K, \mathcal{I} \not\models 1 : \Box\bot$, i.e., $\mathcal{I}(1)$ has at least one child. Assume this is the only child, as in a model depicted on the left:




For a saturation $\Box p, \Box\neg p, [p]_{11}, [\neg p]_{12}$ of this nested sequent, we found an interpolant in SCNF: namely, $1 : \bot \otimes 11 : \bot \otimes 12 : \bot$. A multiworld interpretation $\mathcal{J}$ mapping both 11 and 12 to the only child of $\mathcal{J}(1) := \mathcal{I}(1)$ yields the picture on the right. Clearly, the SCNF is false: $K, \mathcal{J} \not\models 1 : \bot \otimes 11 : \bot \otimes 12 : \bot$. But, without forcing $\mathcal{J}$ to be injective, it is impossible to make $\Box p, \Box\neg p$ false at $\mathcal{J}(1)$: whichever truth value $p$ has at $\mathcal{J}(11)$, it makes one of the boxes true.

### 4.2.3 Uniform interpolation for D and T

The proof for K can be adjusted to prove the same for D and T. For D, if $\Gamma$ is not D-saturated, then we append Table 4.1 with the bottom row of Table 4.2, which is applied only if $\Diamond A$ is not D-saturated in $\Gamma$. Similarly for T, append Table 4.1 with the top row of Table 4.2, which is applied only if $\Diamond A$ is not T-saturated in $\Gamma$. For D- or T-saturated $\Gamma$, we define $\mathcal{A}_p(\Gamma)$ by (4.2) as in the previous section.

| $\Gamma$ matches | $\mathcal{A}_p(\Gamma)$ equals |
|---|---|
| $\Gamma'\{\Diamond A\}$ in logic T | $\mathcal{A}_p(\Gamma'\{\Diamond A, A\})$ |
| $\Gamma'\{\Diamond A\}_{\overline{s}}$ in logic D | $\bigovee_{i=1}^{m} \left(\overline{s} : \Diamond D_i \owedge \bigowedge_{\overline{t}\neq\overline{s}1} \overline{t} : C_{i,\overline{t}}\right)$ where the SDNF of $\mathcal{A}_p(\Gamma'\{\Diamond A, [A]_{\overline{s}1}\})$ is $\bigovee_{i=1}^{m} \left(\overline{s}1 : D_i \owedge \bigowedge_{\overline{t}\neq\overline{s}1} \overline{t} : C_{i,\overline{t}}\right)$ |

**Table 4.2.** Additional recursive rules for constructing $\mathcal{A}_p(\Gamma)$ for $\Gamma$ that are not T-saturated (top row) or not D-saturated (bottom row).

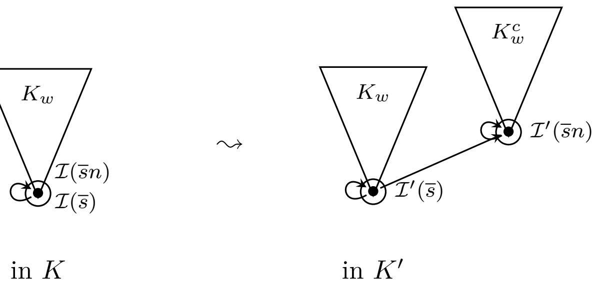



**Figure 4.4.** Cloning reflexive nodes in the construction of T-model $K'$. Circles indicate worlds in the range of $\mathcal{I}$.

### 4.2.18 Theorem

The nested sequent calculus NT has the BNUIP.

Proof. We follow the structure of the proof of Theorem 4.2.14 for K and only describe deviations from it. BNUIP(i) is clearly satisfied by the top row in Table 4.2.

For BNUIP(ii), although T-models are reflexive, this does not affect the reasoning for either saturated nested sequents or non-saturated box formulas. The only new case is the top row of Table 4.2 with an $\overline{s} : \Diamond A$ that is not T-saturated in $\Gamma$. Assume $K, \mathcal{I} \models \mathcal{A}_p(\Gamma'\{\Diamond A, A\}_{\overline{s}})$ for a T-model $K$. By induction, $K, \mathcal{I} \models \Gamma'\{\Diamond A, A\}_{\overline{s}}$. Since $K, \mathcal{I}(\overline{s}) \Vdash A$ implies $K, \mathcal{I}(\overline{s}) \Vdash \Diamond A$ by reflexivity, we have $K, \mathcal{I} \models \Gamma'\{\Diamond A\}_{\overline{s}}$.

For BNUIP(iii)$'$ for T-saturated nested sequents, we have to modify the construction in step (1) on page 124 of an injective multiworld interpretation $\mathcal{J}$ into a new T-model $M$ out of the given $\mathcal{I}$ where $K, \mathcal{I} \not\models \mathcal{A}_p(\Gamma)$. For K, the breadth-first order of injectifying the interpretations of sequent nodes could only yield one situation of $\overline{s}n$ being conflated with some already processed $\overline{t}$: namely, when $\overline{t} = \overline{s}m$ for some $m \neq n$. This can still happen for T-models and is processed the same way. But, due to reflexivity, there is another possibility: conflating with $\overline{t} = \overline{s}$. Here, cloning is used (see Figure 4.4), which yields a bisimilar T-model by Lemma 4.2.9.

Having intransitive trees that are reflexive rather than irreflexive in step (2) on page 124 does not affect the argument. The proof that $K', \mathcal{I}' \not\models \Gamma$ for the given T-saturated $\Gamma$ in step (3) on page 126 requires an adjustment only for the case of $\overline{s} : \Diamond B \in \Gamma$. In addition we have to show that $K', \mathcal{I}'(\overline{s}) \nVdash B$. This is resolved by observing that $\overline{s} : B \in \Gamma$ due to T-saturation and, hence, $B$ must also be false in $\mathcal{I}'(\overline{s})$ by induction hypothesis.

Finally, for BNUIP(iii)$'$ for non-T-saturated nested sequents, we gain a new case when the top row of Table 4.2 is used, but it is clear that $K', \mathcal{I}' \not\models \Gamma'\{\Diamond A, A\}$ obtained by induction hypothesis directly implies $K', \mathcal{I}' \not\models \Gamma'\{\Diamond A\}$. ■

**4.2.19 Theorem**
The nested sequent calculus ND has the BNUIP.

Proof. Again, we follow the structure of the proofs of Theorem 4.2.14 for K. BNUIP(i) is clearly satisfied for the bottom row in Table 4.2.

For BNUIP(ii), the only new case is applying the bottom row of Table 4.2 to a non-D-saturated $\overline{s} : \Diamond A$ in $\Gamma = \Gamma'\{\Diamond A\}_{\overline{s}}$. Let

$$K, \mathcal{I} \models \bigovee_{i=1}^{m} \left( \overline{s} : \Diamond D_i \owedge \bigowedge_{\overline{t} \neq \overline{s}1} \overline{t} : C_{i,\overline{t}} \right)$$

for some multiworld interpretation $\mathcal{I}$ into a D-model $K = (W, R, V)$ where

$$\mathcal{A}_p(\Gamma'\{\Diamond A, [A]_{\overline{s}1}\}) \equiv \bigovee_{i=1}^{m} \left( \overline{s}1 : D_i \owedge \bigowedge_{\overline{t} \neq \overline{s}1} \overline{t} : C_{i,\overline{t}} \right).$$

Then, for some $i$, we have $K, \mathcal{I}(\overline{t}) \Vdash C_{i,\overline{t}}$ for all $\overline{t} \in \mathfrak{L}(\Gamma)$ and $K, \mathcal{I}(\overline{s}) \Vdash \Diamond D_i$. Thus, $K, v \Vdash D_i$ for some $v$ such that $\mathcal{I}(\overline{s}) R v$. Since $\Diamond A$ is not D-saturated in $\Gamma'\{\Diamond A\}_{\overline{s}}$, it follows that $\mathcal{I}_v := \mathcal{I} \sqcup \{(\overline{s}1, v)\}$ is a multiworld interpretation of $\Gamma'\{\Diamond A, [A]_{\overline{s}1}\}$ into $K$ such that $K, \mathcal{I}_v \models \mathcal{A}_p(\Gamma'\{\Diamond A, [A]_{\overline{s}1}\})$. By induction hypothesis, $K, \mathcal{I}_v \models \Gamma'\{\Diamond A, [A]_{\overline{s}1}\}$, from which it follows that $K, \mathcal{I} \models \Gamma'\{\Diamond A\}_{\overline{s}}$.

For BNUIP(iii)$'$ for D-saturated nested sequents, we must change step (1) on page 124 to preserve D-models. By Lemma 4.2.9, duplication used for K preserves D-models when applied to non-leaves of D-models because they are irreflexive. So we first proceed as in step (1) for non-leaves and we denote the obtained model by $N = (W_N, R_N, V_N)$ and the multiworld interpretation by $\mathcal{J}'$. Now we deal with all leaves $w$ in $N$, for which it might happen that $w = \mathcal{J}'(\overline{s})$ and $w = \mathcal{J}'(\overline{t})$ for two different labels $\overline{s}, \overline{t} \in \mathfrak{L}(\Gamma)$. To ensure injectivity, we construct the following model $M = (W_M, R_M, V_M)$. Its construction is depicted in Figure 4.5. Let $W_l$ be the set of leaves of $W_N$ and write $\mathfrak{L}_l := \{\overline{s} \in \mathfrak{L}(\Gamma) \mid \mathcal{J}'(\overline{s}) = w \text{ for some } w \in W_l\}$.

$$\begin{aligned}
W_M &:= W_N \setminus W_l \sqcup \{w_{\overline{s}} \mid \overline{s} \in \mathfrak{L}_l\} \\
R_M &:= R_N \setminus \{(v, w) \mid v \in W_N, w \in W_l\} \sqcup \{(w_{\overline{s}}, w_{\overline{s}n}) \mid \overline{s}, \overline{s}n \in \mathfrak{L}_l\} \\
&\qquad \sqcup \{(w_{\overline{s}}, w_{\overline{s}}) \mid \overline{s} \in \mathfrak{L}_l \text{ and there is no } n \text{ such that } \overline{s}n \in \mathfrak{L}(\Gamma)\}
\end{aligned}$$

$$V_M(v) := \begin{cases} V_N(v) & \text{if } v \in W \setminus W_l; \\ V_N(w) & \text{if } v = w_{\overline{s}} \text{ and } w = \mathcal{I}(\overline{s}). \end{cases}$$

Accordingly, define $\mathcal{J}(\overline{s}) := w_{\overline{s}}$ for $\overline{s} \in \mathfrak{L}_l$ and $\mathcal{J}(\overline{s}) := \mathcal{J}'(\overline{s})$ otherwise. By reasoning similar to the proof of Lemma 4.2.9, it is easy to show that $M$ is a D-model and $(M, \mathcal{J}) \sim^p (K, \mathcal{I})$ with all $w_{\overline{s}}$ being bisimilar to $w$. The replacements of step (2) preserve D-models by Lemma 4.2.9. Step (3) requires no changes either.

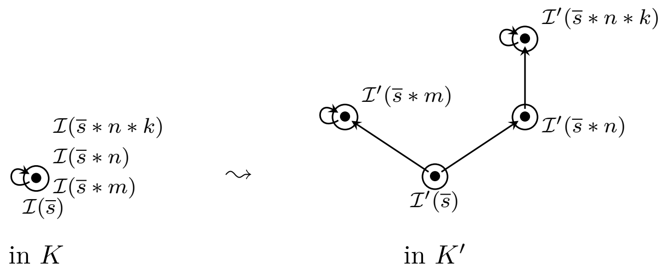



**Figure 4.5.** Additional transformation for constructing D-model $K'$ for reflexive leaves. Circles indicate worlds in the range of $\mathcal{I}$.

The only subtlety in the proof that $K', \mathcal{I}' \not\models \Gamma$ for a D-saturated $\Gamma$ is for $\overline{s} : \Diamond B \in \Gamma$. The argument for $K', \mathcal{I}'(\overline{s}) \not\models \Diamond B$ works the same way as in K for the following reason. Since $\Diamond B$ is D-saturated, node $\overline{s}$ must have a child in the sequent tree. Injectivity of $\mathcal{I}'$ means that $\mathcal{I}'(\overline{s})$ is not a leaf in the D-model $K'$ and, hence, not reflexive.

It remains to show BNUIP(iii)$'$ for non-saturated sequents. The only new case is the application of the bottom row of Table 4.2 for a non-D-saturated $\overline{s} : \Diamond A$, i.e., when node $\overline{s}$ is a leaf of the sequent tree, in BNUIP(iii)$'$. Let

$$K, \mathcal{I} \not\models \bigovee_{i=1}^{m} \left( \overline{s} : \Diamond D_i \owedge \bigowedge_{\overline{t} \neq \overline{s}1} \overline{t} : C_{i,\overline{t}} \right).$$

By seriality, there is a world $v \in W$ such that $\mathcal{I}(\overline{s}) R v$. Define $\mathcal{J} := \mathcal{I}' \sqcup \{(\overline{s}1, v)\}$ which is a multiworld interpretation of $\Gamma'\{\Diamond A, [A]_{\overline{s}1}\}$ into $K$ such that

$$K, \mathcal{J} \not\models \bigovee_{i=1}^{m} \left( \overline{s}1 : D_i \owedge \bigowedge_{\overline{t} \neq \overline{s}1} \overline{t} : C_{i,\overline{t}} \right).$$

By induction hypothesis, there is a multiworld interpretation $\mathcal{J}'$ of $\Gamma'\{\Diamond A, [A]_{\overline{s}1}\}$ into some D-model $K'$ such that $(K', \mathcal{J}') \sim^p (K, \mathcal{J})$ and $K', \mathcal{J}' \not\models \Gamma'\{\Diamond A, [A]_{\overline{s}1}\}$. Similar to the case of $\Box A$ for K, restricting this $\mathcal{J}'$ to the labels of $\Gamma$ yields a multiworld interpretation bisimilar to $\mathcal{I}$ and refuting $\Gamma = \Gamma'\{\Diamond A\}_{\overline{s}}$. ■

Theorems 4.2.18 and 4.2.19 lead to the following analogues of Corollaries 4.2.15 and 4.2.16.

**4.2.20 Corollary**
Logics T and D have the uniform interpolation property.

#### 4.2.21 Corollary

Bisimulation quantifiers are definable over the class of T-models, i.e., finite intransitive reflexive trees. Bisimulation quantifiers are also definable over the class of D-models, i.e., finite intransitive trees with the leaves being the only reflexive worlds.

## 4.3 Uniform interpolation for S5 via hypersequents

Uniform interpolation for hypersequents is defined in the same way as for nested sequents. All definitions and lemmas between Definition 4.2.1 and Corollary 4.2.6 are naturally adapted to the hypersequent setting. Instead of NUIP and BNUIP we now speak of the *hypersequent uniform interpolation property* (*HUIP*) and the *bisimulation hypersequent uniform interpolation property* (*BHUIP*) respectively.

Recall that all definitions and results from Section 4.1.3 also hold in the hypersequent setting with labels $n$ in multiformulas. Therefore, so far the adaptation of nested sequents to the hypersequent setting of logic S5 works smoothly, seemingly with no effect on our method. The crucial difference lies in making the inductive step for the recursive case for saturated hypersequents. The proof of BNUIP(iii)$'$ in Theorem 4.2.14 relies on the fact that in treelike models the truth values of $p$ in a submodel rooted in one child of a world $w$ can be adjusted without affecting the truth value of $p$ in $w$ itself or in submodels rooted in $w$'s other children. By contrast, changing the truth value of $p$ in one world of an S5-model affects all other worlds of the model, making it hard to coordinate the changes across multiple recursive calls.

#### 4.3.1 Remark

In the technical report (van der Giessen *et al.*, 2021b), we sidestepped this difficulty by making use of the fact that every modal formula is S5-equivalent to a formula of modal depth less or equal to one, see, e.g., (Fitting, 1983, Section 5.13). For the modal language restricted to such formulas, recursive calls can only be made for purely propositional formulas, making the coordination possible based on propositional uniform interpolation. Thus, both the interpolant construction used for K, D, and T and the proof of correctness for constructed interpolants applies to this modal fragment with at most cosmetic changes.

Here, we provide a proper construction of uniform interpolants for the full modal language of S5, without relying on *ad hoc* formula transformations. Thus, this method has much more potential to be generalized to other modal logics in a modular and uniform way.

For saturated hypersequents, instead of the recursive case (i.e., interpolant definition (4.2) for logic K), we define an appropriate interpolant transformation based on an application of the hypersequent rule d (defined in (4.1) in Section 4.1.2), This preserves all information about the hypersequent and therefore enables us to satisfy BHUIP(iii)′ by adjusting the truth values of $p$ globally in the whole model.

Unfortunately, with the addition of the rule d, the hypersequent calculus is not terminating anymore. To ensure termination, we employ an external bookkeeping device that shows similarities with the terminating sequent calculus for logic T in (Bílková, 2006). There it is internalized in the calculus, but we choose to only incorporate the bookkeeping in the calculation of the interpolants where we annotate each call to the function $\mathcal{A}_p$ with a set $\Sigma$ containing all modal formulas that have already triggered by an application of rule d. In contrast to the nested sequent setting, we construct the interpolants according to a strict order on rule applications to saturate a hypersequent, giving priority to propositional saturation.

For a hypersequent $\mathcal{G} = \Gamma_1 \mid \cdots \mid \Gamma_n$ we define the set of all boxed and diamond formulas in it:

$$\Box\Diamond\mathcal{G} := \{\Box A \mid \Box A \in \Gamma_i \text{ for some } i \leq n\} \cup \{\Diamond A \mid \Diamond A \in \Gamma_i \text{ for some } i \leq n\}.$$

| $\mathcal{G}$ matches | $\mathcal{A}_p(\Sigma; \mathcal{G})$ equals |
|---|---|
| $\mathcal{G}' \mid \Gamma, \top$ | $1 : \top$ |
| $\mathcal{G}' \mid \Gamma, p, \neg p$ | $1 : \top$ |
| $\mathcal{G}' \mid \Gamma, A \vee B$ | $\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, A \vee B, A, B)$ |
| $\mathcal{G}' \mid \Gamma, A \wedge B$ | $\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, A \wedge B, A) \owedge \mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, A \wedge B, B)$ |
| $\mathcal{G}' \mid \Gamma, \Box A$ | $\bigowedge_{i=1}^{m} \left(1 : \Box D_i \ovee \bigovee_{j=1}^{\lvert\mathcal{G}\rvert} j : C_{ij}\right)$ where the SCNF of $\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, \Box A \mid \{A\}_{\lvert\mathcal{G}\rvert+1})$ is $\bigowedge_{i=1}^{m} \left(\lvert\mathcal{G}\rvert + 1 : D_i \ovee \bigovee_{j=1}^{\lvert\mathcal{G}\rvert} j : C_{ij}\right)$ |
| $\mathcal{G}' \mid \Gamma, \Diamond A$ | $\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, \Diamond A, A)$ |
| $\mathcal{G}' \mid \Gamma, \Diamond A \mid \Delta$ | $\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, \Diamond A \mid \Delta, A)$ |

**Table 4.3.** Recursive construction of $\mathcal{A}_p(\Sigma; \mathcal{G})$ for non-saturated hypersequents $\mathcal{G}$. The notation $\{A\}_{\lvert\mathcal{G}\rvert+1}$ in row 6 specifies the label $\lvert\mathcal{G}\rvert + 1$ to be used for this new sequent component.

For a hypersequent $\mathcal{G}$ we define the interpolant $\mathcal{A}_p(\mathcal{G}) := \mathcal{A}_p(\emptyset, \mathcal{G})$, where the computation of $\mathcal{A}_p(\Sigma; \mathcal{G})$ with annotation $\Sigma$ is defined as follows. We use Table 4.3 in which each row corresponds to a rule from Figure 4.2. On input $\mathcal{G}$ and $\Sigma$ do:

1. If possible, apply rows 1–2 of Table 4.3, stop, and return $1 : \top$.
2. If Step 1 is not applicable and one of the formulas of the form $A \vee B$ or $A \wedge B$ is not saturated, apply rows 3–4 of Table 4.3, to saturate this formula and go to Step 1.
3. If Steps 1–2 are not applicable and one of the formulas of the form $\Box A$ is not saturated or one of the formulas of the form $\Diamond A$ is not saturated w.r.t. some sequent component, apply rows 5–8 of Table 4.3 to saturate this formula (w.r.t. this component) and go to Step 1.
4. If Steps 1–3 are not applicable, i.e., the hypersequent is saturated, then
   (a) if there are no diamond formulas in $\mathcal{G}$, stop and return

$$\mathcal{A}_p(\Sigma; \mathcal{G}) \qquad := \qquad \bigovee_{\substack{k:\ell \in \mathcal{G} \\ \ell \in \mathsf{Lit} \setminus \{p, \neg p\}}} k : \ell. \tag{4.9}$$

   (b) if $\Box\Diamond\mathcal{G} \subseteq \Sigma$, stop and return (4.9);
   (c) otherwise, let $\mathcal{G} = \Gamma_1, \Diamond\Delta_1 \mid \cdots \mid \Gamma_n, \Diamond\Delta_n$ where $\Gamma_i$'s contain no diamond formulas, apply the rule $\mathsf{d}$ as follows:

$$\mathcal{A}_p(\Sigma; \mathcal{G}) \qquad := \qquad \bigovee_{i=1}^{m} \left( 1 : \Diamond D_i \owedge \bigowedge_{j=1}^{n} j : C_{ij} \right), \tag{4.10}$$

   where the SDNF of

$$\mathcal{A}_p(\Box\Diamond\mathcal{G}\ ;\ \mathcal{G} \mid \underbrace{[\Delta_1, \dots, \Delta_n]}_{\mathsf{d}\text{-component}}) \quad = \quad \bigovee_{i=1}^{m} \left( \mathsf{d} : D_i \owedge \bigowedge_{j=1}^{n} j : C_{ij} \right). \tag{4.11}$$

   After that go to Step 1 with $\mathcal{G} := \mathcal{G} \mid [\Delta_1, \dots, \Delta_n]$ and $\Sigma := \Box\Diamond\mathcal{G}$.

The new component that the rule $\mathsf{d}$ creates in (4.11) is called the $\mathsf{d}$-*component* and labeled $\mathsf{d}$ (in addition to the usual numerical label). A hypersequent cannot have more than one $\mathsf{d}$-component, meaning that any new application of the rule $\mathsf{d}$ simultaneously removes the $\mathsf{d}$ status from a preceding $\mathsf{d}$-component if there was one. We enclose the current $\mathsf{d}$-component in brackets [...] in addition to the usual component separator |. Both the label $\mathsf{d}$ and the brackets are purely bookkeeping devices and do not affect how rules are applied.

We will slightly abuse terminology by sometimes speaking about *the proof search tree of* $\mathcal{A}_p$ and applying terms such as *ancestors*, *children*, *leaves*, etc. to the construction of $\mathcal{A}_p$. For instance Steps 1, 4a, and 4b return a value without a recursive call to $\mathcal{A}_p$, corresponding to leaves of the proof search tree of $\mathcal{A}_p$. Before we show termination of the algorithm (aka proof search), we provide an example.

**4.3.2 Example**

Consider the single component hypersequent $\mathcal{G} = \Diamond(p \vee q) \wedge \Box(\neg p \vee r)$. The proof search tree decorated with sets $\Sigma$ is depicted below, where $A = \Diamond(p \vee q) \wedge \Box(\neg p \vee r)$. The right branch terminates when the hypersequent is saturated because there are no diamond formulas. In the left branch the rule d is applied to create a d-component and the only modal formula $\Diamond(p \vee q)$ is placed into the annotation. After saturation, formula $\Diamond(p \vee q)$ remains the only box or diamond formula in the hypersequent, and it is already in the annotation. So here ends the left branch.

$$
\begin{prooftree}
\AxiomC{$\Diamond(p \vee q);\quad A, \Diamond(p \vee q), p \vee q, p, q \mid [p \vee q, p, q]$}
\UnaryInfC{$\Diamond(p \vee q);\quad A, \Diamond(p \vee q), p \vee q, p, q \mid [p \vee q]$}
\LeftLabel{$\mathsf{d}$}
\UnaryInfC{$\emptyset;\quad A, \Diamond(p \vee q), p \vee q, p, q$}
\UnaryInfC{$\emptyset;\quad A, \Diamond(p \vee q), p \vee q$}
\UnaryInfC{$\emptyset;\quad A, \Diamond(p \vee q)$}
\AxiomC{$\emptyset;\quad A, \Box(\neg p \vee r) \mid \neg p \vee r, \neg p, r$}
\UnaryInfC{$\emptyset;\quad A, \Box(\neg p \vee r) \mid \neg p \vee r$}
\RightLabel{$\Box$}
\UnaryInfC{$\emptyset;\quad A, \Box(\neg p \vee r)$}
\BinaryInfC{$\emptyset;\quad \Diamond(p \vee q) \wedge \Box(\neg p \vee r)$}
\end{prooftree}
$$

For the right branch, the interpolant for the leaf is computed according to (4.9),

$$\mathcal{A}_p\Big(\emptyset;\quad A, \Box(\neg p \vee r) \mid \neg p \vee r, \neg p, r\Big) \quad = \quad 1 : \bot \ovee 2 : r.$$

It is already in SCNF, so for the conclusion of the rule $\Box$, the interpolant is

$$\mathcal{A}_p\Big(\emptyset;\quad A, \Box(\neg p \vee r)\Big) \quad = \quad 1 : \bot \ovee 1 : \Box r.$$

For the left branch, the leaf also gets the (4.9) treatment

$$\mathcal{A}_p\Big(\Diamond(p \vee q);\quad A, \Diamond(p \vee q), p \vee q, p, q \mid [p \vee q, p, q]\Big) \quad = \quad 1 : q \ovee \mathsf{d} : q.$$

An SDNF of this interpolant is $(1 : q \owedge \mathsf{d} : \top) \ovee (1 : \top \owedge \mathsf{d} : q)$, thus, the interpolant for the conclusion of the rule d, which is computed by (4.10), is

$$\mathcal{A}_p\Big(\emptyset;\quad A, \Diamond(p \vee q), p \vee q, p, q\Big) \quad = \quad (1 : q \owedge 1 : \Diamond\top) \ovee (1 : \top \owedge 1 : \Diamond q).$$

Now, combining the two branches, we obtain

$$\mathcal{A}_p(\emptyset;\quad \mathcal{G}) \quad = \quad \Big(1 : \bot \ovee 1 : \Box r\Big) \owedge \Big((1 : q \owedge 1 : \Diamond\top) \ovee (1 : \top \owedge 1 : \Diamond q)\Big).$$

After simplifications with Lemma 4.1.34, equivalently, $\mathcal{A}_p(\mathcal{G}) \equiv 1 : \Box r \wedge (q \vee \Diamond q)$. Or, since $\Diamond q$ and $q \vee \Diamond q$ are equivalent in S5, we have $\mathcal{A}_p(\mathcal{G}) \equiv 1 : \Box r \wedge \Diamond q$. Indeed, one can check that it is a uniform interpolant of $\mathcal{G}$ by checking the conditions of BHUIP. So $\Box r \wedge \Diamond q$ is a uniform interpolant w.r.t. $p$ for $\Diamond(p \vee q) \wedge \Box(\neg p \vee r)$ in S5.

**4.3.3 Proposition** (Properties of $\Sigma$)
Consider a proof search tree of $\mathcal{A}_p(\mathcal{H}) = \mathcal{A}_p(\emptyset; \mathcal{H})$. Let $\mathcal{A}_p(\Sigma; \mathcal{G})$ and $\mathcal{A}_p(\Sigma'; \mathcal{G}')$ occur in this proof search such that $\mathcal{A}_p(\Sigma; \mathcal{G})$ is an ancestor of $\mathcal{A}_p(\Sigma'; \mathcal{G}')$. Then

1. $\Sigma \subseteq \Sigma'$;
2. If $\mathcal{A}_p(\Sigma; \mathcal{G})$ is a saturated non-leaf, i.e., Step 4c with rule d is applied to it, and the only other steps applied in between $\mathcal{A}_p(\Sigma; \mathcal{G})$ and $\mathcal{A}_p(\Sigma'; \mathcal{G}')$ are Steps 1 and 2 (propositional axioms and rules), then $i : A \in \mathcal{G}'$ is saturated for any $A \in \Sigma'$ (for $A = \Diamond B$ this means saturation w.r.t. every component of $\mathcal{G}'$).

Proof. For the first property observe that the annotation in the root of the proof search tree is empty. Also note that, from bottom-up, no rule removes formulas. So annotations can only grow.

The second property is due to the fact that $\Sigma' = \Box\Diamond\mathcal{G}$ and $\mathcal{G}' = \mathcal{G} \mid [\Theta]$ where $B \in \Theta$ for every $\Diamond B \in \Sigma'$. If $\Box B \in \Sigma'$, then $j : \Box B \in \mathcal{G}$ for some component $j$, and $k : B \in \mathcal{G}$ for some $k$ by the saturation of $\mathcal{G}$, making any $i : \Box B \in \mathcal{G} \mid [\Theta]$ saturated. If $\Diamond B \in \Sigma'$, then $j : \Diamond B \in \mathcal{G}$ for some component $j$. Therefore, $k : B \in \mathcal{G}$ for all $k = 1, \dots, |\mathcal{G}|$ by the saturation of $\mathcal{G}$, and $B \in \Theta$ by the definition of the rule d, making any $i : \Diamond B \in \mathcal{G} \mid [\Theta]$ saturated w.r.t. every component. ■

**4.3.4 Theorem**
Let $\mathcal{G}$ be a hypersequent. Then $\mathcal{A}_p(\mathcal{G})$ is well defined, that is, the calculation of $\mathcal{A}_p(\emptyset; \mathcal{G})$ terminates.

Proof. Steps 1-3, terminate by the termination of the rules as shown in Theorem 4.1.24. In addition, Step 4b guarantees the termination of the algorithm by curtailing the use of the rule d from Step 4c. Indeed, by Proposition 4.3.3 1 that whenever Step 4c is applied, we have $\Sigma \subsetneq \Box\Diamond\mathcal{G}$. So the annotations $\Sigma$ form a non-decreasing sequence of sets along each branch of the proof search tree from root to leaves. As all formulas appearing in these annotations must be subformulas of the hypersequent at the root of the proof search tree, there can be only finitely many applications of the rule d along each branch. ■

It is possible to show properties BHUIP((i)) and BHUIP((ii)) similarly to the nested sequent case. However, for BHUIP((iii)$'$) the situation is more complex. In particular, for leaves $\mathcal{G}$ of the proof search tree the falsity of $\mathcal{A}_p(\Sigma; \mathcal{G})$ in a model is generally not sufficient to make $\mathcal{G}$ false for some $p$-bisimilar model. This creates non-trivial base cases in the inductive proof of BHUIP(iii)$'$. Whether the statement for the last applications of Step 4c along a proof search branch relies on the induction hypothesis turns out to be model-dependent. Formally, we only prove BHUIP(iii)$'$ for a subset of (annotated) hypersequents in the proof search.

**4.3.5 Definition** (Self-sufficient and insufficient hypersequents)
Given a terminated proof search tree, the set of *self-sufficient* (annotated) hypersequents is the smallest subset of all hypersequents from this tree such that

- every saturated hypersequent without diamond formulas, which must be a leaf, is self-sufficient;
- every saturated hypersequent that is *not* a leaf is self-sufficient;
- if all children of a non-saturated hypersequent in the tree are self-sufficient, then the sequent itself is self-sufficient (this includes leaves from Step 1).

All hypersequents that are not self-sufficient are called *insufficient*.

**4.3.6 Example**
In the proof search tree in Example 4.3.2, all the hypersequents are self-sufficient except for the top two in the left branch:

$$A, \Diamond(p \vee q), p \vee q, p, q \mid [p \vee q, p, q] \quad \text{and} \quad A, \Diamond(p \vee q), p \vee q, p, q \mid [p \vee q], \tag{4.12}$$

where $A = \Diamond(p \vee q) \wedge \Box(\neg p \vee r)$. Recall that for both hypersequents, the function $\mathcal{A}_p$ returns $1 : q \otimes \mathsf{d} : q$. Consider an S5-model $K$ with worlds $w_1$, $w_2$, and $w_3$ such that $q$ is false in $w_1$ and $w_2$ but true in $w_3$, whereas $r$ is true in all three worlds (we do not specify the truth values of $p$ as they can be freely changed in a $p$-bisimilar model). Then $K, w_1, w_2 \not\models 1 : q \otimes \mathsf{d} : q$. It is, however, impossible to construct a $p$-bisimilar model that falsifies the hypersequents from (4.12). Indeed, that would require to falsify $\Diamond(p \vee q)$. However, any $p$-bisimilar model $K'$ contains a world $w_3'$ $p$-bisimilar to $w_3$, meaning that $K', w_3' \Vdash q$. Hence $K', w_3 \Vdash p \vee q$ and $\Diamond(p \vee q)$ is true throughout $K'$. In fact, we chose the truth values of $r$ in $K$ in such a way that the hypersequent $\Diamond(p \vee q) \wedge \Box(\neg p \vee r)$ in the root of the proof search tree is true in every $p$-bisimilar model. This demonstrates why falsity of 'interpolants' of insufficient hypersequents fails to achieve the ultimate goal of the algorithm.

**4.3.7 Lemma**
Consider a proof search tree of $\mathcal{A}_p(\mathcal{H}) = \mathcal{A}_p(\emptyset; \mathcal{H})$. Then

1. $\mathcal{H}$ (annotated with $\Sigma = \emptyset$) is self-sufficient.
2. If Step 4c is applied to $\mathcal{A}_p(\Sigma; \mathcal{G})$, then every branch leading from this hypersequent, after several applications of Steps 1 and 2, results in $\mathcal{A}_p(\Box\Diamond\mathcal{G}; \mathcal{G}|[\Delta])$ such that it is either
   (a) a saturated leaf or
   (b) a propositionally saturated self-sufficient hypersequent.
3. A leaf produced by Step 4b is insufficient.

Proof. The third property is immediate. The first property follows as every branch leading from it leads to one of the following three options. It leads either to a leaf produced by Step 1, or to a saturated leaf without diamond formulas produced by

Step 4a, or to a saturated hypersequent with diamond formulas, which cannot be present in $\Sigma = \emptyset$, to which Step 4c is applied.

Now we will justify the second property. The rule $\mathsf{d}$ applied to $\mathcal{G}$ annotated by $\Sigma$ yields $\mathcal{G} \mid [\Theta]$ for some $\Theta$ annotated by $\Box\Diamond\mathcal{G}$. Due to $\mathcal{G}$ being saturated, applications of propositional rules from Steps 1-2 along any branch lead to hypersequent $\mathcal{G} \mid [\Theta_j]$ for some $\Theta_j \supseteq \Theta$ annotated by $\Box\Diamond\mathcal{G}$. If $\Box\Diamond(\mathcal{G} \mid [\Theta_j]) \not\subseteq \Box\Diamond\mathcal{G}$, then any saturation of this hypersequent still contains new modal formulas and triggers Step 4c. Thus, all saturations are self-sufficient, and so is $\mathcal{G} \mid [\Theta_j]$. Otherwise, $\Box\Diamond(\mathcal{G} \mid [\Theta_j]) = \Box\Diamond\mathcal{G}$, so we stop according to Step 4b making it a leaf. In addition, it follows from Proposition 4.3.3 (2) that in this case, $\mathcal{G} \mid [\Theta_j]$ is saturated. ■

**4.3.8 Theorem**
Logic $\mathsf{S5}$ has the BHUIP.

Proof. We follow the proof of Theorem 4.2.14 showing the three conditions for BHUIP. It is easily seen that $\mathcal{A}_p(\Sigma; \mathcal{G})$ does not contain $p$ and that its labels are from $\mathcal{G}$.

We show BHUIP((ii)) by induction on the finite proof search tree. We will see that $\Sigma$ plays no role and can essentially be ignored. Let $w_1, \dots, w_n$ be a multiworld interpretation of a hypersequent $\mathcal{G}$ in the proof search (annotated by $\Sigma$), and of the multiformula $\mathcal{A}_p(\Sigma; \mathcal{G})$, into an $\mathsf{S5}$-model $K = (W, W \times W, V)$. We use induction on the proof search to show

$$K, w_1, \dots, w_n \models \mathcal{A}_p(\Sigma; \mathcal{G}) \qquad \text{implies} \qquad K, w_1, \dots, w_n \models \mathcal{G}.$$

First we briefly treat the cases from Table 4.3 and then we consider the case where $\mathcal{G}$ is saturated. Cases in rows 1-2 of Table 4.3 are trivial and rows 3-4 for the connectives work the same way as for nested sequents. The case of $\Box A$ is also very similar. The only difference from the nested case for $\mathsf{K}$ is that instead of considering only children of the world to make $\Box A$ true in a treelike model, here we have to consider all worlds in the model. Otherwise, the reasoning is the same. The penultimate row of Table 4.3 can be processed the same way as the row for $\mathsf{T}$ in Table 4.2 because $\mathsf{S5}$-models are similarly reflexive. Finally, the last row of Table 4.3 works the same way as the last row of Table 4.1 because the interpretation of the label with $A$ is in both cases accessible from the interpretation of the label with $\Diamond A$.

The case where $\mathcal{G}$ is saturated and its interpolant was computed according to (4.9) is trivial as the truth of the interpolant implies that some literal from the hypersequent is true. Finally, the case of a saturated $\mathcal{G} = \Gamma_1, \Diamond\Delta_1 \mid \cdots \mid \Gamma_n, \Diamond\Delta_n$ treated via (4.10) is similar in nature to the treatment of row for $\mathsf{D}$ in Table 4.2. However, since the rule is different, let us show this case in full. Let

$$K, w_1, \ldots, w_n \qquad \models \qquad \bigovee_{i=1}^{m} \left( 1 : \Diamond D_i \owedge \bigowedge_{j=1}^{n} j : C_{ij} \right)$$

for some worlds $w_1, \ldots, w_n$ from an S5-model $K = (W, W \times W, V)$ for the interpolant $\mathcal{A}_p(\Sigma; \mathcal{G})$ where

$$\mathcal{A}_p(\Box\Diamond\mathcal{G};\quad \mathcal{G} \mid [\Delta_1, \ldots, \Delta_n]) \qquad \equiv \qquad \bigovee_{i=1}^{m} \left( \mathsf{d} : D_i \owedge \bigowedge_{j=1}^{n} j : C_{ij} \right).$$

Then, for some $i$, we have $K, w_j \Vdash C_{ij}$ for all $j = 1, \ldots, n$ and $K, w_1 \Vdash \Diamond D_i$. Thus, $K, v \Vdash D_i$ for some $v \in W$. Accordingly, due to its $i$th disjunct,

$$K, w_1, \ldots, w_n, v \qquad \models \qquad \mathcal{A}_p(\Box\Diamond\mathcal{G};\quad \mathcal{G} \mid [\Delta_1, \ldots, \Delta_n]).$$

By induction we know, $K, w_1, \ldots, w_n, v \models \mathcal{G} \mid [\Delta_1, \ldots, \Delta_n]$. The case of $K, w_j \Vdash A$ for some $j : A \in \mathcal{G}$ is trivial. If $K, v \Vdash B$ for some $B \in \Delta_j$, then $j : \Diamond B \in \mathcal{G}$ and $K, w_j \Vdash \Diamond B$. Hence, in all cases, $K, w_1, \ldots, w_n \models \mathcal{G}$ as required.

We now show BHUIP(iii)′ for all self-sufficient hypersequents by induction on the finite proof search tree. This works, since the root of the proof search tree is always self-sufficient by Lemma 4.3.7. Let $w_1, \ldots, w_n$ be a multiworld interpretation of a self-sufficient $\mathcal{G}$ (annotated by $\Sigma$) into an S5-model $K = (W, W \times W, V)$ such that $K, w_1, \ldots, w_n \not\models \mathcal{A}_p(\Sigma; \mathcal{G})$. We will find a model $K' = (W', W' \times W', V')$ and an injective interpretation $w'_1, \ldots, w'_n$ of $\mathcal{G}$ into $K'$ such that

$$(K, w_1, \ldots, w_n) \sim^p (K', w'_1, \ldots, w'_n) \tag{4.13}$$

and $K', w'_1, \ldots, w'_n \not\models \mathcal{G}$.

We start with Steps 1-3 of the algorithm of $\mathcal{A}_p$. These cover all rows from Table 4.3 and are shown by induction similarly to the nested sequent setting. Let us only present the case for conjunction and the box belonging to the fourth and fifth row of Table 4.3. For a detailed proof see (van der Giessen *et al.*, 2022).

If $\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, A_1 \wedge A_2)$ is obtained from two self-sufficient hypersequents as

$$\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, A_1 \wedge A_2, A_1) \quad \owedge \quad \mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, A_1 \wedge A_2, A_2),$$

then one of the conjuncts must be false and by induction hypothesis for that self-sufficient hypersequent there is a $p$-bisimilar injective interpretation $w'_1, \ldots, w'_n$ into a model $K'$ such that all formulas from $\mathcal{G} \mid \Gamma, A_1 \wedge A_2, A_i$ are false in their respective worlds for some $i \in \{1, 2\}$. The same interpretation falsifies all formulas in our smaller hypersequent $\mathcal{G} \mid \Gamma, A_1 \wedge A_2$.

If $\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, \Box A)$ is obtained from a self-sufficient hypersequent

$$\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, \Box A \mid \{A\}_{n+1}) \quad = \quad \bigotimes_{i=1}^{m} \left( n+1 : D_i \mathbin{\otimes} \bigovee_{j=1}^{n} j : C_{ij} \right) \tag{4.14}$$

with interpolant in the SCNF as

$$\mathcal{A}_p(\Sigma; \mathcal{G}' \mid \Gamma, \Box A) \quad = \quad \bigotimes_{i=1}^{m} \left( 1 : \Box D_i \mathbin{\otimes} \bigovee_{j=1}^{n} j : C_{ij} \right), \tag{4.15}$$

then its falsity means that there is $i$ such that $K, w_j \nVdash C_{ij}$ for all $i = 1, \ldots, |\mathcal{G}|$ and $K, w_1 \nVdash \Box D_i$. Accordingly, there must exist a world $v$ such that $K, v \nVdash D_i$. The interpretation $w_1, \ldots, w_n, v$ makes the interpolant (4.14) false because of the $i$th conjunct. Hence, by induction hypothesis there is $p$-bisimilar injective interpretation $w'_1, \ldots, w'_n, v'$ into a model $K'$ such that

$$K', w'_1, \ldots, w'_n, v' \not\models \mathcal{G}' \mid \Gamma, \Box A \mid \{A\}_{n+1}.$$

Clearly, (4.13) is fulfilled and $K', w'_1, \ldots, w'_n \not\models \mathcal{G}' \mid \Gamma, \Box A$.

Now we turn to Step 4 of the algorithm in which $\mathcal{G}$ is a self-sufficient saturated hypersequent. We first consider the case where $\mathcal{G}$ has no diamond formulas, i.e., a leaf produced by Step 4a. Let $K, w_1, \ldots, w_n \not\models \mathcal{A}_p(\Sigma; \mathcal{G})$ for $\mathcal{A}_p(\Sigma; \mathcal{G})$ from (4.9). Similarly to saturated nested sequents in Theorem 4.2.14 we construct $p$-bisimilar model $K'$ in several steps.

(1) Whenever $w_i = w_j$, duplicate this world, until all $w_i$'s are distinct. Clearly, this yields a $p$-bisimilar model $M = (W', W' \times W', V_M)$ with $W' \supseteq W$ and an injective multiworld interpretation $w'_1, \ldots, w'_n$ of $\mathcal{G}$ into $M$ such that $M, w'_1, \ldots, w'_n \not\models \mathcal{A}_p(\Sigma; \mathcal{G})$.

(2) Now we define model $K' := (W', W' \times W', V'_p)$ to be the same as model $M$ except for valuations of $p$ as follows:

$$V'_p(w) := \begin{cases} V_M(w) \cup \{p\} & \text{if } w = w'_k \text{ and } k : \neg p \in \mathcal{G}; \\ V_M(w) \setminus \{p\} & \text{otherwise.} \end{cases}$$

Note that (4.13) is clearly fulfilled. This finishes the construction.

Now we prove that $K', w'_k \nVdash A$ whenever $k : A \in \mathcal{G}$ by induction on $A$.

- We leave the cases for $\top$, $\bot$, $B \vee B'$, and $B \wedge B'$ to the reader.
- Let $A = \neg p$, then $p \in V'_p(w'_k)$ and so $K', w'_k \nVdash \neg p$.
- Let $A = p$, then $k : \neg p \notin \mathcal{G}$ (otherwise Step 1 in the algorithm of $\mathcal{A}_p$ would have been used). Hence, $p \notin V'_p(w'_k)$ and so $K', w'_k \nVdash p$.

- Let $A = \ell \in \mathsf{Lit} \setminus \{p, \neg p\}$. It follows from (4.9) that $K, w_k \not\Vdash A$, because $K, w_1, \ldots, w_n \not\models \mathcal{A}_p(\Sigma; \mathcal{G})$. By bisimilarity modulo $p$ this transfers to $K'$ and $w'_k$.
- Let $A = \Box B$, then by saturation, there is a label $l$ such that $l : B \in \mathcal{G}$. By induction hypothesis, $K', w'_l \not\Vdash B$. Therefore, $K', w'_k \not\Vdash \Box A$.
- The case $A = \Diamond B$ cannot occur by the assumption that $\mathcal{G}$ does not contain diamond formulas.

The only remaining case is Step 4c. Note that Step 4b cannot occur, because by Lemma 4.3.7 that hypersequent is insufficient. So suppose $\mathcal{G}$ is a saturated self-sufficient hypersequent that is computed by Step 4c from the hypersequent

$$\mathcal{A}_p(\Box\Diamond\mathcal{G}; \quad \mathcal{G} \mid [\Theta]) \qquad \equiv \qquad \bigovee_{i=1}^{m} \left( \mathsf{d} : D_i \owedge \bigowedge_{j=1}^{n} j : C_{ij} \right), \tag{4.16}$$

with interpolant in the SDNF where $\Theta := \{B \mid \Diamond B \in \Box\Diamond\mathcal{G}\}$ as

$$\mathcal{A}_p(\Sigma; \quad \mathcal{G}) \qquad \equiv \qquad \bigovee_{i=1}^{m} \left( 1 : \Diamond D_i \owedge \bigowedge_{j=1}^{n} j : C_{ij} \right) \tag{4.17}$$

This is the least trivial case, mainly because hypersequent (4.16) may be insufficient. By Lemma 4.3.7, every branch rooted at this hypersequent leads either to a leaf or to a propositionally saturated self-sufficient hypersequent, all of which have the form $\mathcal{A}_p(\Box\Diamond\mathcal{G}; \mathcal{G} \mid [\Theta_j])$ for some $\Theta_j \supseteq \Theta$. Let $\Xi$ denote the multiset of all these interpolants. Then (4.16) is equivalent to $\bigowedge_{\mho \in \Xi} \mho$.

Let us assume that (4.17) is false for some interpretation $w_1, \ldots, w_n$ into a model $K$. Whenever $w_i = w_j$, duplicate this world, until all $w_i$'s are distinct. Clearly, this yields a model $M = (W', W' \times W', V_M)$ with $W' \supseteq W$ and an injective multiworld interpretation $u_1, \ldots, u_n$ of $\mathcal{G}$ into $M$ such that $M, u_1, \ldots, u_n \not\models \mathcal{A}_p(\Sigma; \mathcal{G})$ and $(K, w_1, \ldots, w_n) \sim^p (M, u_1, \ldots, u_n)$. Then, (4.16) is false for the interpretation $u_1, \ldots, u_n, v$ no matter which world $v$ of $M$ the $\mathsf{d}$-component of $\mathcal{G} \mid [\Theta]$ is mapped to. Indeed, for the $i$th disjunct of (4.16), either $M, u_j \not\Vdash C_{ij}$ or $M, u_1 \not\Vdash \Diamond D_i$, in which case $M, v \not\Vdash D_i$. Therefore, for every world $v$, one of interpolants from $\Xi$ is false for the interpretation $u_1, \ldots, u_n, v$. We consider two cases.

If $M, u_1, \ldots, u_n, v \not\models \mho_v$ for some world $v$ and interpolant $\mho_v = \mathcal{A}_p(\Box\Diamond\mathcal{G}; \mathcal{G} \mid [\Theta_v])$ from $\Xi$ of some self-sufficient hypersequent $\mathcal{G} \mid [\Theta_v]$, then by induction hypothesis there is $p$-bisimilar injective interpretation $w'_1, \ldots, w'_n, v'$ into a model $K'$ such that

$$K', w'_1, \ldots, w'_n, v' \not\models \mathcal{G} \mid [\Theta_v].$$

Clearly, (4.13) is fulfilled and $K', w'_1, \ldots, w'_n \not\models \mathcal{G}$.

Otherwise, for every world $v$ in model $M$,

$$M, u_1, \dots, u_n, v \models \bigotimes_{\text{self-sufficient } \mho \in \Xi} \mho, \qquad \text{but} \qquad M, u_1, \dots, u_n, v \not\models \bigotimes_{\mho \in \Xi} \mho.$$

In other words, for every $v$ in $M$ there is an insufficient leaf $\mho_v = \mathcal{A}_p(\Box\Diamond\mathcal{G}; \mathcal{G} \mid [\Theta_v])$ with $M, u_1, \dots, u_n, v \not\models \mho_v$. Note that this falsity means that the leaf interpolant $\mho_v$ could not have been obtained by Step 1. It must be computed according to Step 4b instead. In particular, for any $\ell \in \mathsf{Lit} \setminus \{p, \neg p\}$,

$$k : \ell \in \mathcal{G} \qquad \text{implies} \qquad M, u_k \nVdash \ell; \tag{4.18}$$

$$\ell \in \Theta_v \qquad \text{implies} \qquad M, v \nVdash \ell. \tag{4.19}$$

We use these leaves to construct a requisite $p$-bisimilar interpretation falsifying $\mathcal{G}$: we set $K' := (W', W' \times W', V'_p)$ to be the same as model $M$ except for valuations of $p$ as follows:

$$V'_p(w) := \begin{cases} V_M(w) \cup \{p\} & \text{if } w = w'_k \text{ and } k : \neg p \in \mathcal{G}; \\ V_M(w) \cup \{p\} & \text{if } w \in W' \setminus \{u_1, \dots, u_n\} \text{ and } \neg p \in \Theta_w; \\ V_M(w) \setminus \{p\} & \text{otherwise.} \end{cases}$$

Note that (4.13) is clearly fulfilled.

We now prove by mutual induction on formula $A$ that

$$k : A \in \mathcal{G} \qquad \text{implies} \qquad K', u_k \nVdash A; \tag{4.20}$$

$$A \in \Theta_v \qquad \text{implies} \qquad K', v \nVdash A \qquad \text{for any } v \in W' \setminus \{u_1, \dots, u_n\}. \tag{4.21}$$

- We leave the cases of $\top$, $\bot$, $A = B_1 \vee B_2$, and $A = B_1 \wedge B_2$ to the reader.
- If $A \in \{p, \neg p\}$, then the requisite statements follow directly from the definition of $V'_p$. The only thing to note here is that it is impossible to have both $k : p$ and $k : \neg p$ in $\mathcal{G}$ or both $p$ and $\neg p$ in some $\Theta_v$ because that would have produced an interpolant $\mho_v$ that cannot be falsified.
- Let $A = \ell \in \mathsf{Lit} \setminus \{p, \neg p\}$ be a literal other than $p$ or $\neg p$. The requisite statements follow from (4.18) and (4.19) by $p$-bisimilarity.
- Let $A = \Box B$, then $\Box B \in \mathcal{G}$ or $\Box B \in \Theta_v$ for some $v$. If $\Box B \in \Theta_v$, then the fact that this is a insufficient leaf means that $\Box B \in \Box\Diamond\mathcal{G}$. So $k : \Box B \in \mathcal{G}$ for some component $k$ for either of the cases. By saturation of $\mathcal{G}$, we have $j : B \in \mathcal{G}$ for some component $j$. By induction hypothesis for (4.20), $K', u_j \nVdash B$. Hence, $K', y \nVdash \Box B$ for all worlds $y \in V'$.
- Let $A = \Diamond B$, then $\Diamond B \in \mathcal{G}$ or $\Diamond B \in \Theta_v$ for some $v$. If $\Diamond B \in \Theta_v$, then the fact that this is a insufficient leaf means that $\Diamond B \in \Box\Diamond\mathcal{G}$. So $k : \Diamond B \in \mathcal{G}$ for some component $k$ for either of the cases. By saturation of $\mathcal{G}$, we have $j : B \in \mathcal{G}$ for all components $j$. In addition, by definition of the rule $\mathsf{d}$, we have $B \in \Theta$

and, hence, $B \in \Theta_v$ for all worlds $v \in W'$. By induction hypothesis for (4.20), $K', u_j \nVdash B$ for every component $j$ of $\mathcal{G}$. By induction hypothesis for (4.21), $K', v \nVdash B$ for every world $v \in W' \setminus \{u_1, \ldots, u_n\}$. In other words, $B$ is false in all worlds of $K'$. Hence, $K', y \nVdash \Diamond B$ for all worlds $y$.

This concludes the proof of BHUIP((iii)′), and so we have BHUIP for HS5. ■

Theorem 4.3 leads to the following analogues of Corollaries 4.2.15 and 4.2.16.

**4.3.9 Corollary**
Logic S5 has the uniform interpolation property.

**4.3.10 Corollary**
Bisimulation quantifiers are definable over S5-models, i.e., finite total models.

## 4.4 Conclusion

We have developed a constructive method for proving uniform interpolation based on multisequent calculi such as nested sequents and hypersequents. The method reveals a close connection between a constructive definition of the interpolant and model modifications that are invariant under bisimulation modulo $p$. The uniform interpolants that we define are multiformulas because in this way they closely resemble the multicomponent structure of nested sequents and hypersequents. Our method works well for the non-transitive logics K, D, and T, and we have been able to overcome the difficulties of applying it to hypersequents for the logic S5.

Related work is conducted by Bílková (2011). She has provided a purely syntactic method for uniform interpolation for K via nested sequents. The main difference with our method is that we exploit the treelike structure of nested sequents reflecting the treelike models for K by incorporating semantic arguments while the algorithm for the computation of the interpolants remains fully syntactic. We hope that our method will form a good basis for generalizing to other logics with multicomponent sequent calculi.

Next steps in this line of research would be in finding the right formalism and adapting our method to cover the remaining logics in the so-called modal cube, see (Garson, 2000) between K and S5 with the uniform interpolation. All logics except K4, S4, and D4 have the uniform interpolation property, see, e.g., (Kurahashi, 2020) including results on uniform Lyndon interpolation. Most proofs are semantic, and we hope that our method can be adjusted to provide constructive definitions of the interpolants. A natural start would be taking the grafted hypersequent calculi framework, introduced in (Kuznets and Lellmann, 2016) as a combination of the framework of nested sequents and hypersequents, for K5 and KD5 and generalizing to K45, KD45, and KB5. Furthermore, we would also like to provide a method to prove the uniform interpolation property for the logics KB, KDB, and KTB in a proof-theoretic manner.

It would be interesting to see how our method can be exploited to provide the uniform interpolation property for intuitionistic modal logics via multisequents. Naturally, we could start with the intuitionistic logics iK, iD, and iT, which would lead to a first proof of uniform interpolation for iT (although it might be easier to first try an ordinary sequent calculus for iT). Several nested sequent calculi have been developed for intuitionistic and constructive modal logics with $\Diamond$ by, for example, Marin and Straßburger (2017), Arisaka *et al.* (2015), and Galmiche and Salhi (2015). A first step is to show the termination of the calculi. This is explicitly shown in the latter using a similar method as we employed for the nested sequent calculi in Theorem 4.1.13. However, it might be impossible for some systems when contraction rules are explicitly present in the calculi such as in Arisaka *et al.* (2015). Still, it might be easier to first try ordinary sequent calculi for intuitionistic and constructive modal logics, such as the one developed in (Dalmonte *et al.*, 2021).

We conclude with an intuitive insight gained from our study regarding the failure of uniform interpolation in the logics K4 and S4 by indicating why our method does not work for these logics. They lack the two keys to success: intransitivity and termination. Nested proof systems do not terminate in saturated sequents as explained in Remark 4.1.14. For nested sequents we relied on their treelike structure reflecting *intransitive* treelike models. Intransitivity is crucial in order to modify models by duplications, replacements, and copies. Transitive trees of logics iK4 and iS4 are not suitable for these operations. However, intransitivity is not a necessary condition, as the bisimulation method by Visser (1996) relies on finite transitive trees proving uniform interpolation for K, GL, and S4.Grz. However, this method does not comply with clusters, which again forms a problem for K4 and S4 as observed in Remark 1.2.15.

# Part II

## Admissible Rules and Proof Theory

# 5

# Basics of Admissible Rules

In Part II of the thesis we are interested in the admissible rules of logics. A rule is admissible in a logic when it can be added to the logic without changing its set of theorems. In Part I we encountered admissible rules for concrete proof systems, such as weakening, contraction, and cut. In contrast to these rules, the admissible rules of a logic are not bound to a certain proof system, but reflect the relation between valid formulas of the logic.

Our aim is to provide a first study of admissible rules in the realm of intuitionistic modal logic. It combines results and proof techniques developed in the literature for intuitionistic propositional logic and classical modal logic. This chapter forms an introduction to admissibility and reviews these results. Chapter 6 analyses technical tools on projectivity and its importance in the field of admissible rules. In addition, we discuss its connection to unification theory. We use results from Chapter 6 to show our main contributions in Chapter 7, where we characterize the admissible rules for six intuitionistic modal logics with coreflection.

This chapter is structured as follows. We start with a historical overview where we informally explain different concepts on admissible rules. The subsequent sections present the formal concepts and necessary results from the literature. In short, Section 5.2 introduces consequence relations and admissible rules and Section 5.3 introduces bases for admissible rules and proof systems for admissibility.

## 5.1 History

We provide an overview of the rather short, but very rich, history of the research on admissible rules. An early historical overview can be found in Rybakov (1997).

For some parts of this overview we rely on the detailed historical overview with precise references to primary sources by Goudsmit (2015) and add recent literature. This section is also meant as an informal introduction on admissible rules, where we intuitively define different concepts. As such, it can be useful to already have a look at formal definitions given in next sections to get a better understanding of concepts such as a *rule*, an *admissible rule*, and a *basis* of admissible rules. We also refer to (Iemhoff, 2015) for a technical overview of the field.

### 5.1.1 Derivable and admissible rules

The history on admissible rules can be traced back to Lorenzen (1955).[23] A rule is *admissible* in a formal system if the set of the theorems of that system is closed under the rule. So adding an admissible rule to the system allows us to freely use the rule in new derivations, but keeping the body of the theorems of the logic the same. Admissible rules are interesting to study because they express properties of the logic in question.

*Admissible rules* differ from *derivable* rules. A rule is derivable if its conclusion follows from its premises using specific axioms and rules at hand. So derivable rules are bound to a certain axiomatization of the logic. This is in contrast to admissible rules, that abstract away from axiomatization and form an invariant for the logic. One could say that derivable rules are recognized *within* the logical system itself, whereas for admissible rules we *know* it can be added without changing the set of theorems, but there might be no explanation within the system *why* this is the case.

There are logics for which derivability and admissibility coincide. These logics are called *structurally complete*. CPC is known to be structurally complete and IPC is not. Closely connected are hereditarily structurally complete logics that are logics for which each extension is structurally complete. Citkin (1978) provides a characterization of such intermediate logics and Bezhanishvili and Moraschini (2022) give an alternative proof of this result via duality theory. Rybakov (1995) gives a characterization of hereditarily structurally complete transitive modal logics. A small error in this characterization was corrected via duality techniques by Carr (2022). Structural completeness is broadly studied in the algebraic community (see, e.g., Raftery, 2016; Moraschini *et al.*, 2020).

In this thesis, we are interested in logics that have non-derivable admissible rules.

[23] Lorenzen's work is in German where he introduced the term 'zulässig.' Craig (1957a) translated 'zulässig' into 'admissible' in his review on Lorenzen's work which became the standard terminology. Another term that appears in early works is that of 'permissible' rule translated from the Polish word 'dopuszczalna' introduced by Pogorzelski (1992).

An early example of such a rule is the Kreisel-Putnam rule

$$\neg p \to q \vee r / (\neg p \to q) \vee (\neg p \to r),$$

which is shown to be admissible in IPC (Harrop, 1960), but non-derivable in IPC (Kreisel and Putnam, 1957) (see Example 5.2.25). This is an example of a *single-conclusion* rule, but one can also study rules with *multiple* conclusions. This enables us to study meta-theoretic properties of logics such as the well-known disjunction property for IPC. It states that if formula $A \vee B$ is valid in IPC, then $A$ is valid or $B$ is valid in IPC.

It is useful to have a 'nice' characterization of *all* admissible rules in a given logic. Next sections explore characterizations in terms of decidability, semantics, proof theory, and bases of admissible rules. Most early works investigate the set of single-conclusion admissible rules (in, e.g, the noteworthy textbook (Rybakov, 1997)). The study on multi-conclusion rules is proposed by Kracht (1999) in his review on (Rybakov, 1997) and has become important after (Jeřábek, 2005). Both single- and multi-conclusion rules will be studied in this thesis.

### 5.1.2 Decidability

An early question is posed by Friedman (1975) as one of his one hundred and two problems in mathematical logic: Is the set of admissible rules in IPC decidable? The question was positively answered by Rybakov (1984b), who later addressed the same question for many logics in a remarkable series of papers starting from the 1980's about the admissible rules in, among others, intermediate logics, classical modal logics, pretabular (modal) logics, and most recently in temporal logics. For example, he has shown that admissibility is decidable in many intermediate logics and many modal logics above K4, see, e.g., (Rybakov, 1997). The results are based on semantic criteria further discussed in this historical overview in Section 5.1.4.

Another approach leading to the same affirmative answer to Friedman's problem is provided by Ghilardi (1999). The result follows from a study on projectivity and unification that has close connections to admissibility. This has led to a dominant approach in the field of admissible rules as further discussed in Section 5.1.5 and is of great importance in this thesis. Another more recent approach is from (Jeřábek, 2009) relying on canonical formulas introduced by Zakharyaschev (1992).

Rybakov (1989) asked whether decidability of the logics always guarantees decidability of the admissible rules. This is not the case, as shown by Chagrov (1992) via an explicit construction of a decidable logic for which the admissibility problem is undecidable. Later, Wolter and Zakharyaschev (2008) gave examples of decidable multi-modal logics and description logics in which admissibility is undecidable.

Once the decidability of admissibility is established one might ask about its complexity. This question is taken up by Jeřábek (2007) who proved that admissibility is coNEXP-complete in many intermediate and modal logics. This is interesting as for most of these logics decidability is in PSPACE and, therefore, the admissible rules form a more complex class than the derivable rules for these logics.

### 5.1.3 Bases and proof theory

A *basis* for the admissibility of a logic is a set of admissible rules of the logic that derive all other admissible rules. The idea is that the basis forms an axiomatization to describe *all* the admissible rules. Compare this to axiomatizations of logics that are sets of formulas describing *all* theorems of the logic. A trivial and uninteresting basis is the set of all admissible rules of the logic. The aim is to find a 'nice' basis where the word 'nice' can have different interpretations.

Kuznetsov (1973)[24] asked whether there exists a *finite* basis for the admissible rules in IPC. If so, decidability of admissibility in IPC would follow immediately. However, Rybakov (1985) answered the question in the negative, also for several classical modal logics such as S4 and GL, see (Rybakov, 1985, 1991). The existence of a finite basis is rare among logics, but Rybakov (1984a) shows that each logic extending S4.3 (such as S5)[25] has a finite base for its admissible rules that only contain so-called *passive* rules: rules for which the premises are never satisfied (see also Dzik and Wojtylak, 2016).

A common approach to present bases for admissible rules is by providing a set of *explicitly* defined rules. First explicit bases were conjectured for IPC. Citkin (1979) introduced an infinite sequence of rules and, later, the same rules were also conjectured to be a basis independently by Visser and de Jongh[26] and Skura (1989). The conjecture was confirmed by Iemhoff (2001a) and Rozière (1992), independently. Nowadays, these rules are called *Visser rules* following the terminology from (Iemhoff, 2001a). Jeřábek (2005) introduced *modal Visser rules* to provide bases for the admissible rules for certain transitive modal logics, such as K4, S4, and GL. Visser-like rules will also play an important role in this thesis.

Other explicit bases in the realm of classical modal logic can be found in (Rybakov *et al.*, 2000). Explicit bases are also investigated in for example intermediate logics (Iemhoff, 2005, 2006; Goudsmit and Iemhoff, 2014; Goudsmit, 2018), Łukasiewics logic (Jeřábek, 2010), and $\{\neg, \to\}$-fragment of IPC (Cintula and Metcalfe, 2010).

[24]Kuznetsov posed this question in conversation with Alex Citkin (there is not written record). I thank Alex Citkin for sharing this information with me.

[25]Logic S4.3 is defined over S4 with the axiom of linear frames: (.3) $\Box(\Box p \to q) \vee \Box(\Box q \to p)$.

[26]Personal communication (there is no written record of this).

In addition, in the aforementioned (Jeřábek, 2009) and in (Bezhanishvili *et al.*, 2016) bases are constructed via canonical rules.

Finally, *independent* bases for the admissible rules have also been considered in the literature. In many contexts, a basis already involves some kind of independence, such as in linear algebra. A basis is independent if no rule in the basis can be derived by the other rules in the basis. The Visser rules do not form independent bases. Rybakov *et al.* (1999a) show that all pretabular logics extending IPC or S4 have an independent basis of the admissible rules. Later, Jeřábek (2008) provides independent bases for IPC and classical modal logics including K4, S4, and GL.

Showing that a set forms an explicit basis of the admissible rules requires considerably effort. Quite often it is relatively easy to show that the rules in the basis are admissible, but to show that they derive all other admissible rules turns out to be very complicated. One solution is to develop a *proof theory* for the admissible rules. In contrast to regular proof systems of logics that reason about formulas, these proof systems reason about rules. Proof-theoretic approaches are employed by Rozière (1992); Iemhoff (2001a, 2003b); Jeřábek (2005); Iemhoff and Metcalfe (2009b,a). We discuss these proof systems in more detail in Section 5.3.2. In this thesis we also use a proof theory of admissibility (Chapter 7). Most of these approaches strongly depend on the semantic study of admissible rules.

### 5.1.4 Semantics

One can distinguish between two main approaches in the study of admissible rules. The algebraic perspective investigates quasi-varieties generated by free algebras (e.g., Rybakov, 1997; Ghilardi, 1997) and the syntactic approach treats logics as axiomatic systems (e.g., Iemhoff, 2001a; Jeřábek, 2005). The latter is taken in this thesis.

In both viewpoints, Kripke semantics plays an essential role. There are different approaches, but we would say that they all share the key idea of ‘extensions’ of models. A simple instance of this idea is a semantic proof of the disjunction property in IPC. One starts with two countermodels for formulas $A$ and $B$ and one extends these models with a new root resulting in a countermodel for $A \vee B$.

Rybakov has provided semantic criteria for admissibility in many logics, see, e.g., (Rybakov, 1997). Similarly to the standard completeness of logics, one searches for a class of models so that a rule is admissible if and only if it is valid in every model in a class of Kripke models. Rybakov showed that the admissibility of a rule is equivalent to its validity in a certain characterizing model. Specific properties required on the logic are the *branching below $m$ property* and the *effective $m$-*

*drop property*, which can be seen as certain concepts of 'extensions.' He developed the semantics to prove the decidability of admissibility in many intermediate and classical modal logics, see (Rybakov, 1997). Similar ideas appear in recent work to prove the decidability of admissibility in linear temporal logics (Luk'yanchuk and Rybakov, 2015).

Modern approaches have replaced these interpretations of 'extensions.' In particular, in an analysis of Rybakov's work, Goudsmit (2021) provides a new semantics for admissible rules in IPC in terms of *adequately exact models*, which in IPC correspond to so-called *adequately extendible models* (see also Goudsmit, 2015).

In this thesis we are concerned with two ideas of 'extensions' that arise from the study of projective formulas and unification theory initiated by Ghilardi (1999, 2000): *extendible logics* and the *extension property* (discussed below). These concepts are broadly used to study proof systems for admissibility and bases of admissible rules in for example IPC (Iemhoff, 2001a) and transitive classical modal logics (Jeřábek, 2005). Related semantic criteria were established by Iemhoff (2005, 2006) in the framework of intermediate logics in terms of the (*weak*) *extension property* and *offspring property*, yet other approaches to 'extensions.'

### 5.1.5 Unification theory

Unification theory deals with solving equations in a certain theory, see (Baader and Snyder, 2001) for an introduction and applications. From a logical point of view, one is concerned with substitutions that turn a formula into a theorem of the logic. Such a substitution is called a unifier. One could simply ask whether a given formula is unifiable, but one could also ask whether there is a 'nice' representation of all its unifiers. We study so-called complete minimal sets of unifiers that represent all unifiers in such a way that each unifier is less general than a unifier from the set and all unifiers in the set are incomparable (in a certain way).

The cardinality of complete minimal sets of unifiers indicate how hard unification is for a certain logic. For CPC, unification is 'easy' where each unifiable formula has a most general unifier. This is not the case for IPC and many non-classical logics. However, unification is still 'nice' in logics like IPC, K4, S4, and GL, in which it is shown to be finitary (Ghilardi, 1999, 2000). Moreover, (Ghilardi, 2002) provides a resolution algorithm for IPC to compute the finite complete sets of unifiers.

Ghilardi (1999, 2000) observes the connection between unification theory and the study of admissible rules that we will explore in Chapter 6. Key elements are *projective formulas* and *projective approximations*. Projective formulas form the syntactic counterparts of projective algebras and projective objects in category theory (e.g., projective Heyting algebras, Balbes and Horn, 1970). Ghilardi (1997) studies projective algebras in the general theory of equational unification. Later, he treats logics IPC and certain transitive modal logics in aforementioned papers (Ghilardi, 1999, 2000). Important is the existence of projective approximations for logics that are *extendible* and that allow for the semantic characterization of projective formulas in terms of the *extension property*. Related to this, Bezhanishvili and de Jongh (2012) connect *projective* formulas to so-called *extendible* formulas and *exact* formulas in IPC, the latter introduced in (de Jongh and Visser, 1996).

We end this historical overview by noting that we only discussed admissibility in the context of propositional and modal logics. Admissible rules have also been studied for first-order logics, see, e.g., (Rybakov, 1999) and (Visser, 1999). The latter discusses admissible rules in arithmetical theories (recall page 36 of this thesis where we shortly discussed the literature on admissible rules in arithmetical theories). Finally, one can also consider non-standard rules which are rules that may have variables in the premises that do not occur in the conclusion (this is in contrast to the Visser rules), see (Bezhanishvili *et al.*, 2022).

## 5.2 Rules

Inference rules form the core objects of our study. They come in two flavors: single-conclusion and multi-conclusion rules. We choose to use a uniform definition.

**5.2.1 Definition**
A *rule* is an ordered pair of finite sets of formulas $\Gamma$ and $\Delta$, written $\Gamma/\Delta$. It is called *single-conclusion* if $|\Delta| = 1$ and *multi-conclusion* in general.

By definition, multi-conclusion rules can have a single conclusion. We use the following standard notation for finite set of formulas $\Gamma, \Delta$ and formula $A$: we write $\Gamma/A$ for $\Gamma/\{A\}$, we write $\Gamma, \Delta$ for $\Gamma \cup \Delta$, and $\Gamma, A$ for $\Gamma \cup \{A\}$. Informally, we can think of a single-conclusion rule $\Gamma/A$ as $A$ follows from the formulas in $\Gamma$. For a multi-conclusion rule $\Gamma/\Delta$ it is not straightforward what its intended meaning is. We think of it as there is some $A \in \Delta$ that follows from the formulas in $\Gamma$. Rules for which $\Gamma = \emptyset$ are called *axioms*, and we omit $\emptyset$ in our notation. In rule $\Gamma/\Delta$, $\Gamma$ is called the set of *premises* and $\Delta$ the set of *conclusions*.

**5.2.2 Remark**
When thinking of rules one often thinks of it as 'rule schemes' using a meta-language on top of the object language. For instance, in the *modus ponens* rule

$$(\mathsf{MP}) \quad A, A \to B/B,$$

as used in the definitions of the logics (Definition 1.1.2), $A$ and $B$ are considered as meta-variables for formulas meaning that (MP) can be applied to any formula $A$ and $B$. Here we take another approach. One can avoid the meta-language by representing the rules by propositional variables, such as

$$(\mathsf{MP}) \quad p, p \to q/q,$$

and using substitutions to present any instance of rule (which is in fact a rule itself). This approach is taken in for instance (Goudsmit, 2015; Jeřábek, 2008). This approach seems technically more justified, and will therefore be the approach that we will take throughout this part of the thesis.[27] See (Iemhoff, 2016a) for a technical discussion between the two representations.

### 5.2.1 Consequence relations

Consequence relations provide a foundation of logical entailment. Informally speaking, they determine the overall game on how to apply the rules. Just like with rules, consequence relations come in two flavors: single-conclusion and multi-conclusion. Single-conclusion consequence relations were originally introduced by Tarski (1936), see (Wójcicki, 1988) for a good overview on his work. Multi-conclusion consequence relation are generalizations of the single-conclusion notion and take a variety of forms, see, e.g, (Shoesmith and Smiley, 1978).

So-called *finitary*, *structural* consequence relations are becoming standard practice in the study of admissible rules and are used by, among others, Jeřábek (2005), Cintula and Metcalfe (2010), and Goudsmit (2015). We follow this line of research.

For substitution $\sigma$ and set of formulas $\Gamma$, we write $\sigma(\Gamma)$ to mean $\{\sigma(A) \mid A \in \Gamma\}$.

**5.2.3 Definition** (Single-conclusion consequence relation)
A *finitary structural single-conclusion consequence relation* is a set of single-conclusion rules $\vdash$, where we write $\Gamma \vdash A$ to mean that rule $\Gamma/A$ is contained in $\vdash$, satisfying the following properties for all formulas $A$ and $B$, all finite sets of formulas $\Gamma$ and $\Pi$, and all substitutions $\sigma$:

| | |
|---|---|
| reflexivity: | $A \vdash A$; |
| monotonicity: | if $\Gamma \vdash A$ then $\Gamma, \Pi \vdash A$; |
| transitivity: | if $\Gamma \vdash A$ and $A, \Pi \vdash B$ then $\Gamma, \Pi \vdash B$; |
| structurality: | if $\Gamma \vdash A$ then $\sigma(\Gamma) \vdash \sigma(A)$. |

[27] Chapter 7 is based on (van der Giessen, 2021a). In contrast to this thesis, the paper takes the implicit representation of rules.

**5.2.4 Definition** (Multi-conclusion consequence relation)
A *finitary structural multi-conclusion consequence relation* is a set of rules $\vdash$, where we write $\Gamma \vdash \Delta$ to mean that rule $\Gamma/\Delta$ is contained in $\vdash$, satisfying the following properties for all formulas $A$, all finite sets of formulas $\Gamma, \Pi, \Delta, \Sigma$, and all substitutions $\sigma$:

| | |
|---|---|
| reflexivity: | $A \vdash A$; |
| monotonicity: | if $\Gamma \vdash \Delta$ then $\Gamma, \Pi \vdash \Delta, \Sigma$; |
| transitivity: | if $\Gamma \vdash \Delta, A$ and $A, \Pi \vdash \Sigma$ then $\Gamma, \Pi \vdash \Delta, \Sigma$; |
| structurality: | if $\Gamma \vdash \Delta$ then $\sigma(\Gamma) \vdash \sigma(\Delta)$. |

Monotonicity is sometimes called weakening and transitivity is also known as cut. We only work with finitary consequence relations adopting structurality, so from now on we simply speak of single-conclusion and multi-conclusion consequence relations. Several concepts will be introduced for both single- and multi-conclusion consequence relations. We do so in a uniform manner by simply referring to consequence relation, say $\vdash$, and understanding $\Gamma/\Delta$ and $\Gamma \vdash \Delta$ with $\Delta$ being a singleton in the single-conclusion setting.

**5.2.5 Definition** (Theorems and multi-theorems)
Let $\vdash$ be a consequence relation. Formula $A$ is said to be a *theorem of* $\vdash$ if $\vdash A$ and set $\Delta$ is a *multi-theorem of* $\vdash$ if $\vdash \Delta$. The set of all theorems of $\vdash$ is denoted by $\mathrm{Th}(\vdash)$ and its set of all multi-theorems by $\mathrm{Thm}(\vdash)$.

Note that in the single-conclusion setting all multi-theorems are theorems. The following example is a reformulation of Theorem 2 from (Došen, 1999) as stated in (Iemhoff, 2016a, Lemma 1).

**5.2.6 Example** (From single- to multi-conclusion)
Given a single-conclusion consequence relation $\vdash$ one can define different multi-conclusion consequence relations $\vdash_{\mathsf{m}}$ such that $\Gamma \vdash A$ iff $\Gamma \vdash_{\mathsf{m}} A$. The minimal and maximal one are:

$$\Gamma \vdash_{\mathsf{m}}^{\mathsf{min}} \Delta \text{ iff } \Gamma \vdash A \text{ for some } A \in \Delta; \tag{5.1}$$

$$\begin{aligned} \Gamma \vdash_{\mathsf{m}}^{\mathsf{max}} \Delta \text{ iff for all } \Pi \text{ and } A, \text{ if } \Pi \vdash_{\mathsf{m}}^{\mathsf{min}} \bigwedge \Gamma \text{ and } \bigvee \Delta \vdash_{\mathsf{m}}^{\mathsf{min}} A, \\ \text{then } \Pi \vdash_{\mathsf{m}}^{\mathsf{min}} A. \end{aligned} \tag{5.2}$$

These are multi-conclusion consequence relations and they form the minimal and maximal one in the sense that for each multi-conclusion consequence relation $\vdash_{\mathsf{m}}$ with the same condition that $\Gamma \vdash A$ iff $\Gamma \vdash_{\mathsf{m}} A$ it holds that $\vdash_{\mathsf{m}}^{\mathsf{min}} \subseteq \vdash_{\mathsf{m}} \subseteq \vdash_{\mathsf{m}}^{\mathsf{max}}$.

Consequence relations are associated with abstract derivability.

**5.2.7 Definition** (Derivability)
Given a consequence relation $\vdash$, rule $\Gamma/\Delta$ is said to be *derivable in* $\vdash$ if $\Gamma \vdash \Delta$.

The abstract concept of derivability in structural consequence relations can be made explicit by constructions of specific derivations. For the single-conclusion setting this is well known and is related to Hilbert style derivations for logics (Definition 1.1.2). Informally speaking, $\Gamma \vdash A$ if there is a derivation from formulas in $\Gamma$ resulting in $A$ by only using the rules from $\vdash$. For an algebraic version we refer to the *Łos-Suszko* theorem, see, e.g., (Rybakov, 1997, Theorem 1.5.2). For the multi-conclusion setting this is made explicit by Shoesmith and Smiley (1978). For both kinds of derivations we refer to the technical study in (Iemhoff, 2016a).

We are interested in consequence relations that describe logics. We already encountered the *global* consequence relation $\vdash_{\mathsf{L}}$ for a logic $\mathsf{L}$ defined in terms of derivations in Section 1.1 on page 11 which plays a role in the deduction theorem (Theorems 1.1.4 and 1.3.3). Here we will formally define the *local* and *global* consequence relation.

The following definition applies to both the single- and multi-conclusion context.

**5.2.8 Definition** (Cover)
A consequence relation $\vdash$ *covers* a logic $\mathsf{L}$ if $\mathsf{L} = \mathrm{Th}(\vdash)$.

**5.2.9 Example** (Minimal consequence relation for a logic)
Let $\mathsf{L}$ be a logic. The following defines the smallest single-conclusion consequence relation that covers $\mathsf{L}$:

$$\Gamma \vdash^{\mathsf{L}} A \text{ iff } A \in \Gamma \cup \mathsf{L}. \tag{5.3}$$

We will see that the greatest single-conclusion consequence relation covering logic $\mathsf{L}$ is the consequence relation described by its single-conclusion admissible rules as shown in Lemma 5.2.21. This is not the case for the multi-conclusion setting, see, e.g., (Metcalfe, 2012) and (Iemhoff, 2016a).

The following definition extends a consequence relation with a set of rules. This is used to define the local and global consequence relations and it also plays an important role in the study of bases of the admissible rules.

**5.2.10 Definition**
Let $\mathcal{R}$ be a set of rules and let $\vdash$ be a consequence relation. We define $\vdash^{\mathcal{R}}$ to be the least consequence relation containing $\vdash$ and $\mathcal{R}$.

**5.2.11 Remark**
The fact that $\vdash^{\mathcal{R}}$ is defined as the *least* consequence relation containing $\vdash$ and $\mathcal{R}$,

implies that one can provide inductive proofs along this set of rules, taking reflexivity and rules in $\mathcal{R}$ as the base cases and using monotonicity, transitivity, and structurality as the inductive steps.

**5.2.12 Example** (Global and local consequence relation)
Let $\mathsf{L}$ be a normal modal logic as defined in Definition 1.1.2, where $\mathsf{L}$ is a set of formulas.[28] The *local* and *global* consequence relations, $\vdash^l_{\mathsf{L}}$ and $\vdash^g_{\mathsf{L}}$ respectively, are single-conclusion consequence relations defined as follows, where $\vdash^{\mathsf{L}}$ is defined in (5.3), and the rules ($\mathsf{MP}$) and ($\mathsf{N}$) are in variable form according to Remark 5.2.2:

$$\Gamma \vdash^l_{\mathsf{L}} A \text{ iff } \bigwedge \Gamma \to A \in \mathsf{L}; \tag{5.4}$$

$$\Gamma \vdash^g_{\mathsf{L}} A \text{ iff } \Gamma \vdash^{\mathsf{L},(\mathsf{MP}),(\mathsf{N})} A. \tag{5.5}$$

It is well known that $\vdash^l_{\mathsf{L}}$ is the same as $\vdash^{\mathsf{L},(\mathsf{MP})}$ and thus both are indeed consequence relations by Definition 5.2.10. The difference between the local and global consequence relation for modal logic is usually nicely depicted via the necessitation rule ($\mathsf{N}$) which is derivable in the global consequence relation, but non-derivable in the local one, that is, $p \vdash^g_{\mathsf{L}} \Box p$, but $p \nvdash^l_{\mathsf{L}} \Box p$ (of course without the presence if the coreflection principle). In addition, recall the deduction theorem for transitive logics $\mathsf{L}$ (Theorem 1.1.4) implying

$$\Gamma \vdash^g_{\mathsf{L}} A \quad \text{iff} \quad \boxdot\Gamma \vdash^l_{\mathsf{L}} A.$$

And by Theorem 1.3.3 for logics with the coreflection principle defined in Figure 1.4 we have $\vdash^g_{\mathsf{L}} = \vdash^l_{\mathsf{L}}$.

**5.2.13 Convention**
Note that $\vdash^g_{\mathsf{L}}$ equals $\vdash_{\mathsf{L}}$ as defined in Section 1.1, page 11. Therefore, from now on we write $\vdash_{\mathsf{L}}$ for the global consequence relation $\vdash^g_{\mathsf{L}}$ from (5.5).

**5.2.14 Remark**
Note that in Part I we were only concerned with expressions of the form $\vdash_{\mathsf{L}} A$, which means that we also could have chosen to work with the local consequence relation. Indeed, we focused on the so-called *local* interpolation property instead of the global one.

Example 5.2.6 provides us with multi-conclusion consequence relations that cover a logic $\mathsf{L}$. We will define such a multi-conclusion consequence relation and fix notation for the rest of our work. This is a common choice in the field of admissible rules.

[28]Interesting to note is that in Definition 1.1.2 we take ($\mathsf{Subst}$) as an explicit rule, which is internalized in the definition of a structural consequence relation.

**5.2.15 Definition** (Fixed consequence relations for a logic)
Given logic $\mathsf{L}$, we define multi-conclusion consequence relation $\vdash_{\mathsf{m},\mathsf{L}}$ as

$$\Gamma \vdash_{\mathsf{m},\mathsf{L}} \Delta \text{ iff } \Gamma \vdash^{\mathsf{min}}_{\mathsf{L},\mathsf{m}} \Delta, \tag{5.6}$$

where $\vdash^{\mathsf{min}}_{\mathsf{L},\mathsf{m}}$ is the minimal multi-conclusion consequence relation as defined in (5.1) extending the global consequence relation $\vdash_{\mathsf{L}}$.

When there is no confusion with the single-conclusion setting we drop the subscript $\mathsf{m}$ and simply write $\vdash_{\mathsf{L}}$ instead of $\vdash_{\mathsf{m},\mathsf{L}}$. It is clear that $\vdash_{\mathsf{m},\mathsf{L}}$ indeed covers logic $\mathsf{L}$, i.e., $\mathsf{L} = \mathrm{Th}(\vdash_{\mathsf{m},\mathsf{L}})$.

The following definition applies to the single- and multi-conclusion setting.

**5.2.16 Definition** (Derivability in logic)
Let $\mathsf{L}$ be a logic. Rule $\Gamma/\Delta$ is *derivable in* $\mathsf{L}$ if $\Gamma/\Delta$ is derivable in $\vdash_{\mathsf{L}}$.

Note that derivability in $\mathsf{L}$ completely relies on the definition of $\vdash_{\mathsf{L}}$. For instance, recall from Example 5.2.12 the distinction between the global and local consequence relation with respect to the derivability of rule ($\mathsf{N}$). We will see that admissible rules abstract away from derivability.

### 5.2.2 Admissible rules

The first half of this section introduces well-known definitions, concepts, and results of admissible rules where we point to the relevant literature. The second half of this section examines many examples of specific admissible rules in intuitionistic modal logics.

Two notions of admissibility appear in the literature as pointed out by Metcalfe (2012) and further studied by Iemhoff (2016a). In the latter, these are called the *full* and the *strict* way, which we formally present in Definition 5.2.17 and Definition 5.2.18. In words, the full version states that a rule is admissible if the consequence relation is closed under the rule. And a rule is admissible in the strict sense if each substitution that unifies all premises also unifies the conclusion.

The full version is usually taken as a conceptual definition and the strict version as a formal definition of admissibility. For single-conclusion structural consequence relations, these notions coincide. This is also the case for the specific multi-conclusion consequence relation that we use. However, for the multi-conclusion setting in general it is more subtle (see Iemhoff (2016a) for a new strict definition).

Both definitions apply to the single-conclusion and multi-conclusion setting.

**5.2.17 Definition** (Admissible rule, full version)
Let $\vdash$ be a consequence relation $\vdash$ and let $\Gamma$ and $\Delta$ be finite sets of formulas. Rule $\Gamma/\Delta$ is said to be *admissible in* $\vdash$ if

$$\mathrm{Thm}(\vdash) = \mathrm{Thm}(\vdash^{(\Gamma/\Delta)}). \tag{5.7}$$

Note that for the single-conclusion setting condition (5.7) is equivalent to

$$\mathrm{Th}(\vdash) = \mathrm{Th}(\vdash^{(\Gamma/A)}). \tag{5.8}$$

**5.2.18 Definition** (Admissible rule, strict version)
Let $\vdash$ be a consequence relation, and let $\Gamma$ and $\Delta$ be finite sets of formulas. Rule $\Gamma/\Delta$ is said to be *admissible in* $\vdash$ if for all substitutions $\sigma$,

$$\text{if } \vdash \sigma(B) \text{ for all } B \in \Gamma, \text{ then } \vdash \sigma(A) \text{ for some } A \in \Delta. \tag{5.9}$$

The set of all admissible rules in $\vdash$ is denoted by $\mathrel{|\!\sim}$.

Of course, the substitution $\sigma$ in the above definition is assumed to be defined on all $B \in \Gamma$ and $A \in \Delta$.

These definitions are equivalent for saturated consequence relations.

**5.2.19 Definition** (Saturated)
A multi-conclusion consequence relation $\vdash$ is called *saturated* if

$$\vdash \Delta \text{ iff } \vdash A \text{ for some } A \in \Delta. \tag{5.10}$$

For a proof of the next theorem we refer to (Iemhoff, 2016a, Corollary 2). It shows similarities with the algebraic version in (Rybakov, 1997, Proposition 1.7.4).

**5.2.20 Theorem**
Let $\vdash$ be a saturated multi-conclusion relation. Rule $\Gamma/\Delta$ is admissible in $\vdash$ according to the full sense if and only if it is admissible in $\vdash$ according to the strict sense. Similar result holds for single-conclusion consequence relations.

Note that our fixed multi-conclusion consequence relation $\vdash_{\mathsf{L}}$ (Definition 5.2.15) is saturated. Therefore, Theorem 5.2.20 applies to $\vdash_{\mathsf{L}}$ (both for the single- and multi-conclusion reading of the symbol). From now on when we speak about admissibility we refer to admissibility in the strict sense.

The notation $\mathrel{|\!\sim}$ in Definition 5.2.18 is completely legitimated by the following theorem. For a proof see (Iemhoff, 2016a, Corollaries 3 and 4)

**5.2.21 Theorem**
Let $\vdash$ be a consequence relation. Then $\mid\!\sim$ is a consequence relation. Moreover, when $\vdash$ is single-conclusion, $\mid\!\sim$ is the greatest single-conclusion consequence relation with the same theorems as $\vdash$. And in case $\vdash$ is a saturated multi-conclusion consequence relation, $\mid\!\sim$ is the greatest multi-conclusion consequence relation with the same multi-theorems as $\vdash$.

Again, the next definition applies to the single- and multi-conclusion setting of $\vdash_{\mathsf{L}}$.

**5.2.22 Definition** (Admissibility in logic)
Let $\mathsf{L}$ be a logic. Rule $\Gamma/\Delta$ is *admissible in* $\mathsf{L}$ if $\Gamma/\Delta$ is admissible in $\vdash_{\mathsf{L}}$.

**5.2.23 Remark**
Taking into account previous definitions, we denote the set of all single-conclusion admissible rules in $\mathsf{L}$ by $\mid\!\sim_{\mathsf{L}}$ and the set of all multi-conclusion admissible rules in $\mathsf{L}$ by $\mid\!\sim_{\mathsf{m,L}}$. Similarly to $\vdash_{\mathsf{L}}$, we often write $\mid\!\sim_{\mathsf{L}}$ to mean the set of all multi-conclusion admissible rules in $\mathsf{L}$ when there is no confusion with the single-conclusion setting. Note that $\mid\!\sim_{\mathsf{L}} \subseteq \mid\!\sim_{\mathsf{m,L}}$. However, note that, whereas $\Gamma \vdash_{\mathsf{m,L}} \Delta$ means $\Gamma \vdash_{\mathsf{L}} A$ for some $A \in \Delta$, this is certainly not true in general for $\mid\!\sim_{\mathsf{m,L}}$ and $\mid\!\sim_{\mathsf{L}}$.

Interestingly, the admissibility of a multi-conclusion rule for logic $\mathsf{L}$ can be established by the single-conclusion consequence relation since admissibility only depends on the theorems of $\mathsf{L}$ which is clear from Definition 5.2.18. However, the multi-conclusion consequence relation still plays an important role in the description of *all* admissible rules of $\mathsf{L}$ in terms of a basis (Section 5.3).

Also note that $\mid\!\sim_{\mathsf{L}}$ is the greatest single-conclusion consequence relation that covers $\mathsf{L}$, by Lemma 5.2.21. However, $\mid\!\sim_{\mathsf{m,L}}$ is not the greatest multi-conclusion consequence relation that covers $\mathsf{L}$ (Iemhoff, 2016b).

The following lemma shows that derivability implies admissibility. This result is well known and straightforward.

**5.2.24 Lemma**
Let $\mathsf{L}$ be a logic and let $\Gamma$ and $\Delta$ be finite sets of formulas. If $\Gamma \vdash_{\mathsf{L}} \Delta$, then $\Gamma \mid\!\sim_{\mathsf{L}} \Delta$.

Logics for which admissibility and derivability coincide are called *structurally complete*. In our notation, a logic $\mathsf{L}$ is structurally complete if $\vdash_{\mathsf{L}} = \mid\!\sim_{\mathsf{L}}$.

We are interested in logics that are *not* structurally complete. They all admit non-derivable rules. With an eye on our goal to characterize admissible rules in intuitionistic modal logics let us provide some examples.

We will use Kripke models which are very convenient for recognizing the admissibility of rules. For definitions and semantics of intuitionistic modal logics, recall Figures 1.2 and 1.4 for the logics, Definition 1.2.29 for intuitionistic modal models, Definition 1.3.4 for intuitionistic strong models, and recall the completeness results in Theorems 1.2.37, 1.3.15 and Figure 1.5.

**5.2.25 Example** (Kreisel-Putnam rule)
The *Kreisel-Putnam rule*, also known as *Harrop's rule*, is with no doubt one of the most famous single-conclusion admissible rule in IPC.

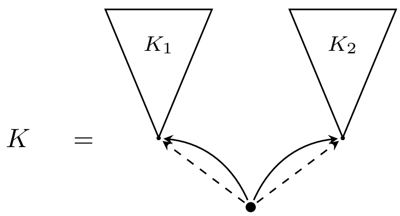
$$(\mathsf{KP}) \quad \neg p \to q \vee r \big/ (\neg p \to q) \vee (\neg p \to r)$$

Harrop (1960) has shown its admissibility by syntactic means. Later, Prucnal (1979) showed (KP) to be admissible in every intermediate logic. That the Kreisel-Putnam rule is non-derivable in IPC was first shown by Kreisel and Putnam (1957) which makes it in a sense a true admissible rule of IPC. Note that its implicational form

$$\big(\neg p \to q \vee r\big) \to \big((\neg p \to q) \vee (\neg p \to r)\big),$$

is the axiom of the intermediate logic KP introduced by Kreisel and Putnam (1957).

Here we show that (KP) is also a non-derivable admissible rule in any intuitionistic modal logic iL from Figures 1.2 and 1.4. Indeed the rule is non-derivable since iL is conservative over IPC as shown in Corollary 1.2.35 for iK, iD, iT, iS4, Corollary 1.3.13 for $\mathsf{iCK4} = \mathsf{IEL}^-$ and IEL, Corollary 1.3.16 for iGL, iSL, KM, mHC, and Corollary 1.3.21 for PLL. To prove admissibility of the rule it suffices to show that for any formulas $A, B, C$, if $\nvdash_{\mathsf{iL}} \neg A \to B$ and $\nvdash_{\mathsf{iL}} \neg A \to C$, then $\nvdash_{\mathsf{iL}} \neg A \to B \vee C$. By completeness there exist rooted countermodels $K_1$ and $K_2$ such that both models satisfy $\neg A$ in the root but $K_1$ refutes $B$ in its root and $K_2$ refutes $C$ in its root. Consider model $K$ below where the dashed arrows represent relation $\leq$ and the other arrows represent modal relation $R$.




This model should be understood as being closed under the frame condition ($\mathsf{R}_\Box$) (Definition 1.2.29). Moreover, it should be considered as the transitive closure for iK4, iS4, iGL, and all logics with the coreflection principle, and as the reflexive closure for iT, iS4, and $\mathsf{iCS4} \equiv \mathsf{IPC}$. It is easy to verify that for logic iL, the model is indeed a model of $\mathsf{iL}$. In particular, for strong models $K_1$ and $K_2$ model $K$ is also strong and if $K_1$ and $K_2$ are serial so is $K$. It easily follows from the monotonicity lemma (Lemma 1.2.31) that $K$ satisfies $\neg A$ but it refutes both $B$ and $C$. Therefore, $(\mathsf{KP})$ is admissible in $\mathsf{iL}$.

**5.2.26 Example** (Converse of necessitation rule)
A simple example of an admissible rule in modal logic is the converse of the necessitation rule.

$$(\mathsf{cN})\quad \Box p/p$$

The admissibility of this rule in the intuitionistic modal logics that we consider can be established by a similar semantic reasoning as in Example 5.2.25. Here we take the moment to note that the construction does not work for modal logics in general, for instance for symmetric models one requires to add an extra $R$-arrow downwards to the root.

**5.2.27 Example** (Disjunction property)
The multi-conclusion admissible rules are notably useful to express meta-theoretic properties such as the well-known disjunction property. A logic has the *disjunction property* if it admits the following multi-conclusion rule.

$$(\mathsf{DP})\quad p \vee q/p, q$$

Gödel (1932) showed that $\mathsf{IPC}$ has the disjunction property. For modal logics there exists a modal analogue of this rule. We refer to it as the *modal disjunction property* that belongs to the following rule.

$$(\mathsf{mDP})\quad \Box p \vee \Box q/p, q$$

We refer to Chapter 15 of (Chagrov and Zakharyaschev, 1997) for discussions on the (modal) disjunction property. Here we show that both rules are admissible in the intuitionistic modal logics $\mathsf{iL}$ that we consider. We proceed by a similar semantic argument from Example 5.2.25. Suppose $\nvdash_{\mathsf{iL}} A$ and $\nvdash_{\mathsf{iL}} B$ for modal formulas $A$ and $B$. By completeness there exist rooted countermodels $K_1$ and $K_2$ refuting respectively $A$ and $B$ in the root. Define model $K$ by the exact same construction from Example 5.2.25. Now by construction, $\Box A$ and $\Box B$ are refuted in $K$. Moreover, by monotonicity (Lemma 1.2.31) $A$ and $B$ are refuted. So $K$ is a countermodel for both $A \vee B$ and $\Box A \vee \Box B$. Therefore, $\mathsf{iL}$ enjoys both the disjunction property and the modal disjunction property.

**5.2.28 Example** (Admissible rules in iGL)
Admissible rule can show equivalence between different axiomatizations for the same logic. Ursini (1979) presents different axiomatizations for the intuitionistic Gödel-Löb logic $\mathsf{iGL}$. He states that the *Löb rule* and a variant of the rule are

admissible in $\mathsf{iGL}$.

$$(\mathsf{LR}) \quad \Box p \to p/p \qquad\qquad (\mathsf{LR}') \quad \Box p \to p/\Box p.$$

The admissibility of $(\mathsf{LR}')$ can easily be established by syntactic means as follows. Suppose $\vdash_{\mathsf{iGL}} \Box A \to A$. By $(\mathsf{N})$ we have $\vdash_{\mathsf{iGL}} \Box(\Box A \to A)$ and together with $(\mathsf{wl\ddot{o}b})$ for $A$ and $(\mathsf{MP})$ we conclude $\vdash_{\mathsf{iGL}} \Box A$. Consequently, admissibility of $(\mathsf{LR})$ follows by the admissibility of rule $(\mathsf{cN})$ from Example 5.2.26. The implicational form of *Löb's rule* is

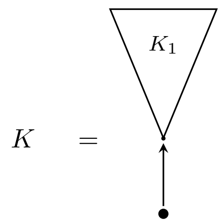
$$(\Box p \to p) \to p,$$

which is exactly the strong Löb axiom $(\mathsf{sl\ddot{o}b})$ in strong Löb logic $\mathsf{iSL}$.

**5.2.29 Example** (Rules in iGL and iSL)
An admissible rule for a given logic $\mathsf{L}$ does not have to be admissible in an extension $\mathsf{L}' \supseteq \mathsf{L}$. This is illustrated in this example. Consider the following rule, which was introduced to the author by Dick de Jongh in an email with the funny title '$\neg\neg$' as a follow up on a discussion on non-trivial admissible rules for $\mathsf{iGL}$:

$$\neg\neg\Box p/\neg\neg p$$

This rule is admissible in $\mathsf{iGL}$. Again we provide a semantic proof (recall completeness Theorem 1.3.15). Suppose $K_1$ is a rooted countermodel of $\neg\neg A$, where $A$ is a formula and $K_1$ is a model of $\mathsf{iGL}$. We construct model $K$ by extending $K_1$ with a new root linked to $K_1$ via the modal relation $R$ depicted as follows, where $R$ should be understood as closed under transitivity. This is certainly a model of $\mathsf{iGL}$.




Since $\neg\neg A$ is refuted in the root of $K$, it also refutes $A$. Since the root of $K$ is 'intuitionistically isolated' it valuates propositional connectives in a classical way, meaning that $\neg\neg\Box A$ is equivalent to $\Box A$ in that world. Thus $\neg\neg\Box A$ is refuted in $K$, concluding that the rule is admissible in $\mathsf{iGL}$. Now note that this construction does not work for $\mathsf{iSL}$, since models of $\mathsf{iSL}$ are required to be strong (Definition 1.3.4) forcing us to also draw a dashed arrow for the intuitionistic relation. In fact, the rule is *not* admissible in $\mathsf{iSL}$ since $\vdash_{\mathsf{iSL}} \neg\neg\Box A$ for each $A$, but of course $\Box A$ does not hold for each formula $A$. This shows that the admissible rules of a certain logic (think of $\mathsf{iGL}$) are not preserved under taking extensions (think of $\mathsf{iSL}$).

## 5.3 Bases

A basis of the admissible rules in a logic provides an axiomatization of all admissible rules of the logic. In words, a basis for the admissible rules in a logic is a set of admissible rules that derive all other admissible rules.

Recall Definition 5.2.15 in which for given logic $\mathsf{L}$ we fixed the single- and multi-conclusion consequence relation $\vdash_{\mathsf{L}}$. We associate it with the set of single- or multi-conclusion admissible rules denoted by $\mathrel{|\!\sim}_{\mathsf{L}}$. Recall Definition 5.2.10, which defines the consequence relation $\vdash_{\mathsf{L}}^{\mathcal{R}}$ for a set of rules $\mathcal{R}$.

**5.3.1 Definition**
Let $\mathsf{L}$ be a logic and let $\mathcal{R}$ be a set of rules. Set $\mathcal{R}$ is called a *basis of the admissible rules in* $\mathsf{L}$ if the relations $\mathrel{|\!\sim}_{\mathsf{L}}$ and $\vdash_{\mathsf{L}}^{\mathcal{R}}$ are equal. That is, if for all finite sets of formulas $\Gamma$ and $\Delta$:

$$\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta \text{ iff } \Gamma \vdash_{\mathsf{L}}^{\mathcal{R}} \Delta.$$

This definition should be understood to define a basis for the multi-conclusion admissible rules in general and a basis for the single-conclusion admissible rules when $\mathcal{R}$ only consists of single-conclusion rules and $|\Delta| = 1$.

Observe that the set of all admissible rules in $\mathsf{L}$ trivially forms a basis for the admissible rules in $\mathsf{L}$. So the existence of a basis is confirmed, but the aim is to give a 'nice' description of the admissible rules in $\mathsf{L}$. In Section 5.1 we have discussed several interpretations for 'nice', namely finite, explicit, and independent bases. We focus on explicit bases via Visser rules.

Next sections provide well-known results for $\mathsf{IPC}$ and transitive classical modal logics, which form the inspiration for our work on intuitionistic modal logics in Chapter 7. We mostly state results without proof, but refer to the relevant literature. Section 5.3.1 discusses the Visser rules. Section 5.3.2 gives an overview of proof systems that can describe all admissible rules and are used in the study of the Visser rules.

### 5.3.1 Visser rules

We start by introducing the Visser rules in $\mathsf{IPC}$. (Variants of) these rules were independently introduced by Citkin (1979), Visser and de Jongh, and Skura (1989). Originally, single-conclusion rules were studied, but multi-conclusion rules can also be defined.

**5.3.2 Definition** (Single-conclusion Visser rules)
The *single-conclusion Visser rules* are defined as follows where $n, m \in \mathbb{N}$.

$$(\widehat{\mathsf{V}}_{nm}) \quad \Big(\bigwedge_{i<n}(p_i \to q_i) \to \bigvee_{n \le j < n+m} p_j\Big) \vee r \Big/ \bigvee_{j<n+m}\Big(\bigwedge_{i<n}(p_i \to q_i) \to p_j\Big) \vee r.$$

The *restricted single-conclusion Visser rules* are defined as follows where $n, m \in \mathbb{N}$.

$$(\widehat{\mathsf{V}}^-_{nm}) \quad \bigwedge_{i<n}(p_i \to q_i) \to \bigvee_{n \le j < n+m} p_j \Big/ \bigvee_{j<n+m}\Big(\bigwedge_{i<n}(p_i \to q_i) \to p_j\Big).$$

The set of all single-conclusion Visser rules is denoted by $\widehat{\mathsf{V}}$.

**5.3.3 Definition** (Multi-conclusion Visser rules)
The *multi-conclusion Visser rules* are defined as follows where $n, m \in \mathbb{N}$.

$$(\mathsf{V}_{nm}) \quad \bigwedge_{i<n}(p_i \to q_i) \to \bigvee_{n \le j < n+m} p_j \Big/ \Big\{\bigwedge_{i<n}(p_i \to q_i) \to p_j \mid j < n+m\Big\}.$$

The set of all multi-conclusion Visser rules is denoted by $\mathsf{V}$.

In the definitions we allow empty conjunctions and disjunctions. Recall that by definition $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$. The Visser rules as presented here are defined for any $m \in \mathbb{N}$. However, the original single-conclusion Visser rules were defined for $m \le 2$. The Visser rules are an infinite collection of rules in the sense that $(\widehat{\mathsf{V}}_{nm})$ is not derivable from $\mathsf{IPC}$ extended by all rules $(\widehat{\mathsf{V}}_{n'm'})$ with $n' < n$ and $m' < m$ as shown by Iemhoff (2001b, Section 4.6.1). She calls this the independence of the Visser rules, but it is not an *independent* basis as discussed in Section 5.1. On the contrary, because $(\widehat{\mathsf{V}}_{nm})$ is derivable from $(\widehat{\mathsf{V}}_{n+1,m})$ and also from $(\widehat{\mathsf{V}}_{n,m+1})$.

The Kreisel-Putnam rule ($\mathsf{KP}$) presented in Example 5.2.25 is a special instance of the single-conclusion Visser rules with $n = 1$, $m = 2$ and $q_0 = \bot$. For the multi-conclusion rules, note that in case $n = m = 0$, rule $(\mathsf{V}_{00})$ is equivalent to rule $\bot/\emptyset$. Recall the disjunction property ($\mathsf{DP}$) from Example 5.2.27. It can be considered as a special instance of the Visser rules with $i = 0$ and $m = 2$. So this means that if a logic admits the multi-conclusion Visser rules, it admits the disjunction property.

By similar semantic constructions as in Examples 5.2.25 and 5.2.27 one can show that the Visser rules are admissible in $\mathsf{IPC}$. One can also employ a syntactic argument using the Aczel-slash[29] as noted in (Iemhoff, 2006).

The single-conclusion Visser rules form a basis for the admissible rules as independently shown by Rozière (1992) and Iemhoff (2001a), i.e., $\mathrel{|\!\sim}_{\mathsf{IPC}} = \vdash^{\widehat{\mathsf{V}}}_{\mathsf{IPC}}$. As the

[29]See (Troelstra and van Dalen, 1988)

rules are admissible we have $\mathrel{|\!\sim}_{\mathsf{IPC}} \supseteq \vdash^{\widehat{\mathsf{V}}}_{\mathsf{IPC}}$, but the other inclusion is much more involved relying on proof theory for admissible rules.

**5.3.4 Theorem** (Rozière, 1992; Iemhoff, 2001a, Theorem 3.20)
Set $\widehat{\mathsf{V}}$ is a basis for the single-conclusion admissible rules in $\mathsf{IPC}$.

Note that the admissibility of the single-conclusion Visser rules follows from the admissibility of the restricted Visser rules by using the disjunction property in $\mathsf{IPC}$ telling us that $A \mathrel{|\!\sim}_{\mathsf{IPC}} C$ and $B \mathrel{|\!\sim}_{\mathsf{IPC}} C$ implies $A \vee B \mathrel{|\!\sim}_{\mathsf{IPC}} C$ for all formulas $A, B$, and $C$. However, they do not form a basis for the admissible rules, but are shown to be a *subbasis*. For details, see (Iemhoff, 2001a).

The multi-conclusion Visser rules show strong similarities with the restricted single-conclusion Visser rules. By the disjunction property we have: $A \mathrel{|\!\sim}_{\mathsf{IPC}} \Delta_1$ and $B \mathrel{|\!\sim}_{\mathsf{IPC}} \Delta_2$ implies $A \vee B \mathrel{|\!\sim}_{\mathsf{IPC}} \Delta_1, \Delta_2$. Although the restricted single-conclusion Visser rules do not form a basis, the multi-conclusion Visser rules do because the aforementioned property is recognized in $\vdash^{\mathsf{V}}_{\mathsf{IPC}}$. That is,

$$A \vdash^{\mathsf{V}}_{\mathsf{IPC}} \Delta_1 \text{ and } B \vdash^{\mathsf{V}}_{\mathsf{IPC}} \Delta_2 \text{ implies } A \vee B \vdash^{\mathsf{V}}_{\mathsf{IPC}} \Delta_1, \Delta_2, \tag{5.11}$$

for all formulas $A$ and $B$, and finite sets of formulas $\Delta_1$ and $\Delta_2$. This is due to the fact that the disjunction property is formalized in the multi-conclusion Visser rules, that is, $A \vee B \vdash^{\mathsf{V}}_{\mathsf{L}} A, B$.

The following results is folklore, but to the best of our knowledge there is no written proof for it. So we present one here.

**5.3.5 Theorem**
Set $\mathsf{V}$ is a basis for the multi-conclusion admissible rules in $\mathsf{IPC}$.

Proof. We have to show that $\mathrel{|\!\sim}_{\mathsf{IPC}} = \vdash^{\mathsf{V}}_{\mathsf{IPC}}$. We only concentrate on $\mathrel{|\!\sim}_{\mathsf{IPC}} \subseteq \vdash^{\mathsf{V}}_{\mathsf{IPC}}$, as the other inclusion follows from the fact that each $(\mathsf{V}_{nm})$ is admissible in $\mathsf{IPC}$ as observed above. So let $\Gamma/\Delta$ be an admissible rule in $\mathsf{IPC}$. Then $\Gamma/\bigvee\Delta$ is a single-conclusion admissible rule. Since $\widehat{\mathsf{V}}$ is a basis of the single-conclusion admissible rules by Theorem 5.3.4, we know $\Gamma \vdash^{\widehat{\mathsf{V}}}_{\mathsf{IPC}} \bigvee\Delta$. Recall Remark 5.2.14 stating that we can proceed by induction on $\vdash^{\widehat{\mathsf{V}}}_{\mathsf{IPC}}$. We will show that $\Gamma \vdash^{\mathsf{V}}_{\mathsf{IPC}} \Delta$ as desired. We only treat two cases, the other cases are left to the reader. If $\Gamma/\bigvee\Delta$ is an instance of $(\widehat{\mathsf{V}}_{nm})$ we have that $\Gamma$ consists of one formula, namely

$$\Big(\bigwedge_{i<n}(p_i \to q_i) \to \bigvee_{n \leq j < n+m} p_j\Big) \vee r,$$

and $\Delta$ contains

$$r \text{ and } \bigwedge_{i<n}(p_i \to q_i) \to p_j \text{ for all } j < n+m.$$

We know $r \vdash_{\mathsf{IPC}} r$ and we have that rule $(\mathsf{V}_{nm})$ is derivable in $\vdash^{\mathsf{V}}_{\mathsf{IPC}}$. By the observation from above in (5.11) we conclude $\Gamma \vdash^{\mathsf{V}}_{\mathsf{IPC}} \Delta$. Now suppose $\Gamma / \bigvee \Delta$ is derived by reflexivity in $\vdash^{\widehat{\mathsf{V}}}_{\mathsf{IPC}}$, that is, $\Gamma = \{\bigvee \Delta\}$. So we have to show that $\bigvee \Delta \vdash^{\mathsf{V}}_{\mathsf{IPC}} \Delta$. This is immediate from $(\mathsf{V}_{0m})$ in $\mathsf{V}$, which can be seen as a generalized disjunction property. ■

Visser rules are also extensively studied in intermediate logics by Iemhoff (2005, 2006), and Goudsmit (2015). As cited in Example 5.2.27, the rule $(\mathsf{KP})$ is admissible in any intermediate logic (Prucnal, 1979). This is definitely not the case for the Visser rules in general. In fact, if a proper extension of $\mathsf{IPC}$ has the disjunction property not all single-conclusion Visser rules are admissible (Iemhoff, 2005). And so, no proper extension of $\mathsf{IPC}$ inherits all multi-conclusion Visser rules. Rybakov (1993) describes all intermediate logics with the finite model property admitting all single-conclusion Visser rules from $\mathsf{IPC}$. Moreover, we have the following important result which we will study for logics in general in Theorem 6.1.19.

**5.3.6 Theorem** (Iemhoff, 2005)
Let $\mathsf{L}$ be an intermediate logic. If all single-conclusion Visser rules from $\widehat{\mathsf{V}}$ are admissible in $\mathsf{L}$, then $\widehat{\mathsf{V}}$ is a basis of the admissible rules in $\mathsf{L}$.

Now we turn to the classical modal setting. The modal Visser rules are introduced by Jeřábek (2005) and studied for transitive modal logics, such as $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{GL}$.

**5.3.7 Definition**
The *single-conclusion modal Visser rules* are defined as follows where $n, m \in \mathbb{N}$.

$$(\widehat{\mathsf{V}}^\bullet_m) \qquad \Box\Big(\Box q \to \bigvee_{j<m} \Box p_j\Big) \vee \Box r \Big/ \bigvee_{j<m} \Box(\boxdot q \to p_j) \vee r,$$

$$(\widehat{\mathsf{V}}^\circ_{nm}) \qquad \Box\Big(\bigwedge_{i<n}(q_i \leftrightarrow \Box q_i) \to \bigvee_{j<m} \Box p_j\Big) \vee \Box r \Big/ \bigvee_{j<m} \Box\Big(\bigwedge_{i<n} \boxdot q_i \to p_j\Big) \vee r.$$

The set of all rules $(\widehat{\mathsf{V}}^\bullet_m)$ is denoted by $\widehat{\mathsf{V}}^\bullet$ and the set of all rules $(\widehat{\mathsf{V}}^\circ_{nm})$ by $\widehat{\mathsf{V}}^\circ$.

**5.3.8 Definition**
The *multi-conclusion modal Visser rules* are defined as follows where $n, m \in \mathbb{N}$.

$$(\mathsf{V}^\bullet_m) \qquad \Box q \to \bigvee_{j<m} \Box p_j \Big/ \big\{\boxdot q \to p_j \mid j < m\big\},$$

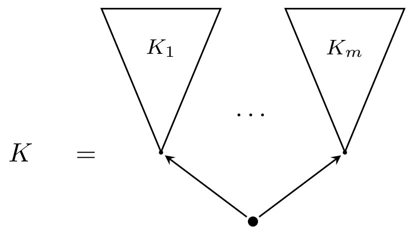
$$(\mathsf{V}^\circ_{nm}) \qquad \bigwedge_{i<n}(q_i \leftrightarrow \Box q_i) \to \bigvee_{j<m} \Box p_j \Big/ \Big\{\bigwedge_{i<n} \boxdot q_i \to p_j \mid j < m\Big\}.$$

The set of all rules $(\mathsf{V}^\bullet_m)$ is denoted by $\mathsf{V}^\bullet$ and the set of all rules $(\mathsf{V}^\circ_{nm})$ by $\mathsf{V}^\circ$.[30]

[30] Jeřábek (2005, 2008) calls these $A^\bullet$ and $A^\circ$ respectively. He denotes single-conclusion rules

As one can see, there are two sorts of modal Visser rules. These depend on the (ir)reflexivity of worlds in the Kripke semantics of the logic. More precisely, the admissibility of the rules can again be shown by extensions in Kripke semantics analogously to Example 5.2.27, where rules from $\mathsf{V}^\bullet$ are admissible for logics in which frames can be extended by an irreflexive world and rules from $\mathsf{V}^\circ$ are admissible in logics in which frames can be extended by a reflexive world. This means that $\mathsf{V}^\bullet$ is relevant for $\mathsf{K4}$ and $\mathsf{GL}$, and $\mathsf{V}^\circ$ for $\mathsf{K4}$ and $\mathsf{S4}$. Let us give an example.

**5.3.9 Example**
We provide a semantic proof for the fact that rule ($\mathsf{V}^\bullet_m$) is admissible in $\mathsf{K4}$ and $\mathsf{GL}$ (recall the completeness Theorem 1.2.8). It is sufficient to show that for any formulas $A$, $B_1, \ldots, B_m$, if $\nvdash_{\mathsf{K4}} \boxdot A \to B_j$ for all $j < m$, then $\nvdash_{\mathsf{K4}} \Box A \to \bigvee_{j<m} \Box B_j$. Having rooted countermodels $K_j$ which satisfy $\boxdot A$ in the root, but refute $B_j$ in the root, we construct the following model $K$ with an irreflexive root, where transitive relation $R$ is indicated by the arrows.




Indeed, this is a countermodel for $\Box A \to \bigvee_{j<m} \Box B_j$, because its root satisfies $\Box A$ since the roots of the models $K$ satisfy $\boxdot A$ and $R$ is transitive. Moreover, $\Box B_j$ is falsified for each $j$, since $B_j$ is false in the root of $K_j$.

Showing that the modal Visser rules form a basis is again much more difficult. In contrast to $\mathsf{IPC}$, the result is first established for multi-conclusion rules from which the result of the single-conclusion rules follows by the modal disjunction property.

**5.3.10 Theorem** (Jeřábek, 2005, Theorem 4.5)
Set $\mathsf{V}^\bullet$ is a basis for the multi-conclusion admissible rules in $\mathsf{S4}$. Set $\mathsf{V}^\circ$ is a basis for the multi-conclusion admissible rules in $\mathsf{GL}$. And $\mathsf{V}^\bullet \cup \mathsf{V}^\circ$ is a basis for the multi-conclusion admissible rules in $\mathsf{K4}$.

**5.3.11 Theorem** (Jeřábek, 2005, Corollary 6.4)
Set $\widehat{\mathsf{V}}^\bullet$ is a basis for the single-conclusion admissible rules in $\mathsf{S4}$. Set $\widehat{\mathsf{V}}^\circ$ is a basis for the single-conclusion admissible rules in $\mathsf{GL}$. And $\widehat{\mathsf{V}}^\bullet \cup \widehat{\mathsf{V}}^\circ$ is a basis for the single-conclusion admissible rules in $\mathsf{K4}$.

by lower case letters.

We can consider the admissible rules in extensions of $\mathsf{K4}, \mathsf{S4}$, and $\mathsf{GL}$. Just like for $\mathsf{IPC}$, there are no proper extensions of $\mathsf{GL}$ that inherits all its multi-conclusion admissible rules (Jeřábek, 2005). But the situation is different for $\mathsf{K4}$ and $\mathsf{S4}$ for which all logics inheriting their multi-conclusion admissible rules are characterized by Jeřábek (2006). For the single-conclusion setting, semantic conditions on inheriting admissible rules of $\mathsf{S4}$ and $\mathsf{K4}$ for logics with the finite model property were already provided by Rybakov *et al.* (1999b), and respectively by Gencer (2002) and Rutskii and Fedorishin (2002). We have the following analogues of Theorem 5.3.6.

**5.3.12 Theorem** (Jeřábek, 2005, Theorem 4.5)
Let $\mathsf{L}$ be a modal logic extending $\mathsf{K4}$ that admits all rules in $\mathsf{V}^\bullet \cup \mathsf{V}^\circ$. Then $\mathsf{V}^\bullet \cup \mathsf{V}^\circ$ is a basis for the multi-conclusion admissible rules in $\mathsf{L}$. Similar statement holds for extensions of $\mathsf{S4}$ with regard to $\mathsf{V}^\circ$.

**5.3.13 Theorem** (Jeřábek, 2005, Corollary 6.4)
Let $\mathsf{L}$ be a modal logic extending $\mathsf{K4}$ that admits all rules in $\widehat{\mathsf{V}}^\bullet \cup \widehat{\mathsf{V}}^\circ$. Then $\widehat{\mathsf{V}}^\bullet \cup \widehat{\mathsf{V}}^\circ$ is a basis for the single-conclusion admissible rules in $\mathsf{L}$. Similar statement holds for extensions of $\mathsf{S4}$ with regard to $\widehat{\mathsf{V}}^\circ$ and extensions of $\mathsf{GL}$ with regard to $\widehat{\mathsf{V}}^\bullet$.

## 5.3.2 Proof theory for admissibility

Showing that the different Visser rules form a basis for the logics $\mathsf{IPC}$, $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{GL}$ in Theorems 5.3.4 and 5.3.10 is complicated. In this section we examine several proof-theoretic approaches that have been studied in the literature. We will only use one concrete proof system for admissibility in this thesis (Chapter 7), but here we would like to provide a brief overview of proof systems that study admissible rules.

Informally speaking, a proof theory for admissibility consists of rules that reason about rules. This in contrast to well-known proof systems of logics that reason on the level of formulas. The aim is to find a proof system that reasons about objects, say $A \rhd B$, and functions as an intermediate step in the proof of a basis. That is, the aim is to show $A \mathrel{|\!\!\sim}_{\mathsf{L}} B$ iff the proof system derives $A \rhd B$ iff $A \vdash^{\mathcal{R}}_{\mathsf{L}} B$. This shows that $\mathcal{R}$ is a basis for the admissible rules in $\mathsf{L}$. Moreover, whereas proof systems for logics provide information about the logic like decidability, the proof system of admissibility might reveal such properties of admissibility.

In general, the design of the proof system of admissibility includes derivability of the logic in question and the corresponding conjectured Visser rules. This makes the connection between $A \rhd B$ and $A \vdash^{\mathcal{R}}_{\mathsf{L}} B$ relatively easy.

However, to show soundness and completeness of the proof system with respect to admissibility is highly complicated and involves difficult semantic arguments. In particular, many approaches rely in one way or another on the semantic study of projective formulas developed by Ghilardi (1999, 2000). In Chapter 6 we investigate projectivity in the field of admissible rules.

Iemhoff (2001a) provides an axiomatic style proof system for the single-conclusion admissible rules in $\mathsf{IPC}$. This proof system is called an $\mathsf{AR}$-system and is a set of sequents of the form $A \rhd B$, where $A$ and $B$ are formulas, which contains all sequents of the form

$$A \rhd B \qquad \begin{array}{l} \text{if } A \vdash_{\mathsf{IPC}} B, \text{ or} \\ \text{if } A/B \text{ is a Visser rule from } \widehat{\mathsf{V}}. \end{array}$$

and is closed under substitution and the rules

$$\frac{C \rhd A \qquad C \rhd B}{C \rhd A \wedge B}\,\text{Conj} \qquad \text{and} \qquad \frac{A \rhd B \qquad B \rhd C}{A \rhd C}\,\text{Cut}.$$

The fact that $A \rhd B$ is derivable in $\mathsf{AR}$ iff $A \vdash_{\mathsf{L}}^{\widehat{\mathsf{V}}} B$ follows almost immediately from the design as shown in (Iemhoff, 2001a, Lemma 3.19). To show that $A \mathrel{|\!\sim}_{\mathsf{L}} B$ iff $A \rhd B$ is derivable in $\mathsf{AR}$ is based on semantic characterization of the $\mathsf{AR}$-system into so-called $\mathsf{AR}$-models.

Jeřábek (2005) generalizes the proof system to multi-conclusion rules for classical modal logics. Also this relies on semantic arguments and applies to transitive *extendible* modal logics which will be introduced in Definition 6.1.21. Logics $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{GL}$ belong to this class. The $\mathsf{AR}$-system for such a logic $\mathsf{L}$ is a set of sequents of the form $\Gamma \rhd \Delta$, where $\Gamma$ and $\Delta$ are finite sets of formulas, which contains all sequents of the form

$$\begin{array}{ll} \Gamma \rhd B & \text{if } \Gamma \vdash_{\mathsf{L}} B, \text{ or} \\ \Gamma \rhd \Delta & \begin{array}{l} \text{if } \Gamma/\Delta \text{ is a Visser rule from } \mathsf{V}^{\bullet} \text{ (in case } \mathsf{L} \text{ is not reflexive), or} \\ \text{if } \Gamma/\Delta \text{ is a Visser rule from } \mathsf{V}^{\circ} \text{ (in case } \mathsf{L} \text{ is not irreflexive),} \end{array} \end{array}$$

and is closed under cut, weakening, and substitution.

The proof systems have an axiomatic flavor and are defined closely to the definition of a consequence relation. This might not come as a surprise since $\mathrel{|\!\sim}_{\mathsf{L}}$ is a consequence relation, but it does not meet many desirable proof-theoretic properties. In particular, the $\mathsf{AR}$-systems contain the cut rule which one would like to eliminate in usual proof systems for logics.

Iemhoff (2003b) attempts to circumvent this problem and proposes a more analytic proof system for the single-conclusion admissibility in $\mathsf{IPC}$ which she calls the

ADM-system. This system contains sequents that have close connections to hypersequents. Although the system is sound with respect to admissibility, a drawback of this system is that it is not complete. We have $A \vdash_{\mathsf{ADM}} B$ implies $A \mathrel{|\!\sim}_{\mathsf{IPC}} B$, but not the converse. However, the system is complete in the sense that for every formula $A$ there exists a unique formula $C_A$ such that $A \vdash_{\mathsf{ADM}} C_A$ and for all formulas $B$:

$$A \mathrel{|\!\sim}_{\mathsf{IPC}} B \text{ iff } C_A \vdash_{\mathsf{IPC}} B.$$

This is inspired by projective approximations developed by Ghilardi (1999) that we will discuss in Chapter 6. In particular it shows similarities with property (6.3) of Lemma 6.1.15.

Analytic proof systems for admissibility are introduced by Iemhoff and Metcalfe (2009b) who introduce Gentzen-style proof systems for the admissible rules in, among others, IPC, K4, S4, and GL. The objects in their proof theory are *sequent rules* and the rules in the system reason about these objects. This can be viewed as a Gentzen system one level higher than regular Gentzen systems where the objects are sequents that represent formulas. Iemhoff and Metcalfe (2009a) have extended these ideas to hypersequent systems to cover intermediate logics and a wider class of modal logics. Still, semantics plays a big role. But one big advantage is that decidability of admissibility follows from the decidability of the logic. These proof systems are exactly the ones that we will study in Chapter 7, where we define Gentzen-style proof systems for the admissible rules for intuitionistic modal logics with coreflection.

We close by mentioning two other interesting proof-theoretic approaches to admissibility. First, Citkin (2010) defines a modal logic for reasoning about the admissible rules in IPC. Second, Condoluci and Manighetti (2018) present a $\lambda$-calculus with proof terms for the Visser rules in which normal forms are just regular intuitionistic proof terms.

# 6

# Projectivity and Admissible Rules

Projectivity plays an essential role in the study of admissible rules since the celebrated work on unification by Ghilardi (e.g., 1997, 1999, 2000, 2002, 2004). It started off from an algebraic and categorical approach on unification via projective objects (Ghilardi, 1997), but has in subsequent papers been translated to a proof-theoretic setting. The approach has proved to be fruitful in the study of finite unification. Moreover, it has led to a new proof of the decidability of the admissibility problem in $\mathsf{IPC}$ in (Ghilardi, 1999) first established by Rybakov (1984b). The same results were established in the realm of transitive classical modal logics in (Ghilardi, 2000) and of intermediate logics in (Ghilardi, 2004) and (Goudsmit and Iemhoff, 2014).

This chapter follows Ghilardi's footsteps. We adopt the proof-theoretic approach and study the concepts of *projective formulas* and *projective approximations*. We will outline their importance in both unification and admissibility theory in Section 6.1. For instance, we will see that projective formulas reduce admissibility to derivability and that projective approximations are used to establish admissibility results in extensions of logics.

A major part of this chapter is devoted to the semantic study of projective formulas presented in Section 6.2. In many logics, they can be characterized via a semantic property called the *extension property*. This crucial fact has been used extensively in the research on admissible rules. For example, it can be used to show that projectivity is decidable in such logics, leading to the decidability of the admissibility problem.

Section 6.2.1, which is based on (van der Giessen, 2021b), analyses the characterization in the framework of classical modal logics. It provides a new explanation of the beautiful proof from Ghilardi (2000). It is an attempt to clarify the proof by indicating its key elements. One key ingredient is bisimulation. We introduce so-called *extension structures* to explain the close relationship between bisimulation and the extension property.

Surprisingly, our analysis reveals an additional benefit in terms of a shortening of the solution. To prove projectivity from the extension property, Ghilardi constructs a unifier that is a composition of substitutions. We will show that this composition can be shortened. We would like to stress that this is a minor simplification and its proof is still strongly based on Ghilardi's proof strategy.

Section 6.2.2, which is based on (van der Giessen, 2021a), takes up the question whether a similar semantic characterization of projective formulas can be established for intuitionistic modal logics. We answer in the affirmative for certain intuitionistic modal logics with coreflection including those from Figure 1.4, i.e., $\mathsf{iCK4}$, $\mathsf{IEL}$, $\mathsf{iCS4} \equiv \mathsf{IPC}$, $\mathsf{iSL}$, $\mathsf{mHC}$, $\mathsf{KM}$, and $\mathsf{PLL}$. The proof strategy is based on the one that Ghilardi (1999) employs for $\mathsf{IPC}$.

We conclude in Section 6.3. Here we discuss among other things promising future work to establish finite unification for intuitionistic modal logics with coreflection.

## 6.1 Unification and admissible rules

In Section 5.1.5 we informally introduced unification theory. It is concerned with solving equations in a certain theory. Its formulation depends on its setting. Here we are interested in the logical point of view where equations can be considered as formulas and the solutions as substitutions mapping the formula in question to a theorem of the logic. See also (Baader and Snyder, 2001) for an introduction to unification theory.

**6.1.1 Definition** (Unifier)
Let $\mathsf{L}$ be a logic and let $A \in \mathsf{Form}(\overline{p})$. A substitution $\sigma : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{q})$ is called a *unifier for $A$ in* $\mathsf{L}$, or an $\mathsf{L}$-*unifier* for short, if $\vdash_{\mathsf{L}} \sigma(A)$. If such a unifier exists we say that formula $A$ is $\mathsf{L}$-*unifiable.*

We often drop the annotation $\mathsf{L}$ and simply say that substitution $\sigma$ is a unifier or that formula $A$ is unifiable when $\mathsf{L}$ is clear from the context.

Unifiers play an important role for admissible rules and were already used in the definition of an admissible rule in Definition 5.2.18. In words, a rule $\Gamma/\Delta$ is admissible if whenever $\sigma$ is a unifier for all formulas in $\Gamma$, it is a unifier for some formula in $\Delta$. The substitutions form a technical tool in the field of admissible rules, but are the objects of study in unification theory.

A frequently addressed problem in unification theory is the representation of all unifiers of one formula. Of course, if a formula is not unifiable its representation is empty. But when the formula is unifiable it is interesting to find a nice representation of its unifiers. Often, such a representation is considered to only contain the 'best' unifiers from which all other unifiers follow. These are called complete minimal sets of unifiers that involve an ordering on unifiers.

**6.1.2 Definition**
Let $\mathsf{L}$ be a logic. A substitution $\sigma : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{q})$ is *less general* than a substitution $\tau : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{r})$, denoted $\sigma \leq_{\mathsf{L}} \tau$, if there exists a substitution $\xi : \mathsf{Form}(\overline{r}) \to \mathsf{Form}(\overline{q})$ such that for all $p \in \overline{p}$:

$$\vdash_{\mathsf{L}} \sigma(p) \leftrightarrow \xi(\tau(p)).$$

Unifiers for a formula $A \in \mathsf{Form}(\overline{p})$ are ordered according to the preorder $\leq_{\mathsf{L}}$.[31]

**6.1.3 Definition** (Minimal complete set of unifiers)
Let $\mathsf{L}$ be a logic and let $A \in \mathsf{Form}(\overline{p})$ be unifiable in $\mathsf{L}$. By $U(A)$ we denote the set of $\mathsf{L}$-unifiers of $A$. Note that $U(A)$ is non-empty. A subset $U \subseteq U(A)$ is said to be

- *complete* if every unifier in $U(A)$ is less general than a unifier from $U$,
- *minimal* if all unifiers in $U$ are incomparable with respect to $\leq_{\mathsf{L}}$.

**6.1.4 Definition** (Most general unifier)
Let $\mathsf{L}$ be a logic and let $A \in \mathsf{Form}(\overline{p})$ be a formula unifiable in $\mathsf{L}$. Substitution $\sigma : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{p})$ is a *most general unifier* for $A$, if $\{\sigma\}$ is a minimal complete set of $\mathsf{L}$-unifiers of $A$.

It is well known that each unifiable formula in $\mathsf{CPC}$ has a most general unifier which follows from the unitarity of Boolean unification, see, e.g., (Martin and Nipkow, 1989). This property does not hold in any classical and intuitionistic modal logic $\mathsf{L}$ that we consider. Recall from Example 5.2.27 that the modal disjunction property holds for these logics. Formula $\Box p \vee \Box\neg p$ has unifiers $\sigma_1, \sigma_2$ defined by

$$\sigma_1(p) := \top \quad \text{and} \quad \sigma_2(p) := \bot.$$

[31] Note that for $A \in \mathsf{Form}(\overline{p})$, every unifier with domain $\mathsf{Form}(\overline{p})$ defines a unifier with a domain $\mathsf{Form}(\overline{q})$ with $\overline{q} \supseteq \overline{p}$, and can in turn not be compared according to Definition 6.1.2 in case $\overline{q}$ strictly extends $\overline{p}$. Therefore we implicitly fix $\overline{p}$ and only consider unifiers with domain $\mathsf{Form}(\overline{p})$.

We can not point to a most general unifier for this formula, because if $\sigma$ is a unifier for the formula, i.e., $\vdash_{\mathsf{L}} \Box\sigma(p) \vee \Box\neg\sigma(p)$, then $\vdash_{\mathsf{L}} \sigma(p)$ or $\vdash_{\mathsf{L}} \neg\sigma(p)$ by the modal disjunction property. In the former case, $\sigma$ is equivalent to $\sigma_1$, but in the latter case it is equivalent to $\sigma_2$. So $\sigma_2 \not\leq_{\mathsf{L}} \sigma$ in the first case and $\sigma_1 \not\leq_{\mathsf{L}} \sigma$ in the second case. Therefore there is no most general unifier for this formula.

From the observation above we could deduce that unification in classical and intuitionistic modal logics is in some sense more difficult than for CPC. This intuition is captured by different unification types.

Given $U(A)$ for a unifiable formulas $A$, there might be different complete minimal sets of $U(A)$, but they must have the same cardinality, see, e.g., (Baader and Snyder, 2001). This gives us the following properties. We say that the *type* of $U(A)$ is *unary, finitary, infinitary* if it has a minimal complete set of unifiers for which its cardinality is 1, non-zero and finite, or infinite, respectively. It might also be the case that no minimal complete set of unifiers exists. Then we say that $U(A)$ has *nullary type.*

**6.1.5 Definition** (Unification types)
Let L be a logic. We say that the unification type of L is

- *nullary*, if there is an L-unifiable formula $A$, such that $U(A)$ has nullary type,
- *unary*, if for every L-unifiable formula $A$, the set $U(A)$ has unary type,
- *finitary*, if for every L-unifiable formula $A$, the set $U(A)$ has unary or finitary type,
- *infinitary*, if for every L-unifiable formula $A$, the set $U(A)$ has unary, finitary, or infinitary type.

The existence of such a minimal complete set of unifiers is not always guaranteed and amounts to the worst unification type, the nullary type, also regularly called type 0. An example is classical modal logic K (Jeřábek, 2013). We will come back to it in Remark 6.1.29, where we see that this has important consequences on how to treat admissibility in K.

In the next sections we introduce the concepts of projective formulas and projective approximations and see in Theorem 6.1.26 that IPC, K4, S4, and GL have finitary unification type, as shown by Ghilardi (1999, 2000).

### 6.1.1 Projective formulas

We recall facts about projective formulas, in particular, they have a most general unifier as shown in Lemma 6.1.7 and they reduce admissibility to derivability in the sense of Lemma 6.1.9. All the concepts introduced here hold for both classical and intuitionistic modal logic.

**6.1.6 Definition** (Projective formula)
Let $\mathsf{L}$ be a logic and let $A \in \mathsf{Form}(\overline{p})$ be a formula. Formula $A$ is called *projective*[32] *in* $\mathsf{L}$, or $\mathsf{L}$-*projective* for short, if there exists a substitution $\sigma : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{p})$ such that:

(i) $\sigma$ is an $\mathsf{L}$-unifier for $A$;
(ii) for all propositional variables $p \in \overline{p}$, we have $A \vdash_{\mathsf{L}} \sigma(p) \leftrightarrow p$.

We call $\sigma$ an $\mathsf{L}$-*projective unifier* for $A$.

Note that an $\mathsf{L}$-projective formula $A$ is also projective in any logic $\mathsf{L}'$ extending $\mathsf{L}$. Also note that the second condition in the definition is equivalent to

$$A \vdash_{\mathsf{L}} \sigma(B) \leftrightarrow B \text{ for all formulas } B \in \mathsf{Form}(\overline{p}). \tag{6.1}$$

Projective formulas play the following special role in unification theory.

**6.1.7 Lemma**
Let $\mathsf{L}$ be a logic and let $A$ be an $\mathsf{L}$-projective formula. Then $A$ has a most general unifier.

Proof. Let $\sigma$ be the $\mathsf{L}$-projective unifier for $A \in \mathsf{Form}(\overline{p})$. We show that this is a most general unifier. So we have to show that $\{\sigma\}$ is a minimal complete set of $\mathsf{L}$-unifiers of $A$. Minimality is trivial. To show completeness suppose that $\tau$ is an $\mathsf{L}$-unifier for $A$. Since $\sigma$ is an $\mathsf{L}$-projective unifier for $A$ we have $A \vdash_{\mathsf{L}} \sigma(p) \leftrightarrow p$ for all $p \in \overline{p}$. It follows that $\vdash_{\mathsf{L}} \tau(\sigma(p)) \leftrightarrow \tau(p)$, showing that $\tau \leq_{\mathsf{L}} \sigma$. ■

**6.1.8 Example**
Formula $p$ is projective in any classical and intuitionistic modal logic $\mathsf{L}$. The projective unifier $\sigma$ is given by $\sigma(p) := \top$ and $\sigma(q) := q$ for all $q \neq p$. Indeed, $\vdash_{\mathsf{L}} \sigma(p)$ and we have $p \vdash_{\mathsf{L}} p \leftrightarrow \top$ and for all $q \neq p$ we have $p \vdash_{\mathsf{L}} q \leftrightarrow q$. Similarly, formula $\neg p$ is projective by changing the definition of $\sigma$ according to $\sigma(p) := \bot$.

Section 6.2 provides more examples. The following lemma is proved analogously to Lemma 6 from (Iemhoff and Metcalfe, 2009b). We additionally consider intu-

[32] In some references, for instance in (Ghilardi, 2000, page 189), projective formulas in modal logic are of the form $\boxdot A$ by definition. The difference is due to the fact that projectivity is defined on the basis of the local consequence relation instead of the global consequence relation. In particular, recall from Example 5.2.12 that $\Gamma \vdash_{\mathsf{L}} A$ iff $\boxdot\Gamma \vdash^{l}_{\mathsf{L}} A$, for transitive modal logics. In addition, note that if $\sigma$ is a unifier for $A$, then so it is for $\boxdot A$. This illustrates the equivalence between the two definitions.

itionistic modal logics, so we give the proof for the sake of completeness.

**6.1.9 Lemma**
Let $\mathsf{L}$ be a logic. Let $A$ be a projective formula in $\mathsf{L}$ and let $\Gamma$ be a finite set of projective formulas in $\mathsf{L}$. For any formula $B$ and finite set of formulas $\Delta$, we have

1. $A \mathrel{|\!\sim}_{\mathsf{L}} \Delta$ if and only if $A \vdash_{\mathsf{L}} \Delta$;
2. $\bigvee \Gamma \mathrel{|\!\sim}_{\mathsf{L}} B$ if and only if $\bigvee \Gamma \vdash_{\mathsf{L}} B$.

Proof. For both statements, the direction from right to left follows immediately from Lemma 5.2.24. For the other direction in (1) suppose that $A \mathrel{|\!\sim}_{\mathsf{L}} \Delta$. Since $A$ is $\mathsf{L}$-projective, there exists an $\mathsf{L}$-projective unifier $\sigma$ for it such that $\vdash_{\mathsf{L}} \sigma(B)$ for some $B \in \Delta$. Because $\sigma$ is an $\mathsf{L}$-projective unifier for $A$ we have $A \vdash_{\mathsf{L}} \sigma(B) \leftrightarrow B$, and so $A \vdash_{\mathsf{L}} B$. And thus $A \vdash_{\mathsf{L}} \Delta$ by definition of $\vdash_{\mathsf{L}}$ (Definition 5.2.15). To show statement (2), suppose $\bigvee \Gamma \mathrel{|\!\sim}_{\mathsf{L}} B$. By definition of admissibility we know that $C \mathrel{|\!\sim}_{\mathsf{L}} B$ for every $C \in \Gamma$. Every $C \in \Gamma$ is $\mathsf{L}$-projective so we apply (1) to obtain $C \vdash_{\mathsf{L}} B$ for each $C \in \Gamma$. Hence $\bigvee \Gamma \vdash_{\mathsf{L}} B$. ■

In Section 6.2 we study the semantic properties of projective formulas and see that projectivity is decidable in many well-known logics.

### 6.1.2 Projective approximations

Projective approximations naturally pop up in the study of finitary unification for logics such as $\mathsf{IPC}$, $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{GL}$ that we will briefly discuss in Section 6.1.3.

The literature captures multiple closely related definitions of such approximations. They originate from the notion of *projective approximation* from Ghilardi (1999) and are adopted by multiple authors, i.e, Iemhoff (2005, 2015), Jeřábek (2005, 2010), and Ghilardi and Lenzi (2022). There are subtle differences between definitions, most importantly the (explicit or implicit) incorporation of admissibility. Other related concepts are *admissible approximations* (Goudsmit, 2015) or *admissibly saturated approximations* (Jeřábek, 2013) that do not *a priori* refer to projectivity. We explicitly introduce admissible projective approximations.

**6.1.10 Definition** (Admissible projective approximation)
Let $\mathsf{L}$ be a logic and let $A \in \mathsf{Form}(\overline{p})$. An $\mathsf{L}$-*projective approximation of* $A$, written $\Pi_A$, is a finite set of formulas in $\mathsf{Form}(\overline{p})$ such that

(i) all formulas in $\Pi_A$ are $\mathsf{L}$-projective;
(ii) for all $B \in \Pi_A$ it holds that $B \vdash_{\mathsf{L}} A$;
(iii) for any $\mathsf{L}$-projective formula $C$ such that $C \vdash_{\mathsf{L}} A$, there is a $B \in \Pi_A$ such that $C \vdash_{\mathsf{L}} B$.

If $\Pi_A$ only satisfies the first two properties, we call it a *weak* $\mathsf{L}$*-projective approximation of* $A$. An $\mathsf{L}$*-admissible projective approximation of* $A$ satisfies instead of (iii) to following property:

(iv) for all substitutions $\sigma$, if $\vdash_{\mathsf{L}} \sigma(A)$, then $\vdash_{\mathsf{L}} \sigma(B)$ for some $B \in \Pi_A$.

We sometimes drop $\mathsf{L}$ in the terminology when $\mathsf{L}$ is clear from the context or when we generally speak about the concepts. Note that in the multi-conclusion setting, property (iv) is often defined by the equivalent definition that $A \mathrel{|\!\!\!\sim}_{\mathsf{L}} \Pi_A$. For single-conclusion rules in transitive logics, it implies $A \mathrel{|\!\!\!\sim}_{\mathsf{L}} A^*$, where $A^* := \bigvee \boxdot \Pi_A$. This follows from Lemma 6.1.15 below.

The existence of admissible approximations are guaranteed for logics $\mathsf{IPC}$, $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{GL}$ by the study on unification, discussed in Section 6.1.3. Uniqueness (up to provable equivalence) follows when only the $\vdash_{\mathsf{L}}$-maximal projective formulas $B$ are taken that satisfy the properties of projective approximation.

Weak projective approximations are not common in the literature, but we would like to use this extra category in order to keep track of the minimal assumptions that are needed in the following results. A weak projective approximation does not have to contain all projective formulas $B$ (up to provable equivalence) maximal in $\vdash_{\mathsf{L}}$ such that $B \vdash_{\mathsf{L}} A$. Note that even the empty set is a weak projective approximation. This might not seem as a fruitful definition, but we have the following useful observation.

**6.1.11 Lemma**
Let $\mathsf{L}$ be a logic, let $\mathsf{L}'$ be an extension of $\mathsf{L}$, and let $A$ be a formula. If $\Pi_A$ is a weak $\mathsf{L}$-projective approximation of $A$, then it is also a weak $\mathsf{L}'$-projective approximation of $A$.

Proof. This follows from the fact that an $\mathsf{L}$-projective formula is also projective in $\mathsf{L}'$. ■

**6.1.12 Lemma**
Let $\mathsf{L}$ be a logic and let $A$ be a formula. Any $\mathsf{L}$-admissible projective approximation of $A$ is an $\mathsf{L}$-projective approximation of $A$.

Proof. Let $\Pi_A$ be an admissible projective approximation of $A$. We have to show that condition (iii) follows from (iv). Suppose $C$ is an $\mathsf{L}$-projective formula such that $C \vdash_{\mathsf{L}} A$. Let $\sigma$ be a projective unifier for $C$. So $\vdash_{\mathsf{L}} \sigma(C)$ and so $\vdash_{\mathsf{L}} \sigma(A)$. By property (iv) there is a $B \in \Pi_A$ such that $\vdash_{\mathsf{L}} \sigma(B)$. Since $\sigma$ is a projective unifier for $C$ we have $C \vdash_{\mathsf{L}} \sigma(B) \leftrightarrow B$. Hence $C \vdash_{\mathsf{L}} B$. ■

**6.1.13 Lemma**
Let $\mathsf{L}$ be a logic and let $A$ be a formula. The following statements hold.

1. If $A$ is not $\mathsf{L}$-unifiable, then $\Pi_A = \emptyset$ is an $\mathsf{L}$-(admissible) projective approximation of $A$.
2. If $A$ is $\mathsf{L}$-projective, then $\Pi_A = \{A\}$ is an $\mathsf{L}$-(admissible) projective approximation of $A$.

Proof. For both, properties (i), (ii), (iv) are trivial, and (iii) follows from the previous lemma. ■

Lemma 6.1.9 can be generalized to admissible approximations where admissibility reduces to derivability as presented in the next lemma. This property is commonly taken as the definition of admissible approximations that do not *a priori* refer to projectivity, see, e.g., (Goudsmit, 2015).

**6.1.14 Lemma**
Let $\mathsf{L}$ be a logic and let $\Gamma$ and $\Delta$ be finite sets of formulas. Let $\Pi_{\bigwedge\Gamma}$ be an $\mathsf{L}$-admissible projective approximation of $\bigwedge\Gamma$. Then

$$\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta \text{ iff } B \vdash_{\mathsf{L}} \Delta \text{ for all } B \in \Pi_{\bigwedge\Gamma}. \tag{6.2}$$

Moreover, the direction from left to right holds when $\Pi_{\bigwedge\Gamma}$ is assumed to be a weak $\mathsf{L}$-projective approximation.

Proof. First suppose $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$ and suppose $B \in \Pi_{\bigwedge\Gamma}$. By definition of projective approximation we have $B \vdash_{\mathsf{L}} \bigwedge\Gamma$. Let $\sigma$ be a projective unifier for $B$. So $\vdash_{\mathsf{L}} \sigma(B)$ and therefore $\vdash_{\mathsf{L}} \sigma(\bigwedge\Gamma)$. Then also $\vdash_{\mathsf{L}} \sigma(C)$ for all $C \in \Gamma$. From $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$ it follows that $\vdash_{\mathsf{L}} \sigma(A)$ for some $A \in \Delta$. As $B \vdash_{\mathsf{L}} \sigma(A) \leftrightarrow A$ we have $B \vdash_{\mathsf{L}} A$ for some $A \in \Delta$. Hence $B \vdash_{\mathsf{L}} \Delta$. For the other direction suppose $B \vdash_{\mathsf{L}} \Delta$ for all $B \in \Pi_{\bigwedge\Gamma}$. Let $\sigma$ be a substitution such that $\vdash_{\mathsf{L}} \sigma(C)$ for all $C \in \Gamma$, so $\vdash_{\mathsf{L}} \sigma(\bigwedge\Gamma)$. By property (iv) of Definition 6.1.10, $\vdash_{\mathsf{L}} \sigma(B)$ for some $B \in \Pi_{\bigwedge\Gamma}$. From $B \vdash_{\mathsf{L}} \Delta$ it follows $\sigma(B) \vdash_{\mathsf{L}} \sigma(\Delta)$ and so $\vdash_{\mathsf{L}} \sigma(\Delta)$. Recall the definition of $\vdash_{\mathsf{L}}$ in Definition 5.2.15. So, $\vdash_{\mathsf{L}} \sigma(A)$ for some $A \in \Delta$ and, thus, $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$. ■

We present a single-conclusion analogue in the following lemma in which the inclusion of $\vdash_{\mathsf{L}}$ in $\mathrel{|\!\!\sim}_{\mathsf{L}}$ has a left adjoint denoted by $A^* := \bigvee \boxdot\Pi_A$. This is called the *maximal admissible consequence* of formula $A$ by Iemhoff (2005). Note that for intuitionistic modal logics with coreflection we can take $A^* := \bigvee \Pi_A$. This fact is folklore but we provide a proof for the sake of completeness.

**6.1.15 Lemma**
Let $\mathsf{L}$ be a logic extending $\mathsf{K4}$ or $\mathsf{iK4}$ and let $A$ and $B$ be formulas. Let $\Pi_A$ be an $\mathsf{L}$-admissible projective approximation of $A$ and let $A^* := \bigvee \boxdot\Pi_A$. Then

$$A \mathrel{|\!\!\sim}_{\mathsf{L}} B \text{ iff } A^* \vdash_{\mathsf{L}} B. \tag{6.3}$$

Moreover, the direction from left to right holds when $\Pi_A$ is assumed to be a weak $\mathsf{L}$-projective approximation.

Proof. Let us show the difficult direction from right to left. So suppose $A^* \vdash_{\mathsf{L}} B$. By the deduction theorem (Theorem 1.1.4) we have $\vdash_{\mathsf{L}} \boxdot A^* \to B$. For transitive modal logics it holds that $\bigvee \boxdot\Pi \to \Box \bigvee \boxdot\Pi$ for any set of formulas $\Pi$. Therefore $\vdash_{\mathsf{L}} A^* \to B$. This means that $\vdash_{\mathsf{L}} \boxdot C \to B$ for all $C \in \Pi_A$ and again by the deduction theorem we obtain $C \vdash_{\mathsf{L}} B$ for all $C \in \Pi_A$. We use Lemma 6.1.14 to conclude $A \mathrel{|\!\sim}_{\mathsf{L}} B$. ■

We have the following immediate consequence of Lemmas 6.1.14 and 6.1.15 as witnessed by Ghilardi (1999).

#### 6.1.16 Corollary

Let $\mathsf{L}$ be a decidable logic and suppose that every formula $A$ has an $\mathsf{L}$-admissible projective approximation. Then the multi-conclusion admissibility problem for $\mathsf{L}$ is decidable. Moreover, the same holds for single-conclusion rules in case $\mathsf{L}$ extends $\mathsf{K4}$ or $\mathsf{iK4}$.

Projective approximations are valuable to present bases of the admissible rules in extensions of logics. The theorems rely on the existence of bases in the base logic. Similar theorems in the context of intermediate logics have been proved by Iemhoff (2005).

#### 6.1.17 Theorem

Let $\mathsf{L}$ be a logic and let $\mathcal{R}$ be a set of multi-conclusion rules admissible in $\mathsf{L}$. Suppose that each formula $A$ has a weak $\mathsf{L}$-projective approximation $\Pi_A$. If

$$A \vdash_{\mathsf{L}}^{\mathcal{R}} \Pi_A \text{ for all formulas } A,$$

then $\mathcal{R}$ is a basis for the multi-conclusion admissible rules in $\mathsf{L}$.

Proof. In order to show that $\mathcal{R}$ is a basis we have to prove that $\mathrel{|\!\sim}_{\mathsf{L}} = \vdash_{\mathsf{L}}^{\mathcal{R}}$. The inclusion $\vdash_{\mathsf{L}}^{\mathcal{R}} \subseteq \mathrel{|\!\sim}_{\mathsf{L}}$ follows by the assumption that $\mathcal{R} \subseteq \mathrel{|\!\sim}_{\mathsf{L}}$. For the other inclusion, let $\Gamma$ and $\Delta$ be finite sets of formulas and suppose $\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta$. By Lemma 6.1.14, we have $B \vdash_{\mathsf{L}} \Delta$ for each $B \in \Pi_{\bigwedge\Gamma}$. Furthermore, $\bigwedge\Gamma \vdash_{\mathsf{L}}^{\mathcal{R}} \Pi_{\bigwedge\Gamma}$ by assumption. Now, transitivity of $\vdash_{\mathsf{L}}^{\mathcal{R}}$ gives us $\bigwedge\Gamma \vdash_{\mathsf{L}}^{\mathcal{R}} \Delta$. Therefore, $\Gamma \vdash_{\mathsf{L}}^{\mathcal{R}} \Delta$. ■

#### 6.1.18 Theorem

Let $\mathsf{L}$ be a logic extending $\mathsf{K4}$ or $\mathsf{iK4}$ and let $\mathcal{R}$ be a set of single-conclusion rules admissible in $\mathsf{L}$. Suppose that each formula $A$ has a weak $\mathsf{L}$-projective approximation $\Pi_A$. If

$$A \vdash_{\mathsf{L}}^{\mathcal{R}} A^* \text{ for all formulas } A,$$

then $\mathcal{R}$ is a basis for the single-conclusion admissible rules in $\mathsf{L}$.

Proof. The proof is analogues to the proof of the previous theorem, but now applying Lemma 6.1.15 instead of Lemma 6.1.14. ■

Note that also the other directions of Theorems 6.1.17 and 6.1.18 hold when we are concerned with admissible projective approximations.

**6.1.19 Theorem**
Let $\mathsf{L}$ be a logic and suppose $\mathcal{R}$ is a basis of the multi-conclusion admissible rules in $\mathsf{L}$. Suppose that for each formula $A$ there exists an $\mathsf{L}$-admissible projective approximation $\Pi_A$. Let $\mathsf{L}'$ be an extension of $\mathsf{L}$ such that all the rules in $\mathcal{R}$ are admissible in $\mathsf{L}'$. Then $\mathcal{R}$ is also a basis of the admissible rules in $\mathsf{L}'$. Moreover, the statement holds for the single-conclusion setting in case $\mathsf{L}$ extends $\mathsf{K4}$ or $\mathsf{iK4}$.

Proof. We would like to apply Theorem 6.1.17 and show that $A \vdash^{\mathcal{R}}_{\mathsf{L}'} \Pi_A$ for all formulas $A$. This is sufficient, because by Lemma 6.1.11, $\Pi_A$ is also a weak $\mathsf{L}'$-projective approximation of $A$. (Actually, it is an admissible projective approximation because $A \mathrel{|\!\sim}_{\mathsf{L}'} \Pi_A$ since $\mathcal{R}$ is a set of admissible rules in $\mathsf{L}'$.) So let $A$ be a formula. Property (iv) of Definition 6.1.10 implies $A \mathrel{|\!\sim}_{\mathsf{L}} \Pi_A$. Since $\mathcal{R}$ is a basis for the admissible rules in $\mathsf{L}$ we have $A \vdash^{\mathcal{R}}_{\mathsf{L}} \Pi_A$. Since $\mathsf{L}'$ is an extension of $\mathsf{L}$ we conclude $A \vdash^{\mathcal{R}}_{\mathsf{L}'} \Pi_A$ as desired. Similar reasoning applies the the single-conclusion setting when applying Theorem 6.1.18. ■

### 6.1.3 Extendible frames

We turn back to unification and see how admissible projective approximations emerge from the study of finitary unification in logics such as $\mathsf{IPC}$, $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{GL}$. For similar results in intermediate logics, see (Goudsmit and Iemhoff, 2014). The proof strategies apply to logics that are sound and complete with respect to a class of frames that are *extendible*.

**6.1.20 Definition**
Let $F_1, \dots, F_n$, with $F_i = (W_i, R_i)$, be a finite list of finite classical rooted transitive frames. Frame $(\sum_{i=1}^n F_i)^\bullet = (W, R)$ is defined as

$$\begin{aligned} W &:= W_1 \sqcup \cdots \sqcup W_n \sqcup \{\rho\}, \\ R &:= R_1 \sqcup \cdots \sqcup R_n \sqcup \{(\rho, w) \mid w \in W_i \text{ for some } 1 \le i \le n\}, \end{aligned}$$

so that $\rho$ is a new irreflexive root extending frames $F_i$. Frame $(\sum_{i=1}^n F_i)^\circ$ is similarly defined, but letting $\rho$ be a reflexive world. A class of frames $\mathcal{F}$ is called *extendible* if for every finite list of frames $F_1, \dots, F_n \in \mathcal{F}$,

- $(\sum_{i=1}^n F_i)^\bullet$ is in $\mathcal{F}$, unless no frame in $\mathcal{F}$ contains an irreflexive world;
- $(\sum_{i=1}^n F_i)^\circ$ is in $\mathcal{F}$, unless no frame in $\mathcal{F}$ contains a reflexive world.

A similar construction can be made for treelike models for IPC and we see that IPC is complete with respect to a class of extendible frames (Iemhoff, 2015).

**6.1.21 Definition**
Let L be a classical modal logic complete with respect to the class $\mathcal{F}$ of all finite rooted transitive frames $F$ such that $F \models A$ for all $A \in \mathsf{L}$. Logic L is called *extendible* if $\mathcal{F}$ is extendible.[33]

Extendible logics are, by definition, classical modal extensions of K4 with the finite model property. Examples are K4, S4, and GL. For instance, S5 is not extendible. Extendible logics satisfy the modal disjunction property, and the other direction is true when dealing with descriptive frames providing a semantic characterization of the modal disjunction property (Chagrov and Zakharyaschev, 1997, Chapter 15).

The following theorems form the main results from Ghilardi (1999, 2000). Roughly speaking, the theorems rely on two aspects. One is the extension property examined in Section 6.2. The second is a (bi)simulation argument that falls outside the scope of this thesis. Therefore we state most of the next results without proof.

Before we state Theorem 6.1.23, recall the definition of modal degree $md$ from Definition 1.2.25. For IPC one is concerned with the implicational degree.[34]

**6.1.22 Definition** (Implicational degree)
The *implicational degree* $id(A)$ of a (box-free) formula $A$ is defined recursively:

$$\begin{aligned} id(p) &= 0, \text{ for } p \in \mathsf{Prop}; \\ id(\bot) &= 0; \\ id(A \cdot B) &= \max(id(A), id(B)), \text{ for } \cdot = \wedge, \vee; \\ id(A \to B) &= \max(id(A), id(B)) + 1. \end{aligned}$$

Similarly to the modal case, there are finitely many non-equivalent formulas over a given finite set of propositional variables in IPC of implicational degree less or equal to $n$.

**6.1.23 Theorem** (Ghilardi, 1999; Ghilardi, 2000)
Let L be an extendible classical modal logic (or let L be IPC). Let $\sigma$ be an L-unifier for $A \in \mathsf{Form}(\overline{p})$. Then there exists an L-projective formula $B \in \mathsf{Form}(\overline{p})$ such that

(i) $\vdash_{\mathsf{L}} \sigma(B)$;
(ii) $B \vdash_{\mathsf{L}} A$; and
(iii) $md(B) \leq md(A)$ (or $id(B) \leq id(A)$ in case of IPC).

[33] In (Jeřábek, 2005, 2006) these are called *extensible* logics.
[34] IPC is defined in the language without $\Box$ and definitions like Form are treated accordingly.

**6.1.24 Theorem** (Ghilardi, 1999; Ghilardi, 2000)
Let L be IPC or an extendible classical modal logic. Then, every formula $A$ has an L-admissible projective approximation.

Proof. If $A$ is not L-unifiable we take $\Pi_A = \emptyset$ by Lemma 6.1.13. Otherwise, for each L-unifier for $A$ take a $B$ as defined in Theorem 6.1.23 and collect them in set $\Pi_A$. Properties (i), (ii), and (iv) are easily proved. ■

We are now able to write down a proof for Theorems 5.3.12 and 5.3.13 presented in the previous chapter. Namely, we have all the conditions to apply Theorem 6.1.19. Indeed, Theorem 5.3.10 and Theorem 5.3.11 give us bases for the admissible rules in K4, S4, and GL, and Theorem 6.1.24 implies the existence of admissible projective approximations for logics K4, S4, and GL.

The combination of Theorem 6.1.24 and Corollary 6.1.16 results in the following.

**6.1.25 Corollary** (Ghilardi, 1999; Ghilardi, 2000)
Let L be IPC or an extendible classical modal logic. If L is decidable, then admissibility in L is decidable.

**6.1.26 Theorem** (Ghilardi, 1999, Theorem 3.5; Ghilardi, 2000, Theorem 3.5)
Let L be IPC or an extendible classical modal logic. Then L has finitary unification type.

Proof. Let us shortly explain the proof. Suppose $A$ is L-unifiable. For each L-unifier for $A$ consider a $B$ as defined in Theorem 6.1.23 and take its most general unifier (which exists by Lemma 6.1.7). Collect all such most general unifiers. This gives us a finite complete set of unifiers of $A$. In particular, each unifier for $A$ is less general than the most general unifier of some $B$, which would also be a unifier for $A$ by Theorem 6.1.23 (ii). The set is guaranteed to be finite by the bounded modal or implicational degree of the formulas $B$, and, in turn, a minimal complete set of unifiers is easily extracted by only keeping the unifiers maximal in $\leq_{\mathsf{L}}$. ■

In addition, Ghilardi (2002) provides an algorithm to compute the minimal complete set of unifiers in IPC.

The presented results about finite unification and the existence of admissible projective approximations apply to IPC and classical extendible logics. For intuitionistic modal logics, the only result that we are aware of is for logic PLL by Ghilardi and Lenzi (2022). We conjecture similar results for the other logics with coreflection from Figure 1.4.

**6.1.27 Theorem** (Ghilardi and Lenzi, 2022)
Every formula $A$ has a PLL-admissible projective approximation.

**6.1.28 Theorem** (Ghilardi and Lenzi, 2022)
Logic PLL has finitary unification type.

Summarizing, projectivity plays an important role in the study of the admissible rules in extendible logics. However, if we would like to go beyond the fixed semantic properties, projectivity can not always give a clear road map on how to deal with admissibility. For instance, transitivity seems a key property.

**6.1.29 Remark**
Consider classical modal logic K. Jeřábek (2013) shows that K has nullary unification type. He does so by showing that our celebrated formula $p \to \Box p$ does not have a minimal complete set of unifiers. In addition, it has no admissible projective approximation. However, there is set of formulas belonging to $p \to \Box p$ satisfying condition (6.2) from Lemma 6.1.14 by showing that it has a so-called *admissibly saturated approximation.* These observations confirm that the outlined method does not work for K. In particular, Jeřábek (2013) sketches the idea why admissibility in K might be undecidable. But until now, that remains one of the major open problems in the area.

## 6.2 Extension property

This section is devoted to the semantic study of projective formulas in terms of the extension property. This is a separate notation next to the extendible frames discussed in the previous section. Theorems 6.2.6 and 6.2.31 present the main results stating that a formula is projective whenever the class of its models has the extension property. Most concepts are similar for classical and intuitionistic modal logics, but the proof techniques are different.

Section 6.2.1 analyses the proof given by Ghilardi (2000) in the setting of transitive classical modal logics. We identify key ingredients and attempt to clarify the close relationship between bisimulation and the extension property by identifying so-called *extension structures.* Section 6.2.2 provides a similar characterization for intuitionistic modal logic with coreflection based on techniques employed for IPC (Ghilardi, 1999).

### 6.2.1 Extension property in classical modal logic

**6.2.1 Convention**
In this section we assume $\mathsf{L}$ to be a classical modal logic extending $\mathsf{K4}$ with the finite model property with respect to the class $\mathcal{F}_{\mathsf{L}}$ of all *finite rooted transitive* frames $F$ such that $F \models A$ for all $A \in \mathsf{L}$. According to Remark 1.2.14, a frame $F = (W, R)$ is called an $\mathsf{L}$-*frame* if $F \in \mathcal{F}_{\mathsf{L}}$ and we call $K = (W, R, V)$ an $\mathsf{L}$-*model* if $K$ is based on an $\mathsf{L}$-frame. So every $\mathsf{L}$-frame and $\mathsf{L}$-model in this section is classical, finite, rooted, and transitive.

Note that in contrast to the results on unification presented Section 6.1.3, we do not assume $\mathsf{L}$ to be extendible. So results presented here do not only apply to for instance $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{GL}$, but also to $\mathsf{S5}$. For these logics, we might have the classes from Theorem 1.2.8 in mind.

**6.2.2 Definition**
For $A \in \mathsf{Form}(\overline{p})$, we define $\mathrm{Mod}_{\mathsf{L}}(A)$ to be the set of all $\mathsf{L}$-models over $\overline{p}$ that satisfy $A$ in the root. And we let $\mathrm{MOD}_{\mathsf{L}}(A)$ be the set of $\mathsf{L}$-models over $\overline{p}$ that satisfy $A$.

Recall Remark 1.2.10 for transitive logics $\mathsf{L}$, which implies $\mathrm{MOD}_{\mathsf{L}}(A) = \mathrm{Mod}_{\mathsf{L}}(\boxdot A)$. In addition, recall Definition 1.2.11 for a cluster and Definition 1.2.12 defining a model to almost satisfy a formula.

**6.2.3 Remark**
Our analysis is published in (van der Giessen, 2021b) and is based on (Ghilardi, 2000). The notation that we use here relies on the projective formulas that are defined using the global consequence relation in Definition 6.1.6. This in contrast to aforementioned work in which projective formulas are defined by the local consequence relation and have the form $\boxdot A$ by definition. In addition, we use $\mathrm{MOD}_{\mathsf{L}}(A)$ instead of $\mathrm{Mod}_{\mathsf{L}}(\boxdot A)$, write $K \models A$ instead of the equivalent $K \models \boxdot A$, and say that model $K$ almost satisfies $A$ instead of the equivalent statement for $\boxdot A$.

**6.2.4 Definition** (Variant)
A *variant* of an $\mathsf{L}$-model $K$ is an $\mathsf{L}$-model $K'$, such that they have the same frame and their valuations agree on all worlds except for possibly worlds in the cluster of the root.

**6.2.5 Definition** (Extension property)
A class $\mathcal{K}$ of $\mathsf{L}$-models is said to have the *extension property* if for every $\mathsf{L}$-model $K$ (with root $\rho$), if $K_w \in \mathcal{K}$ for each $w \notin cl(\rho)$, then there is a variant $K'$ of $K$ such that $K' \in \mathcal{K}$.

The extension property should not be confused with extendible frames form Definition 6.1.20. However, in some contexts the definition of extension property incorporates both notions such as in (Iemhoff, 2001a, 2005).[35]

If $\mathrm{MOD}_{\mathsf{L}}(A)$ has the extension property, it means that every model that almost satisfies $A$ can be turned into a model that satisfies $A$.

**6.2.6 Theorem** (Ghilardi, 2000, Theorem 2.2)
Formula $A$ is projective in $\mathsf{L}$ if and only if $\mathrm{MOD}_{\mathsf{L}}(A)$ has the extension property.

Proof. We show the two directions separately in Theorems 6.2.9 and 6.2.25. ■

A first ingredient in the proof bridges between substitutions in syntax and semantic operations on models.

**6.2.7 Definition**
Let $\sigma : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{q})$ be a substitution. We define the map $\sigma^*$ from $\mathsf{L}$-models over $\overline{q}$ to $\mathsf{L}$-models over $\overline{p}$ as follows. For $\mathsf{L}$-model $K$ over $\overline{q}$, $\sigma^*(K)$ is an $\mathsf{L}$-model over $\overline{p}$ with the same frame as $K$ and its valuation is defined according to:

$$\sigma^*(K), w \Vdash p \text{ iff } K, w \Vdash \sigma(p), \text{ for all } p \in \overline{p}. \tag{6.4}$$

Note that $\sigma^*$ only changes the valuation in the model. From now on we abuse terminology and call $\sigma^*$ a substitution on models. In addition, we are only interested in substitutions where domain and codomain coincide, since we are looking for a projective unifier. This is a first step to connect the extension property to projectivity because the first is a property of semantics and the latter of syntax. We give some properties of $\sigma^*$.

**6.2.8 Lemma** (Ghilardi, 2000, Proposition 1.3)
Let $A \in \mathsf{Form}(\overline{p})$ and let $\sigma : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{p})$ be a substitution. For every $\mathsf{L}$-model $K$ over $\overline{p}$, we have

1. $\sigma^*(K) \models A$ iff $K \models \sigma(A)$,
2. and for every substitution $\tau : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{p})$, $(\tau\sigma)^*(K) = \sigma^*(\tau^*(K))$.

Proof. Point (1) is shown by induction on the structure of $A$. Point (2) follows from (1). ■

Point (2) shows that the order of substitutions $\sigma$ and $\tau$ reverses. The following theorem is shown by (Ghilardi, 2000, Theorem 2.2), but we present the proof here in order to see its analogy with Theorem 6.2.32 in the intuitionistic modal setting.

[35] It is also interesting to mention that Iemhoff (2005) introduces weaker properties of such extension properties, called the *weak extension property* and *offspring property*, in the realm of intermediate logics.

Recall that for model $K = (W, R, V)$ we write $w \in K$ to mean $w \in W$ and $K(w)$ to mean $V(w)$, i.e., the set of propositional variables forced in $w$.

**6.2.9 Theorem** (Ghilardi, 2000, see Theorem 2.2)
If formula $A$ is projective in $\mathsf{L}$, then $\mathrm{MOD}_{\mathsf{L}}(A)$ has the extension property.

Proof. Let $A \in \mathsf{Form}(\overline{p})$ and let $K$ be an $\mathsf{L}$-model over $\overline{p}$ that almost satisfies $A$. Formula $A$ is projective, so we can take a substitution $\sigma$ such that $\vdash_{\mathsf{L}} \sigma(A)$ and $A \vdash_{\mathsf{L}} p \leftrightarrow \sigma(p)$ for each $p \in \overline{p}$. We will show that $\sigma^*(K)$ is a variant of $K$ that satisfies $A$. $\sigma^*(K)$ is a variant, because for all $w \notin cl(\rho)$, $\sigma^*(K)(w) = K(w)$, since $K_w \models A$ and $A \vdash_{\mathsf{L}} p \leftrightarrow \sigma(p)$ for each $p \in \overline{p}$. And $\sigma^*(K) \models A$ follows from $\vdash_{\mathsf{L}} \sigma(A)$, by completeness and Lemma 6.2.8 (1). ■

Now we turn to the other direction of Theorem 6.2.6. Ghilardi defines suitable substitutions that form the building blocks for a projective unifier for $A$.

**6.2.10 Definition** (Simple substitution)
Let $A \in \mathsf{Form}(\overline{p})$ and let $a \subseteq \overline{p}$. The substitution $\sigma_a^A : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{p})$ is defined as:

$$\sigma_a^A(p) = \begin{cases} \boxdot A \to p & \text{if } p \in a, \\ \boxdot A \wedge p & \text{if } p \notin a. \end{cases}$$

We call those substitutions *simple*.

From now, we omit the superscript and just write $\sigma_a$ when $A$ is clear from the context. By using the deduction theorem from Theorem 1.1.4, it is easy to check that condition (6.1) is satisfied for simple substitutions, which is a key condition for projectivity. Moreover, the same condition holds for any composition of simple substitutions.

Now we only have to search for a suitable combination of those $\sigma_a$'s and prove that this is a unifier for $A \in \mathsf{Form}(\overline{p})$. This is immediately a projective unifier by the observation above. However, finding the right concatenation is the hard part of the proof. Ghilardi defines substitution

$$\theta := \sigma_{a_1} \cdots \sigma_{a_s} \tag{6.5}$$

where $a_1, \ldots, a_s$ is any fixed ordering on the subsets of $\overline{p}$. In short, Ghilardi proves that $\theta^{2N}$ is a projective unifier for $A$, where $N$ is the number of $n$-bisimilar equivalence classes, but we will show that it suffices to use $(n-1)$-bisimilar classes. Let $N'$ be the number of different equivalence classes of $(n-1)$-bisimilar models. Number $N'$ is smaller than $N$, so this results in the shorter concatenation $\theta^{2N'}$. If we carefully read the proof of Theorem 6.2.24, we actually conclude that $\theta^{2(N'+1)}$ is the projective unifier for $A$.

The extension property will guide us in the following and gives us the right tools to show that $\theta^{2(N'+1)}$ is a unifier for $A$. The method consists of several steps. We start with relatively simple lemmas.

**6.2.11 Lemma** (Ghilardi, 2000, see Lemma 2.1)
Let $A \in \mathsf{Form}(\overline{p})$ and let $K$ be an $\mathsf{L}$-model over $\overline{p}$. Suppose $a \subseteq \overline{p}$. For every $w \in K$, we have

1. $(\sigma_a^*(K))(w) = K(w)$ if $K_w \models A$,
2. $(\sigma_a^*(K))(w) = a$ if $K_w \not\models A$, and
3. $\sigma_a^* \sigma_a^* = \sigma_a^*$.

Proof. For (1) and (2) see also (Ghilardi, 2000, Lemma 2.1). Point (3) follows from the first two. Let us show (2). By definition, $p \in (\sigma_a^*(K))(w)$ iff $K, w \Vdash \sigma_a(p)$. By definition of $\sigma_a$ and by the assumption that $K_w \not\models A$, or equivalently by Remark 1.2.10 that $K, w \nVdash \boxdot A$, the previous holds if and only if $K, w \Vdash \boxdot A \to p$ and $p \in a$. This shows $(\sigma_a^*(K))(w) = a$. ∎

In words, the first two points of the lemma say that the propositional variables forced in a world $w$ stay the same (in case $K_w \models A$), or become exactly the propositional variables in $a$ (in case $K_w \not\models A$).

**6.2.12 Lemma**
Let $K_1$ be an $\mathsf{L}$-model over $\overline{p}$ with a reflexive root $\rho_1$ such that $cl(\rho_1)$ is a singleton. Let $K_2$ be the result of replacing the root $\rho_1$ in $K_1$ by a cluster with root $\rho_2$ such that for all $w \in cl(\rho_2)$, $K_2(w) = K_1(\rho_1)$. Then $K_1 \models A$ iff $K_2 \models A$ for every formula $A$.

Proof. We show that $K_1$ and $K_2$ are bisimilar, i.e., $K_1 \sim K_2$. Then the desired result follows from Theorem 1.2.21 (having Remark 1.2.10 in mind). The models coincide except for possibly the clusters of the roots. So for $v \neq \rho_1$, $v$ is also a world in $K_2$. We define bisimulation

$$Z := \{(v, v) \mid v \neq \rho_1\} \cup \{(\rho_1, w) \mid w \in cl(\rho_2)\}.$$

It is easy to see that this is a bisimulation, because $\rho_1$ and all $w \in cl(\rho_2)$ are reflexive. ∎

**6.2.13 Lemma** (Ghilardi, 2000, Lemma 2.3)
Let $A \in \mathsf{Form}(\overline{p})$ and suppose that $\mathrm{MOD}_{\mathsf{L}}(A)$ has the extension property. Let $K$ be an $\mathsf{L}$-model over $\overline{p}$ that almost satisfies $A$. Then there is a set $a \subseteq \overline{p}$ such that $\sigma_a^*(K) \models A$.

Proof. Let $\rho$ be the root of $K$. If $cl(\rho)$ is a singleton, the lemma follows by Lemma 6.2.11 (1) and (2), and by definition of the extension property. If not, we construct a new model replacing the cluster by a single reflexive world and apply Lemma 6.2.12. ■

We combine the ingredients so far and sketch a naive idea to find a unifier $\tau$ for $A$ as composition of the simple substitutions $\sigma_a$. We will see that this naive idea is not sufficient and that we need more. The $\tau$ that we are going to construct should satisfy $\vdash_{\mathsf{L}} \tau(A)$. Using the completeness theorem and Lemma 6.2.8, we would like to show that $\tau^*(K) \models A$ for each L-model $K$. For simplicity, one can think of treelike models without any clusters. So let $K$ be such an L-model. We start at the leaves of the model and work our way down to the root. In each step we want to apply a $\sigma_a$ that gives us a model in which more worlds force $A$. Consider a world $w$ that almost satisfies $A$, but such that $A$ is not forced in $w$ itself. By Lemma 6.2.13 there is a valuation $a$ such that $w$ satisfies the variables from $a$ and $\sigma_a^*(K), w \Vdash A$. We pick $\sigma_a$ and apply it to our model. This strategy sounds promising, because we can go through all the worlds and apply a substitution that works for that world. Define $\tau$ on the basis of all those substitutions to yield $\tau^*(K) \models A$. The problem is that the definition of our $\tau$ depends on $K$, so we cannot define a good sequence of $\sigma's$ that works *for all* models $K$. Doing induction on the depth of the model will not solve the problem, because the depth is unbounded.

The key idea is to connect the extension property to bisimulation of models. Recall Definition 1.2.23 of $n$-bisimulation and recall that for a rooted model $K$, we write $[K]_n$ for its equivalence class of $n$-bisimilar models. Ghilardi defines four important ingredients: frontier points, a rank, homogeneous models and the minimal rank (the last is our terminology). Recall Definition 1.2.3 where $R^>$ is defined as $wR^>v$ iff $wRv$ and not $vRw$. Recall also Definition 1.2.25 for the modal degree $md$ of a formula and recall that for each $n$ and given finite list of propositional variables $\overline{p}$, there are finitely many non-equivalent formulas in $\mathsf{Form}(\overline{p})$ of modal degree less or equal to $n$. Let $A \in \mathsf{Form}(\overline{p})$ with $md(A) \leq n$ and let $K$ be an L-model over $\overline{p}$.

- The set of *frontier points* in $K$ is defined as
$$\{w \in K \mid K_w \not\models A \text{ and } \forall v(wR^>v \Rightarrow K_v \models A)\}.$$
- The *rank* $r$ of $K$ is defined as follows, where $|\cdot|$ denotes the cardinality:
$$r(K) := |\{[K_w]_n \mid \rho Rw \text{ and } K_w \models A\}|.$$
- Model $K$ is *homogeneous* if $r(K_w) = r(K_v)$ for each $w, v$ with $K_w \not\models A$ and $K_v \not\models A$.
- $\mu(K) := \min\{r(K_w) \mid K_w \not\models A\}$, which we call the *minimal rank*.

Frontier points are worlds $w$ such that $K_w$ almost satisfies $A$. As observed above, for each frontier point we can use the extension property (Lemma 6.2.13) to find

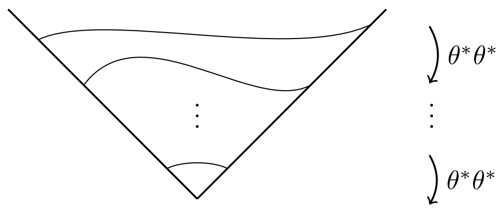



**Figure 6.1.** Lines of frontier points.

a $\sigma_a$ such that $A$ becomes true in that frontier point. For different frontier points there can be different $\sigma_a$'s that work. However, after one application of $\theta$, all frontier points are turned into points that satisfy $A$. The next step is to find the new frontier points and apply $\theta$ again. Ghilardi shows that after two applications of $\theta$, the minimal rank grows strictly. One $\theta$ covers irreflexive worlds and the other $\theta$ reflexive worlds. The minimal rank is bounded by $N$, so $K \models \theta^{2N}(A)$ for all models $K$.

Figure 6.1 sketches the idea of the frontier points in a pointed model. Each curved line represents the set of frontier points, which lowers after two applications of $\theta^*$. There are at most $N$ steps of $\theta^*\theta^*$ in the picture.

We keep the same idea in mind, but we propose to change the definition of the rank and give another approach for the homogeneous models. With our investigation, we want to address the important role of the frontier points and the link between bisimilar models and the extension property. The idea is to identify different so-called *extension structures* in the extension property of $\mathrm{MOD}_{\mathsf{L}}(A)$. Those extension structures are identified using bisimulation. In turn, each extension structure will correspond to a simple substitution $\sigma_a$ which are again the building blocks for $\theta$. We will see that $2(N'+1)$ applications of $\theta$ is enough, where $N'$ is the number of different $(n-1)$-bisimulation equivalence classes.

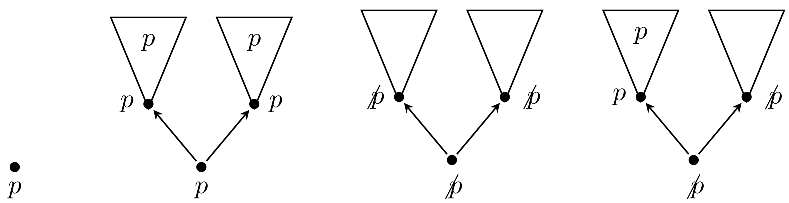
Before we explore the new method, we give some examples to see that in many cases a short substitution suffices to act as a projective unifier for $A$ and that this depends on the nature of the extension property of $\mathrm{MOD}_{\mathsf{L}}(A)$.

**6.2.14 Example**
Consider formula $A = p \to B$ for some formula $B$ and variable $p$. Formula $A$ has the extension property, because for each $\mathsf{L}$-model $K$ that almost satisfies $A$, we can find a variant $K'$ in which no propositional variable is forced in the root. This works independently of the shape of $K$. So $K' \models A$. This means that $\sigma^*_\emptyset(K) \models A$ for each $K$, so $\sigma_\emptyset$ is a projective unifier of $A$.

In general, if the extension property does not depend on the models above the root, one $\sigma$ suffices. So for satisfiable box-free formulas, that is the case.

### 6.2.15 Example

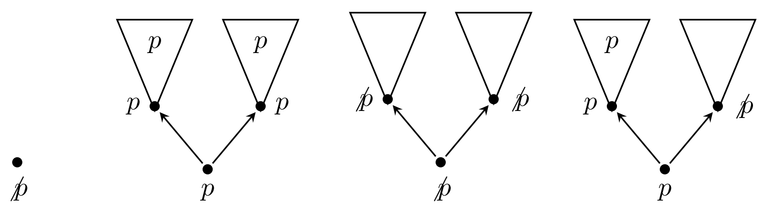
Consider formula $A = \Box p \leftrightarrow p$. For simplicity, we think of treelike models. There are multiple cases of the extension property, schematically depicted below where the arrows indicate the transitive relation $R$. The submodels above the root are assumed to satisfy $A$ (with no submodels in the first picture). (Do not confuse it with the *extendible* constructions such as in Example 5.2.27 where models are extended by a root. Here we start with models that have the depicted treelike form and modify valuations of $p$ in the root.) If all worlds above the root satisfy $p$, then force $p$ in the root, illustrated in the first two pictures. If there is at least one world in which $p$ does not hold, do not force $p$ in the root, illustrated in the last two models.



We want to know which sequence of $\sigma$'s turns each model in a model that satisfies $A$. Let $K$ be a model. We can first apply $\sigma_p^*$ that belongs to the left two pictures. By Lemma 6.2.11, if $K_w \models A$, then the propositional variables forced in $w$ in model $\sigma_p^*(K)$ stay the same, and if $K_w \not\models A$, then the only variable forced in $w$ is $p$. Moreover, for each world $w$ in $\sigma_p^*(K)$ such that $\sigma_p^*(K_w) \not\models A$ we have that there is at least one world $v$ above $w$ such that $\sigma_p^*(K_v) \not\models p$. So all these worlds belong to the third or fourth picture. Now we can take $\sigma_\emptyset^*$ to conclude $\sigma_\emptyset^*\sigma_p^*(K) \models A$. Hence, $\sigma_p\sigma_\emptyset$ is a projective unifier for $A$.

### 6.2.16 Example

Formula $B = \Box\neg p \leftrightarrow \neg p$ is the substitution instance of $A$ from the previous example where $\neg p$ is substituted for $p$. The pictures are now as follows:



We see that $\sigma_p^*\sigma_\emptyset^*$ turns each model in a model that satisfies $B$. Therefore $\sigma_\emptyset\sigma_p$ is a projective unifier for $B$. Note that here the $\sigma$'s depend on $B$, so now $\sigma_p$ means $\sigma_p^B$.

The examples illustrate that the set $a$ of propositional variables forced in the root depends on the structure of the models above it. In addition, we distinguish between the root being reflexive or irreflexive. This results in different *extension structures* defined in Definition 6.2.20.

We will formalise our method. Let us introduce our ingredients. We keep the same notion of frontier points as before.

### 6.2.17 Definition

Let $A \in \mathsf{Form}(\overline{p})$ with $md(A) \leq n$. Let $K$ be an $\mathsf{L}$-model over $\overline{p}$.

- The set of *frontier points* of $K$ is

$$\{w \in K \mid K_w \not\models A \text{ and } \forall v(wR^{>}v \Rightarrow K_v \models A)\}.$$

- The *bisimulation set of* $K$ is defined as

$$B(K) := \{[K_w]_{n-1} \mid \rho Rw \text{ and } K_w \models A\}.$$

- The *rank* $r(K)$ is the cardinality of $B(K)$.
- We call a frontier point $w$ *B-minimal* in $K$, if $r(K_w) \leq r(K_v)$, for all other frontier points $v$ in $K$.

The bisimulation set of $K$ is a subset of the set of all equivalence classes of $(n-1)$-bisimilar models that satisfy $A$. So, the rank is bounded by $N'$, where $N'$ is the number of $(n-1)$-bisimilar equivalence classes. Because we work with transitive models, we have the following important fact following from Lemma 6.2.11 (1): $B(K) \subseteq B(\sigma_a^*(K))$ for every $a \subseteq \overline{p}$. And so $r(K) \leq r(\sigma_a^*(K))$.

### 6.2.18 Remark

Note that if $md(A) = n = 0$, and thus $A$ is box-free, $(n-1)$-bisimulation is undefined. In that case one $\theta$ will suffice. More precisely, only one $\sigma_a$ will be enough, namely its classical propositional valuation making $A$ true (compare to Example 6.2.14). So in the rest we will assume that $n > 0$.

### 6.2.19 Example

Consider Examples 6.2.15 and 6.2.16. The modal degree of formulas $A$ and $B$ is 1. So the different bisimulation sets depend on 0-bisimulation. Therefore we have to examine the propositional variables that are forced in the worlds above the root. There are four bisimulation sets which we intuitively write as $\emptyset$, $\{p\}$, $\{\not p\}$ and $\{p, \not p\}$. They correspond from left to right to the pictures in the examples.

The applied substitution in the examples is the same for reflexive and irreflexive roots, but this is not the case in general.

**6.2.20 Definition**
Let $A \in \mathsf{Form}(\overline{p})$ with $md(A) \leq n$. Let $K$ be an $\mathsf{L}$-model over $\overline{p}$ almost satisfying $A$. The *extension structure* of $K$ (with respect to $A$) is the pair $(B(K), \cdot)$ of its bisimulation set and $\cdot = \mathrm{i}$ if the root of $K$ is irreflexive and $\cdot = \mathrm{r}$ if the root is reflexive.

Each bisimulation set may define two extension structures, depending on the (ir)reflexivity of the root. The following lemma shows that the same substitutions work for models with the same extension structure. We will see that each extension structure gives rise to a *corresponding substitution*.

**6.2.21 Lemma**
Let $A \in \mathsf{Form}(\overline{p})$ with $md(A) \leq n$. Let $K_1, K_2$ be two $\mathsf{L}$-models over $\overline{p}$ that almost satisfy $A$. Assume that they have the same extension structure. Then for each set $a \subseteq \overline{p}$,

$$\sigma_a^*(K_1) \models A \text{ iff } \sigma_a^*(K_2) \models A.$$

Proof. Let $\rho_1$ and $\rho_2$ be the roots of $K_1$ and $K_2$. The models have the same extension structure, so $B(K_1) = B(K_2)$ and $\rho_1$ and $\rho_2$ are both irreflexive or reflexive. By Lemma 6.2.12 together with Lemma 6.2.11 (2), it is enough to consider models $K_i$ with $cl(\rho_i)$ being a singleton. Suppose $\sigma_a^*(K_1) \models A$. We will show that $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$. From this it follows from Theorem 1.2.26 that $\sigma_a^*(K_2) \models A$, since $\sigma_a^*(K_1) \models A$ and $md(A) \leq n$.

We prove $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$ using the characterization in Lemma 1.2.27. We have $\sigma_a^*(K_1) \sim_0 \sigma_a^*(K_2)$ by Lemma 6.2.11 (2). First assume that $\rho_1$ and $\rho_2$ are irreflexive. Suppose $\rho_1 R_1 w$. Root $\rho_1$ is irreflexive so $w \neq \rho_1$ and thus $K_{1,w} \models A$. So

$$[K_{1,w}]_{n-1} \in B(K_1) = B(K_2).$$

Hence there is a $v$ such that $\rho_2 R_2 v$, $K_{2,v} \models A$, and $K_{2,v} \sim_{n-1} K_{1,w}$. We apply Lemma 6.2.11 (1) to obtain $\sigma_a^*(K_{2,v}) = K_{2,v}$ and $\sigma_a^*(K_{1,w}) = K_{1,w}$ and so $\sigma_a^*(K_{2,v}) \sim_{n-1} \sigma_a^*(K_{1,w})$. The other direction is analogous. Therefore we have proved $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$.

Now suppose that $\rho_1$ and $\rho_2$ are reflexive. We show by induction for $k = 0, \ldots, n$ that $\sigma_a^*(K_1) \sim_k \sigma_a^*(K_2)$, . We have $\sigma_a^*(K_1) \sim_0 \sigma_a^*(K_2)$ by Lemma 6.2.11 (2). Take $w$ such that $\rho_1 R_1 w$. If $w \neq \rho_1$ do the same as in the irreflexive case. If $w = \rho_1$, define $v = \rho_2$. By induction hypothesis we have $\sigma_a^*(K_1) \sim_{k-1} \sigma_a^*(K_2)$. Now pick $v$ such that $\rho_2 R_2 v$. This case is symmetric of the previous one, so we can apply a similar argument. Hence $\sigma_a^*(K_1) \sim_k \sigma_a^*(K_2)$ for each $k = 0, \ldots, n$. ■

There can be multiple substitutions that can correspond to an extension structure, but there is at least one by Lemma 6.2.13. For each extension structure we fix such a substitution and call it the *corresponding substitution* to that extension structure. Note that different extension structures can be identified by the same substitution $\sigma_a$. We write $\sigma_{\mathrm{i}}$ and $\sigma_{\mathrm{r}}$ to denote the corresponding substitutions to the irreflexive and, respectively, reflexive extension structure of a bisimulation set.

Lemma 6.2.22 shows the connection between extensions of reflexive and irreflexive worlds under certain criteria. Informally, the substitution $\sigma_{\mathrm{r}}$ corresponding to a reflexive extension also works for the irreflexive extension with the same bisimulation set under these criteria.

### 6.2.22 Lemma

Let $A \in \mathsf{Form}(\overline{p})$ with $md(A) \leq n$. Let $K_1, K_2$ be two $\mathsf{L}$-models over $\overline{p}$, with roots $\rho_1, \rho_2$, that almost satisfy $A$. Let $\rho_1$ be reflexive and $\rho_2$ irreflexive. Suppose $B(K_1) = B(K_2)$. For each $a \subseteq \overline{p}$, if

$$\sigma_a^*(K_1) \models A \text{ and } B(K_1) = B(\sigma_a^*(K_1)),$$

then also $\sigma_a^*(K_2) \models A$.

Proof. Similarly to the proof of the previous lemma, we use the characterization from Lemma 1.2.27 to show that $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$. This implies $\sigma_a^*(K_2) \models A$.

By Lemma 6.2.12 together with Lemma 6.2.11 (2), it is enough to consider $K_1$ with $cl(\rho_1)$ being a singleton. We have $\sigma_a^*(K_1) \sim_0 \sigma_a^*(K_2)$ by Lemma 6.2.11 (2). We must show that for all $w$ such that $\rho_1 R_1 w$ there exists $v$ such that $\rho_2 R_2 v$ and $\sigma_a^*(K_{1,w}) \sim_{n-1} \sigma_a^*(K_{2,v})$ and vice versa. First take $w$ such that $\rho_1 R_1 w$. If $w \neq \rho_1$, we proceed in the same way as for the irreflexive case in the proof of Lemma 6.2.21. If $w = \rho_1$, we use the assumption $\sigma_a^*(K_1) \models A$ to see that

$$[\sigma_a^*(K_1)]_{n-1} \in B(\sigma_a^*(K_1)) = B(K_1) = B(K_2).$$

There is a $v$ such that $\rho_2 R_2 v$, $K_{2,v} \models A$ and $K_{2,v} \sim_{n-1} \sigma_a^*(K_{1,\rho_1})$. We apply Lemma 6.2.11 (1) to see that $K_{2,v} = \sigma_a^*(K_{2,v})$ and so $\sigma_a^*(K_{2,v}) \sim_{n-1} \sigma_a^*(K_{1,\rho_1})$.

Now pick $v$ such that $\rho_2 R_2 v$. This case is easier than the previous one and is left to the reader. Therefore $\sigma_a^*(K_1) \sim_n \sigma_a^*(K_2)$. ■

Now we present the key lemma. Recall (6.5) where we defined $\theta := \sigma_{a_1} \cdots \sigma_{a_s}$ where $a_1, \ldots, a_s$ is any fixed ordering on the subsets of $\overline{p}$. The lemma states that after two applications of $\theta^*$, the $B$-minimal rank of the new frontier points increases. Intuitively, one $\theta$ covers the corresponding irreflexive substitutions $\sigma_{\mathrm{i}}$'s and the other the corresponding reflexive substitutions $\sigma_{\mathrm{r}}$'s. In the following we use the notation $\theta_j^* := \sigma_{a_j}^* \sigma_{a_{j-1}}^* \cdots \sigma_{a_1}^*$, where we define $\theta_0^*$ to be the empty substitution, i.e., $\theta_0^*(K) = K$ for each model $K$.

**6.2.23 Lemma**
Let $A \in \mathsf{Form}(\overline{p})$ with $md(A) \leq n$ and suppose that $\mathrm{MOD}_{\mathsf{L}}(A)$ has the extension property. Let $K$ be an $\mathsf{L}$-model over $\overline{p}$ and let $w$ be a $B$-minimal frontier point in $K$. Then for each frontier point $v$ in $\theta^*\theta^*(K)$ below $w$ we have

$$B(K_w) \subset B(\theta^*\theta^*(K_v)).$$

Consequently, $r(K_w) < r(\theta^*\theta^*(K_v))$.

Proof. Let $K$ be an $\mathsf{L}$-model with $B$-minimal frontier point $w$. Let $v$ be a frontier point in $\theta^*\theta^*(K)$ below $w$. Note that $B(K_w) \subseteq B(\theta^*\theta^*(K_v))$. Suppose that $B(K_w) = B(\theta^*\theta^*(K_v))$. We will prove that it implies $\theta^*\theta^*(K_v) \models A$, and so $v$ cannot be a frontier point in model $\theta^*\theta^*(K)$.

Observe that $B(K_w) \subseteq B(K_v) \subseteq B(\theta^*\theta^*(K_v))$, so $B(K_w) = B(K_v)$. Consider all $v'$ above $v$ such that $K_{v'} \not\models A$ (this includes $w$ itself). Since $w$ is $B$-minimal and $B(K_{v'}) \subseteq B(K_v)$, these $v'$'s satisfy $B(K_w) = B(K_{v'})$ as well. Also note that for each such $v'$ and each index $j$ we have

$$B(K_w) = B(K_{v'}) \subseteq B(\theta_j^*(K_{v'})) \subseteq B(\theta_j^*\theta^*(K_{v'})) \subseteq B(\theta^*\theta^*(K_v)).$$

Therefore, $B(K_w) = B(\theta_j^*(K_{v'})) = B(\theta_j^*\theta^*(K_{v'}))$ for each $j$. We have two cases: all $v'$ are irreflexive or there is at least one that is reflexive.

Let us start with the first case. Here $w$ is irreflexive. By Lemma 6.2.13 we have, $\sigma_{a_j}^*(K_w) \models A$ for some $j$. Note that also $\theta_j^*(K_w) \models A$. This $\sigma_{a_j}$ is the irreflexive substitution $\sigma_{\mathrm{i}}$ corresponding to $B(K_w)$. For each $v'$ above $v$ we will prove $\theta_j^*(K_{v'}) \models A$. We proceed by induction on the maximal length of sequences $v'Rx_1R\ldots Rx_k$ where $x_k$ is a frontier point in $K$. If the length equals 1, then $v'$ is a frontier point in $K$. If $\theta_{j-1}^*(K_{v'}) \models A$, then also $\theta_j^*(K_{v'}) \models A$ by Lemma 6.2.11 (1). If $\theta_{j-1}^*(K_{v'}) \not\models A$, we know that $v'$ is a frontier point in $\theta_{j-1}^*(K)$. We know that $B(K_w) = B(\theta_{j-1}^*(K_{v'}))$, so we can apply Lemma 6.2.21 to conclude $\theta_j^*(K_{v'}) \models A$. Suppose now that the length is $l > 1$. By induction hypothesis we know that $v'$ is an irreflexive point for which all its successors satisfy $A$ in $\theta_j^*(K)$. If $\theta_j^*(K_{v'}) \models A$ we are done. If not, since $B(K_w) = B(\theta_j^*(K_{v'}))$ we know by Lemma 6.2.21 that $\sigma_{a_j}^*\theta_j^*(K_{v'}) \models A$. Hence, by Lemma 6.2.11 (3), $\theta_j^*(K_{v'}) \models A$. Therefore by Lemma 6.2.11 (1), we have $\theta^*\theta^*(K_v) \models A$.

Now we turn to the second case. We consider model $\theta_j^*(K)$, where $j$ is defined in such a way that $\sigma_{a_j}$ is the irreflexive substitution $\sigma_{\mathrm{i}}$ corresponding to $B(K_w)$. In case there is no corresponding irreflexive substitution, we define $j = 0$. If $\theta_j^*(K_v) \models A$ we are done. If not, we will see further in the proof that all frontier points in $\theta_j^*(K)$ above $v$ are reflexive. Fix such a frontier point $w'$. Let $\sigma_{a_h}$ be the corresponding reflexive substitution $\sigma_{\mathrm{r}}$ to $B(K_w)$. Since $B(K_w) = B(\theta_j^*(K_{w'}))$, we have $\sigma_{a_h}^*\theta_j^*(K_{w'}) \models A$. We prove for all $v'$ above $v$ that $\theta_h^*\theta^*(K_{v'}) \models A$. We do so by induction on the maximal length of $v'Rx_1R\dots Rx_{k-1}Rx_k$ where $x_i$'s do not belong to the same cluster and $x_k$ is a frontier point in $\theta_j^*(K)$. If the length equals 1, then $v'$ is a frontier point in $\theta_j^*(K)$ ($v'$ may equal $w'$). Frontier point $v'$ must be reflexive, because suppose $v'$ was irreflexive. Recall that $B(K_w) = B(\theta_j^*(K_{v'}))$. By Lemmas 6.2.21 and 6.2.11 (3) we would have $\theta_j^*(K_{v'}) = \sigma_{a_j}^*\theta_j^*(K_{v'}) \models A$. And so $v'$ would not be a frontier point in $\theta_j^*(K)$. Thus $v'$ is reflexive. If $\theta_{h-1}^*\theta^*(K_{v'}) \models A$, we are done. If not, since $B(\theta_j^*(K_{w'})) = B(\theta_{h-1}^*\theta^*(K_{v'}))$, we can apply Lemma 6.2.21 to conclude $\theta_h^*\theta^*(K_{v'}) \models A$. Now suppose the length is $l > 1$. By induction hypothesis, all the successors of $v'$ not in the cluster of $v'$ satisfy $A$ in $\theta_h^*\theta^*(K)$. Again, if $\theta_h^*\theta^*(K_{v'}) \models A$, we are done. If not, we have two cases. If $v'$ is reflexive we can apply Lemma 6.2.21, because $B(\theta_j^*(K_{w'})) = B(\theta_h^*\theta^*(K_{v'}))$. If $v'$ is irreflexive, we apply Lemma 6.2.22, because $B(\theta_j^*(K_{w'})) = B(\theta_h^*\theta^*(K_{v'}))$ and because $B(\theta_j^*(K_{w'})) = B(\sigma_{a_h}^*\theta_j^*(K_{w'}))$. In both cases we obtain $\sigma_{a_h}^*\theta_h^*\theta^*(K_{v'}) \models A$, hence $\theta_h^*\theta^*(K_{v'}) \models A$. This concludes $\theta^*\theta^*(K_v) \models A$. ■

Consider again Figure 6.1 illustrating the frontier lines. Lemma 6.2.23 shows that the $B$-minimal rank of the frontier lines increases after each step of $\theta^*\theta^*$ in the picture. We show in the final theorem that there are at most $N' + 1$ of these steps. And so a concatenation of $2(N' + 1)$ $\theta$'s forms a projective unifier for $A$. As mentioned before, Ghilardi uses $2N$ $\theta$'s. From a close look at the induction proof of Lemma 2.8 in (Ghilardi, 2000), we think that he would conclude $2(N+1)$ instead of $2N$ $\theta$'s. The rank is indeed bounded by $N$, but it may start at 0, which contributes to an extra application of $\theta$'s. However, this is not so important. We even think that a more clever proof can show that $2N'$ applications is sufficient in our case.

**6.2.24 Theorem**
Let $A \in \mathsf{Form}(\overline{p})$ with $md(A) \leq n$ and suppose that $\mathrm{MOD}_{\mathsf{L}}(A)$ has the extension property. Then $(\theta^*)^{2(N'+1)}(K) \models A$ for all $\mathsf{L}$-models $K$ over $\overline{p}$.

Proof. From Lemma 6.2.23 it follows with induction on $l \leq N'$, that the rank of the $B$-minimal frontier points in $(\theta^*)^{2l}$ is greater than or equal to $l$. Note that the rank can be 0, so the $B$-minimal rank can start at 0. Since the rank is bounded by $N'$, we have that $(\theta^*)^{2(N'+1)}(K)$ does not contain any frontier points. Therefore $(\theta^*)^{2(N'+1)}(K) \models A$. ■

We conclude with the following theorem to finish the proof of Theorem 6.2.6.

**6.2.25 Theorem** (Ghilardi, 2000, see Theorem 2.2)
If $\mathrm{MOD}_{\mathsf{L}}(A)$ has the extension property, then formula $A$ is projective in $\mathsf{L}$.

Proof. Let $A \in \mathsf{Form}(\overline{p})$ and suppose that $\mathrm{MOD}_{\mathsf{L}}(A)$ has the extension property. Let $n$ be such that $md(A) \leq n$. If $n = 0$, it follows from Remark 6.2.18. If $n > 0$, $\theta^{2(N'+1)}$ is a unifier of $A$ by Theorem 6.2.24. It is a projective unifier because it satisfies condition (6.1), since it is a composition of simple substitutions as explained on page 188. So $A$ is a projective formula. ■

We conclude with the following result.

**6.2.26 Corollary** (Ghilardi, 2000)
If $\mathsf{L}$ is decidable, then projectivity is decidable in $\mathsf{L}$.

Proof. Let $A \in \mathsf{Form}(\overline{p})$ and let $n$ be such that $md(A) \leq n$. From Theorem 6.2.6 and Theorem 6.2.24 it follows that $A$ is projective iff $\theta^{2(N'+1)}$ is a unifier for $A$ iff $\mathrm{MOD}_{\mathsf{L}}(A)$ has the extension property. By the decidability of the logic it is decidable to check whether $\theta^{2(N'+1)}$ is a unifier for $A$ or not. Hence it is decidable whether $A$ is projective in $\mathsf{L}$ or not. ■

### 6.2.2 Extension property in intuitionistic modal logic

Here we focus on intuitionistic modal logics with coreflection as discussed in Section 1.3. Recall that strong frames carry the strong condition ($\mathsf{S}$), i.e., $R \subseteq \leq$ (Definition 1.3.4). Since strong models can be considered as Kripke models for $\mathsf{IPC}$ decorated with a modal relation, we do not need to rely on the machinery from the previous section, but we rely on the method for $\mathsf{IPC}$ from (Ghilardi, 1999).

**6.2.27 Convention**
In this section, we assume $\mathsf{iL}$ to be an intuitionistic modal logic extending $\mathsf{iCK4}$ that is complete with respect to the class $\mathcal{F}_{\mathsf{iL}}$ of all *finite rooted strong* frames $F$ such that $F \models A$ for all $A \in \mathsf{iL}$. We apply Remark 1.2.14 and call a frame $F = (W, \leq, R)$ an $\mathsf{iL}$-*frame* if $F \in \mathcal{F}_{\mathsf{iL}}$. We call $K = (W, \leq, R, V)$ an $\mathsf{iL}$-*model* if $K$ is based on an $\mathsf{iL}$-frame.

Recall Figure 1.4 for the particular logics $\mathsf{iCK4}$, $\mathsf{IEL}$, $\mathsf{iCS4} \equiv \mathsf{IPC}$, $\mathsf{iSL}$, $\mathsf{mHC}$, $\mathsf{KM}$, and $\mathsf{PLL}$. For the first six we might consider the Kripke classes of rooted frames listed in Figure 1.5. For $\mathsf{PLL}$, one can take finite rooted strong *dense* frames (aka Goldblatt frames, Theorem 1.3.18).

Although we do not require the frames to be *extendible* (in the sense of Definition 6.1.20), in our investigation of the admissible rules in Chapter 7 we work with such frames. This forms a problem for the Goldblatt frames for $\mathsf{PLL}$, because these are not extendible (see Example 7.3.1). However, the FM-frames are extendible and we will come back to $\mathsf{PLL}$ and FM-frames at the end of this section.

Next definitions are the analogues of the classical Definitions 6.2.2, 6.2.4, and 6.2.5.

**6.2.28 Definition**
For $A \in \mathsf{Form}(\overline{p})$, we define $\mathrm{Mod}_{\mathsf{iL}}(A)$ to be the set of all $\mathsf{iL}$-models over $\overline{p}$ that satisfy $A$ in the root. Let $\mathrm{MOD}_{\mathsf{iL}}(A)$ be the set of $\mathsf{iL}$-models over $\overline{p}$ that satisfy $A$.

Strong monotonicity (Lemma 1.3.5) implies that $\mathrm{MOD}_{\mathsf{iL}}(A) = \mathrm{Mod}_{\mathsf{iL}}(A)$. In addition, recall Definition 1.3.9 defining a model to almost satisfy a formula.

**6.2.29 Definition** (Variant)
A *variant* of an $\mathsf{iL}$-model $K$ is an $\mathsf{iL}$-model $K'$, such that they have the same frame and their valuations agree on all worlds except for possibly the root.

**6.2.30 Definition** (Extension property)
A class $\mathcal{K}$ of $\mathsf{iL}$-models is said to have to *extension property* if for every $\mathsf{iL}$-model $K$ (with root $\rho$), if $K_w \in \mathcal{K}$ for each $w \neq \rho$, then there is a variant $K'$ of $K$ such that $K' \in \mathcal{K}$.

**6.2.31 Theorem**
Formula $A$ is projective in $\mathsf{iL}$ if and only if $\mathrm{MOD}_{\mathsf{iL}}(A)$ has the extension property.

Proof. The two directions are proved separately in Theorems 6.2.32 and 6.2.38. ■

We consider semantic operators $\sigma^*$ which are exactly defined as in Definition 6.2.7, but now for the (intuitionistic) $\mathsf{iL}$-models. Also Lemma 6.2.8 works exactly the same in the intuitionistic setting, so we do not state the lemma here. As one can see, the following theorem has the exact same proof as Theorem 6.2.9.

**6.2.32 Theorem**
If formula $A$ is projective in $\mathsf{iL}$, then $\mathrm{MOD}_{\mathsf{iL}}(A)$ has the extension property.

Proof. Let $A \in \mathsf{Form}(\overline{p})$ and let $K$ be an $\mathsf{iL}$-model over $\overline{p}$ almost satisfying $A$. Formula $A$ is projective, so we can take a substitution $\sigma$ such that $\vdash_{\mathsf{iL}} \sigma(A)$ and $A \vdash_{\mathsf{iL}} p \leftrightarrow \sigma(p)$ for each $p \in \overline{p}$. We will show that $\sigma^*(K)$ is a variant of $K$ that satisfies $A$. $\sigma^*(K)$ is a variant, because for all $w \neq \rho$ we have $\sigma^*(K)(w) = K(w)$, since $K_w \models A$ and $A \vdash_{\mathsf{iL}} p \leftrightarrow \sigma(p)$ for each $p \in \overline{p}$. And $\sigma^*(K) \models A$ follows from $\vdash_{\mathsf{iL}} \sigma(A)$. ■

Now we turn to the other direction of Theorem 6.2.31. Such as for the classical modal case, Ghilardi (1999) defines simple substitutions $\sigma_a$ in the framework of $\mathsf{IPC}$. These are the same from Definition 6.2.10, but observe that $\boxdot A \leftrightarrow A$ for all formulas $A$ for intuitionistic modal logics with coreflection. So the substitutions are defined by:

$$\sigma_a^A(p) = \begin{cases} A \to p & \text{if } p \in a, \\ A \wedge p & \text{if } p \notin a. \end{cases}$$

And of course, they again satisfy the key condition (6.1) of projectivity. The following two lemmas form the analogous of Lemmas 6.2.11 and 6.2.13.

**6.2.33 Lemma**
Let $A \in \mathsf{Form}(\overline{p})$ and let $K$ be an $\mathsf{iL}$-model over $\overline{p}$. Suppose $a \subseteq \overline{p}$. We have

1. $(\sigma_a^*(K))(w) = K(w)$ if $K_w \models A$,
2. $(\sigma_a^*(K))(w) \subseteq a$ if $K_w \not\models A$, more precisely,

$$(\sigma_a^*(K))(w) = \{p \in a \mid \text{ for all } v > w \text{ if } K_v \models A, \text{ then } K_v \models p\},$$

3. $\sigma_a^* \sigma_a^* = \sigma_a^*$.

Proof. Similar proof as Lemma 6.2.11. The difference between points (2) is due to the fact that strong models are monotone. Point (3) is more complicated than for Lemma 6.2.11 so let us present it here. Let $K$ be an $\mathsf{iL}$-model over $\overline{p}$ and let $w$ be a world in $K$. We show that $(\sigma_a^*\sigma_a^*(K))(w) = (\sigma_a^*(K))(w)$. If $\sigma_a^*(K_w) \models A$, then this follows immediately from point (1). If $\sigma_a^*(K_w) \not\models A$, then also $K_w \not\models A$ by (1). So both $(\sigma_a^*\sigma_a^*(K))(w)$ and $(\sigma_a^*(K))(w)$ are defined by point (2) as follows:

$$\begin{aligned}(\sigma_a^*(K))(w) &= \{p \in a \mid \text{ for all } v > w, \text{ if } K_v \models A, \text{ then } K_v \models p\},\\ (\sigma_a^*\sigma_a^*(K))(w) &= \{p \in a \mid \text{ for all } v > w, \text{ if } \sigma_a^*(K_v) \models A, \text{ then } \sigma_a^*(K_v) \models p\}.\end{aligned}$$

Set $(\sigma_a^*\sigma_a^*(K))(w)$ is a subset of $(\sigma_a^*(K))(w)$, because suppose $p \in (\sigma_a^*\sigma_a^*(K))(w)$ and let $v > w$ such that $K_v \models A$. By (1) we have $\sigma_a^*(K_v) = K_v$, and so $K_v \models p$ by the property from $(\sigma_a^*\sigma_a^*(K))(w)$. For the other inclusion suppose $p \in (\sigma_a^*(K))(w)$ and let $v > w$ such that $\sigma_a^*(K_v) \models A$. By monotonicity we immediately have $p \in (\sigma_a^*(K))(v)$, and thus $p \in (\sigma_a^*\sigma_a^*(K))(w)$. ■

**6.2.34 Lemma**
Let $A \in \mathsf{Form}(\overline{p})$ and suppose that $\mathrm{MOD}_{\mathsf{iL}}(A)$ has the extension property. Let $K$ be an $\mathsf{iL}$-model over $\overline{p}$ with root $\rho$ that almost satisfies $A$. Then there is a set $a \subseteq \overline{p}$ such that $\sigma_a^*(K) \models A$ and $\sigma_a^*(K)(\rho) = a$.

Proof. $\mathrm{MOD}_{\mathsf{iL}}(A)$ has the extension property, so there is a variant $K'$ of $K$ that satisfies $A$. Take $a = K'(\rho)$. For all $w$ such that $w \neq \rho$ we have $\sigma_a^*(K_w) = K_w$ by Lemma 6.2.33 (1) and so $\sigma_a^*(K_w) \models A$. For the root $\rho$ we have $\sigma_a^*(K)(\rho) = a$, because of Lemma 6.2.33 (2) and monotonicity of $\leq$. Moreover, we have proved that $\sigma_a^*(K) = K'$ and so $\sigma_a^*(K) \models A$. ■

We use the following notions, where we explicitly define frontier points in contrast to (Ghilardi, 1999). Compare these ingredients to the ones introduced in Definition 6.2.17 for transitive classical modal logics.

**6.2.35 Definition**
Let $A \in \mathsf{Form}(\overline{p})$ and suppose that $\mathrm{MOD}_{\mathsf{iL}}(A)$ has the extension property. Let $K$ be an $\mathsf{iL}$-model over $\overline{p}$.

- The set of *frontier points* of $K$ is defined as

$$\{w \in K \mid K_w \not\models A \text{ and } \forall v(w < v \Rightarrow K_v \models A)\}.$$

- Let $w$ a frontier point in $K$ and let $\sigma_a^*$ be defined as in Lemma 6.2.34 for $K_w$ such that for all $b$ that also satisfy the properties of Lemma 6.2.34 we have $|b| \leq |a|$. We fix such $a$ and call it the *corresponding substitution* for $K_w$.
- Let $w$ be a frontier point in $K$. We define the *rank* of $K_w$ to be $r(K_w) = |a|$, where $\sigma_a^*$ is the corresponding substitution of $K_w$.
- We call a frontier point $w$ *maximal* in $K$, if $r(K_w) \geq r(K_v)$, for all other frontier points $v$ in $K$.

Now we define $\theta$ as in Ghilardi (1999), that is, $\theta := \sigma_{a_1} \ldots \sigma_{a_s}$, where the $a_i$ are subsets of $\overline{p}$ ordered according to $a_i \subseteq a_j$ implies $i \leq j$. We will show that $\theta$ is a projective unifier for $A$. We can divide $\theta$ in parts as follows:

$$\theta = \tau_m \cdots \tau_1 \tau_0, \tag{6.6}$$

where $\tau_j$ contains the $\sigma_a$'s for which $|a| = j$ and $m$ is the number of propositional variables in $\overline{p}$.

**6.2.36 Lemma**
Let $A \in \mathsf{Form}(\overline{p})$ and suppose that $\mathrm{MOD}_{\mathsf{iL}}(A)$ has the extension property. Let $K$ be an $\mathsf{iL}$-model over $\overline{p}$ and let $k$ be the rank of its maximal frontier points. Then, for any frontier point $v$ in $\tau_k^*(K)$, we have $r(\tau_k^*(K_v)) < k$.

Proof. The result follows from induction on the number of simple substitutions occurring in $\tau_k^*$. To see this, let us show that for any $\mathsf{iL}$-model $M$ over $\overline{p}$ with maximal frontier points of rank $k$, for any $\sigma_a^*$ from $\tau_k^*$, for any frontier point $v$ in $\sigma_a^*(M)$, and for any frontier point $w$ in $M$ such that $v \leq w$ we have,

$$r(\sigma_a^*(M_v)) \leq r(M_w). \tag{6.7}$$

We distinguish two cases. If $\sigma_a^*(M_w) \not\models A$, then it holds that $v = w$ and $r(\sigma_a^*(M_w)) = r(M_w) \leq k$ since the corresponding substitution of $M_w$ only depends on the submodels above $w$, which remains the same after application of $\sigma_a^*$ by Lemma 6.2.33 (1). Now suppose $\sigma_a^*(M_w) \models A$. Then by Lemma 6.2.33 (1) and (2), $a' := (\sigma_a^*(M))(w)) \subseteq a$ also satisfies $\sigma_{a'}^*(M_w) \models A$. Let $\sigma_b$ be the corresponding substitution of $M_w$. In particular, $|a'| \leq |b|$. Suppose $\sigma_c^*$ is the corresponding substitution of $\sigma_a^*(M_v)$. So $\sigma_c^*\sigma_a^*(M_v) \models A$, and $(\sigma_c^*\sigma_a^*(M))(v) = c$.

By Lemma 6.2.33 (1) and the monotonicity in $\leq$, we know that $c \subseteq a'$. Therefore $r(\sigma_a^*(M_v)) = |c| \leq |a'| \leq |b| = r(M_w)$.

Now, if $\sigma_a$ is the corresponding substitution of $M_w$, i.e. $r(M_w) = k$, we show that

$$r(\sigma_a^*(M_v)) < r(M_w) = k. \tag{6.8}$$

This is sufficient to show the desired result in the lemma, because (6.7) takes care of frontier points $w$ with rank strictly lower than $k$ and (6.8) takes care of frontier points with rank $k$, since for these there is a corresponding substitution among $\tau_k^*$ reducing the rank of the new frontier points below $k$. We show (6.8) by contradiction by supposing that $r(\sigma_a^*(M_v)) = r(M_w)$. We show that $\sigma_a^*(M_v) \models A$ and so $v$ cannot be a frontier point in $\sigma_a^*(M)$ after all. $\sigma^a$ is the corresponding substitution of $M_w$. So using the observations from above we see that $a' = a$ and so $c \subseteq a$. But $|c| = |a|$ by assumption, so $a = c$. Thus $\sigma_a^*\sigma_a^*(M_v) \models A$ and by Lemma 6.2.33 (3) we have $\sigma_a^*(M_v) \models A$. ■

**6.2.37 Theorem**
Let $A \in \mathsf{Form}(\overline{p})$ and suppose that $\mathrm{MOD}_{\mathsf{iL}}(A)$ has the extension property. Then $\theta^*(K) \models A$ for all $\mathsf{iL}$-models $K$ over $\overline{p}$.

Proof. Let $\overline{p} = \{p_1, \ldots, p_m\}$, so the rank is at most $m$ and $\theta = \tau_m \cdots \tau_1\tau_0$ as defined in (6.6). From Lemma 6.2.36 it follows that for each $m \geq k \geq 0$ the rank of maximal frontier points in $(\tau_k)^* \cdots (\tau_m)^*(K)$ is smaller than $k$. So for $k = 0$, we have that $(\tau_0)^* \cdots (\tau_m)^*(K)$ does not contain any frontier point. Therefore $(\tau_0)^* \cdots (\tau_m)^*(K) \models A$. So $\theta^*(K) \models A$. ■

**6.2.38 Theorem**
If $\mathrm{MOD}_{\mathsf{iL}}(A)$ has the extension property, then formula $A$ is projective in $\mathsf{iL}$.

Proof. This is an immediate consequence from Theorem 6.2.37 and the observation that $\theta$ satisfies property (6.1) making it a projective unifier for $A$. ■

The following corollary follows from the decidability of the logics as discussed in Section 1.3. The proof is similar to Corollary 6.2.26.

**6.2.39 Corollary**
Projectivity is decidable in $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{IEL}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{mHC}, \mathsf{KM}\}$.

We present some examples of (non-)projective formulas using Theorem 6.2.31.

**6.2.40 Example**
Formula $\Box p$ is projective in $\mathsf{iCS4} \equiv \mathsf{IPC}$, but not in any of $\mathsf{iCK4}$, $\mathsf{IEL}$, $\mathsf{iSL}$, $\mathsf{mHC}$, or $\mathsf{KM}$. To show the latter, let $\mathsf{iL}$ be one of these logics except $\mathsf{IEL}$ (for this logic a similar counterexample can be constructed). Consider the following model $K$

with irreflexive worlds that do not force $p$. The dashed arrow represents the intuitionistic relation $\leq$ and the straight arrow represents the modal relation $R$.

$K \quad =$

This is clearly an iL-model. Let us call the root $\rho$ and the upper world $w$. We have $K_w \models \Box p$, but there is no valuation for $\rho$ that makes $\Box p$ true. In other words, there is no variant of $K$ that satisfies $\Box p$, hence $\mathrm{MOD}_{\mathsf{iL}}(\Box p)$ does not have the extension property. Hence $\Box p$ is not projective by Theorem 6.2.31. For $\mathsf{iL} = \mathsf{iCS4}$, this reasoning does not apply, since the worlds in $K$ should then be reflexive. In fact, $\Box p$ is projective in iCS4, since $\vdash_{\mathsf{iCS4}} \Box p \leftrightarrow p$, and $p$ is projective in iCS4 as shown in Example 6.1.8.

### 6.2.41 Example

In Examples 6.2.15 and 6.2.16 we have shown that formulas $A = \Box p \leftrightarrow p$ and $B = \Box\neg p \leftrightarrow \neg p$ are projective in transitive classical modal logics with the finite model property. We show that for any $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{IEL}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{mHC}, \mathsf{KM}\}$, $A$ is projective in iL, by showing that $\mathrm{MOD}_{\mathsf{iL}}(A)$ has the extension property. This can be seen by adopting the idea sketched by the pictures from Example 6.2.15 to the intuitionistic modal case by adding the intuitionistic relation $\leq$. The then obtained models are true strong models, because the defined valuation is indeed monotone. We cannot do the same for formula $B$ from Example 6.2.16, because the valuation in the fourth model sketched there will not be monotone in $\leq$, and hence cannot be an intuitionistic strong Kripke model. However, for irreflexive logics iSL and KM, the second and fourth picture cannot occur since each leaf is irreflexive and should therefore not force $p$. Indeed, one can show that $B$ is projective in these logics.

We finish this section by commenting on logic PLL. The outlined method applies to Goldblatt frames, showing that projectivity is also decidable in PLL. However, we would like to establish a similar semantic characterization in terms of FM-models defined in Definition 1.3.19.

All definitions and results can be translated where we read iL-frame and iL-model as FM-frame and FM-model, respectively. For $A \in \mathsf{Form}(\overline{p})$, we read $\mathrm{MOD}_{\mathrm{FM}}(A)$ as the set of all FM-models over $\overline{p}$ that satisfy $A$. One has to be careful with a *frame*, say $G$. Above it is of the form $G = (W, \leq, R)$, but an FM-frame has the form $G = (W, \leq, R, F)$, with $F \subseteq W$ the set of fallible worlds. The definitions of a *variant* (Definition 6.2.29) and the *extension property* (Definition 6.2.30) follow naturally.

We pay attention to the definition of $\sigma^*$ from Definition 6.2.7, where we have to check that the valuation from (6.4) is well defined. In particular, it is easy to check that the valuation of $\sigma^*(K)$ is full on the set of fallible worlds. In addition, we check the following equivalent to Lemma 6.2.8.

**6.2.42 Lemma**
Let $A \in \mathsf{Form}(\overline{p})$ and let $\sigma : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{p})$ be a substitution. For every FM-model $K$ over $\overline{p}$, we have

1. $\sigma^*(K) \models A$ iff $K \models \sigma(A)$,
2. and for every substitution $\tau : \mathsf{Form}(\overline{p}) \to \mathsf{Form}(\overline{p})$, $(\tau\sigma)^*(K) = \sigma^*(\tau^*(K))$.

Proof. Point (2) easily follows from (1). For (1) we have to take into account the fallible worlds. It is shown with induction to the structure of formula $A$. We only treat $A = \bot$. Let us denote the set of fallible worlds in $K$ by $F$ and in $\sigma^*(K)$ by $\sigma^*(F)$. We use the fact that $\sigma(\bot) = \bot$. We have $\sigma^*(K), w \Vdash \bot$ iff $w \in \sigma^*(F)$ iff $w \in F$ iff $K, w \Vdash \bot$ iff $K, w \Vdash \sigma(\bot)$. ■

We use the same simple substitutions $\sigma_a$ and all results also hold for PLL. The proofs are completely identical, because the interpretation of the connectives $\to$ and $\wedge$ in FM-models remain standard and only these connectives play a role in the substitution $\sigma_a^*$. We take the same definitions of frontier points, corresponding substitution, rank, and maximal frontier points as in Definition 6.2.35. Note that fallible worlds can never be frontier points. Thus we have the following results.

**6.2.43 Theorem**
Formula $A$ is PLL-projective if and only if $\mathrm{MOD}_{\mathrm{FM}}(A)$ has the extension property.

**6.2.44 Theorem**
Projectivity is decidable in PLL.

This finishes our study of projective formulas and their link to the semantic extension property.

## 6.3 Conclusion

The chapter contains both general and detailed technical results about projectivity. The first section of this chapter covers well-known general results about its important role in unification theory and admissible rules. In particular, we made explicit that projective approximations are useful to obtain results about admissibility in extension of logics (Theorem 6.1.19). Although, such results were already present in the literature, may it be hidden in proofs for classical transitive logics (Jeřábek, 2005), and intermediate logics (Iemhoff, 2005, 2006), we present these in general theorems for classical and intuitionistic modal logics, both for multi-conclusion and single-conclusion admissible rules.

Section 6.2.1 contains a detailed examination of the extension property as the semantic characterization of projectivity for several classical modal logics provided by Ghilardi (2000). We explained the close relationship between bisimulation and the extension property on the basis of extension structures. It should be mentioned that the method strongly relies on the transitivity in the classical models. One might expect that the method could easily be adjusted to transitive intuitionistic modal logics. However, many attempts have failed so far, because one has to deal with two relations. Moreover, it might not be surprising if it would turn out to be impossible, because the methods studied in the literature so far do not work for all classical modal logics. Recall from Remark 6.1.29 that a lot of problems are open for K.

We were able to establish the connection between projective formulas and the extension property for intuitionistic modal logics with coreflection including, iCK4, IEL, iCS4 $\equiv$ IPC, iSL, mHC, KM, and PLL in Section 6.2.2. These results will form a technical tool in the characterization of the admissible rules in these logics in Chapter 7. Our method relies on the strong condition of the models that $R \subseteq \leq$ and, again, we do not know how it would work for other intuitionistic modal logics.

Much is unknown about unification in intuitionistic modal logic. The only result that we are aware of is the finite unification of logic PLL by Ghilardi and Lenzi (2022). They also show that each formula has an admissible projective approximation in PLL. Both results rely on a bisimulation argument. We conjecture that bisimulations can be similarly defined for the Kripke semantics in the other intuitionistic modal logics with coreflection and so we conjecture finite unification and the existence of admissible projective approximations for iCK4, IEL, iCS4 $\equiv$ IPC, iSL, mHC, and KM. Theorem 6.1.19 together with the bases for admissible rules constructed in the next chapter widens the scope by providing bases of the admissible rules in extensions of these logics.

We conclude by mentioning a purely syntactic approach to projective formulas provided by Iemhoff (2016b). It provides proof-theoretic proofs of finite unification for certain transitive reflexive modal logics, without any reference to Kripke models. As a proof-theorist, this is exciting, but it is (in the author's own words) believed not to have any proof-theoretic benefit other than showing that it can be done.

# 7

# Admissible Rules in Intuitionistic Modal Logic

This chapter is motivated by the question: how can the methods and results on the admissible rules in IPC and classical modal logics be combined to obtain admissibility results for intuitionistic modal logics? Of course, this is a broad question, specifically, on two aspects. First, the study can focus on different concepts of admissibility such as decidability, semantics, a basis, or proof theory. Second, there are many intuitionistic modal logics, especially if $\Diamond$ is part of the language as well, and for each logic its admissible rules can be studied. Here we present a first characterization of admissible rules in the realm of intuitionistic modal logic.

The candidates of our study are the intuitionistic modal logics with coreflection, $\mathsf{iCK4}, \mathsf{iCS4} \equiv \mathsf{IPC}, \mathsf{iSL}, \mathsf{KM}, \mathsf{mHC}$, and PLL defined in Figure 1.4. We show that a natural combination of the Visser rules for IPC and modal Visser rules for classical modal logics form the bases of admissible rules for these six logics. Recall from Section 5.3.2 that proof theory for admissibility can be useful in the research of bases, which will exactly be the road that we will take.

The main contribution of this chapter is a full description of the admissible rules of these logics using the strategy of Iemhoff and Metcalfe (2009b). They provide Gentzen-style proof systems for admissibility for IPC and several classical transitive modal logics. We combine these systems into a system for admissibility of the intuitionistic modal logics that we study.

The admissibility proof systems have three notable properties. First, in contrast to well-known proof systems for logics that reason about formulas or sequents, these admissibility proof systems contain rules that reason about rules. Second, the shape of the rules in these proof systems is independent of the proof theory of the logics. Third, we can immediately conclude that the admissible rules for the logics are decidable, based on the decidability of the logic.

This chapter is based on (van der Giessen, 2021a) and is structured as follows. Section 7.1 defines the intuitionistic modal Visser rules which will be proved to be a basis for the admissible rules in Section 7.4. This relies on the sequent style proof systems for admissibility presented in Section 7.2 for logics $\mathsf{iCK4}$, $\mathsf{iCS4}$, $\mathsf{iSL}$, $\mathsf{KM}$, and $\mathsf{mHC}$, and separately for $\mathsf{PLL}$ in Section 7.3. We conclude in Section 7.5.

## 7.1 Intuitionistic modal Visser rules

In Section 5.3.1 we introduced the Visser rules for $\mathsf{IPC}$ and the modal Visser rules for transitive modal logics (single-conclusion in respectively Definitions 5.3.2 and 5.3.7 and multi-conclusion in respectively Definitions 5.3.3 and 5.3.8). In this section we define intuitionistic modal Visser rules for six logics with coreflection defined in Figure 1.4. These rules form a fusion of the intuitionistic Visser rules and the modal Visser rules. Whereas the former deal with $\to$ and the latter deal with $\Box$, the intuitionistic modal Visser rules that we define have to consider both.

**7.1.1 Definition** (Intuitionistic modal Visser rules)
Figure 7.1 defines the *multi-conclusion intuitionistic modal Visser rules* and Figure 7.2 the multi-conclusion rules for logic $\mathsf{PLL}$. The rules are defined on the basis of $n, m, k, l, h \in \mathbb{N}$ (that may equal 0). The set of all rules $(\mathsf{V}^{\bullet,i,1}_{nmkl})$ is denoted by $\mathsf{V}^{\bullet,i,1}$, and similarly for the other rules. We define the following sets of rules:

$$\begin{aligned} \mathsf{V}_{\mathsf{iCK4}} &= \mathsf{V}^{\bullet,i,1} \cup \mathsf{V}^{\circ,i}, & \mathsf{V}_{\mathsf{mHC}} &= \mathsf{V}^{\bullet,i,2} \cup \mathsf{V}^{\circ,i}, \\ \mathsf{V}_{\mathsf{iCS4}} &= \mathsf{V}^{\circ,i}, & \mathsf{V}_{\mathsf{KM}} &= \mathsf{V}^{\bullet,i,2}, \\ \mathsf{V}_{\mathsf{iSL}} &= \mathsf{V}^{\bullet,i,1}, & \mathsf{V}_{\mathsf{PLL}} &= \mathsf{V}^{\leq,R} \cup \mathsf{V}^{\leq}. \end{aligned}$$

Recall that in rules, for any finite set of formulas $\Gamma$ and $\Delta$, we write $\Gamma, \Delta$ to mean $\Gamma \cup \Delta$. So the conclusions in the Visser rules in Figures 7.1 and 7.2 should be understood as the union of presented sets. In Section 7.4 we show $\mathsf{V}_{\mathsf{iL}}$ to be a basis for the multi-conclusion admissible rules for the corresponding logic $\mathsf{iL}$.

**7.1.2 Convention**
Recall that the logics $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{KM}, \mathsf{mHC}\}$ are complete with respect to strong intuitionistic modal Kripke models as listed in Figure 1.5. We treat $\mathsf{PLL}$ separately with respect to FM-models (Definition 1.3.19). All these models satisfy the strong condition $(\mathsf{S}) : R \subseteq \leq$. Similarly to Section 6.2.2 we use their rooted versions and according to Remark 1.2.14 we call them $\mathsf{iL}$-models in this chapter.

Let $Y_1 = \bigwedge_{i<n}(p_i \to q_i) \wedge \bigwedge_{i<m}(\Box s_i \to t_i)$.
Let $Y_2 = \bigvee_{n\leq j<n+k} p_j \vee \bigvee_{m\leq j<m+l} \Box s_j$.

**Irreflexive**

$$(\mathsf{V}^{\bullet,i,1}_{nmkl}) \qquad \Box r \wedge Y_1 \to Y_2 \Bigg/ \begin{array}{l}\{\Box r \wedge Y_1 \to p_j \mid j < n+k\},\\ \{r \wedge Y_1 \to s_j \mid j < m+l\}\end{array}$$

$$(\mathsf{V}^{\bullet,i,2}_{nmkl}) \qquad \Box r \wedge Y_1 \to Y_2 \Bigg/ \begin{array}{l}\{r \wedge Y_1 \to p_j \mid j < n+k\},\\ \{r \wedge Y_1 \to s_j \mid j < m+l\}\end{array}$$

**Reflexive**

$$(\mathsf{V}^{\circ,i}_{nmklh}) \qquad \bigwedge_{i<h}(\Box r_i \to r_i) \wedge Y_1 \to Y_2 \Bigg/ \begin{array}{l}\Big\{\bigwedge_{i<h} r_i \wedge Y_1 \to p_j \mid j < n+k\Big\},\\ \Big\{\bigwedge_{i<h} r_i \wedge Y_1 \to s_j \mid j < m+l\Big\}\end{array}$$

**Figure 7.1.** Intuitionistic modal Visser rules

Let

$$Y_1 = \bigwedge_{i<n}(p_i \to q_i) \wedge \bigwedge_{i<m}(\bigcirc s_i \to t_i), \qquad Y_3 = \bigwedge_{i<n}(p_i \to q_i),$$
$$Y_2 = \bigvee_{n\leq j<n+k} p_j \vee \bigvee_{m\leq j<m+l} \bigcirc s_j, \qquad Y_4 = \bigvee_{n\leq j<n+k} p_j.$$

$$(\mathsf{V}^{\leq,R}_{nk}) \qquad \bigcirc r \wedge Y_3 \to Y_4 \Big/ \{\bigcirc r \wedge Y_3 \to p_j \mid j < n+k\}$$

$$(\mathsf{V}^{\leq}_{nmklh}) \qquad \bigwedge_{i<h}(\bigcirc r_i \to r_i) \wedge Y_1 \to Y_2 \Bigg/ \begin{array}{l}\Big\{\bigwedge_{i<h} r_i \wedge Y_1 \to p_j \mid j < n+k\Big\},\\ \Big\{\bigwedge_{i<h} r_i \wedge Y_1 \to s_j \mid j < m+l\Big\}\end{array}$$

**Figure 7.2.** Visser rules for PLL

Semantically, the intuitionistic Visser rules represent extensions in Kripke models for IPC and the modal Visser rules represent extensions in classical modal Kripke models. In turn, the Visser rules in this section will represent extensions in strong models dealing with both the intuitionistic relation $\leq$ and the modal relation $R$. Such as for the classical modal Visser rules from Definition 5.3.8, rules labeled with $\bullet$ match irreflexive extensions and rules labeled with $\circ$ belong to reflexive extensions. Recall Example 5.3.9 in which we used an irreflexive extension to prove the admissibility of rule $(\mathsf{V}^{\bullet}_{m})$ in K4. Similar extensions, but now with both $\leq$ and $R$, can prove the admissibility of the rules from Definition 7.1.1 in the corresponding logic (similarly to the proof of Theorem 7.2.7 in the next section).

Logic PLL is treated a bit different as we work with FM-models defined in Definition 1.3.19. These models are reflexive by definition. In this case, the Visser rules represent extensions with or without modal relation $R$ (see the proof of Theorem 7.3.3).

**7.1.3 Example**

Visser rule $(\mathsf{V}^{\circ,i}_{nmklh})$ can be considered as the reflexive version of $(\mathsf{V}^{\bullet,i,2}_{nmklh})$. One might ask why we do not use a reflexive version of $(\mathsf{V}^{\bullet,i,1}_{nmklh})$ which is the following rule, where $Y_1$ and $Y_2$ are defined as in Figure 7.1.

$$(\mathsf{V}^{\circ,i,1}_{nmklh}) \qquad \bigwedge_{i<h}(\Box r_i \to r_i) \wedge Y_1 \to Y_2 \Bigg/ \begin{array}{l} \Big\{ \bigwedge_{i<h}(\Box r_i \to r_i) \wedge Y_1 \to p_j \mid j < n+k \Big\}, \\ \Big\{ \bigwedge_{i<h} r_i \wedge Y_1 \to s_j \mid j < m+l \Big\} \end{array}$$

The reason is that this rule is equivalent to $(\mathsf{V}^{\circ,i}_{nmklh})$ as shown as follows. For the moment, let us drop $nmklh$ from the subscript. That $(\mathsf{V}^{\circ,i})$ follows from $(\mathsf{V}^{\circ,i,1})$ is due to the fact that $A \vdash_{\mathsf{iL}} \Box A \to A$ for all formulas $A$. For the other direction, we let $r'_i := \Box r_i \to r_i$ for each $i < h$. It can be checked that $\vdash_{\mathsf{iK}} (\Box r'_i \to r'_i) \to r'_i$ for each $i < h$. So we have

$$\bigwedge_{i<h}(\Box r_i \to r_i) \wedge Y_1 \to Y_2 \;\vdash_{\mathsf{iL}}\; \bigwedge_{i<h}(\Box r'_i \to r'_i) \wedge Y_1 \to Y_2.$$

An application of $(\mathsf{V}^{\circ,i})$ to the right-hand side and transitivity of $\vdash^{\mathsf{V}^{\circ,i}}_{\mathsf{iL}}$ gives us

$$\bigwedge_{i<h}(\Box r_i \to r_i) \wedge Y_1 \to Y_2 \;\vdash^{\mathsf{V}^{\circ,i}}_{\mathsf{iL}}\; \begin{array}{l} \Big\{ \bigwedge_{i<h}(\Box r_i \to r_i) \wedge Y \to p_j \mid j < n+k \Big\}, \\ \Big\{ \bigwedge_{i<h}(\Box r_i \to r_i) \wedge Y \to s_j \mid j < m+l \Big\}. \end{array}$$

Since $A \vdash_{\mathsf{iL}} \Box A \to A$ for all formulas $A$ we can replace $\bigwedge_{i<h}(\Box r_i \to r_i)$ by $\bigwedge_{i<h} r_i$ in the second set of the right-hand side resulting in an application of rule $(\mathsf{V}^{\circ,i,1})$ as desired. This shows that $(\mathsf{V}^{\circ,i})$ and $(\mathsf{V}^{\circ,i,1})$ are equivalent.

We now turn to single-conclusion admissible rules. Analogously to (Jeřábek, 2005), we extract the single-conclusion admissible rules from the multi-conclusion admissible rules via the disjunction property (see Example 5.2.27).

**7.1.4 Definition**
Let $\mathsf{iL}$ denote one of $\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{KM}, \mathsf{mHC}, \mathsf{PLL}$. Based on the set of multi-conclusion rules $\mathsf{V}_{\mathsf{iL}}$, we define the following set of single-conclusion rules.

$$\widehat{\mathsf{V}}_{\mathsf{iL}} := \big\{(\bigwedge \Gamma \vee D)/(\bigvee \Delta \vee D) \mid \Gamma/\Delta \in \mathsf{V}_{\mathsf{iL}}\big\}.$$

In Section 7.4 we show that these are bases for the single-conclusion admissible rules.

## 7.2 Proof system for admissible rules

We determine the multi-conclusion admissible rules via a sequent system for admissibility following (Iemhoff and Metcalfe, 2009b). Therefore we introduce sequent versions of multi-conclusion rules, also known as generalized rules.

In the beginning of Section 2.3 we introduced sequents. We use sequents based on sets as in Remark 2.3.2; a sequent $\Gamma \Rightarrow \Delta$ is a pair of finite sets of formulas $\Gamma$ and $\Delta$ with formula interpretation $I(\Gamma \Rightarrow \Delta) := \bigwedge \Gamma \to \bigvee \Delta$. We use letters $S$ to range over sequents. In this chapter we work with finite sets of sequents which we denote by letters $\mathcal{G}$ and $\mathcal{H}$. Note its similarities with, but do not confuse them with, the hypersequents defined in Definition 4.1.18 used in Chapter 4. In particular, here we use $I(\mathcal{G})$ to denote the set of formula interpretations of the sequents occurring in $\mathcal{G}$, i.e.,

$$I(\mathcal{G}) := \{I(S) \mid S \in \mathcal{G}\}.$$

Given a finite set of formulas $\Gamma$, recall the definitions of $\Box\Gamma$, $\boxdot\Gamma$, and $\Box\Gamma \to \Gamma$ from page 9.

Also, recall the definitions of multi-conclusion $\vdash_{\mathsf{iL}}$ and $\mathrel{|\!\!\sim}_{\mathsf{iL}}$ from Definition 5.2.15 and Remark 5.2.23.

**7.2.1 Definition** (Generalized sequent rule)
A *generalized sequent rule* (*gs-rule*) $\mathtt{R}$ is an ordered pair of finite sets of sequents, written $\mathcal{G} \rhd \mathcal{H}$. We say that

- $\mathtt{R}$ is *derivable in* $\mathsf{iL}$, written $\vdash_{\mathsf{iL}} \mathtt{R}$, if $I(\mathcal{G}) \vdash_{\mathsf{iL}} I(\mathcal{H}), \bot$. This means that $I(\mathcal{G}) \vdash_{\mathsf{iL}} I(S)$ for some $S \in \mathcal{H}$ or $I(\mathcal{G}) \vdash_{\mathsf{iL}} \bot$.
- $\mathtt{R}$ is *admissible in* $\mathsf{iL}$, written $\mathrel{|\!\!\sim}_{\mathsf{iL}} \mathtt{R}$, if $I(\mathcal{G}) \mathrel{|\!\!\sim}_{\mathsf{iL}} I(\mathcal{H})$. This means that each $\mathsf{iL}$-unifier for all $I(S)$ with $S \in \mathcal{G}$ is an $\mathsf{iL}$-unifier for $I(S')$ for some $S' \in \mathcal{H}$.

We sometimes simply say that $\mathtt{R}$ is derivable or admissible without referring to logic $\mathsf{iL}$ when the logic is clear from the context.

The reason that we include $\bot$ in the definition of derivability is that it allows us to speak about derivability in case $\mathcal{H}$ is empty. The rules that we use in the proof system of admissibility consist of inferences between gs-rules. Therefore we consider rules that reason about rules in the form

$$\frac{\mathtt{R}_1 \quad \dots \quad \mathtt{R}_n}{\mathtt{R}} \tag{7.1}$$

### 7.2.2 Definition

Consider an inference rule of the form from (7.1). It is called

- $\mathsf{iL}$-*sound* if whenever $\mid\!\sim_{\mathsf{iL}} \mathtt{R}_i$ for all $i$, then $\mid\!\sim_{\mathsf{iL}} \mathtt{R}$,
- $\mathsf{iL}$-*invertible* if whenever $\mid\!\sim_{\mathsf{iL}} \mathtt{R}$, then $\mid\!\sim_{\mathsf{iL}} \mathtt{R}_i$ for all $i$.

Note that this definition of invertibility is with regard to admissibility in $\mathsf{iL}$ and is not invertible in the usual sense with regard to a certain proof system.

Now we define the proof systems for admissibility for five of our logics, denoted by $\mathsf{GAiL}$ for logic $\mathsf{iL}$. Logic $\mathsf{PLL}$ is treated in Section 7.3, so for the rest of this section, let $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{KM}, \mathsf{mHC}\}$. The rules combine the intuitionistic rules and the modal rules from (Iemhoff and Metcalfe, 2009b).

### 7.2.3 Definition

The admissibility proof systems $\mathsf{GAiL}$ contain the rules from Figure 7.3 except that $\mathsf{GAiCS4}$, $\mathsf{GAiSL}$, and $\mathsf{GAKM}$ only contain (AC) and not (AC$_\Box$). In addition, each proof system has logic specific modal Visser rules from Figure 7.4 defined as follows:

$$(\mathrm{V}^{\bullet,i,1}), (\mathrm{V}^{\circ,i}) \in \mathsf{GAiCK4}, \qquad (\mathrm{V}^{\bullet,i,2}), (\mathrm{V}^{\circ,i}) \in \mathsf{GAmHC},$$
$$(\mathrm{V}^{\circ,i}) \in \mathsf{GAiCS4}, \qquad (\mathrm{V}^{\bullet,i,2}) \in \mathsf{GAKM}.$$
$$(\mathrm{V}^{\bullet,i,1}) \in \mathsf{GAiSL},$$

Of course, it is no coincidence that the Visser rules in $\mathsf{GAiL}$ presented in Figure 7.3 show resemblance to the intuitionistic modal Visser rules from Figure 7.1.

### 7.2.4 Remark

In contrast to (Iemhoff and Metcalfe, 2009b), we choose to leave out the right logical rules as presented there and replace it by the one right rule $\rhd()$ that reflects the truth of a sequent in logic $\mathsf{iL}$. This gives us the freedom to use the semantic notions of the logic instead of searching for a sequent calculus that reflects the derivability of the logic. In a sense, this shows that the shape of the rules in the proof systems for admissibility is independent of the proof theory of the logics.

So, the right-hand side of a gs-rule $\mathcal{G} \rhd \mathcal{H}$ reflects derivability/truth in logic $\mathsf{iL}$. The left-hand side of a gs-rule $\mathcal{G} \rhd \mathcal{H}$ captures the admissibility. For each logic, we have logic specific Visser rules depending on their Kripke semantics. We have two rules that reflects irreflexive extensions of models and one rule that reflects reflexive extensions. Informally speaking, these rules are fusions from the modal Visser rules and intuitionistic Visser rule from (Iemhoff and Metcalfe, 2009b). The only rule that connects the left-hand side to the right-hand side is the projection rule (PJ). This rule corresponds to the fact that derivability implies admissibility, see (Iemhoff and Metcalfe, 2009b).

The semantics we have in mind for $\mathsf{GAiL}$ is admissibility in $\mathsf{iL}$. We write $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$ denoting that there is a tree using the rules from $\mathsf{GAiL}$ that ends in gs-rule $\mathcal{G} \rhd \mathcal{H}$ and its leaves are instances of the right rule or one of the Visser rules from $\mathsf{GAiL}$ with no premises. Note that it is decidable whether a gs-rule is a conclusion of the right rule, because of the decidability of logic $\mathsf{iL}$ (Section 1.3).

**7.2.5 Example**

Recall the *disjunction property* from Example 5.2.27. We proved that it is admissible in $\mathsf{iL}$. This means that for all formulas $A$ and $B$, we have $A \vee B \mathrel{|\!\!\sim}_{\mathsf{iL}} A, B$. Once we show soundness and completeness of $\mathsf{GAiL}$, the same claim follows using the following derivation in $\mathsf{GAiL}$, where (V) can be any Visser rule from Figure 7.4:

$$\dfrac{\dfrac{\dfrac{}{(\Rightarrow A,B),(\Rightarrow A)\rhd(A\Rightarrow A),(\Rightarrow B)}\rhd()}{(\Rightarrow A,B),(\Rightarrow A)\rhd(\Rightarrow A),(\Rightarrow B)}\,(\mathrm{PJ}) \qquad \dfrac{\dfrac{}{(\Rightarrow A,B),(\Rightarrow B)\rhd(\Rightarrow A),(B\Rightarrow B)}\rhd()}{(\Rightarrow A,B),(\Rightarrow B)\rhd(\Rightarrow A),(\Rightarrow B)}\,(\mathrm{PJ})}{\dfrac{(\Rightarrow A,B)\rhd(\Rightarrow A),(\Rightarrow B)}{(\Rightarrow A\vee B)\rhd(\Rightarrow A),(\Rightarrow B)}\,(\Rightarrow\vee)\rhd}\,(\mathrm{V})$$

In the realm of admissible rules, it is interesting to think about admissible rules in the proof system $\mathsf{GAiL}$. In (Iemhoff and Metcalfe, 2009b), (W)$\rhd$ and $\rhd$(W) are part of system $\mathsf{GAiL}$, but we show that they are admissible in $\mathsf{GAiL}$.

**7.2.6 Lemma**

The following weakening rules are admissible in $\mathsf{GAiL}$.

$$\dfrac{\mathcal{G}\rhd\mathcal{H}}{\mathcal{G},S\rhd\mathcal{H}}\,(\mathrm{W})\rhd \qquad\qquad \dfrac{\mathcal{G}\rhd\mathcal{H}}{\mathcal{G}\rhd S,\mathcal{H}}\,\rhd(\mathrm{W})$$

Proof. It can be shown by a standard induction on the height of proofs in $\mathsf{GAiL}$. For rules $(\to\Rightarrow)\rhd^i$, $(\Rightarrow\to)\rhd^i$, $(\Box\Rightarrow)\rhd$, and $(\Rightarrow\Box)\rhd$, we have to be careful when sequent $S$ contains propositional variables $p$ and $q$ present in the rules. For these cases we change all $p$ and $q$ in the proof of the premise of these rules into fresh variables $p'$ and $q'$ not occurring in its proof, and also not in $S$. Application of the induction hypothesis implies the desired result. ■

**Right rule** $\dfrac{}{\mathcal{G} \rhd S, \mathcal{H}} \rhd(), I(S) \in \mathsf{iL}$

**Left logical rules**

$$\dfrac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \bot \Rightarrow \Delta) \rhd \mathcal{H}} (\bot \Rightarrow)\rhd \qquad \dfrac{\mathcal{G}, (\Gamma \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \bot, \Delta) \rhd \mathcal{H}} (\Rightarrow \bot)\rhd$$

$$\dfrac{\mathcal{G}, (\Gamma, A, B \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \wedge B \Rightarrow \Delta) \rhd \mathcal{H}} (\wedge \Rightarrow)\rhd \qquad \dfrac{\mathcal{G}, (\Gamma \Rightarrow A, \Delta), (\Gamma \Rightarrow B, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \wedge B, \Delta) \rhd \mathcal{H}} (\Rightarrow \wedge)\rhd$$

$$\dfrac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Gamma, B \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \vee B \Rightarrow \Delta) \rhd \mathcal{H}} (\vee \Rightarrow)\rhd \qquad \dfrac{\mathcal{G}, (\Gamma \Rightarrow A, B, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \vee B, \Delta) \rhd \mathcal{H}} (\Rightarrow \vee)\rhd$$

$$\dfrac{\mathcal{G}, (\Gamma, A \to B \Rightarrow A, \Delta), (\Gamma, B \Rightarrow \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \rhd \mathcal{H}} (\to)\rhd$$

$$\dfrac{\mathcal{G}, (\Gamma, p \to q \Rightarrow \Delta), (p \Rightarrow A), (B \Rightarrow q) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \rhd \mathcal{H}} (\to\Rightarrow)\rhd^i$$

$$\dfrac{\mathcal{G}, (\Gamma \Rightarrow p, \Delta), (p, A \Rightarrow B) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \to B, \Delta) \rhd \mathcal{H}} (\Rightarrow\to)\rhd^i$$

$$\dfrac{\mathcal{G}, (\Gamma, \Box p \Rightarrow \Delta), (A \Rightarrow p) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \Box A \Rightarrow \Delta) \rhd \mathcal{H}} (\Box \Rightarrow)\rhd \qquad \dfrac{\mathcal{G}, (\Gamma \Rightarrow \Box p, \Delta), (p \Rightarrow A) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \Box A, \Delta) \rhd \mathcal{H}} (\Rightarrow \Box)\rhd$$

$p, q$ do not occur in $\mathcal{G}, \mathcal{H}, \Gamma, \Delta, A, B$ in $(\to\Rightarrow)\rhd^i, (\Rightarrow\to)\rhd^i, (\Box \Rightarrow)\rhd$ and $(\Rightarrow \Box)\rhd$.

**Anti-cut, Anti-cut for boxed formulas, and projection rule**

$$\dfrac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma), (\Gamma, \Pi \Rightarrow \Sigma, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma) \rhd \mathcal{H}} (\mathrm{AC})$$

$$\dfrac{\mathcal{G}, (\Gamma, \Theta \Rightarrow \Delta), (\Pi \Rightarrow \Psi, \Sigma), (\Gamma, \Pi, \Box A \to A \Rightarrow \Sigma, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \Theta \Rightarrow \Delta), (\Pi \Rightarrow \Psi, \Sigma) \rhd \mathcal{H}} (\mathrm{AC}_\Box)$$

where $A$ is box-free, $(\Theta \cup \Psi) \subseteq \{A, \Box A\}$ and $\Theta, \Psi \neq \emptyset$.

$$\dfrac{\mathcal{G}, S \rhd (\Gamma, I(S) \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} (\mathrm{PJ}), \Gamma \Rightarrow \Delta \in \mathcal{H} \cup \{\Rightarrow\}$$

**Figure 7.3.** Rules for admissibility

**Irreflexive**

$$\frac{\begin{array}{c}[\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Box\Sigma, \Gamma \Rightarrow D) \rhd \mathcal{H}]_{D \in \Delta} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \rhd \mathcal{H}]_{O \in \Omega} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd (\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta), \mathcal{H}]_{\emptyset \neq \Pi \subseteq \Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd \mathcal{H}} \ (\mathrm{V}^{\bullet,i,1})$$

$$\frac{\begin{array}{c}[\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow D) \rhd \mathcal{H}]_{D \in \Delta} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \rhd \mathcal{H}]_{O \in \Omega} \\ [\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd (\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta), \mathcal{H}]_{\emptyset \neq \Pi \subseteq \Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G}, (\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd \mathcal{H}} \ (\mathrm{V}^{\bullet,i,2})$$

**Reflexive**

$$\frac{\begin{array}{c}[\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow D) \rhd \mathcal{H}]_{D \in \Delta} \\ [\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \rhd \mathcal{H}]_{O \in \Omega} \\ [\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd (\Box\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta), \mathcal{H}]_{\emptyset \neq \Pi \subseteq \Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd \mathcal{H}} \ (\mathrm{V}^{\circ,i})$$

where for all these rules it holds that $\Gamma$ contains only implications, $\Gamma^{\Pi} := \{A \to B \in \Gamma \mid A \notin \Pi\}$ and $\Gamma_{\Box\Omega,\Delta} := \{A \notin \Box\Omega \cup \Delta \mid \exists B(A \to B \in \Gamma)\}$

**Figure 7.4.** Sequent intuitionistic modal Visser rules

### 7.2.1 Soundness

In this section we prove the soundness theorem of proof systems $\mathsf{GAiL}$ for logics $\mathsf{iL}$ in $\{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{KM}, \mathsf{mHC}\}$. The next section treats the completeness theorem. Recall that by $\mathsf{iL}$-model we mean the rooted versions from the semantics presented in Figure 1.5.

**7.2.7 Theorem**
If $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$, then $\mathrel{|\!\sim}_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$.

Proof. We show that each rule in $\mathsf{GAiL}$ is $\mathsf{iL}$-sound. The weakening rules and right rule are clearly $\mathsf{iL}$-sound. Also the soundness of the left logical rules follow easily, see (Iemhoff and Metcalfe, 2009b). Here we treat rule $(\Rightarrow\to)\rhd^i$. Suppose that the premises of the rule are admissible and suppose that $\sigma$ is an $\mathsf{iL}$-unifier for $I(S)$ for all $S \in \mathcal{G}$ and for $I(\Gamma \Rightarrow A \to B, \Delta)$. Propositional variable $p$ does not occur in $\mathcal{G}, \mathcal{H}, \Gamma, \Delta, A, B$, so we can extend $\sigma$ to define a new substitution $\sigma_1$ with $\sigma_1(p) = A \to B$. It follows immediately that $\sigma_1$ is an $\mathsf{iL}$-unifier for $I(\Gamma \Rightarrow p, \Delta)$ and $I(p, A \Rightarrow B)$. Since the premise of the rule is admissible, we conclude that $\sigma_1$ is an $\mathsf{iL}$-unifier for $I(S)$ for some $S \in \mathcal{H}$. Since $p$ does not occur in $\mathcal{H}$, also $\sigma$ is an $\mathsf{iL}$-unifier for this $I(S)$.

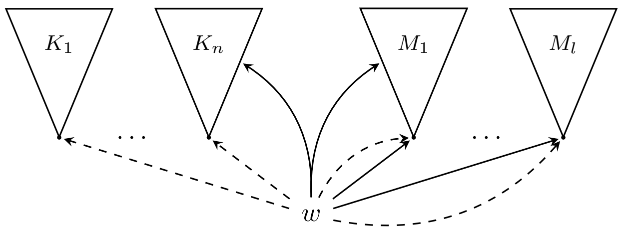
For rule $(\mathrm{AC}_\Box)$, let $\sigma$ be an $\mathsf{iL}$-unifier for $I(S)$ for all $S \in \mathcal{G}$, for $I(\Gamma, \Theta \Rightarrow \Delta)$, and for $I(\Pi \Rightarrow \Psi, \Sigma)$ where $(\Theta \cup \Psi) \subseteq \{A, \Box A\}$, $A$ is box-free, and $\Theta, \Psi \neq \emptyset$. We show that $\sigma$ is also an $\mathsf{iL}$-unifier for $I(\Gamma, \Pi, \Box A \to A \Rightarrow \Sigma, \Delta)$ using Kripke models which immediately implies the desired result. Let $K$ be a strong intuitionistic Kripke model with world $w$. Let $w' \geq w$ such that $K, w' \models \sigma(\bigwedge \Gamma \wedge \bigwedge \Pi \wedge \Box A \to A)$. There are two cases: $K, w' \models \sigma(A)$ or $K, w' \not\models \sigma(A)$. In the first case we have $K, w' \models \sigma(A \wedge \Box A)$ and by the assumption on $(\Gamma, \Theta \Rightarrow \Delta)$ we have $K, w' \models \sigma(\bigvee \Delta)$. In the second case we have $K, w' \not\models \sigma(A)$ and $K, w' \not\models \sigma(\Box A)$. By the assumption on $(\Pi \Rightarrow \Psi, \Sigma)$ we obtain $K, w' \models \sigma(\bigvee \Sigma)$. The soundness of (AC) can be verified in a similar way.

For rule (PJ), suppose that $\sigma$ is an $\mathsf{iL}$-unifier for $I(S')$ for all $S' \in \mathcal{G}$ and for $I(S)$. By the admissibility of the premise, $\sigma$ is an $\mathsf{iL}$-unifier for $I(\Gamma, I(S) \Rightarrow \Delta)$ or for $I(S')$ for some $S' \in \mathcal{H}$. In the latter case we are done. In the former case, since $\sigma$ is an $\mathsf{iL}$-unifier for $I(S)$ and $I(\Gamma, I(S) \Rightarrow \Delta)$, it is also a unifier for $\Gamma \Rightarrow \Delta$. The empty sequent can never be unified, therefore $\Gamma \Rightarrow \Delta \in \mathcal{H}$ by the condition of rule (PJ). Therefore $\sigma$ is an $\mathsf{iL}$-unifier for some sequent in $\mathcal{H}$.

Now we turn to the soundness of the intuitionistic modal Visser rules, which are all shown in a similar way. We start with $(\mathrm{V}^{\bullet,i,1})$ which is a rule in $\mathsf{GAiCK4}$ and $\mathsf{GAiSL}$, so let $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{iSL}\}$. Suppose that $\sigma$ is an $\mathsf{iL}$-unifier for $I(S)$ for all $S \in \mathcal{G}$ and for $I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta)$. Write $\Delta = \{D_1, \ldots, D_n\}$ and $\Omega = \{O_1, \ldots, O_l\}$ (including the cases where the sets are empty). Using the third set of premises, we have for all $\emptyset \neq \Pi \subseteq \Gamma_{\Box\Omega,\Delta}$ that $\sigma$ is either an iL-unifier for some $S \in \mathcal{H}$ or for $I(\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta)$. If there is such a $\Pi$ for which the first case holds we are done. If for all such $\Pi$ we have the second case (or in case there is no such $\Pi$ at all), we will show that $\sigma$ is an iL-unifier for $I(\Box\Sigma, \Gamma \Rightarrow D_i)$ for some $i$, or for $I(\Sigma, \Gamma \Rightarrow O_j)$ for some $j$. This is sufficient, because that implies that $\sigma$ is an iL-unifier for some $S \in \mathcal{H}$ by the first or second set of premises of $(\mathrm{V}^{\bullet,i,1})$. Suppose for a contradiction that this is not the case. Then there exist iL-countermodels $K_1, \ldots, K_n$ and $M_1, \ldots, M_l$ such that

$$K_i \models \sigma(\bigwedge \Box\Sigma \wedge \bigwedge \Gamma) \text{ and } K_i \not\models \sigma(D_i),$$
$$M_j \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } M_j \not\models \sigma(O_j).$$

Consider the following iL-model $M$ with irreflexive world $w$, where $\leq$ is drawn by a dashed line, and $R$ by a straight line. $R$ should be closed under prefixing with $\leq$ as indicated by the lines into model $K_n$ and $M_1$. Model $M$ is a one-node model if $\Delta = \Omega = \emptyset$.

$M$ =


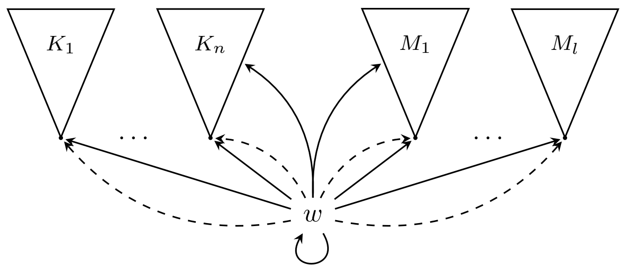

First note that $M \not\models \sigma(A)$ for all $A \in \Box\Omega \cup \Delta$. Also note that $M \models \sigma(\bigwedge(\Box\Sigma))$. Let $\Pi = \{A \in \Gamma_{\Box\Omega,\Delta} \mid M \models \sigma(A)\}$. Thus $M \models \sigma(\bigwedge \Pi)$. We also claim that $M \models \sigma(\bigwedge \Gamma^{\Pi})$. Let $A \to B \in \Gamma^{\Pi}$. Observe that either $A \in \Delta \cup \Box\Omega$ or $M \not\models \sigma(A)$. The first implies the second, so $M \not\models \sigma(A)$. And since $K_i \models \sigma(A \to B)$ for all $i$ and $M_j \models \sigma(A \to B)$ for all $j$, we have $M \models \sigma(A \to B)$. So far we have shown that $M \models \sigma(\bigwedge(\Box\Sigma \cup \Gamma^{\Pi} \cup \Pi))$. If $\Pi = \emptyset$, then $\Gamma^{\Pi} = \Gamma$ and so $M \models \sigma(\bigwedge(\Box\Sigma \cup \Gamma))$. But $\sigma$ is an iL-unifier for $I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta)$. If $\Pi \neq \emptyset$, then $\sigma$ is an iL-unifier for $I(\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta)$ by assumption. In both cases we have $M \models \sigma(\bigvee(\Box\Omega \cup \Delta))$, which is a contradiction with our first observation about model $M$.

For rule $(\mathrm{V}^{\bullet,i,2})$ for logics mHC and KM, the proof is completely analogous, where we construct model $M$ as in the previous picture but now by extending each model with both relations $\leq$ and $R$.

Rule $(\mathrm{V}^{\circ,i})$ is present in the calculi GAiCK4, GAiCS4, and GAmHC, so for the moment let iL $\in \{$iCK4, iCS4, mHC$\}$. Suppose that $\sigma$ is an iL-unifier for $I(S)$ for all $S \in \mathcal{G}$ and for $I(\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta)$. Again, write $\Delta = \{D_1, \ldots, D_n\}$ and $\Omega = \{O_1, \ldots, O_l\}$. By a similar argument as above, it is sufficient to show that $\sigma$ is an iL-unifier for $I(\Sigma, \Gamma \Rightarrow D_i)$ for some $i$, or for $I(\Sigma, \Gamma \Rightarrow O_j)$ for some $j$. Suppose this is not the case. Then there exist iL-countermodels $K_1, \ldots, K_n$ and $M_1, \ldots M_l$ such that

$$K_i \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } K_i \not\models \sigma(D_i),$$
$$M_j \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } M_j \not\models \sigma(O_j).$$

Consider the following iL-model $M$ with reflexive root $w$, drawn in a similar way as above.

$M \quad =$ 


By similar reasoning as above, it leads to a contradiction with $\sigma$ being an iL-unifier for $I(\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta)$ and for $I(\Box\Sigma \to \Sigma, \Gamma^\Pi, \Pi \Rightarrow \Box\Omega, \Delta)$ with $\Pi \neq \emptyset$. ■

**7.2.8 Example**
Recall Example 7.1.3 about equivalent Visser rules. In the proof systems for admissibility GAiL we have the same. Rule $(\mathrm{V}^{\circ,i})$ can be considered as the reflexive version of $(\mathrm{V}^{\bullet,i,2})$ and the following rule as the reflexive version of $(\mathrm{V}^{\bullet,i,1})$:

$$\frac{\begin{array}{c}[\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Box\Sigma \to \Sigma, \Gamma \Rightarrow D) \rhd \mathcal{H}]_{D \in \Delta} \\ [\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \rhd \mathcal{H}]_{O \in \Omega} \\ [\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd (\Box\Sigma \to \Sigma, \Gamma^\Pi, \Pi \Rightarrow \Box\Omega, \Delta), \mathcal{H}]_{\emptyset \neq \Pi \subseteq \Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G}, (\Box\Sigma \to \Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \rhd \mathcal{H}} \; (\mathrm{V}^{\circ,i,1})$$

This rule is equivalent to $(\mathrm{V}^{\circ,i})$.

## 7.2.2 Completeness

In this section we prove the completeness theorem of proof systems GAiL for logics $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{KM}, \mathsf{mHC}\}$.

**7.2.9 Theorem**
If $\mathrel{|\!\!\sim}_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$, then $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$.

The theorem is shown in several steps in the same line of reasoning from (Iemhoff and Metcalfe, 2009b). We first show that derivability is captured by the proof system GAiL as shown in the following lemma. Note the difference between Lemma 19 from (Iemhoff and Metcalfe, 2009b) due to the small difference between the definition of derivability. After that we present some lemmas that show that each gs-rule is derivable in GAiL from a certain class of irreducible gs-rules that have special properties.

**7.2.10 Lemma**
If $\vdash_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$, then $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$

Proof. Suppose $\vdash_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$. By definition, $I(\mathcal{G}) \vdash_{\mathsf{iL}} I(\Gamma \Rightarrow \Delta)$ for some $\Gamma \Rightarrow \Delta \in \mathcal{H}$ or $I(\mathcal{G}) \vdash_{\mathsf{iL}} \bot$. If $\Gamma = \Delta = \emptyset$, then the latter is equivalent to $I(\mathcal{G}) \vdash_{\mathsf{iL}} I(\Gamma \Rightarrow \Delta)$. So assume $I(\mathcal{G}) \vdash_{\mathsf{iL}} I(\Gamma \Rightarrow \Delta)$. The deduction theorem (Theorem 1.3.3) of iL implies $\vdash_{\mathsf{iL}} \bigwedge I(\mathcal{G}) \to I(\Gamma \Rightarrow \Delta)$. By the formula interpretation, we have that $\vdash_{\mathsf{iL}} I(\Gamma, I(\mathcal{G}) \Rightarrow \Delta)$. Apply the right rule $\rhd()$ to obtain $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd (\Gamma, I(\mathcal{G}) \Rightarrow \Delta), \mathcal{H}$. Repeated applications of (PJ) gives us $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$. ∎

Recall Definition 7.2.2 for iL-invertible inference rules.

**7.2.11 Lemma**
All inference rules in GAiL are iL-invertible.

Proof. We only treat a few cases. Consider for example rule $(\vee \Rightarrow)\rhd$. Suppose that the conclusion $\mathcal{G}, (\Gamma, A \vee B \Rightarrow \Delta) \rhd \mathcal{H}$ is admissible in iL. Let $\sigma$ be an iL-unifier for $I(S)$ for all $S \in \mathcal{G}$, for $I(\Gamma, A \Rightarrow \Delta)$, and for $I(\Gamma, B \Rightarrow \Delta)$. By intuitionistic reasoning, $\sigma$ is also an iL-unifier for $I(\Gamma, A \vee B \Rightarrow \Delta)$, hence $\sigma$ is an iL-unifier for $I(S)$ for some $S \in \mathcal{H}$.

For $(\Rightarrow \Box)\rhd$, suppose the conclusion $\mathcal{G}, (\Gamma \Rightarrow \Box A, \Delta) \rhd \mathcal{H}$ is admissible in iL. Let $\sigma$ be an iL-unifier for $I(S)$ for all $S \in \mathcal{G}$, for $I(\Gamma \Rightarrow \Box p, \Delta)$, and for $I(p \Rightarrow A)$. Since we work with normal modal logics, $\sigma$ is also an iL-unifier for $I(\Box p \Rightarrow \Box A)$. Using intuitionistic reasoning we obtain that $\sigma$ is an iL-unifier for $I(\Gamma \Rightarrow \Box A, \Delta)$. Hence, $\sigma$ is an iL-unifier for $I(S)$ for some $S \in \mathcal{H}$.

For (AC), $(\mathrm{AC}_\Box)$, (PJ), and all Visser rules we have that all the sequents in the conclusion appear in the premises, which immediately implies the invertibility of the rules. ∎

**7.2.12 Definition**
We call formulas of the form $\Box p$ *boxed variables* and formulas of the form $p \to q$ *variable implications*, where $p$ and $q$ denote propositional variables.

**7.2.13 Definition**
A sequent $\Lambda \Rightarrow \Phi$ is called

- *semi-modal-implication-irreducible*, if $\Lambda$ contains only variables, boxed variables, variable implications, and implications of the form $\Box p \to p$, and if $\Phi$ contains only variables and boxed variables.
- *modal-implication-irreducible* if it is semi-modal-implication-irreducible without the implications of the form $\Box p \to p$.

A gs-rule $\mathcal{G} \rhd \mathcal{H}$ is called *(semi-)modal-implication-irreducible* if all sequents in $\mathcal{G}$ are *(semi-)modal-implication-irreducible.*

**7.2.14 Lemma**
Every (admissible) gs-rule is derivable in GAiL from an (admissible) modal-implication-irreducible gs-rule only using the left logical rules without using $(\to)\rhd$.

Proof. The fact that every gs-rule can be derived from one modal-implication-irreducible gs-rule follows from the fact that applications of left logical rules (except $(\to)\rhd$) terminate in one modal-implication-irreducible gs-rule. This is seen as follows reading the rules bottom-up. Every left logical rule (except $(\to)\rhd$) replaces a sequent on the left in the conclusion with sequents on the left in the premise that have fewer connectives. For rules $(\Box \Rightarrow)\rhd$ and $(\Rightarrow \Box)\rhd$ we stop if $A$ is a variable. Similarly so for $(\to\Rightarrow)\rhd^i$ if $A \to B$ is of the from $p \to q$. Admissibility follows from the invertibility of the left logical rules shown in Lemma 7.2.11. ■

**7.2.15 Definition**
A gs-rule $\mathcal{G} \rhd \mathcal{H}$ is *full* with respect to a set of gs-rules $X$ if whenever

$$\frac{\mathcal{G}_1 \rhd \mathcal{H}_1 \qquad \dots \qquad \mathcal{G}_n \rhd \mathcal{H}_n}{\mathcal{G} \rhd \mathcal{H}}$$

is an instance of a rule in $X$, then there exists an $i$ such that $\mathcal{G}_i \subseteq \mathcal{G}$ and $\mathcal{H}_i \subseteq \mathcal{H}$.

**7.2.16 Lemma**
Let $X \subset \{(\to)\rhd, (\mathrm{AC}), (\mathrm{AC}_\Box), (\mathrm{V}^{\bullet,i,1}), (\mathrm{V}^{\bullet,i,2}), (\mathrm{V}^{\circ,i})\}$ be rules of GAiL. Every (admissible) gs-rule is derivable in GAiL from (admissible) semi-modal-implication-irreducible gs-rules that are full with respect to $X$.

Proof. First apply Lemma 7.2.14 to backwards reach a gs-rule that is modal-implication-irreducible. Note that bottom-up applications of rules in $X$ to a (semi)-modal-implication-irreducible conclusion yields (semi)-modal-implication-irreducible premises. In particular, only $(\mathrm{AC}_\Box)$ can introduce implications of the form $\Box p \to p$ in the premise, so only $(\mathrm{AC}_\Box)$ can introduce semi-modal-implication-irreducible sequents. Note that in rule $(\mathrm{AC}_\Box)$, $A$ can only be a variable since $A$ is assumed to be box-free. All rules in $X$ do not add new variables into the premises compared to the conclusion and the premises contain more sequents than the conclusion. Since there are only finitely many different semi-modal-implication-irreducible sequents for a fixed set of variables, applying these rules exhaustively backwards terminates with a set of semi-modal-implication-irreducible gs-rules full with respect to these rules. Again, admissibility follows from Lemma 7.2.11. ■

We provide some technical definitions and lemmas that we use in the completeness proof of Theorem 7.2.9 on page 223. We advice the reader to first skip the technical definitions and lemmas and to go immediately to the proof. The lemmas play a role in a resolution refutation argument of the kind also present in the completeness proof in (Iemhoff and Metcalfe, 2009b). Informally, the lemmas show that a 'cut' on $p$ in sequents of the form $S_1 = (\Lambda_1, p \Rightarrow \Phi_1)$ and $S_2 = (\Lambda_2 \Rightarrow \Phi_2, p)$ resulting in $S = (\Lambda_1, \Lambda_2 \Rightarrow \Phi_1, \Phi_2)$ preserves some desirable technical properties.

**7.2.17 Definition**

Define the following property $\bullet$ on pairs consisting of a semi-modal-implication-irreducible sequent $(\Lambda \Rightarrow \Phi)$ and a set of sequents $\mathcal{G}$:

$$\bullet((\Lambda \Rightarrow \Phi), \mathcal{G}) \quad \text{iff} \quad \begin{array}{c} \forall \Pi \subseteq \Lambda_\Phi,\ \exists \Lambda' \subseteq \Pi \cup \Lambda^\Pi \cup \Lambda^a \cup \Lambda^b,\ \exists \Phi' \subseteq \Phi \\ \text{such that } \Lambda' \Rightarrow \Phi' \in \mathcal{G}, \end{array}$$

where $\Lambda_\Phi = \{A \notin \Phi \mid \exists B (A \to B \in \Lambda)\}$, $\Lambda^\Pi = \{A \to B \in \Lambda \mid A \notin \Pi\}$, $\Lambda^a$ is the set of all propositional variables in $\Lambda$, and $\Lambda^b$ is the set of all boxed variables in $\Lambda$.

Note that the property automatically holds for $\Pi = \emptyset$, by taking $(\Lambda \Rightarrow \Phi)$ for $(\Lambda' \Rightarrow \Phi')$. Also note that $\Pi$ may contain variables and boxed variables, since all implications in $\Lambda$ are variable implications or have the form $\Box p \to p$.

**7.2.18 Lemma**

Let $S_1 = (\Lambda_1, p \Rightarrow \Phi_1)$, $S_2 = (\Lambda_2 \Rightarrow \Phi_2, p)$, and $S = (\Lambda_1, \Lambda_2 \Rightarrow \Phi_1, \Phi_2)$ be semi-modal-implication-irreducible sequents. Let $\mathcal{G}, S_1, S_2 \rhd \mathcal{H}$ be a gs-rule full with respect to (AC). Then $\bullet(S_1, \mathcal{G})$ and $\bullet(S_2, \mathcal{G})$ imply $\bullet(S, \mathcal{G})$.

Proof. Suppose $\Pi \subseteq (\Lambda_1 \cup \Lambda_2)_{\Phi_1 \cup \Phi_2}$. Then also $\Pi \subseteq (\Lambda_1)_{\Phi_1} \cup (\Lambda_2)_{\Phi_2}$. Write $\Pi = \Pi_1 \cup \Pi_2$ such that $\Pi_i = \Pi \cap (\Lambda_i)_{\Phi_i}$ for $i = 1, 2$. Note that in this way $\Lambda_i^\Pi = \Lambda_i^{\Pi_i}$ for $i = 1, 2$. So we want to find sets

$$\Lambda' \subseteq \Pi_1 \cup \Pi_2 \cup \Lambda_1^{\Pi_1} \cup \Lambda_2^{\Pi_2} \cup \Lambda_1^a \cup \Lambda_2^a \cup \Lambda_1^b \cup \Lambda_2^b \quad \text{and} \quad \Phi' \subseteq \Phi_1 \cup \Phi_2$$

such that $(\Lambda' \Rightarrow \Phi') \in \mathcal{G}$.

First assume $p \in \Pi_2$. . For this case we only have to use the assumption for $S_1$. (Here we cannot use the assumption for $S_2$, because $\Pi_2 \not\subseteq (\Lambda_2)_{\Phi_2 \cup \{p\}}$). Note that $\Pi_1 \subseteq (\Lambda_1)_{\Phi_1}$, so

$$\exists \Lambda_1' \subseteq \Pi_1 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b \cup \{p\},\ \exists \Phi_1' \subseteq \Phi_1 \text{ such that } (\Lambda_1' \Rightarrow \Phi_1') \in \mathcal{G}.$$

Define $\Lambda' = \Lambda'_1$ and $\Phi' = \Phi'_1$. This is sufficient, because $p \in \Pi_2$.

If $p \notin \Pi_2$, then we have $\Pi_1 \subseteq (\Lambda_1)_{\Phi_1}$ and $\Pi_2 \subseteq (\Lambda_2)_{\Phi_2 \cup \{p\}}$, so by assumption for $S_1$ and $S_2$ we have

$$\exists \Lambda'_1 \subseteq \Pi_1 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b \cup \{p\},\ \exists \Phi'_1 \subseteq \Phi_1 \text{ such that } (\Lambda'_1 \Rightarrow \Phi'_1) \in \mathcal{G},$$
$$\exists \Lambda'_2 \subseteq \Pi_2 \cup \Lambda_2^{\Pi_2} \cup \Lambda_2^a \cup \Lambda_2^b,\ \exists \Phi'_2 \subseteq \Phi_2 \cup \{p\} \text{ such that } (\Lambda'_2 \Rightarrow \Phi'_2) \in \mathcal{G}.$$

We distinguish 3 cases. If $p \notin \Lambda'_1$, we can take $\Lambda' = \Lambda'_1$ and $\Phi' = \Phi'_1$. If $p \notin \Phi'_2$, we can take $\Lambda' = \Lambda'_2$ and $\Phi' = \Phi'_2$. Otherwise, they have the form $\Lambda'_1 = \Lambda''_1 \cup \{p\}$ and $\Phi'_2 = \Phi''_2 \cup \{p\}$. Then $S' = (\Lambda''_1, \Lambda'_2 \Rightarrow \Phi'_1, \Phi''_2) \in \mathcal{G}$ by fullness of (AC). Take $\Lambda' = \Lambda''_1 \cup \Lambda'_2$ and $\Phi' = \Phi'_1 \cup \Phi''_2$. ■

### 7.2.19 Definition

Define the following property $\circ$ on pairs consisting of a semi-modal-implication-irreducible sequent $(\Lambda \Rightarrow \Phi)$ and a set of sequents $\mathcal{G}$:

$$\circ((\Lambda \Rightarrow \Phi), \mathcal{G}) \quad \text{iff} \quad \begin{array}{c} \forall \Pi \subseteq \Lambda^a_\Phi,\ \exists \Lambda' \subseteq \Pi \cup \Box\Pi \cup \Lambda^\Pi \cup \Lambda^a \cup \Lambda^b,\ \exists \Phi' \subseteq \Phi \\ \text{such that } \Lambda' \Rightarrow \Phi' \in \mathcal{G}, \end{array}$$

where $\Lambda^a_\Phi = \{p \notin \Phi \mid \exists q (p \to q \in \Lambda)\}$, $\Lambda^\Pi = \{A \to B \in \Lambda \mid A \notin \Pi\}$, $\Lambda^a$ is the set of all propositional variables in $\Lambda$, and $\Lambda^b$ is the set of all boxed variables in $\Lambda$.

Note that $\Lambda^a_\Phi$ only contains variables by definition, and so does $\Pi$. This means that $\Lambda^\Pi$ contains all implications from $\Lambda$ of the form $\Box p \to p$. Again, the property holds automatically for $\Pi = \emptyset$, by taking $(\Lambda \Rightarrow \Phi)$ for $(\Lambda' \Rightarrow \Phi')$.

### 7.2.20 Lemma

Let $S_1 = (\Lambda_1, \Theta \Rightarrow \Phi_1)$, $S_2 = (\Lambda_2 \Rightarrow \Phi_2, \Psi)$, and $S = (\Lambda_1, \Lambda_2, \Box p \to p \Rightarrow \Phi_1, \Phi_2)$ be semi-modal-implication-irreducible sequents with non-empty $\Theta, \Psi \subseteq \{p, \Box p\}$. Let $\mathcal{G}, S_1, S_2 \rhd \mathcal{H}$ be a gs-rule full with respect to (AC$_\Box$). Then $\circ(S_1, \mathcal{G})$ and $\circ(S_2, \mathcal{G})$ imply $\circ(S, \mathcal{G})$.

Proof. Suppose $\Pi \subseteq (\Lambda_1 \cup \Lambda_2 \cup \{\Box p \to p\})^a_{\Phi_1 \cup \Phi_2}$. Note that $\Pi$ only contains variables. Then also $\Pi \subseteq (\Lambda_1)^a_{\Phi_1} \cup (\Lambda_2)^a_{\Phi_2}$. We write $\Pi = \Pi_1 \cup \Pi_2$ such that $\Pi_i = \Pi \cap (\Lambda_i)^a_{\Phi_i}$ for $i = 1, 2$. Note that in this way $\Lambda_i^\Pi = \Lambda_i^{\Pi_i}$ for $i = 1, 2$. So we want to find sets

$$\Lambda' \subseteq \Pi_1 \cup \Box\Pi_1 \cup \Pi_2 \cup \Box\Pi_2 \cup \Lambda_1^{\Pi_1} \cup \Lambda_2^{\Pi_2} \cup \{\Box p \to p\} \cup \Lambda_1^a \cup \Lambda_2^a \cup \Lambda_1^b \cup \Lambda_2^b,$$
$$\Phi' \subseteq \Phi_1 \cup \Phi_2,$$

such that $(\Lambda' \Rightarrow \Phi') \in \mathcal{G}$.

We distinguish between $p \in \Pi_2$ and $p \notin \Pi_2$. First assume $p \in \Pi_2$. (Here we cannot always use the assumption for $S_2$, because $\Pi_2$ may not be a subset of $(\Lambda_2)^a_{\Phi_2 \cup \Psi}$).

For this case we only have to use the assumption for $S_1$. Note that $\Pi_1 \subseteq (\Lambda_1)^a_{\Phi_1}$, so

$$\exists \Lambda'_1 \subseteq \Pi_1 \cup \Box\Pi_1 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b \cup \Theta,\ \exists \Phi'_1 \subseteq \Phi_1 \text{ such that } (\Lambda'_1 \Rightarrow \Phi'_1) \in \mathcal{G}.$$

Define $\Lambda' = \Lambda'_1$ and $\Phi' = \Phi'_1$. Since we have $\Theta \subseteq \Pi_2 \cup \Box\Pi_2$, it holds that $\Lambda' \subseteq \Pi_1 \cup \Box\Pi_1 \cup \Pi_2 \cup \Box\Pi_2 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b$ and we are done.

If $p \notin \Pi_2$, then we have $\Pi_1 \subseteq (\Lambda_1)^a_{\Phi_1}$ and $\Pi_2 \subseteq (\Lambda_2)^a_{\Phi_2 \cup \Psi}$, so by assumption for $S_1$ and $S_2$ we have

$$\exists \Lambda'_1 \subseteq \Pi_1 \cup \Box\Pi_1 \cup \Lambda_1^{\Pi_1} \cup \Lambda_1^a \cup \Lambda_1^b \cup \Theta,\ \exists \Phi'_1 \subseteq \Phi_1 \text{ such that } (\Lambda'_1 \Rightarrow \Phi'_1) \in \mathcal{G},$$
$$\exists \Lambda'_2 \subseteq \Pi_2 \cup \Box\Pi_2 \cup \Lambda_2^{\Pi_2} \cup \Lambda_2^a \cup \Lambda_2^b,\ \exists \Phi'_2 \subseteq \Phi_2 \cup \Psi \text{ such that } (\Lambda'_2 \Rightarrow \Phi'_2) \in \mathcal{G}.$$

We distinguish 3 cases. If $p, \Box p \notin \Lambda'_1$, we can take $\Lambda' = \Lambda'_1$ and $\Phi' = \Phi'_1$. If $p, \Box p \notin \Phi'_2$, we can take $\Lambda' = \Lambda'_2$ and $\Phi' = \Phi'_2$. Otherwise $\Lambda'_1$ is of the form $\Lambda''_1 \cup \Theta_1$ and $\Phi'_2$ is of the form $\Phi''_2 \cup \Psi_2$ with non-empty sets $\Theta_1, \Psi_2 \subseteq \{p, \Box p\}$. Then $S' = (\Lambda''_1, \Lambda'_2, \Box p \to p \Rightarrow \Phi'_1, \Phi''_2) \in \mathcal{G}$ by fullness of $(\mathrm{AC}_\Box)$. So define $\Lambda' = \Lambda''_1 \cup \Lambda'_2 \cup \{\Box p \to p\}$ and $\Phi' = \Phi'_1 \cup \Phi''_2$. ■

Now we give the proof of completeness from Theorem 7.2.9, i.e., we prove that if $\mathrel{|\!\sim}_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$, then $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$. The idea is that the derivation of $\mathcal{G} \rhd \mathcal{H}$ in GAiL starts with bottom-up applications of left logical rules and rules from the set $\mathsf{GAiL} \cap \{(\to)\rhd, (\mathrm{AC}), (\mathrm{AC}_\Box), (\mathrm{V}^{\bullet,i,1}), (\mathrm{V}^{\bullet,i,2}), (\mathrm{V}^{\circ,i})\}$ resulting in a set of semi-modal-implication-irreducible gs-rules full with respect to that set by Lemma 7.2.16. For each such gs-rule $\mathcal{G}' \rhd \mathcal{H}'$, we show that admissibility $\mathrel{|\!\sim}_{\mathsf{iL}} \mathcal{G}' \rhd \mathcal{H}'$ reduces to derivability $\vdash_{\mathsf{iL}} \mathcal{G}' \rhd \mathcal{H}'$. This results in a derivation of $\vdash_{\mathsf{GAiL}} \mathcal{G}' \rhd \mathcal{H}'$ using the rules (PJ) and $\rhd()$ as shown in Lemma 7.2.10. The proof is based on a distinction between $C := \bigwedge I(\mathcal{G})$ being inconsistent, consistent and projective, and consistent and not projective. Recall the definition of a projective formula in Definition 6.1.6. The latter case is very difficult and relies on the extension property discussed in Section 6.2.2. The proof includes a resolution refutation argument using the technical lemmas presented before.

Proof of Theorem 7.2.9. Suppose $\mathrel{|\!\sim}_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$. Using Lemma 7.2.16, it is sufficient to assume that $\mathcal{G} \rhd \mathcal{H}$ is a semi-modal-implication-irreducible gs-rule that is full with respect to the set of rules

$$X = \mathsf{GAiL} \cap \{(\to)\rhd, (\mathrm{AC}), (\mathrm{AC}_\Box), (\mathrm{V}^{\bullet,i,1}), (\mathrm{V}^{\bullet,i,2}), (\mathrm{V}^{\circ,i})\}.$$

Define formula

$$C := \bigwedge I(\mathcal{G}).$$

We consider three cases. Only for one case we use the assumption $\mathrel{|\!\sim}_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$ to prove $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$. For the other cases $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$ follows immediately. If $C$ is inconsistent, then $I(\mathcal{G}) \vdash_{\mathsf{iL}} \bot$ and so $\vdash_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$ by definition. By Lemma 7.2.10 we have $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$. Now assume $C$ is consistent. For the case that $C$ is projective, we use the assumption $\mathrel{|\!\sim}_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$ and Lemma 6.1.9, to conclude $C \vdash_{\mathsf{iL}} I(S)$ for some $S \in \mathcal{H}$. So $I(\mathcal{G}) \vdash_{\mathsf{iL}} I(S)$ for some $S \in \mathcal{H}$ and so $\vdash_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$ by definition. Again by Lemma 7.2.10, we obtain $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$.

The case remains that $C$ is consistent and not projective. Let the propositional variables of $\mathcal{G}$ and $\mathcal{H}$ be among $\overline{p}$. By Theorem 6.2.31, there is an $\mathsf{iL}$-model over $\overline{p}$, say $K$, with root $\rho$ such that $K_w \models C$ for each $w \neq \rho$ and for each variant $K'$ of $K$ we have $K' \not\models C$. Formula $C$ holds in at least one one-world model, because it is consistent. Therefore there exists at least one $w \neq \rho$ in $K$.

There are finitely many variants of $K$ that only force variables among $\overline{p}$. Let $M_1, \ldots, M_k$ be all such variants of $K$. We have $M_i \not\models C$ for all $i$, and, thus, $M_i \not\models I(S_i)$ for some semi-modal-implication-irreducible sequent $S_i \in \mathcal{G}$. Let us write $S_i = (\Lambda_i \Rightarrow \Phi_i)$, so $M_i \not\models I(\Lambda_i \Rightarrow \Phi_i)$. Since $M_i, w \Vdash I(\Lambda_i \Rightarrow \Phi_i)$ for each $w \neq \rho$, we have $M_i \models \bigwedge \Lambda_i$ and $M_i \not\models \bigvee \Phi_i$. We assume that

$$(p \to q) \in \Lambda_i \quad \Longrightarrow \quad p \in \Phi_i, \tag{7.2}$$

$$(\Box p \to p) \in \Lambda_i \quad \Longrightarrow \quad \Box p \in \Phi_i. \tag{7.3}$$

This is possible because suppose that $(\Box p \to p) \in \Lambda_i$ and $\Box p \notin \Phi_i$. We show that $\Lambda_i \Rightarrow \Phi_i$ can be replaced by another sequent $S \in \mathcal{G}$ that has property (7.3) and $M_i \not\models I(S)$. Since $M_i \models \bigwedge \Lambda_i$ and $M_i \not\models \bigvee \Phi_i$ it follows that $M_i \models \Box p \to p$, which means $M_i, \rho \nVdash \Box p$ or $M_i, \rho \Vdash p$. So either $M_i \not\models I(\Lambda_i \Rightarrow \Box p, \Phi_i)$ or $M_i \not\models I(\Lambda_i \setminus \{\Box p \to p\}, p \Rightarrow \Phi_i)$. Since $\mathcal{G} \rhd \mathcal{H}$ is full with respect to $(\to)\rhd$, both of these sequents belong to $\mathcal{G}$ and can replace $\Lambda_i \Rightarrow \Phi_i$. Similar for property (7.2).

Define the set of propositional variables:

$$P := \{p \in \mathsf{Var}(C) \text{ and } K_w \models p \text{ for all } w \neq \rho\}.$$

There are several possibilities depending on the specific logic.

1. For logics $\mathsf{iCK4}$ and $\mathsf{iSL}$, root $\rho$ can be irreflexive (in case of $\mathsf{iSL}$, $\rho$ must be irreflexive).
2. For logics $\mathsf{mHC}$ and $\mathsf{KM}$, root $\rho$ can be irreflexive and model $K$ satisfies $< \subseteq R$ (in case of $\mathsf{KM}$, $\rho$ must be irreflexive).
3. For logics $\mathsf{iCK4}$, $\mathsf{iCS4}$, and $\mathsf{mHC}$, root $\rho$ can be reflexive (in case of $\mathsf{iCS4}$, $\rho$ must be reflexive).

We treat each case separately using the Visser rules $(\mathrm{V}^{\bullet,i,1})$, $(\mathrm{V}^{\bullet,i,2})$, and $(\mathrm{V}^{\circ,i})$, respectively.

**Case 1:**
Let $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{iSL}\}$ and suppose that $\rho$ is irreflexive. We define formulas for

$i = 1, \ldots, k$ as follows:

$$A_i := \bigwedge_{p \in \Lambda_i} \neg p \wedge \bigwedge_{p \in \Phi_i \cap P} p \quad \text{and} \quad A := \bigvee_{i=1}^{k} A_i.$$

Note that $\mathsf{Var}(\Lambda_i) \subseteq P$, because $M_i \models \bigwedge \Lambda_i$ and so for variant $K$ we have $K_w \models p$ for all $w \neq \rho$ by monotonicity. We show that $A$ is a classical tautology. If the conjuncts in $A_i$ are empty for some $i$, then $A$ is equivalent to $\top$. Otherwise, let $v$ be a classical valuation on $P$. This valuation corresponds to a variant $M$ of $K$ for which

$$M \models p \Leftrightarrow v(p) = 0.$$

Note that this correspondence is well-defined because $M_w \models p$ for each $w \neq \rho$ and $p \in P$. Let $M = M_i$. We have $M \models p$ for all variables $p \in \Lambda_i$ and $M \not\models p$ for all variables $p \in \Phi_i$. Hence $v(p) = 0$ if $p \in \Lambda_i$ and $v(p) = 1$ if $p \in \Phi_i$. Thus $v(A_i) = 1$, hence $A$ is a classical tautology. So $\neg A$ is classically inconsistent and by DeMorgan laws, $\neg A$ is classically equivalent to:

$$\neg A \equiv \bigwedge_{i=1}^{k} ( \bigvee_{p \in \Lambda_i} p \vee \bigvee_{p \in \Phi_i \cap P} \neg p).$$

Therefore there exists a resolution refutation starting with the clauses

$$\{p \mid p \in \Lambda_i\} \cup \{\neg p \mid p \in \Phi_i \cap P\} \text{ for } i = 1, \ldots, k,$$

that ends in the empty clause $\emptyset$. In case the conjuncts of $A_i$ are empty for some $i$, the empty clause is already among the starting clauses. Each clause in the resolution refutation is of the form $\Theta \cup \Psi'$ where $\Theta$ contains only variables and $\Psi'$ contains only negated variables. Define $\Psi := \{p \mid \neg p \in \Psi'\}$.

The refutation resolution can be mimicked by applications of the rule (AC), so that each class $\Theta \cup \Psi'$ corresponds to a semi-modal-implication-irreducible sequent of the form $\Box\Sigma, \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi$, where $\Gamma$ only contains variable implications or implications of the form $\Box p \to p$, sets $\Sigma$ and $\Omega$ only contain variables, and $\Delta$ only contains variables with $\Delta \cap P = \emptyset$. To see this, first note that the starting classes are of this form. Further, each resolution on $\Theta_1 \cup \{p\} \cup \Psi_1'$ and $\Theta_2 \cup \Psi_2' \cup \{\neg p\}$ can be mimicked by (AC) from sequents

$$\Box\Sigma_1, \Gamma_1, \Theta_1, p \Rightarrow \Box\Omega_1, \Delta_1, \Psi_1 \quad \text{and} \quad \Box\Sigma_2, \Gamma_2, \Theta_2 \Rightarrow \Box\Omega_2, \Delta_2, \Psi_2, p$$

by a ‘cut’ on $p$, resulting in sequent

$$\Box\Sigma_1, \Box\Sigma_2, \Gamma_1, \Gamma_2, \Theta_1, \Theta_2 \Rightarrow \Box\Omega_1, \Box\Omega_2, \Delta_1, \Delta_2, \Psi_1, \Psi_2,$$

also of the right form. Moreover, since $\mathcal{G}$ is full with respect to (AC), it is guaranteed that all such $\Box\Sigma, \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi$ are in $\mathcal{G}$.

In addition we have the following properties for each clause $\Theta \cup \Psi'$ in the refutation and its corresponding sequent $\Box\Sigma, \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi \in \mathcal{G}$:

1. $K_w \models \bigwedge \Box\Sigma \wedge \bigwedge \Gamma$ for all $w \neq \rho$,
2. $K_v \models \bigwedge \Sigma \wedge \bigwedge \Gamma$ for all $v$ such that $\rho R v$,
3. $\Box q \in \Box\Omega$ implies $K_v \not\models q$ for some $v$ such that $\rho R v$,
4. $\bullet((\Box\Sigma, \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi), \mathcal{G})$.

We show these properties inductively following the resolution refutation. Properties 1 and 2 are true for the initial clauses $\{p \mid p \in \Lambda_i\} \cup \{\neg p \mid p \in \Phi_i \cap P\}$ with corresponding sequents $\Lambda_i \Rightarrow \Phi_i$, because $M_i \models \bigwedge \Lambda_i$. So $K_w \models \Lambda_i$ for each $w \neq \rho$ since $K$ is a variant of $M_i$ and by monotonicity of $\leq$ in iL-models. Moreover, for $\Box q \in \Lambda_i$ we have $K_v \models q$ for all $v$ such that $\rho R v$. For all other corresponding sequents in the refutation it follows immediately from backwards applications of (AC).

Property 3 holds for initial sequents $\Lambda_i \Rightarrow \Phi_i$, because suppose $\Box q \in \Phi_i$. We know $M_i \not\models \Phi_i$, so there must be a $v$ such that $M_v \not\models q$. We assumed $\rho$ to be irreflexive, so $v \neq \rho$. Since $M_i$ is a variant of $K$ we have $K_v \not\models q$. Again, for all other corresponding sequents in the refutation it follows immediately from backwards applications of (AC).

For property 4 observe that (7.2) implies $\bullet((\Lambda_i \Rightarrow \Phi_i), \mathcal{G})$, because $(\Lambda_i)_{\Phi_i} = \emptyset$ (and note that $\Lambda_i \Rightarrow \Phi_i$ is indeed semi-modal-implication-irreducible by assumption). For the other corresponding sequents in the refutation we use Lemma 7.2.18 to prove the property.

Now we use all those facts for the empty clause $\emptyset$. There is a corresponding semi-modal-implication-irreducible sequent for the empty clause, $\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta$, where $\Gamma$ only contains variable implications or implications of the form $\Box p \to p$, sets $\Sigma$ and $\Omega$ only contain variables, and $\Delta$ only contains variables such that $\Delta \cap P = \emptyset$.

Gs-rule $\mathcal{G} \rhd \mathcal{H}$ is full with respect to ($\mathrm{V}^{\bullet,i,1}$), so we have at least one of the following:

(i) $(\Box\Sigma, \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$,
(ii) $(\Sigma, \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Omega$,
(iii) $(\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta) \in \mathcal{H}$ for some $\emptyset \neq \Pi \subseteq \Gamma_{\Box\Omega\cup\Delta}$.

We will show that the first two lead to a contradiction. For (i), we use the fact that $K_w \models C$ for all $w \neq \rho$, so $K_w \models I(\Box\Sigma, \Gamma \Rightarrow q)$ for all $w \neq \rho$. Since $K_w \models \bigwedge \Box\Sigma \wedge \bigwedge \Gamma$ by property 1, we have $K_w \models q$ for all $w \neq \rho$. But then $q \in P$. This is a contradiction, because $\Delta \cap P = \emptyset$.

For (ii), we also use the fact that $K_w \models C$ for all $w \neq \rho$, so $K_w \models I(\Sigma, \Gamma \Rightarrow q)$

for all $w \neq \rho$. Since $\rho$ is irreflexive, this also holds for all $v$ such that $\rho R v$. By property 2 we have that $K_v \models \bigwedge \Sigma \wedge \bigwedge \Gamma$ for all $v$ such that $\rho R v$. Hence for all these $v$'s, $K_v \models q$. But this contradicts property 3.

For case (iii), we use property 4 saying that $\bullet((\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta), \mathcal{G})$. So

$$\exists \Lambda' \subseteq \Pi \cup \Gamma^{\Pi} \cup \Box\Sigma,\ \exists \Phi' \subseteq \Box\Omega \cup \Delta \text{ such that } \Lambda' \Rightarrow \Phi' \in \mathcal{G}.$$

Clearly $\vdash_{\mathsf{iL}} (\Lambda' \Rightarrow \Phi') \rhd (\Box\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Box\Omega, \Delta)$ by intuitionistic reasoning using Kripke models (or via weakening in a multi-succedent sequent system if available). Since the left sequent is in $\mathcal{G}$ and the right sequent is in $\mathcal{H}$ we have $\vdash_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$. Now we apply Lemma 7.2.10 to conclude $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$.

**Case 2:**
Let $\mathsf{iL} \in \{\mathsf{mHC}, \mathsf{KM}\}$ and suppose that $\rho$ is irreflexive. The proof proceeds in a similar way as for Case 1. The only difference is that we replace property 1 with the following property:

1. $K_v \models \bigwedge \Sigma \wedge \bigwedge \Gamma$ for all $v \neq \rho$.

This property is shown using the fact that $< \subseteq R$ in mHC-models and KM-models. The rest of the proof proceeds the same using Visser rule ($\mathrm{V}^{\bullet,i,2}$) instead of ($\mathrm{V}^{\bullet,i,1}$).

**Case 3:**
Let $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{mHC}\}$ and suppose that $\rho$ is reflexive. Now define formulas for $i = 1, \ldots, k$ as follows, where $l_p$ are new introduced propositional variables:

$$A_i := \bigwedge_{p \in \Lambda_i} \neg l_p \wedge \bigwedge_{\Box p \in \Lambda_i} \neg l_p \wedge \bigwedge_{p \in P, p \in \Phi_i} l_p \wedge \bigwedge_{p \in P, \Box p \in \Phi_i} l_p \quad \text{and} \quad A := \bigvee_{i=1}^{k} A_i.$$

Note that $\mathsf{Var}(\Lambda_i) \subseteq P$, because $M_i \models \bigwedge \Lambda_i$ and so for variant $K$ we have $K_w \models p$ for all $w \neq \rho$ by monotonicity. In addition, if $\Box p \in \Lambda_i$, then $p \in P$ by reflexivity of the root $\rho$.

We show that $A$ is a classical tautology. If the conjuncts in $A_i$ are empty for some $i$, then $A$ is equivalent to $\top$. Otherwise, let $v$ be a classical valuation on the $l_p$'s for $p \in P$. This valuation corresponds to a variant $M$ of $K$ so that

$$M \models p \Leftrightarrow v(l_p) = 0.$$

Note that this correspondence is well-defined because $M_w \models p$ for each $w \neq \rho$ and $p \in P$. Let $M = M_i$. We have $M \models p$ for all variables $p \in \Lambda_i$ and $M \not\models p$ for all variables $p \in \Phi_i \cap P$. Since $\rho$ is reflexive we also have $M \models p$ for all $\Box p \in \Lambda_i$ and $M \not\models p$ for all $p$ such that $p \in P, \Box p \in \Phi_i$. Hence, for all $p \in P$, $v(l_p) = 0$ if $p \in \Lambda_i$ or $\Box p \in \Lambda_1$, and $v(l_p) = 1$ if $p \in \Phi_i$ or $\Box p \in \Phi_i$. Thus $v(A_i) = 1$, hence $A$ is a classical tautology. So $\neg A$ is classically inconsistent and by DeMorgan laws, $\neg A$ is classically equivalent to:

$$\neg A \equiv \bigwedge_{i=1}^{k} ( \bigvee_{p \in \Lambda_i} l_p \vee \bigvee_{\Box p \in \Lambda_i} l_p \vee \bigvee_{p \in \Phi_i \cap P} \neg l_p \vee \bigvee_{p \in P, \Box p \in \Phi_i} \neg l_p).$$

Therefore there exists a resolution refutation starting with the clauses

$$\{l_p \mid p \in \Lambda_i \text{ or } \Box p \in \Lambda_i\} \cup \{\neg l_p \mid p \in P, \text{ and } p \in \Phi_i \text{ or } \Box p \in \Phi_i\} \text{ for } i = 1, \ldots, k,$$

that ends in the empty clause $\emptyset$. In case the conjuncts of $A_i$ are empty for some $i$, the empty clause is already among the starting clauses. Each clause in the resolution refutation is of the form $\Theta' \cup \Psi'$ where $\Theta'$ contains only variables $l_p$ and $\Psi'$ contains only negated variables $\neg l_p$.

Now, the resolution refutation can be mimicked by applications of the rule ($\mathrm{AC}_\Box$), so that each class $\Theta' \cup \Psi'$ corresponds to a semi-modal-implication-irreducible sequent of the form $\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi$, where $\Gamma$ only contains variable implications, $\Sigma$ only variables in $P$, and $\Omega$, $\Delta$ and $\Sigma'$ variables not in $P$. And where $\Theta$ and $\Psi$ are sets of variables and boxed variables satisfying

$$\Theta = \bigcup_{l_p \in \Theta'} \Theta_p, \text{ with } \emptyset \neq \Theta_p \subseteq \{p, \Box p\}, \text{ and}$$
$$\Psi = \bigcup_{\neg l_p \in \Psi'} \Psi_p, \text{ with } \emptyset \neq \Psi_p \subseteq \{p, \Box p\}.$$

To see this, first note that the starting classes are of this form. Further, each resolution on $\Theta'_1 \cup \{l_p\} \cup \Psi'_1$ and $\Theta'_2 \cup \Psi'_2 \cup \{\neg l_p\}$ can be mimicked by ($\mathrm{AC}_\Box$) from sequents with non-empty sets $\Theta_p, \Psi_p \subseteq \{p, \Box p\}$,

$$\Box\Sigma_1 \to \Sigma_1, \Box\Sigma'_1 \to \Sigma'_1, \Gamma_1, \Theta_1, \Theta_p \Rightarrow \Box\Omega_1, \Delta_1, \Psi_1,$$
$$\Box\Sigma_2 \to \Sigma_2, \Box\Sigma'_2 \to \Sigma'_2, \Gamma_2, \Theta_2 \Rightarrow \Box\Omega_2, \Delta_2, \Psi_2, \Psi_p,$$

by a 'cut' on $p, \Box p$, resulting in sequent

$$\begin{array}{r} \Box p \to p, \Box\Sigma_1 \to \Sigma_1, \Box\Sigma_2 \to \Sigma_2, \\ \Box\Sigma'_1 \to \Sigma'_1, \Box\Sigma'_2 \to \Sigma'_2, \Gamma_1, \Gamma_2, \Theta_1, \Theta_2 \end{array} \Rightarrow \Box\Omega_1, \Box\Omega_2, \Delta_1, \Delta_2, \Psi_1, \Psi_2.$$

Moreover, since $\mathcal{G}$ is full with respect to ($\mathrm{AC}_\Box$), we know that all such sequents are in $\mathcal{G}$.

In addition we have the following properties for each clause $\Theta' \cup \Psi'$ in the refutation and its corresponding sequent $(\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi) \in \mathcal{G}$:

1. $K_v \models \bigwedge \Sigma \wedge \bigwedge(\Box\Sigma' \to \Sigma') \wedge \bigwedge \Gamma$ for all $v \neq \rho$,

2. $q \in \Sigma'$ implies $\Box q \in \Box\Omega$,
3. $\Box q \in \Box\Omega$ implies $K_v \not\models q$ for some $v \neq \rho$,
4. $\circ((\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \Box\Omega, \Delta, \Psi), \mathcal{G})$.

We show the properties inductively following the refutation. Property 1 holds for the initial clauses $\{l_p \mid p \in \Lambda_i \text{ or } \Box p \in \Lambda_i\} \cup \{\neg l_p \mid p \in P, \text{ and } p \in \Phi_i \text{ or } \Box p \in \Phi_i\}$ with corresponding sequents $\Lambda_i \Rightarrow \Phi_i$, because $M_i \models \bigwedge \Lambda_i$. So $K_v \models \bigwedge \Lambda_i$ for each $v \neq \rho$ since $K$ is a variant of $M_i$ and by monotonicity of $\leq$ in iL-models. Moreover, for $q \in P$ we have $K_v \models q$ for all $v \neq \rho$, so if $\Box q \to q \in \Lambda_i$ for $q \in P$, then $K_v \models q$ (corresponding to the $\Sigma$ of $\Lambda_i$). For all other corresponding sequents in the refutation it follows immediately from backwards applications of $(\mathrm{AC}_\Box)$. In particular, if the 'cut' is applied to subsets of $\{p, \Box p\}$, we have $p \in P$ so that $\Box p \to p \in \Box\Sigma \to \Sigma$ and $K_v \models p$ for all $v \neq \rho$. In addition, it shows that each $\Box q \to q \in \Box\Sigma' \to \Sigma'$ was already present in the sequents to which the 'cut' was applied, because we do not cut on $\Box q, q$ when $q \notin P$. So indeed the property holds.

Property 2 holds for initial sequents $\Lambda_i \Rightarrow \Phi_i$ because of assumption (7.3). For the other corresponding sequents it follows from the fact that we do not cut on $\Box q, q$ when $q \notin P$.

Property 3 follows immediately from the fact that $q \notin P$.

For property 4 observe that (7.2) implies $\circ((\Lambda_i \Rightarrow \Phi_i), \mathcal{G})$, because $(\Lambda_i)^a_{\Phi_i} = \emptyset$ (and note that $\Lambda_i \Rightarrow \Phi_i$ is indeed semi-modal-implication-irreducible by assumption). For the other corresponding sequents in the refutation we use Lemma 7.2.20 to prove the property.

Now we use all those facts for the empty clause $\emptyset$. There is a corresponding sequent for the empty clause, $\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow \Box\Omega, \Delta$, where $\Gamma$ only contains variable implications, $\Sigma$ only variables in $P$, and $\Omega$, $\Delta$ and $\Sigma'$ variables not in $P$.

Gs-rule $\mathcal{G} \rhd \mathcal{H}$ is full with respect to $(\mathrm{V}^{\circ,i})$, so we have at least one of the following:

(i) $(\Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$,
(ii) $(\Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Omega$,
(iii) $(\Box\Sigma \to \Sigma, (\Box\Sigma' \to \Sigma')^\Pi, \Gamma^\Pi, \Pi \Rightarrow \Box\Omega, \Delta) \in \mathcal{H}$ for some non-empty set $\Pi \subseteq (\Gamma \cup (\Box\Sigma' \to \Sigma'))_{\Box\Omega\cup\Delta}$.

We will show that the first two lead to a contradiction. For (i), we use the fact that $K_v \models C$ for all $v \neq \rho$, so $K_v \models I(\Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow q)$ for all $v \neq \rho$. Since $K_v \models \bigwedge \Sigma \wedge \Box\Sigma' \to \Sigma' \wedge \bigwedge \Gamma$ by property 1, we have $K_v \models q$ for all $v \neq \rho$. But then $q \in P$. This is a contradiction, because $\Delta \cap P = \emptyset$.

For case (ii), we also use the fact that $K_v \models C$ for all $v \neq \rho$, so we have that

$K_v \models I(\Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow q)$ for all $v \neq \rho$. By property 1 it follows that $K_v \models \bigwedge \Sigma \wedge \Box\Sigma' \to \Sigma' \wedge \bigwedge \Gamma$ for all $v \neq \rho$, hence $K_v \models q$ for all $v \neq \rho$. But this contradicts property 3.

For case (iii), we use property 4 saying that $\circ((\Box\Sigma \to \Sigma, \Box\Sigma' \to \Sigma', \Gamma \Rightarrow \Box\Omega, \Delta), \mathcal{G})$. Note that $\Pi$ is a subset of $(\Gamma \cup (\Box\Sigma' \to \Sigma'))_{\Box\Omega\cup\Delta}$. By property 2 we know that $\Pi$ does not contain boxed variables, and so $\Pi \subseteq (\Gamma \cup (\Box\Sigma' \to \Sigma'))^a_{\Box\Omega\cup\Delta}$ and $(\Box\Sigma' \to \Sigma')^\Pi = \Box\Sigma' \to \Sigma'$. So

$$\exists\Lambda' \subseteq \Pi \cup \Box\Pi \cup \Gamma^\Pi \cup (\Box\Sigma \to \Sigma) \cup (\Box\Sigma' \to \Sigma'),\ \exists\Phi' \subseteq \Box\Omega \cup \Delta \text{ s.t. } \Lambda' \Rightarrow \Phi' \in \mathcal{G}.$$

We have $\vdash_{\mathsf{iL}} (\Lambda' \Rightarrow \Phi') \rhd (\Box\Sigma \to \Sigma, (\Box\Sigma' \to \Sigma')^\Pi, \Gamma^\Pi, \boxdot\Pi \Rightarrow \Box\Omega, \Delta)$ by intuitionistic reasoning (i.e., using Kripke models). Recall that $\boxdot A \equiv A$ for all formulas $A$ in intuitionistic modal logics with coreflection. So we can conclude that $\vdash_{\mathsf{iL}} (\Lambda' \Rightarrow \Phi') \rhd (\Box\Sigma \to \Sigma, (\Box\Sigma' \to \Sigma')^\Pi, \Gamma^\Pi, \Pi \Rightarrow \Box\Omega, \Delta)$. Since the left sequent is in $\mathcal{G}$ and the right sequent is in $\mathcal{H}$ we have $\vdash_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$. Now we apply Lemma 7.2.10 to conclude $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$.

This finishes the proof of Theorem 7.2.9. ■

The proof system for admissibility immediately gives rise to the decidability of admissibility.

**7.2.21 Corollary**
Admissibility is decidable for logics $\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{KM}$, and $\mathsf{mHC}$.

Proof. We obtain a terminating procedure to decide the admissibility of a gs-rule $\mathcal{G}\rhd\mathcal{H}$. First apply the left logical rules to reach a modal-implication-irreducible gs-rule (Lemma 7.2.11). After that apply the logic specific rules from the set $X := \mathsf{GAiL} \cap \{(\to)\rhd, (\mathrm{AC}), (\mathrm{AC}_\Box), (\mathrm{V}^{\bullet,i,1}), (\mathrm{V}^{\bullet,i,2}), (\mathrm{V}^{\circ,i})\}$ to obtain leaves $\mathcal{G} \rhd \mathcal{H}$ of the proof search tree that are full with respect to $X$. Similarly to the proof of Theorem 7.2.9, for each such a leaf $\mathcal{G} \rhd \mathcal{H}$ there are three cases: $C := \bigwedge I(\mathcal{G})$ is inconsistent, consistent and projective, or consistent and not projective. By inspection of the proof, in all cases admissibility $\mid\!\sim_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$ reduces to derivability $\vdash_{\mathsf{iL}} \mathcal{G} \rhd \mathcal{H}$ which is decidable for our logics with coreflection $\mathsf{iL}$ as discussed in Section 1.3. ■

# 7.3 Admissibility proof system for PLL

This section is devoted to the admissibility proof system for PLL, where we follow the same line of research from the previous section. Recall Section 1.3.3 with an introduction to this logic. One might wonder why we treat PLL in a separate section. The reason is that we cannot work with the dense strong intuitionistic models from

Theorem 1.3.18. In particular, we cannot apply the reasoning used in the proof of soundness of Theorem 7.2.7. There, we relied on extensions of models (having close connections to the extendible classical modal logics from Definition 6.1.21). Indeed, dense strong intuitionistic modal logics are not extendible shown by the following example.

**7.3.1 Example**
Consider the dense strong frame depicted on the left.

The irreflexive extension depicted on the right is not dense and, thus, does not represent a frame for PLL. In particular, it can be shown that axiom $\bigcirc\bigcirc p \to \bigcirc p$ is invalid in the root by assigning $p$ to the top world in the model.

There is a way out by using another semantics for PLL. Here we will work with finite rooted FM-models defined in Definition 1.3.19. Applying Remark 1.2.14, we call these PLL-models throughout this section. Recall the end of Section 6.2.2 for the discussion about projective formulas in PLL and the extension property for FM-models.

$$\frac{\begin{array}{c}[\mathcal{G}, (\bigcirc\Sigma, \Gamma \Rightarrow \Delta), (\bigcirc\Sigma, \Gamma \Rightarrow D) \triangleright \mathcal{H}]_{D \in \Delta} \\ [\mathcal{G}, (\bigcirc\Sigma, \Gamma \Rightarrow \Delta) \triangleright (\bigcirc\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \Delta), \mathcal{H}]_{\emptyset \neq \Pi \subseteq \Gamma_{\Delta}}\end{array}}{\mathcal{G}, (\bigcirc\Sigma, \Gamma \Rightarrow \Delta) \triangleright \mathcal{H}} \; (\mathrm{V}^{\leq, R})$$

$$\frac{\begin{array}{c}[\mathcal{G}, (\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta), (\Sigma, \Gamma \Rightarrow D) \triangleright \mathcal{H}]_{D \in \Delta} \\ [\mathcal{G}, (\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta), (\Sigma, \Gamma \Rightarrow O) \triangleright \mathcal{H}]_{O \in \Omega} \\ [\mathcal{G}, (\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta) \triangleright (\bigcirc\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta), \mathcal{H}]_{\emptyset \neq \Pi \subseteq \Gamma_{\bigcirc\Omega, \Delta}}\end{array}}{\mathcal{G}, (\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta) \triangleright \mathcal{H}} \; (\mathrm{V}^{\leq})$$

where for all these rules it holds that $\Gamma$ contains only implications,
$\Gamma^{\Pi} := \{A \to B \in \Gamma \mid A \notin \Pi\}$ and $\Gamma_{\bigcirc\Omega, \Delta} := \{A \notin \bigcirc\Omega \cup \Delta \mid \exists B(A \to B \in \Gamma)\}$

**Figure 7.5.** Sequent Visser rules for PLL

**7.3.2 Definition**
We define the proof system for admissibility for PLL, written GAPLL, to be the set of inference rules containing the rules from Figure 7.3, where each $\Box$ is replaced by $\bigcirc$, and the modal Visser rules ($V^{\leq,R}$) and ($V^{\leq}$) from Figure 7.5.

Such as for the other logics, the Visser rules reflect extensions in PLL-models. Rule ($V^{\leq,R}$) reflects extensions with at least one $R$ relation, whereas ($V^{\leq}$) reflects extensions with no $R$ relations. Note that ($V^{\leq}$) is identical to ($V^{\circ,i}$) from Figure 7.4 and can correspond to the reflexive extensions in dense strong models for PLL.

Note that the weakening rules

$$\frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}}\ (\mathrm{W})\rhd \qquad\qquad \frac{\mathcal{G} \rhd \mathcal{H}}{\mathcal{G} \rhd S, \mathcal{H}}\ \rhd(\mathrm{W})$$

are also admissible in GAPLL.

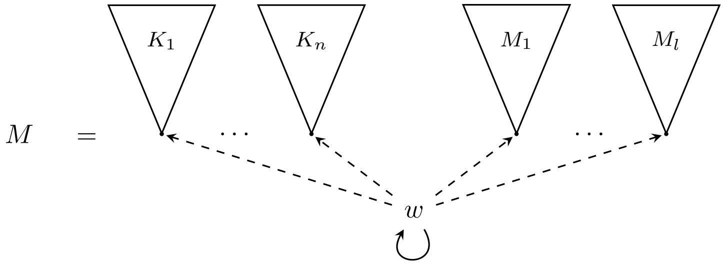
**7.3.3 Theorem**
If $\vdash_{\mathsf{GAPLL}} \mathcal{G} \rhd \mathcal{H}$, then $\mathrel{|\!\sim}_{\mathsf{PLL}} \mathcal{G} \rhd \mathcal{H}$.

Proof. Similar to the proof of Theorem 7.2.7. Here we consider the rules ($V^{\leq,R}$) and ($V^{\leq}$).

For rule ($V^{\leq}$), suppose that $\sigma$ is a unifier for $I(S)$ for all $S \in \mathcal{G}$ and also for $I(\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta)$. We write $\Delta = \{D_1, \ldots, D_n\}$ and $\Omega = \{O_1, \ldots, O_l\}$ (including the cases where the sets are empty). Using the third set of premises, we have for all $\emptyset \neq \Pi \subseteq \Gamma_{\bigcirc\Omega,\Delta}$ that $\sigma$ is either a unifier for some $S \in \mathcal{H}$ or for $I(\bigcirc\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta)$. If there is such a $\Pi$ for which the first case holds we are done. If for all such $\Pi$ we have the second case (or in case there is no such $\Pi$ at all), we will show that $\sigma$ is a unifier for $I(\Sigma, \Gamma \Rightarrow D_i)$ for some $i$ or for $I(\Sigma, \Gamma \Rightarrow O_j)$ for some $j$. This is sufficient, because it implies that $\sigma$ is a unifier for some $S \in \mathcal{H}$ by the first or second set of premises of ($V^{\leq}$). Suppose for a contradiction that this is not the case. Then there exist PLL-countermodels $K_1, \ldots, K_n$ and $M_1, \ldots M_l$ such that

$$K_i \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } K_i \not\models \sigma(D_i),$$
$$M_j \models \sigma(\bigwedge \Sigma \wedge \bigwedge \Gamma) \text{ and } M_j \not\models \sigma(O_j).$$

Consider the following PLL-model $M$ with reflexive root $w$. ($M$ is a one-world model if $\Delta = \Omega = \emptyset$.) Relation $\leq$ is drawn by dashed lines, and $R$ by a straight line. We take $R$ to be reflexive and transitive by definition of PLL-models. Root $w$ is not a fallible world in the model.

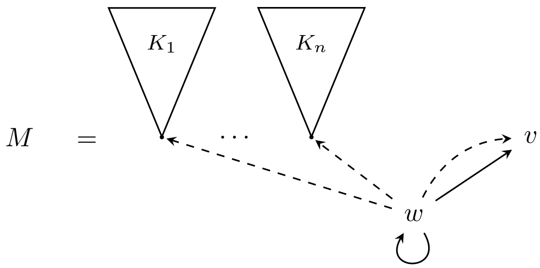


First note that $M \not\models \sigma(D_i)$ for all $i$. We also have $M \not\models \sigma(\bigcirc O_j)$ for all $j$, because for all worlds $v$ such that $wRv$ we have $v \not\models \sigma(O_j)$ (indeed $v$ can only be $w$). Also note that $M \models \sigma(\bigwedge(\bigcirc\Sigma \to \Sigma))$ because the only world that is modal related to $w$ is $w$ itself. Let $\Pi := \{A \in \Gamma_{\bigcirc\Omega,\Delta} \mid M \models \sigma(A)\}$. Thus $M \models \sigma(\bigwedge \Pi)$. We also claim that $M \models \sigma(\bigwedge \Gamma^{\Pi})$. Let $A \to B \in \Gamma^{\Pi}$. Observe that either $A \in \Delta \cup \bigcirc\Omega$ or $M \not\models \sigma(A)$. We already saw that the first implies the second, so $M \not\models \sigma(A)$. And since $K_i \models \sigma(A \to B)$ for all $i$ and $M_j \models \sigma(A \to B)$ for all $j$, we have $M \models \sigma(A \to B)$. So far we have shown that $M \models \sigma(\bigwedge(\bigcirc\Sigma \to \Sigma \cup \Gamma^{\Pi} \cup \Pi))$. In case $\Pi = \emptyset$, we have that $\Gamma^{\Pi} = \Gamma$ and so $M \models \sigma(\bigwedge(\bigcirc\Sigma \to \Sigma \cup \Gamma))$. But $\sigma$ is a unifier for $I(\bigcirc\Sigma \to \Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta)$. In the case that $\Pi \neq \emptyset$, we have that $\sigma$ is a unifier for $I(\bigcirc\Sigma \to \Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta)$ by assumption. In both cases we have $M \models \sigma(\bigvee(\bigcirc\Omega \cup \Delta))$, which is a contradiction with our first observation about model $M$.

For $(\mathrm{V}^{\leq,R})$, suppose that $\sigma$ is a unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\bigcirc\Sigma, \Gamma \Rightarrow \Delta)$. Again, write $\Delta = \{D_1, \ldots, D_n\}$. By a similar argument as above, it is sufficient to show that $\sigma$ is a unifier for $I(\bigcirc\Sigma, \Gamma \Rightarrow D_i)$ for some $i$. Suppose for a contradiction that this is not the case. Then there exist PLL-countermodels $K_1, \ldots, K_n$ such that

$$K_i \models \sigma(\bigwedge \bigcirc\Sigma \wedge \bigwedge \Gamma) \text{ and } K_i \not\models \sigma(D_i).$$

Consider the following PLL-model $M$ with reflexive root $w$ and $v \in F$ is a fallible world. Again, relation $\leq$ is drawn by dashed lines, and $R$ by a straight line. We take $R$ to be reflexive and transitive by definition of PLL-models. Strictly speaking, the fallible world $v$ is only necessary in case $\Delta = \emptyset$, otherwise one $R$-relation to one of $K_i$ will suffice.



First note that $M \not\models \sigma(D_i)$ for all $i$. Also note that $M \models \sigma(\bigwedge(\bigcirc\Sigma))$ because $K_i \models \sigma(\bigwedge(\bigcirc\Sigma))$ for all i, and for $w$ we have fallible world $v$ such that $wRv$ and $v \models \bigwedge\Sigma$ (note that this also holds for $\Delta = \emptyset$). Let $\Pi = \{A \in \Gamma_\Delta \mid M \models \sigma(A)\}$. Thus $M \models \sigma(\bigwedge\Pi)$. Again, $M \models \sigma(\bigwedge\Gamma^\Pi)$, so $M \models \sigma(\bigwedge(\bigcirc\Sigma \cup \Gamma^\Pi \cup \Pi))$. If $\Pi = \emptyset$, then $\Gamma^\Pi = \Gamma$ and so $M \models \sigma(\bigwedge(\bigcirc\Sigma \cup \Gamma))$. But $\sigma$ is a unifier for $I(\bigcirc\Sigma, \Gamma \Rightarrow \Delta)$. If $\Pi \neq \emptyset$, then $\sigma$ is a unifier for $I(\bigcirc\Sigma, \Gamma^\Pi, \Pi \Rightarrow \Delta)$ by assumption. In both cases we have $M \models \sigma(\bigvee(\Delta))$, which is a contradiction with our first observation about model $M$. ■

All results from Lemma 7.2.10 to Lemma 7.2.20 are similarly proved for PLL. So all rules in GAPLL are invertible and each admissible gs-rule is derivable in GAPLL from admissible semi-modal-implication-irreducible gs-rules that are full with respect to $\{(\to)\rhd, (\mathrm{AC}), (\mathrm{AC}_\bigcirc), (\mathrm{V}^{\leq,R}), (\mathrm{V}^{\leq})\}$. This enables us to show completeness.

**7.3.4 Theorem**
If $\mathrel{|\!\sim}_{\mathsf{PLL}} \mathcal{G} \rhd \mathcal{H}$, then $\vdash_{\mathsf{GAPLL}} \mathcal{G} \rhd \mathcal{H}$.

Proof. Same strategy as the proof of Theorem 7.2.9. Suppose $\mathrel{|\!\sim}_{\mathsf{PLL}} \mathcal{G} \rhd \mathcal{H}$. It is sufficient to assume that $\mathcal{G} \rhd \mathcal{H}$ is a semi-modal-implication-irreducible gs-rule that is full with respect to $(\to)\rhd$, $(\mathrm{V}^{\leq,R})$, $(\mathrm{V}^{\leq})$, (AC) and $(\mathrm{AC}_\bigcirc)$.

Define formula $C := \bigwedge I(\mathcal{G})$. We consider three cases. The two cases for which $C$ is inconsistent, or consistent and projective, are shown similarly to the proof for Theorem 7.2.9.

So suppose that $C$ is consistent and not projective. By Theorem 6.2.43, there is a PLL-model $K$ with root $\rho$ such that $K_w \models C$ for each $w \neq \rho$ and for each variant $K'$ of $K$ we have $K' \not\models C$. Since formula $C$ is consistent it holds in at least one non-fallible world of some PLL-model. Therefore, it holds in a model in which all the nodes are fallible worlds except for the root. For model $K$ this means that there exists at least one non-fallible world $w \neq \rho$ in $K$.

Let $M_1, \ldots M_k$ be all the variants of $K$ that only force propositional variables among the variables occurring in $\mathcal{G}$ and $\mathcal{H}$. So $M_i \not\models C$ for all $i$, hence $M_i \not\models I(S_i)$ for some $S_i \in \mathcal{G}$. Write $S_i = (\Lambda_i \Rightarrow \Phi_i)$, then $M_i \not\models I(\Lambda_i \Rightarrow \Phi_i)$. Since $M_i, w \Vdash I(\Lambda_i \Rightarrow \Phi_i)$ for each $w \neq \rho$, we have $M_i \models \bigwedge\Lambda_i$ and $M_i \not\models \bigvee\Phi_i$. Such as for Theorem 7.2.9, since $\mathcal{G} \rhd \mathcal{H}$ is full with respect to $(\to)\rhd$, we can assume that

$$(p \to q) \in \Lambda_i \quad \Longrightarrow p \in \Phi_i. \tag{7.4}$$

$$(\bigcirc p \to p) \in \Lambda_i \quad \Longrightarrow \bigcirc p \in \Phi_i. \tag{7.5}$$

Define the set of propositional variables:

$$P := \{p \mid p \text{ occurs in } C \text{ and } K_w \models p \text{ for all } w \neq \rho\}.$$

We consider two possibilities for the frame of model $K$ (note that each $M_i$ has the same frame): there is some world $w \neq \rho$ such that $\rho R w$ or there is not. In the first case we use the fact that $\mathcal{G} \triangleright \mathcal{H}$ is full with respect to ($\mathrm{V}^{\leq,R}$) and (AC). For the second case we use fullness with respect to ($\mathrm{V}^{\leq}$) and ($\mathrm{AC}_{\bigcirc}$). Case 1 can be compared to the completeness proof for logics with rule ($\mathrm{V}^{\bullet,i,1}$) and Case 2 to logics with rule ($\mathrm{V}^{\circ,i}$).

**Case 1:**
Define formulas for $i = 1, \ldots, k$ as follows:

$$A_i := \bigwedge_{p \in \Lambda_i} \neg p \wedge \bigwedge_{p \in \Phi_i \cap P} p \quad \text{and} \quad A := \bigvee_{i=1}^{k} A_i.$$

Note again that $p \in \Lambda_i$ implies $p \in P$. Using the same proof for Case 1 in Theorem 7.2.9, we have that $A$ is a classical tautology and that

$$\neg A \equiv \bigwedge_{i=1}^{k} ( \bigvee_{p \in \Lambda_i} p \vee \bigvee_{p \in \Phi_i \cap P} \neg p)$$

is classically inconsistent. Therefore there exists a resolution refutation starting with the clauses

$$\{p \mid p \in \Lambda_i\} \cup \{\neg p \mid p \in \Phi_i \cap P\} \text{ for } i = 1, \ldots, k,$$

that ends in the empty clause $\emptyset$. Again, the resolution refutation can be mimicked by rule (AC), where each clause $\Theta \cup \Psi'$ corresponds to a semi-modal-implication-irreducible sequent $\bigcirc\Sigma, \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi \in \mathcal{G}$, where $\Gamma$ only contains variable implications or implications of the form $\bigcirc p \to p$, sets $\Sigma$ and $\Omega$ only contain variables, and $\Delta$ only contains variables with $\Delta \cap P = \emptyset$ (and $\Theta$ and $\Psi$ as in Theorem 7.2.9). We have the following properties for each clause $\Theta \cup \Psi'$:

1. $K_v \models \bigwedge \bigcirc\Sigma \wedge \bigwedge \Gamma$ for all $v \neq \rho$,
2. $\bigcirc q \in \bigcirc\Omega$ implies $K_v \not\models \bigcirc q$ for some $v \neq \rho$,
3. $\bullet((\bigcirc\Sigma, \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi), \mathcal{G})$.

Here we only present the proof for property 2. It holds for initial sequents $\Lambda_i \Rightarrow \Phi_i$, because suppose $\bigcirc q \in \Phi_i$. We know $M_i \not\models \Phi_i$, so there must be a $w \geq \rho$ such that for all $v$ with $wRv$ we have $M_v \not\models q$. If $w \neq \rho$ we are done, since $K$ is a variant of $M_i$. Suppose $w = \rho$, then for all $v$ with $\rho R v$ we have $M_v \not\models q$. We assumed that there is at least one $v \neq \rho$ so that $\rho R v$. For this particular $v$ we know that if $vRy$ then $\rho R y$ by transitivity of relation $R$. Hence $M_v \not\models \bigcirc q$. For all other corresponding sequents in the refutation it follows immediately from backwards applications of (AC), since we do not 'cut' on boxed formulas.

Now we use all those facts for the empty clause $\emptyset$. There is a corresponding sequent for the empty clause, $\bigcirc\Sigma, \Gamma \Rightarrow \bigcirc\Omega, \Delta$, where $\Gamma$ contains variable implications or implications of the form $\bigcirc p \to p$, $\Sigma$ and $\Omega$ only contain propositional variables, and $\Delta$ variables not in $P$.

Gs-rule $\mathcal{G} \rhd \mathcal{H}$ is full with respect to $(\mathrm{V}^{\leq,R})$, so we have at least one of the following:

(i) $(\bigcirc\Sigma, \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$,
(ii) $(\bigcirc\Sigma, \Gamma \Rightarrow \bigcirc q) \in \mathcal{G}$ for some $q \in \Omega$,
(iii) $(\bigcirc\Sigma, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta) \in \mathcal{H}$ for some $\emptyset \neq \Pi \subseteq \Gamma_{\bigcirc\Omega\cup\Delta}$.

The proof proceeds analogous to Case 1 in Theorem 7.2.9, showing that (i) and (ii) lead to a contradiction and that case (iii) implies $\vdash_{\mathsf{GAPLL}} \mathcal{G} \rhd \mathcal{H}$.

**Case 2:**
Now we define formulas for $i = 1, \ldots, k$ as follows, where $l_p$ are new introduced propositional variables:

$$A_i := \bigwedge_{p\in\Lambda_i} \neg l_p \wedge \bigwedge_{\bigcirc p\in\Lambda_i} \neg l_p \wedge \bigwedge_{p\in P, p\in\Phi_i} l_p \wedge \bigwedge_{p\in P, \bigcirc p\in\Phi_i} l_p \quad \text{and} \quad A := \bigvee_{i=1}^{k} A_i.$$

Note that $p \in \Lambda_i$ implies $p \in P$, because $M_i \models \bigwedge\Lambda_i$ and so for variant $K$ we have $K_w \models p$ for all $w \neq \rho$ by monotonicity. Also note that $\bigcirc p \in \Lambda_i$ implies $p \in P$. This follows from the form of model $M_i$ as follows. $M_i \models \bigcirc p$, which means that for all $w \geq \rho$ there exists a $v$ such that $wRv$ and $M_i, v \models p$. For $w = \rho$ we have that $\rho$ itself is the only world $v$ such that $\rho R v$, hence $M_i \models p$ and so $p \in P$. Using a similar argument we have $M_i \not\models p$ for all $p$ such that $p \in P$ and $\bigcirc p \in \Phi_i$.

Again, $A$ is a classical tautology. So

$$\neg A \equiv \bigwedge_{i=1}^{k}\Big( \bigvee_{p\in\Lambda_i} l_p \vee \bigvee_{\bigcirc p\in\Lambda_i} l_p \vee \bigvee_{p\in\Phi_i\cap P} \neg l_p \vee \bigvee_{p\in P, \bigcirc p\in\Phi_i} \neg l_p\Big)$$

is classically inconsistent. Therefore there exists a resolution refutation starting with the clauses

$$\{l_p \mid p \in \Lambda_i \text{ or } \bigcirc p \in \Lambda_i\} \cup \{\neg l_p \mid p \in P, \text{ and } p \in \Phi_i \text{ or } \bigcirc p \in \Phi_i\} \text{ for } i = 1, \ldots, k,$$

that ends in the empty clause $\emptyset$. Let us define $\Theta', \Psi', \Theta, \Psi$ as in Case 3 of Theorem 7.2.9. The resolution refutation can be mimicked by $(\mathrm{AC}_{\bigcirc})$ where each clause $\Theta' \cup \Psi'$ corresponds to a semi-modal-implication-irreducible sequent of the form $\bigcirc\Sigma \to \Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi \in \mathcal{G}$ where $\Gamma$ only contains variable implications, $\Sigma$ only variables in $P$, and $\Omega$, $\Delta$ and $\Sigma'$ variables not in $P$. We have the following properties for each clause $\Theta' \cup \Psi'$ and its corresponding sequent $\bigcirc\Sigma \to \Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi \in \mathcal{G}$:

1. $K_v \models \bigwedge\Sigma \wedge \bigcirc\Sigma' \to \Sigma' \wedge \bigwedge\Gamma$ for all $v \neq \rho$,

2. $q \in \Sigma'$ implies $\bigcirc q \in \bigcirc\Omega$,
3. $\circ((\bigcirc\Sigma \to \Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma, \Theta \Rightarrow \bigcirc\Omega, \Delta, \Psi), \mathcal{G})$.

The properties are similarly shown as in Case 3 of Theorem 7.2.9. Gs-rule $\mathcal{G} \rhd \mathcal{H}$ is full with respect to ($\mathrm{V}^{\leq}$), so we have at least one of the following:

(i) $(\Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$,
(ii) $(\Sigma, \bigcirc\Sigma' \to \Sigma', \Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Omega$,
(iii) $(\bigcirc\Sigma \to \Sigma, (\bigcirc\Sigma' \to \Sigma')^{\Pi}, \Gamma^{\Pi}, \Pi \Rightarrow \bigcirc\Omega, \Delta) \in \mathcal{H}$ for some $\emptyset \neq \Pi \subseteq (\Gamma \cup (\bigcirc\Sigma' \to \Sigma'))_{\bigcirc\Omega\cup\Delta}$.

The proof proceeds analogously to Case 3 in Theorem 7.2.9, showing that (i) and (ii) lead to a contradiction and that case (iii) implies $\vdash_{\mathsf{GAPLL}} \mathcal{G} \rhd \mathcal{H}$. ■

**7.3.5 Corollary**
Admissibility is decidable for PLL.

## 7.4 Bases

For all logics iL among iCK4, iCS4, iSL, mHC, KM, and PLL, recall the definitions of $\mathsf{V}_{\mathsf{iL}}$ defined in Definition 7.1.1 using the intuitionistic modal Visser rules from Figures 7.4 and 7.5. We use the proof systems for admissibility from the previous section to show that these form a basis for the admissible rules for the corresponding logic iL. Recall the definition of a basis from Definition 5.3.1.

The work presented here is not covered (Iemhoff and Metcalfe, 2009b), because the bases for the logics discussed there were already known in the literature (Iemhoff, 2001a; Jeřábek, 2005).

**7.4.1 Theorem**
Let $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{mHC}, \mathsf{KM}, \mathsf{PLL}\}$, then $\mathsf{V}_{\mathsf{iL}}$ forms a basis for the multi-conclusion admissible rules of iL.

Proof. We have to show that $\mathrel{|\!\sim}_{\mathsf{iL}} = \vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}}$. It is easy to prove that the Visser rules from $\mathsf{V}_{\mathsf{iL}}$ are admissible in iL using a similar strategy as the proof of Theorem 7.2.7. Therefore, for all rules $\Gamma/\Delta$, $\Gamma \vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}} \Delta$ implies $\Gamma \mathrel{|\!\sim}_{\mathsf{iL}} \Delta$. For the other direction we prove the following theorem based on the sequent system for admissibility. Note that Theorem 7.2.9 implies the desired result. ■

**7.4.2 Theorem**
If $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$ then $I(\mathcal{G}) \vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}} I(\mathcal{H})$.

Proof. We proceed by induction on the height of the derivation of $\vdash_{\mathsf{GAiL}} \mathcal{G} \rhd \mathcal{H}$.

Almost all rules can be easily checked using intuitionistic reasoning from $\mathsf{iL}$. Let us check $(\Rightarrow\wedge)\rhd$, $(\Box\Rightarrow)\rhd$, and (PJ). After that we discuss the difficult cases of the modal Visser rules.

For $(\Rightarrow\wedge)\rhd$ we have

$$\frac{\mathcal{G},(\Gamma\Rightarrow A,\Delta),(\Gamma\Rightarrow B,\Delta)\rhd\mathcal{H}}{\mathcal{G},(\Gamma\Rightarrow A\wedge B,\Delta)\rhd\mathcal{H}}\,(\Rightarrow\wedge)\rhd\;.$$

By the induction hypothesis, $I(\mathcal{G}),\bigwedge\Gamma\to A\vee\bigvee\Delta,\bigwedge\Gamma\to B\vee\bigvee\Delta\ \vdash^{\mathsf{V_{iL}}}_{\mathsf{iL}} I(\mathcal{H})$. By intuitionistic reasoning we have $\bigwedge\Gamma\to(A\wedge B)\vee\bigvee\Delta\vdash_{\mathsf{iL}}\bigwedge\Gamma\to A\vee\bigvee\Delta$, and similarly with formula $B$. Using transitivity of the consequence relation $\vdash^{\mathsf{V_{iL}}}_{\mathsf{iL}}$ we conclude that $I(\mathcal{G}),I(\Gamma\Rightarrow A\wedge B,\Delta)\vdash^{\mathsf{V_{iL}}}_{\mathsf{iL}} I(\mathcal{H})$.

For $(\Box\Rightarrow)\rhd$ consider

$$\frac{\mathcal{G},(\Gamma,\Box p\Rightarrow\Delta),(A\Rightarrow p)\rhd\mathcal{H}}{\mathcal{G},(\Gamma,\Box A\Rightarrow\Delta)\rhd\mathcal{H}}\,(\Box\Rightarrow)\rhd\;,$$

where $p$ does not occur in any sequent of the conclusion. By induction hypothesis, $I(\mathcal{G}),\bigwedge\Gamma\wedge\Box p\to\bigvee\Delta,A\to p\vdash^{\mathsf{V_{iL}}}_{\mathsf{iL}} I(\mathcal{H})$. Since $p$ is not present in $\mathcal{G},\mathcal{H},\Gamma,\Delta,A$, we can substitute $A$ for $p$ and use the structurality of the consequence relation to show that $I(\mathcal{G}),\bigwedge\Gamma\wedge\Box A\to\bigvee\Delta,A\to A\vdash^{\mathsf{V_{iL}}}_{\mathsf{iL}} I(\mathcal{H})$. Of course, formula $A\to A$ is valid, and therefore $I(\mathcal{G}),I(\Gamma,\Box A\Rightarrow\Delta)\vdash^{\mathsf{V_{iL}}}_{\mathsf{iL}} I(\mathcal{H})$.

The rule (PJ) is

$$\frac{\mathcal{G},S\rhd(\Gamma,I(S)\Rightarrow\Delta),\mathcal{H}}{\mathcal{G},S\rhd\mathcal{H}}\,(\mathrm{PJ})$$

where $(\Gamma\Rightarrow\Delta)\in\mathcal{H}\cup\{\Rightarrow\}$. By induction hypothesis of the premise we have that $I(\mathcal{G}),I(S)\vdash^{\mathsf{V_{iL}}}_{\mathsf{iL}}\bigwedge\Gamma\wedge I(S)\to\bigvee\Delta,I(\mathcal{H})$. By intuitionistic reasoning it holds that $I(S),\bigwedge\Gamma\wedge I(S)\to\bigvee\Delta\vdash_{\mathsf{iL}}\bigwedge\Gamma\to\bigvee\Delta$. Together with transitivity we obtain $I(\mathcal{G}),I(S)\vdash^{\mathsf{V_{iL}}}_{\mathsf{iL}}\bigwedge\Gamma\to\bigvee\Delta,I(\mathcal{H})$. Since $(\Gamma\Rightarrow\Delta)\in\mathcal{H}\cup\{\Rightarrow\}$ we conclude $I(\mathcal{G}),I(S)\vdash^{\mathsf{V_{iL}}}_{\mathsf{iL}} I(\mathcal{H})$.

Now we turn to the Visser rules in $\mathsf{GAiL}$. We treat rule $(\mathrm{V}^{\bullet,i,1})$. The other Visser rules can be handled in the same way and are left to the reader. So consider rule $(\mathrm{V}^{\bullet,i,1})$ and let $\mathsf{iL}=\{\mathsf{iCK4},\mathsf{iSL}\}$.

$$\frac{\begin{array}{c}[\mathcal{G},(\Box\Sigma,\Gamma\Rightarrow\Box\Omega,\Delta),(\Box\Sigma,\Gamma\Rightarrow D)\rhd\mathcal{H}]_{D\in\Delta}\\ [\mathcal{G},(\Box\Sigma,\Gamma\Rightarrow\Box\Omega,\Delta),(\Sigma,\Gamma\Rightarrow O)\rhd\mathcal{H}]_{O\in\Omega}\\ [\mathcal{G},(\Box\Sigma,\Gamma\Rightarrow\Box\Omega,\Delta)\rhd(\Box\Sigma,\Gamma^{\Pi},\Pi\Rightarrow\Box\Omega,\Delta),\mathcal{H}]_{\emptyset\neq\Pi\subseteq\Gamma_{\Box\Omega,\Delta}}\end{array}}{\mathcal{G},(\Box\Sigma,\Gamma\Rightarrow\Box\Omega,\Delta)\rhd\mathcal{H}}\,(\mathrm{V}^{\bullet,i,1})$$

Define sets of formulas

$$\Theta_\Pi := \{D \to (A \to B) \mid A \in \Pi, A \to B \in \Gamma, D \in \Delta\}, \quad \Theta := \Theta_{\Gamma_{\Box\Omega,\Delta}}$$
$$\Psi_\Pi := \{\Box O \to (A \to B) \mid A \in \Pi, A \to B \in \Gamma, O \in \Omega\}, \quad \Psi := \Psi_{\Gamma_{\Box\Omega,\Delta}}.$$

Note that $C \to (A \to B)$ is equivalent to $(C \wedge A) \to B$ for all formulas $A, B, C$. So $\Theta_\Pi$ can be considered as the set of implications from $\Gamma$ not in $\Gamma^\Pi$ whose antecedents are enriched with a formula from $\Delta$. Similarly for $\Psi_\Pi$ and $\Box\Omega$.

We need the following statement which we will show at the end of this proof.

$$\{I(\Box\Sigma, \Gamma^\Pi, \Pi \Rightarrow \Box\Omega, \Delta)\}_{\Pi \subseteq \Gamma_{\Box\Omega,\Delta}} \vdash_{\mathsf{iL}} I(\Box\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow \Box\Omega, \Delta). \tag{7.6}$$

First we use it to show the desired result. Use the induction hypothesis of the third set of premises in $(\mathrm{V}^{\bullet,i,1})$ and apply transitivity together with (7.6) to obtain

$$I(\mathcal{G}), I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}} I(\Box\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow \Box\Omega, \Delta), I(\mathcal{H}). \tag{7.7}$$

Note that $\Gamma^{\Gamma_{\Box\Omega,\Delta}}$ only contains implications of the form $D \to B$ or $\Box O \to B$ with $D \in \Delta$ and $O \in \Omega$. Also all antecedents of implications from $\Theta$ and $\Psi$ are $D$ and $\Box O$, respectively. At this point we apply the basis Visser rule $\mathsf{V}^{\bullet,i,1}$ from Figure 7.1 to $I(\Box\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow \Box\Omega, \Delta)$ to obtain

$$I(\mathcal{G}), I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}} \begin{array}{l} \{I(\Box\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow D)\}_{D \in \Delta}, \\ \{I(\Sigma, \Gamma^{\Gamma_{\Box\Omega,\Delta}}, \Theta, \Psi \Rightarrow O)\}_{O \in \Omega}, \\ I(\mathcal{H}). \end{array} \tag{7.8}$$

Note that $\Gamma \vdash_{\mathsf{iL}} A$ for all $A \in \Gamma^{\Gamma_{\Box\Omega,\Delta}} \cup \Theta \cup \Psi$. This follows from the fact that $A \to B \vdash_{\mathsf{iL}} C \to (A \to B)$ for all formulas $A, B, C$. Therefore

$$I(\mathcal{G}), I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}} \begin{array}{l} \{I(\Box\Sigma, \Gamma \Rightarrow D)\}_{D \in \Delta}, \\ \{I(\Sigma, \Gamma \Rightarrow O)\}_{O \in \Omega}, \\ I(\mathcal{H}). \end{array} \tag{7.9}$$

Now we use the induction hypothesis of the first two sets of premises in $(\mathrm{V}^{\bullet,i,1})$ to conclude the desired result, $I(\mathcal{G}), I(\Box\Sigma, \Gamma \Rightarrow \Box\Omega, \Delta) \vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}} I(\mathcal{H})$.

It remains us to show statement (7.6). To do so, we show by induction on the cardinality of set $\Pi' \subseteq \Gamma_{\Box\Omega,\Delta}$ that the following statement holds for all sets of formulas $\Xi$ and $\Upsilon$:

$$\{I(\Upsilon, \Gamma^{\Pi,\Xi}, \Pi \Rightarrow \Box\Omega, \Delta)\}_{\Pi \subseteq \Pi'} \vdash_{\mathsf{iL}} I(\Upsilon, \Gamma^{\Pi',\Xi}, \Theta_{\Pi'}, \Psi_{\Pi'} \Rightarrow \Box\Omega, \Delta). \tag{7.10}$$

When $\Pi'$ is empty the result follows immediately. Now suppose $|\Pi'| = n+1$. This means that we have $2^{n+1}$ formulas on the left-hand side of (7.10). Let $A \in \Pi'$ be a formula and consider the following $2^n$ derivations for $\Pi \subseteq \Pi' \setminus \{A\}$,

$$\begin{matrix} I(\Upsilon, \Gamma^{\Pi,\Xi}, \Pi \Rightarrow \Box\Omega, \Delta) \\ I(\Upsilon, \Gamma^{\Pi,\Xi,\{A\}}, \Pi, A \Rightarrow \Box\Omega, \Delta) \end{matrix} \vdash_{\mathsf{iL}} I(\Upsilon, \Gamma^{\Pi,\Xi,\{A\}}, \Theta_A, \Psi_A, \Pi \Rightarrow \Box\Omega, \Delta). \tag{7.11}$$

This follows from the fact that $\Upsilon, \Gamma^{\Pi,\Xi,\{A\}}, \Pi \vdash_{\mathsf{iL}} A \leftrightarrow (\bigvee \Delta \vee \bigvee \Box\Omega) \wedge A$ using the second premise in (7.11). So this means that $\Theta_A, \Psi_A \vdash_{\mathsf{iL}} A \to B$ for each $B$ such that $A \to B \in \Gamma$. Using the first premise in (7.11) completes the desired result. Now we can apply the induction hypothesis to $|\Pi' \setminus \{A\}| = n$ which states the following, since $\Theta_{\Pi'} = \Theta_A \cup \Theta_{\Pi' \setminus \{A\}}$ and $\Psi_{\Pi'} = \Psi_A \cup \Psi_{\Pi' \setminus \{A\}}$:

$$\{I(\Upsilon, \Theta_A, \Psi_A, \Gamma^{\Pi,\Xi,\{A\}}, \Pi \Rightarrow \Box\Omega, \Delta)\}_{\Pi \subseteq \Pi' \setminus \{A\}} \vdash_{\mathsf{iL}} I(\Upsilon, \Gamma^{\Pi',\Xi}, \Theta_{\Pi'}, \Psi_{\Pi'} \Rightarrow \Box\Omega, \Delta).$$

Now $2^n$ applications of transitivity on this and (7.11) results in equation (7.10) for $|\Pi'| = n+1$, concluding the induction proof for (7.10). To conclude (7.6), take $\Xi = \emptyset$, $\Upsilon = \Box\Sigma$, and $\Pi' = \Gamma_{\Box\Omega,\Delta}$. ■

Now we turn to the single-conclusion rules. Let us repeat the sets of single-conclusion rules $\widehat{\mathsf{V}}_{\mathsf{iL}}$ from Definition 7.1.4, where $\mathsf{V}_{\mathsf{iL}}$ is the basis of the multi-conclusion admissible rules for $\mathsf{iL}$:

$$\widehat{\mathsf{V}}_{\mathsf{iL}} := \big\{ (\bigwedge \Gamma \vee D)/(\bigvee \Delta \vee D) \mid \Gamma/\Delta \in \mathsf{V}_{\mathsf{iL}} \big\}.$$

Analogously to (Jeřábek, 2005), we prove that these form a basis for the single-conclusion rules of $\mathsf{iL}$.

**7.4.3 Lemma**
Let $\Gamma/A$ be a single-conclusion rule such that $\Gamma \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} A$. Then $\bigwedge \Gamma \vee B \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} A \vee B$ for any formula $B$.

Proof. We proceed by induction on the structure of the single-conclusion consequence relation $\vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}}$, which is by definition the smallest single-conclusion consequence relation containing $\vdash_{\mathsf{iL}}$ and $\widehat{\mathsf{V}}_{\mathsf{iL}}$. If $\Gamma/A$ is derivable in $\mathsf{iL}$, i.e. $\Gamma \vdash_{\mathsf{iL}} A$, then

$$\bigwedge \Gamma \vee B \vdash_{\mathsf{iL}} A \vee B$$

by intuitionistic reasoning. If $\Gamma/A$ is in $\widehat{\mathsf{V}}_{\mathsf{iL}}$, it is of the form $(\bigwedge \Gamma \vee D)/(\bigvee \Delta \vee D)$ for some $(\Gamma/\Delta) \in \mathsf{V}_{\mathsf{iL}}$. We have

$$(\bigwedge \Gamma \vee D) \vee B \vdash_{\mathsf{iL}} \bigwedge \Gamma \vee (D \vee B) \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} \bigvee \Delta \vee (D \vee B) \vdash_{\mathsf{iL}} (\bigvee \Delta \vee D) \vee B.$$

The result for the closure properties of the consequence relation follows easily by induction. ■

**7.4.4 Theorem**
Let $\mathsf{iL} \in \{\mathsf{iCK4}, \mathsf{iCS4}, \mathsf{iSL}, \mathsf{mHC}, \mathsf{KM}, \mathsf{PLL}\}$, then $\widehat{\mathsf{V}}_{\mathsf{iL}}$ forms a basis for the single-conclusion admissible rules of $\mathsf{iL}$.

Proof. We use the fact that $\mathsf{V}_{\mathsf{iL}}$ is a basis for the multi-conclusion rules of $\mathsf{iL}$ by Theorem 7.4.1. We show that $\mathrel{|\!\!\sim}_{\mathsf{iL}} = \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}}$. For the inclusion of the right into the left we prove that each rule in $\widehat{\mathsf{V}}_{\mathsf{iL}}$ is admissible. Let $(\bigwedge \Gamma \vee D)/(\bigvee \Delta \vee D) \in \widehat{\mathsf{V}}_{\mathsf{iL}}$ and let $\sigma$ be a substitution such that $\vdash_{\mathsf{iL}} \sigma(\bigwedge \Gamma \vee D)$. By the disjunction property for $\mathsf{iL}$ shown in Example 5.2.27 we have $\vdash_{\mathsf{iL}} \sigma(\bigwedge \Gamma)$ or $\vdash_{\mathsf{iL}} \sigma(D)$. In the latter case we are done. In the former case we use the fact that $\Gamma/\Delta \in \mathsf{V}_{\mathsf{iL}}$ and that $\mathsf{V}_{\mathsf{iL}}$ is a basis for the multi-conclusion rules. So $\vdash_{\mathsf{iL}} \sigma(B)$ for some $B \in \Delta$. Therefore $\vdash_{\mathsf{iL}} \sigma(\bigvee \Delta \vee D)$ as desired.

For the other inclusion, note that $\Gamma \mathrel{|\!\!\sim}_{\mathsf{iL}} A$ iff $\Gamma \vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}} A$ since $\mathsf{V}_{\mathsf{iL}}$ is a basis for the multi-conclusion rules of $\mathsf{iL}$. It is sufficient to prove

$$\Gamma \vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}} \Delta \text{ implies } \Gamma \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} \bigvee \Delta.$$

We do so by induction on the structure of $\vdash_{\mathsf{iL}}^{\mathsf{V}_{\mathsf{iL}}}$. We only treat two cases, the other cases are left to the reader. If $\Gamma/\Delta$ is in $\mathsf{V}_{\mathsf{iL}}$, we have

$$\Gamma \vdash_{\mathsf{iL}} \bigwedge \Gamma \vee \bot \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} \bigvee \Delta \vee \bot \vdash_{\mathsf{iL}} \bigvee \Delta.$$

The induction step for transitivity is as follows. By induction hypothesis, we have

$$\Gamma_1 \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} \bigvee \Delta_1 \vee C \text{ and } \Gamma_2, C \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} \bigvee \Delta_2.$$

Note that in general we have $F, G \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} F \wedge G$ and together with $\Gamma \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} F$ we have $\Gamma, G \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} F \wedge G$. So for the first we obtain

$$\Gamma_1, \Gamma_2 \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} \bigvee \Delta_1 \vee (\bigwedge \Gamma_2 \wedge C).$$

Lemma 7.4.3 applied to the second yields

$$(\Gamma_2 \wedge C) \vee \bigvee \Delta_1 \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} \bigvee \Delta_2 \vee \bigvee \Delta_1.$$

Using transitivity we have $\Gamma_1, \Gamma_2 \vdash_{\mathsf{iL}}^{\widehat{\mathsf{V}}_{\mathsf{iL}}} \bigvee \Delta_1 \vee \bigvee \Delta_2$. ■

We conclude with the following interesting observation for $\mathsf{PLL}$, which is an analogue of Theorems 5.3.12 and 5.3.13. We use facts of admissible projective approximations discussed in Section 6.1.2. Especially, we use the result from Ghilardi and Lenzi (2022) that each formula has a $\mathsf{PLL}$-admissible projective approximation (Theorem 6.1.27).

**7.4.5 Theorem**
Let $\mathsf{L}$ be an extension of $\mathsf{PLL}$. We have the following results.

1. If all multi-conclusion rules from $\mathsf{V}_{\mathsf{PLL}}$ are admissible in $\mathsf{L}$, then $\mathsf{V}_{\mathsf{PLL}}$ is a basis of the multi-conclusion admissible rules in $\mathsf{L}$.
2. If all single-conclusion rules from $\widehat{\mathsf{V}}_{\mathsf{PLL}}$ are admissible in $\mathsf{L}$, then $\widehat{\mathsf{V}}_{\mathsf{PLL}}$ is a basis of the single-conclusion admissible rules in $\mathsf{L}$.

Proof. We use Theorem 6.1.27 that states that each formula $A$ has a $\mathsf{PLL}$-admissible projective approximation. Theorems 7.4.1 and 7.4.4 show that $\mathsf{V}_{\mathsf{PLL}}$ and $\widehat{\mathsf{V}}_{\mathsf{PLL}}$ form bases for respectively the multi-conclusion and single-conclusion admissible rules in $\mathsf{PLL}$. We apply Theorem 6.1.19 to obtain the desired result. Note that $\mathsf{PLL}$ is indeed an extension of $\mathsf{iK4}$ by definition. ■

## 7.5 Conclusion

We have characterized the admissible rules in six intuitionistic modal logics with coreflection, $\mathsf{iCK4}$, $\mathsf{iCS4} \equiv \mathsf{IPC}$, $\mathsf{iSL}$, $\mathsf{KM}$, $\mathsf{mHC}$, and $\mathsf{PLL}$. Our characterizations are in terms of a Gentzen-style proof system for admissibility and in terms of a basis. In our research, the latter followed from the former. As a consequence we showed admissibility in these logics to be decidable.

We would like to give some comments on the limitations and opportunities of the presented method. After that we provide some directions for future research.

Our Gentzen-style proof systems for admissibility combine the systems for admissibility in $\mathsf{IPC}$ and transitive modal logics from (Iemhoff and Metcalfe, 2009b). They include so-called intuitionistic modal Visser rules. The proof systems are sound and complete with respect to admissibility in the corresponding logic. Both soundness and completeness rely on semantic arguments. In short, soundness is proved via the 'extendibility' of the models (compare to Definition 6.1.20) and completeness relies on the extension property characterizing projective formulas as discussed in Section 6.2.

Therefore, when one tries to investigate other logics, a first step would be to find a suitable semantics that allows both for extensions and for the extension property as the semantic characterization of projective formulas. The former could serve as a guideline to define the Visser-like rules of the considered logic, because, intuitively, the Visser rules represent extensions. One has to be careful with the choice of semantics, because for example for $\mathsf{PLL}$ not all semantics are extendible as discussed at the beginning of Section 7.3. Concerning the latter, it is very difficult to link projectivity to the extension property as discussed in Section 6.3. Indeed, our investigation of the extension property for logics with coreflection relies on the strong condition of the models that $R \subseteq \leq$. For intuitionistic modal logics in general it is an open question whether projective formulas can be connected to the extension property.

The attentive reader might have noticed that we also have discussed a seventh intuitionistic modal logic with coreflection, $\mathsf{IEL}$, in Section 1.3. Indeed, we conjecture that for this logic we can apply the machinery from this chapter.

Let us now look into some ideas for further research. As discussed above, the current method relies on the semantics of projectivity. It would be interesting to see whether the proof system of admissibility is sound and complete with respect to the semantics of admissibility provided by Rybakov (1997) and Goudsmit (2015) as discussed in Section 5.1.4. This would provide a direct link between the presented proof theory and semantics of admissibility.

On the other hand, one might ask whether we can do without semantics at all, and provide a purely syntactic proof of soundness and completeness. As discussed in Section 6.3, a purely syntactic and very technical approach of projective formulas is provided by Iemhoff (2016b) for classical transitive reflexive logics.

A closer proof-theoretic study of the systems themselves could give new results about the admissibility of the considered logics. As the termination of the proof systems result in decidability of admissibility, it would be insightful to see how efficient the proof systems are in terms of complexity. In addition, the proof systems for admissibility are not ordinary proof systems. In particular, they contain rules that introduce new propositional variables in the premises, i.e., the rules $(\to\Rightarrow)\triangleright^i$, $(\Rightarrow\to)\triangleright^i, (\Box \Rightarrow)\triangleright$, and $(\Rightarrow \Box)\triangleright$. These block the analyticity in the usual sense, but the proof systems still appear as analytic proof systems compared to the ones discussed in Section 5.3.2. A better understanding of these rules could justify this intuition. Moreover, standard 'admissibility' questions for ordinary proof systems can also be asked for the admissibility proof systems. For instance, we showed that weakening is admissible (Lemma 7.2.6). Finally, in light of Chapter 4, it would be interesting to investigate multisequent style systems for admissibility such as the ones based on hypersequents in (Iemhoff and Metcalfe, 2009a).

Finally, Theorem 7.4.5 states that if the Visser rules for $\mathsf{PLL}$ are admissible in a logic extending $\mathsf{PLL}$, then these form a basis for the admissible rules in that logic. This is based on the existence of $\mathsf{PLL}$-admissible projective approximations. It is interesting to study the admissible rules of logics extending the other logics treated in this chapter: $\mathsf{iCK4}$, $\mathsf{iCS4} \equiv \mathsf{IPC}, \mathsf{iSL}, \mathsf{KM}$ and $\mathsf{mHC}$. In Section 6.3 we also conjecture the existence of such approximations for these logics, which implies a similar result about the admissibility of extensions of these logics.

# Conclusions and Future Work

Uniform interpolation and admissible rules are the two central topics in this thesis. We investigated these topics in the context of classical and intuitionistic modal logics from a proof-theoretic point of view. We first recall our main results. After that we take a helicopter view and discuss some directions of general future research, some of which relate the two topics.

## Main contributions

The logics of our interest are presented in Figures 1.2 and 1.4. The former presents classical modal logics and their intuitionistic counterparts, and the latter presents intuitionistic modal logics with the coreflection principle.

In our proof-theoretic study of uniform interpolation in Part I we encountered different terminating sequent-style proof systems. In Chapter 3 we have developed (terminating) sequent calculi for logics $\mathsf{iGL}$ and $\mathsf{iSL}$ and investigated the cut-elimination theorem for these systems. We proved Craig interpolation for the logics and we provided a direct countermodel construction for $\mathsf{iSL}$. In Chapter 4 we provided a new method to construct uniform interpolants for logics $\mathsf{K}$, $\mathsf{T}$, $\mathsf{D}$, and $\mathsf{S5}$. The method intertwines proof theory and semantics. The uniform interpolants are constructively defined on the basis of terminating nested sequent and hypersequent calculi. The correctness of the method relies on model modifications that are invariant under bisimulation modulo $p$, providing a direct link to bisimulation quantification.

The main proof-theoretic achievement of Part II is the development, in Chapter 7, of proof systems to describe the admissible rules in six intuitionistic modal logics with coreflection, $\mathsf{iCK4}$, $\mathsf{iCS4} \equiv \mathsf{IPC}$, $\mathsf{iSL}$, $\mathsf{KM}$, $\mathsf{mHC}$, and $\mathsf{PLL}$. The proof systems are used to describe bases of the admissible rules and show that the admissibility problem is decidable for these logics. The results rely on a detailed analyses of well-known results on admissibility in $\mathsf{IPC}$ and classical modal logics discussed in Chapter 5 and a study on projectivity in Chapter 6. In particular, the specific structure of the Kripke models is essential in the semantic study of the extension property in Chapter 6.

Chapters 3, 4, 6, and 7 contain the main contributions and end with a conclusion in which we reflect on the material presented in the respective chapters. We sometimes indicate the limitations of the presented methods and we propose directions for future work. Let us highlight two concrete lines of investigation that we consider interesting future work. First, we would like to establish the uniform interpolation property for $\mathsf{iGL}$. We hope that our terminating sequent calculus could lead to a proof of that statement. Second, a promising line of research could be the investigation of the admissible rules in extensions of $\mathsf{iCK4}$, $\mathsf{iCS4} \equiv \mathsf{IPC}$, $\mathsf{iSL}$, $\mathsf{KM}$, $\mathsf{mHC}$, and $\mathsf{PLL}$. For $\mathsf{PLL}$ we already obtained some results in Theorem 7.4.5 and we conjecture similar results for the aforementioned logics.

## General future directions

Now we reflect on the thesis by taking a broader view on the material. In the second half of this reflection we connect the studies of uniform interpolation and admissible rules.

Let us first make four proposals for further studies regarding intuitionistic modal logics. First, in this thesis we only consider intuitionistic modal logics that only contain $\Box$, but both uniform interpolation and admissible rules are not much studied for intuitionistic modal logics with $\Box$ and $\Diamond$. The only work of this kind that we are aware of is the manuscript by Akbar Tabatabai and Jalali (2021) who investigate the so-called *feasible admissible rules* for a broad class of intuitionistic modal logics. Second, one could explore extensions of intuitionistic modal logics with axioms of intermediate logics, resulting in what we would like to call *intermediate modal logics*. This brings us to a third proposal, already mentioned in Section 6.3, to investigate bisimulation for intuitionistic modal logic in order to show finite unification and explore the admissible rules in extensions of these logics. Finally, we would like to investigate the admissible rules in intuitionistic modal logics without the coreflection principle. As logics without the coreflection principle have more complicated Kripke structures than those including the principle, and as the Kripke models for logics that include $\Diamond$ are even more complicated, the investigation of uniform interpolation and admissible rules for intuitionistic modal logics in general may require some essentially new ideas.

Part I of the thesis can be seen in a broad light of *universal proof theory*, that strives to investigate the generic behaviour of proof systems. The term first appeared in (Akbar Tabatabai and Jalali, 2018a) in which the respective authors thank Masoud Memarzadeh for this terminology. The term hints to its parallel in algebra known as *universal algebra*. Akbar Tabatabai and Jalali (2018a) indicate three pillars of research problems: the *existence problem*, the *equivalence problem*, and the *characterization problem.* Important for us is the first topic studying the existence of proof systems for logics. This is addressed by the negative results, stating that logics without uniform interpolation cannot be described by certain terminating sequent calculi. In addition to uniform interpolation, it would be worthwhile to explore other generic properties of logics with the same function. Examples are *completions* in algebraic theories of the logic (Ciabattoni *et al.*, 2012; Lauridsen, 2019) and the above mentioned *feasible admissible rules* (Akbar Tabatabai and Jalali, 2021).

The uniform interpolation property can be used as a tool in the study of admissible rules. It is used by de Jongh and Visser (1996) to characterize so-called exact formulas in $\mathsf{IPC}$ that in this logic correspond to projective formulas (see also, de Jongh and Chagrova, 1995; Bezhanishvili and de Jongh, 2012). Goudsmit (2015) observes the same for the seven consistent intermediate logics that have uniform interpolation (Ghilardi and Zawadowski, 1997). It would be interesting to see whether a similar approach applies to intuitionistic modal logics. Uniform interpolation is also used to investigate admissible *non-standard* rules. Non-standard rules are rules that may have variables in the premises that do not occur in the conclusion (curiously enough these occur at another 'level' in the proof systems of admissibility in Chapter 7). This in contrast to the (intuitionistic modal) Visser rules in which both premise and conclusion share the same variables. Non-standard rules can express new properties, such as irreflexivity (Gabbay, 1981). Bezhanishvili *et al.* (2022) use *computable* uniform interpolation to characterize admissible non-standard rules for modal systems. Computable uniform interpolation asks for a specific computation of the interpolants which can often be addressed by proof-theoretic methods.

Finally, we would like to discuss the following problem: can we interpolate admissibility? As the admissibility relation $\mathrel{|\!\sim}$ is a consequence relation, we could ask whether it has deductive interpolation, which means, whether given $A \mathrel{|\!\sim} B$, there is a $C$ in the common language of $A$ and $B$ such that $A \mathrel{|\!\sim} C$ and $C \mathrel{|\!\sim} B$. For structurally complete logics, this boils down to the deductive interpolation problem in the logic itself, since for these logics derivability and admissibility coincide. Moreover, as interpolation concerns shared variables, the problem might be more challenging for non-standard rules. In our view it would be interesting to study structural completeness with regard to such rules.

These are just some of the open problems that we think might be interesting to address in the future.

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# Index

**Symbols**

# Samenvatting

De titel van dit proefschrift luidt in het Nederlands: *Uniforme Interpolatie en Toelaatbare Regels. Bewijstheoretisch onderzoek naar (intuïtionistische) modale logica's.* Deze samenvatting zet in vier delen uiteen waar het proefschrift over gaat en licht de verschillende termen toe. Elk deel bouwt op in moeilijkheidsgraad, waarbij het begin van elk deel een introductie is voor niet-wiskundigen en het einde een meer technische bespreking van de resultaten in het proefschrift.

## Logica

Het onderzoeksgebied van de logica bestudeert vormen van redeneren. Hierbij draait het om het beschrijven en analyseren van de deductieve aard van redeneringen: hoe volgt de ene uitspraak uit de ander? Een logica modelleert redeneringen op formele wijze, waarbij men de nadruk legt op de logische vorm van een redenering en niet zozeer op de inhoud ervan.

Neem bijvoorbeeld de uitspraken '*het regent of het regent niet*' en '*mijn jas is geel of mijn jas is niet geel*'. Beide uitspraken gaan over verschillende dingen maar hebben dezelfde vorm die wiskundig weergegeven wordt als '$A \vee \neg A$', waarbij $\vee$ staat voor '*of*', $\neg$ voor '*niet*' en waarbij we voor '$A$' elke bewering kunnen invullen. De geldigheid van zo'n uitspraak hangt af van de vorm maar niet van de inhoud.

Een logica beschrijft redeneringen tussen zulke expressies aan de hand van axioma's (aannames van ware uitspraken) en afleidingsregels (bijv. als '$A$' en '$A$ *impliceert* $B$' waar zijn, dan is '$B$' waar). Het feit dat een uitspraak waar is wordt deductief vastgesteld aan de hand van de vorm van de axioma's en regels.

Er zijn talloze vormen van redeneren en daarmee ook talloze logica's. In dit proefschrift focussen we op zogenoemde *modale logica's*. Modale logica's analyseren uitspraken die een zekere aanduiding bevatten van de waarheid van een bewering. Dit zijn uitspraken als '*het is noodzakelijk dat ...*', '*het is mogelijk dat ...*', '*jij weet dat ...*', '*ik geloof dat ...*', '*het is bewijsbaar dat ...*'. In het algemeen worden modale logica's voor verscheidene toepassingen gebruikt, zoals het modelleren van groepsdynamiek en het implementeren van de communicatie tussen robots. Dit proefschrift richt zich op logica's met toepassingen in de wiskunde, zoals het analyseren van bewijsbaarheid binnen de rekenkunde.

Dit brengt ons bij het verschil tussen de klassieke en intuïtionistische logica, want zoiets als *de* rekenkunde bestaat niet. Er zijn verschillende stromingen, ieder met een eigen vorm van wiskundig redeneren. De bekendste zijn de *klassieke* en *intuïtionistische* wiskunde. L.E.J. Brouwer zette zich begin vorige eeuw af tegen de gangbare klassieke wiskunde en is de grondlegger van het intuïtionisme. Volgens deze stroming moet een wiskundig argument het resultaat zijn van een mentale constructie. Een uitspraak als '*er is een getal $x$ zodanig dat ...*' kan alleen bewezen worden als een specifiek getal wordt geconstrueerd dat aan de uitspraak voldoet. Dit in tegenstelling tot de klassieke wiskunde waarin het volstaat om aan te tonen dat zo'n $x$ moet bestaan zonder het aan te wijzen. Intuïtionistische argumenten zijn constructief en verwant aan algoritmes en spelen daarom een belangrijke rol binnen de theoretische informatica. Deze vorm van redeneren wordt gevangen door de intuïtionistische logica.

In dit proefschrift onderzoeken we zowel klassieke als intuïtionistische modale logica's. Modale uitspraken zoals hierboven weergegeven (i.e., '*het is bewijsbaar dat ...*') worden symbolisch weergegeven met $\Box A$. De intuïtionistische modale logica's die we analyseren in dit proefschrift bevatten het *coreflectie* principe $A \to \Box A$, gebruikt in epistemische logica, bewijsbaarheidslogica en lax logica. Het interessante is dat dit principe geen betekenisvolle klassieke interpretatie kent. Alle logica's die we onderzoeken in dit proefschrift worden gedefinieerd in hoofdstuk 1.

## Bewijstheorie

Een sluitend argument voor een wiskundige uitspraak noemt men een bewijs. De *bewijstheorie* is een breed veld in de wiskundige logica dat bewijzen als wiskundige objecten analyseert. Het behandelt vragen als: 'wanneer zijn twee ogenschijnlijk verschillende bewijzen hetzelfde?' en 'hoe complex is een gegeven bewijs?'.

In dit proefschrift richten we ons op bewijzen in logica's. Zoals hierboven gezegd kenmerkt een logica zich door axioma's en regels, maar om preciezer te zijn is het geheel van axioma's en regels een *bewijssysteem*, en is een logica enkel een verzameling van ware uitspraken die wordt bepaald door zo'n formeel systeem. Een bewijs voor een uitspraak is een geldige afleiding binnen een bewijssysteem en toont aan dat de uitspraak waar is in de logica.

Interessant is dat er verschillende soorten bewijssystemen zijn en dat een enkele logica beschreven kan worden door meerdere systemen. Bewijssystemen die een rol spelen in dit proefschrift zijn sequente calculi geïnspireerd door Gentzen (1935a,b). Het is bekend dat sequente calculi niet altijd toereikend zijn om een gegeven logica te beschrijven en daarom zijn er verschillende varianten zoals de hypersequenten (Avron, 1996; Pottinger, 1983) en de geneste sequenten (Bull, 1992; Brünnler, 2009; Poggiolesi, 2009a). De analyse van zulke bewijssystemen kan iets zeggen over de eigenschappen van een logica, zoals consistentie (er zijn geen tegenspraken) en beslisbaarheid (een algoritme kan bepalen of een uitspraak waar is of niet).

De bewijstheoretische benadering van logica's wordt ook wel syntactisch genoemd. Dit is in contrast met de semantische benadering van logica's waarin de waarheid van uitspraken bepaald wordt door wiskundige structuren als Kripke modellen en algebra's. De bewijstheoretische benadering speelt een belangijke rol in dit proefschrift, maar we maken ook gebruik van semantische methoden.

## Uniforme interpolatie

In deel I van het proefschrift, genaamd *Uniforme Interpolatie in Bewijstheorie*, hebben we de volgende vraag in ons achterhoofd: gegeven een logica, bestaat er een goed bewijssysteem dat de logica beschrijft? Wat 'goed' is hangt af van je doel. Hier zoeken we naar *terminerende* bewijssystemen waarin elke zoektocht naar een bewijs voor een gegeven uitspraak eindigt; in een bewijs (dus de uitspraak is waar), dan wel in een dood spoor (de uitspraak is niet waar).

Onderzoek van Iemhoff (2019a,b) heeft aangetoond dat voor sommige klassieke en intuïtionistische modale logica's *geen* terminerende sequente calculi bestaan van een bepaalde vorm. Deze zogenoemde negatieve resultaten komen voort uit het onderzoek naar uniforme interpolatie.

Uniforme interpolatie is een eigenschap van logica's en sterker dan de bekendere Craig interpolatie. Interpolatie gaat over de informatie waarom de ene uitspraak uit de andere volgt. Een voorbeeld is dat uit '*ik loop in de regen en mijn jas is geel*' volgt '*ik word nat*'. De reden dat de tweede uitspraak volgt uit de eerste heeft niks te maken met de kleur van mijn jas maar berust alleen op de uitspraak '*het regent*'. Die uitspraak noemt men de interpolant. Een logica heeft interpolatie als er voor elke implicatie zo'n interpolant is. In het algemeen worden logica's met deze eigenschappen toegepast in de informatica, zoals het redeneren in databases.

Bewijstheoretisch onderzoek naar uniforme interpolatie begon bij Pitts (1992). Zulk onderzoek heeft twee voordelen. Aan de ene kant kunnen terminerende bewijssystemen de uniforme interpolanten constructief definiëren. Aan de andere kant kan het gebrek aan uniforme interpolatie een rol spelen in het onderzoek naar negatieve resultaten. Hoofstuk 2 geeft een uitgebreide introductie.

Hoofdstuk 3 bespreekt nieuwe resultaten over intuïtionistische modale logica's in relatie tot uniforme interpolatie. Ten eerste ontwikkelen we (terminerende) sequente calculi voor twee intuïtionistische modale logica's met toepassingen in de bewijsbaarheidslogica: *Gödel-Löb logic* en *strong Löb logic*. We bewijzen Craig interpolatie voor beide logica's en vermoeden dat uniforme interpolatie ook geldt. Ten tweede bewijzen we negatieve resultaten voor twee intuïtionistische modale logica's, iK4 en iS4, door aan te tonen dat ze geen uniforme interpolatie hebben.

In hoofdstuk 4 breiden we het bewijstheoretische onderzoek voor uniforme interpolatie uit naar andere bewijssystemen zoals geneste sequente calculi en hypersequente calculi. We ontwikkelen een nieuwe methode om de uniforme interpolatie voor de vier klassieke modale logica's K, T, D, en S5 opnieuw te bewijzen. We construeren uniforme interpolanten via terminerende geneste sequente calculi en hypersequente calculi. Dit levert het eerste constructieve bewijs voor uniforme interpolatie voor de logica S5. Onze methode maakt ook gebruik van semantische argumenten gebaseerd op zogenaamde bisimulatiekwantoren in Kripke modellen.

## Toelaatbare regels

In deel II, genaamd *Toelaatbare Regels en Bewijstheorie*, staat de volgende vraag centraal: gegeven een verzameling van axioma's en regels ter beschrijving van een logica, welke regels kan ik toevoegen zonder dat er nieuwe uitspraken waar worden?

Om deze vraag te illustreren vergelijken we een logica met een routekaart die ons vertelt hoe van $A$ naar $B$ te gaan. Stel als axioma dat we beginnen in Utrecht en als regel dat we te voet op pad moeten. Hieruit volgt dat we Gouda kunnen bereiken. Als we nu als regel toevoegen dat we met de auto mogen, dan zijn nieuwe wegen een optie en zijn we sneller in Gouda. De regel verandert iets aan de manier *waarop* we ergens komen, maar niet *waar* we kunnen komen. Dit is net als met logica's; een toelaatbare regel voegt iets toe aan de manier van rederenen, maar het beïnvloedt niet de uitspraken die waar zijn. Een analogie van een niet-toelaatbare regel is het gebruik van het vliegtuig. Dit stelt ons in staat om naar Australië te gaan wat in eerste instantie niet mogelijk was.

Onderzoek naar toelaatbare regels is nuttig, omdat deze regels de algehele structuur van een logica belichamen. Kenmerkend is dat toelaatbare regels niet gebonden zijn aan een specifiek bewijssysteem, maar dat ze de interactie van geldige uitspraken reflecteren. Daarnaast kan het gebruik van toelaatbare regels leiden tot kortere bewijzen.

In dit proefschrift zijn we geïnteresseerd in karakterisaties van alle toelaatbare regels van een gegeven logica. Een centrale vraag komt van Friedman (1975):

gegeven een logica $\mathsf{L}$, kunnen we beslissen of een regel toelaatbaar is in $\mathsf{L}$ of niet? Een andere vraag is of er een solide beschrijving is van de klasse van toelaatbare regels van een gegeven logica. Deze vragen zijn onderzocht voor vele (modale) logica's door o.a. Rybakov (1997). Zie hoofdstuk 5 voor een uitgebreide introductie over toelaatbare regels.

Dit proefschrift verschaft een eerste studie van de toelaatbare regels voor intuïtionistische modale logica's. In hoofdstuk 7 karakteriseren we alle toelaatbare regels in zes intuïtionistische modale logica's met coreflectie. Daarbij tonen we aan dat het beslisbaar is of een gegeven regel toelaatbaar is in de logica of niet. Onze techniek is geïnspireerd op een bewijstheorie voor toelaatbaarheid geïntroduceerd door Iemhoff and Metcalfe (2009b). Deze bewijssystemen zijn speciaal, omdat ze niet bepalen of een uitspraak waar is in de logica, maar bepalen of een regel toelaatbaar is in de logica.

Het bewijs berust op semantische benaderingen ontwikkeld door Ghilardi (1999, 2000) over de interactie tussen de zogenoemde projectieve formules en de extensie-eigenschap. In hoofdstuk 6 analyseren we het belang van deze in het veld van toelaatbare regels. We analyseren de methode uit (Ghilardi, 2000) voor klassieke modale logica's gebaseerd op een semantisch bisimulatie argument in Kripke modellen. Ook onderzoeken we de interactie tussen beide voor intuïtionistische modale logica's met coreflectie.

Deel I en deel II van het proefschrift kunnen onafhankelijk van elkaar gelezen worden. In het laatste hoofdstuk *Conclusies en Toekomstig Werk* geven we suggesties voor vervolgonderzoek en bespreken we de relatie tussen het onderzoek naar uniforme interpolatie en toelaatbare regels.

# About the Author

Iris van der Giessen was born on October 12th 1995 in Gouda, the Netherlands. She obtained her Bachelor's and Master's degree in Mathematics from the Radboud University Nijmegen, both with the distinction *cum laude*. During her Bachelor's program she enrolled in the Radboud Honours Academy, an extra 30 ECTS program for science students focused on academic skills.

Due to her interest in logic, she took the specialisation in Mathematical Foundations of Computer Science as part of her Master's program, including courses about type theory and algebraic logic. She wrote her Master's thesis under supervision of Prof. Dr. Herman Geuvers on generalized natural deduction systems derived from truth tables.

Immediately after finishing her Master's in August 2018, she started as a PhD candidate at Utrecht University under supervision of Prof. Dr. Rosalie Iemhoff and Dr. Nick Bezhanishvili. Her research is part of the project *Optimal Proofs* funded by the Netherlands Organisation for Scientific Research (NWO).

Starting from September 2022, she will be a postdoctoral researcher at the University of Birmingham collaborating with Dr. Anupam Das in the project *Structure vs. Invariants in Proofs* funded by UK Research and Innovation (UKRI).

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