Stable operations and topological modular forms

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The cover (drawn by Carl Davies) depicts a statue of the Egyptian goddess of wisdom (and mathematics) Seshat amidst a hedge maze or *Heckenlabyrinth*.

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Stabiele operatoren en topologische modulaire vormen

(met een samenvatting in het Nederlands)
(mit einer Zusammenfassung in deutscher Sprache)

Proefschrift

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Abstract

This document expands our structural knowledge of *topological modular forms* TMF in two directions: the first, by extending the functoriality inherent to the definition of TMF, and the second, being tools to calculate the effect that endomorphisms of TMF have on homotopy groups. These structural statements allow us to lift classical operations on modular forms, such as *Adams operations*, *Hecke operators*, and *Atkin–Lehner involutions*, to stable operations on TMF. Some novel applications of these operations are then found, including a derivation of some congruences of Ramanujan in a purely homotopy theoretic manner, improvements upon known bounds of Maeda's conjecture, as well as some applications in homotopy theory. These techniques serve as teasers for the potential of these operations.

Dit document breidt onze structurele kennis van *topologische modulaire vormen* TMF in twee richtingen uit: de eerste, door de functoraliteit uit te breiden die inherent is aan de definitie van TMF, en de tweede, door hulpmiddelen te zijn om het effect te berekenen dat endomorfismen van TMF hebben op homotopiegroepen. Deze structurele verklaringen laten toe om klassieke operaties op modulaire vormen, zoals de operaties van Adams, de operatoren van Hecke en de involuties van Atkin en Lehner, op te heffen naar stabiele operaties op TMF. Vervolgens worden enkele nieuwe toepassingen van deze operaties gevonden, waaronder een afleiding van enkele congruenties van Ramanujan op een zuiver homotopie-theoretische manier, verbeteringen van gekende limieten van Maeda's vermoeden, alsook enkele toepassingen in de homotopie theorie. Deze technieken dienen als teasers voor het potentieel van deze operaties.

This thesis is a combination of [Dav20], [Dav21a], [Dav21b], and [Dav22], with some elaborations and added context.

Contents

Chapter 1

Introduction

For those of us who write, it is necessary to scrutinise not only the truth of what we speak, but the truth of that language by which we speak it. For others, it is to share and spread also those words that are meaningful to us. But primarily for us all, it is necessary to teach by living and speaking those truths which we believe and know beyond understanding. Because in this way alone we can survive, by taking part in a process of life that is creative and continuing, that is growth.

Audre Lorde, The Transformation of Silence into Language and Action

The cohomology theory[1] Tmf of *topological modular forms*, as well as its periodic and connective variants (denoted by TMF and tmf, respectively), have been of intense focus in modern homotopy theory. A lot of attention surrounded its initial construction and connections to modular forms and elliptic genera ([Hop95, Hop02]). This excitement has continued until now, with further uses of these cohomology theories to make conclusions about elements in the stable homotopy groups of spheres ([WX17, IWX20]), study resolutions of the $K(2)$-local sphere ([Beh06, GHMR05]), and construct $\mathbf{E}_\infty$-forms of $\mathrm{BP}\langle 2\rangle$ ([HL10, HM17]). The looming figures of equivariant elliptic cohomology ([GM20]), Lurie's series of papers [EC1, EC2, EC3, SUR09] born out of spectral algebraic geometry, and the hope for a geometric model connected to physics ([ST11]) also promise that this will be an active area of study in the future.

One of the main goals of this thesis is inspired less by the geometric side of TMF, and rather its connections to number theory and arithmetic geometry. The homotopy groups of TMF naturally map into the ring MF_* of *(mero-*

[1] The reader who is not a practicing homotopy theorist may want to start with §C.

morphic) modular forms—a classical object in number theory. The original definition of MF_* is as a complex vector space of holomorphic functions on the half-plane satisfying a certain modular transformation property. However, the algebraic geometry of the mid-twentieth century recast each group MF_k as the global sections of a k-fold tensor product of a line bundle ω on the moduli stack of elliptic curves $\mathcal{M}_{\mathrm{Ell}}$. The connection to TMF comes from the construction of TMF as the global sections of a sheaf $\mathscr{O}^{\mathrm{top}}$ of $\mathbf{E}_\infty$-rings on $\mathcal{M}_{\mathrm{Ell}}$ with the property that each $\pi_{2k}\mathscr{O}^{\mathrm{top}}$ can be identified with $\omega^{\otimes k}$. The map $\pi_{2*}\,\mathrm{TMF} \to \mathrm{MF}_*$ is then the natural comparison map between a limit (global sections) applied before or after taking homotopy groups.

To better study MF_*, in either the classical sense over $\mathbf{C}$ or the neoclassical sense with stacks, one considers many operations and symmetries such as Hecke operators T_n whose simultaneous eigenvectors span MF_*. There has long been a lack of such operations on TMF due to some nontrivial technical hurdles. One can define Hecke operators $\mathrm{T}_n \colon \mathrm{MF}_* \to \mathrm{MF}_*$ using the neoclassical approach above with stacks, but this is not a construction that lies solely in the small étale site of $\mathcal{M}_{\mathrm{Ell}}$ upon which $\mathscr{O}^{\mathrm{top}}$ is traditionally defined. To mimic this construction for TMF, our solution is to expand the functoriality of $\mathscr{O}^{\mathrm{top}}$ using a result of Lurie; originally made public without proof in [BL10, Th.8.1.4]. It is with this result of Lurie that we start the mathematical content of this thesis.

1.1 Summary of results

The powerful statement of Lurie which promises to expose more symmetries of TMF is known as *Lurie's theorem.* This theorem posits the existence of a sheaf $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ of $\mathbf{E}_\infty$-rings on a stack of *p-divisible groups* (also called Barsotti–Tate groups) of height n which resembles the Landweber exact cohomology theories of the 1980s. What follows is a simplified version; a more precise version appears as Th.2.1.7 and is proven throughout Part I of this thesis.

Theorem A (Lurie's theorem (Th.2.1.7))**.** *Let p be a prime and $n \geqslant 1$ a positive integer. Then there is a sheaf of $\mathbf{E}_\infty$-rings $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ from a site over the moduli stack of p-divisible groups of height n such that its value on a p-divisible group $\mathbf{G}$ over a ring R is an $\mathbf{E}_\infty$-ring $\mathcal{E}$ with the following properties:*

1. *$\mathcal{E}$ has a complex orientation and is Landweber exact.*
2. *There is an isomorphism of rings $\pi_0\mathcal{E} \simeq R$.*
3. *The homotopy groups $\pi_*\mathcal{E}$ vanish for all odd integers and otherwise $\pi_{2k}\mathcal{E}$ is the k-fold tensor product of a line bundle on R.*
4. *There is an isomorphism between the formal group of the p-divisible group $\mathbf{G}$ and the formal group of $\mathcal{E}$.*

Our proof of Th.A uses much of Lurie's work from [EC2] as well as important results from [HA, SAG]. Let us reiterate that the above theorem is due to Jacob

Lurie—we only feel obliged to provide a proof as this thesis relies so heavily upon the result and due to the lack of a publicly available proof. This theorem is incredibly powerful and we give some examples of how one can utilise this statement in Chapter 5. The most important example for the rest of this thesis is the following; one can find details in §5.3.

Example 1.1.1. Recall TMF_p is the p-completion of the global sections of the Goerss–Hopkins–Miller, Lurie sheaf $\mathscr{O}^{\mathrm{top}}$ on the moduli stack $\mathcal{M}_{\mathrm{Ell}}$ of elliptic curves. Associated to each elliptic curve E is a p-divisible group $E[p^\infty]$, the collection of p-power torsion for E. This assignment of an elliptic curve to a p-divisible group yields a map of stacks $\mathcal{M}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{BT}_2^p}$. If we restrict our attention to p-complete rings, then the classical Serre–Tate theorem says this map of stacks is formally étale—deformations of elliptic curves are precisely determined by deformations of their associated p-divisible group. We can pullback $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_2^p}$ of Th.A along this map of stacks to a sheaf of $\mathbf{E}_\infty$-rings over the p-completion of $\mathcal{M}_{\mathrm{Ell}}$. This pullback can be identified with the p-completion of $\mathscr{O}^{\mathrm{top}}$ as these sheaves are uniquely determined up to homotopy by (a subset of) the conditions 1-4 of Th.A; we prove this folklore uniqueness statement in Appendix B. This means that TMF_p can be reconstructed from the p-divisible group $\mathscr{E}[p^\infty]$ associated with the universal elliptic curve over $\mathcal{M}_{\mathrm{Ell}}$; this object has many more symmetries than $\mathscr{E}$ itself (which by universality has essentially no symmetries). In particular, for integers n not divisible by p, the n-fold multiplication map $[n]$ on $\mathscr{E}[p^\infty]$ is an equivalence and hence induces an automorphism of $\mathbf{E}_\infty$-rings on TMF_p. We call these operations *stable Adams operations* due to the analogy with those operations on K-theory.

Equipped with Lurie's theorem, we walk straight towards our first result—an integral reinterpretation of the previous example.

Theorem B (Th. 6.1.9)**.** *Write* $\mathrm{Isog}_{\mathrm{Ell}}$ *for the site (Df. 6.1.4) whose objects are those from the small étale site of* $\mathcal{M}_{\mathrm{Ell}}$ *and whose morphisms include those isogenies of elliptic curves of invertible degree. Then there is an étale hypersheaf* $\mathscr{O}^{\mathrm{top}}$ *on* $\mathrm{Isog}_{\mathrm{Ell}}$ *whose restriction to the small étale site of* $\mathcal{M}_{\mathrm{Ell}}$ *is equivalent to the sheaf* $\mathscr{O}^{\mathrm{top}}$ *of [DFHH14].*

These isogenies of elliptic curves in $\mathrm{Isog}_{\mathrm{Ell}}$ then induce extra symmetries on the sections of $\mathscr{O}^{\mathrm{top}}$ including TMF. To construct highly structured operations on TMF using transfer maps, we need to improve the functoriality of $\mathscr{O}^{\mathrm{top}}$ in yet another direction—using *span* ∞*-categories*:

Theorem C (Th.6.2.3)**.** *Write* fin *for the wide subcategory of* $\mathrm{Isog}_{\mathrm{Ell}}$ *spanned by finite morphisms. Then there is a unique functor* $\mathbf{O}^{\mathrm{top}}$ *in the following commutative diagram of* ∞*-categories:*

$$\begin{array}{ccc} \mathrm{Isog}_{\mathrm{Ell}}^{\mathrm{op}} & \xrightarrow{\mathscr{O}^{\mathrm{top}}} & \mathrm{CAlg} \\ \downarrow & & \downarrow \\ \mathrm{Span}_{\mathrm{fin}}(\mathrm{Isog}_{\mathrm{Ell}}) & \xrightarrow{\mathbf{O}^{\mathrm{top}}} & \mathrm{Sp} \end{array}$$

Together, Th.B and Th.C represent the most structured definition of $\mathscr{O}^{\mathrm{top}}$ to date—we also hope to soon apply these techniques to obtain a similar (albeit more subtle and complicated) extension for the sheaf $\mathscr{O}^{\mathrm{top}}$ over the compactification $\overline{\mathcal{M}}_{\mathrm{Ell}}$ of $\mathcal{M}_{\mathrm{Ell}}$; see §7.7 for some preliminary discussion in this direction.[2] With all this extra structure on $\mathscr{O}^{\mathrm{top}}$, we can define (Df. 7.2.1) stable Hecke operators

$$\mathrm{T}_n \colon \mathrm{TMF}[\frac{1}{n}] \to \mathrm{TMF}[\frac{1}{n}]$$

with remarkable formal properties. For instance, these Hecke operators naturally commute with Adams operations (Pr. 7.2.4) and agree with the more classical operations on complex modular forms (Pr.7.5.3). Despite the lack of "calculations with q-expansions" in the topological case, we are also able to prove the following composition formulae for stable Hecke operators by manipulating the stacks involved.

Theorem D (Th. 7.2.7)**.** *Let m and n be positive integers. Then there is a homotopy of morphisms of spectra*

$$\mathrm{T}_m \circ \mathrm{T}_n \simeq \sum_{d|m,n} d\psi^d \mathrm{T}_{\frac{mn}{d^2}} \colon \mathrm{TMF}[\frac{1}{mn\phi}] \to \mathrm{TMF}[\frac{1}{mn\phi}]$$

where $\phi = \gcd(6, \phi(mn))$ and $\phi(mn)$ is Euler's totient function. The above sum ranges over those positive integers d dividing both m and n. In particular, $\mathrm{T}_m \circ \mathrm{T}_n$ is homotopic to $\mathrm{T}_n \circ \mathrm{T}_m$, and if $\gcd(m, n) = 1$ then both are homotopic to T_{mn}.[3]

Combining Th.B and a little inspiration from the classical theory of modular forms, we can also construct stable Fricke and Atkin–Lehner involutions on periodic topological modular forms with various level structures

$$w_Q \colon \mathrm{TMF}_0(N) \to \mathrm{TMF}_0(N)$$

where Q divides N and $\gcd(Q, N/Q) = 1$; see §7.6.

The properties of our stable operations are as good as could be expected (modulo the dream of their existence on Tmf), however, the key step to encourage their usage and applicability is to make some calculations. To this end, we set up some general principles to calculate the effect of endomorphisms of tmf or TMF on their homotopy groups.[4] Let us make two qualitative statements in this direction.

[2]My personal and sincere apologies to those who have read the first version of [Dav21a] or listened to me give talks on the topic where I claimed such a functorial construction on the compactification already exists. I was excited and hasty; see §7.7 or the updated [Dav21a].

[3]The appearance of the number ϕ in this theorem is rather unfortunate, and could be removed if one can show the group $\pi_0\,\mathrm{TMF}_0(\frac{mn}{d})$ contains no torsion for each $d|m, n$; see Rmk.7.4.7. The author has also recently obtained a homotopy between $\mathrm{T}_n \circ \mathrm{T}_m$ and $\mathrm{T}_m \circ \mathrm{T}_n$ over $\mathrm{TMF}[\frac{1}{mn}]$ which will appear in [Dav22].

[4]Recently, Candelori–Salch [CS22a, CS22b] have made some new computational steps in calculations of stable Hecke operators on the elliptic cohomology of spaces which are not necessarily spheres. Some of their work can be translated into our setting over TMF, however, as mentioned in [CS22a, Rmk.3.3], there is not much point in doing so.

Theorem E (Th.8.0.1)**.** *Writing* $\mathfrak{Tors}$ *for the torsion subgroup of* $\pi_* \operatorname{tmf}$*, there is a splitting of abelian groups*

$$\pi_* \operatorname{tmf} \simeq \mathfrak{Free} \oplus \mathfrak{Tors}$$

which is natural with respect to endomorphisms of the spectrum tmf*. In particular, if* $f\colon \operatorname{tmf} \to \operatorname{tmf}$ *is a map of spectra, then one has the containment* $f(\mathfrak{Free}) \subseteq \mathfrak{Free}$*. This result is compatible with localisations and completions at primes, and also holds for* TMF*.*

One can wonder about how general the above phenomenon is. It holds for other likable spectra such as topological K-theories for trivial reasons and some variants of topological modular forms we tried, but it does not hold for the Eilenberg–MacLane spectrum $\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$ nor does it hold if we allow shifts (think of multiplication by a torsion element). The proof of Th.E uses the theory of *synthetic spectra* or $\mathbf{C}$*-motivic homotopy theory*, and a basic application of these tools also yields the following general computational statement.

Theorem F (Th.8.0.2)**.** *Let* $x \in \pi_* \operatorname{tmf}$ *be a homogeneous torsion element with* DSS decomposition $a\cdot t$ *(Df.8.3.1) and* $f\colon \operatorname{tmf} \to \operatorname{tmf}$ *be an morphism of spectra. Then* $f(x)$ *is represented by* $f_{\mathrm{alg}}(a)t$ *on the* E_∞*-page of the descent spectral sequence for* tmf*, where* f_{alg} *is the map* f *induces on* E_2*-pages. Moreover, if* x *is* nearby the Hurewicz image *(Df.8.3.1), then* $f(x) = f(1)x$*. This result is compatible with localisations, completions, and also holds for* TMF*.*

Combining Ths.E and F and applying these statements to the Adams operations and Hecke operators on TMF, we obtain a complete calculation of these operations on homotopy groups.

Theorem G (Ths.9.0.1 and 9.0.2)**.** *Given an integer* k *we have the following equality for every homogeneous element* $x \in \pi_* \operatorname{TMF}[\frac{1}{k}]$*:*

$$\psi^k(x) = \begin{cases} x & x \in \mathfrak{Tors} \\ k^{\frac{|x|}{2}}x & x \in \mathfrak{Free} \end{cases}$$

The same also holds for p*-adic Adams operations. Fix a positive integer* n*.*

- *For each homogeneous element* $x \in \mathfrak{Free} \subseteq \pi_* \operatorname{TMF}[\frac{1}{n}]$ *the image of* x *under* T_n *satisfies* $\mathrm{T}_n(x) = n\mathrm{T}_n^{\mathrm{alg}}(x)$*, where* $\mathrm{T}_n^{\mathrm{alg}}$ *are the classical Hecke operators acting on* x *considered as a classical modular form.*
- *For each homogeneous element* $x \in \mathfrak{Tors} \subseteq \pi_* \operatorname{TMF}[\frac{1}{n}]$ *the element* $\mathrm{T}_n(x)$ *is represented by* $n\mathrm{T}_n^{\mathrm{alg}}(a)t$ *on the* E_∞*-page of the descent spectral sequence, where* at *is a* DSS decomposition *(Df.8.3.1) for* x*.*

We can also use the generic methods of Ths.E and F to show the necessity of inverting k or n when defining the stable Adams operation ψ^k or Hecke operator T_n on TMF, at least in the cases when k and n are powers of 2 and 3; see §9.2.

After we have defined Adams operations and Hecke operators on TMF, checked some of their basic properties, and calculated their effects on homotopy groups, it remains to explore various applications of these operations. The applications we discuss in this thesis are of two varieties: using their existence and the calculations above to obtain various congruences between modular forms, and using inspiration from topological K-theory to make statements in stable homotopy theory. Let us now give examples of both.

1.1.1 Arithmetic applications

First, recall the *normalised Eisenstein series* E_{2k} is a modular form of weight $2k$ with the q-expansion

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{i=1}^{\infty} \sigma_{2k-1}(i) q^i \qquad\qquad q = e^{2\pi i \tau}$$

where B_{2k} are the *Bernoulli numbers* and $\sigma_{2k-1}(i) = \sum_{d|i} d^{2k-1}$ are the *generalised divisor sum functions*. One also defines the *modular discriminant* Δ by the equation and q-expansion

$$\Delta = \frac{E_4^3 - E_6^2}{1728} \qquad\qquad \Delta(\tau) = \sum_{i=1}^{\infty} \tau(i) q^i$$

where $\tau(i)$ is *Ramanujan's τ-function*. The coefficients in the q-expansions of the above modular forms are of much interest in number theory and also naturally appear in our calculations of Hecke operators on the homotopy groups of TMF. In particular, using the torsion inside π_* TMF, we can reprove simple congruences of Ramanujan such as $n\tau(n) \equiv_3 \sigma_1(n)$ for n not divisible by 3 and $n\tau(n) \equiv_8 \sigma(n)$ for all odd n. One can further exploit other simple congruences occurring from our calculations to prove some extra cases of *Maeda's conjecture*.

Write $S_k \subseteq \mathrm{mf}_k^{\mathbf{Q}}$ for the subspace of weight k *cusp forms*, those holomorphic modular forms with vanishing constant term in their q-expansion. The classical Hecke operators $\mathrm{T}_n^{\mathrm{alg}}$ act on mf_k and preserve the subspace of cusp forms S_k. The following is due to Maeda [HM98].

Conjecture 1.1.2. *For every integer $n \geqslant 2$, the characteristic polynomial F of T_n acting on S_k is irreducible over $\mathbf{Q}$ and the Galois group of the splitting field of F is the fully symmetric group on d letters, where $d = \dim_{\mathbf{Q}} S_k$.*

Using some congruences acquired from analysing the action of T_n on the homotopy groups of TMF, we verify the following cases of Maeda's conjecture.

Theorem H (Th. 10.2.5)**.** *Let $k, n \geqslant 2$ be two coprime integers with n not divisible by 3 satisfying the following conditions:*

1. *$k \leqslant 1{,}000$ and for all $1 \leqslant i \leqslant k-1$, the coefficient of q^k in the q-expansion of Δ^i is divisible by 3.*

2. *For each prime factor p of n with exponent e, if $p \equiv_3 1$ then $e \equiv_6 0, 1, 3, 4$, and if $p \equiv_3 2$ then e is even.*

Then T_{kn} acting on S_{12k} satisfies Maeda's conjecture.

Example values of $k \leqslant 20$ are $2, 3, 6, 9$, and 18.

Another more complicated statement is also made at the prime $p = 2$; see Th.10.2.6. We are restricted to the primes 2 and 3 for these kinds of statements, as we rely on the torsion elements in $\pi_* \mathrm{TMF}$. If our stable operations above extend to coherent operations over the compactified moduli stack $\overline{\mathcal{M}}_{\mathrm{Ell}}$, then one could obtain statements at other primes. In fact, we do this in some isolated cases, constructing handicraft stable p-adic Adams operations $\psi^k \colon \mathrm{tmf}_p \to \mathrm{tmf}_p$ for p-adic units $k \in \mathbf{Z}_p^\times$. This leads us to some topological applications.

1.1.2 Topological applications

Inspired by the *Adams summand* ℓ, a p-complete spectrum for odd primes p which is a direct summand of connective p-complete complex K-theory ku_p, we define *height two Adams summands* u and U in the connective and periodic cases, respectively. These constructions share some formal properties with ℓ, but not all of them.

Theorem I (Th.10.3.3)**.** *Let p be an odd prime. The canonical map of $\mathbf{E}_\infty$-rings $\mathrm{U} \to \mathrm{TMF}_p$ witnesses the codomain as a quasi-free module over the domain of rank $\frac{p-1}{2}$. The canonical map of $\mathbf{E}_\infty$-rings $\mathrm{u} \to \mathrm{tmf}_p$ recognises the codomain as a quasi-free module over the domain of rank $\frac{p-1}{2}$ if $p-1$ divides 12, and otherwise tmf_p is never a quasi-free u-module.*

We hope that such splittings may simplify computations of tmf_p-based Adams spectral sequences. In general, it seems that the cofibre of a certain map $\bigoplus_{(p-1)/2} \Sigma^d \mathrm{u} \to \mathrm{tmf}_p$ is equivalent as a u-module to a sum of shifts of ℓ; see Conj.10.3.4. Another speculative application of u is its close relation to forms of $\mathrm{BP}\langle 2\rangle$. Inspired by topological K-theory yet again, we define a *height two image of J spectra* j_2, whose formal properties immediately describe part of the image of the unit map $\mathbf{S} \to \mathrm{j}_2$ in homotopy.

Theorem J (Th.10.4.2)**.** *At the prime 2, all elements $\alpha_{i/j} \in \pi_{2i-1}\mathbf{S}_2$ detected by classes in the 1-line of the Adams–Novikov spectral sequence for the sphere have nontrivial image in $\pi_{2i-1}\mathrm{j}_2$.*

Our applications of stable operations on TMF (and tmf) indicate that even more highly structured operations would yield stronger results, and how stronger results in homotopy theory, such as a calculation of the Adams spectral sequence for j_2, would yield strong number-theoretic congruences. This continues the ongoing theme of this thesis: more sophisticated homotopy theoretic techniques yield stronger number theoretic results.

1.2 Outline

This thesis contains three main parts and three additional small appendices. We have tried to phrase each chapter in isolation, so the reader interested in stable Hecke operators can jump straight to Chapter 7 without having to go through all of Part I, for example. The same goes for the calculations and applications found in Chapters 9 and 10.

1.2.1 Summary of Part I

The first part of this thesis concerns a discussion, proof, and applications of a statement known as *Lurie's theorem* (Th. A). Originally stated in [BL10, Th.8.1.4] without proof, this powerful theorem is crucial to the rest of this thesis, so we conclude that here and now is the correct place to give a proof. We also discuss applications to TMF and beyond.

Chapter 2 – The statement of *Lurie's theorem* is a little involved, so in this chapter, we discuss the precise statement thereof (Th.2.1.7). This involves defining the sites upon which the sheaf $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ of Lurie's theorem is defined (Df.2.1.6) and discussing the motivation behind some of the conditions on the objects defining these sites. We also conclude with an outline of our proof (§2.2).

Chapter 3 – The phrase *formally étale* is not discussed in [SAG], however, it can be used to simplify some arguments revolving around deformation theory in spectral algebraic geometry. In this chapter, we first define what formally étale means for morphisms between presheaves of discrete rings (§3.1), then for presheaves of connective $\mathbf{E}_\infty$-rings (§3.2), as well as discuss their basic properties and some examples. In § 3.3, we show that applying this theory to formally étale maps into the moduli stack of p-divisible groups can be uniquely realised in spectral algebraic geometry (Th.3.3.5).

Chapter 4 – Here we define the sheaf $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ in the statement of Lurie's theorem and prove it satisfies the desired conditions. The construction (Df.4.3.1) of this sheaf applies the deformation theory of Chapter 3 together with a globalisation (§4.1) of the orientation theory of [EC2]. This leads to a natural definition of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ and we use Lurie's orientation theory to further identify the image of affine objects under $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ with an *orientation classifier* (Pr.4.2.4). In §4.3, we use this construction of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ to prove Lurie's theorem.

Chapter 5 – The strength of Lurie's theorem lies in its wide variety of applications. In this chapter, we prove that many of our favourite stable homotopy types can be recovered from $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ and we emphasise the use of Lurie's theorem to obtain operations on these cohomology theories. In particular, we discuss topological K-theory (§5.1) and periodic topological modular forms (§5.3) in-depth, and mention Lubin–Tate theories (which naturally generalise to Barsotti–Tate theories here) (§5.2) and topological automorphic forms (§5.4). We also initiate a general study of stable Adams operations (§5.5) on sections of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$—a precursor for the operations to come in Part II.

1.2.2 Summary of Part II

The middle part of this thesis provides us with the Adams operations, Hecke operators, and Atkin–Lehner involutions on topological modular forms and proves their basic properties.

Chapter 6 – In this chapter, we prove two functoriality results concerning the sheaf $\mathscr{O}^{\mathrm{top}}$ defining TMF. First, we show that $\mathscr{O}^{\mathrm{top}}$ can be extended (Th.B) from the small étale site of $\mathcal{M}_{\mathrm{Ell}}$ to another site $\mathrm{Isog}_{\mathrm{Ell}}$ (Df.6.1.4) which also captures isogenies of elliptic curves of invertible degree. This is done using Lurie's theorem of Part I together with some rational information. Next, we show that $\mathscr{O}^{\mathrm{top}}$ can be further extended to a kind of *spectral Mackey functor* (Th.C) which encodes the homotopy coherence of transfer maps of finite morphisms in $\mathrm{Isog}_{\mathrm{Ell}}$.

Chapter 7 – The titular stable operations are defined in this chapter. In §7.1, we define the stable Adams operations ψ^n on $\mathrm{TMF}[\frac{1}{n}]$ and discuss their basic properties (Th. 7.1.2). In § 7.2, we define stable Hecke operators, show their compatibility with the stable Adams operations (Pr.7.2.4), and state our desired composition formula (Th.7.2.7). This latter statement requires two more sections §7.3 and 7.4 to prove, as we have to carefully study the stacks involved in the definition of composing two Hecke operators—something that ought to be classical, and yet this approach seems to be new. In §7.5, we compare these stable Hecke operators on TMF with the classical Hecke operators arising in number theory (Pr.7.5.3). In §7.6, we define Atkin–Lehner involutions w_Q on $\mathrm{TMF}_0(N)$ and discuss there basic properties. We finish this chapter with a short section explaining a patchwork solution to extending some of these stable operations to the compactified moduli stack $\overline{\mathcal{M}}_{\mathrm{Ell}}$ and hence to Tmf. In particular, we outline how one could obtain stable Adams operations on Tmf, although the answer is far from satisfactory. We only use this final section as an opportunity to emphasise how clean our constructions are over $\mathcal{M}_{\mathrm{Ell}}$ and to provide extra examples to discuss in Part III.

1.2.3 Summary of Part III

The final part of this thesis is concerned with the basic calculations and some applications of the stable operations constructed in Part II. We will assume at this stage that the reader is familiar with some of the basics of topological modular forms, as can be found in [DFHH14] and [Beh20], for example.

Chapter 8 – In this chapter, we discuss general tools for calculating the effect of endomorphisms of the spectra tmf and TMF on homotopy groups. The first is Th.E, which states that a torsion-free element of $\pi_* \operatorname{tmf}$ is sent to a torsion-free element under an endomorphism of tmf, which allows one to carry out calculations by rationalisation. The second is Th.F, which allows us to compute the effect of endomorphisms of tmf on arbitrary elements of $\pi_* \operatorname{tmf}$ using the descent spectral sequence without any fear of extension problems. Both of these statements are proved using the theory of synthetic spectra (or equivalently, **C**-motivic homotopy theory) and the object mmf of *motivic modular forms*. We discuss the basic properties of this object and its bigraded homotopy groups in §8.1, and use these calculations to prove Th.E and Th.F. We also discuss how Anderson and Serre duality can be used to extend our calculations to Tmf (§8.4).

Chapter 9 – Using the general theory of Chapter 8, we can now calculate the effect of our Adams operations and Hecke operators on the homotopy groups of TMF. Apart from Chapters 6 and 7 where we build some technology and define these stable operations, we think this section will be the most useful to the general homotopy theory community, as the formulae found in Th.G are reasonably easy to use. We end this chapter with a conjecture (Conj.9.1.2) on the duals of multiplicative endomorphisms of self-dual ring spectra inspired by our calculations, and some statements (§9.2) on the necessity to invert n to define stable Hecke operators.

Chapter 10 – In this final chapter, we advertise the utility of the stable Adams operations and Hecke operators of Chapter 7. Our first main application is using the existence of Hecke operators on TMF to obtain various congruences of modular forms. This allows us to recover known number-theoretic congruences due to Ramanujan as well as apply some newer-looking congruences to obtain improved bounds of Maeda's conjecture (Th.H). Our second main collection of results is applications in stable homotopy theory. By adapting ideas from topological K-theory due to Adams, we define a summand u of p-complete connective tmf_p at odd primes p, a kind of *height two Adams summand*. We show that, unlike the height one situation, tmf_p rarely splits as a sum of shifts of u (Th.I), although this always holds if we "invert Δ." Connections to the height one image of J spectrum j are also made, and we construct a spectrum j_2 whose homotopy groups contain $\pi_* \mathrm{j}$ as a summand (Th.J). This eclectic collection of applications is hopefully just a warm-up for future applications.

1.2.4 Summary of appendices

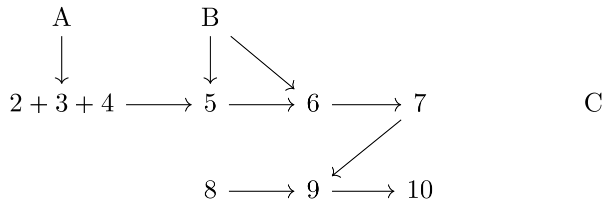
Our three appendices are of three very different flavours and are used to complement other parts of this thesis.

§ A Our first appendix was originally an appendix for Part I, as there we use many facts about *formal* spectral algebraic geometry which cannot (yet) be found in [SAG] but which are obvious extensions of ideas from elsewhere in [SAG].

§B This second appendix concerns another statement that can be found (and is vitally used) in the literature, but for which there is no publicly available proof. The proof is much more straightforward than that of Lurie's theorem, however, it still contains its subtleties. The statement claims that the sheaf $\mathscr{O}^{\mathrm{top}}$ used to define Tmf is uniquely defined (up to homotopy) as a sheaf of $\mathbf{E}_\infty$-rings which takes values in *elliptic cohomology theories*. This theorem is used time and time again in the literature to compare various constructions of Tmf and TMF, and we need to use it in §6.1 for the same reason.

§C The final appendix includes summaries for a general audience.

There is the following logical dependency of our chapters:



1.3 Notation, conventions, and background

Broadly speaking, we use the language of ∞-categories as a framework for our homotopy theory. What follows in this subsection is only particularly relevant to Part I as there we follow a few conventions the reader may not be familiar with.

Higher categories and higher algebra

We will make free and extensive use of the language of ∞-categories, higher algebra, and spectral algebraic geometry, following [HTT09], [HA], [SAG], and especially the conventions listed in [EC2]. In particular:

- For an ∞-category $\mathcal{C}$ and two objects X and Y of $\mathcal{C}$, we write $\mathrm{Map}_{\mathcal{C}}(X, Y)$ for the mapping space and $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for the set of maps if $\mathcal{C}$ happens to be the nerve of a 1-category.
- We will write $X[n]$ for the n-fold suspension of an object X, so $X[1]$ is the pushout of $* \leftarrow X \rightarrow *$ in an ∞-category $\mathcal{C}$ with finite colimits and a terminal object $*$. In stable ∞-categories, n can be negative.

- All rings will be commutative, and commutative rings and abelian groups will be treated as discrete $\mathbf{E}_\infty$-rings and spectra. Moreover, the smash product of spectra will be written as $\otimes$ even if the spectra involved are discrete (this does not mean the output will be discrete). The same goes for completions, and in this case the ∞-categorical completions will be written as $(-)^\wedge_I$ following [SAG, §7].

- All module categories Mod_R refer to the stable ∞-category of R-modules, where R is an $\mathbf{E}_\infty$-ring. In particular, if R is a discrete commutative ring, then Mod_R will be the stable ∞-category of R-module spectra, and **not** the abelian 1-category of R-modules. The same holds for ∞-categories of quasi-coherent sheaves.

- Following [EC2] (and contrary to [SAG] and [EC1]), we will write $\operatorname{Spec} R$ for the nonconnective spectral Deligne–Mumford stack associated to an $\mathbf{E}_\infty$-ring R.

Moreover, all n-categories are $(n,1)$-categories, for $n = 1, 2, \infty$ (except very briefly in Df.6.1.4)

Sites and sheaves

Lurie's theorem concerns sheaves between ∞-categories. The ∞-categories which we want to consider as sites are not necessarily (essentially) small, so we a priori do need to be careful about potential size issues. However, we are interested in constructing particular functors and proving they are sheaves, so we only really need to step into a large universal to quantify the definition of a sheaf.

Definition 1.3.1. Given ∞-category $\mathcal{T}$ with a Grothendieck topology τ ([HTT09, Df.6.2.2.1]) and an ∞-category $\mathcal{C}$ then a functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{C}$ is a *$\mathcal{C}$-valued τ-sheaf on $\mathcal{T}$* if for all τ-sieves $\mathcal{T}^0_{/X} \subseteq \mathcal{T}_{/X}$, the composite

$$\left((\mathcal{T}^0_{/X})^{\triangleright}\right)^{\mathrm{op}} \to \left((\mathcal{T}_{/X})^{\triangleright}\right)^{\mathrm{op}} \to \mathcal{T}^{\mathrm{op}} \xrightarrow{F} \mathcal{C}$$

is a limit diagram inside $\mathcal{C}$.

A *hypercover* is a generalisation of a cover in a Grothendieck site. In general, our sheaves, including the sheaf occurring in the statement of Th. A will be hypersheaves. Following [SAG, §A], this variation on a sheaf comes with a more concrete description.

Definition 1.3.2 ([SAG, Df.A.5.7.1])**.** Let $\Delta_{s,+}$ denote the 1-category whose objects are linearly ordered sets of the form $[n] = \{0 < 1 < \cdots < n\}$ for $n \geqslant -1$, and whose morphisms are strictly increasing functions. We will omit the $+$ when considering the full ∞-subcategory with $n \geqslant 0$. If $\mathcal{T}$ is an ∞-category, we will refer to a functor $X_\bullet\colon \Delta^{\mathrm{op}}_{s,+} \to \mathcal{T}$ as an *augmented semisimplicial object of $\mathcal{T}$*. When $\mathcal{T}$ admits finite limits, then for each $n \geqslant 0$, we can associate to an augmented semisimplicial object $X_\bullet$ the *nth matching object* and its associated *matching map*

$$X_n \to \lim_{[i]\hookrightarrow[n]} X_i = M_n(X_\bullet)$$

where the limit above is taken over all injective maps $[i] \hookrightarrow [n]$ such that $i < n$. Given a collection of morphisms S inside $\mathcal{T}$, we call an augmented semisimplicial object $X_\bullet$ an *S-hypercover* (for $X_{-1} = X$) if the matching maps belong to S for every $n \geqslant 0$. Given a Grothendieck topology τ on $\mathcal{T}$, then a presheaf of spectra F on $\mathcal{T}$ is called a *τ-hypersheaf* if for all τ-hypercovers $X_\bullet$, the natural map

$$F(X_{-1}) \to \lim_{\Delta_s^{\mathrm{op}}} F(X_\bullet)$$

is an equivalence of spectra. Some useful general references for the prefix *hyper* in the homotopy theory of sheaves are [CM21], [DHI04], and [SAG, §A-D].

Given $\mathcal{T}$ and τ from Df.1.3.2, then for each τ-covering family $\{C_i \to C\}$ in $\mathcal{C}$ one can associate a Čech nerve $C_\bullet$ which is a τ-hypercover of C. It is then clear that τ-hypersheaves are τ-sheaves. It is also obvious that if $S \subseteq S'$ then S'-hypersheaves are S-hypersheaves. We find the following diagram of implications useful, and they will often be used implicitly:

$$\begin{array}{ccc} \text{fpqc hypersheaf} & \Longrightarrow & \text{fpqc sheaf} \\ \Downarrow & & \Downarrow \\ \text{étale hypersheaf} & \Longrightarrow & \text{étale sheaf} \end{array}$$

Let us now state two lemmata regarding hypersheaves. Recall the special case $\mathrm{Un}\colon \mathscr{C}\mathrm{at}_{\infty/\mathcal{S}} \to \mathscr{C}\mathrm{at}_\infty$ of the *unstraightening functor* of [HTT09, §3.2].

Lemma 1.3.3. *Let $\mathcal{T}$ be an ∞-category with a Grothendieck topology τ and let $F\colon \mathcal{T}^{\mathrm{op}} \to \mathscr{C}\mathrm{at}_{\infty/\mathcal{S}}$ be a τ-sheaf such that the composite*

$$G\colon \mathcal{T}^{\mathrm{op}} \xrightarrow{F} \mathscr{C}\mathrm{at}_{\infty/\mathcal{S}} \to \mathscr{C}\mathrm{at}_\infty$$

is also a τ-sheaf, where the second functor is the canonical projection. Then the functor H defined by the composite

$$\mathcal{T}^{\mathrm{op}} \xrightarrow{F} \mathscr{C}\mathrm{at}_{\infty/\mathcal{S}} \xrightarrow{\mathrm{Un}} \mathscr{C}\mathrm{at}_\infty$$

is a τ-sheaf. If F and G are τ-hypersheaves, then H is a τ-hypersheaf.

More informally, applying a Grothendieck construction to a sheaf is a sheaf.

Proof. Write $\coprod_\alpha C_\alpha \to C$ for a τ-cover of an object C in $\mathcal{T}$. We then note the following composite of natural equivalences is equivalent to the natural map $H(C) \to \lim H(C_\alpha)$:

$$H(C) = \mathrm{Un}(F(C)\colon G(C) \to \mathcal{S}) \xrightarrow{\simeq} \mathrm{Un}(\lim F(C_\alpha)\colon \lim G(C_\alpha) \to \mathcal{S})$$

$$\xrightarrow{\simeq} \lim \mathrm{Un}(F(C_\alpha)\colon G(C_\alpha) \to \mathcal{S}) = \lim H(C_\alpha)$$

The first equivalence comes from the fact that F and G are both τ-sheaves and the second equivalence is from the fact that Un is a right adjoint. The proof for τ-hypersheaves is the same, with τ-covers replaced by τ-hypercovers. □

Lemma 1.3.4 ([SAG, Cor.D.6.3.4 & Th.D.6.3.5]). *The identity functor on* CAlg *is a hypercomplete* CAlg*-valued sheaf (with respect to the fpqc topology). In particular, for any* $\mathbf{E}_\infty$*-ring* R *and any fpqc hypercover* $R^\bullet$ *of* R*, the following natural map is an equivalence:*

$$R \xrightarrow{\simeq} \lim R^\bullet$$

Notice that if $R \to R^\bullet$ is an fpqc hypercover of an $\mathbf{E}_\infty$-ring R, then there are natural equivalences

$$\tau_{\geqslant 0} R \xrightarrow{\simeq} \tau_{\geqslant 0} \lim R^\bullet \xrightarrow{\simeq} \lim \tau_{\geqslant 0} R^\bullet \tag{1.3.5}$$

from the above lemma and the fact that $\tau_{\geqslant 0}\colon \mathrm{CAlg} \to \mathrm{CAlg}^{\mathrm{cn}}$ commutes with limits as a right adjoint.

Topological rings and formal stacks

With experience, one knows that the study of deformation theory comes hand-in-hand with the study of rings with a topology and the associated algebraic geometry. We will follow the definition of an adic $\mathbf{E}_\infty$-ring from [EC2, Df.0.0.11] except we will only consider the *connective* case.

Definition 1.3.6. An *adic ring* A is a discrete ring with a topology defined by an I-adic topology for some finitely generated ideal of definition $I \subseteq A$. Morphisms between adic rings are continuous ring homomorphisms, defining a subcategory $\mathrm{CAlg}^{\heartsuit}_{\mathrm{ad}}$ of $\mathrm{CAlg}^{\heartsuit}$. An *adic* $\mathbf{E}_\infty$*-ring* is a connective $\mathbf{E}_\infty$-ring A such that $\pi_0 A$ is an adic ring. We define the ∞*-category of adic* $\mathbf{E}_\infty$*-rings* as the following fibre product:

$$\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}} = \mathrm{CAlg}^{\mathrm{cn}} \underset{\mathrm{CAlg}^{\heartsuit}}{\times} \mathrm{CAlg}^{\heartsuit}_{\mathrm{ad}}$$

An adic $\mathbf{E}_\infty$-ring A is said to be *complete* if it is complete with respect to an ideal of definition I; see [SAG, Df.7.3.1.1 & Th.7.3.4.1]. An $\mathbf{E}_\infty$-ring R is *local* if $\pi_0 R$ is a local ring, and we call an adic $\mathbf{E}_\infty$-ring R local if the topology on $\pi_0 R$ is defined by the maximal ideal of $\pi_0 R$. We give $\mathrm{CAlg}^{\heartsuit}_{\mathrm{ad}}$ and $\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}}$ the fpqc and étale topologies via the forgetful functors to $\mathrm{CAlg}^{\heartsuit}$ and $\mathrm{CAlg}^{\mathrm{cn}}$, respectively.

The definition of a formal (spectral) Deligne–Mumford stack follows.

Definition 1.3.7. Let $\mathrm{Spf}\colon \mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}} \to \infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ be the functor described in [SAG, Con.8.1.1.10 & Pr.8.1.2.1]—here $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ is the ∞-category of strictly Henselian spectrally ringed ∞-topoi of [SAG, Con.1.4.1.3 & Df.1.4.2.1]. A spectrally ringed ∞-topos $\mathfrak{X}$ is said to be an *affine formal spectral Deligne–Mumford stack* if it lies in the essential image of Spf. A *formal spectral Deligne–Mumford stack* is a spectrally ringed ∞-topos with a cover by affine formal spectral Deligne–Mumford stacks; see [SAG, Df.8.1.3.1]. Let fSpDM denote the full ∞-subcategory of $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$ spanned by formal spectral Deligne–Mumford stacks. Similarly, one can define a 2-category fDM of classical formal Deligne–Mumford stacks (Df. A.1.1)—in the classical case, we will further assume all formal Deligne–Mumford stacks are *locally Noetherian.*

Definition 1.3.8. Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne-Mumford stack. We call an object U inside $\mathcal{X}$ *affine* if the locally spectrally ringed ∞-topos $(\mathcal{X}_{/U}, \mathscr{O}_{\mathfrak{X}}|_U)$ is equivalent to Spf A for some adic $\mathbf{E}_\infty$-ring A. We will also say that $\mathfrak{X}$ is *locally Noetherian* if for every affine object U of $\mathcal{X}$, the $\mathbf{E}_\infty$-ring $\mathscr{O}_{\mathfrak{X}}(U)$ is Noetherian in the sense of [HA, Df.7.2.4.30].

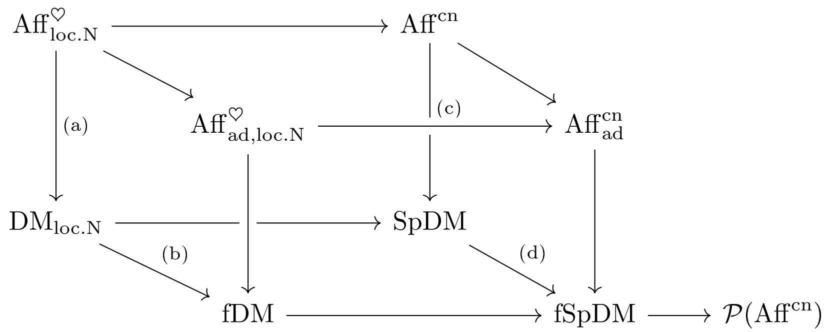
Note that Spf B is locally Noetherian if and only if B itself is a Noetherian $\mathbf{E}_\infty$-ring; see [SAG, Pr.8.4.2.2].

Functor of points

The classical moduli stack $\mathcal{M}^{\heartsuit}_{\mathrm{BT}^p}$ is neither a Deligne–Mumford nor an Artin stack. This necessitates our use of a functorial point of view, for classical, formal, and spectral algebraic geometry.

Notation 1.3.9. Write $\mathrm{Aff} = \mathrm{CAlg}^{\mathrm{op}}$, to which we will add superscripts and subscripts such as $(-)^{\mathrm{cn}}$, $(-)_{\mathrm{ad}}$, and $(-)^{\heartsuit}$ as they apply to CAlg.

When working in $\mathcal{P}(\mathrm{Aff}^{\heartsuit}) = \mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S})$ or $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}}) = \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, we will abuse notation and not distinguish between the objects representing functors and the functors themselves. This is justified by the following commutative diagram of fully faithful functors of ∞-categories:



(1.3.10)

The loc.N subscript denotes those full ∞-subcategories spanned by Noetherian or locally Noetherian objects; see Df.1.3.8. The definitions and fully faithfulness of the functors above are explained in Cor. A.1.5, except the functors (a)-(d), which can be justified as follows:

(a) is fully faithful as this holds without the locally Noetherian hypotheses; see [SAG, Rmk.1.2.3.6] and restrict to the underlying 2-category.

(b) is fully faithful by using part (d) below and Pr.A.1.4. Indeed, if $G \circ F$ and G are fully faithful, then so if F.

(c) is fully faithful by making a connective version of [SAG, Rmk.1.4.7.1]; this is justified by [SAG, Cor.1.4.5.3].

(d) is fully faithful as both SpDM and fSpDM being defined as particular full ∞-subcategories of $\infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$ and because spectral Deligne–Mumford stacks are formal spectral Deligne–Mumford stacks by [SAG, p. 628].

Similarly, we will consider most of classical algebraic geometry as living in the 2-category $\mathrm{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S}_{\leqslant 1})$ which we then embed inside the ∞-category $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ using the inclusion $\mathcal{S}_{\leqslant 1} \to \mathcal{S}$, which preserves limits.

Warning 1.3.11. When we consider quasi-coherent sheaves on a formal spectral Deligne–Mumford stack $\mathfrak{X}$, then what we write as $\mathrm{QCoh}(\mathfrak{X})$ is what Lurie would write as $\mathrm{QCoh}(h_{\mathfrak{X}})$, in other words, we consider the ∞-categories of quasi-coherent sheaves of formal spectral Deligne–Mumford stacks through their functors of points. By [SAG, Cor.8.3.4.6], we see that these two notations are equivalent as long as one restricts to *almost connective* quasi-coherent sheaves on both sides. As all of our quasi-coherent sheaves of interest will be cotangent complexes, which are almost connective by definition ([SAG, Df.17.2.4.2]), this distinction does not matter to us.

Cotangent complexes

Given a natural transformation $X \to Y$ between functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ which admits a cotangent complex ([SAG, Df.17.2.4.2]), we will write this cotangent complex as $L_{X/Y}$ and consider it as an object of $\mathrm{QCoh}(X)$; see [SAG, §6.2]. A few specific cases can be made more explicit.[5]

(1) If $\mathsf{X} \to \mathsf{Y}$ is a morphism of spectral Deligne–Mumford stacks and $X \to Y$ is the associated transformation of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, then $L_{X/Y}$ is equivalent to $L_{\mathsf{X}/\mathsf{Y}}$, defined in [SAG, Df.17.1.1.8] using spectrally ringed ∞-topoi, under the equivalence of categories $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(\mathsf{X})$ by [SAG, Cor.17.2.5.4]. If $\mathsf{X} = \operatorname{Spec} B$ and $\mathsf{Y} = \operatorname{Spec} A$, then we have further identifications of $L_{X/Y}$ with $L_{B/A}$, defined in [HA] using $\mathbf{E}_\infty$-rings, under the equivalence of ∞-categories $\mathrm{QCoh}(\operatorname{Spec} A) \simeq \mathrm{Mod}_A$; see [SAG, Lm.17.1.2.5].

(2) If $\mathfrak{X}$ is a formal spectral Deligne–Mumford stack, and X is the associated functor in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, then L_X is equivalent to $L^{\wedge}_{\mathfrak{X}}$, the *completed cotangent complex* of [SAG, Df.17.1.2.8], under the equivalence of categories

$$\Theta_{\mathfrak{X}} \colon \mathrm{QCoh}(\mathfrak{X})^{\mathrm{acn}} \xrightarrow{\simeq} \mathrm{QCoh}(X)^{\mathrm{acn}}$$

[5] Thank you to an anonymous referee of [Dav20] for vastly simplifying example 3 for us.

of [SAG, Cor.8.3.4.6], where the superscript acn indicates full ∞-subcategories of almost connective objects. If $\mathfrak{X} = \operatorname{Spf} A$ for an adic $\mathbf{E}_\infty$-ring A, then $L_{\operatorname{Spf} A}$ corresponds to $(L_A)^\wedge_I$ (under the equivalence $\mathrm{QCoh}(\operatorname{Spf} A) \simeq \mathrm{Mod}_A^{\mathrm{cpl}}$ of ∞-categories, where I is a finitely generated ideal of definition for the topology on $\pi_0 A$; see [SAG, Ex.17.1.2.9].

(3) If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a morphism of formal Deligne–Mumford stacks and we write $F\colon X \to Y$ for the associated morphism of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, then the cofibre $L^\wedge_{\mathfrak{X}/\mathfrak{Y}}$ of the natural map $f^\star L_{\mathfrak{Y}} \to L_{\mathfrak{X}}$ is naturally equivalent to $L_{X/Y}$ under the equivalence of categories $\Theta_{\mathfrak{X}}\colon \mathrm{QCoh}(\mathfrak{X})^{\mathrm{acn}} \xrightarrow{\simeq} \mathrm{QCoh}(X)^{\mathrm{acn}}$; see [SAG, Df.17.1.2.8] for a definition of $L_{\mathfrak{X}/\mathfrak{Y}}$. Indeed, the naturality of $\Theta_{\mathfrak{X}}$ in $\mathfrak{X}$ ([SAG, Con.8.3.4.1]) yields an equivalence $\Theta_{\mathfrak{X}} \circ f^\star \simeq F^* \circ \Theta_{\mathfrak{Y}}$ of functors. Our desired identification then follows from the existence of the (co)fibre sequences

$$f^\star L_{\mathfrak{Y}} \to L_{\mathfrak{X}} \to L_{\mathfrak{X}/\mathfrak{Y}} \qquad\qquad F^* L_Y \to L_X \to L_{X/Y},$$

the absolute case above§, and the fact that $\mathrm{QCoh}(\mathfrak{X})^{\mathrm{acn}}$ and $\mathrm{QCoh}(X)$ are stable under (co)fibre sequences; see [SAG, Cor.8.2.4.13 & Pr.6.2.3.4], respectively.

Due to the equivalences above, we will drop the completion symbol from our notation for the cotangent complex between formal spectral Deligne–Mumford stacks. The following standard properties of the cotangent complex of functors will be used without explicit reference:

- For a map of connective $\mathbf{E}_\infty$-rings $A \to B$, we have a natural equivalence in $\mathrm{Mod}_{\pi_0 B}$
$$\pi_0 L_{B/A} \simeq \Omega^1_{\pi_0 B/\pi_0 A};$$
see [HA, Pr.7.4.3.9].
- For composable transformations of functors $X \to Y \to Z$ in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, where each functor (or each transformation) has a cotangent complex, we obtain a canonical (co)fibre sequence in $\mathrm{QCoh}(X)$
$$L_{Y/Z}\big|_X \to L_{X/Z} \to L_{X/Y};$$
see [SAG, Pr.17.2.5.2].
- If we have transformations $X \to Y \leftarrow Y'$ of functors inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, where $L_{X/Y}$ exists, then $L_{X \times_Y Y'/Y'}$ exists and is naturally equivalent to $\pi_1^* L_{X/Y}$; see [SAG, Rmk.17.2.4.6].

Warning 1.3.12 (Topological vs algebraic cotangent complexes). The cotangent complexes considered in this article are **not** the same as those developed by André and Quillen; see [Sta, 08P5]. In particular, for an ordinary commutative ring R considered as a discrete $\mathbf{E}_\infty$-ring, then L_R is what some call the *topological cotangent complex*. For more discussion, see [SAG, §25.3].

Deformation theory

We will be using ideas from classical deformation theory as well as Lurie's spectral deformation theory, so we take a moment here to clarify our definitions. What we discuss below is mostly taken from [EC2, §3].

Definition 1.3.13. Let $\mathbf{G}_0$ be a p-divisible group over a commutative ring R_0 and write $\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cpl}}$ for the ∞-subcategory of $\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cn}}$ spanned by complete connective adic $\mathbf{E}_\infty$-rings. Define a functor $\mathrm{Def}_{\mathbf{G}_0}\colon \mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cpl}} \to \mathcal{S}$ by the formula

$$\mathrm{Def}_{\mathbf{G}_0}(A) = \operatorname*{colim}_{I} \left(\mathrm{BT}^p(A) \underset{\mathrm{BT}^p(\pi_0 A/I)}{\times} \mathrm{Hom}_{\mathrm{CRing}}(R_0, \pi_0 A/I) \right)$$

where the colimit is indexed over all finitely generated ideals of definition I for $\pi_0 A$. A priori $\mathrm{Def}_{\mathbf{G}_0}(A)$ is an ∞-category, but [EC2, Lm.3.1.10] states this is an ∞-groupoid. Let $(R, \mathbf{G})$ be a *deformation* of $\mathbf{G}_0$, so an element inside $\mathrm{Def}_{\mathbf{G}_0}(A)$; see [EC2, Df.3.1.4]. We say $\mathbf{G}$ is the *universal spectral deformation* of $\mathbf{G}_0$ with *spectral deformation ring* R if for every A in $\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cpl}}$, the natural map

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cpl}}}(R, A) \xrightarrow{\simeq} \mathrm{Def}_{\mathbf{G}_0}(A)$$

is an equivalence. If R is discrete, we say $\mathbf{G}$ is the *universal classical deformation* of $\mathbf{G}_0$ with *classical deformation ring* R if for every discrete A in $\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cpl}}$, the natural map

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cpl}}}(R, A) \xrightarrow{\simeq} \mathrm{Def}_{\mathbf{G}_0}(A)$$

is an equivalence. If such universal spectral (or classical) deformations $(R, \mathbf{G})$ exist, they are evidently uniquely determined by the pair $(R_0, \mathbf{G}_0)$.

The above definition agrees with [EC2, Df.3.1.11] in the cases that the R above is connective. Indeed, in this case, if A is a nonconnective complete adic $\mathbf{E}_\infty$-ring, the fact connective cover is a right adjoint and $\mathrm{BT}^p(A) = \mathrm{BT}^p(\tau_{\geqslant 0} A)$ by definition yields the following equivalences:

$$\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(R, A) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{ad}}}(R, \tau_{\geqslant 0} A) \simeq \mathrm{Def}_{\mathbf{G}_0}(\tau_{\geqslant 0} A) \simeq \mathrm{Def}_{\mathbf{G}_0}(A)$$

The following will help us identify many classical deformation rings.

Remark 1.3.14. If a spectral deformation ring R exists for a pair $(R_0, \mathbf{G}_0)$, then a classical deformation ring also does, and it can be taken to be $\pi_0 R$. Indeed, if B is a discrete object of $\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cpl}}$ as in Df.1.3.13, then the fact the truncation functor is a left adjoint on connective objects yields the equivalences

$$\mathrm{Def}_{\mathbf{G}_0}(B) \simeq \mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cpl}}}(R, B) \simeq \mathrm{Map}_{\mathrm{CAlg}_{\mathrm{ad}}^{\mathrm{cpl}}}(\pi_0 R, B)$$

showing that $\pi_0 R$ is the classical deformation ring of $(R_0, \mathbf{G}_0)$.

Part I

A proof of Lurie's theorem

Chapter 2

Statement and outline

> Mit der größtmöglichen Deutlichkeit erblickt man die winzigsten Details. Es ist, als schaute man zugleich durch ein umgekehrtes Fernrohr und durch ein Mikroskop.
>
> *W. G. Sebald, Die Ringe des Saturns*

The titular theorem promises the existence of a sheaf $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ on some site over the classical moduli stack of p-divisible groups satisfying certain properties. The idea behind the definition of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ is to construct morphisms of stacks

$$\mathcal{M}^{\heartsuit}_{\mathrm{BT}^p_n} \xrightarrow{\mathcal{D}} \mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n} \xleftarrow{\Omega} \mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$$

and set $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n} = \mathcal{D}^*\Omega_*\mathscr{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ and check this possesses the desired properties. The maps of stacks above do not quite exist in our set-up, but the above formula for $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ is instructive. In this chapter, we state a precise version of Lurie's theorem and give a detailed outline of the proof.

2.1 The precise statement

First, let us recall the definition of a p-divisible group over an $\mathbf{E}_\infty$-ring; see [EC2, Df.2.0.2] for this definition, and [EC1, §6] or [EC3, §2] for a wider discussion.

Definition 2.1.1. Let R be a connective $\mathbf{E}_\infty$-ring. A *p-divisible (Barsotti–Tate) group* over R is a functor $\mathbf{G}\colon \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ with the following properties:

1. For every connective $\mathbf{E}_\infty$-R-algebra B, the $\mathbf{Z}$-module $\mathbf{G}(B)[1/p]$ vanishes.
2. For every finite abelian p-group M, the functor

   $$\mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S} \qquad B \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathbf{G}(B))$$

   is corepresented by a finite flat $\mathbf{E}_\infty$-R-algebra, which we write as $\mathbf{G}(M)$.

3. The map $p\colon \mathbf{G} \to \mathbf{G}$ is locally surjective with respect to the finite flat topology.

A p-divisible group over a general $\mathbf{E}_\infty$-ring R, is a p-divisible group over its connective cover. The ∞-category $\mathrm{BT}^p(R)$ of p-divisible groups over an $\mathbf{E}_\infty$-ring R is the full ∞-subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R}, \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}})$ spanned by p-divisible groups. Let $\mathcal{M}_{\mathrm{BT}^p}$ be the *moduli stack of p-divisible groups*, which is the functor inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ defined on objects by sending R to the ∞-groupoid core $\mathrm{BT}^p(R)^{\simeq}$; see [EC2, Df.3.2.1]. We say a p-divisible group $\mathbf{G}$ has *height n* if the finite $\mathbf{E}_\infty$-R-algebra $\mathbf{G}(\mathbf{Z}/p\mathbf{Z})$ has rank p^n; see [EC1, §6.5]. Using this notion of height, we can further define a subfunctor $\mathcal{M}_{\mathrm{BT}^p_n}$ for all $n \geqslant 1$ consisting of all p-divisible groups of *height n*.

The reader is invited to check for herself that the definition above agrees with that of [Tat67, §2] when R is discrete.

Remark 2.1.2 (Height is an open condition). We claim $\mathcal{M}_{\mathrm{BT}^p_n} \to \mathcal{M}_{\mathrm{BT}^p}$ is an open embedding. Lurie's definition of a commutative finite flat group scheme over R ([EC1, Df.6.1.1]) states that $\pi_0 R \to \pi_0\mathbf{G}(\mathbf{Z}/p\mathbf{Z})$ realises $\pi_0\mathbf{G}(\mathbf{Z}/p\mathbf{Z})$ as a projective $\pi_0 A$-module of finite rank equal to p^n. By [Sta, 00NX], this rank is locally (with respect to the Zariski topology on $|\operatorname{Spec} R| = |\operatorname{Spec} \pi_0 R|$) constant. In particular, if R is a local connective $\mathbf{E}_\infty$-ring then the commutative finite flat group scheme $\mathbf{G}(\mathbf{Z}/p\mathbf{Z})$ has a well-defined height and we obtain the formula:

$$\operatorname{Spec} R \underset{\mathcal{M}_{\mathrm{BT}^p}}{\times} \mathcal{M}_{\mathrm{BT}^p_n} \simeq \begin{cases} \operatorname{Spec} R & \mathrm{ht}(\mathbf{G}) = n \\ \varnothing & \mathrm{ht}(\mathbf{G}) \neq n \end{cases}$$

Definition 2.1.3. Let $\mathfrak{X}$ be a formal spectral Deligne–Mumford stack. A *p-divisible group over* $\mathfrak{X}$ is a natural transformation $\mathbf{G}\colon \mathfrak{X} \to \mathcal{M}_{\mathrm{BT}^p}$ in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. We say $\mathbf{G}$ has *height n* if this map factors through $\mathcal{M}_{\mathrm{BT}^p_n}$. By [EC2, Pr.3.2.2(4)], this is equivalent to a coherent family of p-divisible groups $\mathbf{G}_{B_i}$ on $\operatorname{Spec}(B_i)^{\wedge}_{J_i}$, where the collection $\{\operatorname{Spf} B_i \to \mathfrak{X}\}_i$ form an affine étale cover of $\mathfrak{X}$ and J_i is an ideal of definition for B_i.

Our main object of study is the spectral moduli stack $\mathcal{M}_{\mathrm{BT}^p_n}$, although we are also interested in its relationship to the underlying classical moduli stack.

Notation 2.1.4. For a functor $\mathcal{M}\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, write $\mathcal{M}^{\heartsuit}$ for its restriction along $\mathrm{CAlg}^{\heartsuit} \to \mathrm{CAlg}^{\mathrm{cn}}$. We will only use this notation in Part I. This commutes with finite products:

$$(X \times Y)^{\heartsuit} \xrightarrow{\simeq} X^{\heartsuit} \times Y^{\heartsuit}$$

Given an adic $\mathbf{E}_\infty$-ring B, write $\widehat{\mathcal{M}}_B$ for the product $\widehat{\mathcal{M}}_B = \mathcal{M} \times \operatorname{Spf} B$ in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$—the hat indicates a base-change over Spf, rather than Spec.

Notation 2.1.5 (Fixed adic $\mathbf{E}_\infty$-ring A)**.** Let A denote a fixed complete local Noetherian adic $\mathbf{E}_\infty$-ring with perfect residue field of characteristic p. Write A_0 for $\pi_0 A$, $\mathfrak{m}_A$ for the maximal ideal of A_0, and κ_A for the residue field.

The reader should keep in her mind the initial case of the p-complete sphere $A = \mathbf{S}_p$ with associated A_0 the p-adic integers $\mathbf{Z}_p$. Other choices include the spherical Witt vectors of a perfect field of characteristic p; see [EC2, §5.1].

We can now define the sites occurring in Lurie's theorem. Adjectives used in the definition below will be discussed shortly.

Definition 2.1.6. Recall the conventions of Nt.2.1.5. Let

$$\mathcal{C}_{A_0} \subseteq \mathcal{P}(\mathrm{Aff}^{\heartsuit})_{/\widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}}$$

denote the full ∞-subcategory spanned by those objects $\mathbf{G}_0 \colon \mathfrak{X}_0 \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$ where $\mathfrak{X}_0$ is a locally Noetherian qcqs[6] formal Deligne–Mumford stack with perfect residue fields[7] at all closed points, the cotangent complex[8] $L_{\mathfrak{X}_0/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}}$ is *almost perfect*[9] inside $\mathrm{QCoh}(\mathfrak{X}_0)$, and $\mathbf{G}_0$ is formally étale (§3.1) in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$. Similarly, let

$$\mathcal{C}_{A} \subseteq \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})_{/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}}$$

denote the full ∞-subcategory spanned by those objects $\mathbf{G} \colon \mathfrak{X} \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$ where $\mathfrak{X}$ is a locally Noetherian qcqs formal spectral Deligne–Mumford stack with perfect residue fields at all closed points and $\mathbf{G}$ is formally étale (§3.2) in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. We will endow $\mathcal{C}_{A_0}$ and $\mathcal{C}_A$ with both the fpqc and étale topologies through the forgetful map to $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ and $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, respectively.

A simplified criterion for an object $\mathfrak{X} \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$ to lie in $\mathcal{C}_{A_0}$ is discussed in Pr.2.1.9. For transparency, let us explain the adjectives in the definition of $\mathcal{C}_A$ and $\mathcal{C}_{A_0}$.

(Locally Noetherian) We assume our formal Deligne–Mumford stacks are locally Noetherian (Df.1.3.8) because completions of rings in the classical world and derived world do not necessarily agree; see [SAG, Warn.8.1.0.4]. Moreover, even in the world of spectral algebraic geometry such objects behave better ([SAG, §8.4]). For example, such objects have natural truncations; see Pr.A.2.1.

[6] A locally Noetherian and quasi-compact scheme is called a Noetherian scheme. We choose to keep these two adjectives separate though, as they play different roles in this thesis.

[7] As our fixed A is assumed to be p-complete, all these residue fields are necessarily of characteristic p.

[8] This relative cotangent complex exists as one does for $\mathfrak{X}_0$ and $\widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$—a consequence of [SAG, Pr.17.2.5.1] and [EC2, Pr.3.2.2], respectively.

[9] Paraphrasing [SAG, §6.2.5], recall that a quasi-coherent sheaf $\mathcal{F}$ on a functor $X \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is *almost perfect* if for all connective $\mathbf{E}_\infty$-rings R and all morphisms of presheaves $\eta \colon \operatorname{Spec} R \to X$, the R-module $\eta^* \mathcal{F}$ is almost perfect; see [HA, Df.7.2.4.10 & Pr.7.2.4.17] for the latter definition and a simple criterion for Noetherian $\mathbf{E}_\infty$-rings, respectively.

(Qcqs) This acronym stands for *quasi-compact and quasi-separated*; see Df.A.3.2. When a scheme X is qcqs, then it has an affine Zariski cover $\operatorname{Spec} A \to X$ (qc) and the fibre product $P = \operatorname{Spec} A \times_X \operatorname{Spec} A$ also has a Zariski cover $\operatorname{Spec} B \to P$ (qs). Eventually, we will define an étale (hyper) sheaf $\mathfrak{O}^{\mathrm{aff}}_{\mathrm{BT}^p_n}$ on the affine objects of $\mathcal{C}_A$, and to extend this to a formal Deligne–Mumford stack $\mathfrak{X}$ inside $\mathcal{C}_A$, we will rely upon the adjective qcqs; see Rmk.4.2.3. One could write this thesis again, with the word *separated* replacing the word *quasi-separated* and deleting all occurrences of the prefix *hyper*, although the extra generality of hypersheaves can be useful in practice.

(Formal geometry) One reason we work with the *formal* variety of spectral Deligne–Mumford stacks (§A and [SAG, §8]) is related to topological modular forms. In one interpretation of the classical Serre–Tate theorem, one must work with schemes where p is locally nilpotent, ie, over $\operatorname{Spf} \mathbf{Z}_p$; see Ex.3.1.7. Another reason is for deformation theoretic purposes. As stated in [EC2, Rmk.3.2.7]:

> *"The central idea in the proof of Theorem 3.1.15 (of [EC2]) is (...) to guarantee the representability of $\mathcal{M}_{\mathrm{BT}^p}$ in a* formal *neighborhood of any sufficiently nice R-valued point."*

As our moduli stack of interest is $\mathcal{M}_{\mathrm{BT}^p}$, we embrace formal spectral algebraic geometry.

(Closed points have perfect residue fields) A crucial step in showing our definition of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ satisfies the conditions of Th.2.1.7 is to reduce ourselves to the closed points of the affine objects of $\mathcal{C}_{A_0}$, essentially reducing us to the Lubin–Tate theories of [EC2, §5]. It will also be important that these residue fields are perfect (they will already be of characteristic p as we are working over $\operatorname{Spf} \mathbf{Z}_p$) to apply some of our formal arguments; see Pr.3.3.13.

(Formally étale over $\widehat{\mathcal{M}}_{\mathrm{BT}^p_n}$) Again, one inspiration for Lurie's theorem is the classical Serre–Tate theorem, which posits that $\widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{Ell},\mathbf{Z}_p}$ is formally étale over $\widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_2,\mathbf{Z}_p}$. The phrase formally étale is used in this thesis to control and package our deformation theory; see §3.

(Cotangent complex conditions in $\mathcal{C}_{A_0}$) These conditions are finiteness hypotheses, however, they are necessary to apply a deep existence criterion of Lurie; see Th.3.3.10.

The precise version of Lurie's theorem (Th.A) can now be stated.

Theorem 2.1.7 (Lurie's Theorem)**.** *Given an adic $\mathbf{E}_\infty$-ring A as in Nt.2.1.5, there is an étale hypersheaf of $\mathbf{E}_\infty$-rings $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ on $\mathcal{C}_{A_0}$ such that for a formal affine $\mathbf{G}_0\colon \operatorname{Spf} B_0 \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n}$ in $\mathcal{C}_{A_0}$ the $\mathbf{E}_\infty$-ring $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G}_0) = \mathcal{E}$ has the following properties:*

1. $\mathcal{E}$ *is* complex periodic[10] *and* Landweber exact.[11]

2. *There is a natural equivalence of rings* $\pi_0\mathcal{E} \simeq B_0$ *and* $\mathcal{E}$ *is complete with respect to an ideal of definition for* B_0. *In particular,* $\mathcal{E}$ *is* $\mathfrak{m}_A$*-complete, hence also* p*-complete.*

3. *The groups* $\pi_k\mathcal{E}$ *vanish for all odd integers* k. *Otherwise, there are natural equivalences of* B_0*-modules* $\pi_{2k}\mathcal{E} \simeq \omega_{\mathbf{G}_0}^{\otimes k}$ *where* $\omega_{\mathbf{G}_0}^{\otimes k}$ *is the* dualising line[12] *of the* identity component[13] $\mathbf{G}_0^\circ$ *of* $\mathbf{G}_0$.

4. *There is a natural equivalence of formal groups* $\mathbf{G}_0^\circ \simeq \widehat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0}$ *over* B_0 *where the latter is the* classical Quillen formal group[14] *of* $\mathcal{E}$.

We have included a few more details compared to the original statement [BL10, Th.8.1.4] by incorporating some work of Behrens–Lawson involving Landweber exactness.

Let us now discuss a simple criterion to check if an object lies in $\mathcal{C}_{A_0}$.

Definition 2.1.8. A morphism $f\colon \mathfrak{X}_0 \to \operatorname{Spf} A_0$ of classical formal Deligne–Mumford stacks is *locally of finite presentation* if for all étale maps $\operatorname{Spf} B_0 \to \mathfrak{X}_0$ the induced morphisms of rings $A_0 \to B_0$ are of finite presentation. By the usual arguments, it suffices to check this on a fixed collection of étale morphisms $\operatorname{Spf} B_0 \to \mathfrak{X}_0$ which cover $\mathfrak{X}_0$. We say f is of *finite presentation* if f is locally of finite presentation and quasi-compact (Df.A.3.2).

[10] Recall from [EC2, §4.1], that an $\mathbf{E}_\infty$-ring A is called *complex periodic* if A is *complex orientable* and *weakly 2-periodic*. An object E of $\mathrm{Sp}_{\mathbf{S}/}$ is said to be *complex orientable* if the map given map $e\colon \mathbf{S} \to E$ admits a factorisation $\overline{e}$:

$$\mathbf{S} \simeq S^2[-2] \simeq \mathbf{CP}^1[-2] \to \mathbf{CP}^\infty[-2] \xrightarrow{\overline{e}} E;$$

see [Ada74, §II] or [EC2, §4.1.1]. An $\mathbf{E}_\infty$-ring A is *weakly 2-periodic* if $A[2]$ is a locally free A-module of rank 1, or equivalently, that $\pi_2 A$ is a locally free $\pi_0 A$-module of rank 1 and the natural map $\pi_2 A \otimes_{\pi_0 A} \pi_{-2} A \to \pi_0 A$ is an equivalence. Notice this is a **condition**, not data.

[11] A formal group $\widehat{\mathbf{G}}$ over a ring R is *Landweber exact* if the defining map from $\operatorname{Spec} R$ to the moduli stack of formal groups is flat. A complex periodic $\mathbf{E}_\infty$-ring is *Landweber exact* if its associated Quillen formal group is.

[12] Recall from [EC2, §4.2.5], the *dualising line* of a formal group $\widehat{\mathbf{G}}$ over a commutative ring R is the R-linear dual of its Lie algebra $\mathrm{Lie}(\widehat{\mathbf{G}})$. This *Lie algebra* of a formal group can be defined in multiple ways, but we will define it as the tangent space of $\widehat{\mathbf{G}}$ over R at the unit section $\mathscr{O}_{\widehat{\mathbf{G}}} \to R$; see [Zin84] for a discussion about Lie algebras associated to formal groups or here for an English translation.

[13] Recall from [EC2, Th.2.0.8], for each p-divisible group $\mathbf{G}$ over a p-complete $\mathbf{E}_\infty$-ring R there is a unique formal group $\mathbf{G}^\circ$ over R such that on connective $\mathbf{E}_\infty$-$\tau_{\geqslant 0}R$-algebras A which are truncated and p-nilpotent we can describe $\mathbf{G}^\circ(A)$ as the fibre of $\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}})$ induced by the quotient by the nilradical; see [Tat67, (2.2)] for a classical reference.

[14] Recall from [EC2, Con.4.1.13], that a complex periodic $\mathbf{E}_\infty$-ring A comes with an associated *Quillen formal group* $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ over A. The *classical Quillen formal group* $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$ is the image of $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ under the functor $\mathrm{FGroup}(A) \to \mathrm{FGroup}(\pi_0 A)$, or equivalently as the formal spectrum $\operatorname{Spf} A^0\mathbf{CP}^\infty$. Notice the above definition is **independent** of the choice of complex orientation for A—such a choice would yield a chosen coordinate for our formal group, ie, a *formal group law*; see [Goe08, §2].

Proposition 2.1.9. *Let A be as in Nt. 2.1.5 and $\mathbf{G}_0\colon \mathfrak{X}_0 \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n,A_0}$ be a p-divisible group defined on a formal Deligne–Mumford stack $\mathfrak{X}_0$ of finite presentation over* $\operatorname{Spf} A_0$ *such that the associated map into $\widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n,A_0}$ is formally étale. Then $\mathbf{G}_0$ lies in $\mathcal{C}_{A_0}$.*

These simplified hypotheses are practical, but they do not apply to one of our favourite examples, Lubin–Tate theory, as power series rings $R[\![x]\!]$ are simply **never** of finite presentation over R.

Proof. First, we note that $\mathfrak{X}_0$ is locally Noetherian, qcqs, and has all residue fields corresponding to closed points perfect of characteristic p as the morphism $\mathfrak{X}_0 \to \operatorname{Spf} A_0$ is of finite presentation.[15] It remains to show that the cotangent complex in question,

$$L = L_{\mathfrak{X}_0/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}}$$

is almost perfect. To see this, we consider the composition in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$

$$\mathfrak{X}_0 \xrightarrow{\mathbf{G}_0} \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A} \xrightarrow{\pi_2} \operatorname{Spf} A$$

which induces the following (co)fibre sequence in $\mathrm{QCoh}(\mathfrak{X}_0)$:

$$\mathbf{G}_0^* L_{\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}/\operatorname{Spf} A} \to L_{\mathfrak{X}_0/\operatorname{Spf} A} \to L$$

Abbreviating the above to $\mathbf{G}_0^*L_1 \to L_2 \to L$, we first focus on $\mathbf{G}_0^*L_1$. As a quasi-coherent sheaf on a formal spectral Deligne–Mumford stack $\mathfrak{X}_0$, to see $\mathbf{G}_0^*L_1$ is almost perfect, it suffices to see that $\eta^*\mathbf{G}_0^*L_1$ is almost perfect inside $\mathrm{QCoh}(\mathsf{X})$ for every morphism $\eta\colon \mathsf{X} \to \mathfrak{X}_0$ where X is a spectral Deligne–Mumford stack; see [SAG, Th.8.3.5.2]. Using the base-change equivalence

$$L_1 = L_{\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}/\operatorname{Spf} A} \simeq \pi_1^* L_{\mathcal{M}_{\mathrm{BT}^p_n}}$$

it suffices to show $L_1' = \eta^*\mathbf{G}_0^*\pi_1^*L_{\mathcal{M}_{\mathrm{BT}^p_n}}$ is almost perfect. By [SAG, Cor.8.3.5.3], it suffices to check the affine case of $\mathsf{X} = \operatorname{Spec} R$, where R is a connective $\mathbf{E}_\infty$-ring. Note p is nilpotent in $\pi_0 R$ as $\operatorname{Spec} R$ maps into $\operatorname{Spf} A$, and $p \in \mathfrak{m}_A$ by assumption; see Nt. 2.1.5. Our conclusion that L_1' is almost perfect in Mod_R then follows from [EC2, Pr.3.2.5] and the fact that the adjective almost perfect is preserved under base-change; see [SAG, Cor.8.4.1.6]. Therefore, $\mathbf{G}_0^*L_1$ is almost perfect.

Focusing on L_2 now, we consider the composition $\mathfrak{X}_0 \to \operatorname{Spf} A_0 \to \operatorname{Spf} A$ and the induced (co)fibre sequence of quasi-coherent sheaves over $\mathfrak{X}_0$:

$$L_{\operatorname{Spf} A_0/\operatorname{Spf} A}\big|_{\mathfrak{X}_0} \to L_{\mathfrak{X}_0/\operatorname{Spf} A} = L_2 \to L_{\mathfrak{X}_0/\operatorname{Spf} A_0} \tag{2.1.10}$$

[15] Indeed, for locally Noetherian one can use [Sta, 00FN], for qcqs one can use [GW10, §D], and the residue fields are perfect as finite field extensions of perfect fields are perfect by [Sta, 05DU].

By Pr.A.3.1, we see $L_{\operatorname{Spf} A_0/\operatorname{Spf} A}$ is almost perfect in $\mathrm{QCoh}(\operatorname{Spf} A_0)$, and pullbacks preserve almost perfectness ([SAG, Cor.8.4.1.6]), hence the first term of (2.1.10) is almost perfect. To see the third term of (2.1.10) is almost perfect, we may work locally and replace $\mathfrak{X}_0$ with $\operatorname{Spf} B_0$ where B_0 is a complete discrete adic ring. In this case we use the assumption that $A_0 \to B_0$ is of finite presentation, which implies L_{B_0/A_0} is almost perfect in Mod_{B_0}; see [HA, Th.7.4.3.18]. By [SAG, Pr.7.3.5.7], L_{B_0/A_0} is complete with respect to an ideal of definition J for B_0, and it follows the B_0-module

$$L_{B_0/A_0} \simeq \left(L_{B_0/A_0}\right)^{\wedge}_{J} \simeq L_{\operatorname{Spf} B_0/\operatorname{Spf} A_0}$$

is almost perfect. Therefore L_2 is almost perfect, hence L itself is almost perfect. □

2.2 Outline of the proof

Our proof moves in three distinct, but connected, stages.

(I) First, we move from classical algebraic geometry (in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$) to spectral algebraic geometry (in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$) using deformation theory, presented here through the adjective *formally étale*. Given an object $\mathbf{G}_0 \colon \mathfrak{X}_0 \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$ inside $\mathcal{C}_{A_0}$, we consider the object X inside the following Cartesian diagram in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$:

$$\begin{array}{ccc} X & \longrightarrow & \tau^*_{\leqslant 0}\mathfrak{X}_0 \\ \Big\downarrow{\scriptstyle \mathbf{G}} & & \Big\downarrow{\scriptstyle \mathbf{G}_0} \\ \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A} & \xrightarrow{f} & \tau^*_{\leqslant 0}\widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0} \end{array}$$

The functor $\tau^*_{\leqslant 0} \colon \mathcal{P}(\mathrm{Aff}^{\heartsuit}) \to \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ above is induced by precomposition with $\tau_{\leqslant 0} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{CAlg}^{\heartsuit}$, and the maps $X(R) \to \tau^*_{\leqslant 0}X(R) = X(\pi_0 R)$ are induced by the truncation map $R \to \pi_0 R$. The assumption that $\mathbf{G}_0$ was formally étale in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ implies that X is what Lurie calls the *de Rham space* of the map $\mathfrak{X}_0 \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$ and that $\mathbf{G}$ is formally étale; see Pr.3.3.6. Most of the adjectives defining $\mathcal{C}_{A_0}$ then allow us to employ a powerful representability theorem of Lurie (Th.3.3.10), which identifies X as a formal spectral Deligne–Mumford stack, which we denote as $\mathfrak{X}$. Some analysis shows $\mathbf{G} \colon \mathfrak{X} \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$ lies in $\mathcal{C}_A$ and that the functor

$$\mathcal{D} \colon \mathcal{C}_{A_0} \to \mathcal{C}_A, \qquad (\mathfrak{X}_0, \mathbf{G}_0) \mapsto (\mathfrak{X}, \mathbf{G})$$

is an equivalence of ∞-categories (Th.3.3.5).

(II) Next, we apply the orientation theory of p-divisible groups devised by Lurie in [EC2]. This yields a moduli stack of *oriented p-divisible groups* $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ and a map of presheaves on p-complete $\mathbf{E}_\infty$-rings

$$\Omega \colon \mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n} \to \mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n};$$

see Df.4.1.6. The bulk of this section globalises the work of [EC2, §4]. We then define $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n} : \mathcal{C}_A^{\mathrm{op}} \to \mathrm{CAlg}$ by pushing forward the structure sheaf of $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ along Ω—it will follow rather formally that applying $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ to an affine object of $\mathcal{C}_A$ yields the *orientation classifier* construction of Lurie; see [EC2, §4.3.3].

(III) Finally, we set $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ to be the composition of $\mathcal{D}$ followed by $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}$. In other words, we first send $(\mathfrak{X}_0, \mathbf{G}_0)$ to its spectral deformation $(\mathfrak{X}, \mathbf{G})$ using $\mathcal{D}$, and then take the orientation classifier of the identity component of $\mathbf{G}$; see Df.4.3.1. To check this definition of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ satisfies the properties described in Th.2.1.7, we use descent ideas of Lurie.

The following two chapters carry out these three steps given above.

Chapter 3

Formally étale morphisms

At the heart of spectral algebraic geometry is deformation theory as indicated by the heuristic

$$\{\text{spectral algebraic geometry}\}$$
$$= \{\text{classical algebraic geometry}\} + \{\text{deformation theory}\}$$

from [SAG, p.1385]. The adjective *formally étale* will help us navigate between the two worlds of classical and spectral algebraic geometry. More concretely, given a (nice enough) formally étale morphism $\mathfrak{X}_0 \to \mathcal{M}$, where $\mathfrak{X}_0$ is a classical formal stack, there is a universal spectral deformation of $\mathfrak{X}_0$, say $\mathfrak{X}$, such that $\mathfrak{X}_0$ can be viewed as the 0th truncation of $\mathfrak{X}$. This process allows us to lift objects in classical algebraic geometry to spectral algebraic geometry without changing the underlying classical object; see Th.3.3.5.

3.1 Presheaves on discrete rings

Let us first consider formally étale maps between presheaves of discrete rings.

Definition 3.1.1. A natural transformation $f\colon X \to Y$ of functors in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ is said to be *formally étale* if, for all surjective maps of rings $\widetilde{R} \to R$ whose kernel is square-zero, also called *square-zero extensions* of R, the following natural diagram of spaces is Cartesian:

$$\begin{array}{ccc} X(\widetilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ Y(\widetilde{R}) & \longrightarrow & Y(R) \end{array}$$

Moreover, we say that f is *formally unramified* if the fibres of the map

$$X(\widetilde{R}) \to X(R) \underset{Y(R)}{\times} Y(\widetilde{R})$$

are either empty or contractible.

Let us state some classical formal properties of formally étale morphisms; the reader may enjoy verifying them herself.

Proposition 3.1.2. *Formally étale morphisms in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ are closed under composition. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are composable morphisms in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ such that g is formally unramified and $g \circ f$ is formally étale, then g is formally étale. Formally étale (resp. unramified) morphisms are closed under base-change.*

Let us now relate Df. 3.1.1 to the definitions found in classical algebraic geometry.

Definition 3.1.3. A map $f\colon X \to Y$ between functors in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ is *affine* if for every ring R, and every R-point $\eta \in Y(R)$, the fibre product $\operatorname{Spec} R \times_Y X$ is represented by an affine scheme.

Note that maps between (functors represented by) affines in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ are always affine, as the Yoneda embedding $\mathrm{Aff}^{\heartsuit} \to \mathcal{P}(\mathrm{Aff}^{\heartsuit})$ preserves limits.

Proposition 3.1.4. *Let $f\colon X \to Y$ be a natural transformation of functors in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$. Then f is formally étale if and only if for every ring R, every square-zero extension of rings $\widetilde{R} \to R$, and every commutative diagram of the form*

$$\begin{array}{ccc} \operatorname{Spec} R & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} \widetilde{R} & \longrightarrow & Y \end{array} \tag{3.1.5}$$

the mapping space

$$\operatorname{Map}_{\mathcal{P}(\mathrm{Aff}^{\heartsuit})_{R//Y}}(\operatorname{Spec} \widetilde{R}, X)$$

is contractible.[16] *Moreover, if f is affine, then f is formally étale if and only if for every ring A, and every A-point $\eta \in Y(A)$ such that the fibre product* $\operatorname{Spec} A \times_Y X$ *is equivalent to an affine scheme* $\operatorname{Spec} B$*, the natural projection map $A \to B$ is formally étale as a map of rings.*[17]

Proof. Given a ring R, a square-zero extension $\widetilde{R} \to R$, and a commutative diagram (3.1.5), consider the following commutative diagram of spaces:

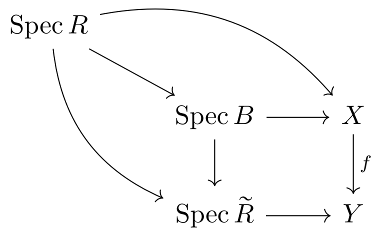
$$\begin{array}{ccccc} \operatorname{Map}_{R//Y}(\widetilde{R}, X) & \longrightarrow & \operatorname{Map}_{/Y}(\widetilde{R}, X) & \longrightarrow & \operatorname{Map}_{/Y}(R, X) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \operatorname{Map}_{R/}(\widetilde{R}, X) & \longrightarrow & \operatorname{Map}(\widetilde{R}, X) & \longrightarrow & \operatorname{Map}(R, X) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \operatorname{Map}_{R/}(\widetilde{R}, Y) & \longrightarrow & \operatorname{Map}(\widetilde{R}, Y) & \longrightarrow & \operatorname{Map}(R, Y) \end{array}$$

[16] Ie, "there exists a unique lift $\operatorname{Spec} \widetilde{R} \to X$ for (3.1.5)."

[17] For the definition of a formally étale map of rings simply apply Df.3.1.1 to the transformation (co)representing this map of rings, or see [Sta, 02HF].

By definition, the rows and columns are fibre sequences,[18] we have abbreviated the categories above to express only the over/under categories, and we suppressed the functor Spec. By the Yoneda lemma, the bottom-right square is naturally equivalent to (3.1.5), hence f is formally étale if and only if this bottom right square is Cartesian. In turn, this is equivalent to the space in the top-left corner being contractible.

For the "moreover" statement, suppose that f is affine. If f is formally étale, then Pr.3.1.2 states that the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is formally étale by base-change. Conversely, suppose we are given a diagram of the form (3.1.5), then by assumption the fibre product $\operatorname{Spec} \widetilde{R} \times_X Y \simeq \operatorname{Spec} B$ is affine and $\operatorname{Spec} B \to \operatorname{Spec} \widetilde{R}$ is formally étale, giving us the following diagram:



One then observes the sequence of natural equivalences of spaces

$$\operatorname{Map}_{R//\widetilde{R}}(\widetilde{R}, B) \simeq \operatorname{Map}_{R/}(\widetilde{R}, B) \underset{\operatorname{Map}_{R/}(\widetilde{R},\widetilde{R})}{\times} \{\operatorname{id}_{\widetilde{R}}\}$$

$$\simeq \operatorname{Map}_{R/}(\widetilde{R}, X) \underset{\operatorname{Map}_{R/}(\widetilde{R},Y)}{\times} \operatorname{Map}_{R/}(\widetilde{R}, \widetilde{R}) \underset{\operatorname{Map}_{R/}(\widetilde{R},\widetilde{R})}{\times} \{\operatorname{id}_{\widetilde{R}}\} \simeq \operatorname{Map}_{R//Y}(\widetilde{R}, X)$$

where we have used the same abbreviations from earlier in the proof. The first space above is contractible as $\operatorname{Spec} B \to \operatorname{Spec} \widetilde{R}$ is formally étale, hence f is formally étale as the last space is contractible. □

Let us see some examples of formally étale morphisms found in classical algebraic geometry.

Example 3.1.6 (Formally étale morphisms of schemes)*.* In the setting of classical algebraic geometry, we usually take the existence of a unique map $\operatorname{Spec} \widetilde{R} \to X$ (under $\operatorname{Spec} R$ and over Y) as the definition of a formally étale maps of rings (or schemes); see Pr. 3.1.4. An object in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ represented by a scheme factors through $\operatorname{Fun}(\mathrm{CAlg}^{\heartsuit}, \mathrm{Set})$, as mapping spaces between classical schemes are discrete, and we see Pr.3.1.4 precisely matches [Sta, 02HG].

Example 3.1.7 (Classical Serre–Tate theorem)*.* The classical Serre–Tate theorem (see [CS15, p.854] for the original source, or [EC1, Th.7.0.1] for statement of the

[18]The fibres in this diagram have been taken with respect to the maps from (3.1.5).

spectral version) states that if $\widetilde{R} \to R$ is a square-zero extension of commutative rings wherein p is nilpotent, then the diagram of 1-groupoids

$$\begin{array}{ccc} \mathrm{AVar}_g(\widetilde{R})^{\simeq} & \longrightarrow & \mathrm{AVar}_g(R)^{\simeq} \\ \downarrow{\scriptstyle [p^\infty]} & & \downarrow{\scriptstyle [p^\infty]} \\ \mathrm{BT}^p_{2g}(\widetilde{R})^{\simeq} & \longrightarrow & \mathrm{BT}^p_{2g}(R)^{\simeq} \end{array} \tag{3.1.8}$$

is Cartesian. This implies the morphism of classical moduli stacks

$$[p^\infty] \colon \mathcal{M}^{\heartsuit}_{\mathrm{AVar}_g} \to \mathcal{M}^{\heartsuit}_{\mathrm{BT}^p_{2g}}$$

sending an abelian variety X to its associated p-divisible group $\mathsf{X}[p^\infty]$ ([Tat67, §2]) is formally étale **after** base-change over $\operatorname{Spf} \mathbf{Z}_p$. This base-change is crucial, as there only exists a map $\operatorname{Spec} R \to \operatorname{Spf} \mathbf{Z}_p$ is when p is nilpotent inside R, as the continuous map of rings $\mathbf{Z}_p \to R$ must send $\{p^i\}_{i\geqslant 0}$ to a convergent sequence in R, where R is equipped with the discrete topology. If we fail to make this base-change, then (3.1.8) may not be Cartesian.[19]

Another classical example of a formally étale map in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ comes from Lubin–Tate theory. The original source for this is [LT66] with respect to formal groups, but we will follow [EC2, §3] as our intended application is for p-divisible groups; see [EC2, Ex.3.0.5] for a statement of the dictionary between deformations of formal and p-divisible groups.

Example 3.1.9 (Classical Lubin–Tate theory). Let $\mathbf{G}_0$ be a p-divisible group of height $0 < n < \infty$ over a perfect field κ of characteristic p. Then there exists a universal classical deformation $\mathbf{G}$ of $\mathbf{G}_0$ over the classical deformation ring $R^{\mathrm{LT}}_{\mathbf{G}_0}$; see [EC2, Df.3.1.4] or the proof of Pr.3.1.10.

This formally implies that the map into $\mathcal{M}^{\heartsuit}_{\mathrm{BT}^p_n}$ defining $\mathbf{G}$ is formally étale. In fact, we generalise the Lubin–Tate case above using [EC2, §3] to formally obtain:

[19] Indeed, consider the elliptic curve E over $\overline{\mathbf{F}}_3$ defined by the equation $y^2 = x^3 + x^2 + x + 1$. The 2^k-torsion subgroups of E are, by [KM85, Th.2.3.1], equivalent to the constant group schemes $\left(\mathbf{Z}/2^k\mathbf{Z}\right)^2$ over $\overline{\mathbf{F}}_3$, hence the associated 2-divisible group $E[2^\infty]$ is equivalent to the constant 2-divisible group $(\mathbf{Q}_2/\mathbf{Z}_2)^2$ over $\overline{\mathbf{F}}_3$. Define two deformations E_1 and E_2 of E over the dual numbers $\overline{\mathbf{F}}_3[\epsilon]$ (augmented by the morphism $\epsilon \mapsto 0$), by the following formulae:

$$E_1 : y^2 = x^3 + x^2 + x + 1 + \epsilon \qquad E_2 : y^2 = x^3 + x^2 + x + 1 - \epsilon$$

We again calculate $E_1[2^\infty]$ and $E_2[2^\infty]$ to both be the constant 2-divisible group $(\mathbf{Q}_2/\mathbf{Z}_2)^2$ over $\overline{\mathbf{F}}_3[\epsilon]$, and hence these 2-divisible groups also base-change to $E[2^\infty]$ over $\overline{\mathbf{F}}_3$. As a final observation, note that E_1 and E_2 are **not** equivalent as elliptic curves over $\overline{\mathbf{F}}_3[\epsilon]$, as one can calculate their j-invariants ([Sil86, §III.1]):

$$j(E_1) = \epsilon - 1 \neq \epsilon + 1 = j(E_2)$$

Hence $[2^\infty] \colon \mathcal{M}^{\heartsuit}_{\mathrm{Ell}} \to \mathcal{M}^{\heartsuit}_{\mathrm{BT}^p_2}$ is **not** formally étale over $\operatorname{Spec} \mathbf{Z}$.

Proposition 3.1.10. *Let R_0 be a discrete $\mathbf{F}_p$-algebra such that L_R is an almost perfect R-module*[20] *and $\mathbf{G}_0$ is a* nonstationary[21] *p-divisible group over R_0 of height n. Then the map $\operatorname{Spf} R_{\mathbf{G}_0} \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, \mathbf{Z}_p}$ induced by the universal classical deformation of $\mathbf{G}_0$ is formally étale. Conversely, if $\mathbf{G}\colon \operatorname{Spf} R \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$ is formally étale for a complete Noetherian discrete ring R and A_0 from Nt.2.1.5, then for every maximal ideal $\mathfrak{m} \subseteq R$ such that the residue field $R/\mathfrak{m} = \kappa$ is perfect, the p-divisible group $\mathbf{G}_{B^\wedge_\mathfrak{m}}$ is the universal classical deformation of $\mathbf{G}_\kappa$.*

Proof. The existence of such an $R_{\mathbf{G}_0}$ follows by taking π_0 of the spectral deformation ring; the spectral deformation ring exists by [EC2, Th.3.4.1] and then we apply Rmk.1.3.14. Let $R \to R/J$ be the quotient map where R is discrete and J is a square-zero ideal. First, we wish to show the following commutative diagram of spaces is Cartesian:

$$\begin{array}{ccc} (\operatorname{Spf} R_{\mathbf{G}_0})(R) & \longrightarrow & (\mathcal{M}^{\heartsuit}_{\mathrm{BT}^p_n})(R) \\ \downarrow l & & \downarrow r \\ (\operatorname{Spf} R_{\mathbf{G}_0})(R/J) & \xrightarrow{b} & (\mathcal{M}^{\heartsuit}_{\mathrm{BT}^p_n})(R/J) \end{array} \tag{3.1.11}$$

Recall the definition of $\mathrm{Def}_{\mathbf{G}_0}(A)$ for a commutative ring A with the discrete topology from Df.1.3.13. As $R_{\mathbf{G}_0}$ is the universal deformation of $\mathbf{G}_0$ one obtains an equivalence of (discrete) spaces

$$\operatorname{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathrm{ad}}}(R_{\mathbf{G}_0}, A) \xrightarrow{\simeq} \mathrm{Def}_{\mathbf{G}_0}(A) =$$

$$\operatorname*{colim}_{I \in \mathrm{Nil}_0(A)} \left(\mathrm{BT}^p(A) \underset{\mathrm{BT}^p(A/I)}{\times} \operatorname{Hom}_{\mathrm{CAlg}^{\heartsuit}}(R_0, A/I) \right) \tag{3.1.12}$$

where the colimit is taken over all finitely generated nilpotent ideals I inside A; see [EC2, Th.3.1.15]. By assumption, the cotangent complex L_{R_0} is almost perfect in Mod_{R_0}, and [EC2, Pr.3.4.3] then implies that the natural map

$$\mathrm{Def}_{\mathbf{G}_0}(A) \xrightarrow{\simeq} \operatorname*{colim}_{I \in \mathrm{Nil}(A)} F_{A,I}$$

is an equivalence, where now the colimit is indexed over *all* nilpotent ideals $I \subseteq A$ and $F_{A,I}$ is the fibre product of (3.1.12). Given a fixed nilpotent ideal $J \subseteq A$, denote by $\mathrm{Nil}_J(A)$ the poset of nilpotent ideals of A which contain J. We obtain a natural inclusion functor $\mathrm{Nil}_J(A) \to \mathrm{Nil}(A)$, which is cofinal, as any nilpotent ideal I lies within the nilpotent ideal $I + J$. Hence the natural map

$$\operatorname*{colim}_{I \in \mathrm{Nil}_J(A)} F_{A,I} \xrightarrow{\simeq} \operatorname*{colim}_{I \in \mathrm{Nil}(A)} F_{A,I}$$

[20] See [EC2, Pr.3.3.7 & Th.3.5.1] for many equivalent conditions to L_R being almost perfect.

[21] Recall the definition of a *nonstationary* p-divisible group $\mathbf{G}_0$ from [EC2, Df.3.0.8], or the equivalent condition for $\mathbf{G}_0$ over a discrete Noetherian $\mathbf{F}_p$-algebra R_0 whose Frobenius is finite, that the cotangent complex $L_{\operatorname{Spec} R/\mathcal{M}_{\mathrm{BT}^p}}$ induced by the defining morphism of $\mathbf{G}_0$ is 1-connective; see [EC2, Rmk.3.4.4 & Th.3.5.1]. In particular, by [EC2, Ex.3.0.10], all p-divisible groups over $\mathbf{F}_p$-algebras R_0 whose Frobenius is surjective are nonstationary.

is an equivalence. The map l of (3.1.11) is then equivalent to

$$\operatorname*{colim}_{I\in \mathrm{Nil}_J(R)} F_{R,I} \xrightarrow{l} \operatorname*{colim}_{I\in \mathrm{Nil}_J(R)} F_{R/J,I/J}$$

where we used the fact that ideals in R/J correspond to ideals in R containing J. If $(\operatorname{Spf} R_{\mathbf{G}_0})(R/J)$ is empty, then so is $(\operatorname{Spf} R_{\mathbf{G}_0})(R)$ and we are done. Otherwise, choose some x in $(\operatorname{Spf} R_{\mathbf{G}_0})(R/J)$ and consider the fibre of l over x. As filtered colimits of spaces commute with finite limits we calculate this fibre as

$$\mathrm{fib}_x(l) \simeq \operatorname*{colim}_{I\in \mathrm{Nil}_J(R)} \mathrm{fib}_{x_I}\left(F_{R,I} \xrightarrow{g} F_{R/J,I/J}\right)$$

using the fact that $\mathrm{Nil}_J(R)$ is filtered. To simplify this further, we contemplate the following diagram in $\mathscr{C}\mathrm{at}_\infty$:

$$\begin{array}{ccc}
\mathrm{BT}^p(R) \underset{\mathrm{BT}^p(R/I)}{\times} \mathrm{Hom}(R_0, R/I) & \longrightarrow & \mathrm{BT}^p(R) \\
\downarrow g & & \downarrow f \\
\mathrm{BT}^p(R/J) \underset{\mathrm{BT}^p(R/I)}{\times} \mathrm{Hom}(R_0, R/I) & \longrightarrow & \mathrm{BT}^p(R/J) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(R_0, R/I) & \longrightarrow & \mathrm{BT}^p(R/I)
\end{array}$$

The lower square and the whole rectangle are Cartesian by definition, so the upper square is also Cartesian. This means the natural map $\mathrm{fib}(g) \to \mathrm{fib}(f)$ is an equivalence in $\mathscr{C}\mathrm{at}_\infty$, hence our fibre of l can be rewritten as follows:

$$\begin{aligned}
\mathrm{fib}_x(l) &\simeq \operatorname*{colim}_{I\in \mathrm{Nil}_J(R)} \left(\mathrm{fib}_{b(x_I)}(\mathrm{BT}^p(R) \xrightarrow{f} \mathrm{BT}^p(R/J))\right) \\
&\simeq \mathrm{fib}_{b(x)}\left(\mathrm{BT}^p(R) \xrightarrow{f} \mathrm{BT}^p(R/J)\right)
\end{aligned}$$

This shows the fibre of f lies in the essential image of $\mathcal{S} \to \mathscr{C}\mathrm{at}_\infty$ as $\mathrm{fib}_x(l)$ is. As r is $f^\simeq$ we obtain a natural equivalence $\mathrm{fib}(l) \simeq \mathrm{fib}(r)$. As the fibres of l and r are naturally equivalent, we see that (3.1.11) is Cartesian, so the composition

$$\operatorname{Spf} R_{\mathbf{G}_0} \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, \mathbf{Z}_p} \to \mathcal{M}_{\mathrm{BT}^p_n} \to \mathcal{M}_{\mathrm{BT}^p}$$

is formally étale. To see the first map in the composition above is formally étale, we use that the last map is open (Rmk.2.1.2) and hence formally étale, the second last map is the base-change of the formally unramified map $\operatorname{Spf} \mathbf{Z}_p \to \operatorname{Spec} \mathbf{Z}$, and the cancellation statement from Pr.3.1.2.

Let us omit a proof of the converse statement; the $\mathbf{E}_\infty$-version is Pr.3.3.13 and the proof strategy is the same in both cases. ☐

3.2 Presheaves on $\mathbf{E}_\infty$-rings

We are now in the position to make a spectral definition. See [HA, §7.4] for the definition of (trivial) square-zero extension of $\mathbf{E}_\infty$-rings, and [SAG, §17.2] for the definition of (infinitesimally) cohesive and nilcomplete functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ and the definition of $L_{X/Y}$.

Definition 3.2.1. Let $f\colon X \to Y$ be a natural transformation of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. For an integer $0 \leqslant n \leqslant \infty$, we say f is *n-formally étale* if for all square-zero extensions of connective n-truncated $\mathbf{E}_\infty$-rings $\widetilde{R} \to R$ the natural diagram of spaces

$$\begin{array}{ccc} X(\widetilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ Y(\widetilde{R}) & \longrightarrow & Y(R) \end{array}$$

is Cartesian. We abbreviate ∞-formally étale to *formally étale.*

Remark 3.2.2. If f is n-formally étale, then f is also m-formally étale for all $0 \leqslant m \leqslant n \leqslant \infty$. In particular, for any $0 \leqslant n \leqslant \infty$, if f is n-formally étale then $X^\heartsuit \to Y^\heartsuit$ is formally étale inside $\mathcal{P}(\mathrm{Aff}^\heartsuit)$.

A converse statement also holds.

Remark 3.2.3. Write $\tau^*_{\leqslant 0}\colon \mathcal{P}(\mathrm{Aff}^\heartsuit) \to \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ for the functor induced by the truncation $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{CAlg}^\heartsuit$. If $X \to Y$ is formally étale in $\mathcal{P}(\mathrm{Aff}^\heartsuit)$, then it follows that $\tau^*_{\leqslant 0}X \to \tau_{\leqslant 0}Y$ is (∞-) formally étale inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. Indeed, for each square-zero extension of connective $\mathbf{E}_\infty$-rings $\widetilde{R} \to R$ we want to show the diagram of spaces

$$\begin{array}{ccc} X(\pi_0\widetilde{R}) & \longrightarrow & X(\pi_0 R) \\ \downarrow & & \downarrow \\ Y(\pi_0\widetilde{R}) & \longrightarrow & Y(\pi_0 R) \end{array}$$

is Cartesian. If we can show the map $\rho\colon \pi_0\widetilde{R} \to \pi_0 R$ is a square-zero extension of classical rings, we are done by our hypotheses. The (co)fibre sequence

$$M \to \widetilde{R} \to R$$

of connective R-modules shows that ρ is surjective. Notice the kernel of ρ is not $\pi_0 M$, but the image of the map $\pi_0 M \to \pi_0\widetilde{R}$. This does not worry us, as the multiplication map $M \otimes_{\widetilde{R}} M \to M$ is nullhomotopic by [HA, Pr.7.4.1.14], hence the image of $\pi_0 M$ in $\pi_0\widetilde{R}$ squares to zero, and we see ρ is a square-zero extension of rings.

Remark 3.2.4. If $\mathfrak{X} \to \mathfrak{Y}$ is a formally étale morphism of (locally Noetherian) classical formal Deligne–Mumford stacks inside $\mathcal{P}(\mathrm{Aff}^\heartsuit)$, then the corresponding morphism inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ is 0-formally étale. This follows by the fully faithfulness of fDM $\to$ fSpDM; see Pr.A.1.4.

Remark 3.2.5. Our definition of formally étale deviates from Lurie's definition of étale morphisms ([HA, Df.7.5.0.4]) as there is no flatness assumption. However, even in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ a formally étale morphism of discrete rings need not be flat.[22] This means there is no inherent descent theory for formally étale morphisms. For more in this direction, the reader is advised to make her way to Rmk.3.2.13.

The basic properties of Pr.3.1.2 also hold in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$.

Proposition 3.2.6. *Let $0 \leqslant n \leqslant \infty$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be composable morphisms in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ where g is n-formally étale. Then f is n-formally étale if and only if h is n-formally étale. Moreover, n-formally étale morphisms are closed under base-change.*

We would now like alternative ways to test if a map $X \to Y$ is formally étale in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$. Although Lurie does not directly discuss the adjective formally étale in [SAG, §17], many of the techniques below follow his ideas.

Proposition 3.2.7. *Let $X \to Y$ be a natural transformation of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ and $0 \leqslant n \leqslant \infty$.*

1. *The map $X \to Y$ is n-formally étale for* finite *n if and only if $X \to Y$ is 0-formally étale and for every connective n-truncated $\mathbf{E}_\infty$-ring R the natural diagram of spaces*

$$\begin{array}{ccc} X(R) & \longrightarrow & X(\pi_0 R) \\ \downarrow & & \downarrow \\ Y(R) & \longrightarrow & Y(\pi_0 R) \end{array}$$

   is Cartesian. If $X \to Y$ is nilcomplete, then the $n = \infty$-case also holds.

2. *If $X \to Y$ is infinitesimally cohesive, then $X \to Y$ is formally étale if and only if for all* trivial *square-zero extensions of connective truncated $\mathbf{E}_\infty$-rings $\widetilde{R} \to R$ the natural diagram of spaces*

$$\begin{array}{ccc} X(\widetilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ Y(\widetilde{R}) & \longrightarrow & Y(R) \end{array}$$

   is Cartesian.

[22] For example, the map of discrete rings $\mathbf{C}\left[t^q | q \in \mathbf{Q}, q > 0\right] \to \mathbf{C}$ sending $t \mapsto 0$ is formally étale but not flat. Indeed, one can always lift square-zero extensions of rings uniquely, as we have all square roots of t in the above ring, hence it is formally étale. To see that this map is not flat, we can tensor it with the exact sequence

$$0 \to (t) \to \mathbf{C}[t^q] \to \mathbf{C}[t^q]/(t) \to 0$$

which yields the following clearly **not** exact sequence:

$$0 \to \mathbf{C} \to \mathbf{C} \to \mathbf{C} \to 0$$

3. *If $X \to Y$ is infinitesimally cohesive and admits a connective cotangent complex $L_{X/Y}$, then $X \to Y$ is formally étale if and only if $L_{X/Y}$ vanishes.*

If $X \to Y$ is infinitesimally cohesive, nilcomplete, and $L_{X/Y}$ exists and is connective, then $X \to Y$ is n-formally étale if certain Ext-groups $\mathrm{Ext}^m_R(\eta^* L_{X/Y}, M)$ vanish in a range, for certain discrete objects (R, η, M) of $\mathrm{Mod}^X_{\mathrm{cn}}$, à la the deformation theory of [Ill71]. There is also a sharpening of part 4 above which deals with an n-connective cotangent complex $L_{X/Y}$, which we note for the readers benefit is **not** equivalent to $X \to Y$ being n-formally étale. These ideas will not be used here though.

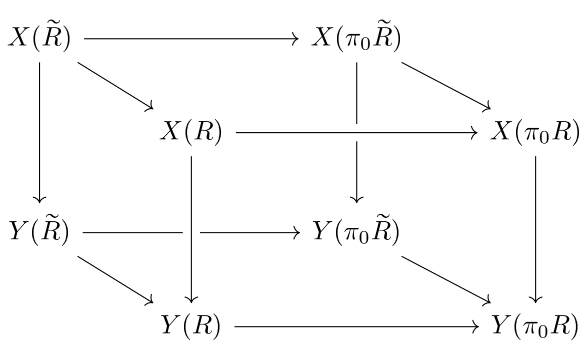
Thank you to an anonymous referee for correcting a previous version of (2) above.

Proof. Write f for the transformation $X \to Y$ in question.

(1) Suppose f is n-formally étale for a finite $n \geqslant 0$, then f is 0-formally étale by Rmk.3.2.2. Given a connective n-truncated $\mathbf{E}_\infty$-ring R, then for any $0 \leqslant m \leqslant n$ we can consider the following diagram:

$$\begin{array}{ccccc} X(\tau_{\leqslant m+1}R) & \longrightarrow & X(\tau_{\leqslant m}R) & \longrightarrow & X(\pi_0 R) \\ \downarrow & & \downarrow & & \downarrow \\ Y(\tau_{\leqslant m+1}R) & \longrightarrow & Y(\tau_{\leqslant m}R) & \longrightarrow & Y(\pi_0 R) \end{array} \tag{3.2.8}$$

Above, the left square is always Cartesian by virtue of f being n-formally étale as $\tau_{\leqslant m+1}R \to \tau_{\leqslant m}R$ is a square-zero extension of $\mathbf{E}_\infty$-rings; see [HA, Cor.7.1.4.28]. To show the outer rectangle Cartesian we use induction. The base-case of $m = 0$ is tautological. For $m \geqslant 1$, the right square is Cartesian by our inductive hypotheses, hence the whole rectangle is Cartesian. Conversely, if the second condition of part 1 holds, we consider a square-zero extension of n-truncated connective $\mathbf{E}_\infty$-rings $\widetilde{R} \to R$ and the following natural diagram of spaces:



The back and front faces are Cartesian by the second condition of part 1, and the rightmost face is Cartesian as the second condition of part 1 also assumes

f is 0-formally étale. Hence by a base-change argument, we see the leftmost square is Cartesian, and we are done. For the $n = \infty$-case, suppose $X \to Y$ is nilcomplete, meaning that for every connective $\mathbf{E}_\infty$-ring R, the diagram of spaces

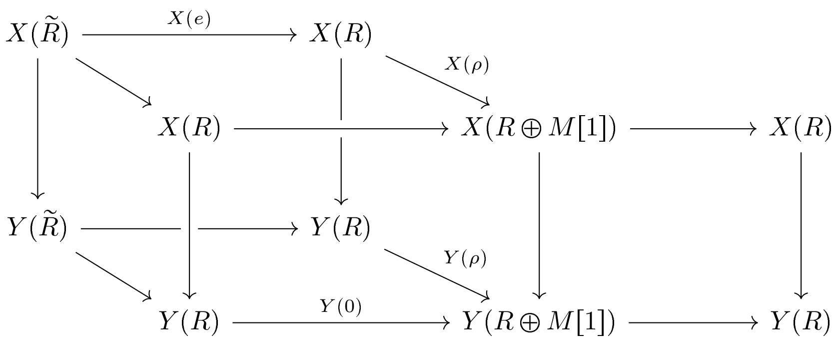
$$\begin{array}{ccc} X(R) & \longrightarrow & \lim X(\tau_{\leqslant n} R) \\ \downarrow & & \downarrow \\ Y(R) & \longrightarrow & \lim Y(\tau_{\leqslant n} R) \end{array}$$

is Cartesian. Combining this diagram with the finite case above yields the desired conclusion.

(2) If f is formally étale, then logic implies the second condition holds. Conversely, let $e\colon \widetilde{R} \to R$ be a square-zero extension of a connective $\mathbf{E}_\infty$-ring R by a connective R-module M and a derivation $d\colon L_R \to M[1]$. By definition ([HA, Df.7.4.1.6]) $\widetilde{R}$ is defined by the Cartesian diagram of connective $\mathbf{E}_\infty$-rings

$$\begin{array}{ccc} \widetilde{R} & \xrightarrow{e} & R \\ \downarrow & & \downarrow{\scriptstyle \rho} \\ R & \xrightarrow{0} & R \oplus M[1] \end{array}$$

where the bottom-horizontal map is induced by the zero map $L_R \to M[1]$ and the right-vertical map is induced by the derivation d. This Cartesian diagram of connective $\mathbf{E}_\infty$-rings then induces the following natural diagram of spaces:



(3.2.9)

The left cube is Cartesian from our assumption that f is infinitesimally cohesive. By assumption the rightmost square is Cartesian, and the only rectangle in the diagram is also Cartesian as the composition $R \to R \oplus M[1] \to R$ is equivalent to the identity, hence the left square in that same rectangle (the front face of the cube) is Cartesian. By a base-change argument[23] we see that the desired

[23] This base-change argument is simple, but let us give an outline. Write I for the poset of nonempty subsets of $\{1, 2, 3\}$, ordered by inclusion, and use this poset to index the cube in (3.2.9) by setting $F_\varnothing = X(\widetilde{R})$, $F_1 = X(R)$ (in the top-right), $F_2 = X(R)$ (in the centre),

back square of the cube (containing $X(e)$ and $Y(e)$) is also Cartesian, and we are done.

(3) Our proof here is roughly that of [SAG, Prs.17.3.9.3-4]. On the one hand, by [SAG, Pr.6.2.5.2(1)] and [SAG, Df.6.2.5.3], we see that for some fixed integer m, an object $\mathcal{F}$ of $\mathrm{QCoh}(X)$ is m-connective if and only if for all connective $\mathbf{E}_\infty$-rings R and all transformations $\eta\colon \operatorname{Spec} R \to X$, the object $\eta^*\mathcal{F}$ is m-connective inside $\mathrm{QCoh}(\operatorname{Spec} R) \simeq \mathrm{Mod}_R$. Furthermore, if $\mathcal{F}$ is connective and $m \geqslant 0$, the object $\eta^*\mathcal{F}$ is m-connective if and only if the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_R^{\mathrm{cn}}}(\eta^*\mathcal{F}, N) \simeq \mathrm{Map}_{\mathrm{Mod}_R^{\mathrm{cn}}}(\tau_{\leqslant m}\eta^*\mathcal{F}, N)$$

is contractible, for all connective $(m-1)$-truncated R-modules N, by the Yoneda lemma. On the other hand, the object $L_{X/Y}$ in $\mathrm{QCoh}(X)$ exists if and only if the functor $F\colon \mathrm{Mod}^X_{\mathrm{cn}} \to \mathcal{S}$, given on objects by

$$F(R,\eta,M) = \mathrm{fib}\left(X(R\oplus M) \to X(R) \underset{Y(R)}{\times} Y(R\oplus M)\right) \tag{3.2.10}$$

is locally almost representable, meaning that we have a (locally almost; see [SAG, Df.17.2.3.1]) natural equivalence for all triples (R,η,M) in $\mathrm{Mod}_X^{\mathrm{cn}}$

$$F(R,\eta,M) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_{X/Y}, M)$$

where R is a connective $\mathbf{E}_\infty$-ring, $\eta\colon \operatorname{Spec} R \to X$ a map in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, and M a connective R-module. If $L_{X/Y}$ vanishes, then we immediately see $F(R,\eta,M)$ is contractible for all triples (R,η,M), which by part 3 implies $X \to Y$ is formally étale, courtesy of the definition (3.2.10) of F. Conversely, if $X \to Y$ is formally étale, then $F(R,\eta,M)$ is contractible for all triples (R,η,M), hence the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_R}(\eta^* L_{X/Y}, M) \simeq F(R,\eta,M)$$

is contractible for all triples (R,η,M) and $L_{X/Y}$ vanishes. □

Let us explore some formally étale maps in spectral algebraic geometry.

Note that all formal spectral Deligne–Mumford stacks are cohesive, nilcomplete, and absolute cotangent complexes always exist, which follows by copying the proof of [SAG, Cor.17.3.8.5] (the same statement for SpDM), as all of the references made there also apply to fSpDM.

$F_3 = Y(\widetilde{R})$, etc. As the whole cube is Cartesian we have $F_\varnothing \simeq \lim_{I_0\in I} F_{I_0}$ and as the front face is also Cartesian we have $F_2 \simeq \lim\,(F_{12} \to F_{123} \leftarrow F_{23})$. These two facts, together with [MV15, Ex.5.3.8] give us the following natural chain of equivalences of spaces

$$F_\varnothing \simeq \lim_{I_0\in I} F_{I_0} \simeq \lim\,(F_2 \to G_{123} \leftarrow G_{13}) \simeq G_{13},$$

where $G_{123} = \lim\,(F_{12} \to F_{123} \leftarrow F_{23}) \simeq F_2$ and $G_{13} = \lim\,(F_1 \to F_{13} \leftarrow F_3)$. This shows the back face of the cube (indexed by $\varnothing$, $\{1\}$, $\{3\}$, and $\{1,3\}$, is Cartesian.

Example 3.2.11 (Étale morphisms of connective $\mathbf{E}_\infty$-rings). Let $A \to B$ be an étale morphism of connective $\mathbf{E}_\infty$-rings, then by [HA, Cor.7.5.4.5] we know $L_{B/A}$ vanishes, hence $A \to B$ is also a formally étale morphism of $\mathbf{E}_\infty$-rings by Pr.3.2.7.

Example 3.2.12 (Relatively perfect discrete $\mathbf{F}_p$-algebras). Another classic example, which will not show up explicitly in this thesis but is at the heart of much of the work done in [EC2], is that a flat relatively perfect map of discrete commutative $\mathbf{F}_p$-algebras has a vanishing cotangent complex ([EC2, Lm.5.2.8]), and hence is formally étale.

Remark 3.2.13. In Rmk.3.2.5, we noted that formally étale morphisms of connective $\mathbf{E}_\infty$-rings were not necessarily flat. However, [EC2, Pr.3.5.5] states that morphisms of (not necessarily connective) *Noetherian* $\mathbf{E}_\infty$-rings with vanishing cotangent complex are flat. Combining this with Pr.3.2.7, we see formally étale morphisms of connective Noetherian $\mathbf{E}_\infty$-rings are flat. It follows (as in classical algebraic geometry [Sta, 02HM]) that formally étale morphisms of almost finite presentation between connective Noetherian $\mathbf{E}_\infty$-rings are étale.

The functor $\mathcal{M}_{\mathrm{BT}^p}$ is cohesive, nilcomplete, and admits a cotangent complex by [EC2, Pr.3.2.2]. It follows that $\mathcal{M}_{\mathrm{BT}^p_n}$ (as well as all base-changes $\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}$) also satisfy these properties as $\mathcal{M}_{\mathrm{BT}^p_n} \to \mathcal{M}_{\mathrm{BT}^p}$ is open (Rmk.2.1.2).

Example 3.2.14 (Spectral Serre–Tate theorem). It follows from the spectral Serre–Tate theorem ([EC1, Th.7.0.1]) and Pr.3.2.7 that the morphism

$$[p^\infty]\colon \widehat{\mathcal{M}}_{\mathrm{AVar}_g,\mathbf{S}_p} \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_{2g},\mathbf{S}_p}$$

is formally étale.

Example 3.2.15 (Spectral Lubin–Tate theory). Given some nonstationary (21) p-divisible group $\mathbf{G}_0$ over a discrete ring R_0 where p is nilpotent and whose absolute cotangent complex L_{R_0} is almost perfect, Lurie uses his de Rham space formalism to construct a map $\mathbf{G}\colon \operatorname{Spf} R \to \mathcal{M}_{\mathrm{BT}^p_n}$ ([EC2, Th.3.4.1]) which is formally étale by [SAG, Cor.18.2.1.11(2)] and Pr.3.2.7. The p-divisible group $\mathbf{G}$ is the universal spectral deformation of $\mathbf{G}_0$ and R its spectral deformation ring; see Df.1.3.13.

Example 3.2.16 (Formal spectral completions). Let X be a spectral Deligne–Mumford stack and $K \subseteq |\mathsf{X}|$ be a cocompact closed subset, then the natural map from the formal completion of X along K ([SAG, Df.8.1.6.1]) $\mathsf{X}^\wedge_K \to \mathsf{X}$ is formally étale by [SAG, Ex.17.1.2.10] and Pr.3.2.7.

Example 3.2.17 (de Rham space). Given a morphism $X \to Y$ of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, one can associate a *de Rham space* $(X/Y)_{\mathrm{dR}}$ inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$, whose value on a connective $\mathbf{E}_\infty$-ring R is(

$$(X/Y)_{\mathrm{dR}}(R) = \operatorname*{colim}_I \left(Y(R) \underset{Y(\pi_0 R/I)}{\times} X(\pi_0 R/I) \right)$$

where the colimit is taken over all nilpotent ideals $I \subseteq \pi_0 R$; see [SAG, §18.2.1]. By [SAG, Cor.18.2.1.11(2)], the natural map $(X/Y)_{\mathrm{dR}} \to Y$ is nilcomplete, infinitesimally cohesive, and admits a vanishing cotangent complex, so by Pr.3.2.7, it is formally étale.

This last example will help us study the moduli stack $\mathcal{M}_{\mathrm{BT}^p_n}$.

3.3 Applied to p-divisible groups

Let us now apply the theory of formally étale natural transformations to the functor $\widehat{\mathcal{M}}_{\mathrm{BT}^p_n}$ and the sites $\mathcal{C}_{A_0}$ and $\mathcal{C}_A$ of Df.2.1.6.

Notation 3.3.1. Write $(-)^\heartsuit \colon \mathrm{CAlg}^\heartsuit \to \mathrm{CAlg}^{\mathrm{cn}}$ for the inclusion (a right adjoint, inducing a left adjoint $(-)^\heartsuit$ on presheaf categories), and $\tau_{\leqslant 0}$ for the truncation functor (a left adjoint, inducing a right adjoint $\tau^*_{\leqslant 0}$ on presheaf categories) $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{CAlg}^\heartsuit$. Also write $\tau_{\leqslant 0}$ for the composition $(-)^\heartsuit \circ \tau_{\leqslant 0}$—this should seldom cause confusion. For each functor $\mathcal{M}$ in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ there is a natural unit $\mathcal{M} \to \tau^*_{\leqslant 0}\mathcal{M}$ induced by the truncation $R \to \pi_0 R$ of a connective $\mathbf{E}_\infty$-ring R. The functor $\tau^*_{\leqslant 0}\mathcal{M}$ is equivalently the right Kan extension of $\mathcal{M}^\heartsuit$ along ι.

Warning 3.3.2. In § A.2 we introduce the truncation of a locally Noetherian formal spectral Deligne–Mumford stack $\tau_{\leqslant 0}\mathfrak{X}$ à la Lurie [SAG, §1.4.6] and we note that this is **not** equivalent to $\tau^*_{\leqslant 0}\mathfrak{X}$.

For mostly formal reasons, we obtain a functor $\mathcal{C}_A \to \mathcal{C}_{A_0}$.

Proposition 3.3.3. *The functor*

$$(-)^\heartsuit \colon \mathcal{C}_A \to \mathcal{P}(\mathrm{Aff}^\heartsuit)_{/\widehat{\mathcal{M}}^\heartsuit_{\mathrm{BT}^p_n, A_0}}$$

factors through $\mathcal{C}_{A_0}$.

Our proof of the above proposition relies on §A.

Proof. By definition, an object $\mathfrak{X}$ of $\mathcal{C}_A$ is qcqs, hence has an affine étale hypercover $\mathcal{U}_\bullet \to \mathfrak{X}$; see Pr.A.3.6. The formal spectral Deligne–Mumford stack $\tau_{\leqslant 0}\mathfrak{X} = \mathfrak{X}_0$ then lies in the essential image of fDM $\to$ fSpDM and hence can be considered as a classical spectral Deligne–Mumford stack. Moreover, $\mathfrak{X}^\heartsuit$ and $\mathfrak{X}_0^\heartsuit$ are naturally equivalent inside $\mathcal{P}(\mathrm{Aff}^\heartsuit)$ by Cor.A.2.5. As each affine formal spectral Deligne–Mumford stack $\mathcal{U}_n$ is locally Noetherian, $\mathfrak{X}^\heartsuit = \mathfrak{X}_0$ has an affine étale hypercover by $\mathcal{U}_\bullet^\heartsuit \to \mathfrak{X}^\heartsuit = \mathfrak{X}_0$ inside fDM. By Pr.A.3.6, we see $\mathfrak{X}_0$ is qcqs. As $\mathfrak{X}^\heartsuit \simeq \mathfrak{X}_0^\heartsuit$ we see $\mathfrak{X}_0$ and $\mathfrak{X}$ have the same closed points. As $\mathcal{U}_0^\heartsuit$ is a Noetherian affine classical formal Deligne–Mumford stack, we also see $\mathfrak{X}_0$ is locally Noetherian. It also follows from Rmk.3.2.2 that $\mathfrak{X}_0 \to \widehat{\mathcal{M}}^\heartsuit_{\mathrm{BT}^p_n, A_0}$ is formally étale inside $\mathcal{P}(\mathrm{Aff}^\heartsuit)$. Finally, to see the cotangent complex L of the map $\mathfrak{X}_0 \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$ is almost perfect inside $\mathrm{QCoh}(\mathfrak{X}_0)$, consider the composition in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$

$$\tau_{\leqslant 0}\mathfrak{X} = \mathfrak{X}_0 \to \mathfrak{X} \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$$

from which we obtain a (co)fibre sequence in $\mathrm{QCoh}(\mathfrak{X}_0)$:

$$L_{\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}}\Big|_{\mathfrak{X}_0} \to L \to L_{\mathfrak{X}_0/\mathfrak{X}}$$

By part 3 of Pr.3.2.7, the first term in the above (co)fibre sequence vanishes as $\mathfrak{X} \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}$ is formally étale. Our desired conclusion follows as $L_{\mathfrak{X}_0/\mathfrak{X}}$ is almost perfect by Pr.A.3.1. $\square$

To see $(-)^\heartsuit \colon \mathcal{C}_A \to \mathcal{C}_{A_0}$ is an equivalence of ∞-categories, we will construct an explicit inverse.

Definition 3.3.4. Define a functor $\mathcal{D} \colon \mathcal{C}_{A_0} \to \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})_{/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}}$ by sending an object $\mathbf{G}_0 \colon \mathfrak{X}_0 \to \widehat{\mathcal{M}}^\heartsuit_{\mathrm{BT}^p_n,A_0}$ of $\mathcal{C}_{A_0}$ to the de Rham space of [SAG, §18.2.1] (and Ex.3.2.17):

$$\mathcal{D}(\mathbf{G}_0) = \left(\mathfrak{X}_0/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}\right)_{\mathrm{dR}}$$

The notation $\mathcal{D}$ is supposed to conjure the word "deformation".

Theorem 3.3.5. *The functor $\mathcal{D}$ factors through $\mathcal{C}_A$, preserves affine objects and étale hypercovers, and is an inverse to $(-)^\heartsuit$.*

This equivalence of ∞-categories fits into the general paradigm of spectral algebraic geometry—a well-behaved site over a classical moduli stack should be equivalent to the same site over the associated spectral moduli stack; see the example of the moduli stack of elliptic curves in [EC1, Rmk.2.4.2] and [EC2, §7], or the affine case in [HA, Th.7.5.0.6].

To prove Th.3.3.5, we will use the interaction of the de Rham space technology of Lurie ([SAG, §18.2.1]) with formally étale morphisms and a representability theorem also due to Lurie.

Proposition 3.3.6. *Recall Nt. 2.1.5. Let $\mathfrak{X}$ be a formal spectral Deligne–Mumford stack and $\mathfrak{X} \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}$ be a 0-formally étale map whose associated cotangent complex is almost perfect. Then the following natural diagram of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$*

$$\begin{array}{ccc} (\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A})_{\mathrm{dR}} & \xrightarrow{u} & \tau^*_{\leqslant 0}\mathfrak{X} \\ \downarrow{\scriptstyle \mathbf{G}_{\mathrm{dR}}} & & \downarrow{\scriptstyle \tau^*_{\leqslant 0}\mathbf{G}} \\ \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A} & \xrightarrow{d} & \tau^*_{\leqslant 0}\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A} \end{array} \qquad (3.3.7)$$

is Cartesian, the natural map $\mathfrak{X} \to (\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A})_{\mathrm{dR}}$ induces an equivalence when evaluated on discrete $\mathbf{E}_\infty$-rings, and $\mathbf{G}_{\mathrm{dR}}$ is formally étale.

The above proposition and its proof generalise to a wider class of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ of which we could not find a neater classification than our leading example—we leave the reader to explore the general example as she wishes.

Proof of Pr.3.3.6. Recall the value of the de Rham space $(X/Y)_{\mathrm{dR}}$ on a connective $\mathbf{E}_\infty$-ring R from Ex.3.2.17

$$(X/Y)_{\mathrm{dR}}(R) = \operatorname*{colim}_I \left(Y(R) \underset{Y(\pi_0 R/I)}{\times} X(\pi_0 R/I) \right) \qquad (3.3.8)$$

where the colimit is taken over all nilpotent ideals of $\pi_0 R$. Define a functor $(X/Y)^0_{\mathrm{dR}}\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ by the same formula as (3.3.8) above but index the colimit over finitely generated nilpotent ideals of $\pi_0 R$. One readily obtains a map of functors

$$(X/Y)^0_{\mathrm{dR}} \to (X/Y)_{\mathrm{dR}}$$

and we claim this map is an equivalence for $X = \mathfrak{X}$ and $Y = \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}$ in our hypotheses. Indeed, one can copy the proof of [EC2, Pr.3.4.3] *mutatis mutandis*, exchanging only R_0 for $\mathfrak{X}$; the crucial step comes at the end and uses the almost perfect assumption on our cotangent complex. Writing $F_{R,I}$ for the fibre product within the colimit of (3.3.8) where $X = \mathfrak{X}$ and $Y = \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}$, we place $F_{R,I}$ into the following commutative diagram of spaces:

$$\begin{array}{ccccc} F_{R,I} & \longrightarrow & \mathfrak{X}(\pi_0 R) & \longrightarrow & \mathfrak{X}(\pi_0 R/I) \\ \downarrow & & \downarrow{\scriptstyle f_{\pi_0 R}} & & \downarrow{\scriptstyle f_{\pi_0 R/I}} \\ \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(R) & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(\pi_0 R) & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(\pi_0 R/I) \end{array}$$

The outer rectangle is Cartesian by definition and we claim that the right square is also Cartesian. Indeed, this follows as I is finitely generated and hence is nilpotent of finite degree n for some integer $n \geqslant 2$, and our 0-formally étale hypotheses can be sequentially applied to the composition of square-zero extensions:

$$R \to R/I^n \to R/I^{n-1} \to \cdots \to R/I^2 \to R/I$$

This implies that the left square above is also Cartesian, so the R-points of the de Rham space in question naturally take the form

$$\operatorname*{colim}_I \left(\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(R) \underset{\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(\pi_0 R)}{\times} \mathfrak{X}(\pi_0 R) \right) \simeq \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(R) \underset{\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(\pi_0 R)}{\times} \mathfrak{X}(\pi_0 R)$$

as the diagram indexing our colimit is filtered. This implies that (3.3.7) is Cartesian. For the second statement, we use the facts that (3.3.7) is Cartesian and d induces equivalences when evaluated on discrete rings to see that u induces an equivalence when evaluated on discrete rings. Noting that the maps

$$(\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A})_{\mathrm{dR}} \to \tau^*_{\leqslant 0}\mathfrak{X} \leftarrow \mathfrak{X}$$

induce equivalences on discrete rings, the natural map $\mathfrak{X} \to (\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A})_{\mathrm{dR}}$ induces an equivalence on discrete rings as well. Finally, to see that the map

$$(\mathfrak{X}/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A})_{\mathrm{dR}} \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}$$

is formally étale we use Rmk.3.2.3 to see $\tau^*_{\leqslant 0}\mathbf{G}$ is formally étale and then base-change Pr.3.2.6. Alternatively, we can refer to Ex.3.2.17 which states such maps from de Rham spaces are always formally étale. ☐

Remark 3.3.9 ($\mathcal{D}$ produces universal spectral deformations). Recall that associated to a classical p-divisible group $\mathbf{G}_0\colon \operatorname{Spf} B_0 \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$, such as those in $\mathcal{C}_{A_0}$, then we can ask if there exists a *universal spectral deformation* of $\mathbf{G}_0$ with associated spectral deformation ring; see Df. 1.3.13. It follows from the proof of Pr.3.3.6 above, that if $\mathbf{G}_0$ lies in $\mathcal{C}_{A_0}$, then the formal spectrum Spf of the spectral deformation ring of $\mathbf{G}_0$ is equivalent to the de Rham space $(\mathbf{G}_0\colon \operatorname{Spf} B_0 \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A})_{\mathrm{dR}}$. By Th.3.3.5, we see that this de Rham space is represented by a formal spectral Deligne–Mumford stack $\operatorname{Spf} B$. This means that $\mathcal{D}(\mathbf{G}_0)$ is represented by the universal spectral deformation of $\mathbf{G}_0$. This is even true in a nonaffine sense, but we will not need to venture further in that direction.

The following representability theorem of Lurie is crucial to prove Th.3.3.5.

Theorem 3.3.10 ([SAG, Th.18.2.3.1])**.** *Let $f\colon \mathfrak{X} \to \mathcal{M}$ be a map of functors in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ such that $\mathfrak{X}$ is a formal spectral Deligne–Mumford stack, $\mathcal{M}$ is nilcomplete, infinitesimally cohesive, admits a cotangent complex, and is an étale sheaf, and $L_{\mathfrak{X}/\mathcal{M}}$ is 1-connective and almost perfect. Then $(\mathfrak{X}/\mathcal{M})_{\mathrm{dR}}$ is represented by a* formal thickening[24] *of $\mathfrak{X}$.*

Importantly, we can apply this theorem to $\mathcal{C}_{A_0}$.

Remark 3.3.11. By definition, the cotangent complex $L = L_{\mathfrak{X}_0/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}}$ corresponding to an object inside $\mathcal{C}_{A_0}$ is almost connective, meaning $L[n]$ is connective for some positive integer n; see [SAG, Var.8.2.5.7 & Rmk.8.2.5.9]. However, we claim that L is actually 1-connective. Indeed, by [SAG, Cor.8.2.5.5] we may check this étale locally on $\mathfrak{X}_0$, so let us replace $\mathfrak{X}_0$ with $\operatorname{Spf} B_0$ for some complete Noetherian discrete adic ring B_0. In particular, L is now an almost perfect J-complete B_0-module, where J is an ideal of definition for B_0. As L is almost perfect, the fibrewise connectivity criterion of [SAG, Cor.2.7.4.3] shows that it suffices to check $L^{\wedge}_{\mathfrak{m}}$ is 1-connective for every maximal ideal $\mathfrak{m} \subseteq B_0$ which contains J. Moreover, considering the maps

$$\operatorname{Spf}(B_0)^{\wedge}_{\mathfrak{m}} \to \operatorname{Spf} B_0 \to \operatorname{Spec} B_0$$

the composition is formally étale in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ by Ex.3.2.16, and hence in $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ by Rmk.3.2.2, and the latter map is unramified, so by Pr.3.1.2 we see the first map is formally étale. We may then assume B_0 is a complete local Noetherian ring. The morphism $\mathbf{G}\colon \operatorname{Spf} B_0 \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$ is formally étale, so by the converse statement in Pr. 3.1.10, we see that B_0 is the classical deformation ring of $\mathbf{G}_\kappa$, where κ is the residue field of B_0, which is necessarily perfect of characteristic p by assumption. For such a pair $(\mathbf{G}_\kappa, \kappa)$, there exists a spectral deformation ring B by [EC2, Th.3.1.15], as κ is perfect and $\mathbf{G}_\kappa$ is nonstationary

[24]Recall from [SAG, Df.18.2.2.1], a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of formal spectral Deligne–Mumford stacks is called a *formal thickening* if the induced map on reductions $\mathfrak{X}^{\mathrm{red}} \to \mathfrak{Y}^{\mathrm{red}}$ is an equivalence ([SAG, §8.1.4]) and the map f is representable by closed immersions which are locally almost of finite presentation.

by [EC2, Ex.3.0.10], which implies $\pi_0 B \simeq B_0$ by Rmk.1.3.14. This means the map $\operatorname{Spf} B_0 \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$ in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ factors as

$$\operatorname{Spf} B_0 \to \operatorname{Spf} B \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A} \tag{3.3.12}$$

where the first map is induced by the truncation. Associated with the above composition is the following (co)fibre sequence of complete B_0-modules:

$$L_{\operatorname{Spf} B/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}}\Big|_{\operatorname{Spf} B_0} \to L_{\operatorname{Spf} B_0/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}} \to L_{\operatorname{Spf} B_0/\operatorname{Spf} B}$$

The first object vanishes as $\operatorname{Spf} B$ is the de Rham space for the composite (3.3.12) and such objects always vanish; see Ex.3.2.17. We then see the middle cotangent complex above is 1-connected and almost perfect as this holds for $L_{\operatorname{Spf} B_0/\operatorname{Spf} B}$ by Pr.A.3.1.

Proof of Th.3.3.5. First, let us check $\mathcal{D}$ factors through $\mathcal{C}_A$. Using Th.3.3.10 and Rmk.3.3.11, we see $\mathcal{D}(\mathbf{G}_0)$ is represented by a formal thickening $\mathfrak{X}$ of $\mathfrak{X}_0$; see [SAG, §18.2.2] or (24). To see $\mathfrak{X}$ satisfies the conditions of Df.2.1.6, we note the following:

- $\mathfrak{X}$ is locally Noetherian, as it is a formal thickening of the locally Noetherian $\mathfrak{X}_0$; see [SAG, Cor.18.2.4.4].
- $\mathfrak{X}$ is qcqs as a formal thickening of a qcqs formal spectral Deligne–Mumford stack is qcqs; see Pr.A.3.8.
- $\mathfrak{X}$ has perfect residue fields at closed points as this is true for $\mathfrak{X}_0$ and $\mathfrak{X}_0 = \tau_{\leqslant 0}\mathfrak{X}$ has the same residue fields as $\mathfrak{X}$.
- $\mathbf{G}$ is formally étale by Pr.3.3.6.

If $\mathfrak{X}_0 \simeq \operatorname{Spf} B_0$ is affine, then the image of any $\mathbf{G}_0\colon \operatorname{Spf} B_0 \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$ in $\mathcal{C}_{A_0}$ under $\mathcal{D}$ is also affine as formal thickenings of affines are affine; see [SAG, Cor.18.2.3.3]. To see $\mathcal{D}$ is inverse to $(-)^{\heartsuit}$, notice the composite $(-)^{\heartsuit}\mathcal{D}$ is equivalent to the identity as $\mathbf{G}_0 \to \mathcal{D}(\mathbf{G}_0)$ induces an equivalence on discrete rings by Pr.3.3.6. For the other composition, part 1 of Pr.3.2.7 states that the following diagram of spaces is Cartesian for every connective $\mathbf{E}_\infty$-ring R:

$$\begin{array}{ccc}
\mathfrak{X}(R) & \longrightarrow & \mathfrak{X}(\pi_0 R) = \tau^*_{\leqslant 0}\mathfrak{X}(R) \\
\big\downarrow{\scriptstyle \mathbf{G}} & & \big\downarrow \\
\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(R) & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(\pi_0 R) = \tau^*_{\leqslant 0}\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}(R)
\end{array}$$

It follows that the natural map $\mathcal{D}((\mathbf{G})^{\heartsuit}) \to \mathbf{G}$ is an equivalence in $\mathcal{C}_A$. Finally, to see $\mathcal{D}$ preserves étale hypercovers, we first note this may be checked étale locally, so take an étale hypercover $\mathfrak{Y}^0_\bullet = \operatorname{Spf} C^\bullet_0 \to \operatorname{Spf} B_0 = \mathfrak{X}_0$ in $\mathcal{C}_{A_0}$ and write $\operatorname{Spf} C^\bullet \to \operatorname{Spf} B$ for its image under $\mathcal{D}$. From the above, we know that

$\operatorname{Spf} C^\bullet \to \operatorname{Spf} B$ is an étale hypercover on zeroth truncations, so it suffices to see each map $B \to C^n$ is étale as a morphism of adic $\mathbf{E}_\infty$-rings. Two applications of Pr.3.3.6 show the commutative diagram in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$

$$\begin{array}{ccccc} \mathfrak{Y}_\bullet & \longrightarrow & \mathfrak{X} & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A} \\ \downarrow & & \downarrow & & \downarrow \\ \tau^*_{\leqslant 0}\mathfrak{Y}^0_\bullet & \longrightarrow & \tau^*_{\leqslant 0}\mathfrak{X}_0 & \longrightarrow & \tau^*_{\leqslant 0}\widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A} \end{array}$$

consists of Cartesian squares, hence the map $\operatorname{Spf} C^n \to \operatorname{Spf} B$ is formally étale by Rmk.3.2.3 and base-change Pr.3.2.6. It follows from Rmk.3.2.13 that $B \to C^n$ is flat and we know its of finite presentation on π_0 by assumption. This follows that $B \to C^n$ is étale by Rmk.3.2.13 again, and we are done. □

Finally, let us solidify some of the connections between formally étale morphisms and universal deformations (Df.1.3.13). The following is analogous to Pr.3.1.10 and our proof follows that of [EC2, Pr.7.4.2].

Proposition 3.3.13. *Recall Nt.2.1.5. Let* $\mathbf{G}\colon \operatorname{Spf} B \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$ *be formally étale map where B is a complete adic Noetherian* $\mathbf{E}_\infty$*-ring with ideal of definition J. Fix a maximal ideal $\mathfrak{m} \subseteq \pi_0 B$ containing J such that $\pi_0 B/\mathfrak{m}$ is perfect of characteristic p. Then the p-divisible group $\mathbf{G}_{B^\wedge_\mathfrak{m}}$ is the universal spectral deformation of $\mathbf{G}_\kappa$, where κ is the residue field of $B_\mathfrak{m}$.*

Proof. As κ is perfect of characteristic p, combining [EC2, Ex.3.0.10] with [EC2, Th.3.1.15] one obtains the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_\kappa} = B^{\mathrm{un}}$ with a universal p-divisible group $\mathbf{G}^{\mathrm{un}}$. By definition, $\mathbf{G}_{B^\wedge_\mathfrak{m}}$ is a deformation over $\mathbf{G}_\kappa$ ([EC2, Df.3.0.3]), so from the universality of $(B^{\mathrm{un}}, \mathbf{G}^{\mathrm{un}})$ we obtain a canonical continuous morphism of adic $\mathbf{E}_\infty$-rings $B^{\mathrm{un}} \xrightarrow{\alpha} B^\wedge_\mathfrak{m} = \widehat{B}$ inducing the identity on the common residue field κ. By [EC2, Th.3.1.15], we see B^{un} belongs to the full ∞-subcategory $\mathcal{C}$ of $(\mathrm{CAlg}^{\mathrm{cn}}_{\mathrm{ad}})_{/\kappa}$ spanned by complete local Noetherian adic $\mathbf{E}_\infty$-rings whose augmentation to κ exhibits κ as its residue field. To see α is an equivalence in this ∞-category, consider an arbitrary object C of $\mathcal{C}$ and the induced map on mapping spaces:

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(\widehat{B}, C) \xrightarrow{\alpha^*} \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(B^{\mathrm{un}}, C)$$

By writing C as the limit of its truncations we are reduced to the case where C is truncated, and by writing $\pi_0 C$ as a limit of Artinian subrings of $\pi_0 C$ we are further reduced to the case when $\pi_0 C$ is Artinian.[25] In this situation, we have a finite sequence of maps

$$C = C_m \to C_{m-1} \to \cdots \to C_1 \to C_0 = k$$

[25] Our conventions demand that local adic $\mathbf{E}_\infty$-rings have their topology determined by the maximal ideal.

where each map is a square-zero extension by an almost perfect connective module. Hence, it would suffice to show that for every $C \to \kappa$ in $\mathcal{C}$, and every square-zero extension $\widetilde{C} \to C$ of C by an almost perfect connective C-module, with $\widetilde{C}$ also in $\mathcal{C}$, the natural diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(\widehat{B}, \widetilde{C}) & \longrightarrow & \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(B^{\mathrm{un}}, \widetilde{C}) \\ \downarrow & & \downarrow \\ \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(\widehat{B}, C) & \longrightarrow & \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(B^{\mathrm{un}}, C) \end{array} \tag{3.3.14}$$

is Cartesian, the $C = \kappa$ case being tautological. As $\widehat{B}$ is the $\mathfrak{m}$-completion of $B_{\mathfrak{m}}$, then for any D in $\mathcal{C}$ (which in particular is complete with respect to the kernel of its augmentation $D \to \kappa$) the map

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(\widehat{B}, D) \xrightarrow{\simeq} \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(B_{\mathfrak{m}}, D)$$

induced by $B_{\mathfrak{m}} \to \widehat{B}$, is an equivalence. Moreover, for any D inside $\mathcal{C}$ we have the following natural identifications:

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(B^{\mathrm{un}}, D) \simeq \underset{B^{\mathrm{un}} \to \kappa}{\mathrm{fib}} \left(\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(B^{\mathrm{un}}, D) \to \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(B^{\mathrm{un}}, \kappa) \right)$$

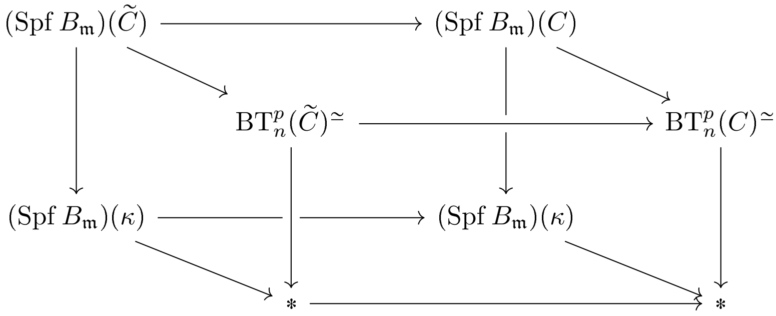
$$\simeq \underset{B^{\mathrm{un}} \to \kappa}{\mathrm{fib}} \left((\mathrm{Spf}\, B^{\mathrm{un}})(D) \to (\mathrm{Spf}\, B^{\mathrm{un}})(\kappa) \right) \simeq \underset{\mathbf{G}^{\mathrm{un}}}{\mathrm{fib}} \left(\mathrm{Def}_{\mathbf{G}_\kappa}(D) \to \mathrm{Def}_{\mathbf{G}_\kappa}(\kappa) \right)$$

$$\simeq \mathrm{Def}_{\mathbf{G}_\kappa}(D, (D \to \kappa)) \simeq \mathrm{BT}^p_n(D) \underset{\mathrm{BT}^p_n(\kappa)}{\times} \{\mathbf{G}_\kappa\}$$

The first equivalence is a categorical fact about over/under categories, the second is the identification of the R-valued points of Spf B^{un} ([SAG, Lm.8.1.2.2]), the third is from universal property of B^{un} ([EC2, Th.3.1.15]), and the fourth and fifth can be taken as two alternative definitions of $\mathrm{Def}_{\mathbf{G}_\kappa}(D, (D \to \kappa))$ ([EC2, Df.3.0.3 & Rmk.3.1.6]). These natural equivalences show (3.3.14) is equivalent to the upper-left square in the following natural diagram of spaces:

$$\begin{array}{ccccc} \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(B_{\mathfrak{m}}, \widetilde{C}) & \longrightarrow & \mathrm{BT}^p_n(\widetilde{C}) \underset{\mathrm{BT}^p_n(\kappa)}{\times} \{\mathbf{G}_\kappa\} & \longrightarrow & \mathrm{BT}^p_n(\widetilde{C})^{\simeq} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_{/\kappa}}(B_{\mathfrak{m}}, C) & \longrightarrow & \mathrm{BT}^p_n(C) \underset{\mathrm{BT}^p_n(\kappa)}{\times} \{\mathbf{G}_\kappa\} & \longrightarrow & \mathrm{BT}^p_n(C)^{\simeq} \\ & & \downarrow & & \downarrow \\ & & \{\mathbf{G}_\kappa\} & \longrightarrow & \mathrm{BT}^p_n(\kappa)^{\simeq} \end{array} \tag{3.3.15}$$

The bottom-right square and right rectangle are both Cartesian by definition, so the upper-right square is Cartesian. It now suffices to see the upper rectangle is Cartesian, so we consider the following natural diagram of spaces:



The top square is Cartesian as $\mathrm{Spf}\, B \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$ (and hence $\mathrm{Spf}\, B_\mathfrak{m} \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$) is formally étale, and the bottom square is trivially Cartesian. Taking the fibres of the vertical morphisms (at the given map $\mathcal{B}_\mathfrak{m} \to \kappa$) we obtain the upper rectangle of (3.3.15), whence this upper rectangle is also Cartesian and we are done. $\square$

Chapter 4

Construction of the structure sheaf

The study of orientations of p-divisible (and formal) groups over $\mathbf{E}_\infty$-rings is the focus of [EC2]. Using Lurie's work, we construct a "derived stack" classifying oriented p-divisible groups, $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$, defined on (not necessarily connective) p-complete $\mathbf{E}_\infty$-rings. The technical complications of this section stem from the necessary movement between presheaves on connective and general $\mathbf{E}_\infty$-rings.

We suggest that the reader keeps a copy of [EC2] in her vicinity when reading this section.

4.1 Orientations on p-divisible groups

Recall the concept of an orientation of a *formal group*[26] over an $\mathbf{E}_\infty$-ring, as detailed in [EC2, §1.6 & 4.3].

Definition 4.1.1. Let R be an $\mathbf{E}_\infty$-ring and $\widehat{\mathbf{G}}$ be a formal group over R. A *preorientation* of $\widehat{\mathbf{G}}$ is an element e of $\Omega^2(\Omega^\infty\widehat{\mathbf{G}})(\tau_{\geqslant 0}R)$. Alternatively, assuming that R is complex periodic (10), then an orientation of $\widehat{\mathbf{G}}$ is a morphism of formal groups $\widehat{\mathbf{G}}^{\mathcal{Q}}_R \to \widehat{\mathbf{G}}$ over R, where $\widehat{\mathbf{G}}^{\mathcal{Q}}_R$ is the Quillen formal group of R; see (14). Such a preorientation $e\colon \widehat{\mathbf{G}}^{\mathcal{Q}}_R \to \widehat{\mathbf{G}}$ is an *orientation* if it is an equivalence of formal groups over R; see [EC2, Pr.4.3.23]. Denote by $\mathrm{OrDat}(\widehat{\mathbf{G}})$ the component of $\Omega^2(\Omega^\infty\widehat{\mathbf{G}})(\tau_{\geqslant 0}R)$ consisting of orientations—by definition this is empty if R is not complex periodic. An *orientation of a p-divisible group* $\mathbf{G}$ over a p-complete $\mathbf{E}_\infty$-ring is an orientation of $\mathbf{G}^\circ$, its identity component (13).

[26] Recall from [EC2, Df.1.6.1 & Var.1.6.2], a that a *formal group* over an $\mathbf{E}_\infty$-ring R is a functor $\widehat{\mathbf{G}}\colon \mathrm{CAlg}^{\mathrm{cn}}_{\tau_{\geqslant 0}R} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}}$ whose postcomposition with $\Omega^\infty\colon \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}} \to \mathcal{S}$ is a formal hyperplane over $\tau_{\geqslant 0}R$ in the sense of [EC2, Df.1.5.10]. The latter can be identified as the essential image in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}}_{\tau_{\geqslant 0}R})$ of the cospectra of smooth coalgebras; see [EC2, §1.5].

Recall that when we associate to a functor $F\colon \mathcal{C} \to \mathscr{C}\mathrm{at}_\infty$ (resp. $\mathcal{C} \to \mathcal{S}$) a coCartesian (resp. left) fibration $\int_\mathcal{C} F \to \mathcal{C}$, or visa versa, we are using the straightening–unstraightening adjunction of [HTT09, Th.3.2.0.1]—the ∞-categorical Grothendieck construction.

Definition 4.1.2. 1. Let $\mathcal{M}^{\mathrm{nc}}_{\mathrm{BT}^p}\colon \mathrm{CAlg} \to \mathcal{S}$ be the composite of the truncation functor $\tau_{\geqslant 0}\colon \mathrm{CAlg} \to \mathrm{CAlg}^{\mathrm{cn}}$ and $\mathcal{M}_{\mathrm{BT}^p}$; see [EC2, Var.2.0.6]. Define a functor $\mathcal{M}^{\mathrm{nc}}_{\mathrm{BT}^p_n}\colon \mathrm{CAlg} \to \mathcal{S}$ analogously.

2. Denote by CAlg^p the full ∞-subcategory of CAlg spanned by p-complete $\mathbf{E}_\infty$-rings and write Aff^p for $(\mathrm{CAlg}^p)^{\mathrm{op}}$. Let $\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p}\colon \mathrm{CAlg}^p \to \mathcal{S}$ be the composition of $\mathcal{M}^{\mathrm{nc}}_{\mathrm{BT}^p}$ with the inclusion $\mathrm{CAlg}^p \to \mathrm{CAlg}$. Define a functor $\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}\colon \mathrm{CAlg}^p \to \mathcal{S}$ analogously.

3. Let R be a p-complete $\mathbf{E}_\infty$-ring. Write $\mathrm{OrBT}^p(R) \to \mathrm{BT}^p(R)^\simeq$ for the left fibration associated to the following functor ([EC2, Rmk.4.3.4]):

$$\mathrm{BT}^p(R)^\simeq \to \mathcal{S} \qquad \mathbf{G} \mapsto \mathrm{OrDat}(\mathbf{G}^\circ) = \mathrm{OrDat}(\mathbf{G})$$

Define $\mathrm{OrBT}^p_n(R)$ analogously.

We restrict to p-complete $\mathbf{E}_\infty$-rings above as we will often use [EC2, Th.2.0.8] to associate to a p-divisible group $\mathbf{G}$ its identity component $\mathbf{G}^\circ$. This is not strictly necessary, as demonstrated by [EC3, §2], however, we only care about the p-complete case to prove Th.2.1.7 anyhow.

Our goal here is to define a moduli functor $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p}\colon \mathrm{CAlg}^p \to \mathcal{S}$ sending R to $\mathrm{OrBT}^p(R)^\simeq$, a sort of iterated Grothendieck construction. To do this honestly in the language of ∞-categories, we will construct the associated left fibration; the reader is invited to skip the following technical construction for now, and only return if she is unconvinced by this heuristic.

Construction 4.1.3. Let $\mathrm{CAlg}^p_{\mathrm{co}}$ be the full ∞-subcategory of CAlg^p spanned by those p-complete complex periodic $\mathbf{E}_\infty$-rings. Using [EC2, Rmk.1.6.4], define the functor $\mathcal{M}_{\mathrm{FGroup}}(-)$

$$\mathrm{CAlg}^p_{\mathrm{co}} \xrightarrow{\mathrm{FGroup}(-)} \mathcal{S} \qquad R \mapsto \mathrm{FGroup}(R)^\simeq$$

sending a p-complete $\mathbf{E}_\infty$-ring to the ∞-groupoid core of its associated ∞-category of formal groups ([EC2, Df.1.6.1]), and write $F\colon \mathcal{M}_{\mathrm{FGroup}} \to \mathrm{CAlg}^p_{\mathrm{co}}$ for the associated left fibration. The functor F has a section $\mathcal{Q}$ which sends a p-complete complex periodic $\mathbf{E}_\infty$-ring R to its Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_R$ ([EC2, Con.4.1.13]); this association is functorial as taking the R-homology and cospectrum operators are functorial. Let $\mathcal{M}_{\mathrm{OrFGroup}}$ be the comma ∞-category $(\mathcal{Q}F \downarrow \mathrm{id}_{\mathcal{M}_{\mathrm{FGroup}}})$, in other words, there is a Cartesian diagram inside $\mathscr{C}\mathrm{at}_\infty$

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{OrFGroup}} & \longrightarrow & (\mathcal{M}_{\mathrm{FGroup}})^{\Delta^1} \\ \downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{M}_{\mathrm{FGroup}} & \xrightarrow{(\mathcal{Q}F\times \mathrm{id})\circ\Delta} & \mathcal{M}_{\mathrm{FGroup}} \times \mathcal{M}_{\mathrm{FGroup}} \end{array} \tag{4.1.4}$$

where Δ^1 is the 1-simplex, Δ is the diagonal map, and (s,t) sends an arrow in $\mathcal{M}_{\mathrm{FGroup}}$ to its source and target. More informally, an object of $\mathcal{M}_{\mathrm{OrFGroup}}$ is a complex periodic p-complete $\mathbf{E}_\infty$-ring R, a formal group $\widehat{\mathbf{G}}$ over R, and a equivalence $\widehat{\mathbf{G}}_R^{\mathcal{Q}} \simeq \widehat{\mathbf{G}}$ of formal groups over R. By [EC2, Pr.4.3.23], such an equivalence of formal groups over R is precisely the data of an orientation of $\widehat{\mathbf{G}}$, hence the name OrFGroup. The functor

$$\mathcal{M}_{\mathrm{OrFGroup}} \to \mathcal{M}_{\mathrm{FGroup}}, \qquad (R, \widehat{\mathbf{G}}, e) \mapsto (R, \widehat{\mathbf{G}}) \tag{4.1.5}$$

is a left fibration with associated functor to spaces

$$\mathcal{M}_{\mathrm{FGroup}} \to \mathcal{S} \qquad (R, \widehat{\mathbf{G}}) \mapsto \mathrm{OrDat}(\widehat{\mathbf{G}}).$$

Indeed, this assignment is a functor by [EC2, Rmk.4.3.10] and the above identification comes by contemplating the fibre product of categories

$$\{(R, \widehat{\mathbf{G}})\} \underset{\mathcal{M}_{\mathrm{FGroup}}}{\times} \mathcal{M}_{\mathrm{OrFGroup}} \simeq \mathrm{Map}_{\mathcal{M}_{\mathrm{FGroup}(R)}}(\widehat{\mathbf{G}}_R^{\mathcal{Q}}, \widehat{\mathbf{G}}) \simeq \mathrm{OrDat}(\widehat{\mathbf{G}}),$$

where the second equivalence again comes from [EC2, Pr.4.3.23]. Now, write $G\colon \mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p} \to \mathrm{CAlg}^p_{\mathrm{co}}$ for the left fibration associated with the following composition:

$$\mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p}(-)\colon \mathrm{CAlg}^p_{\mathrm{co}} \xrightarrow{\text{inc.}} \mathrm{CAlg}^p \xrightarrow{\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p}} \mathcal{S} \qquad R \mapsto \mathrm{BT}^p(R)^{\simeq}$$

The natural assignment sending a p-divisible group $\mathbf{G}$ over a p-complete $\mathbf{E}_\infty$-ring R to its identity component induces a functor $(-)^\circ\colon \mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p} \to \mathcal{M}_{\mathrm{FGroup}}$ between categories over $\mathrm{CAlg}^p_{\mathrm{co}}$. Define an ∞-category $\mathcal{M}_{\mathrm{OrBT}^p}$ by the following Cartesian diagram of ∞-categories:

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{OrBT}^p} & \longrightarrow & \mathcal{M}_{\mathrm{OrFGroup}} \\ \downarrow & & \downarrow \\ \mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p} & \xrightarrow{(-)^\circ} & \mathcal{M}_{\mathrm{FGroup}} \end{array}$$

As (4.1.5) is a left fibration, then $\mathcal{M}_{\mathrm{OrBT}^p} \to \mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p}$ is also a left fibration by base-change. Similarly, we define the ∞-category $\mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p_n}$, which comes with a natural map $\mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p_n} \to \mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p}$ associated to the inclusion $\mathrm{BT}^p_n(R)^{\simeq} \to \mathrm{BT}^p(R)^{\simeq}$. Finally, define a left fibration $\mathcal{M}_{\mathrm{OrBT}^p_n} \to \mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p}$ by the following Cartesian diagram in $\mathscr{C}\mathrm{at}_\infty$:

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{OrBT}^p_n} & \longrightarrow & \mathcal{M}_{\mathrm{OrBT}^p} \\ \downarrow & & \downarrow \\ \mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p_n} & \longrightarrow & \mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p} \end{array}$$

In total, we have a left fibration

$$\mathcal{M}_{\mathrm{OrBT}^p} \to \mathcal{M}_{\mathrm{BT}^p} \to \mathrm{CAlg}^p_{\mathrm{co}}$$

and unravelling the construction above, one can calculate the functor associated with this composition:

$$\mathrm{CAlg}^p_{\mathrm{co}} \to \mathcal{S} \qquad R \mapsto \mathrm{OrBT}^p(R)$$

Similarly, we have $\mathcal{M}_{\mathrm{OrBT}^p_n} \to \mathrm{CAlg}^p_{\mathrm{co}}$ and its associated functor.

Definition 4.1.6. Given a morphism $A \to B$ in CAlg^p, then if A is complex periodic, we see B is also complex periodic; see [EC2, Rmk.4.1.3]. Define a functor $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p}\colon \mathrm{CAlg}^p \to \mathcal{S}$ first on $\mathrm{CAlg}^p_{\mathrm{co}}$ as the functor associated with the composition of left fibrations

$$\mathcal{M}_{\mathrm{OrBT}^p} \to \mathcal{M}^{\mathrm{co}}_{\mathrm{BT}^p} \to \mathrm{CAlg}^p_{\mathrm{co}}$$

defined in Con.4.1.3, and then as the empty space on objects in CAlg^p who are not complex periodic. More informally, $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p}$ is the assignment:

$$R \mapsto \begin{cases} \mathrm{OrBT}^p(R)^{\simeq} & \text{if } R \text{ is complex periodic} \\ \varnothing & \text{if } R \text{ is not complex periodic} \end{cases}$$

Define $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ by the Cartesian square in $\mathcal{P}(\mathrm{Aff}^p)$

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n} & \longrightarrow & \mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p} \\ \downarrow{\scriptstyle \Omega} & & \downarrow{\scriptstyle \Omega} \\ \mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n} & \longrightarrow & \mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p} \end{array}$$

where right Ω is the functor naturally induced by Con.4.1.3.

The notation Ω is reminiscent of the word “orientation”. At present, we have constructed a presheaf $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ on p-complete $\mathbf{E}_\infty$-rings, and a routine check shows this functor is a sheaf.

Proposition 4.1.7. *Let R be an $\mathbf{E}_\infty$-ring and n a positive integer. Then the functors*

$$\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p}, \mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}, \mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p}, \mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}\colon \mathrm{CAlg}^p \to \mathcal{S}$$

are all fpqc (hence also étale) hypersheaves.

As a first step, let us state a slight generalisation of [EC2, Pr.3.2.2(5)]; the proof is exactly as Lurie outlines in *ibid* but with fpqc hypercovers replacing fpqc covers.

Lemma 4.1.8. *The functors $\mathcal{M}_{\mathrm{BT}^p}, \mathcal{M}_{\mathrm{BT}^p_n}\colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ are fpqc hypersheaves.*

Proof of Theorem 4.1.7. To see $\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p}$ in $\mathcal{P}(\mathrm{Aff}^p)$ is an fpqc hypersheaf, it suffices to see $\mathcal{M}^{\mathrm{nc}}_{\mathrm{BT}^p}$ in $\mathcal{P}(\mathrm{Aff})$ is an fpqc hypersheaf as the inclusion $\mathrm{CAlg}^p \to \mathrm{CAlg}$ sends fpqc hypercovers to fpqc hypercovers. To see $\mathcal{M}^{\mathrm{nc}}_{\mathrm{BT}^p}$ is an fpqc hypersheaf, take an $\mathbf{E}_\infty$-ring R and an fpqc hypercover $R \to R^\bullet$. From Lm.1.3.4, (1.3.5), and Lm.4.1.8 we see that the natural maps

$$\mathcal{M}^{\mathrm{nc}}_{\mathrm{BT}^p}(R) = \mathcal{M}_{\mathrm{BT}^p}(\tau_{\geqslant 0}R) \xrightarrow{\simeq} \mathcal{M}_{\mathrm{BT}^p}(\lim{(\tau_{\geqslant 0}R)^\bullet})$$

$$\xrightarrow{\simeq} \lim \mathcal{M}_{\mathrm{BT}^p}((\tau_{\geqslant 0}R)^\bullet) = \lim \mathcal{M}^{\mathrm{nc}}_{\mathrm{BT}^p}(R^\bullet)$$

are all equivalences. Hence $\mathcal{M}^{\mathrm{nc}}_{\mathrm{BT}^p}$, and also $\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p}$, are fpqc hypersheaves. It follows that $\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}$ is also an fpqc hypersheaf as it is an open subfunctor of $\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p}$; see Remark 2.1.2.

By Lm. 1.3.3, to see $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p}$ is a sheaf, it suffices to see that the functor $\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p} \colon \mathrm{CAlg}^p \to \mathcal{S}$ is an fpqc hypersheaf, and that the functor F defined on objects by

$$F \colon \mathrm{CAlg}^p \to (\mathscr{C}\mathrm{at}_\infty)_{/\mathcal{S}}, \qquad R \mapsto \begin{pmatrix} \mathrm{BT}^p(R)^{\simeq} & \to & \mathcal{S} \\ \mathbf{G} & \mapsto & \mathrm{OrDat}(\mathbf{G}^\circ) \end{pmatrix} \tag{4.1.9}$$

is an fpqc hypersheaf; to define this functor honestly, one can use the standard techniques as done in Con.4.1.3. We have just seen $\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p}$ is an fpqc hypersheaf, so it suffices to see that (4.1.9) is an fpqc hypersheaf. Again, write $R \to R^\bullet$ for an fpqc hypercover of R in CAlg^p. As $\mathcal{M}_{\mathrm{BT}^p} \colon \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is an fpqc hypersheaf (Lm.4.1.8), we obtain the following natural equivalence from the definition of $\mathrm{OrDat}(\lim R^\bullet)$:

$$\begin{pmatrix} \mathrm{BT}^p(\lim R^\bullet)^{\simeq} & \to & \mathcal{S} \\ \mathbf{G} & \mapsto & \mathrm{OrDat}(\mathbf{G}^\circ) \end{pmatrix}$$

$$\xrightarrow{\simeq} \begin{pmatrix} \lim \mathrm{BT}^p(R^\bullet)^{\simeq} & \to & \mathcal{S} \\ \mathbf{G}_\bullet & \mapsto & \mathrm{OrDat}((\lim \mathbf{G}_\bullet)^\circ) \end{pmatrix} \tag{4.1.10}$$

Above, we have written $\mathbf{G}_\bullet$ for the base-change of $\mathbf{G}$ over $R^\bullet$. Using the characterising property of the identity component (as seen in [EC2, Th.2.0.8]), we take some $A \in \mathcal{E}$ (using the notation of [EC2, Th.2.0.8] and (13)) and obtain the following sequence of natural equivalences where all fibres are taken over the identity element:

$$(\lim \mathbf{G}_\bullet)^\circ(A) = \mathrm{fib}(\lim{(\mathbf{G}_\bullet)}(A) \to \lim{(\mathbf{G}_\bullet)}(A^{\mathrm{red}}))$$

$$\simeq \lim \mathrm{fib}(\mathbf{G}_\bullet(A) \to \mathbf{G}_\bullet(A^{\mathrm{red}})) = \lim{(\mathbf{G}^\circ_\bullet(A))} \xleftarrow{\simeq} (\lim \mathbf{G}^\circ_\bullet)(A)$$

The first equivalence comes from the fact that fibres commute with small limits and the second equivalence from the fact that limits in functor ∞-categories are computed levelwise. From this we see that (4.1.10) is naturally equivalent to

$$\begin{pmatrix} \lim \mathrm{BT}^p(R^\bullet)^{\simeq} & \to & \mathcal{S} \\ \mathbf{G}_\bullet & \mapsto & \mathrm{OrDat}(\lim{(\mathbf{G}^\circ)_\bullet}) \end{pmatrix} \tag{4.1.11}$$

where $(\mathbf{G}^\circ)_\bullet$ is the base-change of $\mathbf{G}^\circ$ over $R^\bullet$. For a fixed pointed formal hyperplane X over an $\mathbf{E}_\infty$-ring R, the functor

$$\mathrm{CAlg}_R \to \mathcal{S}, \qquad A \mapsto \mathrm{OrDat}(X_A)$$

is representable by [EC2, Pr.4.3.13], hence it commutes with small limits. In particular, this implies that the expression (4.1.11) is naturally equivalent to

$$\begin{pmatrix} \lim \mathrm{BT}^p(R^\bullet)^\simeq & \to & \mathcal{S} \\ \mathbf{G}_\bullet & \mapsto & \lim\ \mathrm{OrDat}(\mathbf{G}^\circ_\bullet) \end{pmatrix} = \lim F(R^\bullet)$$

Combining everything, we obtain the natural equivalence $F(R) \xrightarrow{\simeq} \lim F(R^\bullet)$ hence $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p}$ is an fpqc sheaf. The corresponding statement for $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ follows as it is a fibre product of fpqc hypersheaves. ☐

4.2 Orientation classifiers

It is our goal now to try and understand *universal* orientations and their relation to $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$. We would like to formally construct a presheaf $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ of $\mathbf{E}_\infty$-rings on $\mathcal{C}_A$ which behaves like the pushforward of the structure sheaf on $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ along Ω. We perform this construction by first restricting ourselves to affines.

Definition 4.2.1. Recall Nt. 2.1.5. Write $\mathcal{C}^{\mathrm{aff}}_{A_0}$ (resp. $\mathcal{C}^{\mathrm{aff}}_{A}$) for the full ∞-subcategory of $\mathcal{C}_{A_0}$ (reps. $\mathcal{C}_A$) spanned by affine objects.

We will now define a CAlg-valued presheaf $\mathfrak{O}^{\mathrm{aff}}_{\mathrm{BT}^p_n}$ on $\mathcal{C}^{\mathrm{aff}}_A$ as a composite of certain functors, which we describe now.

Definition 4.2.2. Write $\mathrm{Aff}^p_{/\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}}$ for the ∞-subcategory of $\mathcal{P}(\mathrm{Aff}^p)_{/\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}}$ spanned by affines. Define a functor $a\colon \mathcal{C}^{\mathrm{aff}}_A \to \mathrm{Aff}^p_{/\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}}$ by sending an object[27] $\mathbf{G}\colon \mathrm{Spf}\, B^\wedge_J \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}$ first to the composite with the canonical maps $\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A} \to \mathcal{M}_{\mathrm{BT}^p_n}$, then from $\mathrm{Spf}\, B^\wedge_J \to \mathcal{M}_{\mathrm{BT}^p_n}$ to its *algebraisation*[28]

$$\mathbf{G}^{\mathrm{alg}}\colon\ \mathrm{Spec}\, B^\wedge_J \to \mathcal{M}_{\mathrm{BT}^p_n}$$

which naturally lives in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn},p})$ as $\mathfrak{m}_A$, and hence J, contains p, and then we apply $\tau^*_{\geqslant 0}\colon \mathcal{P}(\mathrm{Aff}^{p,\mathrm{cn}}) \to \mathcal{P}(\mathrm{Aff}^p)$. Define a functor

$$\Gamma(\Omega_*(-))\colon (\mathrm{Aff}^p_{/\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}})^{\mathrm{op}} \to \mathrm{CAlg}$$

by pullback along $\Omega\colon \mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n} \to \mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}$ followed by the global sections functor $(\mathrm{Aff}^p_{/\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}})^{\mathrm{op}} \to \mathrm{CAlg}^p$, which is just a forgetful functor. Let

$$\mathfrak{O}^{\mathrm{aff}}_{\mathrm{BT}^p_n}\colon (\mathcal{C}^{\mathrm{aff}}_A)^{\mathrm{op}} \to \mathrm{CAlg}$$

[27]Recall that the formal spectrum $\mathrm{Spf}\, B$ is equivalent to $\mathrm{Spf}\, B^\wedge_J$ where J is a finitely generated ideal of definition for B; see [SAG, Rmk.8.1.2.4].

[28]Recall from [EC2, Th.3.2.2(4)], the map $\mathcal{M}_{\mathrm{BT}^p}(\mathrm{Spec}\, B) \to \mathcal{M}_{\mathrm{BT}^p}(\mathrm{Spf}\, B)$ is an equivalence of spaces if B is complete with respect to its ideal of definition. We call any $\mathbf{G}^{\mathrm{alg}}\colon \mathrm{Spec}\, B \to \mathcal{M}_{\mathrm{BT}^p}$ the *algebraisation* of the corresponding $\mathbf{G}\colon \mathrm{Spf}\, B \to \mathcal{M}_{\mathrm{BT}^p}$. This also implies the natural map $\mathcal{M}_{\mathrm{BT}^p_n}(\mathrm{Spec}\, B) \to \mathcal{M}_{\mathrm{BT}^p_n}(\mathrm{Spf}\, B)$ is an equivalence for B which are complete with respect to their ideal of definition, and we likewise use the phrase *algebraisation.*

be the composition of a followed by $\Gamma(\Omega_*(-))$, which is an étale hypersheaf as a sends étale hypercovers to étale hypercovers by construction, and $\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}$ and $\mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ are étale hypersheaves by Pr.4.1.7. We also define $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n} \colon \mathcal{C}_A^{\mathrm{op}} \to \mathrm{CAlg}$ by right Kan extension along the inclusion $(\mathcal{C}_A^{\mathrm{aff}})^{\mathrm{op}} \to \mathcal{C}_A^{\mathrm{op}}$. As a right Kan extensions preserve limits, we see $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ is an étale hypersheaf.

Remark 4.2.3. The right Kan extension defining $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ on $\mathcal{C}_A$ can be made more explicit. Indeed, by assumption, each object $\mathfrak{X}$ in $\mathcal{C}_A$ is qcqs, so by Pr.A.3.6, we have an étale hyper cover $\mathfrak{Y}_\bullet \to \mathfrak{X}$ such that each $\mathfrak{Y}_n = \operatorname{Spf} B_n$ is affine. The fact that $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ is an étale hypersheaf (as this is true étale locally on affines) then gives us a formula for $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}(\mathfrak{X})$:

$$\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}(\mathfrak{X}) \simeq \lim \left(\mathfrak{O}^{\mathrm{aff}}_{\mathrm{BT}^p_n}(\operatorname{Spf} B^0) \rightrightarrows \mathfrak{O}^{\mathrm{aff}}_{\mathrm{BT}^p_n}(\operatorname{Spf} B^1) \Rrightarrow \cdots\right)$$

By Pr.4.2.4 below, the terms in the above limit take a known form.

Proposition 4.2.4. *Given a p-complete $\mathbf{E}_\infty$-ring R and an associated p-divisible group $\mathbf{G}$ of height n, then there is a natural equivalence of p-complete $\mathbf{E}_\infty$-rings*

$$\Gamma(\Omega_*(\mathbf{G})) \simeq \widehat{\mathfrak{O}}_{\mathbf{G}^\circ}$$

where the latter is the p-completion of $\mathfrak{O}_{\mathbf{G}^\circ}$, the orientation classifier[29] *for $\mathbf{G}^\circ$.*

Proof. From the definition of $\Gamma(\Omega_*(-))$, it suffices to show that the following natural square of presheaves of p-complete $\mathbf{E}_\infty$-rings is Cartesian:

$$\begin{array}{ccc} \operatorname{Spec} \widehat{\mathfrak{O}}_{\mathbf{G}^\circ} & \longrightarrow & \mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n} \\ \downarrow & & \downarrow \Omega \\ \operatorname{Spec} R & \xrightarrow{\mathbf{G}} & \mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n} \end{array} \tag{4.2.5}$$

Fix a p-complete $\mathbf{E}_\infty$-ring A and evaluate the above diagram at A. If there are no maps of p-complete $\mathbf{E}_\infty$-rings $R \to A$, then the two left-most spaces are empty and we are done, so let us then fix a map $\psi \colon R \to A$. We then note the following chain of natural equivalences between the fibres of the vertical morphisms from left to right:

$$\operatorname{Spec} \widehat{\mathfrak{O}}_{\mathbf{G}^\circ}(A) \simeq \mathrm{Map}_{\mathrm{CAlg}}(\mathfrak{O}_{\mathbf{G}^\circ}, A) \simeq \mathrm{OrDat}(\mathbf{G}^\circ_A) \simeq \{\psi\} \underset{\mathcal{M}^{\mathrm{un}}_{\mathrm{BT}^p_n}(A)}{\times} \mathcal{M}^{\mathrm{or}}_{\mathrm{BT}^p_n}(A)$$

The first equivalence follows as p-completion is a left adjoint, the second from [EC2, Pr.4.3.13], and the third from the construction of Ω; see Con.4.1.3. As these equivalences are natural in A, this shows (4.2.5) is Cartesian. □

Now that we can calculate $\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}$ when restricted to affines, so we are ever so close to definition $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ and a proof of Th.2.1.7.

[29] Recall from [EC2, §4.3.3], for a formal group $\widehat{\mathbf{G}}$ over an $\mathbf{E}_\infty$-ring R, the *orientation classifier* of $\widehat{\mathbf{G}}$ is the corepresenting R-algebra for the functor $\mathrm{CAlg}_R \to \mathcal{S}$ mapping A to $\mathrm{OrDat}(\widehat{\mathbf{G}}_A)$.

4.3 The sheaf of Lurie's theorem

The definition of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ mirrors Lurie's definition of $\mathscr{O}^{\mathrm{top}}$ ([EC2, §7.3]) and the proof that this definition satisfies Th.2.1.7 also follows Lurie's ideas.

Definition 4.3.1. Fix an adic $\mathbf{E}_\infty$-ring A as in Nt.2.1.5. Let $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ be the étale hypersheaf on $\mathcal{C}_{A_0}$ defined by the composition

$$\mathcal{C}^{\mathrm{op}}_{A_0} \xrightarrow{\mathcal{D}^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}_{A} \xrightarrow{\mathfrak{O}^{\mathrm{or}}_{\mathrm{BT}^p_n}} \mathrm{CAlg} \tag{4.3.2}$$

or in other words, first one calculates the universal spectral deformation of $\mathbf{G}_0 \colon \mathfrak{X}_0 \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$ giving $\mathcal{D}(\mathbf{G}_0) = \mathbf{G}$ (Rmk.3.3.9), then the identity component $\mathbf{G}^\circ$ of $\mathbf{G}$, and $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G}_0)$ is then the p-completion orientation classifier of $\mathbf{G}^\circ$; we will see in the proof of Th.2.1.7 below that this p-completion is unnecessary. It follows from Th.3.3.5 and Df.4.2.2 that $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ is an étale hypersheaf.

With our sheaf in hand, we can prove Lurie's theorem; our proof follows the outline of the proof of [EC2, Th.7.0.1].

Proof of Th.2.1.7. We have an étale hypersheaf of $\mathbf{E}_\infty$-rings $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ on $\mathcal{C}_{A_0}$ from Df.4.3.1. It remains to show that on objects $\mathbf{G}_0 \colon \operatorname{Spf} B_0 \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$ in $\mathcal{C}^{\mathrm{aff}}_{A_0}$, where we may assume B_0 is complete with respect to its ideal of definition J, the $\mathbf{E}_\infty$-ring $\mathcal{E} = \mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G}_0)$ has the expected properties 1-4 of Th.2.1.7. Under $\mathcal{D}$, the object $\mathbf{G}_0$ is sent to the affine object $\mathbf{G}_{\mathrm{un}} \colon \operatorname{Spf} B \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n, A}$ of $\mathcal{C}^{\mathrm{aff}}_A$ such that $\pi_0 B \simeq B_0$ and $\mathbf{G}_{\mathrm{un}}$ is equivalent to $\mathbf{G}_0$ over $\operatorname{Spf} B_0$; see Df.3.3.4. By Pr.4.2.4 and (4.3.2), we see $\mathcal{E}$ is the p-completion of the orientation classifier of the identity component $\mathbf{G}^\circ$ of $\mathbf{G}$, denoted by $\mathfrak{O}_{\mathbf{G}^\circ}$. First, we will argue that the $\mathbf{E}_\infty$-ring $\mathfrak{O}_{\mathbf{G}^\circ}$ satisfies the desired properties 1-4, and then for $\mathcal{E}$.

Firstly, note that as $\mathfrak{O}_{\mathbf{G}^\circ}$ is an orientation classifier, [EC2, Pr.4.3.23] states that $\mathfrak{O}_{\mathbf{G}^\circ}$ is complex periodic (we will discuss Landweber exactness at the very end). It follows that $\mathcal{E}$ is complex periodic as it receives an $\mathbf{E}_\infty$-ring homomorphism $\mathfrak{O}_{\mathbf{G}^\circ} \to \mathcal{E}$; see [EC2, Rmk.4.1.10].

To see conditions 2 and 3 (except for the identification of $\pi_{2k}\mathcal{E}$), it suffices to show the formal group $\mathbf{G}^\circ$ is *balanced*[30] over B. Indeed, as we have proven condition 1 of Th. 2.1.7, we know that $\mathfrak{O}_{\mathbf{G}^\circ}[2]$ is a locally free of rank 1 so each $\pi_{2k}\mathfrak{O}_{\mathbf{G}^\circ}$ is a line bundle over $\pi_0\mathfrak{O}_{\mathbf{G}^\circ}$. If $\mathbf{G}^\circ$ is balanced over B, then each $\pi_k\mathfrak{O}_{\mathbf{G}^\circ}$ is complete with respect to the ideal of definition J of $\pi_0\mathfrak{O}_{\mathbf{G}^\circ} \simeq B_0$, so $\mathfrak{O}_{\mathbf{G}^\circ}$ itself is J-complete, hence also $\mathfrak{m}_A$-complete and p-complete. This would also imply that $\mathcal{E} \simeq \mathfrak{O}_{\mathbf{G}^\circ}$. To show that $\mathbf{G}^\circ$ is balanced over B, we use [EC2, Rmk.6.4.2] (twice) to reduce ourselves to showing that $\mathbf{G}^\circ_{B^\wedge_{\mathfrak{m}}}$ is balanced over $B^\wedge_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subseteq \pi_0 B \simeq B_0$; these ideals contain J as B_0 is

[30] Recall from [EC2, §6.4.1], that a formal group $\widehat{\mathbf{G}}$ over a connective $\mathbf{E}_\infty$-ring R is *balanced* if the unit map $R \to \mathfrak{O}_{\widehat{\mathbf{G}}}$ induces an equivalence on π_0 and the homotopy groups of $\mathfrak{O}_{\widehat{\mathbf{G}}}$ are concentrated in even degree.

J-complete. By Pr.3.3.13, we see $\mathbf{G}_{B^\wedge_{\mathfrak{m}}}$ is the universal spectral deformation of $\mathbf{G}_\kappa$, where κ is the residue field of $B^\wedge_{\mathfrak{m}}$. A powerful statement of Lurie [EC2, Th.6.4.6] then implies the identity component $\mathbf{G}^\circ_{B^\wedge_{\mathfrak{m}}}$ of $\mathbf{G}_{B^\wedge_{\mathfrak{m}}}$ is balanced. Hence $\mathfrak{O}_{\mathbf{G}^\circ} \simeq \mathcal{E}$ satisfies conditions 2 and 3 (except for the identification of $\pi_{2k}\mathcal{E}$).

For condition 4, [EC2, Pr.4.3.23] states that the canonical orientation of the p-divisible group $\mathbf{G}$ over $\mathcal{E}$ supplies us with an equivalence $\widehat{\mathbf{G}}^{\mathcal{Q}}_{\mathcal{E}} \xrightarrow{\simeq} \mathbf{G}^\circ$ of formal groups over $\mathcal{E}$ between the Quillen formal group of $\mathcal{E}$ and the identity component of $\mathbf{G}$. In particular, this implies the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathcal{E}}$ is isomorphic to the formal group $\mathbf{G}^\circ_0$ after an extension of scalars along the unit map $B_0 \simeq \pi_0 B \to \pi_0\mathcal{E}$. As $\mathbf{G}^\circ$ is a balanced formal group over B, this unit map is an isomorphism, giving us property 4.

To round off condition 3 and calculate $\pi_{2k}\mathcal{E}$, we note this follows from the facts that $\mathcal{E}$ is weakly 2-periodic, the p-divisible group $\mathbf{G}$ over $\mathcal{E}$ comes equipped with a canonical orientation and hence a chosen equivalence of locally free $\mathcal{E}$-modules of rank 1 $\beta\colon \omega_{\mathbf{G}} \to \mathcal{E}[-2]$, and the equivalence of $\pi_0\omega_{\mathbf{G}} \simeq \omega_{\mathbf{G}_0}$:

$$\pi_{2k}\mathcal{E} \simeq (\pi_2\mathcal{E})^{\otimes k} \simeq (\pi_0\omega_{\mathbf{G}})^{\otimes k} \simeq \omega_{\mathbf{G}_0}^{\otimes k}$$

Finally, to finish condition 1 and the Landweber exactness of $\mathcal{E}$, we appeal directly to Behrens–Lawson's arguments in [BL10, Lm.8.1.6 & Cor.8.1.7], as they are checking the same conditions on a sheaf with the same properties as ours above. □

Remark 4.3.3. Let us close this section by stating that there have been other iterations of Lurie's theorem; see [BL10, Th.8.1.4] and [Beh20, §6.7]. The statements made there are certainly not aesthetically identical[31] to our Th. 2.1.7, however, we believe that the chapter to follow, detailing applications of Lurie's theorem, justifies that all available statements of Lurie's theorem apply to the same set of examples. In particular, as we can construct Lubin–Tate theories, TMF, and TAF, all using Th.2.1.7, we do not find any reason to compare all available statements in too much detail—neither would we know how to.

[31] Phrases such as "(locally) fibrant in the Jardine model structure" can be translated to "étale hypersheaf", and compatibility between checking fibres are universal deformation spaces and the adjective "formally étale" is explained in [Beh20, Rmk.6.7.5]; see Pr.3.1.10 for a similar iteration of that idea.

Chapter 5

Applications of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$

To advertise Lurie's theorem to a wider audience and lay (known) groundwork for future applications, let us now discuss how this theorem (Th.2.1.7) can be used. A vast majority of the applications below can be found in either [Beh20, §6.7], [BL10], or [EC2].

5.1 Topological K-theory

As our first application of Th. 2.1.7, we would like to prove that one of the simplest p-divisible groups gives us an example of an $\mathbf{E}_\infty$-ring near and dear to stable homotopy theory: complex topological K-theory. To define the $\mathbf{E}_\infty$-ring KU, we will follow the construction of [EC2, §6.5], which we will repeat here for the readers' convenience.

Construction 5.1.1. Denote by $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ the 1-category of finite dimensional complex vector spaces and complex linear isomorphisms. Considering this as a topologically enriched category with a symmetric monoidal structure given by the direct sum of vector bundles, the (topological) coherent nerve $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$ is a Kan complex with an $\mathbf{E}_\infty$-structure. The inclusion

$$\coprod_{n\geqslant 0} \mathrm{BU}(n) \to \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$$

classified on each summand $\mathrm{BU}(n)$ by the universal n-dimensional complex vector bundle, is an equivalence of spaces and the $\mathbf{E}_\infty$-structure restricts to one on the domain. The group completion of this $\mathbf{E}_\infty$-space is the zeroth space of a connective spectrum ku, *connective complex topological K-theory*. The natural group completion map can be identified with the map

$$\xi\colon \coprod_{n\geqslant 0} \mathrm{BU}(n) \simeq \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq}) \to \Omega^\infty \mathrm{ku} \simeq \mathbf{Z}\times \mathrm{BU}$$

sending each $\mathrm{BU}(n)$ component to $\{n\} \times \mathrm{BU}$ via the canonical inclusion, which represents the universal complex vector bundle ξ_n over $\mathrm{BU}(n)$. There is also a multiplicative $\mathbf{E}_\infty$-structure on $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$ given by the tensor product of vector bundles, which also gives the connective spectrum ku the structure of a connective $\mathbf{E}_\infty$-ring; see [GGN15, Ex.5.3(ii)]. The map ξ is also a morphism of $\mathbf{E}_\infty$-spaces with respect to this multiplicative $\mathbf{E}_\infty$-structure. By identifying $\mathbf{CP}^\infty \simeq \mathrm{BU}(1)$ as a summand of $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$, then space $\mathbf{CP}^\infty$ inherits the multiplicative $\mathbf{E}_\infty$-structure, as the tensor product of line bundles is a line bundle. As ξ restricted to $\mathbf{CP}^\infty$ lands in the identity component of $\Omega^\infty \mathrm{ku}$, that is $\{1\} \times \mathrm{BU}$, we obtain a map of $\mathbf{E}_\infty$-spaces $\mathbf{CP}^\infty \to \mathbf{GL}_1(\mathrm{ku})$. Under the adjunction[32]

$$\Sigma^\infty_+ \colon \mathrm{CMon} \rightleftarrows \mathrm{CAlg} \colon \mathbf{GL}_1$$

we obtain a morphism of $\mathbf{E}_\infty$-rings $\rho\colon \Sigma^\infty_+ \mathbf{CP}^\infty \to \mathrm{ku}$. Furthermore, the based inclusion

$$\iota\colon S^2 \simeq \mathbf{CP}^2 \to \mathbf{CP}^\infty$$

postcomposed with the unit

$$\eta\colon \mathbf{CP}^\infty \to \Omega^\infty \Sigma^\infty \mathbf{CP}^\infty$$

followed by Ω^∞ of the inclusion into the first summand

$$j\colon \Sigma^\infty \mathbf{CP}^\infty \to \Sigma^\infty \mathbf{CP}^\infty \oplus \mathbf{S} \simeq \Sigma^\infty_+ \mathbf{CP}^\infty$$

gives us an element β inside $\pi_2 \Sigma^\infty_+ \mathbf{CP}^\infty$. The image of β under the map ρ is also called $\beta \in \pi_2\,\mathrm{ku}$, which one can identify with the element $[\gamma_1] - 1$ inside $\widetilde{\mathrm{ku}}^0(\mathbf{CP}^1)$, where γ_1 is the tautological line bundle over $\mathbf{CP}^1$—this is a consequence of Pr.5.5.6. We define the $\mathbf{E}_\infty$-ring of *periodic complex topological K-theory* as the localisation $\mathrm{KU} = \mathrm{ku}[\beta^{-1}]$; see [EC2, Pr.4.3.17] for a discussion about localising line bundles over $\mathbf{E}_\infty$-rings, and [HA, §7.2.3] for the $\mathbf{E}_1$-ring case.

With this geometric definition of KU, let us define an algebraic object to compare it to.

Definition 5.1.2. Let $\mu^\heartsuit_{p^\infty}$ denote the *multiplicative p-divisible group* over $\mathrm{Spec}\,\mathbf{Z}$. For each positive integer n, the R-valued points (for a discrete ring R) are defined as

$$\mu^\heartsuit_{p^n}(R) = \left\{x \in R \mid x^{p^n} = 1\right\}.$$

This lifts to a p-divisible group μ_{p^∞} over $\mathrm{Spec}\,\mathbf{S}$ by [EC2, Pr.2.2.11].

[32]Recall the $(\Sigma^\infty_+, \mathbf{GL}_1)$-adjunction (see [ABG+14, §2] for a modern reference) is the composite of two adjunctions:

$$\mathrm{CMon}^{\mathrm{grp}} \underset{\mathbf{GL}_1}{\overset{\mathrm{inc.}}{\rightleftarrows}} \mathrm{CMon} \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \mathrm{CAlg}$$

The superscript $(-)^{\mathrm{grp}}$ denotes those $\mathbf{E}_\infty$-spaces whose π_0 is a group. The functor $\mathbf{GL}_1\colon \mathrm{CMon} \to \mathrm{CMon}^{\mathrm{grp}}$ sends an $\mathbf{E}_\infty$-space X to the subspace $\mathbf{GL}_1 X$ spanned by those path components of X with inverses in $\pi_0 X$.

Proposition 5.1.3. *The object $\mu_{p^\infty}^\heartsuit$ over $\operatorname{Spf}\mathbf{Z}_p$ is an object of $\mathcal{C}_{\mathbf{Z}_p}$ (for $n=1$), and there is a natural equivalence of $\mathbf{E}_\infty$-rings*

$$\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_1^p}(\mu_{p^\infty}^\heartsuit) \simeq \mathrm{KU}_p.$$

One can view this proposition as a special case of the Lubin–Tate example Pr.5.2.1, but a more direct comparison to the geometric discussion above is also possible using Lurie's machinery from [EC2, §6.5]. Our argument below is a combination of [EC2, §3-4 & 6.5].

Proof. The fact that $\mu_{p^\infty}^\heartsuit$ lies in $\mathcal{C}_{\mathbf{Z}_p}$ follows immediately from Pr. 2.1.9 and Pr.3.1.10. Alternatively, one can view this as a special case of Pr.5.2.1.

We now follow the argument of Lurie from [EC2]. First, notice the natural equivalence $\mathcal{D}(\mu_{p^\infty}^\heartsuit/\operatorname{Spf}\mathbf{Z}_p) \simeq (\mu_{p^\infty}/\operatorname{Spf}\mathbf{S}_p)$. Indeed, [EC2, Cor.3.1.19] states that the universal spectral deformation of $\mu_{p^\infty}^\heartsuit$ over $\mathbf{Z}_p$ is μ_{p^∞} over $\mathbf{S}_p$, and this is identified with $\mathcal{D}(\mu_{p^\infty}^\heartsuit/\operatorname{Spf}\mathbf{Z}_p)$ via Rmk.3.3.9.

By [EC2, Pr.2.2.12], we see that the identity component of μ_{p^∞} over $\operatorname{Spf}\mathbf{S}_p$ is precisely the multiplicative formal group $\widehat{\mathbf{G}}_m$ over $\operatorname{Spf}\mathbf{S}_p$. Our desired $\mathbf{E}_\infty$-ring then takes the form of the orientation classifier $\mathcal{E}$ of $\widehat{\mathbf{G}}_m$ over $\mathbf{S}_p$. By the p-completion of [EC2, Pr.4.3.25], the preorientation classifier of $\widehat{\mathbf{G}}_m$ over $\mathbf{S}_p$ is $\Sigma^\infty_+\mathbf{CP}^\infty_p$. Taking a p-completion in Con.5.1.1, we obtain a map of $\mathbf{E}_\infty$-rings $\rho_p\colon \Sigma^\infty_+\mathbf{CP}^\infty_p \to \mathrm{ku}_p$. Similarly, by [EC2, Cor.4.3.27], the localisation $\Sigma^\infty_+\mathbf{CP}^\infty_p[\beta^{-1}]$, where $\beta \in \pi_2\Sigma^\infty_+\mathbf{CP}^\infty$ is the Bott element of Con. 5.1.1 from [EC2, §6.5], is the orientation classifier $\mathcal{E}$ we are after. By construction, this naturally admits a map of $\mathbf{E}_\infty$-rings $\rho_p[\beta^{-1}]\colon \mathcal{E} \to \mathrm{KU}_p$. We claim this map is an equivalence.

Now we follow [EC2, §6.5]. As $\mu_{p^\infty}/\mathbf{S}_p$ is the universal spectral deformation of both $\mu_{p^\infty}^\heartsuit/\operatorname{Spf}\mathbf{Z}_p$ and $\mu_{p^\infty}^\heartsuit/\operatorname{Spec}\mathbf{F}_p$ (Pr.3.3.13), it follows from [EC2, Th.6.4.6] that $\widehat{\mathbf{G}}_m$ is balanced (30) over $\mathbf{S}_p$. This and the complex periodicity of $\mathcal{E}$ yield an isomorphism of graded rings $\mathbf{Z}_p[\beta^\pm] \simeq \pi_*\mathcal{E}$ defined by the invertible element $\beta \in \pi_2\mathcal{E}$. The p-completion of the classical Bott periodicity theorem then states the composite $\mathbf{Z}_p[\beta^\pm] \to \pi_*\mathrm{KU}_p$ through $\rho_p[\beta^{-1}]$ is an equivalence, hence $\rho_p[\beta^{-1}]$ is an equivalence. □

There is a standard trick to obtain the integral $\mathbf{E}_\infty$-ring KU from the collection $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_1^p}(\mu_{p^\infty}^\heartsuit)$ for all primes p by purely algebraic methods.

Remark 5.1.4. Consider the symmetric monoidal Schwede–Shipley equivalence of ∞-categories

$$\mathrm{Mod}_R \simeq \mathcal{D}(R) \tag{5.1.5}$$

where R is a discrete commutative ring; see [SS03] or [HA, Th.7.1.2.13]. Replacing R with $\mathbf{Q}$, we note that the $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra $\mathrm{ku}_\mathbf{Q}$, the rationalisation of ku, has homotopy groups $\pi_*\,\mathrm{ku}_\mathbf{Q} \simeq \mathbf{Q}[\beta]$, for $|\beta| = 2$. Define a map of $\mathbf{Q}$-cdgas

$\Lambda_{\mathbf{Q}}[x_2] \to \mathrm{ku}_{\mathbf{Q}}$ from the free $\mathbf{Q}$-cdga on one element in degree 2 to $\mathrm{ku}_{\mathbf{Q}}$, defined by the element β. This is easily seen to be an equivalence of $\mathbf{Q}$-cdgas, and one obtains an equivalence upon localisations at x_2

$$\Lambda_{\mathbf{Q}}[x_2^{\pm 1}] \xrightarrow{\simeq} \mathrm{ku}_{\mathbf{Q}}[\beta^{-1}] \simeq \mathrm{KU}_{\mathbf{Q}}$$

where $\mathrm{KU}_{\mathbf{Q}}$ is the rationalisation of KU. Carrying out the same construction in CAlg, we obtain a morphism $\Lambda[x_2^{\pm 1}] \to \mathrm{KU}_p$ of $\mathbf{E}_\infty$-rings from the free $\mathbf{E}_\infty$-ring on a single invertible generator in degree two to KU_p defined by $\beta \in \pi_2 \mathrm{KU}_p$. Taking the product of these morphisms over all primes p and rationalising gives a morphism in $\mathrm{CAlg}_{\mathbf{Q}}$

$$\theta\colon \Lambda_{\mathbf{Q}}[x_2^{\pm 1}] \to \left(\prod_p \mathrm{KU}_p \right)_{\mathbf{Q}}$$

where we note that $(\Lambda[x_2^{\pm 1}])_{\mathbf{Q}}$ is naturally equivalent to $\Lambda_{\mathbf{Q}}[x_2^{\pm 1}]$. One then obtains KU from the following *Hasse* Cartesian square of $\mathbf{E}_\infty$-rings:

$$\begin{array}{ccc} \mathrm{KU} & \longrightarrow & \prod_p \mathrm{KU}_p \\ \downarrow & & \downarrow \\ \Lambda_{\mathbf{Q}}[x_2^{\pm 1}] & \xrightarrow{\theta} & (\prod_p \mathrm{KU}_p)_{\mathbf{Q}} \end{array}$$

where the two products are taken over all prime numbers p; see [Bau14].

The $\mathbf{E}_\infty$-ring KO can also be obtained through these means. The following is a carbon copy of Con.5.1.1, replacing $\mathbf{C}$ with $\mathbf{R}$.

Construction 5.1.6. Write $\mathrm{Vect}_{\mathbf{R}}^{\simeq}$ for the topological category of real vector spaces of finite-dimension and real linear isomorphisms. As this category has two symmetric monoidal structures given by the direct sum and tensor product of vector bundles, the (topological) coherent nerve $\mathrm{N}(\mathrm{Vect}_{\mathbf{R}}^{\simeq})$ is a commutative monoid object in the ∞-category of $\mathbf{E}_\infty$-spaces. Moreover, the functor

$$c\colon \mathrm{Vect}_{\mathbf{R}}^{\simeq} \to \mathrm{Vect}_{\mathbf{C}}^{\simeq} \qquad V \mapsto V \otimes_{\mathbf{R}} \mathbf{C}$$

is symmetric monoidal with respect to both monoidal structures, hence we obtain a morphism of commutative monoid objects in $\mathbf{E}_\infty$-spaces:

$$c\colon \mathrm{N}(\mathrm{Vect}_{\mathbf{R}}^{\simeq}) \to \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$$

The group completion (with respect to the direct sum $\mathbf{E}_\infty$-structure) of $\mathrm{N}(\mathrm{Vect}_{\mathbf{R}}^{\simeq})$ is the zeroth space of the connective $\mathbf{E}_\infty$-ring ko, *connective real topological K-theory*, and c induces a morphism $\mathrm{ko} \to \mathrm{ku}$ of $\mathbf{E}_\infty$-rings. There is an element $\beta_{\mathbf{R}}$ inside $\pi_8\,\mathrm{ko}$ and $c(\beta_{\mathbf{R}}) = \beta^4$ inside $\pi_8\,\mathrm{ku}$; see [Ada74, §III] for example. We define the $\mathbf{E}_\infty$-ring of *periodic real K-theory* as the localisation $\mathrm{KO} = \mathrm{ko}[\beta_{\mathbf{R}}^{-1}]$, and we notice this induces a morphism $c\colon \mathrm{KO} \to \mathrm{KU}$. By [HS14], the map c can be identified with the $\mathbf{E}_\infty$-inclusion of the C_2-fixed points of KU through the C_2-action given by complex conjugation of vector bundles.

Definition 5.1.7. Let $\mathfrak{B}C_2$ be the quotient stack $\operatorname{Spf} \mathbf{Z}_p/C_2$ with respect to the trivial action on $\operatorname{Spf} \mathbf{Z}_p$. This formal spectral Deligne–Mumford stack has a cover $\operatorname{Spf} \mathbf{Z}_p \to \mathfrak{B}C_2$ given by the canonical quotient map. By [LN14, A.3-4], this is the base-change over $\operatorname{Spf} \mathbf{Z}_p$ of the *moduli stack of forms of the multiplicative group scheme* $\mathbf{G}_m$. The reason for the quotient by C_2 is to remove the automorphism on $\mathbf{G}_m$ given by inversion. Moreover, the multiplicative p-divisible group $\mu_{p^\infty}^\heartsuit$ lives over $\mathfrak{B}C_2$, so we obtain a map $\mathfrak{B}C_2 \to \widehat{\mathcal{M}}^\heartsuit_{\mathrm{BT}_1^p, \mathbf{Z}_p}$.

Proposition 5.1.8. *The map $\mathfrak{B}C_2 \to \widehat{\mathcal{M}}^\heartsuit_{\mathrm{BT}_1^p, \mathbf{Z}_p}$ lives in $\mathcal{C}_{\mathbf{Z}_p}$, and there is a natural equivalence of $\mathbf{E}_\infty$-rings*

$$\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_1^p}(\mathfrak{B}C_2) \simeq \mathrm{KO}\,.$$

Moreover, the map $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_1^p}(\operatorname{Spf} \mathbf{Z}_p \to \mathfrak{B}C_2)$ is homotopic (as maps of spectra) to the p-completion of the map $c\colon \mathrm{KO} \to \mathrm{KU}$.

The proof below uses some results about stable Adams operations which we discuss in §5.5.

Proof. As $\operatorname{Spf} \mathbf{Z}_p \to \mathfrak{B}C_2$ is a finite étale cover and the composite

$$\operatorname{Spf} \mathbf{Z}_p \to \mathfrak{B}C_2 \to \widehat{\mathcal{M}}^\heartsuit_{\mathrm{BT}_1^p, \mathbf{Z}_p}$$

lies in $\mathcal{C}_{A_0}$, then so does $\mathfrak{B}C_2$. It suffices now to show that $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_1^p}(\mathfrak{B}C_2) = \mathcal{E}$ is the inclusion of the C_2-fixed points of KU_p with respect to the complex conjugation action on KU_p. We can rewrite $\mathcal{E}$ using the fact that $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_1^p}$ is an étale sheaf:

$$\mathcal{E} \simeq \lim \left(\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_1^p}(\mu_{p^\infty}^\heartsuit / \operatorname{Spf} \mathbf{Z}_p) \rightrightarrows \mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_1^p}(\mu_{p^\infty}^\heartsuit / \operatorname{Spf} \mathbf{Z}_p \underset{\mathfrak{B}C_2}{\times} \operatorname{Spf} \mathbf{Z}_p) \Rrightarrow \cdots \right)$$

As $\operatorname{Spf} \mathbf{Z}_p \to \mathfrak{B}C_2$ is a C_2-torsor by construction and using Pr. 5.1.3, we can rewrite the above limit as

$$\lim \left(\mathrm{KU}_p \rightrightarrows \prod_{C_2} \mathrm{KU}_p \Rrightarrow \prod_{C_2 \times C_2} \mathrm{KU}_p \cdots \right)$$

which is simply the homotopy fixed points $\mathrm{KU}_p^{hC_2}$. We are only left to check that this C_2-action on KU_p is homotopic as maps of spectra to that given by complex conjugation. By the construction of $\mathfrak{B}C_2$, we see that $\operatorname{Spf} \mathbf{Z}_p \to \mathfrak{B}C_2$ is the quotient by the inversion action on the multiplicative group scheme $\mathbf{G}_m$, hence $\mathcal{E} \to \mathrm{KU}_p$ is the inclusion of the C_2-homotopy fixed point of KU_p with action given by $[-1]^*$. These C_2-homotopy fixed points are equivalently given by the $[-1]^*$-fixed points. As we will discuss in Pr. 5.5.5, the map $[-1]^*$ is homotopic to the stable Adams operation ψ^{-1}. Following arguments of *ibid* we see this is determined as a map of spectra by what it does on line bundles on

KU_p-cohomology of finite spaces.[33] We now refer to [MS74, p.168], which states that the complex conjugate of a complex line bundle L is isomorphic to the dual of a complex line bundle L, the latter also being given by $\psi^{-1}(L)$. This finishes the proof. □

5.2 Lubin–Tate and Barsotti–Tate theories

The above example of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_1}(\mu^{\heartsuit}_{p^\infty}) \simeq \mathrm{KU}_p$ can be extended to arbitrary heights. The following is a combination of [EC2, §5-6]. Recall the Lubin–Tate deformation theory of Ex.3.1.9.

Proposition 5.2.1. *Let $\widehat{\mathbf{G}}_0$ be a formal group of exact height n over a perfect field κ and $\mathbf{G}_0$ for a p-divisible group over κ whose identity component is equivalent to $\widehat{\mathbf{G}}_0$; see [EC2, Pr.4.4.22]. Write $\mathbf{G}$ for the classical universal deformation of $\mathbf{G}_0$, which is a p-divisible group over the discrete ring $R^{\mathrm{LT}}_{\widehat{\mathbf{G}}_0}$. The object $\mathbf{G}\colon \operatorname{Spf} R^{\mathrm{LT}}_{\widehat{\mathbf{G}}_0} \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n,\mathbf{Z}_p}$ lies in $\mathcal{C}_{\mathbf{Z}_p}$. Moreover, there is an equivalence of $\mathbf{E}_\infty$-rings $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G}) \simeq E_n$ where $E_n = E(\widehat{\mathbf{G}}_0)$ is the Lubin–Tate $\mathbf{E}_\infty$-ring of $\widehat{\mathbf{G}}_0$ (also known as Morava E-theory); see [EC2, §5].*

This will follow from a more general statement.

Proposition 5.2.2. *Let R_0 be a discrete Noetherian $\mathbf{F}_p$-algebra such that the Frobenius endomorphism on R_0 is finite and $\mathbf{G}_0$ be a nonstationary (21) p-divisible group of height n over R_0. Write R for the universal spectral deformation adic $\mathbf{E}_\infty$-ring of $\mathbf{G}_0$ from [EC2, Th.3.4.1] and assume the residue fields of $\pi_0 R$ are perfect of characteristic p. Then $\mathbf{G}\colon \operatorname{Spf} \pi_0 R \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n,\mathbf{Z}_p}$, the morphism defined by the base-change of the universal spectral deformation of $\mathbf{G}_0$ along $R \to \pi_0 R$, lies in $\mathcal{C}_{A_0}$. Moreover, there is a natural equivalence of $\mathbf{E}_\infty$-rings $\mathcal{D}(\mathbf{G}) \simeq R$.*

The $\mathbf{E}_\infty$-rings produced by applying $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ to the p-divisible groups $\mathbf{G}$ occurring in Pr.5.2.2 seem interesting enough to name.

Definition 5.2.3. Let R_0, $\mathbf{G}_0$, and $\mathbf{G}$ be as in Pr.5.2.2. We call $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G})$ the *Barsotti–Tate $\mathbf{E}_\infty$-ring* associated to $(R_0, \mathbf{G}_0)$.

Proof of Pr.5.2.2. Let us first see that $\mathbf{G}$ lies in $\mathcal{C}_{A_0}$ by checking the conditions of Df.2.1.6. It is shown in Pr.3.1.10 that the morphism $\mathbf{G}$ is formally étale. As R_0 is Noetherian, then [EC2, Th.3.4.1(6)] tells us that R and hence also $\pi_0 R$ are also Noetherian. Consider the maps in $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$

$$\operatorname{Spf} \pi_0 R \to \operatorname{Spf} R \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}$$

[33] Recall from [HA, Nt.1.4.2.5], that the ∞-category of *finite spaces* is the full ∞-subcategory of $\mathcal{S}$ generated by the terminal object under finite colimits.

and the associated (co)fibre sequence of complete $\pi_0 R$-modules:

$$L_{\operatorname{Spf} R/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}}\Big|_{\operatorname{Spf}\pi_0 R} \to L_{\operatorname{Spf}\pi_0 R/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}} \to L_{\operatorname{Spf}\pi_0 R/\operatorname{Spf} R}$$

By construction ([EC2, Pr.3.4.3]), R corepresents the de Rham space of the map $\operatorname{Spec} R_0 \to \mathcal{M}_{\mathrm{BT}^p}$, or equivalently, the de Rham space of $\operatorname{Spec} R_0 \to \widehat{\mathcal{M}}_{\mathrm{BT}^p_n,\mathbf{Z}_p}$, as R_0 is an $\mathbf{F}_p$-algebra and $\mathbf{G}_0$ is of height n. Identifying R as representing this de Rham space and using Ex.3.2.17, we see that $L_{\operatorname{Spf} R/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}}$ vanishes. Hence $L_{\operatorname{Spf}\pi_0 R/\widehat{\mathcal{M}}_{\mathrm{BT}^p_n,A}}$ is almost perfect as $L_{\operatorname{Spf}\pi_0 R/\operatorname{Spf} R}$ is almost perfect; see Pr.A.3.1. Rmk.3.3.9 identifies R with $\mathcal{D}(\mathbf{G})$. □

Proof of Pr.5.2.1. The fact that $\mathbf{G}$ lies in $\mathcal{C}_{\mathbf{Z}_p}$ follows from Pr.5.2.2. The fact that $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p}(\mathbf{G})$ is equivalent to E_n follows as the universal spectral deformation of $\mathbf{G}_0$ is given by $\mathcal{D}(\mathbf{G})$ (Pr.5.2.2) and the orientation classifier of $\mathcal{D}(\mathbf{G})$ is E_n ([EC2, Cor.6.0.6]). □

From the functoriality of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ we obtain an action of the automorphism group of the pair $(\kappa, \widehat{\mathbf{G}}_0)$ on the $\mathbf{E}_\infty$-ring E_n. In other words, E_n obtains an action of the *extended Morava stabiliser group*; see [EC2, §5] and [GH04, §7]. It is not clear from these techniques that these account for all $\mathbf{E}_\infty$-automorphisms of E_n; this requires a dash of chromatic homotopy theory as done in [EC2, §5].

5.3 Topological modular forms

Another exciting application of Th.2.1.7 is to construct the $\mathbf{E}_\infty$-ring TMF of *periodic topological modular forms.* Of course, this also uses the ideas of Lurie from [EC2] and [SUR09], but reinterpreting TMF_p as a section of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_2}$ yields additional endomorphisms to those previously known—that is the topic of Part II.

Proposition 5.3.1. *The map* $[p^\infty]\colon \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{Ell},\mathbf{Z}_p} \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_2,\mathbf{Z}_p}$*, sending an elliptic curve* E *to its associated* p*-divisible group* $E[p^\infty]$*, lies inside* $\mathcal{C}_{\mathbf{Z}_p}$*.*

Proof. Using Pr.2.1.9, we only need to show that the map $[p^\infty]$ above is formally étale inside $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ and that $\widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{Ell},\mathbf{Z}_p}$ is finitely presented over $\operatorname{Spf}\mathbf{Z}_p$. The former follows from Ex.3.1.7; a consequence of the classical Serre–Tate theorem. The latter follows by base-change from [Ols16, Th.13.1.2], which states that $\mathcal{M}^{\heartsuit}_{\mathrm{Ell}}$ is smooth, separated, and of finite type over $\operatorname{Spec}\mathbf{Z}$, hence it is finitely presented over $\operatorname{Spec}\mathbf{Z}$. □

As promised in the introduction, we should relate $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_2}$ to a more classical object:

Definition 5.3.2. Let $\mathscr{O}^{\mathrm{top}}$ denote the Goerss–Hopkins–Miller sheaf of $\mathbf{E}_\infty$-rings on the étale site $\mathrm{DM}^{\text{ét}}_{/\mathcal{M}^{\heartsuit}_{\mathrm{Ell}}}$ of $\mathcal{M}^{\heartsuit}_{\mathrm{Ell}}$; see [EC2, Th.7.0.1], or [Beh14] and [Goe10, Th.1.2] for versions over the compactification of $\mathcal{M}^{\heartsuit}_{\mathrm{Ell}}$. The global sections $\mathscr{O}^{\mathrm{top}}(\mathcal{M}^{\heartsuit}_{\mathrm{Ell}})$ are the $\mathbf{E}_\infty$-ring TMF of *periodic topological modular forms.*

We also have functors $\mathrm{DM}^{\text{ét}}_{/\mathcal{M}^{\heartsuit}_{\mathrm{Ell}}} \to \mathrm{fDM}^{\text{ét}}_{/\widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{Ell},\mathbf{Z}_p}}$ defined by base-change along the canonical map $\operatorname{Spf}\mathbf{Z}_p \to \operatorname{Spec}\mathbf{Z}$ and $\mathrm{fDM}^{\text{ét}}_{/\widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{Ell},\mathbf{Z}_p}} \to \mathcal{C}_{\mathbf{Z}_p}$ by postcomposition along the map $[p^\infty]$ of Pr.5.3.1.

Theorem 5.3.3. *The following diagram of ∞-categories commutes up to homotopy:*

$$\begin{array}{ccccc}
\left(\mathrm{DM}^{\text{ét}}_{/\mathcal{M}^{\heartsuit}_{\mathrm{Ell}}}\right)^{\mathrm{op}} & & \xrightarrow{\quad\mathscr{O}^{\mathrm{top}}\quad} & & \mathrm{CAlg} \\
\downarrow & & & & \downarrow{\scriptstyle (-)^{\wedge}_p} \\
\left(\mathrm{fDM}^{\text{ét}}_{/\widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{Ell},\mathbf{Z}_p}}\right)^{\mathrm{op}} & \longrightarrow & \left(\mathcal{C}_{\mathbf{Z}_p}\right)^{\mathrm{op}} & \xrightarrow{\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_2}} & \mathrm{CAlg}
\end{array}$$

In particular, there is an equivalence of $\mathbf{E}_\infty$-rings:

$$\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_2}\left([p^\infty]\colon \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{Ell},\mathbf{Z}_p} \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_2,\mathbf{Z}_p}\right) \simeq \mathrm{TMF}_p$$

The following proof is essentially that of [EC2, Th.7.0.1] which proves an integral statement.

Proof. As done in the proof of [EC2, Th.7.0.1], we will conclude the proof by checking that for each affine object $E_0\colon \operatorname{Spec} B_0 \to \mathcal{M}^{\heartsuit}_{\mathrm{Ell}}$ of $\mathrm{DM}^{\text{ét}}_{/\mathcal{M}^{\heartsuit}_{\mathrm{Ell}}}$, the $\mathbf{E}_\infty$-ring $\mathcal{E} = \mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(E[p^\infty])$ satisfies the following conditions:

1. $\mathcal{E}$ is weakly 2-periodic; see (10).
2. The homotopy groups $\pi_k\mathcal{E}$ vanish in odd degrees, so in particular, $\mathcal{E}$ is complex orientable.
3. There is a natural (in $\mathrm{DM}^{\text{ét}}_{/\mathcal{M}^{\heartsuit}_{\mathrm{Ell}}}$) isomorphism of rings $(B_0)^{\wedge}_p \simeq \pi_0\mathcal{E}$.
4. There is a natural (in affines in $\mathrm{DM}^{\text{ét}}_{/\mathcal{M}^{\heartsuit}_{\mathrm{Ell}}}$) isomorphism of formal groups $\widehat{\mathbf{E}}_{(B_0)^{\wedge}_p} \simeq \widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathcal{E}}$ over $\operatorname{Spf}(B_0)^{\wedge}_p$.

Once one applies [EC2, Pr.7.4.1] to identify $\widehat{E} \simeq E[p^\infty]^\circ$, these conditions above are precisely the properties of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_2}$ by Th. 2.1.7, hence they hold. The p-completion of the étale sheaf of $\mathbf{E}_\infty$-rings $\mathscr{O}^{\mathrm{top}}$ is determined up to homotopy by the four conditions above, the desired diagram of ∞-categories commutes up to homotopy. This is a folklore statement, see [EC2, Rmk.7.0.2] or [Goe10, Th.1.2], but a precise statement and proof can be found in Th.B.0.2. ☐

As done in [Beh20, §6], we can use the collection of all p-complete $\mathbf{E}_\infty$-rings TMF_p and a little rational information to construct integral TMF, similar to Rmk.5.1.4. This is generalised in §6.1.

5.4 Topological automorphic forms

The first examples of new cohomology theories constructed with Th.2.1.7 come from Behrens–Lawson [BL10]. The main idea is that the Serre–Tate theorem, which was vital in our construction of TMF_p from $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_2}$, actually applies to the moduli stack of dimension g abelian varieties for any $g \geqslant 1$; the $g = 1$ case recovers the moduli stack of elliptic curves. A new problem now arises: we need our p-divisible groups to be of dimension 1, and then and only then can they have an orientation. To obtain a 1-dimensional p-divisible group from an abelian variety A of dimension $g \geqslant 2$, one needs more structure on A to split its associated p-divisible group into one of dimension 1 and another of dimension $g-1$ (which we forget about). This comes in the form of **p**olarisations, **e**ndomorphisms, and **l**evel structure, leading us *PEL-Shimura varieties*; for a full introduction to the subject and the intended application to stable homotopy theory, see [BL10]. What appears below is simply a restatement of [BL10] and [Beh20].

Notation 5.4.1. Fix an integer $n \geqslant 1$. Let F be a quadratic imaginary extension of $\mathbf{Q}$, such that p splits as $u\overline{u}$, and write $\mathscr{O}_F$ for the ring of integers of F. Let V be an F-vector space of dimension n equipped with a $\mathbf{Q}$-valued nondegenerate Hermitian alternating form of signature $(1, n-1)$. Finally, fix an $\mathscr{O}_F$-lattice L in V such that the alternating form above takes integer values on L and makes $L_{(p)}$ self-dual.

Definition 5.4.2. Write $\mathcal{X}_{V,L}$ for the formal Deligne–Mumford stack over $\operatorname{Spf} \mathbf{Z}_p$ (of [BL10, Th.6.6.2] with $K^p = K^p_0$) where a point in $\mathcal{X}_{V,L}(S)$ for a locally Noetherian formal scheme S over $\operatorname{Spf} \mathbf{Z}_p$, is a triple (A, i, λ) where:

- A is an abelian scheme over S of dimension n.
- $\lambda \colon A \to A^\vee$ is a polarisation (principle at p), with Rosati involution $\dagger$ on $\mathrm{Endo}(A)_{(p)}$.
- $i \colon \mathscr{O}_{F,(p)} \to \mathrm{Endo}(A)_{(p)}$ is an inclusion of rings satisfying $i(\overline{z}) = i(z)^\dagger$.

These triples have to satisfy two additional conditions assuring they are locally modelled by V and L; see [Beh20, §6.7].

In the situation above, the splitting $p = u\overline{u}$ induces a splitting of p-divisible groups

$$A[p^\infty] \simeq A[u^\infty] \oplus A[\overline{u}^\infty]$$

and our assumptions on (A, i, λ) ensure that $A[u^\infty]$ is a 1-dimensional p-divisible group. This yields a morphism of stacks $[u^\infty] \colon \mathcal{X}_{V,L} \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, \mathbf{Z}_p}$ which sends (A, λ, i) to $A[u^\infty]$.

Proposition 5.4.3. *Given V and L as in Nt.5.4.1, then the morphism*

$$[u^\infty] \colon \mathcal{X}_{V,L} \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, \mathbf{Z}_p}$$

defines an object of $\mathcal{C}_{\mathbf{Z}_p}$.

Proof. Pr.2.1.9 reduces us to show that $[u^\infty]$ is formally étale inside $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ and that $\mathcal{X}_{V,L}$ is of finite presentation over $\operatorname{Spf} \mathbf{Z}_p$. The first statement follows straight from the definitions of a formally étale morphism and [BL10, Th.7.3.1], which itself is a consequence of the classical Serre–Tate theorem. We now use [BL10, Cor.7.3.3] to see $\mathcal{X}_{V,L}$ is of locally finite presentation over $\operatorname{Spf} \mathbf{Z}_p$, so it suffices to show now that $\mathcal{X}_{V,L}$ is qcqs. To do this, we first use [BL10, Th.6.6.2], which states that $\mathcal{X}_{V,L}$ has an étale cover by a quasi-projective scheme. As a quasi-projective formal scheme X is separated and qc, we see X itself has a Zariski cover by an affine formal scheme $\operatorname{Spf} B$, meaning $\mathcal{X}_{V,L}$ has an étale cover by $\operatorname{Spf} B$. By Pr.A.3.6, this implies $\mathcal{X}_{V,L}$ is qcqs. ☐

We can now define the $\mathbf{E}_\infty$-rings of topological automorphic forms as done in [BL10, §8.3].

Definition 5.4.4. Let V and L be as in Nt. 5.4.1. Define the $\mathbf{E}_\infty$-ring of *topological automorphic forms*

$$\mathrm{TAF}_{V,L} = \mathcal{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}\left(\mathcal{X}_{V,L} \xrightarrow{[u^\infty]} \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, \mathbf{Z}_p}\right)$$

One can also define variants of $\mathrm{TAF}_{V,L}$ which incorporate level structures. Such extra structure can then be used to define restriction maps, transfers, and Hecke operators on $\mathrm{TAF}_{V,L}$; see [BL10, §11].

5.5 Stable Adams operators

Let us now explore the simplest functoriality intrinsic to $\mathcal{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$.

Definition 5.5.1. Let $k = (k_1, k_2, \ldots)$ be a p-adic integer and $\mathbf{G}$ a p-divisible group over an arbitrary scheme (or stack) S. Write

$$[k] \colon \mathbf{G} \to \mathbf{G}$$

for the endomorphism of $\mathbf{G}$ given on p^n-torsion by the k_n-fold multiplication $[k_n] \colon \mathbf{G}_n \to \mathbf{G}_n$. These assemble to an endomorphism of $\mathbf{G}$ as the sequence $(k_1, k_2, \ldots)$ represents a p-adic integer and the closed immersions $\mathbf{G}_n \to \mathbf{G}_{n+1}$ witness the equality $\mathbf{G}_n = \mathbf{G}_{n+1}[p^n]$. If k is a unit inside $\mathbf{Z}_p$ then each $[k_n]$ is an isomorphism of finite flat group schemes on S, hence $[k]$ is an automorphism of $\mathbf{G}$. If $\mathbf{G}$ defines a morphism $S \to \widehat{\mathcal{M}}^{\heartsuit}_{\mathrm{BT}^p_n, A_0}$ inside $\mathcal{C}_{A_0}$ and $k \in \mathbf{Z}_p^\times$, then write

$$\psi^k \colon [k]^* \colon \mathcal{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G}) \to \mathcal{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G})$$

for the induced automorphism of $\mathbf{E}_\infty$-rings. These are the (p-adic) *stable Adams operations* $\mathcal{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G})$; we will justify this name shortly.

Many properties expected of Adams operations are formal.

Proposition 5.5.2. *Let l, k be two units in $\mathbf{Z}_p$, $\mathbf{G}$ be an object of $\mathcal{C}_{A_0}$, and write $\mathcal{E} = \mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G})$. Then ψ^1 is canonically homotopic to the identity map on the $\mathbf{E}_\infty$-ring $\mathcal{E}$, and the maps of $\mathbf{E}_\infty$-rings $\psi^l\psi^k$ and ψ^{lk} on $\mathcal{E}$ are canonically homotopic.*

The homotopy H between $\psi^l\psi^k$ and ψ^{lk} is canonical with respect to morphisms in $\mathcal{C}_{A_0}$. For example, if j is another p-adic unit, then the homotopy between $\psi^j\psi^l\psi^k$ and ψ^{jlk} factors through H. This follows straight from the fact that $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n} : \mathcal{C}^{\mathrm{op}}_{A_0} \to \mathrm{CAlg}$ is first and foremost a functor of ∞-categories, and the calculations $[l][k] = [lk]$ hold up to equality in $\mathcal{C}_{A_0}$.

Proof. As these facts hold for $[k]$ in $\mathcal{C}_{A_0}$ and $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ is a functor, we are done. □

Using the information we already have at hand, we can calculate $[k]^*$ on the homotopy groups of the $\mathbf{E}_\infty$-rings $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G})$ over affine objects of $\mathcal{C}_{A_0}$.

Proposition 5.5.3. *Let k be a unit in $\mathbf{Z}_p$ and $\mathbf{G}$ be a p-divisible group defining an affine object in $\mathcal{C}_{A_0}$. Then for every integer n, we have the following equality of morphisms of $\mathbf{Z}_p$-modules:*

$$[k]^* = k^n : \pi_{2n}\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G}) \to \pi_{2n}\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G})$$

Proof. Using Th.2.1.7, we see that $\pi_{2n}\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G})$ is naturally isomorphic to the line bundle $\omega_{\mathbf{G}}^{\otimes n}$ over $\pi_0\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G}) = B$. It then suffices to calculate the $n = 1$ case. As $\omega_{\mathbf{G}}$ is the dualising line for the identity component $\mathbf{G}^\circ$ of $\mathbf{G}$, we see the B-module $\omega_{\mathbf{G}}$ is naturally equivalent to the dual of the Lie algebra $\mathrm{Lie}(\mathbf{G}^\circ)$ (12), so it now suffices to calculate $[k]^*$ on this Lie algebra. This is quite elementary, but let us recall some details. The question can be answered by localising at each minimal ideal $\mathfrak{m}$ of B containing its ideal of definition J. Over $B_{\mathfrak{m}}$ the 1-dimensional formal group $\mathbf{G}^\circ$ has coordinate t and an associated formal group law G—the choice of coordinates forms a line bundle over $B_{\mathfrak{m}}$ and line bundles over local rings are trivial; see [Goe08, §2]. Assume B is local then. If k is an integer, can write $[k]$ on $B[\![t]\!]$, the global sections of $\mathbf{G}^\circ$ using the coordinate t, as the composite

$$[k] \colon B[\![t]\!] \xrightarrow{c_k} B[\![t_1, \ldots, t_k]\!] \xrightarrow{\mu} B[\![t]\!] \tag{5.5.4}$$

where the first map is the k-fold comultiplication on $B[\![t]\!]$ induced by G and the second is the completed multiplication map sending each t_i to t. As we have the congruence

$$c_k(t) \equiv t_1 + \cdots + t_k$$

modulo higher degree terms, then $[k](t) \equiv kt$ modulo higher powers of t. Finally, the Lie algebra $\mathrm{Lie}(\mathbf{G}^\circ)$ can be written as a Zariski tangent space:

$$\mathrm{Lie}(\mathbf{G}^\circ) \simeq \mathrm{Hom}_{\mathrm{Mod}_B}(tB[\![t]\!]/(tB[\![t]\!])^2, B)$$

It is now clear that $[k]^* \colon \mathrm{Lie}(\mathbf{G}^\circ) \to \mathrm{Lie}(\mathbf{G}^\circ)$ is simply multiplication by k if k is an integer. For a general p-adic unit k, we approximate k by integers using its p-adic expansion, and our conclusion then follows in this more general case by taking the limit. □

Let us justify why we call the operations $[k]^*$ stable Adams operations.

Proposition 5.5.5. *For integers k not divisible by p, the map of $\mathbf{E}_\infty$-rings $[k]^*\colon \mathrm{KU}_p \to \mathrm{KU}_p$ is homotopic to classical stable Adams operation ψ^k; see [Ati67, §3.2].*

Using a slight variant of Rmk. 5.1.4, one can construct maps of $\mathbf{E}_\infty$-rings $[n]^*\colon \mathrm{KU}[\frac{1}{n}] \to \mathrm{KU}[\frac{1}{n}]$ for every integer n. It is well-known ([Ada74, §II.13]) that to construct a stable Adams operation ψ^n as a map of spectra, one **must** invert n. To prove Pr. 5.5.5, we need some preliminaries.

Proposition 5.5.6. *The composition*

$$\mathbf{CP}^\infty \xrightarrow{\eta} \Omega^\infty\Sigma^\infty\mathbf{CP}^\infty \xrightarrow{\Omega^\infty j} \Omega^\infty\Sigma^\infty_+\mathbf{CP}^\infty \xrightarrow{\Omega^\infty \rho} \Omega^\infty \mathrm{ku}$$

represents the class $[\xi_1] - 1$ in $\widetilde{\mathrm{ku}}^0(\mathbf{CP}^\infty)$, where ξ_1 is the universal line bundle over $\mathbf{CP}^\infty$.

Let us recall that for a spectrum E and a based space X, one defines the unreduced and reduced E cohomology groups of X as the abelian groups:

$$E^0(X) = \pi_0 \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ X, E) \simeq \pi_0 \operatorname{Map}_{\mathcal{S}}(X, \Omega^\infty E)$$

$$\widetilde{E}^0(X) = \pi_0 \operatorname{Map}_{\mathrm{Sp}}(\Sigma^\infty X, E) \simeq \pi_0 \operatorname{Map}_{\mathcal{S}_*}(X, \Omega^\infty E)$$

Let us also state a lemma we will use regarding the $(\Sigma^\infty_+, \mathbf{GL}_1)$-adjunction; we only state it to keep track of base-points.

Lemma 5.5.7. *If R is an $\mathbf{E}_\infty$-ring, then the composite*

$$\mathbf{GL}_1(R) \to \mathbf{GL}_1(R)_+ \xrightarrow{\eta_+} \Omega^\infty\Sigma^\infty_+\mathbf{GL}_1(R) \xrightarrow{\Omega^\infty\epsilon} \Omega^\infty R$$

is homotopic to the inclusion $\mathbf{GL}_1(R) \to \Omega^\infty R$, where $\epsilon\colon \Sigma^\infty_+\mathbf{GL}_1(R) \to R$ is the counit of the $(\Sigma^\infty_+, \mathbf{GL}_1)$-adjunction.

Note that the unit and the counit appearing in the lemma above do not come from the same adjunction.

Proof. The $(\Sigma^\infty_+, \mathbf{GL}_1)$-adjunction is a composite of the adjunctions

$$\mathrm{CMon}^{\mathrm{grp}} \underset{\mathbf{GL}_1}{\overset{\mathrm{inc.}}{\rightleftarrows}} \mathrm{CMon} \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \mathrm{CAlg}$$

so the counit $\epsilon\colon \Sigma^\infty_+\mathbf{GL}_1(R) \to R$ factors as the map induced by the defining inclusion $\mathbf{GL}_1(R) \to \Omega^\infty R$ and the counit of the $(\Sigma^\infty_+, \Omega^\infty)$-adjunction. This implies the diagram of spaces

$$\begin{array}{ccccccc}
\mathbf{GL}_1(R) & \longrightarrow & \mathbf{GL}_1(R)_+ & \xrightarrow{\eta_+} & \Omega^\infty\Sigma^\infty_+\mathbf{GL}_1(R) & \searrow^{\Omega^\infty\epsilon} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega^\infty R & \longrightarrow & (\Omega^\infty R)_+ & \longrightarrow & \Omega^\infty\Sigma^\infty_+\Omega^\infty R & \longrightarrow & \Omega^\infty R
\end{array}$$

commutes, where the vertical maps are all induced by the defining inclusion. Similarly, the first two maps in the bottom composition compose of the unit of the $(\Sigma^\infty_+, \Omega^\infty)$-adjunction on $\Omega^\infty R$, and by the triangle identity for this adjunction, the bottom horizontal composite is the identity. This is what we wanted to prove. $\square$

Proof of Pr.5.5.6. Consider the natural commutative diagram of spaces

$$\begin{array}{ccccccc} \mathbf{CP}^\infty & \longrightarrow & \mathbf{CP}^\infty_+ & \xrightarrow{\eta_+} & \Omega^\infty\Sigma^\infty_+\mathbf{CP}^\infty & \xrightarrow{\Omega^\infty\rho} & \\ \big\downarrow{\scriptstyle \xi|_{\mathrm{BU}(1)}} & & \big\downarrow{\scriptstyle \xi|_{\mathrm{BU}(1)+}} & & \big\downarrow{\scriptstyle \Omega\Sigma^\infty_+\xi|_{\mathrm{BU}(1)}} & & \big\downarrow \\ \mathbf{GL}_1(\mathrm{ku}) & \longrightarrow & \mathbf{GL}_1(\mathrm{ku})_+ & \xrightarrow{\eta_+} & \Omega^\infty\Sigma^\infty_+\mathbf{GL}_1(\mathrm{ku}) & \xrightarrow{\Omega^\infty\epsilon} & \Omega^\infty\,\mathrm{ku} \end{array}$$

where ϵ is the counit of the $(\Sigma^\infty_+, \mathbf{GL}_1)$-adjunction. By Lm.5.5.7, the bottom horizontal composite is the inclusion $\mathbf{GL}_1(\mathrm{ku}) \to \Omega^\infty\,\mathrm{ku}$. Hence the composition $\mathbf{CP}^\infty \to \Omega^\infty\,\mathrm{ku}$ above corresponds to the maps $\xi|_{\mathrm{BU}(1)}\colon \mathbf{CP}^\infty \to \Omega^\infty\,\mathrm{ku}$ which lands in $\{1\}\times\mathrm{BU}$ defining the universal line bundle ξ_1 over $\mathbf{CP}^\infty$. As this morphism represents $[\xi_1]$ in $\mathrm{ku}^0(\mathbf{CP}^\infty)$, it follows by the $(\Sigma^\infty_+, \Omega^\infty)$-adjunction that ρ also represents the element $[\xi_1]$. Our desired composite is then represented by the image of ρ under the map

$$j^*\colon\ \mathrm{ku}^0(\mathbf{CP}^\infty) \to \widetilde{\mathrm{ku}}^0(\mathbf{CP}^\infty).$$

To identify j^* we write down the split (co)fibre sequence of spectra

$$\Sigma^\infty\mathbf{CP}^\infty \xrightarrow{j} \Sigma^\infty_+\mathbf{CP}^\infty \simeq \Sigma^\infty\mathbf{CP}^\infty \oplus \mathbf{S} \xrightarrow{q} \mathbf{S}$$

where q is induced by the unique map of pointed spaces $\mathbf{CP}^\infty_+ \to S^0$ which is surjective on π_0. We can calculate $q^*\colon\ \mathrm{ku}^0(*) \to \mathrm{ku}^0\,\mathbf{CP}^\infty$—it induces a map of rings on ku^0-cohomology, and $\mathrm{ku}^0(*) \simeq \mathbf{Z}$, so q^* is the unique map. More explicitly, q^* sends an integer n to the n-dimensional virtual vector bundle on $\mathrm{ku}^0\,\mathbf{CP}^\infty$. One can also calculate that the splitting i of q induces a map $i^*\colon\ \mathrm{ku}^0\,\mathbf{CP}^\infty \to \mathbf{Z}$ sending a virtual vector bundle to its dimension. Indeed, this can be seen geometrically, as a class $x\colon \mathbf{CP}^\infty \to \mathbf{Z}\times\mathrm{BU}$ is sent to the composition $* \to \mathbf{CP}^\infty \to \mathbf{Z}\times\mathrm{BU}$ which only remembers which $\mathbf{Z}$-component the original x landed in, ie, its virtual dimension. We can then identify the map p^* induced by the splitting p of j with the inclusion of the kernel of i^*, ie, the inclusion of those virtual vector bundles over $\mathbf{CP}^\infty$ with dimension 0. It follows that j^* can then be identified by the formula:

$$j^*(x) = x - q^*i^*(x) = x - \dim(x)$$

Back to the question at hand, we wish to calculate $j^*(\rho)$. Using the above yields our desired conclusion:

$$j^*(\rho) = j^*([\xi_1]) = [\xi_1] - 1$$

$\square$

A consequence of the above is that we obtain the usual complex orientation on KU.

Remark 5.5.8. The map $j\colon \Sigma^\infty \mathbf{CP}^\infty \to \Sigma^\infty_+ \mathbf{CP}^\infty$ defines a class

$$j \in (\widetilde{\Sigma^\infty_+ \mathbf{CP}^\infty})^0(\mathbf{CP}^\infty).$$

Let us also write $j \in \widetilde{\mathcal{E}}^0(\mathbf{CP}^\infty)$ for the image of the above element under the localisation map:

$$\Sigma^\infty_+ \mathbf{CP}^\infty \to \Sigma^\infty_+ \mathbf{CP}^\infty[\beta^{-1}] = \mathcal{E}$$

A complex orientation $x_\mathcal{E}$ can be defined as $x_\mathcal{E} = \frac{j}{\beta} \in \widetilde{\mathcal{E}}^2(\mathbf{CP}^\infty)$ as we have $\iota^*(x_\mathcal{E}) = \beta \cdot \beta^{-1} = 1$ where $\iota\colon \mathbf{CP}^1 \to \mathbf{CP}^\infty$ is the canonical inclusion. It follows that the image of $x_\mathcal{E}$ inside KU under the map

$$\rho[\beta^{-1}]\colon \mathcal{E} = \Sigma^\infty_+ \mathbf{CP}^\infty[\beta^{-1}] \to \mathrm{ku}[\beta^{-1}] = \mathrm{KU}$$

is a complex orientation x_{KU} of KU, as complex orientations are sent to complex orientations by morphisms of $\mathbf{E}_\infty$-rings; see [EC2, Rmk.4.1.3]. This complex orientation on KU is also the orientation we were all expecting, as by Pr.5.5.6 we obtain the equalities

$$x_{\mathrm{KU}} = \rho_*(x_\mathcal{E}) = \frac{[\xi_1] - 1}{\beta} \in \widetilde{\mathrm{KU}}^0(\mathbf{CP}^\infty)$$

where ξ_1 is the universal line bundle over $\mathbf{CP}^\infty$.

Proof of Pr.5.5.5. By restricting ourselves to the case of an integer k not divisible by p, we have assured that $[k]\colon \mu_{p^\infty}^\heartsuit \to \mu_{p^\infty}^\heartsuit$ is an automorphism of p-divisible groups.

Let us write $\mathcal{E} = \mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_1^p}(\mu_{p^\infty}^\heartsuit)$. We claim that $[k]^*$ can be calculated on the universal line bundle over $\mathbf{CP}^\infty$ using just the algebraic geometry of $\widehat{\mathbf{G}}_m$. By (the proof of) Pr.5.1.3, the map $\rho_p[\beta^{-1}]\colon \mathcal{E} \to \mathrm{KU}_p$ is an equivalence of $\mathbf{E}_\infty$-rings, and Rmk.5.5.8 states this equivalence sends the canonical complex orientation $x_\mathcal{E}$ of $\mathcal{E}$ to the usual complex orientation x_{KU} of KU_p. We obtain orientations (now in the sense of Df.4.1.1) $e_\mathcal{E}$ and e_{KU} of the formal multiplicative group $\widehat{\mathbf{G}}_m$ over $\mathcal{E}$ and KU_p, respectively, ([EC2, Ex.4.3.22]) with the additional property that $\rho(e_\mathcal{E}) = e_{\mathrm{KU}}$. As these orientations of $\widehat{\mathbf{G}}_m$ determine morphisms from the associated Quillen formal group to $\widehat{\mathbf{G}}_m$ ([EC2, Pr.4.3.23]) and $\rho(e_\mathcal{E}) = e_{\mathrm{KU}}$, we obtain the commutative diagram of equivalences of formal groups:

$$\begin{array}{ccc} \widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathrm{KU}_p} & \xrightarrow{\rho^*} & \widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathcal{E}} \\ & \searrow \quad \swarrow & \\ & \widehat{\mathbf{G}}_{m,\mathbf{Z}_p} & \end{array}$$

Focusing on KU_p now, let us rewrite the above diagram of equivalences of formal groups over $\operatorname{Spf} \mathbf{Z}_p$:

$$\operatorname{Spf} \mathrm{KU}_p^0(\mathbf{CP}^\infty) \xrightarrow{\simeq} \operatorname{Spf} \mathbf{Z}_p[\![t]\!] = \widehat{\mathbf{G}}_m \tag{5.5.9}$$

By definition, $[k]$ acts by taking k-fold multiplication, which on the multiplicative formal group is an operation represented by the map of rings:

$$[k]^* \colon \mathbf{Z}_p[\![t]\!] \xrightarrow{c^k} \mathbf{Z}_p[\![t_1, \ldots, t_k]\!] \xrightarrow{\mu} \mathbf{Z}_p[\![t]\!], \qquad t \mapsto (t+1)^k - 1$$

Recall from (5.5.4) that the first map is the k-fold iteration of the comultiplication[34], and the second map is the completed multiplication map. As the map (5.5.9) induces a map of adic rings sending t to βx_{KU}, we then obtain the same formulae for $[k]^*$ in $\mathrm{KU}_p^0(\mathbf{CP}^\infty)$:

$$[k]^*(\beta x_{\mathrm{KU}}) = (\beta x_{\mathrm{KU}} + 1)^k - 1$$

As $\beta x_{\mathrm{KU}} \in \mathrm{KU}_p^0(\mathbf{CP}^\infty)$ is represented by $[\xi_1] - 1$ one obtains:

$$[k]^*([\xi_1]) = [k]^*(\beta x_{\mathrm{KU}} + 1) = [k]^*(\beta x_{\mathrm{KU}}) + 1 = (\beta x_{\mathrm{KU}} + 1)^k - 1 + 1 = [\xi_1^{\otimes k}]$$

It follows that for any finite space (33) X and any complex line bundle $\mathcal{L}$ over X with corresponding map $g \colon X \to \mathbf{CP}^\infty$, the inherent naturality of $[k]^*$ gives us the formula:

$$[k]^*([\mathcal{L}]) = [k]^*([g^*\xi_1]) = g^*([k]^*(\xi_1)) = g^*[\xi_1]^k = [\mathcal{L}^{\otimes k}]$$

It follows from [Ati67, Pr.3.2.1(3)] that the operations $[k]^*$ on $\mathrm{KU}_p^0(X)$ are the Adams operations ψ^k as maps of cohomology theories.

To lift this statement from one about cohomology theories to one about the spectra that represent them, we need to see there are no phantom maps of spectra $\mathrm{KU}_p \to \mathrm{KU}_p$—this is the only obstacle to the fully faithfulness of the functor

$$\mathrm{hSp} \to \mathrm{CohomTh} \qquad E \mapsto E^*(-)$$

where CohomTh denotes the 1-category cohomology theories on finite spaces; see [HS99, §2 & Cor.2.15] and [CHT10, Lec.17]. As KU_p represents an even periodic Landweber exact cohomology theory, it follows there exist no phantom endomorphisms of KU_p; see [CHT10, Cor.7, Lec.17]. □

Remark 5.5.10. There is a C_2-action on the sections of $\mathcal{O}^{\mathrm{top}}_{\mathrm{BT}_n^p}$ coming from the inversion action on p-divisible groups, ie, from ψ^{-1}. Any C_2-action on an $\mathbf{E}_\infty$-ring $\mathcal{E}$ can be used to upgrade $\mathcal{E}$ to a genuinely commutative C_2-ring spectrum

[34] The comultiplication on the ring $\mathbf{Z}_p[\![t]\!]$ representing the multiplicative formal group is given by

$$\mathbf{Z}_p[\![t]\!] \to \mathbf{Z}_p[\![x, y]\!], \qquad t + 1 \mapsto xy + x + y + 1 = (x+1)(y+1).$$

(the kind with norms); see [HM17, Th.2.4]. When $p = 2$, this has interesting results, for example, the C_2-structure on sections of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ can be used to obtain a C_2-equivariant refinement of part 1 of Th. 2.1.7: the complex orientability and Landweber exactness of affine sections of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$ can be upgraded to *Real orientability* and *Real Landweber exactness* à la [HM17, §3]. This essentially follows from the *regular homotopy fixed point spectral sequences* of [Mei22], the descent theory developed by Lurie in [EC2, §6], and the analogous result of Hahn–Shi [HS20] for Lubin–Tate spectra.

Remark 5.5.11. Let p be an odd prime. Using the Teichmüller character, a map of groups $\mathbf{F}_p^\times \to \mathbf{Z}_p^\times$, which sends d to the limit of the Cauchy sequence $\{d^{p^n}\}_{n\geqslant 0}$, one obtains an action of $\mathbf{F}_p^\times \simeq C_{p-1}$ on any sections of $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}$. In particular, for any $\mathbf{G}$ in $\mathcal{C}_{A_0}$ (it need not be just an affine object), the $\mathbf{E}_\infty$-ring $\mathcal{E} = \mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_n}(\mathbf{G})$ has an $\mathbf{E}_\infty$-$\mathbf{F}_p^\times$-action, and the homotopy fixed points $\mathcal{E}^{h\mathbf{F}_p^\times}$ split off a summand of $\mathcal{E}$ using the idempotent map:

$$\frac{1}{p-1}\sum_{d\in\mathbf{F}_p^\times\subseteq\mathbf{Z}_p^\times}\psi^d\colon\mathcal{E}\to\mathcal{E}$$

In particular, if $\mathcal{E} = \mathrm{KU}_p$ as in § 5.1, this summand is the *periodic Adams summand.*

As some obvious foreshadowing for the rest of this thesis, let us concretely define stable p-adic Adams operations on TMF_p using what we have seen so far.

Definition 5.5.12. For any p-adic unit $k \in \mathbf{Z}_p^\times$ we obtain an automorphism of $\mathbf{E}_\infty$-rings $\psi^k\colon \mathrm{TMF}_p \to \mathrm{TMF}_p$ by specialising Df.5.5.1 to Th.5.3.3.

Part II

The constructions

Chapter 6

The functoriality of $\mathscr{O}^{\mathrm{top}}$

> The nihilist recognizes that the highest conceivable ideals that command our respect and admiration exist nowhere except in our own minds.
>
> *John Marmysz, Laughing at Nothing*

In this chapter, we prove Th. B and Th. C, in §6.1 and §6.2, respectively, which show that $\mathscr{O}^{\mathrm{top}}$ has more structure than previously known.

Warning 6.0.1. In the previous chapter, we wrote $\mathcal{M}$ for an object of $\mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ and $\mathcal{M}^{\heartsuit}$ for an object of $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$. As there will not be any more explicit work with derived or spectral stacks, let us drop the $(-)^{\heartsuit}$ from our notation for the rest of the thesis. In other words, our study of stacks takes place inside $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$.

6.1 Isogenies of invertible degree

In Th.5.3.3, we showed that the sheaf of $\mathbf{E}_\infty$-rings $\mathscr{O}^{\mathrm{top}}_p$ on the small étale site on the moduli stack of elliptic curves, can be factored as follows:

$$\left(\mathrm{DM}^{\text{ét}}_{/\mathcal{M}_{\mathrm{Ell}}}\right)^{\mathrm{op}} \to \left(\mathrm{fDM}^{\text{ét}}_{/\mathcal{M}_{\mathrm{Ell}}\times \operatorname{Spf}\mathbf{Z}_p}\right)^{\mathrm{op}} \xrightarrow{[p^\infty]} \mathcal{C}^{\mathrm{op}}_{\mathbf{Z}_p} \xrightarrow{\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_2}} \mathrm{CAlg} \qquad (6.1.1)$$

Above, the first functor in induced by base-change with $\operatorname{Spf}\mathbf{Z}_p$, $[p^\infty]$ sends an elliptic curve E over a p-complete ring to its associated p-divisible group $E[p^\infty]$, and $\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}^p_2}$ is a sheaf of $\mathbf{E}_\infty$-rings. One consequence of this is that $\mathscr{O}^{\mathrm{top}}_p$ is functorial with respect to morphisms $\phi\colon E \to E'$ of elliptic curves which induce an isomorphism on the associated p-divisible group. Such morphisms of elliptic curves have a simple classification.

Proposition 6.1.2. *Let $\phi\colon E \to E'$ be a morphism of elliptic curves over a formal Deligne–Mumford stack $\mathfrak{X}$ over* $\operatorname{Spf}\mathbf{Z}_p$*. Then ϕ induces an isomorphism on the associated p-divisible group if and only if ϕ is an* isogeny of degree prime to p*, meaning that ϕ is finite locally free of rank prime to p.*

Proof. Note that by [KM85, Th.2.4.2], a morphism of elliptic curves $E \to E'$ is either an isogeny or the zero map, so we immediately see ϕ must be an isogeny. For to calculate its rank, we notice this question is local on $\mathfrak{X}$, so we may work over $\operatorname{Spf} A$ for some adic ring A with adic topology generated by an ideal I. Writing $\operatorname{Spf} A = \operatorname{colim} \operatorname{Spec} A/I^n$, we are reduced to the case of a classical (non formal) affine scheme X. It is a general fact that we have the following exact sequence of étale sheaves over X

$$0 \to K_p \to E[p^\infty] \xrightarrow{\phi[p^\infty]} E'[p^\infty] \to 0$$

where K_p is the component of the finite group scheme $K = \ker(E \to E')$ of maximal p-power order. The vanishing of K_p is then equivalent to the degree of $E \to E'$ being prime to p. $\square$

Using the language of isogenies of degree prime to p, our reinterpretation of Th. 5.3.3 states $\mathscr{O}_p^{\mathrm{top}}$ is functorial with respect to such isogenies of elliptic curves. We will often work over $\operatorname{Spec} \mathbf{Z}$ rather than $\operatorname{Spf} \mathbf{Z}_p$, and in this case, we can still use p-divisible groups to test if an isogeny of elliptic curves has degree invertible on X.

Corollary 6.1.3. *Let $\phi\colon E \to E'$ be a morphism of elliptic curves over a Deligne–Mumford stack X. Then ϕ induces an isomorphism on the associated p-divisible group base-changed over* $\operatorname{Spf} \mathbf{Z}_p$ *for every prime p if and only if ϕ is an* isogeny of invertible degree.

Proof. If p is invertible in X, then $\mathsf{X} \times \operatorname{Spf} \mathbf{Z}_p$ vanishes and the condition is vacuous. If p is not invertible on X, then we apply Pr.6.1.2 to $\mathsf{X} \times \operatorname{Spf} \mathbf{Z}_p$. $\square$

The two statements above inspire the following definition.

Definition 6.1.4. Let $\mathcal{I}\mathrm{sog}_{\mathrm{Ell}}$ be the following $(2,2)$-category:

- Objects are those in the small étale site of $\mathcal{M}_{\mathrm{Ell}}$, so pairs (X, E) where X is a Deligne–Mumford stack and E is an elliptic curve over X defining an étale morphism $\mathsf{X} \to \mathcal{M}_{\mathrm{Ell}}$.
- 1-Morphisms are pairs $(f, \phi)\colon (\mathsf{X}, E) \to (\mathsf{X}', E')$ where $f\colon \mathsf{X} \to \mathsf{X}'$ is a morphism of Deligne–Mumford stacks and $\phi\colon E \to f^*E'$ is an isogeny of elliptic curves over X of invertible degree.
- 2-Morphisms $\alpha\colon (f, \phi) \to (g, \psi)$ are isogenies $\alpha\colon f^*E' \to g^*E'$ of elliptic curves over X of invertible degree such that $\alpha \circ \phi = \psi$.

Write $\mathrm{Isog}_{\mathrm{Ell}}$ for the initial $(2,1)$-category which receives a functor of $(2,2)$-categories from $\mathcal{I}\mathrm{sog}_{\mathrm{Ell}}$. This can be obtained by formally inverting all of the 2-morphisms inside $\mathcal{I}\mathrm{sog}_{\mathrm{Ell}}$. Also, define categories $\widehat{\mathcal{I}\mathrm{sog}}_{\mathrm{Ell}}$ and $\widehat{\mathrm{Isog}}_{\mathrm{Ell}}$ by replacing $\mathcal{M}_{\mathrm{Ell}}$ with $\widehat{\mathcal{M}}_{\mathrm{Ell}} = \mathcal{M}_{\mathrm{Ell}} \times \operatorname{Spf} \mathbf{Z}_p$ in the definition above, and working with formal Deligne–Mumford stacks and isogenies of elliptic curves of degree prime to p.

Remark 6.1.5 (Grothendieck topologies). There is a forgetful functor $\mathrm{Isog}_{\mathrm{Ell}}$ to DM sending a pair (X, E) to the underlying Deligne–Mumford stack X. Give $\mathrm{Isog}_{\mathrm{Ell}}$ the étale topology through this forgetful functor. We claim this forgetful functor is a morphism of sites following [Sta, 00X0]. The only condition left to check is that this forgetful functor preserves fibre products, so it suffices to calculate fibre products inside $\mathrm{Isog}_{\mathrm{Ell}}$, which we detail now. Consider the span

$$(\mathsf{X}_0, E_0) \xrightarrow{(f,\phi)} (\mathsf{X}_{01}, E_{01}) \xleftarrow{(g,\psi)} (\mathsf{X}_1, E_1)$$

inside $\mathrm{Isog}_{\mathrm{Ell}}$, so $\phi\colon E_0 \to f^*E_{01}$ and $\psi\colon E_1 \to g^*E_{01}$ are isogenies of invertible degree on X_0 and X_1, respectively. Let X and E denote the following fibre products of stacks:

$$\mathsf{X}_0 \underset{\mathsf{X}_{01}}{\times} \mathsf{X}_1 = \mathsf{X} \qquad E_0 \underset{E_{01}}{\times} E_1 = E$$

Above, the map $E_0 \to E_{01}$ is the composition

$$E_0 \xrightarrow{\phi} f^*E_{01} \to E_{01}$$

of maps of stacks over X_{01}, and similarly for $E_1 \to E_{01}$. Let us write $f'\colon \mathsf{X} \to \mathsf{X}_1$ for the base-change of f, and similarly for g'. To see E is an elliptic curve over X and the maps $E \to E_i$ induce maps $\alpha'\colon E \to f'^*E_1$ and $\beta'\colon E \to g'^*E_0$ which are isogenies of invertible degree, consider the following Cartesian diagram of stacks over X:

$$\begin{array}{ccc} E & \xrightarrow{\alpha'} & f'^*E_1 \\ \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle f'^*\beta} \\ g'^*E_0 & \xrightarrow{g'^*\alpha} & g'^*f^*E_{01} \simeq f'^*g^*E_{01} \end{array}$$

This witnesses that the canonical maps $E \to \mathsf{X}$ and $\mathsf{X} \to E$ give E the structure of an elliptic curve over X, and that β' and α' are isogenies of invertible degree over X. Likewise, the forgetful functor $\widehat{\mathrm{Isog}}_{\mathrm{Ell}}$ to the category of formal Deligne–Mumford stacks gives $\widehat{\mathrm{Isog}}_{\mathrm{Ell}}$ the étale topology and is also continuous.

Remark 6.1.6 (Small étale site of $\mathcal{M}_{\mathrm{Ell}}$). Consider the $(2,1)$-subcategory of $\mathcal{I}\mathrm{sog}_{\mathrm{Ell}}$ with the same objects but with those 1-morphisms (f,ϕ) where ϕ is an isomorphism, which forces any 2-morphisms α to be isomorphisms as well. This $(2,1)$-category is precisely the small étale site $\mathrm{DM}^{\text{ét}}_{/\mathcal{M}_{\mathrm{Ell}}}$ of $\mathcal{M}_{\mathrm{Ell}}$ inside in $\mathcal{I}\mathrm{sog}_{\mathrm{Ell}}$. Let us write ι for the composition $\mathrm{DM}^{\text{ét}}_{\mathcal{M}_{\mathrm{Ell}}} \to \mathrm{Isog}_{\mathrm{Ell}}$. The same comment holds over $\widehat{\mathcal{M}}_{\mathrm{Ell}}$.

The above discussion yields the following consequence of Th.5.3.3.

Corollary 6.1.7. *Fix a prime p. There is an étale hypersheaf $\mathcal{O}_p^{\mathrm{top}}$ of $\mathbf{E}_\infty$-rings on $\widehat{\mathrm{Isog}}_{\mathrm{Ell}}$ such that its restriction along ι is equivalent to the sheaf $\mathcal{O}_p^{\mathrm{top}}$ of [Beh14].*

The prove the above statement and many similar ones, we need to discuss *elliptic cohomology theories*. Recall that $\overline{\mathcal{M}}_{\mathrm{Ell}}$ denotes the Deligne–Mumford compactification of the moduli stack $\mathcal{M}_{\mathrm{Ell}}$. The moduli description of $\overline{\mathcal{M}}_{\mathrm{Ell}}$ involves the use of *generalised elliptic curves*; see [Ces17], [Con07], or [DR73] for some background.

Definition 6.1.8. Let E be a generalised elliptic curve over a ring R with irreducible geometric fibres, which is equivalent data to a morphism of stacks $\operatorname{Spec} R \to \overline{\mathcal{M}}_{\mathrm{Ell}}$. We say that a homotopy commutative ring spectrum $\mathcal{E}$ is an *elliptic cohomology theory*[35] for E (or $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$) if we have the following data:

1. $\mathcal{E}$ is weakly 2-periodic (10) and even, meaning that $\pi_k\mathcal{E}$ vanishes for all odd integers k (so, in particular, $\mathcal{E}$ is complex orientable), and the $\mathcal{E}$-module $\mathcal{E}$ is locally free of rank 1;
2. There is a chosen isomorphism of rings $\pi_0\mathcal{E} \simeq R$; and
3. There is a chosen isomorphism of formal groups $\widehat{E} \simeq \widehat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0}$ over R, between the formal group of E and the classical Quillen formal group of $\mathcal{E}$; see [Lur18, §4].

We say a collection of such $\mathcal{E}$ is *natural* if the isomorphisms in parts 3-4 above are natural with respect to some subcategory of affine schemes over $\mathcal{M}_{\mathrm{Ell}}$, for example, affine objects of $\mathrm{Isog}_{\mathrm{Ell}}$.

The folklore theorem Th.B.0.2 states that as a sheaf of $\mathbf{E}_\infty$-rings on the small étale site of $\overline{\mathcal{M}}_{\mathrm{Ell}}$, the sheaf $\mathscr{O}^{\mathrm{top}}$ is determined up to homotopy by the fact that it produces natural elliptic cohomology theories; see §B. This is already used to prove Th.5.3.3, and we will find a similar use for this statement in the proof of Cor.6.1.7 and Th.6.1.9.

Proof of Cor.6.1.7. By Pr.5.3.1, which states that the functor $[p^\infty]$ of (6.1.1) is well-defined, and Pr.6.1.2, there is a functor from $\widehat{\mathrm{Isog}}_{\mathrm{Ell}}^{\mathrm{op}}$ to the site $\mathcal{C}_{\mathbf{Z}_p}$ of Df.2.1.6 (and of height $n = 2$) sending $(\mathfrak{X}, E)$ to $(\mathfrak{X}, E[p^\infty])$. The desired hypersheaf $\mathscr{O}_p^{\mathrm{top}}$ can then be given as the following composite:

$$\widehat{\mathrm{Isog}}_{\mathrm{Ell}}^{\mathrm{op}} \to \mathcal{C}_{\mathbf{Z}_p}^{\mathrm{op}} \xrightarrow{\mathscr{O}^{\mathrm{top}}_{\mathrm{BT}_2^p}} \mathrm{CAlg}$$

Restriction along ι yields $\mathscr{O}_p^{\mathrm{top}}$ as this sheaf is characterised up to homotopy by the fact it defines an elliptic cohomology theory (Th.B.0.2), a property also satisfied by the restriction of the above sheaf to the small étale site over $\mathcal{M}_{\mathrm{Ell}}$—we have already seen this argument in Th.5.3.3. ☐

With a little extra care, we obtain the integral statement and Th.B.

[35] These were previously called *generalised elliptic cohomology theories* in [Dav21a, §2.2], and other variations on this theme can be found elsewhere; see [AHS01, Df.1.2], [Beh14, §6] or [Lur09a, Df.1.2], for example.

Theorem 6.1.9. *There is an étale hypersheaf of $\mathbf{E}_\infty$-rings $\mathscr{O}^{\mathrm{top}}$ on $\mathrm{Isog}_{\mathrm{Ell}}$ whose restriction to the small étale site of $\mathcal{M}_{\mathrm{Ell}}$ is equivalent to the sheaf $\mathscr{O}^{\mathrm{top}}$ from Behrens' construction [Beh14]. In particular, $\mathscr{O}^{\mathrm{top}}$ defines a* natural elliptic cohomology theory *on affine objects inside* $\mathrm{Isog}_{\mathrm{Ell}}$.

The fact that $\mathscr{O}^{\mathrm{top}}$ on $\mathrm{Isog}_{\mathrm{Ell}}$ restricts to something well-known on the wide subcategory $\mathrm{DM}^{\text{ét}}_{/\mathcal{M}_{\mathrm{Ell}}}$ means that many classical statements about $\mathscr{O}^{\mathrm{top}}$ still hold. For example, evaluating $\mathscr{O}^{\mathrm{top}}$ on affines in $\mathrm{Isog}_{\mathrm{Ell}}$ produces *elliptic cohomology theories* which are *Landweber exact*. Most importantly for us, the global sections of $\mathscr{O}^{\mathrm{top}}$ on $\mathrm{Isog}_{\mathrm{Ell}}$ still yield TMF (up to homotopy). To prove this theorem, we will use the following two standard lemmata.

Lemma 6.1.10. *Write $\mathrm{Isog}^{\mathrm{aff}}_{\mathrm{Ell}}$ for the full subcategory spanned by affine objects. Then for any complete ∞-category $\mathcal{D}$, the inclusion $i\colon \mathrm{Isog}^{\mathrm{aff}}_{\mathrm{Ell}} \to \mathcal{C}$ induces the following equivalence of ∞-categories of $\mathcal{D}$-valued sheaves:*

$$i^*\colon \mathcal{S}\mathrm{hv}_{\mathcal{D}}(\mathrm{Isog}_{\mathrm{Ell}}) \to \mathcal{S}\mathrm{hv}_{\mathcal{D}}(\mathrm{Isog}^{\mathrm{aff}}_{\mathrm{Ell}})$$

Moreover, the same holds for hypersheaves if $\mathcal{D} = \mathrm{Sp}$ or CAlg.

Proof. Using the "comparison lemma" of [Hoy14, Lm.C.3], which applies as $\mathcal{M}_{\mathrm{Ell}}$ is a qcqs Deligne–Mumford stack, we obtain the middle equivalence in the following chain of equivalences of ∞-categories:

$$\mathcal{S}\mathrm{hv}_{\mathcal{D}}(\mathrm{Isog}_{\mathrm{Ell}}) \xleftarrow{\simeq} \mathcal{S}\mathrm{hv}_{\mathcal{D}}(\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Isog}_{\mathrm{Ell}}))$$

$$\xrightarrow{i^*_{\mathcal{S}},\simeq} \mathcal{S}\mathrm{hv}_{\mathcal{D}}(\mathcal{S}\mathrm{hv}_{\mathcal{S}}(\mathrm{Isog}^{\mathrm{aff}}_{\mathrm{Ell}})) \xrightarrow{\simeq} \mathcal{S}\mathrm{hv}_{\mathcal{D}}(\mathrm{Isog}^{\mathrm{aff}}_{\mathrm{Ell}}).$$

The first and last equivalences follow by [SAG, Pr.1.3.1.7]; all of the inverses to the above equivalences are given by the evident right Kan extensions. The naturality of [SAG, Pr.1.3.1.7] show the above composite is equivalent to i^*. For the hypersheaf statement, by [CM21, Ex.2.5] a sheaf is a hypersheaf if it is *hypercomplete* in the sense of [SAG, Df.C.1.2.12], and this adjective is preserved by the equivalences above. ☐

The following statement is essentially [HL16, Lm.4.5].

Lemma 6.1.11. *Let $E\colon \operatorname{Spec} R \to \overline{\mathcal{M}}_{\mathrm{Ell}}$ be a morphism of stacks, and*

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E}_0 \\ \downarrow & & \downarrow \\ \mathcal{E}_1 & \longrightarrow & \mathcal{E}_{01} \end{array} \tag{6.1.12}$$

be a Cartesian diagram of $\mathbf{E}_\infty$-rings such that $\pi_0\mathcal{E} \simeq R$, and each $\mathbf{E}_\infty$-ring $\mathcal{E}_i$ is an elliptic cohomology theory for $E|_{\operatorname{Spec}\pi_0\mathcal{E}_i}$ such that the isomorphisms $\alpha_i\colon \widehat{E}|_{\operatorname{Spec}\pi_0\mathcal{E}_i} \simeq \widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathcal{E}_i}$ both agree with the isomorphism $\alpha_{01}\colon \widehat{E}|_{\operatorname{Spec}\pi_0\mathcal{E}_{01}} \simeq \widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathcal{E}_{01}}$ of formal groups over $\pi_0\mathcal{E}_{01}$. Suppose that $\pi_0\mathcal{E}_0 \oplus \pi_0\mathcal{E}_1 \to \pi_0\mathcal{E}_{01}$ is surjective. Then $\mathcal{E}$ obtains the unique structure of an elliptic cohomology theory for E such that the isomorphism $\widehat{E} \simeq \widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathcal{E}}$ base changes to the isomorphisms α_i over $\pi_0\mathcal{E}_i$ for $i = 0, 1$.

Proof. By assumption, we have an isomorphism $\pi_0\mathcal{E} \simeq R$, so $\mathcal{E}$ satisfies criterion 3 of Equation (6.1.12). The Mayer–Vietoris sequence associated to (6.1.11) and the assumption that $\mathcal{E}_i$ are elliptic cohomology theories shows that $\pi_{2k}\mathcal{E} \simeq \omega_E^{\otimes k}$ for every integer k. This latter fact, and the assumption that $\mathcal{E}_0 \to \mathcal{E}_{01} \leftarrow \mathcal{E}_1$ are jointly surjective on π_0, imply that these maps are jointly surjective on π_*, which shows that $\mathcal{E}$ satisfies criterion 2. This same Mayer–Vietoris sequence combined with the fact that each $\mathcal{E}_i$ is complex orientable gives $\mathcal{E}$ complex orientable restricting to those on $\mathcal{E}_i$ and shows $\mathcal{E}$ is weakly 2-periodic, giving us criterion 1. Using the compatibility of α_0 and α_1, one obtains the commutative diagram

$$\begin{array}{ccccc} \hat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0} \otimes \pi_0\mathcal{E}_0 & \longleftarrow & \hat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0} \otimes \pi_0\mathcal{E}_{01} & \longrightarrow & \hat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0} \otimes \pi_0\mathcal{E}_1 \\ \simeq\downarrow\alpha_0 & & \simeq\downarrow\alpha_{01} & & \simeq\downarrow\alpha_1 \\ \hat{E} \otimes \pi_0\mathcal{E}_0 & \longleftarrow & \hat{E} \otimes \pi_0\mathcal{E}_{01} & \longrightarrow & \hat{E} \otimes \pi_0\mathcal{E}_1 \end{array} \tag{6.1.13}$$

and the fact that π_0 of (6.1.12) yields a Cartesian diagram of discrete rings, we obtain an isomorphism of formal groups $\hat{\mathbf{G}}_{\mathcal{E}}^{\mathcal{Q}_0} \to \hat{E}$ over R restricting to those given by (6.1.13). This yields criterion 4, and we are done. $\square$

Proof of Th.6.1.9. Let us first produce a rational sheaf $\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}}$ on $\mathrm{Isog}_{\mathrm{Ell}}$. By Lm.6.1.10, it suffices to define $\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}}$ on the subcategory of $\mathrm{Isog}_{\mathrm{Ell}}$ spanned by affine objects. Recall from (12) that $\omega_E = p_*\Omega_{E/S}$ is the dualising line of $\hat{E}$ and the equivalence of symmetric monoidal ∞-categories $\mathrm{Mod}_{\mathbf{Q}} \simeq \mathcal{D}(\mathbf{Q})$ of (5.1.5). Following [Beh14, §9] and [HL16, Df.5.13], we define $\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}}(R, E)$ for a pair (R, E) by the formal $\mathbf{Q}$-cdga $\omega_E^* \otimes \mathbf{Q}$ defined by placing the invertible R-module $\omega_E^{\otimes n} \otimes \mathbf{Q}$ in degree $2n$ for all $n \in \mathbf{Z}$. This defines an étale hypersheaf as each of the discrete sheaves $\omega_E^{\otimes m}$ have this property. Using this definition, the functoriality of $\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}}$ with respect to $\mathrm{Isog}_{\mathrm{Ell}}$ is clear as nonzero isogenies of rational elliptic curves induce isomorphisms on the associated dualising lines ω_E.

For each prime p, there is a morphism of sites

$$\pi^p \colon \mathrm{Isog}_{\mathrm{Ell}} \to \widehat{\mathrm{Isog}}_{\mathrm{Ell}}$$

induced by base change along the projection $\mathrm{Spf}\,\mathbf{Z}_p \to \mathrm{Spec}\,\mathbf{Z}$. The construction $\mathcal{O}_p^{\mathrm{top}}$ of Cor. 6.1.7 yields an étale hypersheaf $\pi_*^p\mathcal{O}_p^{\mathrm{top}}$ on $\mathrm{Isog}_{\mathrm{Ell}}$. Let us now construct a map of étale hypersheaves $\alpha \colon \mathcal{O}_{\mathbf{Q}}^{\mathrm{top}} \to (\prod_p \pi_*^p\mathcal{O}_p^{\mathrm{top}})_{\mathbf{Q}}$. By Lm.6.1.10, we again restrict our attention to affines, so for each étale morphism $E \colon \mathrm{Spec}\,R \to \mathcal{M}_{\mathrm{Ell}}$ we want a map of $\mathbf{E}_\infty$-rings

$$\mathcal{O}_{\mathbf{Q}}^{\mathrm{top}}(R, E) \to \prod_p (\pi_*^p\mathcal{O}_p^{\mathrm{top}}(R, E)) \otimes \mathbf{Q} \tag{6.1.14}$$

which is natural in $\mathrm{Isog}_{\mathrm{Ell}}$. By definition, the left-hand side is the formal $\mathbf{Q}$-cdga $\omega_E^{\otimes *} \otimes \mathbf{Q}$. By [Mei21, Pr.4.8], we see the sheaf $(\pi_*^p\mathcal{O}_p^{\mathrm{top}})[p^{-1}]$ on $\mathrm{Spec}\,R$ is formal for each p, hence the right-hand side is the formal $R \otimes \mathbf{Q}$-cdga $\prod_p (\omega_E^{\otimes *})_p^\wedge \otimes \mathbf{Q}$. We then define our desired α as the rationalisation of the product over all primes of the natural completion map $\omega_E^{\otimes *} \to (\omega_E^{\otimes *})_p^\wedge$. This clearly yields a collection of maps (6.1.14) natural in the category $\mathrm{Isog}_{\mathrm{Ell}}$.

The sheaf $\mathscr{O}^{\mathrm{top}}$ is then defined by the following Cartesian diagram of étale hypersheaves:

$$\begin{array}{ccc} \mathscr{O}^{\mathrm{top}} & \longrightarrow & \prod_p \pi^p_* \mathscr{O}^{\mathrm{top}}_p \\ \downarrow & & \downarrow \\ \mathscr{O}^{\mathrm{top}}_{\mathbf{Q}} & \xrightarrow{\alpha} & \left(\prod_p \pi^p_* \mathscr{O}^{\mathrm{top}}_p\right)_{\mathbf{Q}} \end{array}$$

By virtue of Lm.6.1.11, $\mathscr{O}^{\mathrm{top}}$ defines a natural elliptic cohomology theory on $\mathrm{Isog}_{\mathrm{Ell}}$. In particular, this allows us to apply Th. B.0.2 and conclude that restriction to the small étale site of $\mathcal{M}_{\mathrm{Ell}}$ is homotopy equivalent to $\mathscr{O}^{\mathrm{top}}$ of [Beh14]. □

It is claimed in our first version of [Dav21a] that Th.6.1.9 extends over the cusp of $\overline{\mathcal{M}}_{\mathrm{Ell}}$, however, there is a mistake in the argumentation there—we give some partial fixes in §7.7.

Let us end with an application of Th.6.1.9 to produce Behrens $Q(N)$ $\mathbf{E}_\infty$-rings of [Beh06]. In *loc. cit.*, even though some intuition with a modular definition is given, these spectra are constructed explicitly using $K(2)$-local chromatic methods. Using Th.6.1.9, we can make this modular definition precise, which yields an almost integral description of the $\mathbf{E}_\infty$-rings $Q(N)$, and §7 gives these spectra Adams operations, Hecke operators, and a kind of Atkin–Lehner involution.

Example 6.1.15. Let N be a positive integer, $\mathcal{M}_0(N)$ be the moduli stack of elliptic curves with chosen cyclic subgroup H of order N (Df.7.1.3 and Df.7.3.1), and $p, q\colon \mathcal{M}_0(N) \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{N}]}$ be the structure and quotient maps given by $(E,H) \mapsto E$ and $(E,H) \mapsto E/H$, respectively, of Df. 7.3.1. Moreover, write $\tau\colon \mathcal{M}_0(N) \to \mathcal{M}_0(N)$ for the involution given by $(E,H) \mapsto (E/H, H^\vee)$—this is discussed in the proof of Pr.7.3.3. We then obtain the following diagram of stacks:

$$\mathcal{M}_0(N) \overset{\xrightarrow{\tau}}{\underset{\xrightarrow{\mathrm{id}}}{\xrightarrow{p}}} \mathcal{M}_{\mathrm{Ell}} \sqcup \mathcal{M}_0(N) \underset{\mathrm{id} \sqcup p}{\overset{\mathrm{id} \sqcup q}{\rightrightarrows}} \mathcal{M}_{\mathrm{Ell}}$$

We can enhance the above diagram to one in $\mathrm{Isog}_{\mathrm{Ell}}$ as follows. Equip each object $\mathcal{M}$ and $\mathcal{M}_0(N)$ with their respective universal elliptic curves $\mathscr{E}$ and $\mathscr{E}_0(N)$. Equip each occurrence of p with the canonical isomorphism $\mathscr{E}_0(N) \simeq p^*\mathscr{E}$, the left and upper-right occurrences of id with the identity on the associated elliptic curves, the lower-right occurrence of id with the N-fold multiplication map $[N]\colon \mathscr{E} \to \mathscr{E}$, and finally equip both τ and q with the canonical quotient map $Q\colon \mathscr{E}_0(N) \to \mathscr{E}_0(N)/\mathcal{H}$ of the universal elliptic curve by its universal cyclic subgroup $\mathcal{H}$. This produces the following diagram in $\mathrm{Isog}_{\mathrm{Ell}}$:

$$\begin{array}{ccc} (\mathcal{M}_0(N), \mathscr{E}_0(N)) & \begin{array}{c} \xrightarrow{(\tau,Q)} \\ \xrightarrow{(p,\mathrm{id})} \\ \xrightarrow[(\mathrm{id},\mathrm{id})]{} \end{array} & (\mathcal{M}_{\mathrm{Ell}}, \mathscr{E}) \sqcup (\mathcal{M}_0(N), \mathscr{E}_0(N)) \\ & & {\scriptstyle (\mathrm{id},\mathrm{id})\sqcup(p,\mathrm{can})}\downarrow\ \downarrow{\scriptstyle (\mathrm{id},[N])\sqcup(q,Q)} \\ & & (\mathcal{M}_{\mathrm{Ell}}, \mathscr{E}) \end{array}$$

As in [Beh06, §1.1], these morphisms satisfy the semisimplicial relations up to homotopy in $\mathrm{Isog}_{\mathrm{Ell}}$—the n-simplices for $n \geqslant 3$ are all degenerate. Moreover, all of the morphisms above lie in $\mathrm{Isog}_{\mathrm{Ell}}$, so we can apply $\mathscr{O}^{\mathrm{top}}$ to the above diagram by Th. 6.1.9 and obtain a semicosimplicial $\mathbf{E}_\infty$-ring $Q(N)^\bullet$. Behrens defines $Q(N)$ as the limit of this diagram of $\mathbf{E}_\infty$-rings.

6.2 Spectral Mackey functors

In § 6.1 we extended the functoriality of the sheaf $\mathscr{O}^{\mathrm{top}}$ to include isogenies of elliptic curves of invertible degree. To define stable Hecke operators with as much homotopy coherence as possible, we extend the functoriality of $\mathscr{O}^{\mathrm{top}}$ again, this time to encode the homotopy coherence of certain *transfer maps*. These kinds of coherence and functoriality questions for transfer maps have long been a source of technical complications in homotopy theory, but luckily for us, the rather general framework of [BH21] suits us perfectly; see the proof of Th.6.2.3. First a little set-up; see [Bar17, §5] or [BH21, §C] for details on the following constructions.

Definition 6.2.1. Given an ∞-category $\mathcal{C}$ with pullbacks equipped with a class of morphisms M closed under composition and pullback, then there is an ∞-category $\mathrm{Span}_M(\mathcal{C})$ of *M-spans in $\mathcal{C}$*. The objects of $\mathrm{Span}_M(\mathcal{C})$ are those of $\mathcal{C}$, 1-morphisms from X to Y are spans

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

where f is any map in $\mathcal{C}$ and g belongs to M, and composition is given by pullback. A functors $\mathrm{Span}_M(\mathcal{C}) \to \mathrm{Sp}$ is called *spectral Mackey functors* à la Barwick; see [Bar17].

Informally, a functor $F\colon \mathrm{Span}_M(\mathcal{C}) \to \mathcal{D}$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$ which associates to each $f \in M$ a forward map $f_!$ which satisfies a kind of Beck–Chevalley formula. We are only interested in one example of a span ∞-category.

Definition 6.2.2. Let fin be the collection of morphisms in $\mathrm{Isog}_{\mathrm{Ell}}$ of the form $(f, \alpha)\colon (\mathsf{X}, E) \to (\mathsf{X}', E')$ where $f\colon \mathsf{X} \to \mathsf{X}'$ is a finite morphism of stacks.

The condition above that α be an equivalence means that fin is more accurately a collection of morphisms in the small étale site of $\mathcal{M}_{\mathrm{Ell}}$. We can now state our second extension theorem for $\mathscr{O}^{\mathrm{top}}$ which also appeared as Th.C.

Theorem 6.2.3. *There is a unique functor* $\mathbf{O}^{\mathrm{top}}$ *in the following commutative diagram of* ∞*-categories:*

$$\begin{array}{ccc} \mathrm{Isog}_{\mathrm{Ell}}^{\mathrm{op}} & \xrightarrow{\mathscr{O}^{\mathrm{top}}} & \mathrm{CAlg} \\ \downarrow & & \downarrow \\ \mathrm{Span}_{\mathrm{fin}}(\mathrm{Isog}_{\mathrm{Ell}}) & \xrightarrow{\mathbf{O}^{\mathrm{top}}} & \mathrm{Sp} \end{array}$$

The same holds for $\widehat{\mathrm{Isog}}_{\mathrm{Ell}}$ *and* $\mathscr{O}_p^{\mathrm{top}}$ *at any prime* p.

In particular, for a morphism $(f,\alpha)\colon (\mathsf{X},E) \to (\mathsf{X}',E')$ in fin, we obtain a transfer map of spectra

$$(f,\alpha)_!\colon \mathscr{O}^{\mathrm{top}}(\mathsf{X},E) \to \mathscr{O}^{\mathrm{top}}(\mathsf{X}',E')$$

and this association of transfer maps is functorial and commutes with base change. In particular, this is an $\mathscr{O}^{\mathrm{top}}(\mathsf{X}',E')$-module map.

Proof. Let $\mathcal{C}$ be either $\mathrm{Isog}_{\mathrm{Ell}}$ or $\widehat{\mathrm{Isog}}_{\mathrm{Ell}}$ for a prime p. Our goal is to apply [BH21, Cor.C.13] to the above situation. This means that $\mathcal{C} = \mathcal{C}$, t will be the étale topology, $m = \mathrm{fin}$, $F = \mathscr{O}^{\mathrm{top}}$, and $\mathcal{D} = \mathrm{Sp}$. Let us now check the required hypotheses:

- First, we need to show $\mathcal{C}$ is *extensive*, meaning $\mathcal{C}$ admits finite coproducts which are disjoint[36] and where finite coproduct decompositions are stable under pullback; see [BH21, Df.2.3].
- Next, we want to show $\mathcal{S}\mathrm{hv}^{\text{ét}}(\mathcal{C}) \subseteq \mathcal{S}\mathrm{hv}^{\sqcup}(\mathcal{C})$, where the latter is the ∞-category of sheaves with respect to the Grothendieck topology with covers given by finite fold maps.
- Then we want to see that all finite fold maps are contained in fin, written as fold $\subseteq$ fin.
- Finally, we want to show every morphism in fin is étale locally in fold.

To see the first condition, consider that finite coproducts in $\mathrm{Isog}_{\mathrm{Ell}}$ are given in the stack and elliptic curve variables separately, and it quickly follows that $\mathrm{Isog}_{\mathrm{Ell}}$ is extensive. The second and third conditions are clear, and the fourth is classical and uses that isogenies of invertible degree are in particular étale morphisms; see [Sta, 04HN]. We can now apply [BH21, Cor.C.13] which yields an essentially unique dashed arrow in the commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \xrightarrow{\mathscr{O}^{\mathrm{top}}} & \mathrm{CAlg} \\ \downarrow & & \downarrow \\ \mathrm{Span}_{\mathrm{fin}}(\mathcal{C}) & \overset{\mathbf{O}^{\mathrm{top}}}{\dashrightarrow} & \mathrm{CAlg}^{\times}(\mathrm{Sp}) \end{array}$$

[36] Recall that in an ∞-category with finite coproducts, we say coproducts are *disjoint* if for all objects X, Y, the fibre product $X \times_{X \sqcup Y} Y$ exists and is the initial object.

where $\mathrm{CAlg}^{\times}(\mathrm{Sp})$ is the ∞-category of $\mathbf{E}_{\infty}$-objects in Sp with the Cartesian monoidal structure. This is what we want, even though not aesthetically. Indeed, there are equivalences of ∞-categories

$$\mathrm{CAlg}^{\times}(\mathrm{Sp}) \xrightarrow{\simeq} \mathrm{CAlg}^{\sqcup}(\mathrm{Sp}) \xleftarrow{\simeq} \mathrm{Sp};$$

the first as finite products and coproducts agree in Sp, and the second is an equivalence as every object in a cocomplete ∞-category has an essentially unique $\sqcup$-monoidal structure. ☐

These transfer maps are not so mysterious after taking homotopy groups.

Remark 6.2.4. By [BH21, Cor.C.13], the spectral Mackey functor $\mathbf{O}^{\mathrm{top}}$ of Th.6.2.3 is the right Kan extension of the functor

$$\mathbf{O}^{\mathrm{top}}_{\mathrm{fold}} \colon \mathrm{Span}_{\mathrm{fold}}(\mathrm{Isog}_{\mathrm{Ell}}) \to \mathrm{Sp}$$

whose value on fold maps $\mathsf{X}^{\sqcup n} \to \mathsf{X}$ is given by the addition of spectra map:

$$\mathbf{O}^{\mathrm{top}}_{\mathrm{fold}}(\mathsf{X}^{\sqcup n}) \simeq \mathbf{O}^{\mathrm{top}}_{\mathrm{fold}}(\mathsf{X})^n \to \mathbf{O}^{\mathrm{top}}_{\mathrm{fold}}(\mathsf{X})$$

On homotopy groups, this is also the n-fold addition map. A finite étale morphism $f \colon \mathsf{X} \to \mathsf{Y}$ in the small étale site over $\mathcal{M}_{\mathrm{Ell}}$ is étale locally on Y a fold map, so the induced transfer map

$$f_! = \mathbf{O}^{\mathrm{top}}(\mathsf{X} \xleftarrow{=} \mathsf{X} \xrightarrow{f} \mathsf{Y}) \colon \mathbf{O}^{\mathrm{top}}(\mathsf{X}) \to \mathbf{O}^{\mathrm{top}}(\mathsf{Y})$$

is étale locally on Y given by the sum $\mathbf{O}^{\mathrm{top}}(\mathsf{Y}^{\sqcup n}) \simeq \mathbf{O}^{\mathrm{top}}(\mathsf{Y})^n \to \mathbf{O}^{\mathrm{top}}(\mathsf{Y})$. In the affine case where $\mathsf{X} = \operatorname{Spec} B$ and $\mathsf{Y} = \operatorname{Spec} A$ and the finite étale map $f \colon A \to B$ has constant rank n, then there is an étale cover $\operatorname{Spec} C \to \operatorname{Spec} A$ such that $C \otimes_A B \simeq C^n$. The morphism of $\mathbf{O}^{\mathrm{top}}(A)$-modules $f_! \colon \mathbf{O}^{\mathrm{top}}(B) \to \mathbf{O}^{\mathrm{top}}(A)$ is given étale locally on $\pi_0 \mathbf{O}^{\mathrm{top}}(A) \simeq A$ as the fold map of spectra

$$f'_! \colon \mathbf{O}^{\mathrm{top}}(C^n) \simeq \mathbf{O}^{\mathrm{top}}(C)^n \to \mathbf{O}^{\mathrm{top}}(C)$$

and we can recover $f_!$ as the limit of the Čech nerve of $f'_!$. As étale morphisms of $\mathbf{E}_{\infty}$-rings are nicely behaved on homotopy groups and $\mathscr{O}^{\mathrm{top}}$ (hence also $\mathbf{O}^{\mathrm{top}}$) has controllable homotopy groups on affines, we see that for each integer k the induced map $\pi_k \mathbf{O}^{\mathrm{top}}(B) \to \pi_k \mathbf{O}^{\mathrm{top}}(A)$ induced by $f_!$ is the map of modules over $A = \pi_0 \mathbf{O}^{\mathrm{top}}(A)$ which étale locally looks like a fold map. In fact, this is a characterisation of this map of A-modules; see [AGV$^+$73, Exposé IX, §5] or [Sta, 03SH].

There is not much extra work to extend the Hill–Lawson construction of $\mathscr{O}^{\mathrm{top}}$ on the small log étale site of $\overline{\mathcal{M}}_{\mathrm{Ell}}$ to a spectral Mackey functor. Indeed, just apply [BH21, Cor.C.13] and the fact that log étale morphisms are log étale locally fold maps; see [HL16, Df.2.26]. The same would hold for many variations of Df.6.1.4 over the small log étale site of $\overline{\mathcal{M}}_{\mathrm{Ell}}$.

We will need the following lemma concerning the above transfer maps—the argument is standard; see [HM17, Lm.4.9].

Lemma 6.2.5. *Let $(f,\alpha)\colon (\mathsf{X},E) \to (\mathsf{X}',E')$ be a morphism in* fin *within* $\mathrm{Isog}_{\mathrm{Ell}}$ *and assume f has constant rank d. Suppose that in the descent spectral sequence for $\mathscr{O}^{\mathrm{top}}(\mathsf{X}',E')$, the $E_\infty^{s,s}$ column (abutting to $\pi_0\mathscr{O}^{\mathrm{top}}(\mathsf{X}',E')$) is concentrated in filtration zero, so $E_\infty^{s,s}=0$ if $s>0$. Then there is a homotopy*

$$(f,\alpha)_! \circ (f,\alpha)^* \simeq d\colon \mathbf{O}^{\mathrm{top}}(\mathsf{X}',E') \to \mathbf{O}^{\mathrm{top}}(\mathsf{X}',E')$$

of maps of $\mathbf{O}^{\mathrm{top}}(\mathsf{X}',E')$-modules, where d indicates multiplication by d.

The above homotopy is not necessarily natural in any sense—we only claim that these two maps agree in the homotopy category $\mathrm{hMod}_{\mathbf{O}^{\mathrm{top}}(\mathsf{X}',E')}$.

Remark 6.2.6. The descent spectral sequence condition above applies in many cases of interest. The following three examples cover everything we will use in this thesis.

- If X' is affine the descent spectral sequence collapses on the E_2-page, so the condition holds.
- If $\mathsf{X}'=\mathcal{M}_\Gamma$ is a moduli stack of elliptic curves with Γ-level structure and we further assume that Γ is *tame* as defined in [Mei22, Df.2.2], then by part 4 of [Mei22, Pr.2.5], the descent spectral sequence for $\mathrm{TMF}(\Gamma)$ is concentrated in filtration zero. This applies to the $\mathbf{E}_\infty$-rings $\mathrm{TMF}(n)$ and $\mathrm{TMF}_1(n)$ for $n \geqslant 2$, as well as $\mathrm{TMF}_0(n)$ if we invert $\gcd(6,\phi(n))$, where $\phi(n)$ is *Euler's totient function.*[37]
- If $\mathsf{X}'=\mathcal{M}_{\mathrm{Ell},R}$ where R is any localisation of the integers, then by the calculations of [Bau08] (which applies to TMF as explained in [Kon12] or Chapter 13 of [DFHH14]), we see the E_∞-page of the associated descent spectral sequence is concentrated in filtration zero in the desired column.

We will often use the second condition above, which is why Th.D demands that $\gcd(6,\phi(n))$ is inverted.

Proof of Lm.6.2.5. Let us drop the elliptic curves from our notation for brevity. Let $\operatorname{Spec} A \to \mathsf{X}'$ be an affine étale cover, and as $\mathsf{X}\to\mathsf{X}'$ is finite and hence also affine, we see that $\operatorname{Spec} B \simeq \operatorname{Spec} A \times_{\mathsf{X}'} \mathsf{X}$ is also affine. As $\operatorname{Spec} B \to \operatorname{Spec} A$ is finite étale by base change, there is a further étale cover $\operatorname{Spec} C \to \operatorname{Spec} A$ such that $C \otimes_A B = D$ is isomorphic to C^d and the map $h\colon \operatorname{Spec} D \to \operatorname{Spec} C$ is isomorphic to the canonical fold map. As in Rmk. 6.2.4, it follows from [BH21, Cor.C.13] that $\mathbf{O}^{\mathrm{top}}$ is right Kan extended from $\mathbf{O}^{\mathrm{top}}_{\mathrm{fold}}$, so in particular we see that $h_!\colon \mathbf{O}^{\mathrm{top}}(D) \simeq \mathbf{O}^{\mathrm{top}}(C)^n \to \mathbf{O}^{\mathrm{top}}(C)$ is homotopic to the d-fold addition map of spectra. Moreover, the compatibility of these coherent transfer maps with base change, we see $h_!$ is homotopic d-fold addition map as a map of $\mathbf{O}^{\mathrm{top}}(C)$-modules. This shows that the base-change of $f_! \circ f^*$ along $\operatorname{Spec} C \to \mathsf{X}'$

[37] Recall Euler's totient function $\phi\colon \mathbf{N}\to\mathbf{N}$ is defined as setting $\phi(n)$ to be the number of positive integers less than n which are relatively prime to n. This can be characterised as the multiplicative function satisfying $\phi(p^e)=p^e-p^{e-1}$ for primes p and $e \geqslant 1$, with $\phi(1)=1$.

is homotopic to multiplication by d.

Write $A \to C^\bullet$ for the Čech nerve of the étale cover $A \to C$. The descent spectral sequence for $\mathbf{O}^{\mathrm{top}}(\mathsf{X}') = \mathcal{E}$ also abuts to the internal mapping spectrum $F_{\mathcal{E}}(\mathcal{E}, \mathcal{E}) \simeq \mathcal{E}$. Recall there is an edge map $e_r \colon \pi_0 \mathcal{E} \to E_r^{0,0}$ to the E_r-page for all $1 \leqslant r \leqslant \infty$. As this spectral sequence is concentrated in the upper half-plane, there are inclusions $E_{r+1} \to E_r$ which commute with these edge maps, hence we have the following commutative diagram of abelian groups:

$$\begin{array}{ccccccc} & & & & & & \pi_0 F_{\mathcal{E}}(\mathcal{E}, \mathcal{E}) \\ & & {\scriptstyle e_\infty} \swarrow & & {\scriptstyle e_2} \swarrow & & \downarrow {\scriptstyle e_1} \\ E_\infty^{0,0} & \longrightarrow & \cdots & \longrightarrow & E_2^{0,0} & \longrightarrow & E_1^{0,0} \end{array}$$

By assumption, the map e_∞ is an equivalence, and as the horizontal maps above are injective, it suffices to show that the image of $f_! \circ f^*$ and the multiplication by d map under e_1 agree. The E_1-page of the descent spectral sequence is given by the groups $\pi_t \mathcal{E}^s$ where $\mathcal{E}^s = \mathbf{O}^{\mathrm{top}}(C^s)$, which can also be written in terms of internal mapping spectra $\mathcal{E}^s \simeq F_{\mathcal{E}^s}(\mathcal{E}^s, \mathcal{E}^s)$. Under this identification, the map

$$e_1 \colon \pi_0 F_{\mathcal{E}}(\mathcal{E}, \mathcal{E}) \to \pi_0 F_{\mathcal{E}^0}(\mathcal{E}^0, \mathcal{E}^0)$$

is induced by base-change along $\mathcal{E} \to \mathcal{E}^0 = \mathbf{O}^{\mathrm{top}}(C)$, hence $f_! \circ f^*$ (resp. the multiplication by d map on $\mathcal{E}$) is sent to $h_! \circ h^*$ (resp. the multiplication by d map on $\mathcal{E}^0$). The first paragraph of this proof shows these maps agree up to homotopy, and hence agree in $\pi_0 F_{\mathcal{E}^0}(\mathcal{E}^0, \mathcal{E}^0)$ on the nose. $\square$

Chapter 7

Our stable operators

In this chapter, we use the two structural statements (Th.6.1.9 and Th.6.2.3) proven in the previous chapter to define a variety of stable operations on TMF. We start with Adams operations ψ^n, followed by a detailed look at Hecke operators T_n, and we finish with Atkin–Lehner involutions w_Q and some discussions towards extensions to Tmf.

7.1 Stable Adams operations

There are only two isogenies of the universal elliptic curve $\mathscr{E}$ over $\mathcal{M}_{\mathrm{Ell}}$ of invertible degree: the identity and the inversion map $[-1]$. If we invert n, then the n-fold multiplication map $[n]\colon \mathscr{E} \to \mathscr{E}$ is also an isogeny of invertible degree over $\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}$, as it has degree is n^2 ([KM85, Th.2.3.1]). This gives us our first family of morphisms in $\mathrm{Isog}_{\mathrm{Ell}}$.

Definition 7.1.1. Let n be an integer. Define the *nth stable Adams operation* ψ^n on $\mathrm{TMF}[\frac{1}{n}]$ by applying $\mathscr{O}^{\mathrm{top}}$ (Th.6.1.9) to the following morphism inside $\mathrm{Isog}_{\mathrm{Ell}}$:

$$(\mathrm{id}, [n])\colon (\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \mathscr{E}) \to (\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \mathscr{E})$$

Recall Df.5.5.12, where we defined p-adic Adams operations on TMF_p.

One might have been able to squeeze out the above operations on TMF using Goerss–Hopkins obstruction theory, but this approach would lack the functoriality of our construction above. For instance, using Df.7.1.1, one immediately obtains the following statement.

Theorem 7.1.2. *The maps of* $\mathbf{E}_\infty$*-rings* ψ^1 *and* ψ^{-1} *are both homotopic to the identity morphism on* TMF. *Given two integers* m *and* n*, then there are the following natural homotopies of morphisms of* $\mathbf{E}_\infty$*-rings:*

$$\psi^m \circ \psi^n \simeq \psi^{mn} \simeq \psi^n \circ \psi^m\colon\ \mathrm{TMF}[\frac{1}{mn}] \to \mathrm{TMF}[\frac{1}{mn}]$$

Given a prime p and two p-adic units $\ell, k \in \mathbf{Z}_p^\times$, then there is the following natural homotopies of $\mathbf{E}_\infty$-rings:

$$\psi^\ell \circ \psi^{\ell k} \simeq \psi^k \simeq \psi^k \circ \psi^\ell \colon \mathrm{TMF}_p \to \mathrm{TMF}_p$$

By *natural homotopy*, we mean that there is a natural choice of homotopy $H_{m,n} \colon \psi^m \circ \psi^n \simeq \psi^n \circ \psi^m$ for the above, and given another integer k, there is a natural higher homotopy between $\psi^k \star H_{m,n}$ and $H_{k,m} \star \psi^n$, where $\star$ denotes whiskering, and another pair of natural homotopies between these homotopies and $H_{k,m,n}$. These are all a consequence of applying $\mathcal{O}^{\mathrm{top}}$ to higher morphisms in $\mathrm{Isog}_{\mathrm{Ell}}^{\text{ét}}$ which is a 2-category where these higher morphisms are given by equalities.

Proof. All of the above follows from the functoriality of $\mathcal{O}^{\mathrm{top}}$ and $\mathcal{O}_p^{\mathrm{top}}$, but let us explain the identification of ψ^{-1} in more detail.[38] The trivial C_2-action $(\mathrm{id}, \mathrm{id})$ of $(\mathcal{M}_{\mathrm{Ell}}, \mathscr{E})$ is homotopic to the C_2-action $(\mathrm{id}, [-1])$ of $(\mathcal{M}_{\mathrm{Ell}}, \mathscr{E})$ in $\mathrm{Isog}_{\mathrm{Ell}}^{\text{ét}}$. In fact, this homotopy exists in the small étale site of $\mathcal{M}_{\mathrm{Ell}}$, and is given by the invertible 2-morphism $[-1] \colon (\mathrm{id}, \mathrm{id}) \to (\mathrm{id}, [-1])$. Applying $\mathcal{O}^{\mathrm{top}}$ we obtain a natural homotopy $\mathrm{id} \simeq \psi^{-1} \colon \mathrm{TMF} \to \mathrm{TMF}$. □

Recall that for every positive integer N and each subgroup $\Gamma \leqslant \mathbf{GL}_2(\mathbf{Z}/N\mathbf{Z})$, called a *congruence subgroup of level N*, there is a moduli stack $\mathcal{M}(\Gamma)$ of elliptic curves with Γ-*level structure*; see [KM85] for more background on the cases we are interested in, where $\Gamma = \Gamma(N)$, $\Gamma_1(N)$, or $\Gamma_0(N)$. The structure morphism $\mathcal{M}(\Gamma) \to \mathcal{M}_{\mathrm{Ell}, \mathbf{Z}[\frac{1}{N}]}$ is étale, hence we can define $\mathrm{TMF}(\Gamma) = \mathcal{O}^{\mathrm{top}}(\mathcal{M}(\Gamma))$. If Γ is one of the classical congruence subgroups $\Gamma(N)$, $\Gamma_1(N)$, or $\Gamma_0(N)$, then we will write $\mathrm{TMF}(N)$, $\mathrm{TMF}_1(N)$, and $\mathrm{TMF}_0(N)$, respectively. Note that $\mathrm{TMF}(\Gamma)$ is by definition an $\mathbf{E}_\infty$-$\mathrm{TMF}[\frac{1}{N}]$-algebra.

Definition 7.1.3. Given integer n and N with $N \geqslant 1$, then the n-fold multiplication map $[n]$ on the universal elliptic curve $\mathscr{E}(\Gamma)$ over $\mathcal{M}(\Gamma)$ is an isogeny of invertible degree. We define the *nth stable Adams operation*

$$\psi^n \colon \mathrm{TMF}(\Gamma)[\frac{1}{n}] \to \mathrm{TMF}(\Gamma)[\frac{1}{n}]$$

as $\mathcal{O}^{\mathrm{top}}$ applied to the endomorphism $(\mathrm{id}, [n])$ of $(\mathcal{M}(\Gamma) \otimes \mathbf{Z}[\frac{1}{n}], \mathscr{E}(\Gamma))$.

The functoriality of this definition, combined with the fact that $\mathscr{E}(\Gamma)$ is the pullback along $\mathcal{M}(\Gamma) \to \mathcal{M}_{\mathrm{Ell}}$ of the universal elliptic curve, we see the following diagram of $\mathbf{E}_\infty$-rings commutes up to natural homotopy:

$$\begin{array}{ccc} \mathrm{TMF}[\frac{1}{n}] & \xrightarrow{\psi^n} & \mathrm{TMF}[\frac{1}{n}] \\ \downarrow & & \downarrow \\ \mathrm{TMF}(\Gamma)[\frac{1}{n}] & \xrightarrow{\psi^n} & \mathrm{TMF}(\Gamma)[\frac{1}{n}] \end{array}$$

[38] We learned this argument from Lennart Meier—see [Mei22, Ex.6.12] for the same argument with respect to $\mathcal{M}_{\mathrm{Ell}}$ with level structures. The added generality of level structures produces a less opaque proof, in our opinion.

The same can be said in the p-complete world for the stable p-adic Adams operations.

7.2 Stable Hecke operators

A vitally important tool in the study of classical modular forms is the existence of Hecke operators. As stated by Zagier in Section 2 of Chapter 4 in [WMLI92]:

The key to the rich internal structure of the theory of modular forms is the existence of (...) operators T_n *(...).*

Our goal of this section is then to introduce a collection of stable Hecke operators on the $\mathbf{E}_\infty$-ring TMF. First, let us define the stacks that we need for our construction.

For any positive integer n, write $\mathcal{M}_n$ for the moduli stack over $\mathbf{Z}[\frac{1}{n}]$ of elliptic curves E with a chosen finite subgroup H of order n. Note that $\mathcal{M}_n$ is **not** the same as $\mathcal{M}_0(n)$ unless n is squarefree. Let us write $p, q\colon \mathcal{M}_n \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}$ for the maps of stacks defined by $p(E,H) = E$ and $q(E,H) = E/H$; see §7.3 for more about these stacks, including the fact that both p and q are étale.

Definition 7.2.1. Let n be a positive integer. Define the *nth stable Hecke operator* $\mathrm{T}_n\colon \mathrm{TMF}[\frac{1}{n}] \to \mathrm{TMF}[\frac{1}{n}]$ by applying $\mathbf{O}^{\mathrm{top}}$ (Th.6.2.3) to the span

$$(\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \mathscr{E}) \xleftarrow{(q,Q)} (\mathcal{M}_n, \mathscr{E}_n) \xrightarrow{(p,\mathrm{can})} (\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \mathscr{E})$$

where $Q\colon \mathscr{E}_n \to \mathscr{E}_n/\mathcal{H}$ is the canonical quotient map of the universal elliptic curve over $\mathcal{M}_n$ by the universal order n subgroup and can is the canonical equivalence $\mathscr{E}_n \simeq p^*\mathscr{E}$. In other words, T_n is given by pullback along (q,Q) followed by the transfer $p_!$ along (p,can). Make the same definition for $\mathrm{T}_n\colon \mathrm{TMF}(\Gamma) \to \mathrm{TMF}(\Gamma)$ by pulling back the above span along the structure map $\mathcal{M}_\Gamma \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}$, as long as the level N of Γ is coprime to n; see the discussion before the proof of Pr.7.5.4 for more details.

Warning 7.2.2. The stable Hecke operators above differ by the unit $\frac{1}{n}$ from the conventional Hecke operators upon taking homotopy groups; see Pr.7.5.3. This is by design though, as we find division by n to be an unnecessary extra step and we prefer the formula $\mathrm{T}_n(1) = \sigma(n)$ to $\mathrm{T}_n^{\mathrm{alg}}(1) = \frac{\sigma(n)}{n}$ implied by Pr.7.5.3, where $\sigma(n)$ is the divisor function.[39] None of the results of this section depend on the choice to divide by n or not, however, one should be careful when considering the calculations made Part III.

[39]Recall the *generalised divisor function* $\sigma_k(-)\colon \mathbf{N} \to \mathbf{N}$ for a nonnegative integer k is defined by the formula $\sigma_k(n) = \sum_{d|n} d^k$ ranging over all positive divisors d of n. When $k = 1$ we drop it from our notation.

Remark 7.2.3. Inspired by Venkatesh [Ven19], we would like to define *derived stable Hecke operators* for each positive integer n and each $\alpha \in \pi_d \operatorname{TMF}_n$, where $\operatorname{TMF}_n = \mathscr{O}^{\mathrm{top}}(\mathcal{M}_n)$, by the following composite:

$$\operatorname{TMF}[\frac{1}{n}][d] \xrightarrow{q^*} \operatorname{TMF}_n[d] \xrightarrow{\cdot\alpha} \operatorname{TMF}_n \xrightarrow{p_!} \operatorname{TMF}[\frac{1}{n}]$$

These have the conceptual advantage of being stable cohomology operations of degree d and may relate to a kind of *derived spectral Hecke algebra*. This is currently wild speculation which the author plans to follow up on. Such operations will not be discussed elsewhere in this thesis.

The stable Hecke operators on TMF (and $\operatorname{TMF}(\Gamma)$) have a nice relationship with the stable Adams operations of §7.1.

Proposition 7.2.4. *Let m, n be integers with $n \geqslant 1$. Then we have the following natural homotopy of morphisms of spectra:*

$$\mathrm{T}_n \circ \psi^m \simeq \psi^m \circ \mathrm{T}_n \colon \operatorname{TMF}[\frac{1}{mn}] \to \operatorname{TMF}[\frac{1}{mn}]$$

Similarly, if p is a prime not dividing n and k is a p-adic unit, we have the following natural homotopy of morphisms of spectra:

$$\mathrm{T}_n \circ \psi^k \simeq \psi^k \circ \mathrm{T}_n \colon \operatorname{TMF}_p \to \operatorname{TMF}_p$$

The same holds for stable Hecke operators with Γ-level structure where the level N of Γ is coprime to n.

This proposition implies that the C_2-equivariant version of $\operatorname{TMF}(\Gamma)$ studied in [HM17] carry stable Hecke operators. Similarly, the height two periodic Adams summand U and the height two periodic image of J spectrum S of §10.3 and §10.4, respectively, also carry stable Hecke operators.

Proof. Both T_n and ψ^m are the application of $\mathbf{O}^{\mathrm{top}}$ to morphisms in $\mathrm{Isog}_{\mathrm{Ell}}$, so by functoriality of $\mathbf{O}^{\mathrm{top}}$, it suffices to provide a natural homotopy between the two spans in $\mathrm{Span}_{\mathrm{fin}}(\mathrm{Isog}_{\mathrm{Ell}})$

$$\begin{array}{ccccccc} & & \mathcal{M}_n & & \mathcal{M} & & \\ & \swarrow^{(q,Q)} & & \searrow_{(p,\mathrm{can})} \; \swarrow^{(\mathrm{id},[m])} & & \diagdown\!\!\diagdown & \\ \mathcal{M} & & & \mathcal{M} & & & \mathcal{M} \end{array} \tag{7.2.5}$$

$$\begin{array}{ccccccc} & & \mathcal{M} & & \mathcal{M}_n & & \\ & \swarrow^{(\mathrm{id},[m])} & & \diagdown\!\!\diagdown \; \swarrow^{(q,Q)} & & \searrow^{(p,\mathrm{can})} & \\ \mathcal{M} & & & \mathcal{M} & & & \mathcal{M} \end{array} \tag{7.2.6}$$

where $\mathcal{M}$ is an abbreviation for $\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{mn}]}$ and we have suppressed the elliptic curves in our notation for objects in $\mathrm{Isog}_{\mathrm{Ell}}$. Recall that to compose morphisms in $\mathrm{Span}_{\mathrm{fin}}(\mathrm{Isog}_{\mathrm{Ell}})$ we take fibre products. The pullbacks of both (7.2.5) and

(7.2.6) are easily identified with $\mathcal{M}_n$, and the associated morphisms on the right are both naturally equivalent to the map $(p, \mathrm{can})\colon \mathcal{M}_n \to \mathcal{M}$. We are left to compare the two compositions of the left legs, which involves contemplating the following diagram in $\mathrm{Isog}_{\mathrm{Ell}}$:

$$\begin{array}{ccc} \mathcal{M}_n & \xrightarrow{(\mathrm{id},[m])} & \mathcal{M}_n \\ {\scriptstyle (q,Q)}\downarrow & & \downarrow{\scriptstyle (q,Q)} \\ \mathcal{M} & \xrightarrow{(\mathrm{id},[m])} & \mathcal{M} \end{array}$$

The upper-right composition comes from (7.2.5) and the lower-left composition from (7.2.6). Unwinding the definitions, the fact that there exists a natural 2-cell in the above diagram boils down to the fact that the quotient map $Q\colon \mathscr{E}_n \to \mathscr{E}_n/\mathcal{H}_n$ of the universal elliptic curve over $\mathcal{M}_n$ commutes with multiplication by m, which is clear as Q is a homomorphism of elliptic curves.

The proof for stable Hecke operators on $\mathrm{TMF}(\Gamma)$ is the same after pullback along $\mathcal{M}_\Gamma \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{nN}]}$. The p-adic proof is also the same, as isogenies of elliptic curves of invertible degree also induce homomorphisms on their associated p-divisible groups. $\square$

We showed that there exists a homotopy $\psi^m \circ \psi^n \simeq \psi^{mn}$ as endomorphisms of $\mathrm{TMF}[\frac{1}{mn}]$. Most of the rest of this chapter is now focused on proving the analogue for stable Hecke operators.

Theorem 7.2.7. *Let m and n be positive integers. Then there is a homotopy of morphisms of spectra*

$$\mathrm{T}_m \circ \mathrm{T}_n \simeq \sum_{d|m,n} d\psi^d \mathrm{T}_{\frac{mn}{d^2}}\colon \mathrm{TMF}[\frac{1}{mn\phi}] \to \mathrm{TMF}[\frac{1}{mn\phi}]$$

where $\phi = \gcd(6, \phi(mn))$ and $\phi(mn)$ is Euler's totient function. The above sum ranges over those positive integers d dividing both m and n. In particular, $\mathrm{T}_m \circ \mathrm{T}_n$ is homotopic[40] *to $\mathrm{T}_n \circ \mathrm{T}_m$, and if $\gcd(m, n) = 1$ then both are homotopic to T_{mn}.*

As one will see in our proof of the above theorem, the homotopy above is close to being natural. It will end up being a composition of a bunch of natural homotopies and one non-canonical homotopy—the singular non-canonical homotopy comes from Lm.6.2.5. We do not claim it is impossible to obtain a natural homotopy, but this seems beyond our reach for now.

Unlike in the classical world of modular forms, we cannot prove the above theorem by calculating q-expansions. Instead, we show that in $\mathrm{Isog}_{\mathrm{Ell}}$ the stacks involved in the definition of the composite of T_m and T_n split (Th.7.4.1), and one then obtains Th.7.2.7 by applying $\mathscr{O}^{\mathrm{top}}$ to this splitting and carefully analysing the result.

[40] As to-be mentioned in Rmk.7.4.8, the author has recently proven that there is a homotopy between $\mathrm{T}_m \circ \mathrm{T}_n$ and $\mathrm{T}_n \circ \mathrm{T}_m$ over $\mathrm{TMF}[\frac{1}{mn}]$.

7.3 Moduli stacks of subgroups

We need to define and discuss the stacks used in Df.7.2.1 in more detail as well as some auxiliary objects. We refer the reader to [KM85, §1] for the background on moduli problems for subgroups of curves of a given type, and particularly [KM85, §1.5] for what it means for a subgroup of an elliptic curve to have a type of a finite abelian group A.

Definition 7.3.1. Let A, B be finite abelian groups and $d, e, m, n \geqslant 1$ be positive integers.

1. Let $\mathcal{M}_A$ denote the *moduli stack of elliptic curves with chosen subgroup H of type A* over $\mathbf{Z}[\frac{1}{|A|}]$. In particular, we have $\mathcal{M}_{C_n} = \mathcal{M}_0(n)$, where C_n is the cyclic group of order n. We will also write $\mathcal{M}_{(m,n)}$ for $\mathcal{M}_{C_m \times C_n}$.

2. Let $\mathcal{M}_n$ denote the *moduli stack of elliptic curves with chosen étale subgroup of order n* over $\mathbf{Z}[\frac{1}{n}]$.

3. Let $\mathcal{M}_{A \leqslant B}$ denote the *moduli stack of elliptic curves with a chosen pair of nested subgroups $K \leqslant H$ of type A and B, respectively*, over $\mathbf{Z}[\frac{1}{|A| \cdot |B|}]$. If $A = C_d \times C_m$ and $B = C_e \times C_n$, then we will write

$$\mathcal{M}_{C_d \times C_m \leqslant C_e \times C_n} = \mathcal{M}_{(d,m) \leqslant (e,n)}.$$

4. Let $\mathcal{M}_{m \leqslant n}$ denote the *moduli stack of elliptic curves with a chosen pair of nested subgroups $K \leqslant H$ of order m and n, respectively*, over $\mathbf{Z}[\frac{1}{n}]$.

In the definitions above, we leave open the possibility that these moduli stacks are empty. For example, if A is not isomorphic to a finite subgroup of $S^1 \times S^1$ in part 1, or $m \nmid n$ in part 4. Each of the stacks above come with structure maps p to $\mathcal{M}_{\text{Ell}}$, and those of the form $\mathcal{M}_A$ and $\mathcal{M}_n$ also come with quotient maps q to $\mathcal{M}_{\text{Ell}}$ which quotient by the given subgroup.

We will spend the rest of this section discussing some basic facts about these stacks, starting with something simple.

Proposition 7.3.2. *Let $m, n \geqslant 1$ be positive integers. There are the following decompositions of stacks, first over $\mathbf{Z}[\frac{1}{n}]$ and then over $\mathbf{Z}[\frac{1}{mn}]$:*

$$\mathcal{M}_n \simeq \coprod_{|A|=n} \mathcal{M}_A \qquad \mathcal{M}_{m \leqslant n} \simeq \coprod_{\substack{|A|=m \\ |B|=n}} \mathcal{M}_{A \leqslant B}$$

Proof. This follows straight from the definitions. □

Let us show these stacks lie in Isog_{Ell}.

Proposition 7.3.3. *All of the stacks of Df.7.3.1 are Deligne–Mumford stacks. All of the structure p and quotient q morphisms to $\mathcal{M}_{\text{Ell}}$ are finite étale.*

Proof. Fix two finite abelian groups A and B. By [KM85, Pr.1.6.4], we see that for a fixed scheme S and an S-point E inside $\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{|A|}]}(S)$, the moduli scheme of A-structures on E is finite étale (although potentially empty) over S. Therefore the structure map $p\colon \mathcal{M}_A \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{|A|}]}$ is finite étale and $\mathcal{M}_A$ is a Deligne–Mumford stack. The same argument used to prove [KM85, Pr.1.6.4] shows that the moduli schemes of $A \leqslant B$-structures on elliptic curves are finite étale over S, and we see the structure map $p\colon \mathcal{M}_{A\leqslant B} \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{|A|\cdot|B|}]}$ is finite étale and $\mathcal{M}_{A\leqslant B}$ is also Deligne–Mumford. This covers the stacks and structure maps in parts 1 and 3 of Df.7.3.1 and parts 2 and 4 follow by the decompositions of Pr. 7.3.2. To see the quotient morphisms are étale, note there exists an involution of stacks

$$\tau\colon \mathcal{M}_A \to \mathcal{M}_A \qquad (E,H) \mapsto (E/A, H^\vee)$$

and that the quotient maps $\mathcal{M}_A \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{|A|}]}$ are a composition of τ with the finite étale projections. Back to τ, we have written $H^\vee$ for the kernel of the isogeny $E/A \to E$ dual to the quotient isogeny, which can be identified with the Cartier dual of H by [KM85, (2.8.2.1)]. Notice that $H^\vee$ has the same type as H as are working where $|A|$ is invertible. We also see that τ^2 is naturally equivalent to the identity on $\mathcal{M}_A$ as the composition of isogenies $E \to E/A \to E$ is equal to the $|A|$-fold multiplication map ([KM85, Th.2.6.1]) which induces a natural equivalence $E/E[|A|] \simeq E$, and $(H^\vee)^\vee$ is naturally equivalent to H. □

We will now focus on some relations between the stacks defined above. The first relation states one can shave off common factors in products of cyclic groups.

Proposition 7.3.4. *Let $a, b, c \geqslant 1$ be integers. The morphism of stacks over* $\mathbf{Z}[\frac{1}{abc}]$

$$\mathcal{M}_{(ab,ac)} \to \mathcal{M}_{(b,c)} \qquad (E,H) \mapsto (E,[a]H)$$

is an equivalence whose inverse is given by the following map:

$$\mathcal{M}_{(b,c)} \to \mathcal{M}_{(ab,ac)} \qquad (E,K) \mapsto (E,[a]^*K)$$

To be clear, $[a]H$ is the image of H in E under the a-fold multiplication map $[a]$ on E, and $[a]^*K$ is the pullback of a subgroup $K \leqslant E$ along $[a]$.

Proof. To justify that the second map is well-defined, note that when given a pair (E,K) inside $\mathcal{M}_{(b,c)}(S)$ for a scheme S, we have the following commutative diagram of schemes:

$$\begin{array}{ccccc} E[a] & \longrightarrow & [a]^*K & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle [a]} & & \downarrow{\scriptstyle [a]} \\ S & \longrightarrow & K & \longrightarrow & E \end{array}$$

The right and outer rectangles are Cartesian by definition, hence the left square is also Cartesian, so we have the short exact sequence of finite étale group schemes over S

$$0 \to E[a] \to [a]^*K \xrightarrow{[a]} K \to 0$$

occurring inside $E[a^2bc]$. Suppose for a moment that S is connected. Choosing a geometric point $s\colon \operatorname{Spec}\kappa \to S$, consider the equivalence of categories between finite étale commutative group schemes over S and finite abelian groups with an action of $\pi_1^{\text{ét}}(S,s)$; see [Sta, 03VD] or [Gro03]. From this equivalence, we see that $[a]^*K$ must have type $C_{ab} \times C_{ac}$, as that is the unique subgroup of $C_{a^2bc} \times C_{a^2bc}$ whose image under the a-fold multiplication map is $C_b \times C_c$. For a general scheme S we may apply this argument over each connected component, and we see that $[a]^*K$ has the correct type.

It is clear that $[a][a]^*K$ is isomorphic to K, given a point (E,K) in $\mathcal{M}_{(b,c)}(S)$ for any scheme S. Conversely, take a point (E,H) in $\mathcal{M}_{(ab,ac)}(S)$ for a general scheme S. The finite étale subscheme $[a]^*[a]H$ of E is defined by the following Cartesian diagram of schemes:

$$\begin{array}{ccc} [a]^*[a]H & \longrightarrow & [a]H \\ \downarrow & & \downarrow \\ E & \xrightarrow{[a]} & E \end{array}$$

There is also a natural diagram of schemes

$$\begin{array}{ccccc} H & \longrightarrow & [a]H & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{[a]} & E & \longrightarrow & E/[a]H \end{array} \tag{7.3.5}$$

so it suffices to see the left square above is Cartesian. The right square is Cartesian by inspection, so it suffices to show the whole rectangle above is Cartesian, ie, we need to show the kernel of the composition

$$E \xrightarrow{[a]} E \to E/[a]H$$

is exactly H. There is a short exact sequence of finite étale group schemes over S

$$0 \to E[a] \to H \xrightarrow{[a]} [a]H \to 0$$

from the definition of $[a]H$. Indeed, by definition the latter map is surjective whose kernel K is contained in $E[a]$, and as $E[a] \leqslant H$ as H has type $C_{ab} \times C_{ac}$, we see $K = E[a]$. In particular, this short exact sequence yields an isomorphism $H/E[a] \xrightarrow{[a],\simeq} [a]H$ compatible with the classical isomorphism $E/E[a] \xrightarrow{[a],\simeq} E$. We then identify the bottom composite of (7.3.5) as the following composite of quotient maps:

$$E \to E/E[a] \to (E/E[a])/(H/E[a])$$

By one of the numbered isomorphism theorems from algebra, the above is naturally equivalent to the quotient map $E \to E/H$, and we are done. □

Our second relation states that the moduli stacks $\mathcal{M}_A$ are not much of a generalisation of the stacks $\mathcal{M}_0(n)$ after all.

Proposition 7.3.6. *Let A be a finite abelian group and denote the order of A by n. If A is not isomorphic to a subgroup of $C_n \times C_n$, then $\mathcal{M}_A = \varnothing$, and otherwise there exists a unique positive integer d such that $A \simeq C_d \times C_{\frac{n}{d}}$ and $d^2|n$. In this case, the equivalence of Pr.7.3.4 takes the following form:*

$$\mathcal{M}_A \simeq \mathcal{M}_{(d,\frac{n}{d})} \to \mathcal{M}_{(1,\frac{n}{d^2})} = \mathcal{M}_0\left(\frac{n}{d^2}\right)$$

Proof. The first statement is clear. For the second statement, it is clear there exists an d such that $A \simeq C_d \times C_{\frac{n}{d}}$. The minimal such d has the property that $\gcd(d, \frac{n}{d}) = d$ which is equivalent to the condition that $d^2|n$. The last statement is a special case of Pr.7.3.4 with $a = d$, $b = 1$, and $c = \frac{n}{d}$. □

To any of these stacks of Df.7.3.1, we can associate a stable Hecke operator on TMF, which will come in handy when proving Th.7.2.7.

Definition 7.3.7. Write $\mathcal{M}_\Gamma$ for any of the stacks of Df.7.3.1 defined by a single subgroup, $p_\Gamma, q_\Gamma\colon \mathcal{M}_\Gamma \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}$ for the associated structure and quotient maps, respectively, $\mathscr{E}_\Gamma$ for the universal elliptic curve over $\mathcal{M}_\Gamma$ which is canonically isomorphic to $p_\Gamma^*\mathscr{E}$, and $Q_\Gamma\colon \mathscr{E}_\Gamma \to \mathscr{E}_\Gamma/\mathcal{H}_\Gamma$ for the quotient by the universal subgroup. We define the *stable* Γ*-Hecke operator* $\mathrm{T}_\Gamma\colon \mathrm{TMF}[\frac{1}{n}] \to \mathrm{TMF}[\frac{1}{n}]$ as the image under $\mathbf{O}^{\mathrm{top}}$ of the following span:

$$(\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \mathscr{E}) \xleftarrow{(q_\Gamma, Q_\Gamma)} (\mathcal{M}_\Gamma, \mathscr{E}_\Gamma) \xrightarrow{(p_\Gamma,\mathrm{can})} (\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \mathscr{E})$$

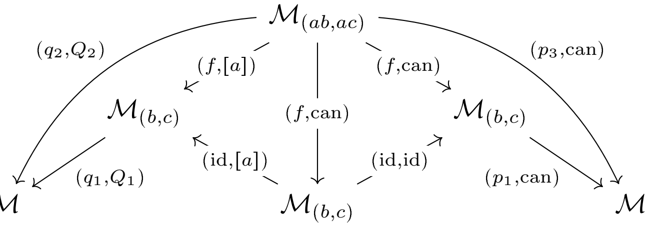
When $\mathcal{M}_\Gamma = \mathcal{M}_{(a,b)}$ we will write $\mathrm{T}_{(a,b)}\colon \mathrm{TMF}_{(a,b)} \to \mathrm{TMF}_{(a,b)}$.

A corollary of the identifications above are the following canonical identifications of varying types of stable Hecke operators—we will only list those we will use later.

Corollary 7.3.8. *Let a, b, c be positive integers. Then we have the following natural homotopies between endomorphisms of* $\mathrm{TMF}[\frac{1}{a}]$ *and* $\mathrm{TMF}[\frac{1}{abc}]$*, respectively:*

$$\sum_{d^2|a} \mathrm{T}_{(d,\frac{a}{d})} \simeq \mathrm{T}_a \qquad\qquad \mathrm{T}_{(ab,ac)} \simeq \psi^a \circ \mathrm{T}_{(b,c)}$$

Proof. The first statement follows directly from Pr.7.3.2, so let us move on to the second. Consider the following diagram in $\mathrm{Isog}_{\mathrm{Ell}}$:



(7.3.9)

Above we have suppressed the universal elliptic curves from our notation and abbreviated $\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{abc}]}$ as $\mathcal{M}$, the maps (q_i, Q_i) and (p_i, can) are the obvious maps, and f is the equivalence of Pr. 7.3.4. Both of the two middle triangles tautologically commute. It is also tautological that the right-most region commutes, so we are left with the left-most region. This commutes by inspection:

1. First, take a point (E, H) inside $\mathcal{M}_{(ab,ac)}(S)$ for some fixed scheme S. Applying $q_1 \circ f$ yields $(q_1 \circ f)(E, H) = E/[a]H$. As in the proof of Pr. 7.3.4, $[a]H$ fits into the exact sequence of finite étale commutative groups schemes over S

$$0 \to E[a] \to H \xrightarrow{[a]} [a]H \to 0$$

which provides an isomorphism $[a]H \simeq H/E[a]$ compatible with the identification of E with $E/E[a]$. Using this, we obtain the first natural isomorphism

$$E/[a]H \xleftarrow{\simeq,[a]} \frac{E/E[a]}{H/E[a]} \simeq E/H$$

and the second natural isomorphism above is one of the usual numbered isomorphism theorems from algebra. This yields a canonical isomorphism $q_1 \circ f \simeq q_2$.

2. Next, we consider the maps of universal elliptic curves. This is comparing the quotient $Q_2 \colon \mathscr{E} \to \mathscr{E}/\mathcal{H}$ of the universal elliptic curve over $\mathcal{M}_{(ab,ac)}$ by the universal subgroup $\mathcal{H}$ with the following composite:

$$\mathscr{E} \xrightarrow{[a]} \mathscr{E} \to \mathscr{E}/[a]\mathcal{H}$$

As in the proof of Pr.7.3.4, we can naturally identify the above composite with Q_2, which shows the morphisms of universal elliptic curves are also equivalent.

Unwinding the definitions, we see the two paragraphs above imply the left-most region of (7.3.9) commutes up to natural equivalence. The fact that the vertical arrow (f, can) in (7.3.9) is an equivalence by Pr.7.3.4 shows the span defining $\mathrm{T}_{(ab,ac)}$ is naturally equivalent to the lower span in (7.3.9). The same argument as in the proof of Pr.7.2.4 shows this second span is naturally equivalent to the composition of spans defining $\psi^a \circ T_{(b,c)}$. This finishes the proof. □

The last two statements in this section are purely combinatorial but are key ingredients in our proof of Th.7.2.7.

Proposition 7.3.10. *Let $m, n \geqslant 1$ be positive integers and $d, e \geqslant 1$ be positive integers such that $d^2|m$ and $e^2|mn$.*

1. *If $d \nmid e$, then $\mathcal{M}_{(d,\frac{m}{d})\leqslant(e,\frac{mn}{e})} = \varnothing$. If both $m \nmid de$ and $e \nmid dn$, then this moduli stack is also empty.*
2. *If $d|e|dn$ and $m \nmid de$, then the structure map*
$$\mathcal{M}_{(d,\frac{m}{d})\leqslant(e,\frac{mn}{e})} \to \mathcal{M}_{(e,\frac{mn}{e})} \tag{7.3.11}$$
is a finite étale surjection of degree $\frac{e}{d}$.
3. *If $d|e$ and $m|de$, then the structure map (7.3.11) is a finite étale surjection of degree $\psi\left(\frac{m}{d^2}\right)$ where ψ denotes the* Dedekind ψ function.[41]

Proof. For part 1, note that if an injection of groups

$$i\colon C_d \times C_{\frac{m}{d}} \to C_e \times C_{\frac{mn}{e}}$$

exists, then either $d|e$ or $d|\frac{mn}{e}$, and either $\frac{m}{d}|e$ or $\frac{m}{d}|\frac{mn}{e}$. This last condition immediately reveals that either $m|de$ or $e|dn$, so now we will focus on if d divides e or not. Suppose $d \nmid e$, then y_1 from $i(1,0) = (x_1, y_1)$ must generate the standard cyclic subgroup of $C_{\frac{mn}{e}}$ of order d. Likewise, as $d \nmid e$ then $\frac{m}{d} \nmid e$, and y_2 from $i(0,1) = (x_2, y_2)$ must generate the standard cyclic subgroup of order $\frac{m}{d}$. As $d|\frac{m}{d}$, we see there is an a such that $ay_1 = y_2$. Our assumption that $d \nmid e$ implies $(e, -ae)$ is nonzero in $C_d \times C_{\frac{m}{d}}$, however, we have

$$i(e, -ae) = (e(x_1 - ax_2), e(y_1 - ay_2)) = (0,0) \in C_e \times C_{\frac{mn}{e}}$$

a contradiction, hence $d|e$ must hold if such an injection i exists.

For parts 2 and 3, we note that the maps in question are finite étale as they jointly form a disjoint union of maps which are finite étale by Pr.7.3.3. Moreover, as long as the domain of (7.3.11) is nonempty this map is clearly an étale cover, which will follow from the nowhere vanishing of the degree of this map, which we discuss now. Denote the degree of (7.3.11) by $c_{m,n}(d,e)$, which is exactly the number of subgroups of $C_e \times C_{\frac{mn}{e}}$ which have the isomorphism type of $C_d \times C_{\frac{m}{d}}$. By [Tó14, Th.4.5], there is the closed formula for $c_{m,n}(d,e)$

$$c_{m,n}(d,e) = \sum_{\substack{i|e,\, j|\frac{mn}{e} \\ m|ij \\ \mathrm{lcm}(i,j)=\frac{m}{d}}} \phi\left(\frac{ij}{m}\right)$$

[41] Recall the Dedekind ψ function is a multiplicative function $\psi\colon \mathbf{N} \to \mathbf{N}$ defined on prime powers by $\psi(p^e) = p^e + p^{e-1}$ where $e \geqslant 1$ and with $\psi(1) = 1$. It can equivalently defined as the degree of the map of stacks $\mathcal{M}_0(n) \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}$.

ranging over all permitted positive integers i and j, and where ϕ is Euler's totient function. It is well-known that ϕ and the formulae given in the statement of this proposition are multiplicative with respect to prime decompositions, hence we may fix a prime ℓ and replace d, e, m, n with $\ell^d, \ell^e, \ell^m, \ell^n$, respectively. Doing this, our old assumptions that $d^2|m$ and $e^2|mn$ are replaced by the inequalities $2d \leqslant m$ and $2e \leqslant m+n$, and the condition $d|e$ is replaced with $d \leqslant e$. In this case, the formula above reads:

$$c_{\ell^m,\ell^n}(\ell^d, \ell^e) = \sum_{\substack{0\leqslant a\leqslant e \\ 0\leqslant b\leqslant m+n-e \\ m\leqslant a+b \\ \max(a,b)=m-d}} \phi\left(\ell^{a+b-m}\right) \tag{7.3.12}$$

We will often split the above sum into two parts depending on if a or b achieves the maximum $m-d$. For part 2, we assume that $e < m-d$, then $a \leqslant e$ cannot reach the maximum $m-d$, so $b = m-d$. We also know $m \leqslant a+b = a+m-d$, so $d \leqslant a \leqslant e$. The formula for (7.3.12) then becomes

$$\sum_{d\leqslant a\leqslant e} \phi\left(\ell^{a-d}\right) = 1 + \sum_{d+1\leqslant a\leqslant e} \left(\ell^{a-d} - \ell^{a-d-1}\right) = \ell^{e-d}$$

as desired, using the expression of Euler's totient function for prime powers. For part 3, we have the added assumption that $m-d \leqslant e$. Consider the half of (7.3.12) where the maximum is obtained by a (and potentially also b), so we have $a = m-d$ and $b \leqslant m-d$, and the variable b ranges over $0 \leqslant b \leqslant m+n-e$. From the fact that $m \leqslant a+b = m-d+b$, we see that $d \leqslant b \leqslant m-d$. The assumptions that $m \leqslant d+e$ and $2e \leqslant m+n$ show that $2e \leqslant d+e+n$ and so $e \leqslant d+n$.[42] Adding m to both sides we see $m+e \leqslant m+n+d$, hence $m-d \leqslant m+n-e$, and so the range of the variable b is seen to be $d \leqslant b \leqslant m-d$. The first half of (7.3.12) is then given by

$$\sum_{d\leqslant b\leqslant m-d} \phi(\ell^{b-d}) = \ell^{m-2d}$$

as in part 2. For the second half of (7.3.12) the maximum is obtained by $b = m-d$ and is *strictly* greater than $a \leqslant m-d-1$. As with the argument for the variable b above, we see the variable a ranges over the values $d \leqslant a \leqslant m-d-1$, and by the assumption that $m-d \leqslant e$, this is a necessarily tighter bound than $a \leqslant e$. This second half of (7.3.12) produces ℓ^{m-2d-1}, and we obtain the desired result:

$$c_{\ell^m,\ell^n}(\ell^d, \ell^e) = \ell^{m-2d} + \ell^{m-2d-1} = \psi(\ell^{m-2d}) \qquad \square$$

The following is essentially a piece of bookkeeping involving the numbers $c_{m,n}(d,e)$ which occurred in the proof above—to us it signifies that these numbers are a little magical.

[42] This shows that in part 3 we can also assume $e \leqslant d+n$, but this will not factor into what follows.

8+7	6+5	4+3	2+1	0
8+7	6+5	4+3	2+1	0
8+7	6+5	4+3	2+1	0
7	6+5	4+3	2+1	0
6	5	4+3	2+1	0
5	4	3	2+1	0
4	3	2	1	0
3	2	1	0	
2	1	0		
1	0			
0				

				4+3	2+1	0
			4	3	2+1	0
		4	3	2	1	0
	4	3	2	1	0	
4	3	2	1	0		
3	2	1	0			
2	1	0				
1	0					
0						

Table 7.1: Values of $c^\ell(d,e)$ for $(m,n) = (8,12)$ and $(12,4)$, respectively. An x (resp. $x+y$) above refers to $c^\ell(d,e) = \ell^x$ (resp. $\ell^x + \ell^y$) where d runs along the horizontal axis and e along the vertical.

Proposition 7.3.13. *Let $m,n \geqslant 1$ be positive integers. Define a function $c_{m,n}\colon \mathbf{N}^2 \to \mathbf{N}$ by setting $c_{m,n}(d,e)$ to be the degree of the map of stacks (7.3.11) where we set $c_{m,n}(d,e) = 0$ if the domain is empty. Then the following equality of polynomials in x holds:*

$$\sum_{\substack{d^2|m\\ e^2|mn}} c_{m,n}(d,e)x^e = \sum_{\substack{b|m,n\\ a^2|\frac{mn}{b^2}}} bx^{ab}$$

Proof. As in the proof of Pr.7.3.10, the function $c_{m,n}(d,e)$ is multiplicative, so it suffices to prove the following:

Claim 7.3.14. For any prime ℓ and integers m and n, we have the equality of polynomials in x

$$\sum_{\substack{0\leqslant 2d\leqslant m\\ 0\leqslant 2e\leqslant m+n}} c^\ell(d,e)x^{\ell^e} = \sum_{\substack{0\leqslant t\leqslant \min(m,n)\\ 0\leqslant 2u\leqslant m+n-2t}} \ell^t x^{\ell^{t+u}}$$

where $c^\ell(x,y) = c_{\ell^m,\ell^n}(\ell^x,\ell^y)$.

This equality is clear, once one analyses the following $\mathbf{N}^2$-table of values of $c^\ell(-,-)$; see Table 7.1 for two examples tables, and Tables 7.2 and 7.3 at the end of this section for the general case. In a bit more detail, by Pr.7.3.10, note that for $e < d$ we have $c^\ell(d,e) = 0$ so our tables vanish below the diagonal. The conditions $0 \leqslant 2d \leqslant m$ and $0 \leqslant 2e \leqslant m+n$ further confine our table to a bounded region of $\mathbf{N}^2$. If $n \leqslant m$, we also see a vanishing triangle in the top left corner, corresponding to the constraint that $e \leqslant n+d$ in this case. The desired formula appears by summing together the monomials of $\sum c^\ell(d,e)x^{\ell^e}$ of the form $\ell^t x^{\ell^e}$ for each fixed t, and note the range of such t is precisely $0 \leqslant t \leqslant \min(m,n)$. Moreover, for every fixed t, the range of possible powers of x with coefficient ℓ^t are $x^{\ell^{t+u}}$ for $0 \leqslant 2u \leqslant m+n-2t$. $\square$

$m+(m-1)$	$(m-2)+(m-3)$	$(m-4)-(m-5)$	$\cdots$	$6+5$	$4+3$	$2+1$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m+(m-1)$	$(m-2)+(m-3)$	$(m-4)-(m-5)$					
$m+(m-1)$	$(m-2)+(m-3)$	$(m-4)-(m-5)$					
$m-1$	$(m-2)+(m-3)$	$(m-4)-(m-5)$					
$m-2$	$m-3$	$(m-4)-(m-5)$					
$m-3$	$m-4$	$m-5$					
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
				$6+5$	$4+3$	$2+1$	0
				5	$4+3$	$2+1$	0
				4	3	$2+1$	0
				3	2	1	0
				2	1	0	
				1	0		
				0			
$\vdots$	$\vdots$	$\vdots$	$\iddots$				
2	1	0					
1	0						
0							

Table 7.2: Values of $c^\ell(d,e)$ where $m \leqslant n$, concentrated in $0 \leqslant 2d \leqslant m$ and $0 \leqslant 2e \leqslant m+n$. We have also assumed m is even for the above picture, however, the m is odd case simply has $1+0$ in the final column instead of simply 0's, and the other columns are shifted appropriately. Each x above corresponds to $c^\ell(d,e) = \ell^x$ and $x+y$ to $c^\ell(d,e) = \ell^x + \ell^y$, where the horizontal axis is the d-axis and the vertical axis is the e-axis.

					$n+(n-1)$	$(n-2)+(n-3)$	$\cdots$	$4+3$	$2+1$	0
				n	$(n-1)$	$(n-2)+(n-3)$		$\vdots$	$\vdots$	$\vdots$
			n	$(n-1)$	$(n-2)$	$(n-3)$	$\cdots$	$4+3$	$2+1$	0
		$\iddots$	$(n-1)$	$(n-2)$	$(n-3)$	$(n-4)$	$\cdots$	3	$2+1$	0
	n	$\iddots$	$(n-2)$	$(n-3)$	$(n-4)$	$\vdots$	$\iddots$	2	1	0
n	$(n-1)$	$\iddots$	$(n-3)$	$(n-4)$	$\vdots$	2	$\iddots$	1	0	
$(n-1)$	$(n-2)$	$\iddots$	$(n-4)$	$\vdots$	2	1	$\iddots$	0		
$(n-2)$	$(n-3)$	$\iddots$	$\vdots$	2	1	0				
$(n-3)$	$(n-4)$		2	1	0					
$(n-4)$	$\vdots$	$\iddots$	1	0						
$\vdots$	2	$\iddots$	0							
2	1	$\iddots$								
1	0									
0										

Table 7.3: Values of $c^{\ell}(d,e)$ where $n \leqslant m$, which is concentrated in $0 \leqslant 2d \leqslant m$ and $0 \leqslant 2e \leqslant m+n$. As in Table 7.2, we have assumed m and n are even, but the other cases are similar.

7.4 Hecke composition formula with stacks

The formula of Th.7.2.7 will boil down to the following algebro-geometric statement and the combinatorics of these numbers $c_{m,n}(d,e)$ discussed above.

Theorem 7.4.1. *Let m, n be positive integers. Then there exists the following Cartesian diagram of stacks over* $\operatorname{Spec} \mathbf{Z}[\frac{1}{mn}]$*:*

$$\begin{array}{ccc} \coprod \mathcal{M}_{\left(d,\frac{m}{d}\right)\leqslant\left(e,\frac{mn}{e}\right)} & \longrightarrow & \mathcal{M}_m \\ \downarrow & & \downarrow q \\ \mathcal{M}_n & \xrightarrow{p} & \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{mn}]} \end{array} \tag{7.4.2}$$

The above coproduct ranges over positive integers d, e such that $d^2|m$, $e^2|mn$, $d|e$, and either $m|de$ or $e|dn$.

Proof. Define the stack $F_{m,n}$ over $\mathbf{Z}[\frac{1}{mn}]$ as the pullback in the following diagram:

$$\begin{array}{ccc} F_{m,n} & \xrightarrow{q'} & \mathcal{M}_n \\ \downarrow p' & & \downarrow p \\ \mathcal{M}_m & \xrightarrow{q} & \mathcal{M}_{\mathbf{Z}[\frac{1}{mn}]} \end{array}$$

This has a natural modular interpretation: for a fixed scheme S, we can identify the groupoid $F_{m,n}(S)$ with that of pentuples

$$(E_m, H_m, E_n, H_n, \alpha)$$

where E_m, E_n are elliptic curves over S with finite closed subgroups $H_m \leqslant E_m$ and $H_n \leqslant E_n$ of order m and n, respectively, and $\alpha \colon E_n \simeq E_m/H_m$ is an isomorphism of elliptic curves over S. Given such a pentuple, one can consider the following commutative diagram of schemes over our fixed S:

$$\begin{array}{ccccc} H_m & \longrightarrow & \pi^* H_n & \longrightarrow & E_m \\ \downarrow & & \downarrow & & \downarrow \pi \\ S & \longrightarrow & H_n & \longrightarrow & E_n \overset{\alpha}{\simeq} E_m/H_m \end{array}$$

The right square above is Cartesian by construction, the whole rectangle is Cartesian by inspection, so we see the left square is also Cartesian. This left square then witnesses the following short exact sequence of finite étale groups schemes over S:

$$0 \to H_m \to \pi^* H_n \to H_n \to 0$$

From the above short exact sequence, it is clear that $\pi^* H_n$ has order mn. These observations justify the well-definedness of the following map of stacks over $\operatorname{Spec} \mathbf{Z}[\frac{1}{mn}]$:

$$F_{m,n} \to \mathcal{M}_{m \leqslant mn}, \qquad (E_m, H_m, E_n, H_n, \alpha) \mapsto (E_m, H_m \leqslant Q^* H_n)$$

We claim this map is an equivalence of stacks, which is easy to check using the following inverse:

$$M_{m \leqslant mn} \to F_{m,n} \qquad (E, K \leqslant H) \mapsto (E, K, E/K, H/K, \mathrm{id})$$

Indeed, all of the homotopies used to show these two functors are inverse to each other are canonical. By Pr.7.3.2, the stack $\mathcal{M}_{m \leqslant mn}$ decomposes as

$$\mathcal{M}_{m \leqslant mn} \simeq \coprod_{\substack{|A|=m \\ |B|=mn}} \mathcal{M}_{A \leqslant B}$$

where $A \leqslant C_m \times C_m$ and $B \leqslant C_{mn} \times C_{mn}$. By elementary group theory (see the proof of Pr.7.3.6, for example), for each A (resp. B) there exists a unique positive integer d (resp. e) such that $d^2 | m$ and $A \simeq C_d \times C_{\frac{m}{d}}$ (resp. $e^2 | mn$ and $B \simeq C_e \times C_{\frac{mn}{e}}$). Notice that $d | e$ and either $e | nd$ or $m | de$ by Pr.7.3.10, which gives us the desired indexing set for our coproduct. □

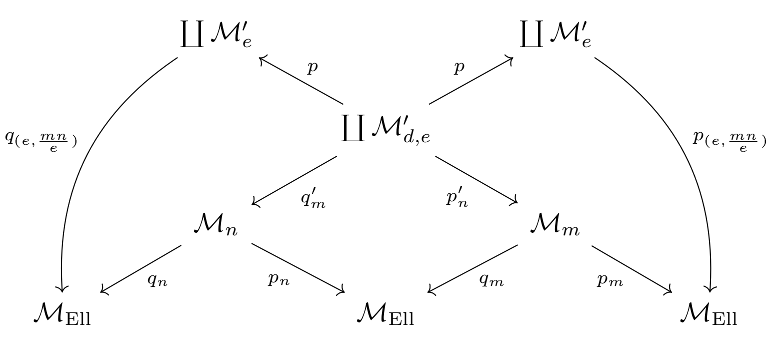
Finally, we can now prove Th.D which we restate for convenience.

Theorem 7.4.3 (Th.D and Th.7.2.7)**.** *Let m and n be positive integers. Then there is a homotopy of morphisms of spectra*

$$\mathrm{T}_m \circ \mathrm{T}_n \simeq \sum_{d | m,n} d\psi^d \mathrm{T}_{\frac{mn}{d^2}} : \mathrm{TMF}[\frac{1}{mn\phi}] \to \mathrm{TMF}[\frac{1}{mn\phi}]$$

where $\phi = \gcd(6, \phi(mn))$ and $\phi(mn)$ is Euler's totient function.

Proof. Consider the diagram of stacks over $\mathbf{Z}[\frac{1}{mn}]$

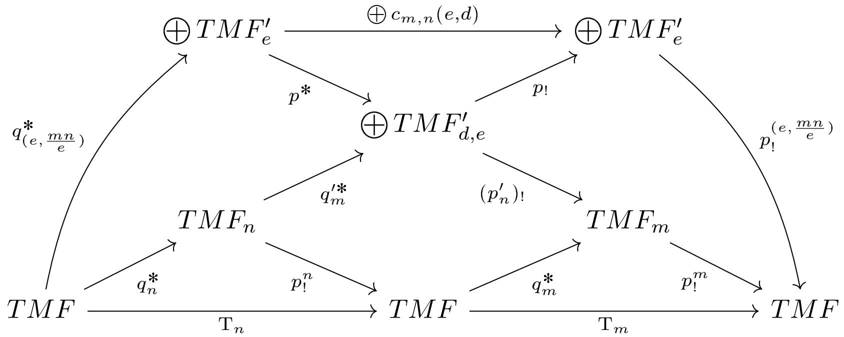


(7.4.4)

where we have abbreviated $\mathcal{M}_{(d,\frac{m}{d}) \leqslant (e,\frac{mn}{e})}$ as $\mathcal{M}'_{d,e}$, $\mathcal{M}_{(e,\frac{mn}{e})}$ as $\mathcal{M}'_e$, written p for the structure map (7.3.11), the two coproducts are indexed as in Th.7.4.1, and mn is implicitly inverted. To see this diagram commutes, we only need to check the left and right regions—the centre is given by Th.7.4.1. Clearly, the right region commutes: given a scheme S and an S-valued point in the middle coproduct given by a triple $(E, K \leqslant H)$, then we see both composites send this triple to E in $\mathcal{M}_{\mathrm{Ell}}(S)$. For the left region, first take such a triple

$(E, K \leqslant H)$. The upper composition sends this triple first to the pair (E, H), then to E/H. The lower composition sends this triple first to $(E/K, H/K)$ and then to $(E/K)/(H/K)$ which is naturally isomorphic to E/H by one of the usual numbered isomorphism theorems from algebra.

We can enhance (7.4.4) to a diagram in $\mathrm{Isog}_{\mathrm{Ell}}$—just pair the universal elliptic curves with all the stacks above, and pair each map labelled with a "p" with a canonical equivalence of elliptic curves, and each map labelled with a "q" with the obvious quotient (as in Df.7.2.1 for T_n, for example). We can now apply $\mathbf{O}^{\mathrm{top}}$ to this diagram, meaning that all of the maps going to the left are realised by the induced maps of $\mathbf{E}_\infty$-rings, and the maps going to the right are realised by transfer maps. This yields the diagram of spectra



(7.4.5)

where we used analogous notation to (7.4.4) and suppressed inverting m and n. Above we display the definitions of T_n and T_m, so the bottom two triangles commute by definition. The centre diamond commutes up to natural homotopy by the functoriality of $\mathbf{O}^{\mathrm{top}}$ as a spectral Mackey functor on $\mathrm{Isog}_{\mathrm{Ell}}$, and similarly, the left and right regions also commute up to natural homotopy, as they did so in $\mathrm{Isog}_{\mathrm{Ell}}$. We are left with the composition p_* with $p_!$, which is **not** the identity—by Lm. 6.2.5 and Pr. 7.3.10 it is (non-canonically) homotopic to the constant functor $c_{m,n}(d, e)$ for this number is the degree of the map of stacks $\mathcal{M}'_{d,e} \to \mathcal{M}'_e$. To apply Lm.6.2.5 we crucially use Rmk.6.2.6 which applies to us as $\mathcal{M}'_e \simeq \mathcal{M}_0\left(\frac{mn}{e^2}\right)$ by Pr.7.3.6, hence $\mathrm{TMF}'_e \simeq \mathrm{TMF}_0\left(\frac{mn}{e^2}\right)$ these cases are covered by Rmk.6.2.6 if we invert ϕ.

The commutativity of (7.4.5) gives us the first homotopy between maps of the spectrum $\mathrm{TMF}[\frac{1}{mn}]$

$$\mathrm{T}_m \circ \mathrm{T}_n \simeq \sum c_{m,n}(d,e)\mathrm{T}_{\left(e, \frac{mn}{e}\right)} \simeq \sum_{\substack{b \mid m,n \\ a^2 \mid \frac{mn}{b^2}}} b\mathrm{T}_{(ab, \frac{mn}{ab})}$$

where the first sum is indexed as the sums in (7.4.5) are, and the second natural homotopy comes from the bookkeeping in Pr.7.3.13 by setting $x^e = \mathrm{T}_{(e, \frac{mn}{e})}$. Using the natural homotopies provided in part 2 followed by part 1 of Cor.7.3.8, we obtain the desired conclusion:

$$\sum_{\substack{b|m,n \\ a^2|\frac{mn}{b^2}}} b\mathrm{T}_{(ab,\frac{mn}{ab})} \simeq \sum_{\substack{b|m,n \\ a^2|\frac{mn}{b^2}}} b\psi^b\mathrm{T}_{(a,\frac{mn}{ab^2})} \simeq \sum_{b|m,n} b\psi^b\mathrm{T}_{\frac{mn}{b^2}} \qquad \square$$

Remark 7.4.6. Let us work with the classical algebraic Hecke operators $\mathrm{T}_n^{\mathrm{alg}}$ of Df.7.5.2 for a moment, and observe that Th.7.2.7 implies the classical formula

$$\mathrm{T}_m^{\mathrm{alg}} \circ \mathrm{T}_n^{\mathrm{alg}} = \sum_{d|m,n} d^{k-1}\mathrm{T}_{\frac{mn}{d^2}}$$

on the space MF_k of meromorphic modular forms of weight k—the same statement then holds on the space mf_k of holomorphic modular forms as the classical Hecke operators preserve the inclusion $\mathrm{mf}_k \to \mathrm{MF}_k$. Indeed, using the comparison result Pr.7.5.3 below, and the fact that on MF_k the Adams operation ψ^d acts by multiplication by d^k, we obtain the following equalities of homomorphism on $\mathrm{MF}_k^{\mathbf{Z}[\frac{1}{mn}]}$:

$$\mathrm{T}_m^{\mathrm{alg}} \circ \mathrm{T}_n^{\mathrm{alg}} = \frac{1}{mn}\mathrm{T}_n \circ \mathrm{T}_m = \frac{1}{mn}\sum_{d|m,n} d\psi^d\mathrm{T}_{\frac{mn}{d^2}} = \sum_{d|m,n} d^{k-1}\mathrm{T}^{\mathrm{alg}}_{\frac{mn}{d^2}}$$

In fact, the comparison result Pr.7.5.3 is not necessary—just apply $H^0(-,\omega^{\otimes k})$ to (7.4.4) and argue as in the proof of Th.7.4.3 above.

Remark 7.4.7. There is a singular moment in our proof of Th.7.4.3 where we use the fact that $f_! \circ f^*$ is homotopic the degree of f, in other words, where we use Lm.6.2.5. This is also the only moment where we need to invert ϕ. If we could apply Lm.6.2.5 directly to $\mathrm{TMF}_0\left(\frac{mn}{e^2}\right)$ without inverting ϕ, then we would be done. In other words, if one could show that in general $\pi_0\,\mathrm{TMF}_0(n)$ has no torsion, then one would obtain a version of Th.7.4.3 without inverting ϕ. We are currently investigating this.

Remark 7.4.8. The author has recently shown that $\mathrm{T}_m \circ \mathrm{T}_n$ is homotopic to $\mathrm{T}_n \circ \mathrm{T}_m$ over $\mathrm{TMF}[\frac{1}{mn}]$, skipping the splitting of Th.7.4.3 over $\mathrm{TMF}[\frac{1}{mn\phi}]$. Our proof is very much in the spirit of the rest of this thesis, but as we came across this argument as the thesis committee was reviewing this thesis, let us delay this until [Dav22].

7.5 Comparison of Hecke operators

There is another kind of Hecke operator that we claim our operations on TMF generalise—the classical Hecke operators of Hecke himself.

Definition 7.5.1. Let R be a subring of $\mathbf{C}$ and k an integer. Define the space MF_k^R of *weight k meromorphic modular forms* over R by the sheaf cohomology group

$$\mathrm{MF}_k^R = H^0(\mathcal{M}_{\mathrm{Ell},R}, \omega^{\otimes k})$$

where ω is the line bundle on $\mathcal{M}_{\mathrm{Ell}}$ defined by the pushforward of the sheaf of differential $p_*\Omega_{\mathscr{E}/\mathcal{M}_{\mathrm{Ell}}}$ on the universal elliptic curve. Write MF_*^R for the associated graded R-algebra. If $R = \mathbf{Z}$ we will drop it from our notation. Define the space mf_k^R of *weight k holomorphic modular forms* over R by the analogous sheaf cohomology group over $\overline{\mathcal{M}}_{\mathrm{Ell},R}$.

By [DI95, Th.12.3.7], we see the q-expansion map $\mathrm{MF}_k^R \to R((q))$ is injective whose image are those meromorphic modular forms over $\mathbf{C}$ whose q-expansion has coefficients in R. The canonical quotient map $\mathscr{E}_n \to \mathscr{E}_n/\mathcal{H}_n$ of the universal elliptic curve over $\mathcal{M}_n$ induces an isomorphism $\xi\colon \omega_{\mathscr{E}_n/\mathcal{H}_n} \to \omega_{\mathscr{E}_n}$, which we will use in our definitions of classical Hecke operators now.

Definition 7.5.2. Let k and n be integers with $n \geqslant 1$. The *nth algebraic Hecke operator* $\mathrm{T}_n^{\mathrm{alg}}$ on $\mathrm{MF}_k^{\mathbf{Z}[\frac{1}{n}]}$ is defined as the composition

$$\omega^{\otimes k}(\mathcal{M}_{\mathbf{Z}[\frac{1}{n}]}) \xrightarrow{q^*} q^*\omega^{\otimes k}(\mathcal{M}_n) \xrightarrow{\xi} p^*\omega^{\otimes k}(\mathcal{M}_n) \xrightarrow{\frac{1}{n}\mathrm{Tr}_p} \omega^{\otimes k}(\mathcal{M}_{\mathbf{Z}[\frac{1}{n}]})$$

where $\mathrm{Tr}_p\colon p_*p^*\omega^{\otimes k} \to \omega^{\otimes k}$ is the transfer map associated to finite locally free morphism p; see [AGV$^+$73, Exposé IX, §5] or [Sta, 03SH].

By [Kat73, §1.11] (or [Con07, §4.5]), we see the above definition agrees with the even more classical definition over $\mathbf{C}$.

Proposition 7.5.3. *Let n be a positive integer. Writing e for the edge map in the descent spectral sequence for* $\mathrm{TMF}[\frac{1}{n}]$*, then the following diagram of graded abelian groups commutes:*

$$\begin{array}{ccc} \pi_{2*}\,\mathrm{TMF}[\frac{1}{n}] & \xrightarrow{\mathrm{T}_n} & \pi_{2*}\,\mathrm{TMF}[\frac{1}{n}] \\ \downarrow e & & \downarrow e \\ \mathrm{MF}_*^{\mathbf{Z}[\frac{1}{n}]} & \xrightarrow{n\mathrm{T}_n^{\mathrm{alg}}} & \mathrm{MF}_*^{\mathbf{Z}[\frac{1}{n}]} \end{array}$$

In particular, we have an equality $\mathrm{T}_n = n\mathrm{T}_n^{\mathrm{alg}}$ of homomorphisms on $\mathrm{MF}_*^{\mathbf{Z}[\frac{1}{6n}]}$.

We will make more explicit calculations in Chapter 9. The following is mostly a formal exercise.

Proposition 7.5.4. *Let n be a positive integer. The stable Hecke operator T_n on $\mathrm{TMF}[\frac{1}{n}]$ induces a map of descent spectral sequences*

$$\mathrm{T}_n\colon\ \mathrm{DSS}(\mathrm{TMF}[\frac{1}{n}]) \to \mathrm{DSS}(\mathrm{TMF}[\frac{1}{n}])$$

whose effect on the E_2-page

$$\mathrm{T}_n\colon H^0(\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \omega^{\otimes t}) \to H^0(\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \omega^{\otimes t})$$

can be identified with the multiple $n\mathrm{T}_n^{\mathrm{alg}}$ of the algebraic Hecke operators.

For use in the following proof, let us note that for every object (X, E) of $\mathrm{Isog}_{\mathrm{Ell}}$, we can define a stable Hecke operator $\mathrm{T}_n^{(\mathsf{X},E)}$ on $\mathscr{O}^{\mathrm{top}}(\mathsf{X}, E)$ by pullback. This means we consider the Cartesian diagrams inside $\mathrm{Isog}_{\mathrm{Ell}}$

$$\begin{array}{ccc} (\mathsf{X}_n, E_n) & \xrightarrow{p',\mathrm{can}} & (\mathsf{X}, E) \\ \downarrow{\scriptstyle f',\alpha'} & & \downarrow{\scriptstyle (f,\alpha)} \\ (\mathcal{M}_n, \mathscr{E}_n) & \xrightarrow{(p,\mathrm{can})} & (\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \mathscr{E}) \end{array} \qquad \begin{array}{ccc} (\mathsf{X}_n, E_n) & \xrightarrow{q',Q'} & (\mathsf{X}, E) \\ \downarrow{\scriptstyle f',\alpha'} & & \downarrow{\scriptstyle (f,\alpha)} \\ (\mathcal{M}_n, \mathscr{E}_n) & \xrightarrow{(q,Q)} & (\mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}, \mathscr{E}) \end{array}$$

and then define $\mathrm{T}_n^{(\mathsf{X},E)}$ as the following composition:

$$\mathscr{O}^{\mathrm{top}}(\mathsf{X}, E) \xrightarrow{(q',Q')^*} \mathscr{O}^{\mathrm{top}}(\mathsf{X}_n, E_n) \xrightarrow{(p',\mathrm{can})_!} \mathscr{O}^{\mathrm{top}}(\mathsf{X}, E)$$

The functoriality of pullback and transfers combine to give us sheafy stable Hecke operators

$$\mathscr{T}_n \colon \mathscr{O}^{\mathrm{top}}_{\mathbf{Z}[\frac{1}{n}]} \to \mathscr{O}^{\mathrm{top}}_{\mathbf{Z}[\frac{1}{n}]}$$

as maps of sheaves of spectra on $\mathrm{Isog}_{\mathrm{Ell}}$.

Proof. Let $S = \mathrm{Spec}\, A \to \mathcal{M}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{n}]}$ be an étale cover, and let us suppress inverting n in our notation from now. Write $S_\bullet$ for the Čech nerve of this cover and $\mathcal{E}^\bullet$ for the augmented cosimplicial $\mathbf{E}_\infty$-ring resulting from applying $\mathscr{O}^{\mathrm{top}}$ to $S_\bullet$:

$$S_m = (\mathrm{Spec}\, A)^{\times_{\mathcal{M}_{\mathrm{Ell}}} m} \qquad \mathcal{E}^m = \mathscr{O}^{\mathrm{top}}(S_m)$$

Using these $\mathbf{E}_\infty$-$\mathrm{TMF}[\frac{1}{n}]$-algebras $\mathcal{E}^m$ and the functorial stable Hecke operators $\mathrm{T}_n^{S_m}$ defined above, we obtain the stable Hecke operators $\mathrm{T}_n^\bullet$ on the cosimplicial spectrum $\mathcal{E}^\bullet$, the case $\bullet = -1$ being the stable Hecke operator T_n on TMF. The Bousfield–Kan spectral sequence associated with $\mathcal{E}^\bullet$ is by definition the descent spectral sequence for TMF, and the functoriality of this spectral sequence implies our morphisms $\mathrm{T}_n^\bullet$ of cosimplicial spectra induce a morphism T_n between descent spectral sequences

$$\mathrm{T}_n \colon \mathrm{DSS}(\mathrm{TMF}) \to \mathrm{DSS}(\mathrm{TMF}).$$

We are trying to identify T_n on the zeroth line on the E_2-page of this spectral sequence, so it suffices to show that these operations agree on the E_1-page, which is given by the groups $\pi_*\mathcal{E}^\bullet$. In fact, each row $E_1^{*,2t}$ of the E_1-page can be identified with the Čech complex for $\omega^{\otimes t}$ using the cover $\mathrm{Spec}\, A \to \mathcal{M}_{\mathrm{Ell}}$ we started with. It then suffices to show that for any affine étale $\mathrm{Spec}\, B \to \mathcal{M}_{\mathrm{Ell}}$, the induced map $\mathrm{T}_n \colon \pi_{2t}\mathscr{O}^{\mathrm{top}}(B) \to \pi_{2t}\mathscr{O}^{\mathrm{top}}(B)$ agrees with the pullback of the classical Hecke operator $\mathrm{T}_n^{\mathrm{alg}}$ to $\omega_B^{\otimes t}$ multiplied by n, now as maps of sheaves of abelian groups on $\mathrm{Spec}\, B$. A slight rewriting of the stable Hecke operators as the composite

$$\begin{array}{ccccc} \mathscr{O}^{\mathrm{top}}(\mathcal{M}, \mathscr{E}) & \xrightarrow{(q,\mathrm{can})^*} & \mathscr{O}^{\mathrm{top}}(\mathcal{M}_n, \mathscr{E}_n/\mathcal{H}_n) & & \\ & & \downarrow{\scriptstyle (\mathrm{id},Q)^*} & & \\ & & \mathscr{O}^{\mathrm{top}}(\mathcal{M}_n, \mathscr{E}) & \xrightarrow{(p,\mathrm{can})_!} & \mathscr{O}^{\mathrm{top}}(\mathcal{M}, \mathscr{E}) \end{array}$$

shows that it suffices to calculate the composition for all $t \in \mathbf{Z}$

$$\begin{array}{ccc} \pi_{2t}\mathscr{O}^{\mathrm{top}}(B,\mathscr{E}_B) \xrightarrow{(q_B,\mathrm{can})^*} \pi_{2t}\mathscr{O}^{\mathrm{top}}(B_n,\mathscr{E}_{B_n}/\mathcal{H}) & & \\ \downarrow{\scriptstyle (\mathrm{id},Q_{B_n})^*} & & \\ \pi_{2t}\mathscr{O}^{\mathrm{top}}(B_n,\mathscr{E}_{B_n}) \xrightarrow{(p_B,\mathrm{can})_!} \mathscr{O}^{\mathrm{top}}(B,\mathscr{E}_B) & & \end{array} \tag{7.5.5}$$

where $\operatorname{Spec} B_n = \mathcal{M}_n \times_{\mathcal{M}_{\mathrm{Ell}}} \operatorname{Spec} B$. Using the fact that q_B is étale, and hence $(q_B, \mathrm{can})^*$ is an étale morphism of $\mathbf{E}_\infty$-rings, we can identify the first map of (7.5.5) with the base change of the map q^* of Df.7.5.2 along $\operatorname{Spec} B \to \mathcal{M}_{\mathrm{Ell}}$. Similarly, the map Q_{B_n} is the quotient of elliptic curves $\mathscr{E}_{B_n} \to \mathscr{E}_{B_n}/\mathcal{H}$. By Th. 6.1.9, $\mathscr{O}^{\mathrm{top}}$ defines a natural elliptic cohomology theory on $\mathrm{Isog}_{\mathrm{Ell}}$ so the isomorphism $\pi_{2t}\mathscr{O}^{\mathrm{top}}(B) \simeq \omega_B^{\otimes t}$ commutes with the morphisms induced by such quotients of elliptic curves. In other words, $(\mathrm{id}, Q_{B_n})^*$ agrees with ξ of Df.7.5.2. Finally, the spectral transfer agrees with the classical transfer as both are defined étale locally as summing fold maps; see Rmk.6.2.4. Importantly, Df.7.5.2 now divides by n to obtain $\mathrm{T}_n^{\mathrm{alg}}$, which explains the discrepancy between stable Hecke operators and their classical counterparts. □

Proof of Pr. 7.5.3. This is simply a statement of the existence of the natural edge map of the descent spectral sequence and the E_2-identification from Pr.7.5.4. □

Remark 7.5.6. Using the notation Th. 8.0.2, we can rephrase Pr. 7.5.4 as the identification of the morphisms on E_2-pages $(\mathrm{T}_n)_{\mathrm{alg}}$ and $n\mathrm{T}_n^{\mathrm{alg}}$.

It also seems likely that our stable operators T_n agree (up to homotopy) with Baker's stable Hecke operators $n\mathrm{T}_n^{\mathrm{Baker}}$ over the Landweber exact theory $\mathcal{E}\ell\ell = \mathrm{TMF}[\frac{1}{6}]$. To show this, one should reduce to the prime case $n = p$ and explicitly compare the two constructions; a task we have not been able to complete yet.

7.6 Stable Fricke and Atkin–Lehner involutions

In the realm of classical modular forms, the *Fricke involution* is an endomorphism

$$w_N \colon \mathrm{MF}_0(N)_* \to \mathrm{MF}_0(N)_*$$

where $\mathrm{MF}_0(N)$ is the ring of modular forms of level $\Gamma_0(N)$ defined by the sheaf cohomology groups:

$$\mathrm{MF}_0(N) = H^0(\mathcal{M}_0(N), p^*\omega^{*\otimes})$$

There is a modular interpretation of this map. Recall the S-valued points of $\mathcal{M}_0(N)$ can be equivalently described as pairs (E, H) of an elliptic curve E over S and a cyclic subgroup $H \leqslant E$ of order N, or pairs (E, ϕ) where $\phi \colon E \to E'$ is an isogeny of elliptic curves whose kernel is cyclic of order N. The bijection between these pairs is given by the following maps:

$$(E, H) \mapsto (E, E \to E/H)$$

$$(E, \ker(\phi)) \leftarrow\!\shortmid (E, \phi)$$

Definition 7.6.1. Let $N \geqslant 2$ be an integer and Q be a positive divisor of N with $\gcd(Q, N/Q) = 1$. Define the *Atkin–Lehner involution* $w_Q \colon \mathcal{M}_0(N) \to \mathcal{M}_0(N)$ of stacks on S-valued points as follows: given a pair (E, ϕ) in $\mathcal{M}_0(N)(S)$ then the kernel K of ϕ uniquely splits into a product of the subgroups $K_Q \times K_{N/Q}$, where K_m has order m. The isogeny ϕ can then be factored in two ways:

$$\begin{array}{ccccc} & & E/K_Q & & \\ & \overset{\phi_Q}{\nearrow} & & \searrow & \\ E & & \xrightarrow{\quad\phi\quad} & & E' = E/K \\ & \underset{\phi_{N/Q}}{\searrow} & & \nearrow & \\ & & E/K_{N/Q} & & \end{array}$$

We then define $w_Q(E, \phi) = (E/K_Q, \phi_{N/Q} \circ \phi_Q^\vee)$ where $(-)^\vee$ denotes taking the dual isogeny—it is easy to check the kernel of $\phi_{N/Q} \circ \phi_Q^\vee$ is cyclic of order N. If $Q = N$, then we call w_N the *Fricke involution* of $\mathcal{M}_0(N)$.

This algebraic construction combined with Th.6.1.9 immediately leads us to a spectral definition.

Definition 7.6.2. Let $N \geqslant 2$ be an integer and Q be a positive divisor of N with $\gcd(Q, N/Q) = 1$. Define the *Atkin–Lehner involution* on $\mathrm{TMF}_0(N)$ by applying $\mathscr{O}^{\mathrm{top}}$ to the morphism

$$(w_Q, \phi_{\mathcal{K}_Q}) \colon (\mathcal{M}_0(N), \mathscr{E}_0(N)) \to (\mathcal{M}_0(N), \mathscr{E}_0(N))$$

in $\mathrm{Isog}_{\mathrm{Ell}}$, where $\phi_{\mathcal{K}_Q} \colon \mathscr{E}_0(N) \to \mathscr{E}_0(N)/\mathcal{K}_Q = w_Q^* \mathscr{E}_0(N)$ is the natural quotient map of the universal elliptic curve over $\mathcal{M}_0(N)$ with $\mathcal{K}_Q$ the Q-primary part of the universal cyclic subgroup $\mathcal{K}$ of order N.

Calling these operations "involutions" is a little misleading.

Proposition 7.6.3. *Let $N \geqslant 2$ be an integer and Q be a positive divisor of N with* $\gcd(Q, N/Q) = 1$. *Then we have the following natural homotopy of* $\mathbf{E}_\infty$*-rings:*

$$w_Q \circ w_Q \simeq \psi^Q \colon \mathrm{TMF}_0(N) \to \mathrm{TMF}_0(N)$$

In other words, the Atkin–Lehner involutions are a square root of the Adams operations on $\mathrm{TMF}_0(N)$.

Proof. By functoriality of $\mathscr{O}^{\mathrm{top}}$ we are left to show the composition

$$(\mathcal{M}_0(N), \mathscr{E}_0(N)) \xrightarrow{(w_Q, \phi_Q)} (\mathcal{M}_0(N), \mathscr{E}_0(N)) \xrightarrow{(w_Q, \phi_Q)} (\mathcal{M}_0(N), \mathscr{E}_0(N))$$

in $\mathrm{Isog}_{\mathrm{Ell}}$ is homotopic to $(\mathrm{id}, [Q])$. This follows rather easily if one remembers how to compose in $\mathrm{Isog}_{\mathrm{Ell}}$. Indeed, $w_Q \circ w_Q$ is naturally equivalent to the identity on $\mathcal{M}_0(N)$ by construction and the fact that the dual of a dual isogeny is naturally equivalent to the original isogeny; see [KM85, Cor.2.6.1.1]. For the maps of elliptic curves over $\mathcal{M}_0(N)$, we note that the composition

$$\mathscr{E}_0(N) \xrightarrow{\phi_Q} \mathscr{E}_0(N)/\mathcal{K}_Q \xrightarrow{w_Q^*\phi_Q} \mathscr{E}_0(N)$$

can be identified with $\phi_Q^\vee \circ \phi_Q$ which is naturally equivalent to $[Q]$ by [KM85, Th.2.6.1]. Applying $\mathscr{O}^{\mathrm{top}}$ to the map $(\mathrm{id}, [Q])$ on $(\mathcal{M}_0(N), \mathscr{E}_0(N))$ is our definition of the Qth Adams operation on $\mathrm{TMF}_0(N)$. ☐

These operations also behave well with respect to themselves and all of the other operations we have seen so far.

Proposition 7.6.4. *Let k,n, and N be integers, with $n, N \geqslant 2$ and coprime, and Q and R be two positive divisors of N with $\gcd(Q, N/Q) = \gcd(R, N/R) = 1$. Then there exist the following natural homotopies of morphisms of $\mathbf{E}_\infty$-rings:*

$$w_Q \circ \psi^k \simeq \psi^k \circ w_Q \colon \mathrm{TMF}_0(N)[\frac{1}{n}] \to \mathrm{TMF}_0(N)[\frac{1}{n}]$$

$$w_Q \circ w_R \simeq w_R \circ w_Q \colon \mathrm{TMF}_0(N)[\frac{1}{n}] \to \mathrm{TMF}_0(N)[\frac{1}{n}]$$

There also exists the following natural homotopy of morphisms of spectra:

$$w_Q \circ \mathrm{T}_n \simeq \mathrm{T}_n \circ w_Q \colon \mathrm{TMF}_0(N)[\frac{1}{n}] \to \mathrm{TMF}_0(N)[\frac{1}{n}]$$

Proof. The natural homotopy witnessing $w_Q \circ \psi^k \simeq \psi^k \circ w_Q$ follows as in the proof of Pr.7.2.4—the map $\phi_{\mathcal{K}_Q}$ is one of elliptic curves, hence it commutes with the k-fold multiplication map $[k]$. For the second family of homotopies, note that if $Q = R$ we are done; this is also covered by Pr.7.6.3. Otherwise, if $Q \neq R$ we see that $\gcd(Q, R) = 1$ from our assumptions, hence the universal cyclic subgroup $\mathcal{K}$ of order N splits uniquely into $\mathcal{K}_Q \times \mathcal{K}_R \times \mathcal{K}_M$, where $N = QRM$ and neither Q nor R divide M. Using this fact, we quickly see that $w_Q \circ w_R$ is naturally equivalent to $w_R \circ w_Q$ as maps of stacks. The fact that the diagram of elliptic curves

$$\begin{array}{ccc} \mathscr{E}_0(N) & \longrightarrow & \mathscr{E}_0(N)/\mathcal{K}_Q \\ \downarrow & & \downarrow \\ \mathscr{E}_0(N)/\mathcal{K}_R & \longrightarrow & \mathscr{E}_0(N)/(\mathcal{K}_Q \times \mathcal{K}_R) \end{array}$$

naturally commutes, where all of the maps are the expected quotients, we see that the two composites $(w_Q, \phi_{\mathcal{K}_Q}) \circ (w_R, \phi_{\mathcal{K}_R})$ and $(w_R, \phi_{\mathcal{K}_R}) \circ (w_Q, \phi_{\mathcal{K}_Q})$ are naturally equivalent in $\mathrm{Isog}_{\mathrm{Ell}}$. This gives the second collection of natural homotopies of $\mathbf{E}_\infty$-rings. The final case of the natural homotopy between $w_Q \circ \mathrm{T}_n$ and $\mathrm{T}_n \circ w_Q$ follows for similar reasons, where we again crucially use that $\gcd(n, N) = 1$. ☐

As with everything we do here, we should clarify how these operations induce those studied classically.

Proposition 7.6.5. *Let $N \geqslant 2$ be an integer and Q a positive divisor of N with $\gcd(Q, N/Q) = 1$. Then the edge map e for the descent spectral sequence induces a commutative diagram of graded abelian groups*

$$\begin{array}{ccc} \pi_{2*}\,\mathrm{TMF}_0(N) & \xrightarrow{w_Q} & \pi_{2*}\,\mathrm{TMF}_0(N) \\ \downarrow e & & \downarrow e \\ \mathrm{MF}_0(N) & \xrightarrow{w_Q^{\mathrm{alg}}} & \mathrm{MF}_0(N) \end{array}$$

where the bottom horizontal map is the classical Atkin–Lehner involution on $\mathrm{MF}_0(N)$.

The definition of the algebraic operations $w_Q^{\mathrm{alg}}\colon \mathrm{MF}_0(N) \to \mathrm{MF}_0(N)$ can be found in the introduction of [Xue09], and by design is given by applying $\pi_{2t}\mathscr{O}^{\mathrm{top}}$ to the morphism $(w_Q, \phi_{\mathcal{K}_Q})$ of Df.7.6.2. This fact alone essentially gives Pr.7.6.5.

Proof. As in the proof of Pr. 7.5.3 and Pr. 7.5.4, we have to show that after base change along an étale map $\operatorname{Spec} B \to \mathcal{M}_0(N)$, taking π_{2t} of the morphism of E_∞-rings $(w_Q)_B\colon \mathscr{O}^{\mathrm{top}}(B) \to \mathscr{O}^{\mathrm{top}}(B)$ agrees with a classical definition of the Atkin–Lehner involution w_Q^{alg} on $\pi_{2t}\mathscr{O}^{\mathrm{top}}(B) \simeq \omega_B^{\otimes t}$. This follows as in the proof of Pr. 7.5.4, as w_Q is étale (even an automorphism) and the isomorphism $\pi_{2t}\mathscr{O}^{\mathrm{top}}(B) \simeq \omega_B^{\otimes t}$ is natural with respect to isogenies of elliptic curves of invertible degree, such as $\phi_{\mathcal{K}_Q}\colon \mathscr{E}_0(N) \to \mathscr{E}_0(N)/\mathcal{K}_Q$. ☐

It can be very hard to obtain a generators-and-relations expression for the graded rings $\pi_*\,\mathrm{TMF}_0(N)$ and even for $\mathrm{MF}_0(N)$. There are calculations how Atkin–Lehner involutions act on these coefficient rings in the cases of $Q = N = p$ for $p = 2$, 3, and 5 in the literature; see [Beh06, §1.3], [MR09], and [BO16, §1.4], respectively.

7.7 Handicraft operations on Tmf

In this section we construct some underwhelming operations ψ^k on Tmf_p for each p-adic integer $k \in \mathbf{Z}_p^\times$. The constructions below do not show any compatibility of these operations as k varies nor does it show compatibility with potential Hecke operators—each construction is done for each k insolation. This is much less satisfying than what we have already seen for TMF, and we expect to prove coherence in the long run with methods other than what is described below; see the final paragraph of this section. We do **not** claim the construction of ψ^k on Tmf_p that follows is the “correct” definition, but it will nevertheless be useful to study TMF_p as done in §10.3 and 10.4.

The details and proofs for the outline that follows can be found in the updated version of [Dav21a].

In our first version of [Dav21a], it was claimed that Th.6.1.9 also holds over $\overline{\mathcal{M}}_{\mathrm{Ell}}$. This statement might still be true, but our proof skips an important step, so we cannot claim to provide a complete proof here. At least two remedies that can be made though, and we would like to discuss them now. To avoid bloat, we will use much of the language and statements of [Dav21a] without hesitation, such as *generalised elliptic curves* and *logarithmic geometry.*

Sites of fractured quasi p-divisible groups A minor error in [Dav21a] claims that there is a multiplication map $\overline{\mathscr{E}} \to \overline{\mathscr{E}}$ on the universal generalised elliptic curve. The construction outlined there was to define this isogeny as the quotient $\overline{\mathscr{E}} \to \overline{\mathscr{E}}/\overline{\mathscr{E}}^{\mathrm{sm}}[n]$ and then identify this quotient with $\overline{\mathscr{E}}$ as this is clear over $\mathcal{M}_{\mathrm{Ell}}$ and also holds over $\mathcal{M}_{\mathrm{Tate}}$ using a uniqueness property of the Tate curve from [Con07, Th.2.5.2]. This does not work though, as [Ces17] only guarantees the existence of a quotient E/H of a generalised elliptic curve by a subgroup H when H is finite locally free, and in particular, is flat. The subgroup $\overline{\mathscr{E}}^{\mathrm{sm}}[n] \subseteq \overline{\mathscr{E}}^{\mathrm{sm}}$ is **not** flat, as one can see in the case of the Tate curve T over $\operatorname{Spec} \mathbf{Z}[\![q]\!]$: upon specialisation to $q = 0$, the degree of the subgroup scheme $T^{\mathrm{sm}}[n]$ jumps from n^2 to n, meaning $T^{\mathrm{sm}}[n]$ is not flat.

The above reason does not change the proof of [Dav21a, Th.A], it simply means that this specific theorem cannot be used to define Adams operations on $\mathrm{Tmf}[\frac{1}{n}]$. There is an easy, albeit ugly, fix. Rather than study a site whose objects are log étale morphisms of log stacks $\mathsf{X} \to \overline{\mathcal{M}}_{\mathrm{Ell}}$ and whose morphisms are isogenies of generalised elliptic curves (a concept taken from [Ces17, Df.2.2.8]) of invertible degree (analogous to our $\mathrm{Isog}_{\mathrm{Ell}}$), one should use a site with the same objects but whose morphisms can be defined as follows: a morphism between affine objects $(\operatorname{Spec} R, E) \to (\operatorname{Spec} R', E')$ are a system of isomorphisms $\alpha_p\colon E[p^\infty] \simeq f^*E'[p^\infty]$ of quasi p-divisible groups[43] and an isomorphism $\alpha_0\colon \widehat{E} \simeq f^*\widehat{E}'$ of formal groups over $\operatorname{Spec} R$, such that for each prime p, the identity component of α_p over $\operatorname{Spec} R_p^\wedge$ agrees with α_0 over $\operatorname{Spec} R_p^\wedge[p^{-1}]$. One then shows the stack classifying these morphisms satisfies étale descent, and we obtain an expression for morphisms between not necessarily affine objects with log étale maps to $\overline{\mathcal{M}}_{\mathrm{Ell}}$. Write $\mathrm{qBT}_{\mathrm{Ell}}$ for this site, equipped with the log étale topology. One can then extend the proof of [Dav21a, Th.A] to this new site with ease, and define Adams operations on Tmf using the evident multiplication maps on the quasi p-divisible groups (and formal groups) associated with the universal generalised elliptic curve.

[43] As mentioned above, the n-torsion of a generalised elliptic curve is not necessarily flat over the base, meaning the collection of p-power torsion is not a p-divisible group, but rather a *quasi p-divisible group*—an Ind collection of finite (not necessarily flat) group schemes with the expected p-divisibility property.

This new site is necessary to try and obtain the desired functoriality of $\mathscr{O}^{\mathrm{top}}$ over $\overline{\mathcal{M}}_{\mathrm{Ell}}$, however, it is not strictly necessary, as our updated version of [Dav21a] shows. There is a more subtle problem with the proof of [Dav21a, Th.A].

Naturality of the gluing map In the proof of [Dav21a, Pr.2.35], an important step in the proof of [Dav21a, Th.A], we followed Hill–Lawson ([HL16, Pr.5.9]) and used Goerss–Hopkins obstruction theory to construct the following natural gluing map for an affine $\operatorname{Spec} R \to \overline{\mathcal{M}}_{\mathrm{Ell}}$ is Weierstraß form:

$$\phi\colon \mathscr{O}^{\mathrm{sm}}(R[\Delta^{-1}])_p^\wedge \to \mathscr{O}^{\mathrm{Tate}}(R_\Delta^\wedge)[\Delta^{-1}]_p^\wedge \tag{7.7.1}$$

The naturality of this gluing map in the small log étale site of $\overline{\mathcal{M}}_{\mathrm{Ell}}$ is evident in Hill–Lawson's proof, as they construct ϕ as a morphism of $K(1)$-local $\mathbf{E}_\infty$-$\mathrm{tmf}_{K(1)}$-algebras. We do not want to construct $\mathrm{tmf}_{K(1)}$-algebra maps, as isogenies of the universal generalised elliptic curve of invertible degree, should also act nontrivially on $\mathrm{tmf}_{K(1)}$. For example, it is clear from the fact that ψ^n is not the identity (using the calculation of §9, for example) that the morphism of $\mathbf{E}_\infty$-ring $\psi^n\colon \mathrm{TMF}[\frac{1}{n}] \to \mathrm{TMF}[\frac{1}{n}]$ is **not** a TMF-algebra map, and the same should hold over the cusp.

We can think of (at least) two alternative ways to construct the desired ϕ: more obstruction theory or constructive methods; these points are explained in detail in the updated version of [Dav21a].

(More obstruction theory I) To fix the proof of [Dav21a, Pr.2.35], fix an log étale morphism $E\colon \operatorname{Spec} R \to \overline{\mathcal{M}}_{\mathrm{Ell}}$, and write the domain and codomain of (7.7.1) as $\mathcal{E}_{\mathrm{sm}}$ and $\mathcal{E}_{\mathrm{Tate}}$. By the arguments of [HL16, Pr.5.9], we obtain a morphism $\phi\colon \mathcal{E}_{\mathrm{sm}} \to \mathcal{E}_{\mathrm{Tate}}$ which commutes with the tmf-algebra structures. On the domain, this algebra structure comes from the natural maps

$$\mathrm{tmf}_p \to \mathrm{TMF}_p \to \mathcal{E}_{\mathrm{sm}}$$

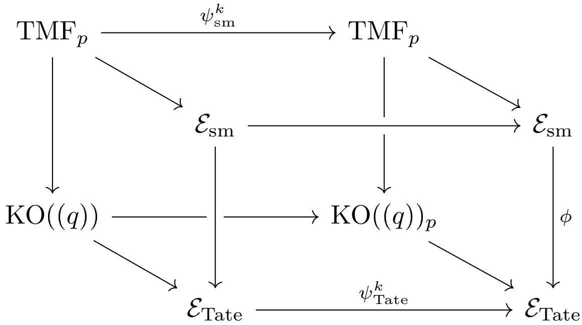
the first the canonical localisation map and the second from the construction of $\mathscr{O}^{\mathrm{sm}}$ and the identification of its global sections from Th.5.3.3. On the codomain, this algebra structure comes from the natural maps

$$\mathrm{tmf} \to \mathrm{KO}[\![q]\!] \to \mathcal{E}_{\mathrm{Tate}}$$

the first now coming from [HL16, §A] and the second from the construction of $\mathscr{O}^{\mathrm{Tate}}$ and the identification of the global sections of $\mathscr{O}^{\mathrm{mult}}$ as KO from Prs.5.1.3 and 5.1.8. As Δ is inverted in $\mathcal{E}_{\mathrm{Tate}}$, we see this algebra structure naturally factors as follows:

$$\mathrm{tmf} \to \mathrm{TMF} \to \mathrm{KO}((q)) \to \mathcal{E}_{\mathrm{Tate}}$$

In other words, ϕ is a TMF_p-algebra map. Fix a p-adic unit $k \in \mathbf{Z}_p^\times$. Let $[k]\colon \overline{\mathscr{E}} \to \overline{\mathscr{E}}$ be the morphism in $\mathrm{qBT}_{\mathrm{Ell}}$ on the universal generalised elliptic curve over $\overline{\mathcal{M}}_{\mathrm{Ell}} \times \operatorname{Spf} \mathbf{Z}_p$ defined as the k-fold multiplication map on quasi p-divisible groups and formal groups—these maps clearly glue where necessary. This map $[k]$ yields two maps of $\mathbf{E}_\infty$-ring $\psi^k_{\mathrm{sm}} \colon \mathcal{E}_{\mathrm{sm}} \to \mathcal{E}_{\mathrm{sm}}$ and $\psi^k_{\mathrm{Tate}} \colon \mathcal{E}_{\mathrm{Tate}} \to \mathcal{E}_{\mathrm{Tate}}$, and the question is whether or not ϕ naturally commutes with these maps. Consider the following diagram of morphisms of $\mathbf{E}_\infty$-rings:



(7.7.2)

Our goal is to naturally construct a homotopy for the front face of the above cube. The right and left faces commute by the construction of ϕ as a tmf-, and hence TMF-, algebra map. The top and bottom faces commute by the functoriality of $\mathscr{O}^{\mathrm{sm}}$ and $\mathscr{O}^{\mathrm{Tate}}$, respectively.

Claim 7.7.3. For each p-adic unit $k \in \mathbf{Z}_p^\times$, the back face of (7.7.2) can be chosen to commute up to homotopy.

Assuming this claim, for now, it follows from Hill–Lawson's arguments that the front face naturally commutes. Indeed, the commutativity of the rest of (7.7.2) shows that all the maps of $\mathbf{E}_\infty$-rings in the front face are TMF_p-algebra maps, where we view the right hand $\mathcal{E}_{\mathrm{sm}}$ and $\mathcal{E}_{\mathrm{Tate}}$ as TMF_p-algebras through (any of) the obvious composites. By precomposition, this also yields a diagram of tmf-algebra maps we wish to see commutes. As $\mathcal{E}_{\mathrm{Tate}}$ is $K(1)$-local, we may $K(1)$-localise $\mathcal{E}_{\mathrm{sm}}$ and when then find ourselves in the position to use the Goerss–Hopkins obstruction theory of [HL16, Pr.4.49], which states that the mapping space

$$\mathrm{Map}_{\mathrm{CAlg}^{K(1)}_{L_{K(1)}\mathrm{tmf}}}(L_{K(1)}\mathcal{E}_{\mathrm{sm}}, \mathcal{E}^{\psi}_{\mathrm{Tate}})$$

is discrete where we use the superscript ψ to denote a twist in the $L_{K(1)}$ tmf-algebra structure on $\mathcal{E}_{\mathrm{Tate}}$ by Adams operations. In fact, *loc. cit.* shows the above mapping space is equivalent to the set of V-θ-algebra maps commuting with Adams operations between the p-adic K-theories of the above $\mathbf{E}_\infty$-rings; Hill–Lawson write V for what Behrens [Beh14] writes as $V^\wedge_\infty$. The p-adic K-theory of an $\mathbf{E}_\infty$-ring comes equipped with natural stable Adams operations, which we will call the *algebraic Adams operations* ψ^ℓ_{alg} for every $\ell \in \mathbf{Z}_p^\times$. As the map on p-adic K-theory induced by ϕ commutes with these algebraic Adams operations ψ^ℓ_{alg} for every $\ell \in \mathbf{Z}_p^\times$, it is a map of $\mathbf{E}_\infty$-rings, it suffices to show that ψ^k_{sm} and ψ^k_{Tate} induce the algebraic Adams operations ψ^k_{alg} on p-adic K-theory. For ψ^k_{sm}, this follows by construction and the identification of $\mathscr{O}^{\mathrm{sm}}$ with

$\mathscr{O}^{\mathrm{top}}$ restricted to $\mathcal{M}_{\mathrm{Ell}}$; see [Beh14, Rmk.6.3]. For ψ^k_{Tate}, we make two simple observations:

1. There is an isomorphism of $\mathbf{Z}_p$-algebras $W_{\mathrm{Tate}} \simeq \mathrm{KU}_0^\wedge \mathcal{E}_{\mathrm{Tate}}$ where W is the adic ring defined by the Cartesian diagram of formal schemes

$$\begin{array}{ccc} \mathrm{Spf}\, W_{\mathrm{Tate}} & \longrightarrow & \overline{\mathcal{M}}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty) \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, R_{\mathrm{Tate}} & \longrightarrow & \overline{\mathcal{M}}_{\mathrm{Ell}} \times \mathrm{Spf}\, \mathbf{Z}_p \end{array}$$

   where $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty) \simeq \mathrm{Spf}\, V$ classifies generalised elliptic curves with ordinary reduction modulo p and a chosen isomorphism with $\widehat{\mathbf{G}}_m$; see the proof of Lm.B.3.1 or [Beh14, p.14]. Indeed, this follows by [Beh14, Pr.6.1]. Note that the p-adic K-theory of tmf is V.

2. The algebraic Adams operations ψ^k_{alg} on $\mathrm{Spf}\, W_{\mathrm{Tate}}$ come from the universal operations on V, which in turn come from the k-fold multiplication map on $\widehat{\mathbf{G}}_m$ using the chosen isomorphism.

By construction, ψ^k_{Tate} induces the k-fold multiplication map on $\widehat{E}_{\mathrm{Tate}}$, from the construction of $\mathscr{O}^{\mathrm{Tate}}$. Hence ψ^k_{Tate} induces the same map on p-adic K-theory as ψ^k_{alg}, and the front face of (7.7.2) commutes—modulo our claim from earlier.

Proof of Clm.7.7.3. Copying the proof of [HL16, Pr.A.6], but replacing tmf with TMF, the path components of the mapping space of $K(1)$-local maps of $\mathbf{E}_\infty$-rings from TMF_p to $\mathrm{KO}((q))$ is equivalent to the C_2-fixed points of the set of V-θ-algebra maps commuting with Adams operations from the p-adic K-theory of TMF_p to that of $\mathrm{KU}((q))$. We have to only check now that the two composites on the back face of Equation (7.7.2) induce the same map on p-adic K-theory, but this follows from the same arguments made above about what the stable Adams operations on TMF_p and $\mathrm{KO}((q))$ induce on p-adic K-theory. △

(More obstruction theory II) Fixing an odd prime p, rather than asking for ϕ to commute with a particular stable Adams operation as done above, one can ask ϕ to be $\mathbf{F}_p^\times$-equivariant with respect to the p-adic Adams operations ψ^k where $k \in \mathbf{F}_p^\times \subseteq \mathbf{Z}_p^\times$. As discussed in [Sto12, §5.1], there is an equivariant form of Goerss–Hopkins obstruction theory, and if the order of our group is not divisible by p, which happens to be the case for $\mathbf{F}_p^\times$, then the arguments by Hill–Lawson [HL16, Pr.5.9] follow through with little change.

In total, we have outlined a proof of the following unsatisfying statement—a preliminary version of an extension of Th.6.1.9 to $\overline{\mathcal{M}}_{\mathrm{Ell}}$; see [Dav21a] for more details.

Theorem 7.7.4. *Fix a prime p and a p-adic unit $k \in \mathbf{Z}_p^\times$. There exists a morphism of $\mathbf{E}_\infty$-rings $\psi^k \colon \mathrm{Tmf}_p \to \mathrm{Tmf}_p$ which commutes with both $\mathbf{E}_\infty$-maps $\mathrm{Tmf}_p \to \mathrm{TMF}_p$ and $\mathrm{Tmf}_p \to \mathrm{KO}$ where the ψ^k acts on TMF_p and KO_p as in §5.5. If p is odd, this can be further enhanced to an $\mathbf{F}_p^\times$-action of $\mathbf{E}_\infty$-rings on Tmf_p which restricts to the actions of the p-adic Adams operations when restricted to TMF_p and the map $\mathbf{E}_\infty$-map $\mathrm{Tmf}_p \to \mathrm{KO}[\![q]\!]$ to be $\mathbf{F}_p^\times$-equivariant.*

Let us reiterate:

*These methods above do **not** prove any compatibility for varying k.*

This highlights the moral difference between the obstruction theoretic construction of TMF and the construction using spectral algebraic geometry given by Lurie. In the first case, one proves that TMF exists and is unique up to homotopy by showing the vanishing of various obstruction groups. In the second case, one does not know if TMF is uniquely defined, but the construction does provide a canonical model; see [EC2, Rmk.7.0.2]. It seems likely that one could work harder to obtain maps $\mathrm{Tmf} \to \mathrm{Tmf}_0(p)$ which act as the quotient map appearing in the definition of stable Hecke operators Df.7.2.1 (although these quotient maps over the cusp pose a whole new set of problems; see [Ces17]). As everything else we have done so far avoids obstruction theory, it seems morally bankrupt to appeal to it now, when we have come so far already.

Constructive methods We do not yet know of a direct way to use Lurie's constructive methods to construct ϕ, however, remain optimistic that [EC2] holds the key…somewhere. Indeed, by construction, we see that $\mathcal{E}_{\mathrm{sm}}$ is the orientation classifier of the identity component of the universal spectral deformation of $\operatorname{Spec} R_{\mathrm{sm}} \to \mathcal{M}_{\mathrm{Ell}}$, where $\operatorname{Spec} R_{\mathrm{sm}}$ is the pullback of $\operatorname{Spec} R$ along the affine inclusion $\mathcal{M}_{\mathrm{Ell}} \to \overline{\mathcal{M}}_{\mathrm{Ell}}$. Mapping out of such an $\mathbf{E}_\infty$-ring with such a description is simple, as highlighted by the proof of [EC2, Th.5.1.5], however, we cannot follow *loc. cit.* verbatim, as $E_{R_{\mathrm{sm}}}$ does not have a formally connected p-divisible group and it defines an ordinary (as opposed to a supersingular) elliptic curve on most of $\operatorname{Spec} R_{\mathrm{sm}}$. This seems to be the right way to go to obtain the map ϕ in a way that naturally commutes with maps induced by isogenies of generalised elliptic curves of invertible degree or compatible collections of automorphisms of quasi p-divisible groups. The author is currently experimenting with these ideas.

Part III

Computations and applications

Chapter 8

Generalities on endomorphisms of tmf

> Maintenant le principal est fait. Je tiens quelques évidences dont je ne peux me détacher. Ce que je sais, ce qui est sûr, ce que je ne peux nier, ce que je ne peux rejeter, voilà ce qui compte.
>
> *Albert Camus, The Myth of Sisyphus*

To calculate the effect of Adams operations or Hecke operators on the homotopy groups of Tmf, we appeal to a general paradigm first discussed in [Dav21a, §3.2]. This is the idea that the homotopy groups of tmf (and TMF) can be partitioned into two subgroups $\mathfrak{Tors} \oplus \mathfrak{Free}$: the first summand containing all torsion elements, and the second generated by certain torsion free elements. This decomposition is *natural* with respect to endomorphisms of spectra on tmf, meaning that for such an endomorphism f, then both $\mathfrak{Tors}$ and $\mathfrak{Free}$ are preserved by f. This fact is obvious for $\mathfrak{Tors}$ but highly nontrivial for $\mathfrak{Free}$. The following is a generalisation of [Dav21a, Cor.3.17] and has appeared as Th.D.

Theorem 8.0.1 (Splitting of $\pi_* \operatorname{tmf}$)**.** *Writing* $\mathfrak{Tors}$ *for the torsion subgroup of* $\pi_* \operatorname{tmf}$*, there is a splitting of abelian groups*

$$\pi_* \operatorname{tmf} \simeq \mathfrak{Tors} \oplus \mathfrak{Free}$$

which is natural with respect to endomorphisms of the spectrum tmf *and is compatible with localisations and completions, and also holds for* TMF*.*

The above theorem states that we can calculate the effect of endomorphisms of tmf on $\mathfrak{Free}$ after rationalisation as $f(\mathfrak{Free}) \subseteq \mathfrak{Free}$. Using similar arguments used to prove the above theorem, we can also produce a formula for calculating the effect of endomorphisms on $\mathfrak{Tors}$. The following is Th.8.0.2.

Theorem 8.0.2 (Calculations using DSS representatives)**.** *Let* $x \in \pi_* \operatorname{tmf}$ *be a torsion element with* DSS decomposition $a \cdot t$ *(Df.8.3.1) and* $f\colon \operatorname{tmf} \to \operatorname{tmf}$ *be an morphism of spectra. Then* $f(x)$ *is represented by* $f_{\mathrm{alg}}(a)t$ *on the* E_∞*-page, where* f_{alg} *is the map* f_{alg} *induces on the* E_2*-page. Moreover, if* x *is* nearby the Hurewicz image *(Df.8.3.1), then* $f(x) = f(1)x$*. The same result holds for* TMF*, as well as after localisation and completions at primes.*

Combined, these two theorems allow anyone to boil down a calculation of an endomorphism of tmf on homotopy groups to a rational algebraic calculation. Both theorems can be viewed as solving extension problems in a spectral sequence: given an element $x \in \pi_* \operatorname{tmf}$, we can represent x by an element y in the descent spectral sequence (read: Adams–Novikov spectral sequence) for tmf, and we ask if $f(y)$ jumps in filtration or not. A reinterpretation of both Ths.8.0.1 and 8.0.2 is that these jumps **never** occur, at least, not for endomorphisms $f\colon \operatorname{tmf} \to \operatorname{tmf}$ of spectra.

Remark 8.0.3. The above theorems also work for other nice spectra like **S** and KO for tautological reasons. As mentioned in the introduction, statements such as Th.8.0.1 do not work for Eilenberg–MacLane spectra such as $\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. We suspect the same is true for Tmf—we are currently mulling the details.

In §8.1, we discuss the *synthetic spectra* and the synthetic $\mathbf{E}_\infty$-ring of *motivic modular forms* mmf. These modern techniques are crucial to our proofs of Ths.8.0.1 and 8.0.2 which are carried out in §8.2 and §8.3. In §8.4, we discuss how Anderson and Serre duality can help us with some calculations involving $\pi_* \operatorname{Tmf}$.

8.1 Synthetic spectra and mmf

The collection of tools we will use to prove Th.8.0.1 comes from motivic homotopy theory. In [GIKR18], the authors define a candidate for (the 2-completion of) the motivic $\mathbf{E}_\infty$-ring mmf of *(connective) motivic modular forms*. This is done by constructing an artificial ∞-category $\mathcal{C}$ equivalent to the 2-completion of the cellular $\mathbf{C}$-motivic stable homotopy category $\mathrm{Sp}_{\mathbf{C}}$ and a lax monoidal functor $\mathrm{Sp} \to \mathcal{C}$. One then defines mmf as the image of tmf under this functor to $\mathcal{C}$. Our approach to studying a version of motivic modular forms is very much inspired by [GIKR18] and [Isa09]. Recently an alternative construction of an artificial ∞-category equivalent to $\mathrm{Sp}_{\mathbf{C}}$ after p-completion has been achieved by Pstrągowski; see [Pst19]. As the framework of Pstrągowski betters fits our applications, we will use his approach called *synthetic spectra*, which we now axiomatise.

Theorem 8.1.1 ([Pst19, §4-7])**.** *There is a symmetric monoidal stable* ∞*-category* Syn *of* synthetic spectra *and a subcategory* $\mathrm{Syn}^{\mathrm{ev}}$ *together with functors* $\iota\colon \mathrm{Sp} \to \mathrm{Syn}$, $L\colon \mathrm{Syn} \to \mathrm{Sp}$, $\Theta\colon \mathrm{Syn}^{\mathrm{ev}} \to \mathrm{Sp}_{\mathbf{C}}$ *such that the following conditions hold true:*

1. *The functor* ι *is lax symmetric monoidal.*

2. *The functor L is symmetric monoidal and a left adjoint.*
3. *The composite $L \circ \iota$ is naturally equivalent to the identity functor on* Sp.
4. *The functor Θ is an equivalence on p-complete objects.*

Furthermore, let us write $\mathbf{S}^{t,w} = \Sigma^{t-w}\iota\mathbf{S}^w$ inside Syn *for any pair of integers t, w, and $\pi_{t,w}Y$ for the associated bigraded homotopy groups of a synthetic spectrum Y. Then for any spectrum X, there is a natural map*

$$\pi_{t,w}\iota X \to \pi_t X \tag{8.1.2}$$

which exhibits $\pi_t X$ as the τ-localisation of $\pi_{t,*}\iota X$, where τ is a canonical element of $\pi_{0,-1}\mathbf{S}^{0,0}$.

Proof. In [Pst19], Syn is denoted by $\mathrm{Syn}_{\mathrm{MU}}$, $\mathrm{Syn}^{\mathrm{ev}}$ by $\mathrm{Syn}^{\mathrm{ev}}_{\mathrm{MU}}$, ι is ν, L is τ^{-1}, and Θ is Θ^*. Part 1 is then [Pst19, Lm.4.4], part 2 is [Pst19, Pr.4.32], part 3 is [Pst19, Cor.4.33], and part 4 is [Pst19, Th.7.34]. The "furthermore" statement is [Pst19, Rmk.4.40]. □

The following definition is essentially that of [GIKR18].

Definition 8.1.3. The synthetic spectrum mmf of *connective motivic modular forms* is $\iota(\mathrm{tmf})$. This has a canonical $\mathbf{E}_\infty$-structure as ι is lax symmetric monoidal.

One can calculate the homotopy groups of mmf just as Isaksen did in [Isa09, §5]—the added advantage of this setup with synthetic spectra is we have an integral definition before we make local calculations.

Proposition 8.1.4. *The bigraded homotopy groups of* $\mathrm{mmf}[\frac{1}{6}]$ *are given by the formula*

$$\pi_{*,*}\,\mathrm{mmf}[\frac{1}{6}] \simeq (\pi_*\,\mathrm{tmf}[\frac{1}{6}])[\tau] \simeq \mathrm{mf}_{\frac{*}{2}}^{\mathbf{Z}[\frac{1}{6}]}.$$

The bigraded homotopy groups of the 2-localisation of mmf *are described in [Isa09, §5] (up to faithfully flat base-change), and the 3-localisation is described in Figs.8.1 and 8.2. In this latter diagram, empty squares are a copy of $\mathbf{Z}_{(3)}[\tau]$, black circles are a copy of $\mathbf{F}_3[\tau]$, red circles are a copy of $\mathbf{F}_3[\tau]/\tau^2$, yellow circles are a copy of $\mathbf{F}_3[\tau]/\tau^4$, dashed lines are exotic multiplication (filtration jumps), red lines are multiplication by α, and blue lines are multiplication by β. If one ignores the coloured dots, one obtains the homotopy groups of* tmf_3*, as* $L\iota\,\mathrm{tmf} \simeq L\,\mathrm{mmf} \simeq \mathrm{tmf}$.

Proof. By [BHS19, Th.A.6], the E_2-page for the ιMU-based Adams–Novikov spectral sequence for mmf has E_2-page the classical MU-based Adams–Novikov spectral sequence for tmf tensored with $\mathbf{Z}[\tau]$, and all synthetic d_r-differentials are equal to $\tau^{r-1}d_r^{\mathrm{cl}}$ for a classical d_r^{cl}-differential. We can now determine $\pi_{*,*}\,\mathrm{mmf}[\frac{1}{6}]$ as the ιMU-based Adams–Novikov spectral sequence is concentrated in filtration zero. At the primes 2 and 3, we refer to [Bau08] to obtain

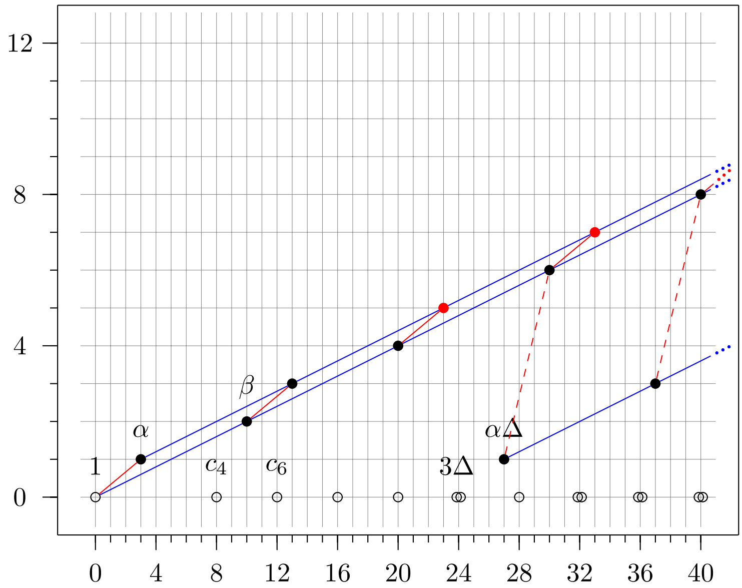


Figure 8.1: Homotopy groups of $\mathrm{mmf}_{(3)}$ in the range $0 \leqslant s \leqslant 40$.

the classical E_2-page and the classical differentials which in turn gives us the calculation of $\pi_{*,*}\,\mathrm{mmf}_2$ and $\pi_{*,*}\,\mathrm{mmf}_{(3)}$. This is precisely how Isaksen calculations $\pi_{*,*}\,\mathrm{mmf}_2$ in [Isa09, §5], so his calculations apply here (using the faithfully flat map $\mathbf{Z}_{(2)} \to \mathbf{Z}_2$). The simpler calculation of $\pi_{*,*}\,\mathrm{mmf}_{(3)}$ appears in Figs.8.1 and 8.2. □

Using the bigraded homotopy groups of mmf calculated above at the primes 2 and 3, we can now prove our first main result of this section.

8.2 Proof of the splitting theorem

In this section, we will prove Th.8.0.1. We first have to define our subgroup $\mathfrak{Free}$ of $\pi_*\,\mathrm{tmf}$. Intuitively, this subgroup is generated by the "modular forms" in $\pi_*\,\mathrm{tmf}$, which we make precise now. We will follow the notation of [Bau08], [DFHH14, §13], and [Kon12]—the relationship with [BR21] is also mentioned below at the prime 2.

Notation 8.2.1. The elements of $\mathfrak{Tors} \subseteq \pi_*\,\mathrm{tmf}$ are simply the torsion elements, which can also be interpreted as elements in strictly positive filtration in the descent spectral sequence; see [Bau08]. The elements of $\mathfrak{Free}$ in nonnegative degree are then described in the following three cases:

- When 6 is inverted, $\mathfrak{Free}$ is multiplicatively generated by the classes c_4 and c_6 corresponding to the normalised Eisenstein series of weight 4 and

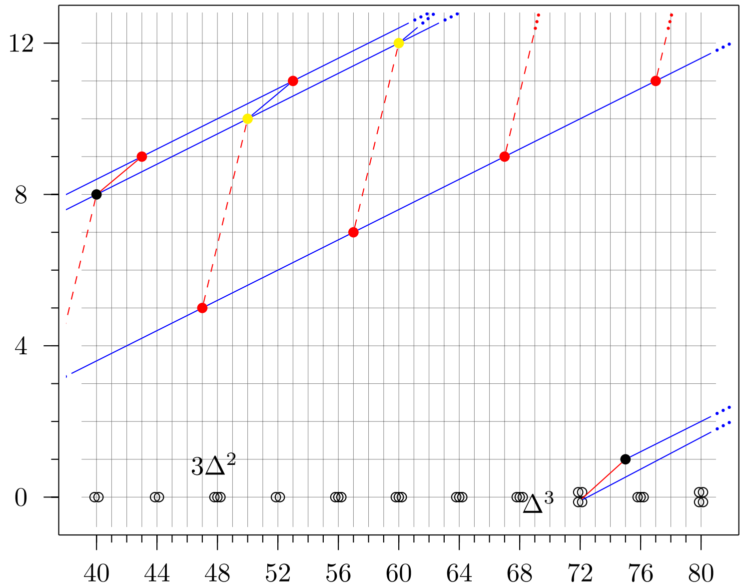


Figure 8.2: Homotopy groups of $\mathrm{mmf}_{(3)}$ in the range $40 \leqslant s \leqslant 80$.

6, respectively, which are uniquely determined by the collapsing descent spectral sequence. In this case $\mathfrak{Free} = \pi_* \operatorname{tmf}[\frac{1}{6}]$.

- Using the notation of [DFHH14, §13], when localised at 3, $\mathfrak{Free}$ is multiplicatively generated by the classes

$$c_4, c_6, [3\Delta], [c_4\Delta], [c_6\Delta], [3\Delta^2], [c_4\Delta^2], [c_6\Delta^2], \Delta^3.$$

- When localised at 2, $\mathfrak{Free}$ is multiplicatively generated by the classes[44]

$$c_4, [2c_6], [8\Delta^{2i+1}], [4\Delta^{4j+2}], [2\Delta^4], [c_4\Delta^{k+1}], [2c_6\Delta^{k+1}], \Delta^8$$

for $i \in \{0, 1, 2, 3\}$, $j \in \{0, 1\}$, and $k \in \{0, 1, 2, 3, 4, 5, 6\}$. Using the notation of [BR21, §9.1], these correspond to the following elements:

$$B, C, D_{2i+1}, D_{4j+2}, D_4, B_{k+1}, C_{k+1}, M$$

Define the splitting $\mathfrak{Tors} \oplus \mathfrak{Free} \simeq \pi_* \operatorname{TMF}$ by inverting Δ^{24}.

[44] We define c_4 as the unique class in $\pi_8 \operatorname{tmf}_{(2)}$ which is mapped to the well-defined c_4 in $\pi_8 \operatorname{tmf}_{\mathbf{Q}}$ and which is also κ-torsion. The same goes for the classes $[c_4\Delta^{k+1}]$ below, which we furthermore take to be both κ- and $\overline{\kappa}$-torsion. We also define $[2c_6\Delta^2]$ similarly by demanding it is $\overline{\kappa}$-torsion. For $[2c_6\Delta^6]$, we can equivalently define this as the element C_6 as in [BR21, §9.1], or using mmf as the image under the localisation map $\pi_{156,*} \operatorname{mmf} \to \pi_* \operatorname{tmf}$ of the element $\underline{[2c_6\Delta^6]} \in \pi_{156,0} \operatorname{mmf}$, itself defined to hit $2c_6\Delta^6$ in $\operatorname{mf}_{78}^{\mathbf{Z}_2}$ and also by demanding it to be ν-torsion; see [Isa09, §5].

Proof of Th.8.0.1. From the definition of $\mathfrak{Free}$ (Nt. 8.2.1) it suffices to prove Th.8.0.1 after completing at the primes 2 and 3—the case away from 6 follows as $\pi_* \operatorname{tmf}[\frac{1}{6}]$ has no torsion.

The $p = 3$ case At the prime 3, we note that the only degrees d where $\mathfrak{Free}$ and $\mathfrak{Tors}$ have a nontrivial intersection are those congruent to 20 or 40 modulo 72. Due to Δ^3-periodicity, the argument that follows works equally as well for any choice of positive d congruent to 20 (resp. 40) modulo 72, so let us only prove the $d = 20$ (resp. $d = 40$) case explicitly.

For a class x in $\mathfrak{Free}_{20}^{\mathbf{Z}_3}$, we see that $x\beta = 0$ from the multiplicative structure of $\pi_* \operatorname{tmf}_3$. Using this, and the fact that f is a map of spectra and hence commutes with elements in the image of the unit map $\mathbf{S} \to \operatorname{tmf}$, we obtain the equality $f(x)\beta = f(x\beta) = 0$. All of the nonzero classes in $\mathfrak{Tors}_{20}^{\mathbf{Z}_3}$ support nonzero multiplication by β, so we see that $f(x)$ must lie in $\mathfrak{Free}_{20}^{\mathbf{Z}_3}$ in this case. For the $d = 40$ case, consider the commutative diagram from the "furthermore" part of Th.8.1.1

$$\begin{array}{ccc} \pi_{k,w} \operatorname{mmf}_3 & \xrightarrow{\iota f} & \pi_{k,w} \operatorname{mmf}_3 \\ \big\downarrow l & & \big\downarrow l \\ \pi_k \operatorname{tmf}_3 & \xrightarrow{f} & \pi_k \operatorname{tmf}_3 \end{array} \tag{8.2.2}$$

where k is an integer and the vertical maps are the τ-localisations. The associated graded of the homotopy group $\pi_{40,*} \operatorname{mmf}_3$ can be found in the 40th column of Fig.8.1 and $\pi_{40,w} \operatorname{mmf}_3$ is the summand with weight w, ie, with τ-degree w. Using the degree 3 calculation, we can also determine that the generator α of $\pi_3 \operatorname{tmf}_3 \simeq \mathbf{Z}/3\mathbf{Z}$ has a lift (under l) in $\pi_{3,0} \operatorname{mmf}_3$, which we will also denote by α. Using this calculation of $\pi_{40,0} \operatorname{mmf}_3$, we see that every element x in $\mathfrak{Free}_{40}^{\mathbf{Z}_3}$ has a lift y inside $\pi_{40,0} \operatorname{mmf}_3$ such that y is α-torsion. It is clear $\iota f(y)$ is also α-torsion—this is same argument made above inside $\pi_{20} \operatorname{tmf}_3$. The next key observation is that α-torsion elements inside $\pi_{40,0} \operatorname{mmf}_3$ are sent to α-torsion—this is the same argument as the $d = 20$ case inside $\pi_{20} \operatorname{tmf}_3$. By Fig.8.2, we see α-torsion elements of $\pi_{40,0} \operatorname{mmf}_3$ are sent to $\mathfrak{Free}_{40}^{\mathbf{Z}_3}$ inside $\pi_{40} \operatorname{tmf}_3$.

Altogether, the commutativity of (Equation (8.2.2)) then shows that the element $f(x) = (l \circ \iota f)(y)$ lies in $\mathfrak{Free}_{40}^{\mathbf{Z}_3}$, as desired.

The $p = 2$ case By Δ^8-periodicity we only consider those degrees between 0 and 191. For those degrees d equal to $8, 28, 32, 52, 104, 124, 128, 136$, and 148 we note that $\mathfrak{Free}_d$ contains only κ-torsion and the map $\cdot\kappa\colon \mathfrak{Tors}_d \to \mathfrak{Tors}_{d+14}$ is injective; one can check this on any copy of the 2-primary homotopy groups of tmf. We conclude that for $x \in \mathfrak{Free}_d$ in these degrees, we have $f(x) \in \mathfrak{Free}_d$, if not $\kappa f(x)$ would be nonzero, contradicting the fact that $\kappa f(x) = f(\kappa x) = 0$. For d equal to 80, we apply the same trick with κ replaced by $\overline{\kappa}$. The only d

between 0 and 191 left to check, where $\mathfrak{Tors}$ and $\mathfrak{Free}$ are both nonzero, are those in the following set of numbers:

$$D_1 = \{20, 40, 60, 68, 100, 116, 156, 164\}$$

Now we appeal to mmf_2, the calculations found in [Isa09], and mimic the arguments from the $p = 3$ case. For d equal to 20 or 116, consider the $\overline{\kappa}$-torsion elements of $\pi_{d,0}\,\mathrm{mmf}_2$, and for all other $d \in D_1$, consider the ν-torsion inside $\pi_{d,0}\,\mathrm{mmf}_2$. It is clear that ιf preserves both $\overline{\kappa}$- and ν-torsion elements, and the tables for $\pi_{*,*}\,\mathrm{mmf}_2$ found in [Isa09] show that each $x \in \mathfrak{Free}_d^{\mathbf{Z}_2}$ lifts to an y in $\pi_{d,0}\,\mathrm{mmf}$ which is either $\overline{\kappa}$- or ν-torsion. Observing that in these degrees the $\overline{\kappa}$-torsion and ν-torsion elements of $\pi_{d,0}\,\mathrm{mmf}_2$ are sent to $\mathfrak{Free}_d^{\mathbf{Z}_2}$ as argued for $\mathfrak{Free}_{40}^{\mathbf{Z}_3}$ above, then an application of the 2-complete version of (8.2.2) concludes our proof.

For TMF one runs the same arguments as for tmf above and uses Δ^{24}-periodicity. The only difference is that $\pi_4\,\mathrm{TMF}$ is nonzero, however, this group is torsion free so we have no additional problems. ☐

8.3 Torsion in homotopy groups

Let us now discuss how morphisms of spectra $f\colon \mathrm{tmf} \to \mathrm{tmf}$ (or endomorphisms of TMF) behave on the torsion classes of $\pi_*\,\mathrm{tmf}$ (or $\pi_*\,\mathrm{TMF}$).

Definition 8.3.1. Let x be a homogeneous element of $\mathfrak{Tors}$. We say x is *nearby the Hurewicz image* if it can be written as a linear combination of any of the following three families of elements:

1. The image of the map induced by the unit $\mathbf{S} \to \mathrm{tmf}$ on homotopy groups. We call this the *Hurewicz image*.
2. Those torsion elements x such that for some elements y in the Hurewicz image, the product xy is also in the Hurewicz image, and the product map
$$\cdot y\colon \pi_{|x|}\,\mathrm{tmf} \to \pi_{|x|+|y|}\,\mathrm{tmf}$$
is injective.
3. Those torsion elements $x = yz$, where y is in the Hurewicz image and z is in the second case above.

Let x be a torsion element of $\pi_*\,\mathrm{tmf}$. If x is represented on the E_2-page of the descent spectral sequence (DSS) for tmf by a permanent cycle $a \cdot t$ where a lies in $E_2^{|a|,0}$ and t is a permanent cycle of strictly positive filtration representing an element in the Hurewicz image, then we call $a \cdot t$ a *DSS decomposition* of x.

It is clear that every class $x \in \mathfrak{Tors}$ is a linear combination of DSS decompositions. Using these decompositions, one can delegate the task of calculating endomorphisms evaluated on torsion classes to calculate endomorphisms on classes in $\mathfrak{Free}$.

Proposition 8.3.2. *Let f be an endomorphism of the spectrum* tmf*,* TMF*, or any of their localisations at a set of primes or completion at a particular prime. Then for any homogeneous torsion element x which is nearby the Hurewicz image, we have the equality $f(x) = xf(1)$ in homotopy groups.*

Proof. If x is in the Hurewicz image, then the **S**-linearity of f implies that $f(x) = xf(1)$. For x in the second case of Df.8.3.1, we see the map

$$\cdot y \colon \pi_{|x|} \operatorname{tmf} \to \pi_{|x|+|y|} \operatorname{tmf}$$

is injective and f commutes with this map by **S**-linearity, so it suffices to compute $f(xy)$, which reduces us to the first case. For the third case, we use **S**-linearity and the second case. □

We can now prove the second main result of this chapter.

Proof of Th.8.0.2. The "moreover" statement is precisely Pr.8.3.2, so let us focus on the first statement. As in the proof of Th.8.0.1, the cases for TMF follow more easily than those for tmf, so let us focus on the latter.

First, note the functoriality of the Adams–Novikov spectral sequence with respect to MU for tmf; see [Rav04, Th.2.2.3]. It follows that $f_{\mathrm{alg}}(a)t$ detects $f(x)$ up to higher filtration on the E_∞-page, so our goal is now to show that there are no elements in higher filtration to worry about. We know that $f_{\mathrm{alg}}(a)t + w$ detects $f(x)$, where w is a permanent cycle of filtration strictly higher than $a \cdot t$. This implies that $f(x) = [f_{\mathrm{alg}}(a)t] + y$ inside the homotopy groups of mmf, where y is detected by w and $f_{\mathrm{alg}} \colon E_2^{s,t} \to E_2^{s,t}$ is the morphism on E_2-pages induced by f. To show $y = 0$ (and hence also $w = 0$), we will work case-by-case, starting in nonnegative degrees. The argument is rather like the proof of Th.8.0.1.

As is often the case, a glance at the homotopy groups of tmf shows that the only time a DSS decomposition $a \cdot t$ is not the class in highest filtration on the E_∞-page of the DSS is at the prime 2. Let us then implicitly complete at the prime 2. The only nonnegative degrees of x such that a DSS decomposition $a \cdot t$ is not the class in highest filtration are those degrees d congruent modulo 192 to an element in the following list D_2:

$$3, 9, 17, 27, 33, 34, 41, 42, 51, 54, 57, 65, 66, 90, 99, 105$$

$$110, 113, 123, 130, 137, 138, 147, 150, 153, 161, 162$$

By Δ^8-periodicity, we must only consider classes with degree **equal** to an element in D_2. Let us detail the cases of $d = 17$ and $d = 3$, and only outline the other cases where the same arguments should be made.

($d = 17$) In this case we have $f(x) = [f_{\text{alg}}(a)t] + y$, where y is a multiple of the class $\nu\kappa \in \pi_{17}\,\text{tmf}$. The morphism f commutes with multiplication by ν, as the latter lies in the Hurewicz image inside $\pi_*\,\text{tmf}$, so this would imply

$$0 = f(\nu x) = \nu f(x) = \nu\left([f_{\text{alg}}(a)t] + y\right) = \nu y$$

where we used that $\nu[f_{\text{alg}}(a)t] = 0$ which follows from the ring structure on $\pi_*\,\text{tmf}$. As νy vanishes in $\pi_{20}\,\text{tmf}$ only if $y = 0$, we see that $f(x) = [f_{\text{alg}}(a)t]$.

($d = 3$) In this case we have $f(x) = [f_{\text{alg}}(a)t] + y$, where y is a multiple of the class $4\nu \in \pi_3\,\text{tmf}$. Rather than looking at the ring structure on $\pi_*\,\text{tmf}$, we use $\pi_{*,*}\,\text{mmf}$ (completed at the prime 2). The functor ι applied to f yields a map $\iota f\colon \text{mmf} \to \text{mmf}$ which in turn induces a map $\pi_{3,*}\,\text{mmf} \to \pi_{3,*}\,\text{mmf}$. Recall the elements $\overline{\kappa} \in \pi_{20,0}\mathbf{S}$ and $\tau \in \pi_{0,-1}\mathbf{S}$, and that both of these elements have no zero image inside $\pi_{*,*}\,\text{mmf}$; see [Isa09, §5]. Let us now play the same game as the $d = 17$ case above, but applied to $\overline{\kappa}\tau^2$ as opposed to ν. Inside $\pi_{*,*}\,\text{mmf}$ we have the equalities:

$$\overline{\kappa}\tau^2 x = 0 \in \mathbf{Z}/4\mathbf{Z}[\tau]/\tau^2 \qquad \overline{\kappa}\tau^2\nu \neq 0 \in \mathbf{Z}/2\mathbf{Z}[\tau]/\tau^3 \tag{8.3.3}$$

As $f(x) = [f_{\text{alg}}(a)t]+y$ inside $\pi_*\,\text{tmf}$, then inside $\pi_{*,*}\,\text{mmf}$ we have the equality $f(x) = [f_{\text{alg}}(a)t] + y + z$, where z is τ-power torsion, in particular, it is linearly independent from y. We then consider the equalities

$$0 = \iota f(\overline{\kappa}\tau^2 x) = \overline{\kappa}\tau^2 \iota f(x) = \overline{\kappa}\tau^2\left([f_{\text{alg}}(a)t] + y + z\right) = \overline{\kappa}\tau^2(y + z).$$

From (8.3.3), the above only holds if $y = 0$, as y and z are linearly independent, which yields the desired result.

($d \in D_2$) For the rest of the degrees, let us mention if our intended argument requires mmf or not, and what class in the (synthetic) Hurewicz image can be used as we used ν and $\overline{\kappa}\tau^2$ above.

d	9	27	33	34	41	42	51	54	57	65	66
mmf	✓	✓			✓	✓	✓	✓			
Hurewicz	$\overline{\kappa}$	$\overline{\kappa}\tau^2$	$\overline{\kappa}$	$\overline{\kappa}$	$\overline{\kappa}$	η	$\overline{\kappa}\tau^2$	$\overline{\kappa}\tau^2$	ν	$\overline{\kappa}$, ν	★

d	90	99	105	110	113	123	129	130
mmf	✓	✓	✓	✓		✓		✓
Hurewicz	$\overline{\kappa}$	$\overline{\kappa}\tau^2$	$\overline{\kappa}, \overline{\kappa}$	$\overline{\kappa}^3\tau^3$	ν	$\overline{\kappa}\tau^2$	$\overline{\kappa}$	$\overline{\kappa}^2\tau^6$, ★

d	137	138	147	150	153	161	162
mmf	✓		✓	✓			
Hurewicz	$\overline{\kappa}$	★	$\overline{\kappa}\tau^2$	$\overline{\kappa}\tau^2, \overline{\kappa}\tau^6$	ν	ν	★

Take $d = 9$ for example, we use the fact that $\overline{\kappa}\eta c_4 = 0$ and $\overline{\kappa}\nu^3$ is nonzero in $\pi_{24,0}\,\text{mmf}$ and apply the same arguments as above. The cases marked with a ★ follow from the previous case by multiplication by η. In degrees 65, 105, 130, and 150, there are two higher filtrations to consider, and they can be dealt with using the elements that appear in the table. This completes our proof. □

8.4 Anderson and Serre duality

The following section is superfluous for those interested solely in TMF and tmf, however, Tmf satisfies a kind of duality that can help in calculations.

Definition 8.4.1. For an injective abelian group J, we write I_J for the spectrum represented by the following cohomology theory:

$$\mathrm{Sp} \to \mathrm{Ab}_* \qquad X \mapsto \mathrm{Hom}_{\mathrm{Ab}_*}(\pi_{-*}X, J)$$

For a general abelian group A, take an injective resolution $0 \to A \to J_1 \to J_2$ which induces a morphism of spectra $I_{J_1} \to I_{J_2}$. The fibre of this morphism we denote by I_A, and for a spectrum X, we define the *Anderson dual of* X to be the function spectrum $I_A X = F(X, I_A)$.

From the definition above one can calculate

$$\pi_* I_J X \simeq \mathrm{Hom}_{\mathbf{Z}}(\pi_{-*}X, J)$$

for an injective abelian group J. When A is a general abelian group, we obtain the following natural exact sequence of abelian groups for all $k \in \mathbf{Z}$

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(\pi_{-k-1}X, A) \to \pi_k I_A X \to \mathrm{Hom}_{\mathbf{Z}}(\pi_{-k}X, A) \to 0 \qquad (8.4.2)$$

which non canonically splits when A is a subring of $\mathbf{Q}$. More basic facts about Anderson duality, such as the fact that the natural map $X \to I_A I_A X$ is an equivalence when X has finitely generated homotopy groups, can be found in [SAG, §6.6], under the guise of *Grothendieck duality* in spectral algebraic geometry. Anderson duality is of interest to us as Tmf is *Anderson self-dual*; see Ex.8.4.4.

Definition 8.4.3. Let X be a spectrum and A an abelian group. We say that X is *Anderson self-dual* if there is an integer d and an equivalence of spectra

$$\phi\colon \Sigma^d X \xrightarrow{\simeq} I_A X.$$

We also want to define a stricter form of self-duality for ring spectra. Let R be an $\mathbf{E}_1$-ring with $\pi_0 R \simeq A$ such that $\pi_{-d}R$ is a free A-module of rank one. We say an element $D \in \pi_{-d}R$ *witnesses the Anderson self-duality of* R if the following holds: the isomorphism $\phi_D\colon \pi_{-d}R \to A$ sending $D \mapsto 1$ which identifies D as an A-module generator of $\pi_{-d}R$, lifts to an element $D^\vee \in \pi_d I_A R$ under the surjection of (8.4.2), and the representing map of left R-modules $D^\vee : \Sigma^d R \to I_A R$ is an equivalence.

Example 8.4.4. There are some famous examples of Anderson self-duality.

- The class $1 \in \pi_0\mathrm{KU}$ witnesses the Anderson self-duality of KU, ie,

$$1^\vee : \mathrm{KU} \xrightarrow{\simeq} I_{\mathbf{Z}}\mathrm{KU}$$

  is an equivalence. This is originally due to Anderson [And69], and is an immediate consequence of the fact that $\mathrm{Hom}_{\mathbf{Z}}(\pi_*\mathrm{KU}, \mathbf{Z})$ is a free $\pi_*\mathrm{KU}$-module; see [HS14, p.3].

- The class $vu_{\mathbf{R}}^{-1} \in \pi_{-4}\,\mathrm{KO}$ witnesses the Anderson self-duality of KO, ie,
$$(vu_{\mathbf{R}}^{-1})^{\vee} : \Sigma^4\,\mathrm{KO} \xrightarrow{\simeq} I_{\mathbf{Z}}\,\mathrm{KO}$$
is an equivalence. This result is also due to Anderson, and an accessible modern proof (with an eye towards spectral algebraic geometry) can be found in [HS14, Th.8.1].

- The class $D = [c_4^{-1}c_6\Delta^{-1}] \in \pi_{-21}\,\mathrm{Tmf}$ witnesses the Anderson self-duality of Tmf, ie,
$$D^{\vee} : \Sigma^{21}\,\mathrm{Tmf} \xrightarrow{\simeq} I_{\mathbf{Z}}\,\mathrm{Tmf}$$
is an equivalence. This result is due to Stojanoska; see [Sto12, Th.13.1] for the case with 2 inverted and [Sto14] where it is announced in general.

- The class $\frac{1}{\lambda_1\lambda_2} \in \pi_{-9}\,\mathrm{Tmf}(2)$ witnesses the Anderson self-duality of Tmf(2), ie,
$$\left(\frac{1}{\lambda_1\lambda_2}\right)^{\vee} : \Sigma^9\,\mathrm{Tmf}(2) \xrightarrow{\simeq} I_{\mathbf{Z}[\frac{1}{2}]}\,\mathrm{Tmf}(2)$$
is an equivalence. This is also due to Stojanoska; see [Sto12, Th.9.1].

- There are classes D_m in $\pi_{l_m}\,\mathrm{Tmf}_1(m)$, with m and l_m taking the values

m	2	3	4	5	6	7	8	11	14	15	23
l_m	13	9	7	5	5	3	3	1	1	1	−1

which witnesses the Anderson self-duality of these particular $\mathrm{Tmf}_1(m)$, ie, the map
$$D_m^{\vee} : \Sigma^{l_m}\,\mathrm{Tmf}_1(m) \xrightarrow{\simeq} I_{\mathbf{Z}[\frac{1}{m}]}\,\mathrm{Tmf}_1(m)$$
is an equivalence. This result is due to Meier [Mei22, Th.5.14], where it is also shown the above are the only $m \geqslant 2$ such that $\mathrm{Tmf}_1(m)$ is Anderson self-dual.

Studying endomorphisms of Anderson self-dual spectra leads us to *dual endomorphisms.*

Definition 8.4.5. Let A be an abelian group, X an Anderson self-dual spectrum, and $F: X \to X$ an endomorphism of X. Define the *dual endomorphism* of F as the composite
$$\check{F}: X \xrightarrow{\phi,\simeq} \Sigma^{-d}I_AX \xrightarrow{\Sigma^{-d}I_AF} \Sigma^{-d}I_AX \xleftarrow{\phi,\simeq} X.$$

Given A, X, and F from the above definition, then the naturality of (8.4.2) yields the following commutative diagram of abelian groups with exact rows for all $k \in \mathbf{Z}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^1_{\mathbf{Z}}(\pi_{-k-1-d}X, A) & \longrightarrow & \pi_kX & \longrightarrow & \mathrm{Hom}_{\mathbf{Z}}(\pi_{-k-d}X, A) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{Ext}^1_{\mathbf{Z}}(F,A)=F_1^*} & & \Big\downarrow{\scriptstyle \check{F}} & & \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Z}}(F,A)=F_0^*} & & \\ 0 & \longrightarrow & \mathrm{Ext}^1_{\mathbf{Z}}(\pi_{-k-1-d}X, A) & \longrightarrow & \pi_kX & \longrightarrow & \mathrm{Hom}_{\mathbf{Z}}(\pi_{-k-d}X, A) & \longrightarrow & 0 \end{array} \tag{8.4.6}$$

Our calculations of ψ^n on Tmf in negative degrees will rest upon explicit calculations of $\check{\psi}^n$ and using (8.4.6).

When working with 6 inverted, these also exists a kind of algebro-geometric duality on $\overline{\mathcal{M}}_{\mathrm{Ell}}$ called *Serre duality.* The following can be found in [Mei20, Ap.A] using the well-known identification of $\overline{\mathcal{M}}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{6}]}$ with the weighted projective line $\mathcal{P}_{\mathbf{Z}[\frac{1}{6}]}(4,6)$; see [Mei20, Ex.2.1].

Theorem 8.4.7. *The dualising sheaf for $\overline{\mathcal{M}}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{6}]}$ is ω^{-10}. In particular, for any integer k the natural cup product map*

$$H^0(\overline{\mathcal{M}}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{6}]},\omega^k)\otimes H^1(\overline{\mathcal{M}}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{6}]},\omega^{-k-10})\to H^1(\overline{\mathcal{M}}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{6}]},\omega^{-10})\simeq \mathbf{Z}[\frac{1}{6}]$$

is a perfect pairing of $\mathbf{Z}[\frac{1}{6}]$-modules.

Let us note that the stack $\overline{\mathcal{M}}_{\mathrm{Ell}}$ certainly has **no** Serre duality before inverting 6, which can be seen through the cohomology calculations of ω^* over $\overline{\mathcal{M}}_{\mathrm{Ell}}$ from [Kon12].

Remark 8.4.8. A simple consequence of the above theorem is that one can immediately see the $\mathbf{E}_\infty$-ring $\mathrm{Tmf}[\frac{1}{6}]$ is Anderson self-dual. Indeed, as discussed on [Sto12, p.8], the Serre duality statement of Th.8.4.7, the calculation of $H^*(\overline{\mathcal{M}}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{6}]},\omega^*)$ in [Kon12, §3], and a collapsing descent spectral sequence, immediately implies the Anderson self-duality of $\mathrm{Tmf}[\frac{1}{6}]$ as in Ex.8.4.4.

When 6 is inverted, dual endomorphisms on Tmf defined using Anderson duality can be computed directly using Serre duality.

Lemma 8.4.9. *Let k be a positive integer, and $\mathcal{P}$ be a set of primes containing both 2 and 3 and implicitly localise everywhere away from $\mathcal{P}$. If $F\colon \mathrm{Tmf}\to\mathrm{Tmf}$ is a morphism of spectra, then $\check{F}$ can be written as follows:*

$$\check{F}\colon \pi_k\,\mathrm{Tmf}\simeq H^1(\overline{\mathcal{M}}_{\mathrm{Ell}},\omega^{-\frac{k}{2}-10})^\vee \xrightarrow{F^\vee} H^1(\overline{\mathcal{M}}_{\mathrm{Ell}},\omega^{-\frac{k}{2}-10})^\vee\simeq \pi_k\,\mathrm{Tmf}$$

$$\check{F}\colon \pi_{-k}\,\mathrm{Tmf}\simeq H^0(\overline{\mathcal{M}}_{\mathrm{Ell}},\omega^{\frac{k-1}{2}-10})^\vee \xrightarrow{F^\vee} H^0(\overline{\mathcal{M}}_{\mathrm{Ell}},\omega^{\frac{k-1}{2}-10})^\vee\simeq \pi_{-k}\,\mathrm{Tmf}$$

Above, we have implicitly used the Serre duality isomorphism from Th.8.4.7.

Proof. This follows immediately from the definitions, as in this case, the Anderson duality equivalence comes directly from Serre duality; see Rmk.8.4.8. □

Chapter 9

Fundamental calculations

The previous two chapters defined endomorphisms of TMF and discussed how to calculate their effect on homotopy groups—this proves Th.G. In this chapter, we carry out these calculations explicitly. First, is our calculation of (p-adic) stable Adams operations. Let us use the handicraft operations ψ^k on Tmf_p as well as make a statement for TMF_p.

Theorem 9.0.1. *Given a prime p and a p-adic unit $k \in \mathbf{Z}_p^\times$ we have the following equality for every homogeneous element $x \in \pi_* \mathrm{Tmf}_p$:*

$$\psi^k(x) = \begin{cases} x & x \in \mathfrak{Tors} \\ k^{\lceil \frac{|x|}{2} \rceil} x & x \in \mathfrak{Free} \end{cases}$$

In particular, for ψ^k acting on TMF_p, this states that $\psi^k(x) = x$ if x is torsion and $k^{\frac{|x|}{2}} x$ if x lies in $\mathfrak{Free}$.

Our result for Hecke operators is less explicit as it relies on some arithmetic input: the algebraic Hecke operators $\mathrm{T}_n^{\mathrm{alg}}$ of Df.7.5.2.

Theorem 9.0.2. *Given a positive integer $n \geqslant 1$ then:*

- *For each homogeneous element $x \in \mathfrak{Free} \subseteq \pi_* \mathrm{TMF}[\frac{1}{n}]$ the image of x under T_n satisfies $\mathrm{T}_n(x) = n\mathrm{T}_n^{\mathrm{alg}}(x)$, where $\mathrm{T}_n^{\mathrm{alg}}$ are the classical Hecke operators acting on x considered as a classical modular form.*

- *For each homogeneous element $x \in \mathfrak{Tors} \subseteq \pi_* \mathrm{TMF}[\frac{1}{n}]$ the element $\mathrm{T}_n(x)$ is represented by $n\mathrm{T}_n^{\mathrm{alg}}(a)t$ on the E_∞-page of the descent spectral sequence, where at is a* DSS decomposition *(Df.8.3.1) for x.*

The above theorem is stated abstractly, but with some knowledge of the classical Hecke operators (Df.7.5.2), one can perform many calculations. In §10, we see many examples of such calculations, but let us list a few important ones here inside $\pi_* \operatorname{TMF}[\frac{1}{n}]$. First, recall from (39) the *generalised divisor function* $\sigma_k(n)$, as well as the *Ramanujan's τ-function* $\tau(n)$.[45]

$$\mathrm{T}_n(1) = \sigma_1(n) \qquad \mathrm{T}_n(c_4) = \sigma_3(n)c_4 \qquad \mathrm{T}_n([24\Delta]) = n\tau(n)[24\Delta]$$

Using the DSS decompositions (8.3.1), it is also easy to do calculations on torsion elements. Let us list some examples inside $\pi_* \operatorname{TMF}_2$, so assume that n is odd:

$$\mathrm{T}_n(\eta) = \sigma(n)\eta \qquad \mathrm{T}_n(\eta c_4) = \sigma_3(n)\eta c_4 \qquad \mathrm{T}_n([2\nu\Delta]) = n\tau(n)[2\nu\Delta]$$

We can prove our calculation for Hecke operators immediately using the results we have already seen.

Proof. If $x \in \mathfrak{Free}$, then by Th.8.0.1 we may work rationally, in which case the edge map e is an isomorphism and the result follow from Pr.7.5.3. For $x \in \mathfrak{Tors}$, apply Th.8.0.2 and use Rmk.7.5.6 to identify $(\mathrm{T}_n)_{\mathrm{alg}}$ with $n\mathrm{T}_n^{\mathrm{alg}}$. □

Our calculation for the stable Adams operations is more explicit, so the proof is a little longer, and will occupy the following section.

9.1 Calculation of Adams operations

Proof of Th.9.0.1. First consider homogeneous elements x in $\pi_* \operatorname{Tmf}_p$ lying in $\mathfrak{Free}$. As the natural localisation map

$$\pi_* \operatorname{Tmf}_p \to \pi_* \operatorname{Tmf}_p[\frac{1}{p}]$$

is injective on the submodule $\mathfrak{Free}$ of $\pi_* \operatorname{Tmf}_p$, and the morphism ψ^n preserves $\mathfrak{Free}$ by Th.8.0.1, then it suffices to work inside $\pi_* \operatorname{Tmf} \otimes \mathbf{Q}_p$. If the degree of x is nonnegative, the descent spectral sequence collapses immediately (as 6 is inverted) and we see that $\psi^k(x) = k^{\frac{|x|}{2}} x$, as this what the k-fold multiplication map induces on ω; see Pr.5.5.3 for the case over $\mathcal{M}_{\mathrm{Ell}}$. The ceiling function, in this case, is unnecessary.

If the degree of x is negative we have to compute the morphism

$$\psi^n \colon H^1(\overline{\mathcal{M}}_{\mathrm{Ell},\mathbf{Q}_p}, \omega^k) \to H^1(\overline{\mathcal{M}}_{\mathrm{Ell},\mathbf{Q}_p}, \omega^k)$$

for all $k < 0$. This we can do with a calculation of the cohomology of the stack with graded structure sheaf $(\overline{\mathcal{M}}_{\mathrm{Ell},\mathbf{Q}_p}, \omega^*)$, which is equivalent to the weighted projective line $\mathcal{P}_{\mathbf{Q}_p}(4,6)$; see [Mei20, Ex.2.1]. In this case we can use the fact that groups $H^*(\mathcal{P}_{\mathbf{Q}_p}(4,6), \omega^*)$ are isomorphic to the groups $H^*(\widetilde{P}(4,6), \mathscr{O})$,

[45] Define the *Ramanujan τ-function* as the coefficients in the q-expansion of Δ:

$$\sum_{n \geqslant 1} \tau(n) q^n = q \prod_{n \geqslant 1} (1 - q^n)^{24}$$

where $(\widetilde{P}(4,6), \mathscr{O})$ is $(\operatorname{Spec} A - \{0\}, \mathscr{O})$, where $A = \mathbf{Q}_p[c_4, c_6]$, together with the $\mathbf{G}_m$-action given by the gradings $|c_4| = 4$ and $|c_6| = 6$. As discussed for $\overline{\mathcal{M}}(2)$ in [Sto12, §7], one can use the long exact sequence on cohomology induced by the inclusions $\widetilde{P}(4,6) \subseteq \operatorname{Spec} A \supseteq \{0\}$ [Har83, Exercise III.2.3], and the fact that $R\Gamma_{\{0\}}(\operatorname{Spec} A, \mathscr{O})$ can be computed via the Koszul complex

$$A \to A[\frac{1}{c_4}] \times A[\frac{1}{c_6}] \to A[\frac{1}{c_4 c_6}]$$

to obtain the following exact sequence

$$0 \to A \to H^0(\widetilde{P}(4,6), \mathscr{O}) \to 0 \to 0 \to H^1(\widetilde{P}(4,6), \mathscr{O}) \to A/(c_4^\infty, c_6^\infty) \to 0$$

Using this, we can explicitly calculate ψ^k on $H^1(\widetilde{P}(4,6), \mathscr{O}) \simeq A/(c_4^\infty, c_6^\infty)$:

$$\psi^k(\frac{1}{c_4^i c_6^j}) = k^{-4i-6j} \frac{1}{c_4^i c_6^j}$$

As $\frac{1}{c_4^i c_6^j}$ represents a class in $\pi_* \operatorname{Tmf} \otimes \mathbf{Q}_p$ of topological degree $-8i - 12j - 1$, this gives us the desired result.

Let us now consider an element $x \in \pi_* \operatorname{Tmf}_p$ inside $\mathfrak{Tors}$, and leave our p-completion implicit for the rest of this proof. It suffices to consider $p = 2$ or 3, otherwise $\mathfrak{Tors} = 0$. If x has nonnegative degree, then using Th.8.0.2 we see $\psi^k(x)$ is represented by $k^{\frac{|a|}{2}} at$ for a DSS decomposition $a \cdot t$ of x, in other words, we are multiplying by $k^{\frac{|a|}{2}}$, where $|a|$ denotes the degree of a. By inspection, at the primes 2 and 3, all elements x have DSS decomposition at where a has degree divisible by 8. Using *Euler's theorem*,[46] we see that modulo 8 or 3 the number $k^{\frac{|a|}{2}}$ is congruent to 1. This means that in the torsion of $\pi_* \operatorname{Tmf}$, which is at most 3-torsion or 8-torsion, the element $\psi^k(x)$ is represented by $a \cdot t$, hence $\psi^k(x) = x$.

If x is an element of $\mathfrak{Tors}$ of negative degree, then we will consider (8.4.6) for Tmf, which yields the following commutative diagram of abelian groups for every integer n:

$$\begin{array}{ccccc} \operatorname{Ext}^1_{\mathbf{Z}}(\pi_{-n-22} \operatorname{Tmf}) & \longrightarrow & \pi_n \operatorname{Tmf} & \longrightarrow & \operatorname{Hom}_{\mathbf{Z}}(\pi_{-n-21} \operatorname{Tmf}) \\ \downarrow (\psi^k)^*_1 & & \downarrow \check{\psi}^k & & \downarrow (\psi^n)^*_0 \\ \operatorname{Ext}^1_{\mathbf{Z}}(\pi_{-n-22} \operatorname{Tmf}) & \longrightarrow & \pi_n \operatorname{Tmf} & \longrightarrow & \operatorname{Hom}_{\mathbf{Z}}(\pi_{-n-21} \operatorname{Tmf}) \end{array} \qquad (9.1.1)$$

The Ext- and Hom-groups above have $\mathbf{Z}$ as a codomain and the rows are short exact—the zeroes on the ends have been dropped. As ψ^k induces a map of abelian groups on homotopy groups, we can then detect the effect of ψ^k on

[46] Recall Euler's theorem states that for coprime positive integers m, n, the value $m^{\phi(n)}$ is congruent to 1 modulo n, where $\phi(n)$ is Euler's totient function (37).

$\mathfrak{Tors} \subseteq \pi_* \mathrm{Tmf}$ by the effect of $(\psi^k)_1^*$ on the above Ext-groups. Using (9.1.1), we are left to calculate $\check{\psi}^k$ on elements in $\mathfrak{Tors}$ of nonnegative degree, which follows a similar pattern to our calculation of ψ^k for torsion elements in nonnegative degree. Again, we first note that $\check{\psi}^k$ on torsion-free elements can be calculated rationally, so we then apply Lm.8.4.9 and the above calculations of ψ^n to obtain an equality

$$\check{\psi}^k(f) = k^{-10-\frac{|f|}{2}} f$$

for every $f \in \mathfrak{Free}$ in nonnegative degree. As mentioned above, we can now calculate $\check{\psi}^k(x) = k^{-10}x$ on torsion elements in positive degree, using the DSS decompositions mentioned above. One then notices that k^{-10} is congruent to 1 modulo 24 as this is true modulo 3 and modulo 8 separately. Using (9.1.1), we see that $\psi^k(x) \equiv x$ for $x \in \mathfrak{Tors}$ of negative degree, and we are done. $\square$

The calculations of $\check{\psi}^k$ above lead us to a conjecture.

Conjecture 9.1.2. *Let R be an $\mathbf{E}_1$-ring and write $A = \pi_0 R$. Suppose that there is a class $D \in \pi_{-d}R$ such that D witnesses the Anderson self-duality of R; see Df.8.4.3. Then, for any endomorphism $F\colon R \to R$ of algebra objects in* hSp *such that $F(D) = \lambda D$ for some $\lambda \in A$, the composites $F \circ \check{F}$ and $\check{F} \circ F$ are equivalent to multiplication by λ on $\pi_* R$, where $\check{F}$ is the dual endomorphisms of F; see Df.8.4.5.*

Perhaps this equality is witnessed by a homotopy of spectra—although this is mostly careless optimism. One can validate this conjecture in the following cases:

- For $\mathrm{KU}[\frac{1}{n}]$ and ψ^n one has $D = 1$ and $\lambda = 1$. In this case, the above conjecture can be checked using (8.4.6).
- For $\mathrm{KO}[\frac{1}{n}]$ and ψ^n one has $D = vu_{\mathbf{R}}^{-1}$ and $\lambda = n^{-2}$. In this case, the above conjecture can be checked using (8.4.6) again. Furthermore, Heard–Stojanoska verified that in the stable homotopy category localised at the first Morava K-theory at the prime 2, there is a homotopy between $\check{\psi}^l$ and $\Sigma l^{-2}\psi^{1/l}$, where l is a topological generator of $\mathbf{Z}_2^\times/\{\pm 1\}$; see [HS14, Lm.9.2].
- For Tmf_p and ψ^k for a p-adic unit $k \in \mathbf{Z}_p^\times$, one has $D = [c_4^{-1}c_6\Delta^{-1}]$ and $\lambda = n^{-10}$. In this case, the above conjecture can be checked (in a range of degrees) using the proof of Th.9.0.1.

Remark 9.1.3. Let us note a counter-example if we do not assume F is multiplicative, as mentioned to us by Lennart Meier. Consider $F = \mathrm{id} + \psi^{-1}$ as an endomorphism of KU. Then $\lambda = 2$, however $F(u) = u - u = 0$ on the usual generator $u \in \pi_2\mathrm{KU}$, so Conj.9.1.2 cannot possibly hold in this case.

9.2 Nonexistence of integral operators

In Chapter 7 we constructed an array of stable operations on TMF and $\mathrm{TMF}_0(N)$ lifting the analogous operations from number theory, however, one cannot expect all operations of modular forms to lift to stable operations on TMF. Baker [Bak94, §12] gives the important operator ∂ as an example that cannot lift to a stable operation on elliptic cohomology.[47] For stable Adams operations and Hecke operators on topological modular forms we have a direct computational argument that stable operators do not exist integrally. Let us start with Adams operations.

Proposition 9.2.1. *Let $p = 2, 3$. There exists a map $\psi^k\colon \mathrm{TMF}_p \to \mathrm{TMF}_p$ of spectra which agrees with the operation $\psi^k\colon \mathrm{TMF}[\frac{1}{k}] \to \mathrm{TMF}[\frac{1}{k}]$ of Df.7.1.1 on rational homotopy groups if and only if $p \nmid k$.*

Proof. If $p \nmid k$, then we are done by Df.7.1.1. Conversely, suppose that $k = p^n m$, where $p \nmid m$ and $n \geqslant 1$, and now work prime-by-prime:

($p = 2$ case) We know that on rational homotopy groups $\psi^k(f) = 2^{nd}m^d$ where d is the weight of the modular form f. In particular, we see that $\psi^{2^n m}(1) = 1$ and $\psi^{2^n m}(c_4^2) = 2^{8n}m^8 c_4^2$ inside $\pi_* \mathrm{TMF}_{(2)}$. Consider the element $x = \eta^2 c_4^2 \in \pi_{18}\mathrm{TMF}_{(2)} \simeq \mathbf{Z}/2\mathbf{Z}$. As this element lies in the Hurewicz image of $\mathbf{S} \to \mathrm{TMF}$, we obtain the following equality:

$$\psi^k(x) = x\psi^k(1) = x \tag{9.2.2}$$

The operation ψ^k also induces a morphism on the E_2-page of the descent spectral sequence for $\mathrm{TMF}_{(2)}$, so we can calculate the effect of ψ^k on the class $h_1^2 c_4^2$ in $E_2^{2,18}$ which represents x in homotopy. We know ψ^k acts on the E_2-page as

$$\psi^k(h_1^2 c_4^2) = \psi^{2^n m}(h_1^2 c_4^2) = h_1^2 \psi^{2^n m}(c_4^2) = 2^{8n}m^8 h_1^2 c_4^2 = 0$$

where we use that ψ^k is linear with respect to the Adams–Novikov spectral sequence for the sphere (as $\psi^{2^n m}$) is a morphism of spectra. As there are no classes in higher filtration in $E_2^{2,*}$, this calculation survives to the $\mathbf{E}_\infty$-page and shows that $\psi^k(x) = 0$, a contradiction to (9.2.2).

($p = 3$ case) The same argument works at the prime 3, using the class $\alpha[\alpha\Delta] = \beta^3 \in \pi_{30}\mathrm{TMF}_{(3)}$. □

Now onto stable Hecke operators and the simpler $p = 3$-case.

Proposition 9.2.3. *Let e be a positive integer. There is no map of spectra $\mathrm{T}_{3^e}\colon \mathrm{TMF}_{(3)} \to \mathrm{TMF}_{(3)}$ (or on $\mathrm{Tmf}_{(3)}$) which agrees with the stable Hecke operator $\mathrm{T}_{3^e}\colon \mathrm{TMF}[\frac{1}{3}] \to \mathrm{TMF}[\frac{1}{3}]$ of Df.7.2.1 on rational homotopy groups.*

[47] Interestingly enough, Baker, just like us, can only show nonexistence *stably*. This leaves the door open for constructions of *unstable* operations. For Hecke operators, this could mean an unstable Hecke operator T_p on TMF_p-cohomology, which one might hope to prove is congruent to $F + V$ modulo p—a lift of the famous *Eichler–Shimura relation.*

Recall the *generalised divisor function* $\sigma_k(n)$ from (39) and the *Ramanujan's* τ*-function* $\tau(n)$ from (45).

Proof. Suppose such an operator did exist, then we choose to calculate such an T_{3^e} on the group $\pi_{27}\,\mathrm{TMF}_{(3)} \simeq \mathbf{Z}/3\mathbf{Z}\{[\alpha\Delta]\}$, where α is the 3-primary part of the Hopf map $\nu\colon \mathbf{S}^3 \to \mathbf{S}$ detected in $\pi_3\,\mathrm{TMF}$. In this case, we can use Th.8.0.2, our hypotheses, and the fact that classically one has the equality $\mathrm{T}_n^{\mathrm{alg}}(\Delta) = \tau(n)\Delta$, to calculate $\mathrm{T}_{3^e}([\alpha\Delta]) = 3\tau(3^e)[\alpha\Delta] = 0$. Alternatively, as $[\alpha\Delta]$ is nearby the Hurewicz, as $\alpha \cdot [\alpha\Delta] = \beta^3$ is in the image of the Hurewicz, then Pr. 8.3.2 states that $\mathrm{T}_{3^e}([\alpha\Delta]) = \mathrm{T}_{3^e}(1)[\alpha\Delta] = \sigma(3^e)[\alpha\Delta] \equiv [\alpha\Delta]$ as $\sigma(3^e) = 3^e + 3^{e-1} + \cdots + 3 + 1$, which is congruent to 1 modulo 3.

This calculation also holds in $\mathrm{Tmf}_{(3)}$ and we obtain the same contradiction. □

Notice that the factor of n, the difference between our stable Hecke operators T_n and the algebraic operators $n\mathrm{T}_n^{\mathrm{alg}}$ (Pr.7.5.3), is not the problem here as $\tau(3^e)$ is also divisible by 3.[48]

A similar argument holds at the prime 2.

Proposition 9.2.4. *Let e be a positive integer. There is no map of spectra* $\mathrm{T}_{2^e}\colon \mathrm{TMF}_{(2)} \to \mathrm{TMF}_{(2)}$ *(or on* $\mathrm{Tmf}_{(2)}$*) which agrees with the stable Hecke operator* $\mathrm{T}_{2^e}\colon \mathrm{TMF}[\frac{1}{2}] \to \mathrm{TMF}[\frac{1}{2}]$ *of Df.7.2.1 on rational homotopy groups.*

Proof. Consider such a hypothetical operation T_{2^e} on $\pi_{25}\,\mathrm{TMF}_2 \simeq (\mathbf{Z}/2\mathbf{Z})^2$ where one summand is generated by $[2\eta\Delta]$. Using the fact that $\tau(2^e)$ is always divisible by 2 (see §9.2), then Th.8.0.2 shows that $\mathrm{T}_{2^e}([2\eta\Delta) = 2^e\tau(2^e)[2\eta\Delta] = 0$ in $\pi_{27}\,\mathrm{TMF}_2$.[49] However, we also note that $\nu \cdot [2\eta\Delta]$ is equal to κ^2, which lies in the Hurewicz image, so $\mathrm{T}_{2^e}(\nu \cdot [2\eta\Delta]) = \mathrm{T}_{2^e}(1)\kappa^2$. As $\mathrm{T}_{2^e}(1) = \sigma(2^e)$ is even, we see that T_2 acting on $\nu \cdot [2\eta\Delta]$ is nonzero in $\pi_{28}\,\mathrm{TMF}_2$. As T_2 is $\mathbf{S}$-linear, we arrive at our contradiction, as $\mathrm{T}_2(\nu[2\eta\Delta]) = \nu\mathrm{T}_2([2\eta\Delta]) = 0$ from earlier in the proof.

This calculation also holds in $\mathrm{Tmf}_{(2)}$, so we obtain the same contradiction there. □

[48]This is a classical result of Ramanujan but also follows from the basic theory of modular forms. Indeed, the Hecke operators on Δ show that if $\tau(p) \equiv 0$ modulo p, then $\tau(pn) \equiv 0$ modulo p. On then only has to calculate $\tau(3) = 252$ to see that $\tau(3^e)$ is always divisible by 3. The same goes at the prime two, as $\tau(2) = -24$ is divisible by 2.

[49]We emphasis again that had we asked that T_{2^e} agrees with $\mathrm{T}_{2^e}^{\mathrm{alg}}$ on rational homotopy groups, we would still obtain a contradiction as $\tau(2^e)$ is divisible by 2.

Chapter 10

Applications of operations on TMF

Our goal for this chapter is to demonstrate how a cursory glance at the calculations of Ths.9.0.1 and 9.0.2 combined with either a little number theory (§10.1 and §10.2) or homotopy theory (§10.3 and §10.4) produces interesting results. Our general motto is:

More sophisticated homotopy theoretic techniques
lead to stronger number theoretic statements.

For example, we will easily calculate $n\tau(n) \equiv \sigma(n)$ modulo 2 using the existence of stable Hecke operators T_n on TMF (Pr.10.1.2), but employing the use of Toda brackets we can improve this to a congruence modulo 8 (Pr.10.1.3). This theme is repeated throughout this chapter. For definiteness, let us write a clear basis of (meromorphic) modular forms we will use.

Notation 10.0.1. Let k be an even integer. Write $\mathfrak{B}_k$ for the basis of MF_k given by

$$\{\Delta^l E_{k'} j^m\}_{m \geqslant 0}$$

where k is uniquely written as $k = 12l + k'$ for k' in the set $\{0, 4, 6, 8, 10, 14\}$, the symbol $j = \frac{x^3}{\Delta}$ denotes the j-invariant, and $E_{k'}$ is the weight k' normalised (meaning with linear term 1) Eisenstein series which can be summarised by the following formulae:

$$E_0 = 1 \qquad E_4 = c_4 \qquad E_6 = c_6 \qquad E_8 = c_4^2 \qquad E_{10} = c_4 c_6 \qquad E_{14} = c_4^2 c_6$$

The fact that $\mathfrak{B}_k$ is a basis follows by direct inspection—one could alternatively refer to the basis $\{\Delta^l E_{k'} F_{k,D}(j)\}$ used in [DJ08], as $F_{k,D}(x)$ is a monic polynomial with integer coefficients of degree $D = l + m'$ for $m' \geqslant -l$.

Notation 10.0.2. For a modular form f, be it meromorphic or holomorphic, we will write $a_n(f)$ for the coefficient of q^n in the q-expansion of f.

Let us also assume the reader is familiar with the homotopy groups of $\mathrm{tmf}, \mathrm{Tmf}$, and TMF. The calculation for $\pi_* \mathrm{tmf}$ can be found in [Bau08], for Tmf in [Kon12], and the results for TMF follow by inverting Δ^{24} in either of the previous two calculations. Another good resource is [DFHH14, §13]. We will also assume results about the Hurewicz image, so the image of the unit map **S** into any of the above $\mathbf{E}_\infty$-rings. This image is denoted by the colourful classes in [DFHH14, §13], and a proof this is exactly the Hurewicz image can be found in [BMQ20] and [BS21], for the cases at the prime 2 and 3, respectively.

10.1 Congruences of modular forms

The simplest applications of our stable Hecke operators to classical number theory come in the form of *divisibility results.*

Proposition 10.1.1. *Let d be a positive integer and f an integral meromorphic modular form of weight $12d$ inside MF_{12d} which is in the image of the edge map $e\colon \pi_{24d}\,\mathrm{TMF} \to \mathrm{MF}_{12d}$.*

1. *If $d \not\equiv 0$ modulo 3, then for any positive integer n not divisible by 3, the Δ^d-coefficient of $\mathrm{T}_n^{\mathrm{alg}} f$ is divisible by 3.*

2. *If $d \not\equiv 0$ modulo 8, then for any positive integer n not divisible by 2, the Δ^d-coefficient of $\mathrm{T}_n^{\mathrm{alg}} f$ is divisible by $2^{e_2(d)}$, where $e_2(d) = 3 - v_2(\overline{d}_8)$, $v_2(-)$ is 2-adic valuation, and $\overline{d}_8$ is the mod 8 reduction of d.*

If $d \not\equiv 0$ modulo 24, then for any positive integer n not divisible by 2 nor 3, the Δ^d-coefficient of $\mathrm{T}_n^{\mathrm{alg}} f$ is divisible by $2^{e_2(d)}3^{e_3(d)}$ where $e_3(d) = 1 - v_3(\overline{d}_3)$.

Proof. For Part 1 first. By Pr.7.5.3, which expresses the compatibility of stable Hecke operators with their algebraic counterparts, we have the following commutative diagram of abelian groups:

$$\begin{array}{ccc} \pi_{24d}\,\mathrm{TMF}[\tfrac{1}{n}] & \xrightarrow{\frac{1}{n}\mathrm{T}_n} & \pi_{24d}\,\mathrm{TMF}[\tfrac{1}{n}] \\ \downarrow e & & \downarrow e \\ \mathrm{MF}_{12d}^{\mathbf{Z}[\frac{1}{n}]} & \xrightarrow{\mathrm{T}_n^{\mathrm{alg}}} & \mathrm{MF}_{12d}^{\mathbf{Z}[\frac{1}{n}]} \end{array}$$

As f lies in the image of e we can use the above diagram to compute $\mathrm{T}_n^{\mathrm{alg}}(f)$ upstairs in $\pi_{24d}\,\mathrm{TMF}[\frac{1}{n}]$. As $3 \nmid n$, we notice that Δ^d is **not** in the image of e (in the descent spectral sequence for $\mathrm{TMF}_{(3)}$ the element $\Delta^d \in E_2^{0,24d}$ supports a d_5-differential), only the element $[3\Delta^d]$; see [Bau08, §6]. This implies that in $\mathrm{MF}_{12d}^{\mathbf{Z}[\frac{1}{n}]}$, the Δ^d-coefficient of the element $\mathrm{T}_n^{\mathrm{alg}}(f)$ must be divisible by 3. A similar story happens at the prime 2, and the mixed case is a combination of those at 2 and 3. $\square$

As the proof above shows, the differentials in the descent spectral sequence for TMF can contribute to calculations between modular forms. Let us further explore this.

Given a torsion element x in $\pi_* \mathrm{TMF}$ which is *nearby the Hurewicz image* (Df. 8.3.1), then Th. 8.0.2 and Pr. 8.3.2 together give us two different ways of calculating $\mathrm{T}_n(x)$. The discrepancy between these two techniques allows us to make congruency statements about *divisor functions* and *Ramanujan's τ function*—recall the definition of these functions from (39) and (45), respectively.

Proposition 10.1.2. *If n is odd, then $n\tau(n) \equiv_2 \sigma(n)$. If n is not divisible by 3, then $n\tau(n) \equiv_3 \sigma(n)$.*

Recall that $\mathrm{T}_n^{\mathrm{alg}}(1) = \frac{\sigma(n)}{n}$, so by Pr. 7.5.3 we see that $\mathrm{T}_n(1) = \sigma(n)$ for our stable Hecke operators.

Proof. Let us explain the $p = 2$ case—the $p = 3$ is entirely similar. Consider $[2\nu\Delta] \in \pi_{27} \mathrm{TMF}_2 \simeq \mathbf{Z}/4\mathbf{Z}$ and an odd n; an even n would kill the torsion in $\pi_* \mathrm{TMF}_2$ and hence the following argument too. We know that $\eta[2\nu\Delta] = \overline{\kappa}\epsilon$ by looking at [Bau08, Pr.8.4(2)], which implies that $\mathrm{T}_n([2\nu\Delta])$ is congruent to $\sigma(n)[2\nu\Delta]$ modulo 2 as $\overline{\kappa}\epsilon$ is 2-torsion and our stable Hecke operators are $\mathbf{S}$-linear. Using Th. 8.0.2, the obvious DSS decomposition (Df. 8.3.1) for $[2\nu\Delta]$, and the fact that Δ is a Hecke eigenform with eigenvalue $\tau(n)$, we also obtain the following equalities inside $\pi_{27} \mathrm{TMF}_2$:

$$\mathrm{T}_n([2\nu\Delta]) = [2\nu n \mathrm{T}_n^{\mathrm{alg}}(\Delta)] = [2\nu n\tau(n)\Delta] = n\tau(n)[2\nu\Delta]$$

Combining the two computations above, we conclude that $n\tau(n) \equiv_2 \sigma(n)$ for odd n. The same goes for the argument at the $p = 3$ using $[\alpha\Delta]$. □

If we use slightly more sophisticated techniques in homotopy theory such as Toda brackets, we can improve one of the congruences above.

Proposition 10.1.3. *If n is odd, then $n\tau(n) \equiv_8 \sigma(n)$.*

To prove the above proposition, we will use the following statement about Toda brackets; see [Sch12, Pr.IV.2.3].

Lemma 10.1.4. *Let $f\colon A \to B$ be a morphism of $\mathbf{E}_\infty$-rings, N a B-module, and $\phi\colon M \to f_*N$ a morphism of A-modules. If $\langle x, y, z\rangle$ is a well-defined Toda bracket in π_*M with $x \in \pi_*M$ and $y, z \in \pi_*A$, then we have the containment*

$$\phi(\langle x, y, z\rangle) \subseteq \langle \phi(x), f(y), f(z)\rangle$$

of subsets inside π_*N.

Proof. Using Th. 8.0.1 and the classical fact that Δ is a Hecke eigenform of eigenvalue $\tau(n)$, we see that $\mathrm{T}_n([8\Delta]) = n\tau(n)[8\Delta]$ inside $\pi_{24} \mathrm{TMF}_2$. We also have a Toda bracket expression $\langle 8, \nu, \overline{\kappa}\rangle$ for $[8\Delta]$, which we will explain in some detail now. One can calculate the Massey product $\langle 8, h_2, g\rangle$ in the E_5-page of the descent spectral sequence for TMF_2, where h_2 (resp. g) represents ν (resp. $\overline{\kappa}$) in homotopy. Indeed, for degree reasons we see $\langle 8, h_2, g\rangle$ is the set $\{8x \in E_5^{24,0} \mid d_5(x) = h_2 g\}$. Bauer's calculation of the descent spectral sequence [Bau08, §8.3] shows the collection of such x is $\Delta + ac_4 + 8b\Delta$, where $a, b \in \mathbf{Z}_2$. In particular, working modulo 8 and c_4, we see that $\langle 8, h_2, g\rangle \equiv_{8,c_4} 8\Delta$. The Moss convergence theorem (see [Mos70] for the original statement for the Adams spectral sequence and [BK21] for the adaptation to other multiplicative spectral sequences) for the descent spectral sequence of TMF_2 then shows our desired Toda bracket $\langle 8, \nu, \overline{\kappa}\rangle$ is congruent to $[8\Delta]$ modulo 8 and c_4. From this and Lm.10.1.4, we obtain the following chain of containments:

$$\mathrm{T}_n([8\Delta]) \equiv_{8,c_4} \mathrm{T}_n(\langle 8, \nu, \overline{\kappa}\rangle) \subseteq \langle \mathrm{T}_n(8), \nu, \overline{\kappa}\rangle \subseteq \sigma(n)\langle 8, \nu, \overline{\kappa}\rangle \equiv_{8,c_4} \sigma(n)[8\Delta]$$

These containments are in fact equalities of a one-element subset of the quotient

$$\pi_{24}\,\mathrm{TMF}_2 \,/\, (8 \cdot \pi_{24}\,\mathrm{TMF}_2 \oplus c_4 \cdot \pi_{16}\,\mathrm{TMF}_2)$$

as $\sigma(n)\langle 8, \nu, \overline{\kappa}\rangle$ has trivial indeterminacy in this group. It follows that, modulo 8 and c_4, we have $n\tau(n)[8\Delta] \equiv \sigma(n)[8\Delta]$. As Δ and c_4 are linearly independent, we see that $n\tau(n) \equiv_8 \sigma(n)$. □

The above does not quite match Ramanujan's congruence $\tau(n) \equiv_8 \sigma(n)$ for odd n, but the extra factor of n does not change the value of $\tau(n)$ modulo 8. Indeed, for odd n we see from Pr.10.1.2 that $\tau(n)$ is odd if and only if n is a square, and in this case $n \equiv_8 1$, hence it does not contribute. Otherwise, $\tau(n)$ is even. If $\tau(n)$ is divisible by 4, then we are also done, as $4k \equiv_8 4$ for all odd k, so we are left in the case where 2 is the largest power of 2 dividing $\tau(n)$. Using Ramanujan's congruence $\tau(n) \equiv_8 \sigma(n)$ for odd n, we notice that $\tau(n) \equiv_4 2$ if and only if n has odd primes factors whose exponents are all even except precisely one p with exponent e, and for this pair we have $p \equiv_4 e \equiv_4 1$ (see Pr.10.2.3). In this case, we notice that $n \equiv_8 p^e$ is congruent to 1 or 5 modulo 8, and if 2 is the largest power of 2 dividing $\tau(n)$, we have $\tau(n) \equiv_8 n\tau(n)$ in either situation.

Ramanujan's classical congruence $\tau(n) \equiv_8 \sigma(n)$ for odd n cannot be extended to higher powers of 2 without more restrictions on n, for example:

$$\sigma(5) = 6 \not\equiv_{16} 14 \equiv_{16} 4830 = \tau(5)$$

Curiously, it seems that our congruence $n\tau(n) \equiv_8 \sigma(n)$ does actually hold modulo 16—a quick `SAGE` check shows this holds for odd $n \leqslant 10^6$. One might hope this equivalence could be strengthened to one modulo 16 using variations on our homotopy theoretic arguments above.

We can generalise Prs. 10.1.2 and 10.1.3 to other torsion-free elements in $\pi_*\,\mathrm{TMF}$ which support nontrivial multiplication by torsion elements. Recall the basis $\mathfrak{B}$ of Nt.10.0.1.

Proposition 10.1.5. *Let d be a nonnegative integer and n a positive integer, and write b_n^e for the Δ^e-coefficient of $\mathrm{T}_n^{\mathrm{cl}}(\Delta^e)$ with respect to the basis $\mathfrak{B}_{12e}$.*

1. *If n is odd, then $\sigma(n)$ is congruent to:*
   (a) *nb_n^{8d} and nb_n^{8d+1} modulo 8;*
   (b) *nb_n^{8d+2}, nb_n^{8d+4}, nb_n^{8d+5}, and nb_n^{8d+6} modulo 4; and*
   (c) *nb_n^{8d+3} and nb_n^{8d+7} modulo 2.*
2. *If n is not divisible by 3, then $\sigma(n)$ is congruent to nb_n^d modulo 3.*

Proof. Let us start with the $p = 2$ case. We will use two arguments repeatedly, so let us call them the **Hurewicz argument** and the **Toda argument**. Let us demonstrate these two arguments in the two cases of (a), respectively.

(Hurewicz argument) Notice that $\mathfrak{B}_{96d}$ is also a basis for $\pi_{192}\,\mathrm{TMF}_{(2)}$, and the only element in this basis supporting multiplication by $\overline{\kappa}$ is Δ^{8d}. Consider the following equalities in $\pi_{192d+20}\,\mathrm{TMF}_{(2)}$:

$$\sigma(n)\overline{\kappa}\Delta^{8d} = \overline{\kappa}\Delta^{8d}\mathrm{T}_n(1) = \mathrm{T}_n(\overline{\kappa}\Delta^{8d}) = [n\mathrm{T}_n^{\mathrm{cl}}(g\Delta^{8d})] = \overline{\kappa}\cdot[n\mathrm{T}_n^{\mathrm{cl}}(\Delta^{8d})]$$

The first equality comes from the classical calculation $\mathrm{T}_n^{\mathrm{cl}}(1) = \frac{\sigma(n)}{n}$, the second from the fact that $\overline{\kappa}\Delta^{8d} \in \pi_{192d+20}\,\mathrm{TMF}_{(2)}$ lies in the image of the unit map $\mathbf{S} \to \mathrm{TMF}$ ([BMQ20]), also called the Hurewicz image, and T_n is $\mathbf{S}$-linear, and the third and forth from Th.8.0.2 combined with the facts that $[g]$ represents $\overline{\kappa}$ on the E_2-page of the descent spectral sequence (DSS) for $\mathrm{TMF}_{(2)}$ and that no nonzero classes live in higher filtration than $g\Delta^{8d}$ in that degree on the $\mathbf{E}_\infty$-page. As $\mathfrak{B}_{96d}$ is also a basis for $\pi_{192}\,\mathrm{TMF}_{(2)}$, and the only element in this basis supporting multiplication by $\overline{\kappa}$ is Δ^{8d}, we see that nb_n^{8d} is equal to $\sigma(n)$ inside $\mathbf{Z}/8\mathbf{Z}$. This is the first case of (a).

(Toda argument) For the second case of (a), consider the Toda bracket $\langle 8, \nu, \overline{\kappa}\Delta^{8d}\rangle$ as a subset of $\pi_{192d+48}\,\mathrm{TMF}_{(2)}$. We will now discuss this bracket in some detail. One can calculate the Massey product $\langle 8, h_2, g\Delta^{8d}\rangle$ on the E_5-page of the DSS, where h_2 (resp. g) represent ν (resp. $\overline{\kappa}$). Indeed, on the E_5-page we have $8h_2 = 0$ so we have the following equality of sets:

$$\langle 8, h_2, g\Delta^{8d}\rangle = \{8x \in E_5^{24+192d,0} | d_5(x) = h_2 g\Delta^{8d}\}$$

Bauer's calculation of the homotopy groups of tmf ([Bau08]) shows that the above set is equivalent to those elements

$$a\Delta^{8d+1} + a_1\Delta^{8d}c_4^3 + a_2\Delta^{8d-1}c_4^6 + \cdots + a_{8d+1}c_4^{24d+3}$$

where $a \equiv 1$ modulo 8. In particular, modulo 8 and all the elements in the basis $\mathfrak{B}_{96d+12}$ **not** of the form Δ^{8d+1} the Massey product $\langle 8, h_2, g\rangle$ is congruent to the singleton set of 8Δ. The Moss convergence theorem (see [Mos70] for the original statement for the Adams spectral sequence and [BK21] for the adaptation to other multiplicative spectral sequences) shows our desired Toda bracket $\langle 8, \nu, \overline{\kappa}\Delta^{8d}\rangle$ is congruent to $[8\Delta]$ modulo 8 and the elements of the complement $\mathfrak{B}_{96d+12} - \{\Delta^{8d+1}\}$. By Lm.10.1.4 and the above, we obtain the following containment of sets:

$$\mathrm{T}_n([8\Delta]\Delta^{8d}) \equiv \mathrm{T}_n(\langle 8, \nu, \overline{\kappa}\Delta^{8d}\rangle) \subseteq \langle \mathrm{T}_n(8), \nu, \overline{\kappa}\Delta^{8d}\rangle$$

$$\subseteq \sigma(n)\langle 8, \nu, \overline{\kappa}\Delta^{8d}\rangle \equiv \sigma(n)[8\Delta]\Delta^{8d}$$

The first and last elements are singleton sets giving us the equality

$$\mathrm{T}_n([8\Delta]\Delta^{8d}) \equiv \sigma(n)[8\Delta]\Delta^{8d}$$

modulo 8 and the elements of $\mathfrak{B}_{96d+12} - \{\Delta^{8d+1}\}$. We can also use Th.8.0.2 to see $\mathrm{T}_n([8\Delta]\Delta^{8d})$ is represented by the class $[n\mathrm{T}_n^{\mathrm{cl}}(8\Delta^{8d+1})]$. In total, this shows that the Δ^{8d+1}-coefficient of $\mathrm{T}_n^{\mathrm{cl}}(8\Delta^{8k+1})$ multiplied by n is congruent to $\sigma(n)$—our desired result.

(Remaining 2-local cases) The justifications for parts (b) and (c) follow from slight variations on the Hurewicz and Toda arguments given above, so let us only outline the differences in all the cases. First, by Δ^8 periodicity, let us only consider the $d = 0$ cases.

For (b), we use the Hurewicz argument with respect to the element $[2\Delta^4]\overline{\kappa}$ for the b_n^4-case. This is straightforward, as $[2\Delta^4]\overline{\kappa}$ lies in the Hurewicz image, is 4-torsion, and no nonzero elements exist in higher filtration on the E_∞-page of the DSS in the 116$^{\mathrm{th}}$ column. We can also use the Hurewicz argument with respect to the elements $[\nu\Delta^2]$ and $[\nu\Delta^6]$ for the b_n^2- and b_n^6-cases, respectively, however, we should be more careful. Neither of these elements lie in the Hurewicz image, but in both cases their image under multiplication by ν does, and this image is 4- and 8-torsion, respectively. This implies that these elements are T_n-eigenvectors modulo 4 with eigenvalue $\sigma(n)$. There are also classes in higher filtration that both of these classes, but we can kill these classes of higher filtration and work modulo 4, which yields our result. The last case for (b) is to apply the Toda argument to $\langle 4, e_{116}, \nu\rangle$ inside $\pi_{120}\,\mathrm{TMF}_{(20)}$, where e_{116} is represented by $[2g\Delta^4]$ on the E_2-page of the DSS. One then calculates this Toda bracket is equal to $[8\Delta^5]$ modulo 4 and elements in $\mathfrak{B}_{60}$ not of the form Δ^5. This yields the b_n^5-case.

For (c), first consider the class $x = \eta\epsilon\Delta^8 = \nu^3\Delta^8$. We would like to apply the Hurewicz argument here, but we will use a slight modification. Indeed, this lies in the Hurewicz image, leading us to the following equality:

$$\mathrm{T}_n(\nu^3\Delta^8) = \sigma(n)\nu^3\Delta^8$$

This class has representation $ch_1\Delta\Delta^7$ on the E_2-page of the DSS where $ch_1\Delta$ represents the class q in homotopy, which also lies in the Hurewicz image. Using this, and the fact that there are no nonzero classes in filtration higher than x in this column of the DSS, we obtain

$$\mathrm{T}_n(\nu^3\Delta^8) = [n\mathrm{T}_n^{\mathrm{cl}}(ch_1\Delta\Delta^7)] = [nch_1\Delta\mathrm{T}_n^{\mathrm{cl}}(\Delta^7)]$$

using Th.8.0.2, as per usual. On the E_2-page of the DSS, we see the only classes in degree 168 and filtration zero (where Δ^7 lives) the support multiplication by $ch_1\Delta$ are multiples of Δ^7, leading us the equality

$$\mathrm{T}_n(\nu^3\Delta^8) = [nch_1\Delta\mathrm{T}_n^{\mathrm{cl}}(\Delta^7)] = nb_n^7[ch_1\Delta\Delta^7] = nb_n^7\nu^3\Delta^8$$

and our desired congruence $nb_n^7 \equiv \sigma(n)$ modulo 2. For the b_n^3-case, apply the Toda argument to $\langle 2, e_{70}, \eta\rangle$, where e_{70} is the unique nonzero class in $\pi_{70}\,\mathrm{TMF}_{(2)}$. This Toda bracket is then calculated to be equal to $[8\Delta^3]$ modulo 8 and the elements of $\mathfrak{B}_{36}$ not of the form Δ^3.

(The 3-local cases) In the 3-local world, we use the Hurewicz argument with respect to the elements $\beta\Delta^{3d}$ and $\alpha[\alpha\Delta]\Delta^{3d}$ to obtain the b_n^{3d}- and b_n^{3d+1}-cases, respectively. For the b_n^{3d+2}-case, use the slightly altered Hurewicz argument (as done in the b_n^7-case at the prime 2) applied to the element $\alpha\beta\Delta^{3d+3}$, as this class lies in the Hurewicz image and has E_2-representation $[\alpha\beta\Delta\Delta^{3d+2}]$ and $\alpha\beta\Delta$ represents a class in the Hurewicz image. □

For some values d and n, one can calculate b_n^d by hand. For example, if n is not divisible by 3, then $b_n^3 \equiv \tau(n)^3 \equiv \tau(n)$ modulo 3 which gives another proof of Pr.10.1.2 at the prime 3. For higher d, we have to do a little more bookkeeping. We give a few examples now, although more can be produced with more time and willpower.

Corollary 10.1.6. *Let n be a positive integer.*

1. *If n is odd, then we have the following congruences:*

$$\sigma(n) \equiv_4 n\left(\tau(n)^2 + 2\sum_{i=1}^{n-1}\tau(i)\tau(2n-i)\right) \tag{10.1.7}$$

$$\sigma(n) \equiv_8 n\left(\tau(n)^4 + 6\sum_{i=1}^{n-1}\tau(i)^4\tau(2n-i)^4\right) \tag{10.1.8}$$

2. *If n is not divisible by 3, then $\sigma(n)$ is congruent modulo 3 to the following expressions:*

$$\begin{cases} n\left(\tau(n)^2 + 2\sum_{i=1}^{n-1}\tau(i)\tau(2n-i)\right) & n \equiv_2 1 \\ n\left(\tau(n)^2 + 2\left(\sum_{i=1}^{n-1}\tau(i)\tau(2n-i) + \sum_{j=1}^{\frac{n}{4}}\tau(j)\tau(\frac{n}{2}-j)\right)\right) & n\equiv_4 2 \\ n(\tau(n)^2 + 2\tau(\frac{n}{4}) & \\ \quad +2(\sum_{i=1}^{n-1}\tau(i)\tau(2n-i) + \sum_{j=1}^{\frac{n}{4}}\tau(j)\tau(\frac{n}{2}-j))) & n \equiv_4 0 \end{cases} \tag{10.1.9}$$

We can also combine some of the above congruences. For example, if n is divisible by neither 2 nor 3, then one obtains the following:

$$\sigma(n) \equiv_{12} n \left(\tau(n)^2 + 2\sum_{i=1}^{n-1} \tau(i)\tau(2n-i) \right)$$

Before we prove Cor.10.1.6, let us mention a small combinatorial lemma to help us calculate the powers of power series.

Lemma 10.1.10. *Let $r \geqslant 1$ be an integer, R a ring, and $f(x) = \sum_{m\geqslant 0} a_m x^m$ a power series with coefficients in R. Then we have the equality*

$$f(x)^r = \sum_{m\geqslant 0} \left(\sum_{\substack{I=\{i_1,\ldots,i_r\} \\ i_1,\ldots,i_r\geqslant 0 \\ i_1+\cdots+i_r=m}} \frac{r!}{R_I} a_{i_1}\cdots a_{i_r} \right) x^m$$

where the interior sum is taken over all unordered r-tuples of nonnegative integers summing to m, R_I is the product of $n_x!$ indexed over each unique integer x appearing in I, and n_x is the number of times x occurs in I.

Proof. For some integer $m \geqslant 0$, the x^m-coefficient of $f(x)^r$ is given by

$$\sum_{\substack{(i_1,\ldots,i_r) \\ i_1,\ldots,i_r\geqslant 0 \\ i_1+\cdots i_r=m}} a_{i_1}\cdots a_{i_r}$$

where the sum is taken over all *ordered* r-tuples $(i_1,\ldots,i_r)$ of nonnegative integers summing to m. As R is commutative the order of these r-tuples does not matter so we are left to count how many ordered r-tuples there are of the form $(i_1,\ldots,i_r)$ for each choice of r-many integers $i_1,\ldots,i_r$—call this unordered r-tuple I. We claim this number is $c_I = \frac{r!}{R_I}$ given in the statement of this lemma. Indeed, this is now just the statement of the *multinomial coefficients* from ones school days; see [DLMF, §26.4]. □

Proof of Cor.10.1.6. First, suppose n is odd, and consider $[\nu\Delta^2] \in \pi_{51}\,\mathrm{TMF}_2$ which we denote by x. Now $\nu\cdot[\nu\Delta^2]$ is 4-torsion inside the image of the unit map $\pi_*\mathbf{S} \to \pi_*\,\mathrm{TMF}_2$, so x is *almost* nearby the Hurewicz—as multiplication by ν sends x, which is 8-torsion, to something that is 4-torsion, we only have $\mathrm{T}_n(x) \equiv \sigma(n)x$ modulo 4. One can also calculate $\mathrm{T}_n(x)$ by applying Th.8.0.2 to $x = [\nu\Delta^2]$ with the obvious DSS decomposition (Df.8.3.1):

$$\mathrm{T}_n(x) = \mathrm{T}_n([\nu\Delta^2]) = [\nu n \mathrm{T}_n^{\mathrm{alg}}(\Delta^2)] = nb_n^2[\nu\Delta^2]$$

We can easily calculate this b_n^2 modulo 8. Classical formulae for Hecke operators give the coefficient of q and q^2 in the q-expansion of $\mathrm{T}_n^{\mathrm{alg}}(\Delta^2)$ as the following two numbers:

$$\sum_{d|1,n} d^{23} a_{\frac{n}{d^2}}(\Delta^2) = a_n(\Delta^2)$$

$$\sum_{d|2,n} d^{23} a_{\frac{2n}{d^2}}(\Delta^2) = a_{2n}(\Delta^2)$$

Consider the following basis for the cusp forms in $\mathrm{mf}_{24}^{\mathbf{Z}_2}$:

$$\{e_2 = \Delta^2, e_1 = c_4^3\Delta - a_2(c_4^3\Delta)\Delta^2\}$$

Using this basis, the above values for the first two coefficients in the q-expansion for $\mathrm{T}_n^{\mathrm{alg}}(\Delta^2)$, and the equality $a_2(c_4^3\Delta) = 216$, which one can do by hand, we obtain the following equations:

$$\mathrm{T}_n^{\mathrm{alg}}(\Delta^2) = a_n(\Delta^2)e_1 + a_{2n}(\Delta^2)e_2 = a_n(\Delta^2)c_4^3\Delta + (a_{2n}(\Delta^2) - 216a_n(\Delta^2))\Delta^2$$

Importantly, we see that modulo 4, the coefficient of Δ^2 is $a_{2n}(\Delta^2)$ which can be given as the sum $\sum_{i=1}^{2n-1}\tau(i)\tau(2n-1)$. Summarising the above and a little simplification yields the desired result (10.1.7) for all odd n:

$$\sigma(n) \equiv_4 n\left(\tau(n)^2 + 2\sum_{i=1}^{n-1}\tau(i)\tau(2n-i)\right)$$

For (10.1.8), we use the same tactics applied to the element $y = \overline{\kappa}\Delta^8$ inside $\pi_{212}\,\mathrm{TMF}_2$. Indeed, one easily obtains the congruence $nb_n^8 \equiv \sigma(n)$ modulo 8. Because the q-expansions for Δ and c_4 take the form

$$\Delta(\tau) = q + O(q^2) \qquad c_4(\tau) = 1 + 240(O(q))$$

then for all $1 \leqslant d \leqslant 7$ and $d+1 \leqslant i \leqslant 8$, the coefficient $a_i(c_4^{3d}\Delta^{8-d})$ is congruent to 0 modulo 8. This implies that $\sigma(n) \equiv_8 a_{8n}(\Delta^8)$. Using Lm.10.1.10 and the fact that $x^8 \equiv_8 x^4$ for all x, we obtain the desired congruence for all odd n:

$$\sigma(n) \equiv_8 na_{8n}(\Delta^8) \equiv_8 n\left(\tau(n)^8 + 6\sum_{i=1}^{n-1}\tau(i)^4\tau(2n-i)^4\right)$$

Finally, for (10.1.9) suppose that n is not divisible by 3 and consider the element $z = \beta\Delta^6 \in \pi_{154}\,\mathrm{TMF}_3 \simeq \mathbf{Z}/3\mathbf{Z}$. This element lies in the Hurewicz image, and using the evident DSS decomposition as well as the arguments made above, we quickly see that $\sigma(n)$ is congruent to $na_6(\mathrm{T}_n^{\mathrm{alg}}(\Delta^6))$ modulo 3. The classical formula for Hecke operators takes the form

$$a_6(\mathrm{T}_n^{\mathrm{alg}}(\Delta^6)) \equiv_3 \sum_{d|6,n} d^{71} a_{\frac{6n}{d^2}}(\Delta^6) \equiv_3 \begin{cases} a_{6n}(\Delta^6) & 2 \nmid n \\ a_{6n}(\Delta^6) + 2a_{\frac{3n}{2}}(\Delta^6) & 2 \nmid n \end{cases}$$

and we are left to compute the various values of $a_m(\Delta^6)$ modulo 3. Using Lm.10.1.10, we obtain the congruence

$$a_{6n}(\Delta^6) \equiv_3 \tau(n)^6 + 2\sum_{i=1}^{n-1}\tau(i)^3\tau(2n-i)^3 \equiv_3 \tau(n)^2 + 2\sum_{i=1}^{n-1}\tau(i)\tau(2n-i)$$

as the only I for which $\frac{6!}{R_I}$ is not divisible by 3 are

$$\{n,n,n,n,n,n\} \qquad \{i,i,i,2n-i,2n-i,2n-i\}$$

for $1 \leqslant i \leqslant n-1$. Similarly, for $n \equiv_4 2$ we obtain the congruence

$$a_{\frac{3n}{2}}(\Delta^6) \equiv_3 2 \sum_{j=1}^{\frac{n}{2}-1} \tau(j)\tau(\frac{n}{2}-j)$$

and in this case the only I for which $\frac{6!}{R_I}$ is not divisible by 3 is

$$\{j,j,j,\frac{n}{2}-j,\frac{n}{2}-j,\frac{n}{2}-j\} \qquad 1 \leqslant j \leqslant \frac{n}{2}-1$$

as $\frac{n}{4}$ is not an integer. Similarly again, for $n \equiv_4 0$, we obtain the congruence

$$a_{\frac{3n}{2}}(\Delta^6) \equiv_3 \tau(\frac{3n}{2}) + 2 \sum_{j=1}^{\frac{n}{2}-1} \tau(j)\tau(\frac{n}{2}-j)$$

modulo 3. This finishes our proof. □

One can push the above techniques as far as one has the patience and necessities, however, with more advanced homotopical methods, one can obtain congruences using the above ideas which do not just relate $\sigma(n)$ and $\tau(n)$. Indeed, the only modular forms in π_* TMF which support nontrivial multiplication by torsion elements and this multiplication yields something in the Hurewicz image are powers of c_4 and powers of Δ. However, other spectra do have a more intimate relationship with elements coming from the homotopy groups of $\mathbf{S}$, for example, the height 2 Adams summands and height 2 image of J spectra in §10.3 and §10.4, and Behrens' $Q(N)$ spectra of Ex.6.1.15.

There is another algebraic application of Pr. 10.1.5 that we would like to explore.

10.2 An expanded range of Maeda's conjecture

Let us first recall Maeda's conjecture from Conj.1.1.2. Write S_k for the subspace of $\mathrm{mf}_k^{\mathbf{Q}}$ spanned by *cusp forms*, so weight k *holomorphic* modular forms with vanishing constant term in their q-expansion.

Conjecture 10.2.1 (Maeda's conjecture)**.** *For every even integer $k \geqslant 4$ and every positive integer n, the characteristic polynomial $\mathrm{T}_{n,k}(X)$ for the operation $\mathrm{T}_n^{\mathrm{alg}}\colon S_k \to S_k$ is irreducible over $\mathbf{Q}$ and has Galois group the full symmetric group Σ_{d_k}, where $d_k = \dim S_k$.*

There is experimental evidence for this conjecture—[GM12] summarises the state of affairs in 2012. In *loc. cit.*, it is shown that the above conjecture holds for all n, k where $k \leqslant 12{,}000$ and either $n \leqslant 10{,}000$ or n is a prime not congruent to ± 1 modulo 5 or 7. We can add a few n to this list, using the following result of Ahlgren.

Theorem 10.2.2 ([Ahl08, Th.1.4])**.** *Write S'_k for the subspace of S_k spanned by modular forms with vanishing constant and linear term in their q-expansion. Let k be a positive integer such that $\dim_{\mathbf{Q}} S_k \geqslant 2$ and suppose there exists an m such that Maeda's conjecture holds for $\mathrm{T}_{m,k}(X)$. Then for any positive integer $n \geqslant 2$, the following are equivalent:*

1. *Maeda's conjecture holds for $\mathrm{T}_{n,k}(X)$.*
2. *There exists a modular form $f \in S'_k$ with $a_n(f) \neq 0$.*

Using this result, we are left to find a few modular forms of particular weights such that specific coefficients in their q-expansions do not vanish. To do this, we will use Pr.10.1.5 and the following elementary non vanishing results.

Proposition 10.2.3. *Let n be an odd positive integer.*

1. *Then $\sigma(n) \not\equiv_2 0$ if and only if n is a square.*
2. *Then $\sigma(n) \not\equiv_4 0$ if and only if there is at most one prime factor p of n whose exponent e is odd, and in that case we demand that both e and p are congruent to 1 modulo 4.*
3. *Then $\sigma(n) \not\equiv_8 0$ if and only if exactly one of the following two statements is true:*
   (a) *There are at most two prime factors of n whose exponents are odd, and in this case, we demand such primes and their exponents are congruent to 1 modulo 4.*
   (b) *There is at most one prime factor p of n with odd exponent e such that either $e \equiv_8 1$ and $p \equiv_8 3$, or $e \equiv_8 3$ and $p \equiv_4 1$, or $e \equiv_8 5$ and $p \equiv_8 7$.*

Proof. Fix our odd n with odd prime factors p_i and exponents e_i. For part 1, note that $\sigma(n) = \prod_i \sigma(p_i^{e_i})$ is odd if and only if each $\sigma(p_i^{e_i})$ is odd. In this case, we have

$$\sigma(p^e) = 1 + p + p^2 + \cdots + p^e \equiv_2 (e+1)$$

using Fermat's little theorem, which is odd if and only if e is even. In other words, every exponent e_i in n is even, so n is an odd square. For part 2, we note that for $\sigma(n)$ to be nonzero modulo 4, we can have at most one $\sigma(p^e)$ with odd exponent, and in this case we need $\sigma(p^e) \equiv_4 2$. Assuming e is odd, then have the congruences

$$\sigma(p^e) = 1 + p + p^2 + \cdots + p^e \equiv_4 (1+p)\frac{e+1}{2}$$

using Euler's theorem (46). We see the above vanishes modulo 4 if $p \equiv_4 3$ so we must have $p \equiv_4 1$ and also $e \equiv_4 1$, hence part 2. Part 3 is simply a more subtle version of the previous parts, but is still elementary. We see that for $\sigma(n) \not\equiv_8 0$ to hold, we need either at most two primes p_1, p_2 with exponents e_1, e_2, respectively, such that $\sigma(p_i^{e_i}) \equiv_4 2$, or at most a single prime p with exponent e such that $\sigma(p^e) \equiv_8 4$. The first case is covered by the argument for part 2 above, so we focus on the second case, classifying odd primes p and exponents e such that $\sigma(p^e) \equiv_8 4$. Writing $\sigma(p^e)$ and using Euler's theorem, we obtain the following:

$$\sigma(p^e) = \sum_{i=0}^{e} p^i \equiv_8 \begin{cases} \frac{e}{4}(1+p+p^2+p^3)+1 & e \equiv_4 0 \\ \frac{e-1}{4}(1+p+p^2+p^3)+1+p & e \equiv_4 1 \\ \frac{e-2}{4}(1+p+p^2+p^3)+1+p+p^2 & e \equiv_4 2 \\ \frac{e+1}{4}(1+p+p^2+p^3) & e \equiv_4 3 \end{cases}$$

Note the sum $1+p+p^2+p^3$ is always divisible by 4. If $e \equiv_4 0$, we see $\sigma(p^e) \not\equiv_8 4$ for any choice of p, so we ignore this case. Likewise for the $e \equiv_4 2$ case. If $e \equiv_4 1$, then we are divided into two further cases: if $e \equiv_8 1$ then $\frac{e-1}{4}$ is even, and we demand $p \equiv_8 3$; if $e \equiv_8 5$, then we similarly demand $p \equiv_8 7$. In the last case, when $e \equiv_4 3$, the fact the sum $1+p+p^2+p^3$ is always divisible by 4 forces $\frac{e+1}{4}$ to be odd so $e \equiv_8 3$. We also want the sum $1+p+p^2+p^3$ to be congruent to 4 modulo 8, so we also require $p \equiv_4 1$. This finishes part 3. □

Proposition 10.2.4. *Let n be a positive integer not divisible by 3. Then $\sigma(n) \not\equiv_3 0$ if and only if for each prime factor p of n with exponent e, if $p \equiv_3 1$ then $e \equiv_6 0, 1, 3$ or 4, and if $p \equiv_3 2$ then e is even.*

Proof. As in the proof of Pr.10.2.3, write $n = \prod p_i^{e_i}$ where each prime is different than 3. For $\sigma(n) \not\equiv_3 0$, we need all $\sigma(p^e) \not\equiv_3 0$. Using Fermat's little theorem, we obtain the following congruences:

$$\sigma(p^e) = 1+p+p^2+\cdots+p^e \equiv_3 \begin{cases} (p+1)\frac{e+1}{2} & \text{for odd } e \\ (p+1)\frac{e}{2}+1 & \text{for even } e \end{cases}$$

For odd e, we see that for $\sigma(p^e)$ not to vanish modulo 3, we need $p \equiv_3 1$ and $e \not\equiv_6 5$. If e is even, then we either require $p \equiv_3 1$ and $e \not\equiv_3 2$, or $p \equiv_3 2$ in which case e can be an arbitrary even number. Collecting these observations, we see obtain our claimed result. □

We then have the following general two statements concerning Maeda's conjecture. Let us begin with the simpler statement at the prime 3. Thank you to Gerd Laures for pointing out a mistake in a previous argument here.

Theorem 10.2.5. *Let $k, n \geqslant 2$ be two coprime integers with n not divisible by 3 satisfying the following conditions:*

1. *$k \leqslant 1{,}000$ and for all $1 \leqslant i \leqslant k-1$, the coefficient of q^k in the q-expansion of Δ^i is divisible by 3.*

2. *For each prime factor p of n with exponent e, if $p \equiv_3 1$ then $e \equiv_6 0, 1, 3, 4$, and if $p \equiv_3 2$ then e is even.*

Then $\mathrm{T}_{kn,12k}(X)$ satisfies Maeda's conjecture.

In particular, as there are an infinite number of primes p with $p \equiv_3 1$, we see $\mathrm{T}_{kp}^{\mathrm{alg}}$ satisfies Maeda's conjecture on S_{12k} for valid k. A computer check shows that the only k satisfying the condition 1 above in the range $2 \leqslant k \leqslant 500$ are $k = 2$, 3, 6, 9, 18, 27, 54, 81, 162, 243, and 486.

The situation at the prime 2 is similar, albeit more complicated.

Theorem 10.2.6. *Let $k, n \geqslant 1$ be coprime integers where n is odd satisfying the following conditions:*

1. *$k \leqslant 1{,}000$ and for all $1 \leqslant i \leqslant k-1$, then writing e_i for the 2-adic valuation of the coefficient of q^k in the q-expansion of Δ^i, we require that*

$$e_i \geqslant \begin{cases} 1 & k \equiv_8 3, 7 \\ 2 & k \equiv_8 2, 4, 5, 6 \\ 3 & k \equiv_8 0, 1 \end{cases}$$

2. *Setting $e = \min(e_i)$, then we require that if:*
   (a) *$e = 1$ then n is a square.*
   (b) *$e = 2$ then n has at most one prime factor with odd exponent, and in this case, the prime and the exponent are congruent to 1 modulo 4.*
   (c) *$e \geqslant 3$ then n satisfies the equivalent conditions in part 3 of Pr.10.2.3.*

Then $\mathrm{T}_{kn,12k}(X)$ satisfies Maeda's conjecture.

In particular, as there are infinitely many primes $p \equiv_4 1$, there are infinitely many $\mathrm{T}_{kp}^{\mathrm{alg}}$ satisfying Maeda's conjecture on S_{12k} for valid k. A computer check shows that the only k satisfying the condition 1 above in the range $2 \leqslant k \leqslant 500$ are $k = 2$, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 128, 192, 256, and 384. This leaves those $2 \leqslant k \leqslant 500$ satisfying both conditions 1 and 2 as the following set:

$$\{2, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 128, 192, 256, 384\}$$

Both Ths. 10.2.5 and 10.2.6 seem to be the first examples that show Maeda's conjecture holds for infinite families of Hecke operators $\mathrm{T}_n^{\mathrm{alg}}$ on a space of cusp forms of a fixed weight k where n is not assumed to be prime.

The strategies to prove both Ths. 10.2.5 and 10.2.6 are the same: we use Th. 10.2.2 to reduce ourselves to show certain coefficients of particular modular forms do not vanish, and then we use Pr. 10.1.5 in tandem with Prs. 10.2.3 and 10.2.4 to make some approximations modulo 3 or powers of 2, respectively. An eagle-eyed reader will notice that a more careful proof that pays close attention to the q-expansions of powers of Δ could lead to a sharper result; here we prefer to focus on the results obtains from our congruences.

Proof of Th.10.2.5. Consider the following basis for S_{12k}:

$$\{\Delta^k = e_k, c_4^3\Delta^{k-1} = e_{k-1}, \ldots, c_4^{3(k-1)}\Delta = e_1\}$$

Notice that c_4 has q-expansion $1 + 240(O(q))$, which implies that modulo 3 the above basis is congruent to $\{\Delta, \Delta^2, \ldots, \Delta^k\}$. By assumption, the q^k-coefficient in the q-expansions of Δ^i vanish for all $1 \leqslant i \leqslant k-1$. This means that for each cusp form $f \in S_{12k}$, the q^k-coefficient in its q-expansion is congruent to its Δ^k-coefficient using the above basis, modulo 3. Now consider the cusp form $f = \mathrm{T}_n^{\mathrm{alg}}(\Delta^k)$ using the classical Hecke operator $\mathrm{T}_n^{\mathrm{alg}}$. There is the following elementary formula for the coefficient of q^k in the q-expansion of this f:

$$\sum_{d|k,n} d^{12k-1} a_{\frac{nk}{d^2}}(\Delta^k) = a_{nk}(\Delta^k) \tag{10.2.7}$$

Above, we used that $\gcd(k, n) = 1$. We now have the following chain of congruences modulo 3:

$$a_{nk}(\Delta^k) \equiv_3 b_n^k \equiv_3 \frac{\sigma(n)}{n}$$

The first congruence comes from (10.2.7) and our discussion above, and the second from Pr.10.1.5. By Pr.10.2.4, we know that for our chosen n, the above quantity does not vanish modulo 3. This implies the coefficient of q^{kn} in the q-expansion of $\Delta^k \in S'_{12k}$ does not vanish modulo 3. By Th. 10.2.2, we see that Maeda's conjecture holds for $\mathrm{T}_{kn}^{\mathrm{alg}}$ acting on S_{12k} as this is true for $\mathrm{T}_2^{\mathrm{alg}}$ by [GM12, Th.1.5]. ☐

Proof of Th.10.2.6. We leave the proof to the reader—the eclectic conditions reflect the hypotheses of Pr.10.1.5 and Pr.10.2.3. ☐

In the proof of Cor.10.1.6, we played with some more explicit congruences, and a continued study of these kinds of congruences promises to further improve on the known range where Maeda's conjecture is valid.

10.3 Height 2 Adams summands

By §5.5, we see that KU_p and TMF_p both have p-adic Adams operations ψ^k for each $k \in \mathbf{Z}_p^\times$. When p is odd, then $\mathbf{Z}_p^\times$ splits as $\mathbf{F}_p^\times \times \mathbf{Z}_p$ (the latter is now viewed as a group additively). This implies that both KU_p and TMF_p have $\mathbf{E}_\infty$-actions of the group $\mathbf{F}_p^\times$, which by Gauß is isomorphic to the cyclic group of order $p-1$. A classical construction in homotopy theory is called the *Adams summand* $\mathrm{KU}_p^{h\mathbf{F}_p^\times}$ and is usually denoted by L, and with connective cover ℓ. Both L and ℓ have simple homotopy groups as we are working with p-complete spectra and the group $\mathbf{F}_p^\times$ has order prime to p. In particular, we have isomorphisms

$$\pi_*\ell \simeq \mathbf{Z}_p[v_1] \qquad \pi_*L \simeq \mathbf{Z}_p[v_1^{\pm}]$$

where $v_1 = u^{p-1}$ is the first Hasse invariant from chromatic homotopy theory. When written like this, it is clear that ℓ is an $\mathbf{E}_\infty$-form of p-complete $\mathrm{BP}\langle 1\rangle$. These $\mathbf{E}_\infty$-rings L and ℓ are summands of KU_p and ku_p, respectively, associated to the idempotent map

$$\frac{1}{p-1}\sum_{k\in\mathbf{F}_p^\times}\psi^k \tag{10.3.1}$$

which reveals why they are called Adams *summands*. In fact, more is true, as one can easily check that the canonical maps of $\mathbf{E}_\infty$-rings $L \to \mathrm{KU}_p$ and $\ell \to \mathrm{ku}_p$ recognise the codomain as a *quasi-free*[50] module over the source of rank $p-1$. Given we have the same p-adic Adams operations on TMF_p (and tmf_p from Th.7.7.4), we would like to see how much of the above works at the height two—the results are not what one might first expect; see Th.10.3.3.

Definition 10.3.2. For an odd prime p, recall the $\mathbf{F}_p^\times$ action on the $\mathbf{E}_\infty$-rings TMF_p and tmf_p given by Th.7.1.2 and Th.7.7.4, respectively. Define the $\mathbf{E}_\infty$-rings $\mathrm{u} = \mathrm{tmf}_p^{h\mathbf{F}_p^\times}$ and $\mathrm{U} = \mathrm{TMF}_p^{h\mathbf{F}_p^\times}$ and called them the *height two Adams summands.* By Th.7.7.4, the natural map $\mathrm{tmf}_p \to \mathrm{TMF}_p$ factors through a map of $\mathbf{E}_\infty$-rings $\mathrm{u} \to \mathrm{U}$, and $\mathrm{tmf}_p \to \mathrm{ku}_p$ factors through a map of $\mathbf{E}_\infty$-rings $\mathrm{u} \to \ell$.

We choose these names as u is to tmf_p as ℓ is to ku_p—we are open to other conventions. The homotopy groups of u and U as still simple to write down if $p \geqslant 5$:

$$\pi_*\mathrm{u} \simeq (\mathbf{Z}_p[x,y])^{\mathbf{F}_p^\times} \simeq \mathbf{Z}_p\left[x^iy^j \mid i,j \geqslant 0 \text{ such that } 4i+6j \equiv_{p-1} 0\right]$$

$$\pi_*\mathrm{U} \simeq \mathbf{Z}_p\left[x^iy^j\Delta^k \mid i,j \geqslant 0, k\in\mathbf{Z} \text{ such that } 4i+6j+12k \equiv_{p-1} 0\right]$$

where $x = c_4$ has degree 8, $y = c_6$ has degree 12, and $\Delta = \frac{x^3-y^2}{1728}$. Both u and U are summands of tmf_p and TMF_p, respectively, using the same idempotent (10.3.1) as the height one case. What is curious, is that the inclusions $\mathrm{u} \to \mathrm{tmf}_p$ and $\mathrm{U} \to \mathrm{TMF}_p$ behave differently to the height one case.

Theorem 10.3.3. *Let p be an odd prime. The map $\mathrm{U} \to \mathrm{TMF}_p$ recognises the codomain as a rank $\frac{p-1}{2}$ quasi-free module over the domain. The map $\mathrm{u} \to \mathrm{tmf}_p$ recognises the target as a rank $\frac{p-1}{2}$ quasi-free module if $p-1$ divides 12 and for all other odd primes tmf_p is never a quasi-free u-module.*

The proof of this theorem is rather elementary and consists of formal stable homotopy theory and some dimension formulae for spaces of (meromorphic) modular forms.

Proof. Let us start with the connective case—it is a little simpler. For $p = 3$, the map $\mathrm{u} \to \mathrm{tmf}_3$ is an equivalence, as $\mathbf{F}_3^\times$ act trivially on $\pi_*\mathrm{tmf}_3$ and the

[50]Recall from [HA, Df.7.2.1.16] that for an $\mathbf{E}_\infty$-ring R and an R-module M, we say M is *quasi-free* if there exists an equivalence $M \simeq \bigoplus_\alpha R[n_\alpha]$, and M is *free* if all of the n_α can be taken to be zero.

order of this group is invertible in $\pi_0 \operatorname{tmf}_3 \simeq \mathbf{Z}_3$ so the associated homotopy fixed point spectral sequence collapses. Let us focus on larger primes then. At $p = 5$, we claim the map of u-modules

$$\mathrm{u} \oplus \mathrm{u}[12] \xrightarrow{1 \oplus y} \operatorname{tmf}_5$$

defined by the elements $1, y \in \pi_* \operatorname{tmf}_5$, is an equivalence. This is clear as the first summand contains all the monomials $x^i y^j$ where j is even, and the second summand those where j is odd. Similarly, we can define maps of u-modules

$$\mathrm{u} \oplus \mathrm{u}[8] \oplus \mathrm{u}[16] \xrightarrow{1 \oplus x \oplus x^2} \operatorname{tmf}_7$$

$$\mathrm{u} \oplus \mathrm{u}[8] \oplus \mathrm{u}[12] \oplus \mathrm{u}[16] \oplus \mathrm{u}[20] \oplus \mathrm{u}[28] \xrightarrow{1 \oplus x \oplus y \oplus x^2 \oplus xy \oplus x^2 y} \operatorname{tmf}_{13}$$

at the primes 7 and 13, respectively. As in the $p = 5$ case, one easily checks these maps are equivalences on homotopy groups. For the negative cases now. For $p = 11$, we notice that $\pi_* \mathrm{u}$ is precisely the summand of $\pi_* \operatorname{tmf}_{11}$ supported in nonnegative degrees divisible by 20. Any potential splitting of tmf_{11} into sums of u would have to start by hitting generators in degrees 0, 8, 12, 16, and 24. The problem is that we need two summands $\mathrm{u}[24]$ to hit both y^2 and x^3 in degree 24, which would mean that our sum of u's has dimension at least 4 in degree 64. This contradicts the fact that the dimension of the $\mathbf{Z}_{11}$-module $\pi_{64} \operatorname{tmf}_{11}$ has dimension 3. Similar problems happen for primes $p \geqslant 17$. Indeed, for each of these primes, $\pi_* \mathrm{u}$ is the summand of $\pi_* \operatorname{tmf}_p$ supported in nonnegative degrees divisible by $2(p-1)$. A potential splitting of tmf_p into sums of u would have to hit the two generators in degree 24, as $2(p-1) \geqslant 2(16) = 32$ is greater than 24, so $\pi_{24} \mathrm{u} = 0$. However, writing d for the dimension

$$d = \dim_{\mathbf{Z}_p} \left(\pi_{2(p-1)} \operatorname{tmf}_p\right) = \dim_{\mathbf{Z}_p} \left(\pi_{2(p-1)} \mathrm{u}\right) \geqslant 2$$

where the inequality comes from the fact that $2(p-1) \geqslant 32$, we obtain the following:

$$\dim_{\mathbf{Z}_p} \left(\pi_{2(p-1)+24} \operatorname{tmf}_p\right) = d + 1 < 2d = \dim_{\mathbf{Z}_p} \left(\pi_{2(p-1)+24} \mathrm{u}[24] \oplus \mathrm{u}[24]\right)$$

This shows that there can be no splitting of tmf_p purely in terms of suspensions of u.

Onto the periodic case. For primes $p = 3, 5, 7$, and 13, we can simply take the connective equivalence and invert $\Delta \in \pi_{24} \mathrm{U}$. For primes $p \geqslant 17$ and $p = 11$, consider the basis $\mathfrak{B}$ of Nt.10.0.1. Let us write $f_k = \Delta^l E_{k'}$ for the generators of $\mathrm{MF}_k^{\mathbf{Z}_p}$ as a module over $\mathbf{Z}_p[j] \simeq \mathrm{MF}_0^{\mathbf{Z}_p}$. Note these basis elements have some multiplicativity properties which we will implicitly use in what follows:

$$f_{k_1} \cdot f_{12k_2}^r = f_{k_1} \cdot f_{12rk_2} = f_{k_1} \cdot \Delta^{rk_2} = f_{k_1 + 12rk_2}$$

We now have four cases to consider depending on the remainder of p modulo 12. Essentially, $f_{p-1} \in \pi_{2(p-1)} \mathrm{U}$ is the first nonzero generator of $\pi_* \mathrm{U}$ after $\pi_0 \mathrm{U}$. Our splitting of TMF_p will depend on if f_{p-1} is purely a power of Δ, or a power of Δ multiplied by $x^2 y$, x, or y. These are precisely the four cases below, respectively.

(The $p \equiv_{12} 1$ case) Consider the following map of U-modules:

$$\phi_1 \colon \mathrm{U}[2p] \oplus \bigoplus_{\substack{0 \leqslant 2d < p-1 \\ d \neq 1}} \mathrm{U}[4d] \xrightarrow{f_p \oplus \bigoplus f_{2d}} \mathrm{TMF}_p$$

We claim ϕ_1 is an equivalence. First note the map is injective on homotopy groups, as $\pi_* \mathrm{U}$ is concentrated in degrees divisible by $2(p-1)$ and each summand in the domain of the map ϕ_1 only hits elements in $\mathrm{MF}_*^{\mathbf{Z}_p}$ in degrees which are pairwise distinct modulo $2(p-1)$. In the range $0 \leqslant k \leqslant p-2$, every f_k is hit by ϕ_1 by construction—the only case up for debate is f_2, however, $f_{p-1} = \Delta^{\frac{p-1}{12}}$ lies in $\pi_{2(p-1)}\mathrm{U}$ with inverse $\Delta^{\frac{1-p}{12}}$ inside $\pi_{2(1-p)}\mathrm{U}$, and we then obtain the following:

$$f_p \cdot f_{1-p} = \Delta^{\frac{p-1}{12}-1} x^2 y \cdot \Delta^{\frac{1-p}{12}} = \frac{x^2 y}{\Delta} = f_2$$

It then follows that all other f_k are hit, for all even $k \in 2\mathbf{Z}$. Indeed, for each such k, there is an integer r such that $k + r(p-1)$ lies in the range between 0 and $p-2$. As $f_{k+r(p-1)} = f_k \cdot f_{p-1}^r$ is hit by ϕ_1, and f_{p-1}^r and its inverse lies in $\pi_{2r(p-1)}\mathrm{U}$, we see that the $\pi_*\mathrm{U}$-module map induced by ϕ_1 hits f_k.

(The $p \equiv_{12} 11$ case) Consider the following map of U-modules:

$$\phi_{11} \colon \bigoplus_{0 \leqslant 2d < p-1} \mathrm{U}[24d] \xrightarrow{\bigoplus f_{12d} = \Delta^d} \mathrm{TMF}_p$$

We claim this map is an equivalence. As in the $p \equiv_{12} 1$ case above, we see the induced map on π_* is injective. To see each f_k in $\mathrm{MF}_*^{\mathbf{Z}_p}$ is hit by ϕ_{11}, we first note that $f_{6(p-1)} = f_{p-1}^6 = \Delta^{\frac{p-1}{2}}$ lies in $\pi_{12(p-1)}\mathrm{U}$, and the above map hits every power of Δ less than $f_{6(p-1)}$ by construction. Given an even integer k, then clearly f_k is hit by ϕ_{11} if k is divisible by 12. Also, note the following equalities inside $\pi_*\mathrm{U}$:

$$f_{p-1} = \Delta^{\frac{p-11}{12}} xy \qquad f_{2(p-1)} = \Delta^{\frac{p-11}{6}+1} x^2$$

$$f_{3(p-1)} = \Delta^{\frac{p-11}{4}+2} y \qquad f_{4(p-1)} = \Delta^{\frac{p-11}{3}+3} x$$

$$f_{5(p-1)} = \Delta^{5\frac{p-11}{12}+3} x^2 y$$

If f_k is of the form $\Delta^l E_{k'}$ for k not divisible by 12, then the equations above show there exists an integer r and an i in the range $1 \leqslant i \leqslant 5$ such that $f_k = \Delta^r f_{i(p-1)}$ simply because this range of $f_{i(p-1)}$ contain the five remaining possible $E_{k'}$. Writing $r = a + b\frac{p-1}{2}$ for an integer b and $0 \leqslant a < \frac{p-1}{2}$, we see that f_k is in the image of ϕ_{11}

$$f_k = \Delta^r \cdot f_{i(p-1)} = \Delta^a \cdot \Delta^{b\frac{p-1}{2}} \cdot f_{i(p-1)} = \Delta^a \cdot f_{6(p-1)}^b \cdot f_{i(p-1)}$$

as Δ^a lies in the image of ϕ_{11} and the other elements lie in $\pi_*\mathrm{U}$.

The following two cases are a mixture of the previous two—let us only detail the first.

(The $p \equiv_{12} 5$ case) Consider the following map of U-modules:

$$\phi_5 \colon \bigoplus_{0 \leqslant 2d < p-1} \mathrm{U}[12d] \xrightarrow{\bigoplus f_{6d}} \mathrm{TMF}_p$$

As previously discussed, the induced map on homotopy groups is injective, so it suffices to see ϕ_5 hits all the generators of $\mathrm{MF}_*^{\mathbf{Z}_p}$. By inspection, we see that ϕ_5 hits all f_k of the form Δ^i and $\Delta^i y$ for all $0 \leqslant i \leqslant \frac{p-5}{4}$. Moreover, note the following equalities in $\pi_*\mathrm{U}$:

$$f_{p-1} = \Delta^{\frac{p-5}{12}} x \qquad f_{2(p-1)} = \Delta^{\frac{p-5}{6}} x^2 \qquad f_{3(p-1)} = \Delta^{\frac{p-5}{4}+1}$$

It follows that every f_k of the form Δ^i and $\Delta^i y$ is hit by ϕ_5, for all integers i now. As in the $p \equiv_{12} 11$ case above, the f_k's of the form $\Delta^l x$, $\Delta^l x^2$, $\Delta^l xy$, and $\Delta^l x^2 y$, are then hit by ϕ_5 as every one of these $E_{k'}$'s is a product of elements in the image of ϕ_5 by construction or in $\pi_*\mathrm{U}$. This shows that ϕ_5 is an equivalence of U-modules.

(The $p \equiv_{12} 7$ case) The map of U-modules

$$\phi_7 \colon \bigoplus_{0 \leqslant 2d < p-1} \mathrm{U}[8d] \xrightarrow{\bigoplus f_{4d}} \mathrm{TMF}_p$$

is an equivalence by an analogous argument to the previous case—we omit the proof. ☐

The negative fact that tmf_p is not a quasi-free u-module for primes $p \geqslant 17$ and $p = 11$ seems salvageable.

Conjecture 10.3.4. *For primes $p \geqslant 17$ and $p = 11$, there exists a cofibre sequence of the following form:*

$$\bigoplus_{0 \leqslant 2k < p-1} \mathrm{u}[?] \xrightarrow{\phi_{\overline{p}}} \mathrm{tmf}_p \to \bigoplus \ell[?]$$

The map of u*-modules $\phi_{\overline{p}}$ localises to $\phi_{\overline{p}}$ from the proof of Th.10.3.3, and only depends on the mod* 12 *reduction $\overline{p}$ of p.*

The only real mathematical hurdle left in proving the above conjecture seems to be a combinatorial argument involving the known dimensions of spaces of modular forms of a fixed weight. Let us now see the example for $p = 11$ in more detail, and quote the results for $p = 17, 19, 23$, and 37.

Fix $p = 11$ and recall we have the following commutative diagram of $\mathbf{E}_\infty$-rings, a consequence of Th.7.7.4:

$$\begin{array}{ccc} \mathrm{u} & \longrightarrow & \mathrm{tmf}_{11} \\ \downarrow & & \downarrow \\ \ell & \longrightarrow & \mathrm{ku}_{11} \end{array}$$

Consider the map of u-modules

$$y^{10}\colon \Sigma^{120}\mathrm{u} \to \mathrm{u}$$

and its cofibre, which we write as u/y^{10}. If one inspects the homotopy groups of u/y^{10}, one will find they look just like those of the following u-module:

$$\bigoplus \ell = \ell \oplus \ell[40] \oplus \ell[60] \oplus \ell[80] \oplus \ell[120]$$

To prove that these u-modules u/y^{10} and $\bigoplus \ell$ are equivalent, and more importantly, to later obtain a morphism of u-modules from ℓ to a quotient of tmf_{11}, consider the cohomological Ext-spectral sequence

$$E_2^{s,t} \simeq \mathrm{Ext}^{s,t}_{\mathrm{u}*}(\pi_{-*}M, \pi_{-*}N) \Longrightarrow \pi_{-s-t}F_{\mathrm{u}}(M,N)$$

for any pair of u-modules M and N. Setting $M = \mathrm{u}/y^{10}$, the short exact sequence defining $\pi_*\mathrm{u}/y^{10}$ shows it has projective dimension 1 as a $\pi_*\mathrm{u}$-module, meaning the above spectral sequence is supported in $s = 0, 1$ and immediately collapses. This degeneration yields a surjection of groups

$$\pi_0 F_{\mathrm{u}}(\mathrm{u}/y^{10}, N) \to \mathrm{Ext}^{0,0}_{\mathrm{u}*}(\pi_{-*}\mathrm{u}/y^{10}, N).$$

Setting $N = \bigoplus \ell$ and using the desired isomorphism on homotopy groups as $\pi_*\mathrm{u}$-modules, we obtain an equivalence of u-modules

$$\mathrm{u}/y^{10} \simeq \bigoplus \ell.$$

The u-module ℓ then naturally maps into u/y^{10} as the first summand of $\bigoplus \ell$, and with this inclusion, we will study a quotient of tmf_p. Consider the map of u-modules

$$\phi_{11}\colon \bigoplus_{d=0}^{4} \mathrm{u}[24d] \to \mathrm{tmf}_{11}$$

defined by the elements y^{2d} for $d = 0, \ldots, 4$—this is the the connective version of the map ϕ_{11} from the proof of Th.10.3.3. Write tmf_{11}/ϕ for the cofibre of this map. Consider the map of u-modules

$$x\colon \mathrm{u}[8] \to \mathrm{tmf}_{11}$$

given by $x \in \pi_8 \mathrm{tmf}_p$ and the following diagram of u-modules:

$$\begin{array}{ccccc} \mathrm{u}[128] & \xrightarrow{y^{10}} & \mathrm{u}[8] & \longrightarrow & \mathrm{u}/y^{10}[8] \\ & & \downarrow{\scriptstyle x} & & \\ & & \mathrm{tmf}_{11} & \longrightarrow & \mathrm{tmf}_{11}/\phi \end{array}$$

The composite $\mathrm{u}[128] \to \mathrm{tmf}_{11}/\phi$ vanishes. Indeed, this map of u-modules is represented by the class xy^{10} in $\pi_{128}\mathrm{tmf}_{11}/\phi$ and $y^{10} = 0 \in \pi_{120}\mathrm{tmf}_{11}/\phi$ by the construction of ϕ_{11}. Hence, we obtain a map $\mathrm{u}/y^{10}[8] \to \mathrm{tmf}_{11}/\phi$ which induces multiplication by x on homotopy groups. Precomposing this map with the inclusion $\ell \to \bigoplus \ell$ and the equivalence $\mathrm{u}/y^{10} \simeq \bigoplus \ell$, we obtain the following map of u-modules:

$$i_x \colon \ell[8] \to \mathrm{tmf}_{11}/\phi$$

Repeating this process for the classes $z \in \pi_{|z|}\,\mathrm{tmf}_{11}$ in the set

$$Z_{11} = \{y, x^2, x^3, x^4, y^3, x^6, x^7, x^9, x^{12}\}$$

we obtain corresponding maps of u-modules $i_z \colon \ell[|z|] \to \mathrm{tmf}_{11}/\phi$. These morphisms sum to give the following map of u-modules:

$$i_{11} \colon \bigoplus_{z \in Z_{11}} \ell[|z|] \to \mathrm{tmf}_{11}/\phi$$

It is now a purely combinatorial exercise to check that this is an equivalence. Altogether, this yields the following cofibre sequence of u-modules:

$$\bigoplus_{d=0}^{4} \mathrm{u}[24d] \xrightarrow{\phi_{11}} \mathrm{tmf}_{11} \to \bigoplus_{z \in Z_{11}} \ell[|z|]$$

Other examples validating Conj.10.3.4 are the cofibre sequences:

$$\bigoplus_{d=0}^{7} \mathrm{u}[12d] \xrightarrow{\bigoplus y^d} \mathrm{tmf}_{17} \to \bigoplus_{12} \ell[?]$$

$$\bigoplus_{d=0}^{8} \mathrm{u}[8d] \xrightarrow{\bigoplus x^d} \mathrm{tmf}_{19} \to \bigoplus_{9} \ell[?]$$

$$\bigoplus_{d=0}^{10} \mathrm{u}[24d] \to \mathrm{tmf}_{23} \xrightarrow{\bigoplus \Delta^d} \bigoplus_{55} \ell[?]$$

$$u \oplus \bigoplus_{d=0}^{16} \mathrm{u}[4d+8] \oplus \Sigma^{76}\mathrm{u} \xrightarrow{1 \oplus \bigoplus f_{2d+4} \oplus f_{38}} \mathrm{tmf}_{37} \to \bigoplus_{18} \ell[?]$$

The question marks above signify our lack of understanding of the pattern behind the types of shifts of ℓ that occur, although everything above seems to only truly depend upon the residue of the prime modulo 12.

10.4 Height 2 image of J spectra

Fix a prime p and a generator g of $\mathbf{Z}_p^\times/F$ where F is the maximal finite subgroup of $\mathbf{Z}_p^\times$. At the prime $p = 2$ (take $g = 3$ in this case), a classical construction in homotopy theory is that of the *connective image of J* spectrum j, defined by the following cofibre sequence of spectra:

$$\mathrm{j} \to \mathrm{ko}_2 \xrightarrow{\psi^g - 1} \tau_{\geqslant 4}\,\mathrm{ko}_2$$

This is to be thought of as the $\mathbf{Z}_p^\times$-fixed points of ku_2, or the $\mathbf{Z}_p^\times/\{\pm 1\}$-fixed points of ko. The map $\psi^g - 1\colon \mathrm{ko}_2 \to \mathrm{ko}_2$ factors through $\tau_{\geqslant 4}\,\mathrm{ko}_2$ as it induces the zero map on the first three non vanishing homotopy groups of ko_2 as a quick calculation shows.

We will use Th.7.7.4 to copy this construction, and compare the results to the classical case—we restrict ourselves to the $p = 2$ case for simpler exposition; see others primes in [Dav21a, §3].

Definition 10.4.1. Define j_2 as the fibre of the following map:

$$\mathrm{tmf}_2 \xrightarrow{\psi^3 - 1} \tau_{\geqslant 8}\,\mathrm{tmf}_2$$

This comes with a unit $\mathbf{S} \to \mathrm{j}_2$ as the unit $\mathbf{S} \to \mathrm{tmf}_2$ vanishes in $\tau_{\geqslant 8}\,\mathrm{tmf}_2$. For an odd prime p, fix a generator g of $\mathbf{Z}_p^\times/\mathbf{F}_p^\times$. By Th.7.1.2, the $\mathbf{E}_\infty$-ring U of Df.10.3.2 has an action of $\mathbf{Z}_p^\times/\mathbf{F}_p^\times$. Writing g for a fixed generator of $\mathbf{Z}_p^\times/\mathbf{F}_p^\times$, define J_2 as the fibre of the following map:

$$\mathrm{U} \xrightarrow{\psi^g - 1} \mathrm{U}$$

Again, there is a unit $\mathbf{S} \to \mathrm{J}_2$ as $\psi^g(1) = 1$.

Note that j_2 and J_2 are **not** just the 2-completions of j and $\mathrm{J} = \mathrm{j}[\alpha^{-1}]$, but rather the 2 indicates height. The notation s and S is also tempting, but it seems too close to the sphere $\mathbf{S}$. We also want to make a connective definition of j_2 at odd primes, but in this case, we would like to first take the $\mathbf{Z}_p^\times/F$-fixed points of u from Df.10.3.2, but Th.7.7.4 does not yet give us compatible Adams operations on tmf_p.

Due to the classical results and the fact we have a map $\mathrm{j}_2 \to \mathrm{j}$ factoring the units from $\mathbf{S}$ to j and j_2, one obtains the following.

Theorem 10.4.2. *All elements $\alpha_{i/j} \in \pi_{2i-1}\mathbf{S}$ detected by elements in the 1-line of the Adams–Novikov spectral sequence for the sphere have nontrivial image in $\pi_{2i-1}\mathrm{j}_2$.*

We do not obtain a similar result for J_2 at odd primes as we lack a map to K-theory from TMF—this is one reason why we would like to extend our constructions of coherent stable Adams operations "over the cusp" to tmf.

Proof. The elements $\alpha_{i/j}$ are all nonzero in $\pi_*\mathrm{j}$ and the map $\pi_*\mathbf{S} \to \pi_*\mathrm{j}$ is a split surjection with image exactly these $\alpha_{i/j}$—the image of J; see [Mah75] for the original construction of j and [Koc90, §4] for statement mentioned here. This implies that $\pi_*\mathrm{j}_2 \to \pi_*\mathrm{j}$ is split surjective, as splitting can be given by the composite

$$\pi_*\mathrm{j} \to \pi_*\mathbf{S} \to \pi_*\mathrm{j}_2$$

which implies the result. ∎

If we had stable Hecke operators T_n on tmf_p commuting with stable Adams operations à la Pr.7.2.4, then we could obtain stronger congruences than those obtained in §10.1 using j_2 in place of tmf_p. For example, one could show that given odd positive integers k and n with $k \geqslant 3$, then $\sigma_1(n)$ is congruent to $n\sigma_k(n)$ modulo a_k, where a_k is defined 8 if $k \equiv 1$ modulo 4 and the 2-primary part of the denominator of $B_{k+1/2}/k+1$ if $k \equiv 3$ modulo 4. This would be proven using the techniques discussed in §10.1 combined with Th.10.4.2.

This is just the tip of the iceberg. For example, we have only used formal properties of j_2 to analyse the map $\pi_*\mathbf{S} \to \pi_*\mathrm{j}_2$, but one could try an Adams spectral sequence to further calculate the Hurewicz image inside $\pi_*\mathrm{j}_2$ which would yield further congruences in number theory via Hecke operators. One could also replace j_2 with Behrens $Q(N)$ spectra, or perhaps even connective versions thereof, and play the same tricks. This could give us further cases of Maeda's conjecture, or might shed some light on *Lehmer's conjecture*[51] on the non vanishing of $\tau(n)$. These ideas are all future research directions for the author and anyone else in the community.

[51] Recall Lehmer's conjecture states that Ramanujan's τ function never vanishes and was originally stated in [Leh34].

Appendices

Appendix A

Formal spectral algebraic geometry

All work and no play makes Jack a dull boy.

English proverb

Throughout Part I we have used basic properties of formal spectral Deligne–Mumford stacks that are not explicitly contained in [SAG] (at least not obviously to the author), so we have arranged this appendix to prove these statements. Every single statement in this subsection is an extension of a proof in [SAG] and the author claims no originality for the ideas below.

A.1 The embedding fDM $\to$ fSpDM

To formalise the relationship between the classical and spectral worlds of formal algebraic geometry, we need a functor fDM $\to$ fSpDM. Let us begin by defining these categories.

Definition A.1.1. Let A be a classical adic Noetherian ring with finitely generated ideal of definition $I \subseteq A$, cutting out a closed subset $V \subseteq |\operatorname{Spec} A|$.

1. Define the topos $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_{\mathrm{Set}}(\mathrm{CAlg}^{\mathrm{\acute{e}t}}_A)$ as the full ∞-subcategory of $\mathcal{S}\mathrm{hv}^{\mathrm{\acute{e}t}}_{\mathrm{Set}}(\mathrm{CAlg}^{\mathrm{\acute{e}t}}_A)$ spanned by those étale sheaves $\mathcal{F}$ such that if the space $V \times_{|\operatorname{Spec} A|} |\operatorname{Spec} B|$ is empty, then $\mathcal{F}(B)$ is a point.
2. One has a sheaf of discrete rings $\mathscr{O}_{\operatorname{Spec} A}$ on $\mathcal{S}\mathrm{hv}^{\mathrm{\acute{e}t}}_{\mathrm{Set}}(\mathrm{CAlg}^{\mathrm{\acute{e}t}}_A)$ as in [SAG, Df.1.2.3.1], which we complete at I to obtain a sheaf $\widehat{\mathscr{O}}$. As $\widehat{\mathscr{O}}(B) \simeq B^\wedge_I$ vanishes whenever the image of I generates the unit ideal of B, we can regard $\widehat{\mathscr{O}}$ as a sheaf on $\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_{\mathrm{Set}}(\mathrm{CAlg}^{\mathrm{\acute{e}t}}_A)$.

Define the ringed topos $\operatorname{Spf} A = (\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_{\mathrm{Set}}(\mathrm{CAlg}^{\mathrm{\acute{e}t}}_A), \widehat{\mathscr{O}})$, the *formal spectrum of* A, leaving the dependency on the specific topology on A implicit. A *locally*

Noetherian formal Deligne–Mumford stack is a ringed topos $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ such that $\mathcal{X}$ has a cover U_α such that each ringed topos $\mathfrak{X}_{/U_\alpha}$ is equivalent (in the 2-category of ringed topoi of [SAG, Df.1.2.1.1]) to Spf A_α for some discrete adic Noetherian ring A_α. Write fDM for the full 2-category of $1\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^\heartsuit}$ spanned by locally Noetherian formal Deligne–Mumford stacks.

The ∞-category of formal spectral Deligne–Mumford stacks fSpDM can be defined similarly; see [SAG, Df.8.1.3.1].

As in [SAG, §8], when dealing with classical formal Deligne–Mumford stacks, we restrict ourselves to the locally Noetherian case *by definition*, as opposed to the spectral case, when we only add this assumption when we need it. As mentioned in [SAG, Warn.8.1.0.4], this is due to the incompatibility between completions in the classical and derived worlds.

Remark A.1.2. If an adic discrete ring A has a nilpotent ideal of definition, then $\mathrm{Spf}\, B$ is naturally equivalent to $\mathrm{Spec}\, B$ by definition. In this way, we can see (Noetherian) affine Deligne–Mumford stacks as affine formal Deligne–Mumford stacks with ideal of definition (0). It immediately follows from the definitions that $\mathrm{DM}_{\mathrm{loc.N}}$ is a full 2-subcategory of fDM.

The following is [SAG, Rmk.1.4.1.5].

Construction A.1.3. There is a fully faithful embedding of ∞-categories from classical ringed topoi to spectrally ringed ∞-topoi

$$1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^\heartsuit} \hookrightarrow \infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}} \qquad (\mathcal{X}, \mathscr{O}_{\mathcal{X}}) \mapsto (\mathcal{S}\mathrm{hv}(\mathcal{X}), \mathscr{O}).$$

In other words, it associates to a classical Grothendieck topos $\mathcal{X}$ the associated ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{X})$ (this is done using [HTT09, Pr.6.4.5.7]) and by [SAG, Rmk.1.3.5.6] we obtain a connective 0-truncated structure sheaf on $\mathcal{S}\mathrm{hv}(\mathcal{X})$, denoted as $\mathscr{O}$. In fact, the essential image of the above embedding is spanned by the spectrally ringed ∞-topoi $(\mathcal{X}, \mathscr{O}_{\mathcal{X}})$ where $\mathcal{X}$ is 1-localic, meaning the canonical geometric morphism $\mathcal{X} \to \mathcal{S}\mathrm{hv}(\mathcal{X}^\heartsuit)$ is an equivalence, and $\mathscr{O}_{\mathcal{X}}$ is connective and 0-truncated.

By [SAG, Rmk.1.4.8.3], the fully faithful embedding of Con.A.1.3 restricts to the full-faithful embedding DM → SpDM. Let us show that the same holds for *formal* Deligne–Mumford stacks.

Proposition A.1.4. *The functor of Con.A.1.3, when restricted to* fDM *factors through* fSpDM*. Moreover, the essential image of this fully faithful functor* fDM → fSpDM *consists of those locally Noetherian formal spectral Deligne–Mumford stacks* $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ *for which the* ∞*-topos* $\mathcal{X}$ *is 1-localic ([HTT09, Df.6.4.5.8]) and the structure sheaf* $\mathscr{O}_{\mathfrak{X}}$ *is 0-truncated.*

Proof. The fully faithful functor of Con.A.1.3 descends to a fully faithful functor between (not full) ∞-subcategories of local topoi:

$$1\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}^\heartsuit} \hookrightarrow \infty\mathcal{T}\mathrm{op}^{\mathrm{loc}}_{\mathrm{CAlg}}$$

Indeed, we say $\mathscr{X} = (\mathcal{X}, \mathscr{O}_{\mathscr{X}})$ in $\infty\mathcal{T}\mathrm{op}_{\mathrm{CAlg}}$ is local if $\pi_0\mathscr{O}_{\mathscr{X}}$ is local on $\mathcal{X}^{\heartsuit}$ ([SAG, Df.1.4.2.1]), and given $\mathscr{X}_0 = (\mathcal{X}_0, \mathscr{O}_0)$ in $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^{\heartsuit}}$, then the ringed topos $(\mathcal{S}\mathrm{hv}(\mathcal{X})^{\heartsuit}, \pi_0\mathscr{O})$ is naturally equivalent to $\mathscr{X}_0$ by [HTT09, Pr.6.4.5.7]. Local morphisms between local spectrally ringed ∞-topoi are morphisms of spectrally ringed ∞-topoi whose underlying morphism of ringed topoi is local.

Let $\mathfrak{X}_0 = (\mathcal{X}_0, \mathscr{O}_0)$ be a classical formal Deligne–Mumford stack, and write $\mathfrak{X} = (\mathcal{X}, \mathscr{O})$ for the image of $\mathfrak{X}_0$ under Con.A.1.3, so $\mathcal{X} = \mathcal{S}\mathrm{hv}(\mathcal{X}_0)$. By [SAG, Pr.8.1.3.3], the property of being a formal spectral Deligne–Mumford stack is a local one, so it suffices to show that there exists a cover U_α of $\mathcal{X}$ such that each $\mathfrak{X}_{/U_\alpha}$ is in fSpDM. Consider a formal affine cover of $\mathfrak{X}_0$ in $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^{\heartsuit}}$, so a collection of U_α inside $\mathcal{X}_0$ such that $\coprod U_\alpha \to \mathbf{1}_{\mathcal{X}_0}$ is an effective epimorphism and $(\mathfrak{X}_0)_{/U_\alpha}$ is equivalent in $1\mathcal{T}\mathrm{op}_{\mathrm{CAlg}^{\heartsuit}}$ to $\mathrm{Spf}\, A_\alpha$. Considering U_α as a discrete object V of $\mathcal{X}$ (as in [HTT09, Pr.6.4.5.7]), then [SAG, Lm.1.4.7.7(2)] states that $\mathcal{X}_{/V}$ is 1-localic, as $\mathcal{X}$ is 1-localic and V is 0-truncated in $\mathcal{X}$. One then notes the following natural equivalences:

$$\mathcal{X}_{/V} \xrightarrow{\simeq} \mathcal{S}\mathrm{hv}((\mathcal{X}_{/V})^{\heartsuit}) \simeq \mathcal{S}\mathrm{hv}((\mathcal{X}_0)_{/U_\alpha})$$

$$\simeq \mathcal{S}\mathrm{hv}(\mathcal{S}\mathrm{hv}^{\mathrm{ad}}_{\mathrm{Set}}(\mathrm{CAlg}^{\mathrm{\acute{e}t}}_{A_\alpha})) \xleftarrow{\simeq} \mathcal{S}\mathrm{hv}^{\mathrm{ad}}(\mathrm{CAlg}^{\mathrm{\acute{e}t}}_{A_\alpha})$$

The first equivalence holds as $\mathcal{X}_{/V}$ is 1-localic, the second by identifying $\mathcal{X}_0$ as the underlying discrete objects of $\mathcal{X}$ (and then [HTT09, Rmk.7.2.2.17]), the third from the choice of U_α as an affine object of $\mathcal{X}_0$, and the forth from the fact that affine formal spectral Deligne–Mumford stacks are 1-localic; see [SAG, Rmk.8.1.1.9]. Furthermore, as $\mathscr{O}$ was defined as the sheaf of connective 0-truncated $\mathbf{E}_\infty$-rings on $\mathcal{X}$ associated to the commutative ring object $\mathscr{O}_0$ on $\mathcal{X}_0$, we claim that by [SAG, Rmk.1.3.5.6] the spectrally ringed ∞-topos $\mathfrak{X}_{/U_\alpha}$ is equivalent to $\mathrm{Spf}\, A_\alpha$. To see this, one notes that $\mathscr{O}(\mathrm{Spf}\, B) = B_I^\wedge$ for some étale morphism $\mathrm{Spf}\, B \to \mathrm{Spf}\, A_\alpha$ in $\mathcal{X}_0 \subseteq \mathcal{X}$, and one also has a natural equivalence $\mathscr{O}_{\mathrm{Spf}\, A_\alpha}(\mathrm{Spf}\, B) \simeq B_I^\wedge$ by [SAG, Con.8.1.1.10]. The "moreover" statement follows by [SAG, Rmk.1.4.1.5]. □

Combining the functor of points approach with the above, we obtain the following:

Corollary A.1.5. *The following diagram of ∞-categories and fully faithful functors commutes:*

$$\begin{array}{ccccccc}
\mathrm{Aff}^{\heartsuit}_{\mathrm{loc.N}} & \xrightarrow{a} & \mathrm{Aff}^{\heartsuit}_{\mathrm{ad,loc.N}} & \xrightarrow{b} & \mathrm{fDM} & & \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} & & \\
\mathrm{Aff}^{\mathrm{cn}} & \xrightarrow{f} & \mathrm{Aff}^{\mathrm{cn}}_{\mathrm{ad}} & \xrightarrow{g} & \mathrm{fSpDM} & \xrightarrow{h} & \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})
\end{array}$$

Warning A.1.6. One might want to place $\mathcal{P}(\mathrm{Aff}^{\heartsuit})$ in the top-right corner of the diagram above, however, we do not see a functor $\mathcal{P}(\mathrm{Aff}^{\heartsuit}) \to \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ such that the diagram above commutes. Indeed, the obvious right Kan extension along $\mathrm{CAlg}^{\heartsuit} \to \mathrm{CAlg}^{\mathrm{cn}}$ does not commute with the other constructions above by inspection and a left Kan extension would not necessarily preserve sheaves. The existence of the functors c, d, and e above, are all due to nontrivial theorems of Lurie, and the lack of a similar functor $\mathcal{P}(\mathrm{Aff}^{\heartsuit}) \to \mathcal{P}(\mathrm{Aff}^{\mathrm{cn}})$ indicates one reason why we restrict our attention to (formal) Deligne–Mumford stacks.

Proof of Cor.A.1.5. The functors a, b, f, and g are all the inclusions of full ∞-subcategories, c and d are the inclusions of ∞-subcategories as shown by Lurie ([HA, Pr.7.1.3.18]), e is Con.A.1.3, and h is the functor-of-points functor. The diagram commutes as c and d are restrictions of e. To see why each functor is fully faithful, we have:

- By definition, we see that a, b, f, and g are fully faithful.
- By [HA, Pr.7.1.3.18], we see c and hence d are fully faithful.
- Pr.A.1.4 shows e is fully faithful.
- The fact that h is fully faithful is the content of [SAG, Th.8.1.5.1]. □

A.2 Truncations

We would like to show that for locally Noetherian formal spectral Deligne–Mumford stacks, there is a well-defined truncation functor. The following is a generalisation of [SAG, Pr.1.4.6.3] to formal spectral Deligne–Mumford stacks; we will even use the same proof and notation.

Proposition A.2.1. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ be a locally Noetherian formal spectral Deligne–Mumford stack. For each $n \geqslant 0$, the object $\tau_{\leqslant n}\mathfrak{X} = (\mathcal{X}, \tau_{\leqslant n}\mathcal{O}_{\mathfrak{X}})$ is a locally Noetherian formal spectral Deligne–Mumford stack. Moreover, for every $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ inside $\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}$, if $\mathcal{O}_{\mathcal{Y}}$ is connective and n-truncated, then the canonical map $\tau_{\leqslant n}\mathfrak{X} \to \mathfrak{X}$ induces an equivalence of spaces*

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \tau_{\leqslant n}\mathfrak{X}) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \mathfrak{X}).$$

Proof. The first half of the proof of [SAG, Pr.1.4.6.3] applies *mutatis mutandis*. That is, by copying that proof we see that for every strictly Henselian spectrally ringed ∞-topos $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ which is connective and n-truncated, the canonical map

$$\mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \tau_{\leqslant n}\mathfrak{X}) \to \mathrm{Map}_{\infty\mathcal{T}\mathrm{op}^{\mathrm{sHen}}_{\mathrm{CAlg}}}((\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}), \mathfrak{X}) \tag{A.2.2}$$

is an equivalence of spaces. Hence, we are left to show that $\tau_{\leqslant n}\mathfrak{X} = (\mathcal{X}, \tau_{\leqslant n}\mathcal{O}_{\mathfrak{X}})$ is a locally Noetherian formal spectral Deligne–Mumford stack. By [SAG, Prs.8.1.3.3 & 8.4.2.7], being a formal spectral Deligne–Mumford stack and being locally Noetherian are local conditions, hence we may assume $\mathfrak{X} = \operatorname{Spf} A$ for a complete Noetherian adic $\mathbf{E}_\infty$-ring A. Set $B = \tau_{\leqslant n}A$, equipped with the same topology as A induced by $I \subseteq \pi_0 A$ using the isomorphism $\pi_0 A \simeq \pi_0 B$. We now need to show $\operatorname{Spf} B$ is connective, n-truncated, and construct an equivalence with $\tau_{\leqslant n}\mathfrak{X}$.

By [SAG, Pr.8.1.1.13], we see $\operatorname{Spf} B = (\mathcal{X}_{\operatorname{Spf} B}, \mathscr{O}_{\operatorname{Spf} B})$ is connective. For n-truncatedness, one can argue as follows: for affine objects U of $\mathcal{X}_{\operatorname{Spf} B}$ we have $\mathscr{O}_{\operatorname{Spf} B}(U) \simeq C_I^\wedge$ for some étale B-algebra C. As C is an étale $\mathbf{E}_\infty$-B-algebra, then it is almost of finite presentation, and as B is Noetherian (as a truncation of the Noetherian $\mathbf{E}_\infty$-ring A), then the spectral Hilbert basis theorem ([HA, Pr.7.2.4.31]) implies that C is also Noetherian. It then follows from [SAG, Cor.7.3.6.9] that the natural map of $\mathbf{E}_\infty$-A-algebras $C \to C_I^\wedge$ is flat. As the composition

$$B \to C \to C_I^\wedge \simeq \mathscr{O}_{\operatorname{Spf} B}(U)$$

is flat, we see $\mathscr{O}_{\operatorname{Spf} B}(U)$ is n-truncated as B is so. The ∞-topos $\mathcal{X}_{\operatorname{Spf} B}$ is generated by affine objects under small colimits ([SAG, Pr.8.1.3.7]) and the structure sheaf $\mathscr{O}_{\operatorname{Spf} B}\colon \mathcal{X}_{\operatorname{Spf} B}^{\operatorname{op}} \to \operatorname{CAlg}$ preserves limits, so it follows that $\mathscr{O}_{\operatorname{Spf} B}(X)$ is n-truncated for all $X \in \mathcal{X}_{\operatorname{Spf} B}$, hence $\operatorname{Spf} B$ is n-truncated; see [SAG, Rmk.1.3.2.6]. By (A.2.2), the natural map $\operatorname{Spf} B \to \operatorname{Spf} A = \mathfrak{X}$ factors as:

$$\operatorname{Spf} B \xrightarrow{\phi} \tau_{\leqslant n}\mathfrak{X} = (\mathcal{X}, \tau_{\leqslant n}\mathscr{O}_{\mathfrak{X}}) \to (\mathcal{X}, \mathscr{O}_{\mathfrak{X}}) = \mathfrak{X}$$

Using [SAG, Rmk.8.1.1.9], we see the map of underlying ∞-topoi induced by $\phi\colon A \to \tau_{\leqslant n}A = B$ is an equivalence,

$$\mathcal{S}\mathrm{hv}^{\text{ét}}_{\pi_0 B/I} \simeq \mathcal{S}\mathrm{hv}^{\mathrm{ad}}_B \xrightarrow{\phi_*} \mathcal{S}\mathrm{hv}^{\mathrm{ad}}_A \simeq \mathcal{S}\mathrm{hv}^{\text{ét}}_{\pi_0 A/I}$$

where we used the notation of [SAG, Nt.8.1.1.8]. Under this map, the structure sheaf of $\operatorname{Spf} B$ is sent to the functor

$$\phi_*\mathscr{O}_{\operatorname{Spf} B}\colon \operatorname{CAlg}^{\text{ét}}_A \to \operatorname{CAlg}^{\mathrm{cn}} \qquad D \mapsto (D \otimes_A B)_I^\wedge \simeq (\tau_{\leqslant n}D)_I^\wedge. \tag{A.2.3}$$

The equivalence above comes from the facts that $A \to D$ is étale and a degenerate Tor-spectral sequence calculation; see [HA, Pr.7.2.1.19]. To see ϕ is an equivalence, it therefore suffices to see that (A.2.3) is equivalent to $\tau_{\leqslant n}\mathscr{O}_{\operatorname{Spf} A}$. This is slight variation on an argument made above. As D is étale over the Noetherian $\mathbf{E}_\infty$-ring A, then the spectral Hilbert basis theorem implies that D is also Noetherian. It follows straight from the definition that the $\mathbf{E}_\infty$-ring $\tau_{\leqslant n}D$ is Noetherian, so the natural completion map of $\mathbf{E}_\infty$-A-algebras

$$\tau_{\leqslant n}D \to (\tau_{\leqslant n}D)_I^\wedge$$

is flat. This implies that $(\tau_{\leqslant n}D)_I^\wedge$ is n-truncated. As $\tau_{\leqslant n}(D_I^\wedge)$ is I-complete by [SAG, Cor.7.3.4.3], there is a natural equivalence of $\mathbf{E}_\infty$-A-algebras:

$$(\tau_{\leqslant n}D)_I^\wedge \simeq \tau_{\leqslant n}(D_I^\wedge)$$

Hence ϕ is an equivalence of spectrally ringed ∞-topoi. ☐

The following is a formal generalisation of [SAG, Cor.1.4.6.4]:

Corollary A.2.4. *For each integer $n \geqslant 0$, write* $\mathrm{fSpDM}_{\mathrm{loc.N}}^{\leqslant n}$ *for the full ∞-subcategory of* $\mathrm{fSpDM}_{\mathrm{loc.N}}$ *spanned by those n-truncated locally Noetherian formal spectral Deligne–Mumford stacks. The inclusion*

$$\mathrm{fSpDM}_{\mathrm{loc.N}}^{\leqslant n} \hookrightarrow \mathrm{fSpDM}_{\mathrm{loc.N}}$$

has a right adjoint, given on objects by

$$\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}}) \mapsto \tau_{\leqslant n}\mathfrak{X} = (\mathcal{X}, \tau_{\leqslant n}\mathscr{O}_{\mathfrak{X}}).$$

Proof. This follows straight from the universal property of Pr. A.2.1 and the observation that truncations of locally Noetherian formal spectral Deligne–Mumford stacks remain locally Noetherian. □

Corollary A.2.5. *Let $\mathfrak{X}$ be a locally Noetherian formal spectral Deligne–Mumford stack. Then for any integer $n \geqslant 0$ the truncation $\tau_{\leqslant n}\mathfrak{X}$ and $\mathfrak{X}$ represent the same functor on n-truncated $\mathbf{E}_\infty$-rings.*

Proof. Follows straight from Pr. A.2.1, as $\operatorname{Spec} R$ is a connective n-truncated spectrally ringed ∞-topos when R is a connective n-truncated $\mathbf{E}_\infty$-ring; see [SAG, Ex.1.4.6.2]. □

A.3 Finiteness and compactness

Next, let us discuss finiteness and compactness conditions in fSpDM.

Proposition A.3.1. *Let $\mathfrak{X}$ be a locally Noetherian formal spectral Deligne–Mumford stack. Then for any $n \geqslant 0$ the natural map $\tau_{\leqslant n}\mathfrak{X} \to \mathfrak{X}$ admits an $(n+1)$-connective and almost perfect cotangent complex.*

Proof. These are local conditions, so we may take $\mathfrak{X} = \operatorname{Spf} A$ for a complete Noetherian adic $\mathbf{E}_\infty$-ring A with finitely generated ideal of definition $I \subseteq \pi_0 A$. By the Hilbert basis theorem for connective $\mathbf{E}_\infty$-rings ([HA, Pr.7.2.4.31]) we see $\tau_{\leqslant n}A$ is almost finitely presented as an $\mathbf{E}_\infty$-A-algebra and the cofibre of the map $A \to \tau_{\leqslant n}A$ is $(n+1)$-connective. By [HA, Cor.7.4.3.2] and [HA, Th.7.4.3.18], we then see $L = L_{\tau_{\leqslant n}A/A}$ is $(n+1)$-connective and almost perfect inside $\mathrm{Mod}_{\tau_{\leqslant n}A}$. It follows from [SAG, Pr.7.3.5.7] that L is in fact I-complete, hence we have a natural equivalence $L \simeq L_{\operatorname{Spf}\tau_{\leqslant n}A/\operatorname{Spf}A}$ by [SAG, Df.17.1.2.8], and we are done. □

Definition A.3.2. A formal spectral Deligne–Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ is *quasi-compact* (qc) if the underlying ∞-topos $\mathcal{X}$ is quasi-compact, ie, every cover of $\mathcal{X}$ has a finite subcover; see [SAG, Df.A.2.0.12]. A morphism of formal spectral Deligne–Mumford stacks

$$f\colon \mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}}) \to \mathfrak{Y} = (\mathcal{Y}, \mathscr{O}_{\mathfrak{Y}})$$

is *qc* if for any qc object U of $\mathcal{Y}$, the pullback $f^*(U)$ is qc in $\mathcal{X}$, meaning $\mathcal{X}_{/f^*U}$ is qc. A morphism of formal spectral Deligne–Mumford stacks is called *quasi-separated* (qs) if the diagonal map $\Delta\colon \mathfrak{Y} \to \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ is qc. We say $\mathfrak{X}$ is qs if $\mathfrak{X} \to \operatorname{Spec}\mathbf{S}$ is qs.

It is a purely formal exercise that qc (and qs) maps are stable under base-change—a fact we will use without further reference.

Proposition A.3.3. *Let A be an adic $\mathbf{E}_\infty$-ring. Then* $\operatorname{Spf} A$ *is qc.*

Proof. By [SAG, Rmk.8.1.1.9], we see the underlying ∞-topos of $\operatorname{Spf} A$ is equivalent to $\mathcal{S}\mathrm{hv}^{\text{ét}}_{\pi_0 A/I}$ where I is a finitely generated ideal of definition for the topology on $\pi_0 A$. As this is the same underlying ∞-topos of $\operatorname{Spec}(\pi_0 A/I)$, it follows from [SAG, Pr.2.3.1.2] that $\operatorname{Spf} A$ is qc. □

The following is a formal generalisation of a special case of [SAG, Pr.2.3.2.1].

Proposition A.3.4. *Let $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$ be a formal spectral Deligne–Mumford stack. Then the following are equivalent.*

1. *$\mathfrak{X}$ is qs.*
2. *For all qc objects U, V of $\mathcal{X}$, the product $U \times V$ in $\mathcal{X}$ is qc.*
3. *For all affine objects U, V of $\mathcal{X}$, the product $U \times V$ is qc.*

Proof. It is clear that 1 implies 2 as $U \times V = \Delta^*(U, V)$ inside $\mathcal{X} \times \mathcal{X}$, and 2 also implies 1 as the quasi-compact objects of $\mathcal{X} \times \mathcal{X}$ are all of the form (U, V) for U and V quasi-compact in $\mathcal{X}$. Pr.A.3.3 shows that 2 implies 3. Conversely, for two arbitrary qc objects U and V of $\mathcal{X}$, using the fact they are qc, there exists two effective epimorphisms $U' \to U$ and $V' \to V$ where U' and V' are affine. It then follows that $U \times V$ is qc as there is an effective epimorphism $U' \times V' \to U \times V$ from a qc object of $\mathcal{X}$. □

Corollary A.3.5. *Let A be an adic $\mathbf{E}_\infty$-ring. Then* $\operatorname{Spf} A$ *is qcqs.*

Proof. By Pr.A.3.3, we see $\operatorname{Spf} A$ is qc, and by Pr.A.3.4 it suffices to see that for all affine objects $U = \operatorname{Spf} B$ and $V = \operatorname{Spf} C$ inside $\mathcal{X}_{\operatorname{Spf} A}$, that the product $U \times V$ in $\mathcal{X}_{\operatorname{Spf} A}$ is qc. This product can be recognised as the fibre product ([SAG, Lm.8.1.7.3])

$$\operatorname{Spf} B \underset{\operatorname{Spf} A}{\times} \operatorname{Spf} C \simeq \operatorname{Spf} \left(B \underset{A}{\otimes} C \right)^{\wedge}_{I}$$

where I is an ideal of definition for the topology on $\pi_0 A$, which is qc by Pr.A.3.4. □

The following statement is why we care about the adjectives of Df.A.3.2.

Proposition A.3.6. *Let $\mathfrak{X}$ be a formal spectral Deligne–Mumford stack. Then $\mathfrak{X}$ is qcqs if and only if there exists an étale hypercover $\mathfrak{U}_\bullet$ of $\mathfrak{X}$ such that each $\mathfrak{U}_n$ is an affine formal spectral Deligne–Mumford stack for every $n \geqslant 0$. In particular, the same holds for classical Deligne–Mumford stacks.*

Proof. First, let us assume $\mathfrak{X}$ is qcqs, write $\mathfrak{X} = (\mathcal{X}, \mathscr{O}_{\mathfrak{X}})$, and set $\mathfrak{U}_{-1} = \mathfrak{X}$. As a formal spectral Deligne–Mumford stack, there exists a collection of affine objects U_α in $\mathcal{X}$ such that $\coprod_\alpha U_\alpha$ cover $\mathcal{X}$, and as $\mathfrak{X}$ is qc, this collection can be taken to be finite. As $\mathcal{X}_{/U_\alpha} \simeq \operatorname{Spf} A_\alpha$ for some adic $\mathbf{E}_\infty$-ring A_α, we see the fact that $\coprod U_\alpha$ covers $\mathcal{X}$ is equivalent to the statement that

$$\operatorname{Spf} A_0 = \operatorname{Spf}\left(\prod A_\alpha\right) \simeq \coprod \operatorname{Spf} A_\alpha \to \mathfrak{X}$$

is an étale surjection, where we have used the finiteness of the above (co)product. Set $\mathfrak{U}_0 = \operatorname{Spf} A_0$ and $\mathfrak{U}_0 \to M_0(\mathfrak{U}_\bullet^{\leqslant -1}) \simeq \mathfrak{U}_{-1} = \mathfrak{X}$ to be the étale surjection above. The rest of this direction of the proof can be summarised informally by inductively calculating the matching objects $M_n(\mathfrak{U}_\bullet^{\leqslant n-1})$ which must be affine as a finite limit of affine formal spectral Deligne–Mumford stacks. Using that affines are qcqs (Cor. A.3.5) we find $\mathfrak{U}_{n+1}$ by taking an affine étale cover of $M_n(\mathfrak{U}_\bullet^{\leqslant n-1})$. To formalise this outline, we will need to play around with these matching objects more carefully.

Inductively, let us assume the following three hypotheses:

1. Suppose we have the nth stage of an étale hypercover $\mathfrak{U}_\bullet^{\leqslant n}$ such that $\mathfrak{U}_m \simeq \operatorname{Spf} A_m$ is affine for each $0 \leqslant m \leqslant n$.
2. Suppose that for every $0 \leqslant m \leqslant n$, $M_m(\mathfrak{U}_\bullet^{\leqslant m-1})$ is affine. The base case that $\mathfrak{U}_0 \simeq \operatorname{Spf} A_0$ is affine holds by construction.

For every $1 \leqslant k \leqslant m \leqslant n$, write $\mathfrak{U}_{\bullet+k}^{\leqslant m-k}$ for the functor defined by precomposition with the shift functor, itself defined on objects by

$$\Delta_{s,+}^{\leqslant m-k} \to \Delta_{s,+}^{\leqslant m}, \qquad [i] \mapsto [i+k]$$

and on morphisms by sending $\phi\colon [i] \to [j]$ to $\phi'\colon [i+k] \to [j+k]$ which sends $a + k \mapsto \phi(a) + k$ for $a \geqslant 0$ and $-1 \mapsto -1$. The third hypothesis is then:

3. Suppose that for every $1 \leqslant k \leqslant m \leqslant n$, $M_{m-k+1}(\mathfrak{U}_{\bullet+k}^{\leqslant m-k})$ is affine. This condition is vacuous in the base-case.

We claim that there is a natural equivalence

$$M = M_{n+1}(\mathfrak{U}_\bullet^{\leqslant n}) = \lim_{[i]\hookrightarrow[n+1]} \mathfrak{U}_i^{\leqslant n} \simeq \mathfrak{U}_n \underset{M_n(\mathfrak{U}_\bullet^{\leqslant n-1})}{\times} M_n(\mathfrak{U}_{\bullet+1}^{\leqslant n-1}) \tag{A.3.7}$$

which occurs in fSpDM, as [SAG, Pr.8.1.7.1] states the ∞-category fSpDM has finite limits. To see (A.3.7) is an equivalence, recall that our diagram 1-category above is the poset of proper subsets of $[n+1]$. Using notation from [MV15, §5.1], we see the opposite of this poset is precisely the 1-category $\mathcal{P}_0(\underline{n+2})$ of nonempty subsets S of $\{1, \ldots, n+2\}$. This yields the equivalence:

$$M \simeq \lim_{S \in \mathcal{P}_0(\underline{n+2})} \mathfrak{U}_{n-|S|+1}^{\leqslant n}$$

Using the cubical limit manipulations of [MV15, Lm.5.3.6], we obtain the natural equivalence of (A.3.7):

$$M \simeq \lim_{S \in \mathcal{P}_0(\underline{n+2})} \mathfrak{U}^{\leqslant n}_{n-|S|+1} \simeq \mathfrak{U}_n \underset{M_n(\mathfrak{U}^{\leqslant n-1}_\bullet)}{\times} M_n(\mathfrak{U}^{\leqslant n-1}_{\bullet+1})$$

Now, the map $\mathfrak{U}_n \to M_n(\mathfrak{U}^{\leqslant n-1}_\bullet)$ is an étale cover by our first inductive hypothesis and the natural map

$$M_n(\mathfrak{U}^{\leqslant n-1}_{\bullet+1}) \to M_n(\mathfrak{U}^{\leqslant n-1}_\bullet)$$

is an étale cover by base-change—each $\mathfrak{U}_{m+1} \to \mathfrak{U}_m$ is an étale cover and $M_n(-)$ is a finite limit of such covers. We also note that M is qcqs, which follows from (A.3.7), our inductive assumptions 1-3, and the fact that affines are qcqs (Cor.A.3.5). This guarantees the existence of an étale cover $\mathfrak{U}_{n+1} \to M_{n+1}(\mathfrak{U}^{\leqslant n}_\bullet)$ with $\mathfrak{U}_{n+1}$ an affine formal spectral Deligne–Mumford stack, from which we obtain our first inductive conclusion for $(n+1)$. As $M = M_{n+1}(\mathfrak{U}^{\leqslant n}_\bullet)$ is qcqs, we also have our second inductive conclusion for $(n+1)$. For the third inductive hypothesis, we consider $M(k) = M_{n-k+2}(\mathfrak{U}^{\leqslant n+1-k}_{\bullet+k})$, the only case left to consider; the others fall under part 3 of the the previous inductive step. We claim that $M(k)$ is affine. To see this, use an index shift of (A.3.7) to obtain:

$$M(k) \simeq \mathfrak{U}_{n-k+2} \underset{M_{n-k+1}(\mathfrak{U}^{\leqslant n-k}_{\bullet+k})}{\times} M_{n-k+1}(\mathfrak{U}^{\leqslant n-k}_{\bullet+k+1})$$

The left and bottom objects in the fibre product above are affine by our inductive hypotheses 1 and 2, respectively, so it suffices to show the right object in the above fibre product is affine. This can be done by applying (A.3.7) again, noting the left and bottom objects are affine by inductive hypotheses 1 and 2 again and again considering the right factor. Applying this process $(n-k)$-many times, it suffices to show $M_0(\mathfrak{U}^{\leqslant -1}_{\bullet+n+2}) \simeq \mathfrak{U}_{n+1}$ is affine, which follows from our construction above.

Conversely, assume that $\mathfrak{X}$ has an étale hypercover $\mathfrak{U}_\bullet \to \mathfrak{X}$ where each $\mathfrak{U}_n$ is affine, which we write as $U_\bullet \to \mathbf{1}$ when considered as objects in the ∞-topos $\mathcal{X}$. Given an arbitrary cover $\{V_\alpha\}_{\alpha \in I}$ of $\mathfrak{X}$, so an effective epimorphism $\coprod V_\alpha \to \mathbf{1}$ inside $\mathcal{X}$, then we can consider the following Cartesian square inside $\mathcal{X}$:

$$\begin{array}{ccc} W & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ \coprod_I V_\alpha & \longrightarrow & \mathbf{1} \end{array}$$

All of the maps above are effective epimorphisms either by assumption or by base-change; see [HTT09, Pr.6.2.3.15]. Products commute with colimits in an ∞-topos as colimits in ∞-topoi are universal,[52] hence we have a natural equivalence in $W \simeq \coprod_I W_\alpha$ in $\mathcal{X}$, where $W_\alpha = V_\alpha \times U_0$. As U_0 is quasi-compact

[52]We say that colimits in a presentable ∞-category $\mathcal{C}$ are *universal* if pullbacks commute with all small colimits; see [HTT09, Df.6.1.1.2]. This holds in an ∞-topos due to the ∞-categorical version of Giraud's axioms; see [HTT09, Th.6.1.0.6].

(as an affine object of $\mathcal{X}$; see Pr.A.3.3), we can choose a finite subset of I, say I_0, such that $\coprod_{I_0} W_\alpha \to U_0$ is an effective epimorphism. We then consider the commutative diagram inside the ∞-topos $\mathcal{X}$:

$$\begin{array}{ccc} \coprod_{I_0} W_\alpha & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ \coprod_{I_0} V_\alpha & \longrightarrow & \mathbf{1} \end{array}$$

The top and right maps are effective epimorphisms by assumption, and the bottom map is an effective epimorphism by [HTT09, Cor.6.2.3.12(2)], hence $\mathcal{X}$ is qc. To see $\mathcal{X}$ is qs, we look at the Cartesian diagram of formal spectral Deligne–Mumford stacks:

$$\begin{array}{ccc} \mathfrak{U}_0 & \xrightarrow{\Delta_{\mathfrak{U}_0}} & \mathfrak{U}_0 \times \mathfrak{U}_0 \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times \mathcal{X} \end{array}$$

As $\mathfrak{U}_\bullet \to \mathfrak{X}$ is an étale hypercover, the map $U_0 \times U_0 \to \mathbf{1} \times \mathbf{1}$ is an effective epimorphism in $\mathcal{X}$. As U_0 is the ∞-topos of an affine formal Deligne–Mumford stack, then by Cor.A.3.5 we see $\mathfrak{U}_0$ is qs and the map $\Delta_{\mathfrak{U}_0}$ is qc. It follows from [SAG, Cor.A.2.1.5] that $\Delta_{\mathcal{X}}$ is qc; in *ibid*, a qc morphism is called *relatively 0-coherent*. Hence, $\mathcal{X}$, and therefore $\mathfrak{X}$, is qs. □

Let us now show the *formal thickenings* of [SAG, §18.2.2] preserve the adjective qcqs.

Proposition A.3.8. *Let $\mathfrak{X}_0$ be a qcqs formal spectral Deligne–Mumford stack and $\mathfrak{X}_0 \to \mathfrak{X}$ a formal thickening. Then $\mathfrak{X}$ is qcqs.*

Proof. The adjective qcqs depends only on the underlying ∞-topoi, so it suffices to show that $\mathfrak{X}_0 \to \mathfrak{X}$ is an equivalence of ∞-topoi. To see this, consider the *reduction* of a formal spectral Deligne–Mumford stack of [SAG, Pr.8.1.4.4]. From this one obtains the following commutative diagram of formal spectral Deligne–Mumford stacks:

$$\begin{array}{ccc} \mathfrak{X}_0^{\mathrm{red}} & \longrightarrow & \mathfrak{X}_0 \\ \downarrow & & \downarrow \\ \mathfrak{X}^{\mathrm{red}} & \longrightarrow & \mathfrak{X} \end{array}$$

We know the natural map from the reduction of a formal spectral Deligne–Mumford stack $\mathfrak{X}$ back into $\mathfrak{X}$ is an equivalence of underlying ∞-topoi (by [SAG, Pr.8.1.4.4]), and the underlying ∞-topoi of the reduction of a formal thickening is also an equivalence (by [SAG, Pr.18.2.2.6]). Hence the horizontal and the left vertical maps are equivalences of underlying ∞-topoi, hence the right vertical map is as well. □

Appendix B

Uniqueness of $\mathscr{O}^{\mathrm{top}}$

It's hell on Earth and the city's on fire,
Inhale, in hell there's heaven.
There's a bull and a matador dueling in the sky,
Inhale, in hell there's heaven.

Frank Ocean, Solo

For this chapter, which is based on [Dav21b], let us write $\mathcal{M}_{\mathrm{Ell}}$ for the compactification of the moduli stack of elliptic curves, and $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}$ for the moduli stack of smooth elliptic curves (written elsewhere as $\mathcal{M}_{\mathrm{Ell}}$). The main character in this chapter will be $\mathcal{M}_{\mathrm{Ell}}$, which classifies *generalised elliptic curves* (with irreducible geometric fibres); see [Ces17], [Con07], or [DR73] for some background on these objects.

The following is a simple uniqueness statement for the Goerss–Hopkins–Miller sheaf $\mathscr{O}^{\mathrm{top}}$ on the small étale site of $\mathcal{M}_{\mathrm{Ell}}$ (of [DFHH14]) as a functor valued in homotopy commutative ring spectra.

Proposition B.0.1. *The functor* $\mathrm{h}\mathscr{O}^{\mathrm{top}}\colon \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{hSp})$*, from the small affine étale site* $\mathcal{U}$ *of* $\mathcal{M}_{\mathrm{Ell}}$ *to the 1-category of homotopy commutative ring spectra, is uniquely defined up to isomorphism by the property that it defines natural elliptic cohomology theories (Df.6.1.8) on* $\mathcal{U}$*.*

The proof of the above statement follows from the fact that each section $\mathscr{O}^{\mathrm{top}}(R)$ is Landweber exact; see [Beh14, Rmk.1.6]. A remarkable fact about $\mathscr{O}^{\mathrm{top}}$ is that the property that it defines a natural elliptic cohomology theory characterises this sheaf with values in the ∞-category CAlg of $\mathbf{E}_\infty$-rings. The following is stated (without proof) in [Lur09a, Th.1.1] and [Goe10, Th.1.2].

Theorem B.0.2. *The sheaf of* $\mathbf{E}_\infty$*-rings* $\mathscr{O}^{\mathrm{top}}$ *on the small étale site of* $\mathcal{M}_{\mathrm{Ell}}$ *is uniquely defined up to homotopy by the property that it defines natural elliptic cohomology theories on the small affine étale site of* $\mathcal{M}_{\mathrm{Ell}}$*. The same holds for the restriction* $\mathscr{O}^{\mathrm{top}}_{\mathrm{sm}}$ *of* $\mathscr{O}^{\mathrm{top}}$ *to the small étale site of* $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}$*.*

The difference between the Pr.B.0.1 and Th.B.0.2 is two-fold: firstly, as the former concerns presheaves of homotopy commutative ring spectra, rather than the more structured $\mathbf{E}_\infty$-rings of the latter, and secondly, the natural transformations in the former exist in a 1-category and in the latter such a natural equivalence exists in an ∞-category of sheaves of $\mathbf{E}_\infty$-rings. At the end of the day, both statements only show uniqueness up to some form of homotopy.

The utility of Th.B.0.2 is evident. For example, it retroactively shows that the various constructions of $\mathscr{O}^{\mathrm{top}}$ found in [Beh14], [HL16], and [Dav21a, §2] (and also [Lur18, §7] and [Dav20, §2.3] over the moduli stack of smooth elliptic curves) all agree up to homotopy—we have used Th.B.0.2 several times in this thesis already, such as in the proofs of Th.5.3.3, Cor.6.1.7, and Th.6.1.9. Importantly, Th.B.0.2 constructs noncanonical (see Rmk.B.1.2) equivalences of $\mathbf{E}_\infty$-rings between all available definitions of Tmf; a conclusion which does **not** follow directly from Pr.B.0.1. The author also finds the proof long enough to warrant a publicly available write-up, even if the steps involved are mostly predictable.

To prove Th.B.0.2, we will first reduce the question to one of the connectedness of a certain moduli space (Th.B.1.1) where we also formulate and prove a statement about spaces of natural transformations which we will often use (Pr.B.1.5). We then prove Th.B.1.1 which follows Behrens' construction of Tmf rather closely: first we work with the separate chromatic layers, before gluing things together in both a transchromatic sense and then an arithmetic sense. The $K(1)$-local case in this section requires a statement about p-adic Adams operations on p-adic K-theory, which is the focus of Lm.B.3.1.

B.1 A reduction

Recall from Pr.B.0.1 that $\mathrm{h}\mathscr{O}^{\mathrm{top}}$ is uniquely defined inside $\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{hSp}))$ by the fact it defines an elliptic cohomology theory.

Theorem B.1.1. *Write $\mathcal{U}$ (resp. $\mathcal{U}_{\mathrm{sm}}$) for the (2-) category of affine schemes with étale maps to $\mathcal{M}_{\mathrm{Ell}}$ (resp. $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}$). Then the spaces*

$$\mathscr{Z} = \mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg}) \underset{\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg}(\mathrm{hSp}))}{\times} \{\mathrm{h}\mathscr{O}^{\mathrm{top}}\}$$

$$\mathscr{Z}^{\mathrm{sm}} = \mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg}) \underset{\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}_{\mathrm{sm}}, \mathrm{CAlg}(\mathrm{hSp}))}{\times} \{\mathrm{h}\mathscr{O}^{\mathrm{top}}\}$$

are connected.

Remark B.1.2. As mentioned in [Lur18, Rmk.7.0.2], the moduli space $\mathscr{Z}^{\mathrm{sm}}$ is **not** contractible. In other words, Th.B.1.1 states that $\mathscr{O}^{\mathrm{top}}$ is unique as a CAlg-valued presheaf of elliptic cohomology theories on $\mathcal{U}^{\mathrm{sm}}$ only up to homotopy, and **not** up to contractible choice. We would like to guide the reader to an explanation for this fact given by Tyler Lawson on `mathoverflow.net`; see [Law].

Proof of Th.B.0.2 from Th.B.1.1. The ∞-category of sheaves of $\mathbf{E}_\infty$-rings on the étale site of $\mathcal{M}_{\mathrm{Ell}}$ is equivalent, by restriction and right Kan extension, to the ∞-category of sheaves of $\mathbf{E}_\infty$-rings on the affine étale site of $\mathcal{M}_{\mathrm{Ell}}$; see Lm.6.1.10 for a similar argument. Note that the latter is an ∞-subcategory of $\mathrm{Fun}(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg})$, and that if a functor $F\colon \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}$ defines a natural elliptic cohomology theory and there is an equivalence $F \simeq G$, then G also defines a natural elliptic cohomology theory. These two observations show that it suffices to prove the space $\mathscr{Z}'$ is connected, where $\mathscr{Z}'$ is the component of $\mathrm{Fun}\,(\mathcal{U}^{\mathrm{op}}, \mathrm{CAlg})^{\simeq}$ spanned by those functors which define natural elliptic cohomology theories. There is a map $\mathscr{Z} \to \mathscr{Z}'$ as both $\mathscr{O}^{\mathrm{top}}$ (defined by [Beh14], for example) and any presheaf of $\mathbf{E}_\infty$-rings equivalent to $\mathscr{O}^{\mathrm{top}}$ as a diagram of homotopy commutative ring spectra, defines a natural elliptic cohomology theory. The map $\mathscr{Z} \to \mathscr{Z}'$ induces an equivalence on π_0 as Pr.B.0.1 states that any functor $\mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{hSp})$ which defines an elliptic cohomology theory is isomorphic to $\mathrm{h}\mathscr{O}^{\mathrm{top}}$. Th.B.1.1 then implies that the moduli space $\mathscr{Z}'$, and hence also $\mathscr{Z}$, is connected. The same argument can be made for $\mathscr{Z}^{\mathrm{sm}}$. □

Remark B.1.3. Write $\mathcal{U}_{\mathbf{Q}}$ for the small affine étale site of $\mathcal{M}_{\mathrm{Ell}} \times \operatorname{Spec}\mathbf{Q}$ and for each prime p write $\mathcal{U}_p$ for the small affine étale site of $\mathcal{M}_{\mathrm{Ell}} \times \operatorname{Spf}\mathbf{Z}_p$. The construction of $\mathscr{O}^{\mathrm{top}}$ as found in [Beh14] for example proceeds first with a rational construction $\mathscr{O}^{\mathrm{top}}_{\mathbf{Q}}$ over $\mathcal{U}_{\mathbf{Q}}$, and a p-complete construction $\mathscr{O}^{\mathrm{top}}_p$ over $\mathcal{U}_p$. The methods of our proof for Th.B.1.1 show that the moduli spaces $\mathscr{Z}_{\mathbf{Q}}$ and $\mathscr{Z}_p$, of realisations of $\mathrm{h}\mathscr{O}^{\mathrm{top}}_{\mathbf{Q}}$ and $\mathrm{h}\mathscr{O}^{\mathrm{top}}_p$ over these aforementioned sites, are also connected. This means that analogs of Th.B.0.2 also holds for both $\mathscr{O}^{\mathrm{top}}_{\mathbf{Q}}$ and $\mathscr{O}^{\mathrm{top}}_p$. The same holds for the p-completed and rational version of $\mathscr{O}^{\mathrm{top}}_{\mathrm{sm}}$ for similar reasons. Moreover, following the "arithmetic compatibility" discussed in the proof of Th.B.1.1, it follows that the localisations $\mathscr{O}^{\mathrm{top}}[\mathcal{P}^{-1}]$ and $\mathscr{O}^{\mathrm{top}}_{\mathrm{sm}}[\mathcal{P}^{-1}]$ satisfy their own version of Th.B.0.2, where $\mathcal{P}$ is any set of primes.

The following is a short remark on the homotopy groups of elliptic cohomology theories which will be important later.

Remark B.1.4. Let $\mathcal{E}$ be an elliptic cohomology theory associated with a morphism $E\colon \operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$. It follows that there is a natural isomorphism $\pi_{2k}\mathcal{E} \simeq \omega_E^{\otimes k}$ for all integers k, where ω_E is the dualising line for the formal group $\widehat{E}$ (12). Indeed, as the odd homotopy groups of $\mathcal{E}$ vanish, we see $\mathcal{E}$ possesses a complex orientation which yields the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathcal{E}}$ over $\pi_0\mathcal{E}$ (14), for example. From this we see $\mathcal{E}$ is *complex periodic*, meaning it has a complex orientation and is weakly 2-periodic (10), and [Lur18, Ex.4.2.19] then implies that $\pi_2\mathcal{E}$ is isomorphic to the dualising line for the formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_{\mathcal{E}}$. Part 4 of Df.6.1.8 states that $\pi_2\mathcal{E}$ is naturally isomorphic to ω_E, and part 1 gives us the claim above.

To prove Th.B.1.1, we will show that any two functors $\mathscr{O}$ and $\mathscr{O}'$ in $\mathscr{Z}$ can be connected by a path in $\mathscr{Z}$. In particular, we would like effective tools for studying spaces of natural transformations between functors of ∞-categories. The following is known to experts, and a model categorical interpretation can be found in [DKS89].

Proposition B.1.5. *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. Suppose that for all objects X, Y in $\mathcal{C}$ the mapping space $\mathrm{Map}_{\mathcal{D}}(FX, GY)$ is discrete, meaning the natural map*

$$\mathrm{Map}_{\mathcal{D}}(FX, GX) \to \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(\mathrm{h}FX, \mathrm{h}GY)$$

is an equivalence of spaces. Then the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$ is also discrete, so the natural map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \to \mathrm{Hom}_{\mathrm{h}\,\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{h}\mathcal{D})}(\mathrm{h}F, \mathrm{h}G)$$

is an equivalence of spaces, where an h *before a functor denotes post composition with the natural map $\mathcal{D} \to \mathrm{h}\mathcal{D}$; the unit of the homotopy category-nerve adjunction of [Lur09b, Pr.1.2.3.1].*

Proof. By [GHN17, Pr.5.1], the space of natural transformations from F to G is naturally equivalent to the limit of the diagram

$$\mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \xrightarrow{H} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{F^{\mathrm{op}} \times G} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{Map}_{\mathcal{D}}(-,-)} \mathcal{S}, \tag{B.1.6}$$

where $\mathrm{Tw}(\mathcal{C})$ is the *twisted arrow category* of $\mathcal{C}$ (see [GHN17, Df.2.2]), and H is the natural right fibration (see *loc. cit.*).[53] The limit of (B.1.6) is by definition the *end* of the composition $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$. Consider the following not a priori commutative diagram of ∞-categories:

$$\begin{array}{ccccccc}
\mathrm{Tw}(\mathcal{C})^{\mathrm{op}} & \xrightarrow{T} & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{F^{\mathrm{op}} \times G} & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} & \xrightarrow{\mathrm{Map}_{\mathcal{D}}(-,-) = M} & \mathcal{S} \\
\downarrow & & \downarrow & & \downarrow & & \uparrow \\
\mathrm{h}\,\mathrm{Tw}(\mathcal{C})^{\mathrm{op}} & \xrightarrow[T']{} & \mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{h}\mathcal{C} & \xrightarrow[\mathrm{h}F^{\mathrm{op}} \times \mathrm{h}G]{} & \mathrm{h}\mathcal{D}^{\mathrm{op}} \times \mathrm{h}\mathcal{D} & \xrightarrow[\mathrm{Hom}_{\mathrm{h}\mathcal{D}}(-,-) = H]{} & \mathcal{S}_{\leqslant 0}
\end{array} \tag{B.1.7}$$

Above, the vertical functors are the obvious ones, hence the left and middle squares commute. Our hypotheses dictate that the dashed arrow above exists, which we will now denote by P, such that the top-right and bottom left triangles commute. As the inclusion of ∞-subcategories $\mathcal{S}_{\leqslant 0} \subseteq \mathcal{S}$, from the ∞-category of discrete spaces, preserves limits, we note it suffices to compute the limit of (B.1.6) as the limit of $P \circ T$ inside $\mathcal{S}_{\leqslant 0}$. As this limit lands in $\mathcal{S}_{\leqslant 0}$, which is equivalent to the nerve of the 1-category of sets, we see the limit of $P \circ T$ can be calculated as the limit of the lower-horizontal composition of (B.1.7). We then obtain the following natural equivalences, twice employing [GHN17, Pr.5.1], first for general ∞-categories, and again in the classical 1-categorical case:

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \simeq \lim_{\mathrm{Tw}(\mathcal{C})} M(F^{\mathrm{op}} \times G)T \simeq \lim_{\mathrm{Tw}(\mathcal{C})} PT$$

[53] We will stick to the notation and conventions of [GHN17], which is a particular choice out of a possible two; see [GHN17, Wrn.2.4].

$$\simeq \lim_{\mathrm{Tw}(\mathrm{h}\mathcal{C})} H(\mathrm{h}F^{\mathrm{op}} \times \mathrm{h}G)T' \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{h}\mathcal{C},\mathrm{h}\mathcal{D})}(\mathrm{h}F, \mathrm{h}G).$$

The final (discrete) space above is naturally equivalent to the set

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{h}\mathcal{D})}(\mathrm{h}F, \mathrm{h}G)$$

from the natural equivalence of ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathrm{h}\mathcal{D}) \simeq \mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{h}\mathcal{D})$. □

B.2 Proof of uniqueness

Our proof will follow the construction of $\mathscr{O}^{\mathrm{top}}$ presented in [Beh14] rather closely—the reader may want to keep a copy at hand.

Let $\mathscr{O}\colon \mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}$ be an object of $\mathcal{Z}$, hence it comes equipped with an equivalence $\mathrm{h}\phi\colon \mathrm{h}\mathscr{O}^{\mathrm{top}} \to \mathrm{h}\mathscr{O}$ of functors $\mathcal{U}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{hSp})$. To see $\mathcal{Z}$ is connected, it suffices to show $\mathrm{h}\phi$ can be lifted to an equivalence $\phi\colon \mathscr{O}^{\mathrm{top}} \to \mathscr{O}$ of presheaves of $\mathbf{E}_\infty$-rings on $\mathcal{U}$. Fix such an $\mathrm{h}\phi$ for the remainder of this proof. Let us work section-wise, so we also fix an object $\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{Ell}}$ inside $\mathcal{U}$, and write

$$\mathrm{h}\phi\colon \mathcal{E}^{\mathrm{top}} = \mathscr{O}^{\mathrm{top}}(R) \to \mathscr{O}(R) = \mathcal{E}$$

for the given natural equivalence of homotopy commutative ring spectra. To naturally lift this map to one of $\mathbf{E}_\infty$-rings we will work through the layers of chromatic homotopy theory. This means we will first work $K(2)$-locally, $K(1)$-locally, and then $K(0)$-locally, where $K(n)$ denotes the *nth Morava K-theory spectrum at a prime p*, before gluing these cases together with a p-complete statement followed by an arithmetic statement.

($K(2)$-local case) Fix a prime p. Writing $\widehat{(-)}$ for base-change over $\mathrm{Spf}\,\mathbf{Z}_p$, we define $\mathrm{Spf}\, R^{\mathrm{ss}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ss}}$ as the base-change of $\mathrm{Spf}\,\widehat{R} \to \widehat{\mathcal{M}}_{\mathrm{Ell}}$ over $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{ss}}$, where the latter is the completion of $\widehat{\mathcal{M}}_{\mathrm{Ell}}$ at the moduli stack $\mathcal{M}_{\mathrm{Ell},\mathbf{F}_p}^{\mathrm{ss}}$ of supersingular elliptic curves over $\mathbf{F}_p$. This pullback $\mathrm{Spf}\, R^{\mathrm{ss}}$ is affine by [Beh14, Rmk.8.7]. Write E^{ss} for the elliptic curve defined by $\mathrm{Spf}\, R^{\mathrm{ss}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ss}}$. Serre– and Lubin–Tate theory yield another description of R^{ss}. Indeed, as $\mathcal{M}_{\mathrm{Ell},\mathbf{F}_p}^{\mathrm{ss}}$ is zero-dimensional and smooth over $\mathrm{Spec}\,\mathbf{F}_p$, it follows that $\mathrm{Spec}\, R^{\mathrm{ss}}/I$ is étale over $\mathbf{F}_p$, where I is the finitely generated ideal generating the topology on R^{ss}. This implies R^{ss}/I splits as a finite product $\prod_i \kappa_i$ where each κ_i is a finite separably field extension of $\mathbf{F}_p$. This provides a splitting of E_0, the reduction of E^{ss} over R/I, into $E_0 \simeq \coprod E_0^i$. Writing $R_i \simeq W(\kappa_i)[\![u_1]\!]$ for the universal deformation ring of the pair $(\kappa_i, \widehat{E}_0^i)$ with associated universal formal group $\widehat{E}_{R_i}^{\mathrm{ss}}$, we obtain a natural equivalence $R^{\mathrm{ss}} \simeq \prod_i R_i$ as $E^{\mathrm{ss}}\colon \mathrm{Spf}\, R^{\mathrm{ss}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ss}}$ was étale; see [Beh14, Cor.4.3], and we have seen this for p-divisible groups in Pr.3.1.10.

By [Beh14, Pr.4.4], the $K(2)$-localisations $\mathcal{E}_{K(2)}^{\mathrm{top}}$ and $\mathcal{E}_{K(2)}$ are elliptic cohomology theories for R^{ss}, and also split into products $\mathcal{E}_i^{\mathrm{top}}$ and $\mathcal{E}_i$. It follows from

[GH04, §7] (also see [Lur18, Rmk.5.0.5] or [PV19, §7]), that these $K(2)$-local $\mathbf{E}_\infty$-rings $\mathcal{E}^{\mathrm{top}}_{K(2)}$ and $\mathcal{E}_{K(2)}$ are naturally equivalent to the product of Lubin–Tate $\mathbf{E}_\infty$-rings associated to the formal groups $\widehat{E}^{\mathrm{ss}}_{\kappa_i}$ over the (finite and hence also) perfect fields κ_i. By *idem*, we see that morphisms between these Lubin–Tate $\mathbf{E}_\infty$-rings are defined by the associated morphisms on the pairs $(\kappa_i, \widehat{E}^{\mathrm{ss}}_{\kappa_i})$. As $\mathrm{h}\phi_{K(2)}$ yields an equivalence on π_0 as well as an equivalence on associated Quillen formal groups, we see $\mathrm{h}\phi_{K(2)}$ lifts to a morphism $\phi_{K(2)}\colon \mathcal{E}^{\mathrm{top}}_{K(2)} \to \mathcal{E}_{K(2)}$ of $K(2)$-local $\mathbf{E}_\infty$-rings, which is unique up to contractible choice. This uniqueness allows us to use Pr.B.1.5 to conclude that the collection of morphisms of $\mathbf{E}_\infty$-rings define a natural morphism $\phi_{K(2)}\colon \mathcal{O}^{\mathrm{top}}_{K(2)} \to \mathcal{O}_{K(2)}$ of presheaves of $\mathbf{E}_\infty$-rings on $\mathcal{U}$; here $(-)_{K(2)}$ denotes $K(2)$-localisation.

($K(1)$-local case) Fix a prime p. Consider the $K(1)$-localisation

$$\mathrm{h}\phi_{K(1)}\colon \mathcal{E}^{\mathrm{top}}_{K(1)} = \mathcal{O}^{\mathrm{top}}_{K(1)}(R) \to \mathcal{O}_{K(1)}(R) = \mathcal{E}_{K(1)}$$

of the map $\mathrm{h}\phi$ of homotopy commutative ring spectra. Recall from [Beh14, §6], that the p-adic K-theory[54] of an $\mathbf{E}_\infty$-ring has the structure of a θ-$\pi_*\mathrm{KU}_p$-algebra, which is functorial in maps of $\mathbf{E}_\infty$-rings. Let us write $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}$ for the moduli of generalised elliptic curves over p-complete rings with ordinary reduction modulo p (see [Beh14, p.3]), and $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty)$ for the moduli stack of generalised elliptic curves E over p-complete rings and level structures given by an isomorphism $\widehat{\mathbf{G}}_m \simeq \widehat{E}$ of formal groups.

Claim B.2.1. The following facts hold for the p-adic K-theory of $\mathcal{E}^{\mathrm{top}}_{K(1)}$ and $\mathcal{E}_{K(1)}$:

1. Both are isomorphic in degree zero to the pullback of $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$ with the composite
$$\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty) \to \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}} \to \widehat{\mathcal{M}}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{Ell}};$$
2. Both are concentrated in even degrees;
3. Both are ind-étale over R^{ord}, the base-change of $\operatorname{Spec} R$ over $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}$, in degree zero; and
4. Both degree zero components have vanishing higher continuous group cohomologies, with respect to its $\mathbf{Z}_p^\times$-action inherited from part 1.

Proof of Clm.B.2.1. Part 1 is obtained from [Beh14, Pr.6.1] by base-change. Part 2 also follows from a graded version of [Beh14, Pr.6.1]. Parts 3 and 4 follow from part 1, as the map $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty) \to \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}$ is not only ind-étale, but a $\mathbf{Z}_p^\times$-torsor; see [Beh14, Lm.5.1]. △

[54] Recall that for a spectrum X, one defines its *p-adic K-theory* as the homotopy groups of the localisation $\mathrm{K}^\wedge_* X = \pi_* L_{K(1)}(X \otimes \mathrm{KU}_p)$, or equivalently $\pi_*((X \otimes \mathrm{KU}_p)^\wedge_p)$.

For another object $\operatorname{Spec} R' \to \mathcal{M}_{\mathrm{Ell}}$ inside $\mathcal{U}$, consider the map induced by the p-adic K-theory functor

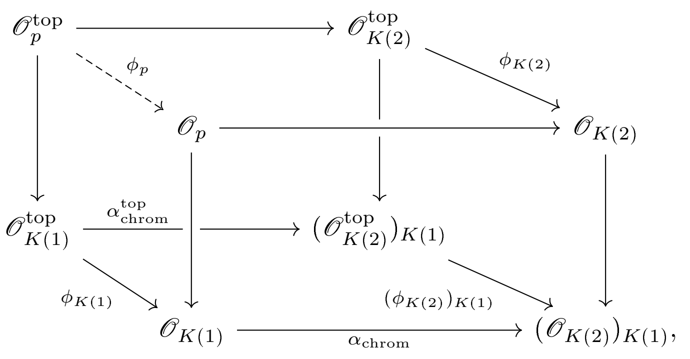
$$\mathrm{Map}_{\mathrm{CAlg}_{K(1)}}(\mathcal{E}^{\mathrm{top}}_{K(1)}, \mathcal{E}'_{K(1)}) \to \mathrm{Hom}_{\theta\mathrm{Alg}_{\mathrm{K}^\wedge_*}}(\mathrm{K}^\wedge_* \mathcal{E}^{\mathrm{top}}_{K(1)}, \mathrm{K}^\wedge_* \mathcal{E}'_{K(1)}), \qquad \text{(B.2.2)}$$

where $\mathcal{E}'_{K(1)} = \mathcal{O}_{K(1)}(R')$. Combining Clm. B.2.1 and the fact that R^{ord} is smooth over $\mathbf{Z}_p$ with [Beh14, Lm.7.5] implies that all the André–Quillen cohomology groups in [Beh14, Th.7.1] vanish. By *ibid*, it then follows that the above map is an equivalence of (discrete) spaces. Despite the fact that each $\mathrm{h}\phi_{K(1)}(R)$ is currently just a morphism of homotopy commutative ring spectra, Lm. B.3.1 states that its zeroth p-adic K-theory is a morphism of θ-algebras. As $\mathbf{Z}$-graded p-adic K-theory obtains a θ-algebra structure from that in degree zero, the p-adic K-theory of $\mathrm{h}\phi_{K(1)}$ defines an element inside the codomain of the equivalence (B.2.2) when $R' = R$. By Pr. B.1.5, we can therefore lift $\mathrm{h}\phi_{K(1)} \colon \mathrm{h}\mathcal{O}^{\mathrm{top}}_{K(1)} \to \mathrm{h}\mathcal{O}_{K(1)}$ to a morphism $\phi_{K(1)} \colon \mathcal{O}^{\mathrm{top}}_{K(1)} \to \mathcal{O}_{K(1)}$ of presheaves of $\mathbf{E}_\infty$-rings on $\mathcal{U}$.

($K(0)$-local case) The Morava K-theory spectrum $K(0)$ is equivalent to $\mathbf{Q}$, the Eilenberg–MacLane spectrum of the rational numbers. We can lift $\mathrm{h}\phi_{\mathbf{Q}}$ globally, meaning we will not have to work section-by-section. Consider post-composing the functors $\mathcal{O}^{\mathrm{top}}$ and $\mathcal{O}$ with the rationalisation functor from CAlg to $\mathrm{CAlg}_{\mathbf{Q}}$, and denote the resulting presheaves with a subscript $\mathbf{Q}$. By construction (also see [HL16, Pr.4.47]) the functor $\mathcal{O}^{\mathrm{top}}_{\mathbf{Q}}$ is formal, and by [Mei21, Pr.4.8] the sheaf $\mathcal{O}_{\mathbf{Q}}$ is also formal. This yields the following chain of equivalences lifting $\mathrm{h}\phi_{\mathbf{Q}}$:

$$\phi_{\mathbf{Q}} \colon \mathcal{O}^{\mathrm{top}}_{\mathbf{Q}} \xrightarrow{\simeq} \pi_* \mathcal{O}^{\mathrm{top}}_{\mathbf{Q}} \xrightarrow{\mathrm{h}\phi^{\mathbf{Q}}_*, \simeq} \pi_* \mathcal{O}_{\mathbf{Q}} \xleftarrow{\simeq} \mathcal{O}_{\mathbf{Q}}$$

(Transchromatic compatibility) Fix a prime p. We now have morphisms fitting into the following not a priori commutative solid diagram of presheaves of p-complete $\mathbf{E}_\infty$-rings on $\mathcal{U}$:



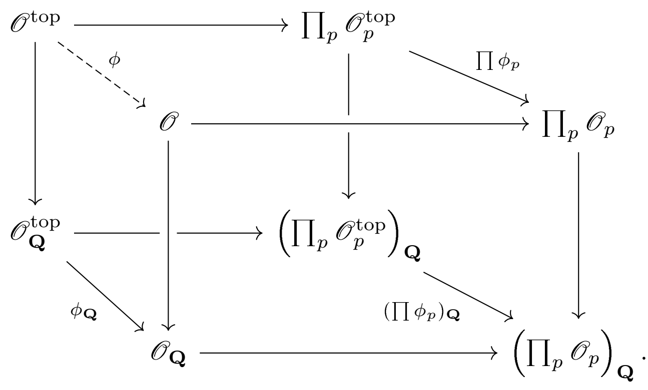
(B.2.3)

The right face commutes from the naturality of the unit of the $K(1)$-localisation functor. We also claim that the bottom face commutes. In other words, we claim that for each $\operatorname{Spec} R \to \mathcal{M}$ inside $\mathcal{U}$, there is a natural path $\gamma(R)$ between $\alpha_{\mathrm{chrom}} \circ \phi_{K(1)}$ and $(\phi_{K(2)})_{K(1)} \circ \alpha^{\mathrm{top}}_{\mathrm{chrom}}$ as maps of $\mathbf{E}_\infty$-tmf-algebras. Note that the $\mathbf{E}_\infty$-tmf-algebra structure on $(\mathscr{O}_{K(2)})_{K(1)}$ can come from either one of these maps (and a posteriori these two choices will agree up to homotopy). By [Beh14, p.44], the p-adic K-theory functor induces an equivalence of discrete spaces

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathrm{tmf}}}(\mathcal{E}^{\mathrm{top}}, \mathcal{E}_{K(2)}) \xrightarrow{\simeq} \mathrm{Hom}_{\theta\mathrm{Alg}_{(V^{\wedge}_{\infty})_*}}(\mathrm{KU}^{\wedge}_*\mathcal{E}^{\mathrm{top}}, \mathrm{KU}^{\wedge}_*\mathcal{E}_{K(2)}) \qquad \text{(B.2.4)}$$

where everything above is implicitly $K(1)$-localised (for typographical reasons). As $\alpha_{\mathrm{chrom}} \circ \phi_{K(1)}$ and $(\phi_{K(1)})_{K(1)} \circ \alpha^{\mathrm{top}}_{\mathrm{chrom}}$ are isomorphic as functors into CAlg(hSp), by assumption, their effects on p-adic K-theory are equal. By Lm.B.3.1, we see these morphisms of homotopy commutative ring spectra induce morphisms of θ-algebras on p-adic K-theory, and these morphisms agree by the previous sentence. This yields two equal elements in the codomain of (B.2.4), and hence a homotopy of $K(1)$-local $\mathbf{E}_\infty$-tmf-algebras between these maps. By virtue of Pr.B.1.5, we obtain a homotopy between $\alpha_{\mathrm{chrom}} \circ \phi_{K(1)}$ and $(\phi_{K(2)})_{K(1)} \circ \alpha^{\mathrm{top}}_{\mathrm{chrom}}$ as morphisms of presheaves of $K(1)$-local $\mathbf{E}_\infty$-$\mathrm{tmf}_{K(1)}$-algebras from $\mathscr{O}^{\mathrm{top}}_{K(1)}$ to $(\mathscr{O}_{K(2)})_{K(1)}$. Using the fact that the front and back faces of (B.2.3) are Cartesian, we obtain a natural morphism of presheaves of p-complete $\mathbf{E}_\infty$-rings $\phi_p\colon \mathscr{O}^{\mathrm{top}}_p \to \mathscr{O}_p$ on $\mathcal{U}$, lifting $\mathrm{h}\phi_p$, as indicated by the dashed morphism in (B.2.3).

(Arithmetic compatibility) Currently, we have morphisms $\phi_{\mathbf{Q}}$ and ϕ_p fitting into the not a priori commutative solid diagram of presheaves of $\mathbf{E}_\infty$-rings on $\mathcal{U}$:



(B.2.5)

Similar to the transchromatic compatibilities, the right face naturally commutes, so we are left to argue why the bottom face commutes. To study the bottom face, let us first work on the open substacks $\mathcal{M}_{\mathrm{Ell}}[c_4^{-1}]$ and $\mathcal{M}_{\mathrm{Ell}}[\Delta^{-1}]$ of $\mathcal{M}_{\mathrm{Ell}}$, which themselves form a cover of $\mathcal{M}_{\mathrm{Ell}}$; see [Beh14, §9]. We then follow an analogous argument to the transchromatic situation above; see [Beh14, p.51] which shows the discreteness of the desired mappings spaces. Indeed, as the two homotopies witnessing the commutativity of the bottom face of (B.2.5) restricted to the substacks $\mathcal{M}_{\mathrm{Ell}}[c_4^{-1}]$ and $\mathcal{M}_{\mathrm{Ell}}[\Delta^{-1}]$ agree on their intersection

$\mathcal{M}_{\mathrm{Ell}}[c_4^{-1}, \Delta^{-1}]$ (as the mapping spaces in question are discrete) these homotopies then glue to a homotopy on $\mathcal{M}_{\mathrm{Ell}}$. This yields a homotopy witnessing the commutativity of the bottom face of (B.2.5). As the front and back faces of (B.2.5) are Cartesian, we obtain our final natural equivalence of presheaves of $\mathbf{E}_\infty$-rings $\phi\colon \mathcal{E}^{\mathrm{top}} \to \mathcal{E}$ on $\mathcal{U}$, lifting $\mathrm{h}\phi$.

Therefore, $\mathcal{Z}$ is connected. The same argument can be made for $\mathcal{Z}^{\mathrm{sm}}$. $\square$

B.3 Compatibility of θ structures

The above proof of Th.B.1.1 is contingent on Lm.B.3.1, whose proof we find rather delicate. Recall from [Beh14, §6] that the p-adic K-theory of an $\mathbf{E}_\infty$-ring has the structure of a θ-algebra, a $\mathbf{Z}_p$-algebra with stable Adams operations ψ^ℓ and a θ-operator (see [GH04, §2.2]), and this structure is functorial in morphisms of $\mathbf{E}_\infty$-rings.

Lemma B.3.1. *Fix a prime p. Let $\mathcal{O}$ be an object of $\mathcal{Z}$ and $\mathrm{h}\phi\colon \mathrm{h}\mathcal{O}^{\mathrm{top}} \xrightarrow{\simeq} \mathrm{h}\mathcal{O}$ be the given equivalence of diagrams of homotopy commutative ring spectra. Then for any étale $\mathrm{Spec}\, R \to \mathcal{M}_{\mathrm{Ell}}$, the map induced by*

$$\mathrm{h}\phi\colon \mathcal{F}^{\mathrm{top}} = \mathcal{O}^{\mathrm{top}}_{K(1)}(R) \to \mathcal{O}_{K(1)}(R) = \mathcal{F}$$

on the zeroth p-adic K-theory ring is a morphism of θ-algebras.

In general, it is not true that a morphism of homotopy commutative ring spectra between $\mathbf{E}_\infty$-rings should induce a morphism of θ-algebras upon taking their p-adic K-theory. However, in the situation above the sections of the $K(2)$-localisation of the sheaf of $\mathbf{E}_\infty$-rings $\mathcal{O}$ have a prescribed $\mathbf{E}_\infty$-structure given by Lubin–Tate spectra (also called Morava E-theory); see the $K(2)$-local case in the proof of Th.B.1.1 above. The comparison map in the chromatic fracture square between the $K(1)$-localisation of $\mathcal{O}$ and the $K(1)$-localisation of its $K(2)$-localisation is a map of $\mathbf{E}_\infty$-rings, and if we can show it induces an injection on p-adic K-theory we would obtain Lm.B.3.1. This is first done for an explicit étale morphism into $\mathcal{M}_{\mathrm{Ell}}$, which has the properties that it covers $\mathcal{M}^{\mathrm{sm}}_{\mathrm{Ell}}$ and each of its connected components is an integral domain. A little descent and deformation theory then generalises this result for a general étale morphism.

Proof. To show $\lambda\colon \mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}} \to \mathrm{K}_0^\wedge \mathcal{F}$, the map induced by $\mathrm{h}\phi$ on p-adic K-theory, is a morphism of θ-algebras, one must check it commutes with the stable p-adic Adams operations ψ^ℓ for every $\ell \in \mathbf{Z}_p^\times$ as well as the action of the operator θ. The stable p-adic Adams operations ψ^ℓ are constructed on the spectrum KU_p, so we automatically have compatibility with them for any map of spectra. It will be shown shortly that both rings above are étale over the ring $V_\infty^\wedge$, hence they are $V_\infty^\wedge$-torsion free. In particular, this implies that both $\mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}}$ and $\mathrm{K}_0^\wedge \mathcal{F}$ are $\mathbf{Z}_p$-torsion free, in which case the operator θ is equivalent datum to the unstable

p-adic Adams operator ψ^p; see [GH04, Rmk.2.2.5]. Therefore, suffices to show that the following diagram of $\mathbf{Z}_p$-algebras commutes:

$$\begin{array}{ccc} \mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}} & \xrightarrow{\lambda} & \mathrm{K}_0^\wedge \mathcal{F} \\ \downarrow \psi^p_{\mathrm{top}} & & \downarrow \psi^p_{\mathrm{top}} \\ \mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}} & \xrightarrow{\lambda} & \mathrm{K}_0^\wedge \mathcal{F} \end{array} \tag{B.3.2}$$

We will write ψ^p_{top} for these unstable Adams operations on p-adic K-theory to differentiate them from what will come. Let us write R^{ord} for the base-change of $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$ over $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}} \to \widehat{\mathcal{M}}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{Ell}}$, where $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}$ is the moduli stack of generalised elliptic curves over p-complete rings whose reduction modulo p is ordinary. By [Beh14, Pr.7.16], we see $\mathcal{F}^{\mathrm{top}}_{K(1)}$ is an elliptic cohomology theory for $\operatorname{Spf} R^{\mathrm{ord}} \to \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}$, and we can also consider $\mathcal{F}_{K(1)}$ as an elliptic cohomology theory for $\operatorname{Spf} R^{\mathrm{ord}} \to \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}$ using $\mathrm{h}\phi_{K(1)}$. Define W using the Cartesian diagram of formal stacks

$$\begin{array}{ccc} \operatorname{Spf} W & \longrightarrow & \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty) \\ \downarrow & & \downarrow \\ \operatorname{Spf} R^{\mathrm{ord}} & \longrightarrow & \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}} \end{array}$$

where $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty)$ is the formal stack of generalised elliptic curves E over $\operatorname{Spf} R$ with ordinary reduction modulo p with a given isomorphism $\eta\colon \widehat{\mathbf{G}}_m \to \widehat{E}$; see [Beh14, §5]. The stack $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty)$ is represented by the formal affine scheme $\operatorname{Spf} V_\infty^\wedge$ which is ind-étale over $\mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}$; see [Beh14, p.14-5]. This W also has the structure of a θ-algebra (see [Beh14, §6]), and we denote the p-adic Adams operation on W by ψ^p_{alg}. By [Beh14, Pr.6.1], or rather its proof, we obtain isomorphisms of $\mathbf{Z}_p$-algebras $v^{\mathrm{top}}\colon \mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}} \simeq W$ and $v\colon \mathrm{K}_0^\wedge \mathcal{F} \simeq W$, which are natural in complex orientation preserving morphisms in $\mathrm{CAlg}(\mathrm{hSp})$. These isomorphisms are **not** a priori isomorphisms of θ-algebras; see [Beh14, §6.2]. As $\mathcal{F}$ obtains the structure of an elliptic cohomology theory for R^{ord} from the equivalence $\mathrm{h}\phi_{K(1)}$, we see that the following diagram of isomorphisms of $\mathbf{Z}_p$-algebras commutes:

$$\begin{array}{ccccc} \mathrm{K}_0^\wedge \mathcal{F}^{\mathrm{top}} & & \xrightarrow{\lambda} & & \mathrm{K}_0^\wedge \mathcal{F} \\ & {\scriptstyle v^{\mathrm{top}}} \searrow & & \swarrow {\scriptstyle v} & \\ & & W & & \end{array}$$

By the construction of $\mathscr{O}^{\mathrm{top}}$ (see [Beh14, Rmk.6.3]), we see v^{top} is an isomorphism of θ-algebras. To show λ is a morphism of θ-algebras, it suffices to show v is a morphism of θ-algebras, or in other words: (B.3.2) commutes if and only if the following diagram of $\mathbf{Z}_p$-algebras commutes:

$$\begin{array}{ccc} \mathrm{K}_0^\wedge \mathcal{F} & \xrightarrow{v} & W \\ \downarrow{\scriptstyle \psi^p_{\mathrm{top}}} & & \downarrow{\scriptstyle \psi^p_{\mathrm{alg}}} \\ \mathrm{K}_0^\wedge \mathcal{F} & \xrightarrow{v} & W \end{array} \tag{B.3.3}$$

Let us now prove this is the case for a specific étale map $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$.

(Choosing a particular étale morphism) Recall the moduli stack $\mathcal{M}_1^{\mathrm{sm}}(N)$ of smooth elliptic curves with $\Gamma_1(N)$-level structure, from [Beh20, (1.3.12)] for example or the discussion before Df. 7.1.3. Importantly, recall the map $\mathcal{M}_1^{\mathrm{sm}}(N) \to \mathcal{M}^{\mathrm{sm}}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{N}]}$ is an étale cover and that for $N \geqslant 4$ the moduli stack $\mathcal{M}_1^{\mathrm{sm}}(N)$ is in fact affine. This implies that the morphism of stacks

$$\operatorname{Spec} A = \mathcal{M}_1^{\mathrm{sm}}(4) \sqcup \mathcal{M}_1^{\mathrm{sm}}(5) \to \mathcal{M}^{\mathrm{sm}}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{2}]} \sqcup \mathcal{M}^{\mathrm{sm}}_{\mathrm{Ell},\mathbf{Z}[\frac{1}{5}]} \to \mathcal{M}^{\mathrm{sm}}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{Ell}}$$

is étale, and the restriction to $\mathcal{M}^{\mathrm{sm}}_{\mathrm{Ell}}$ is an étale cover. By base-change over $\operatorname{Spf} \mathbf{Z}_p$ we obtain an étale map $E\colon \operatorname{Spf} \widehat{A} \to \widehat{\mathcal{M}}_{\mathrm{Ell}}$. Following [Beh14, p.42-3], write A_* for the graded ring defined by $A_{2k} = \omega_E^{\otimes k}(\operatorname{Spf} \widehat{A})$ were E is the elliptic curve over $\widehat{A}$ defined by the map of formal stacks above. Note that the Hasse invariant v_1 for E lives in $A_{2(p-1)}$. Let us also make the following definitions:

$$A_*^{\mathrm{ord}} = (A_*)[v_1^{-1}]_p^\wedge \qquad A_*^{\mathrm{ss}} = (A_*)^\wedge_{(v_1)} \qquad (A_*^{\mathrm{ss}})^{\mathrm{ord}} = (A_*^{\mathrm{ss}})[v_1^{-1}]_p^\wedge \tag{B.3.4}$$

If we omit the subscript $*$ we are implicitly considering the ring in degree zero. By [Beh14, (8.6)], there is a canonical map $\alpha_*\colon A_*^{\mathrm{ord}} \to (A_*^{\mathrm{ss}})^{\mathrm{ord}}$ as v_1 is invertible in $(A_*^{\mathrm{ss}})^{\mathrm{ord}}$, and we now define W^{ss} using the diagram of stacks

$$\begin{array}{ccccc} \operatorname{Spf} W^{\mathrm{ss}} & \xrightarrow{\widetilde{\alpha}} & \operatorname{Spf} W & \longrightarrow & \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spf}(A^{\mathrm{ss}})^{\mathrm{ord}} & \xrightarrow{\alpha} & \operatorname{Spf} A^{\mathrm{ord}} & \longrightarrow & \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}, \end{array} \tag{B.3.5}$$

where all squares are Cartesian. The ring W^{ss} obtains a θ-algebra structure from the above diagram, and in such a way that $\widetilde{\alpha}\colon W \to W^{\mathrm{ss}}$ is a morphism of θ-algebras; see [Beh14, p.40]. We claim that $\widetilde{\alpha}$ comes from a map of $\mathbf{E}_\infty$-rings.

Claim B.3.6. The zeroth p-adic K-theory of the canonical map of $\mathbf{E}_\infty$-rings

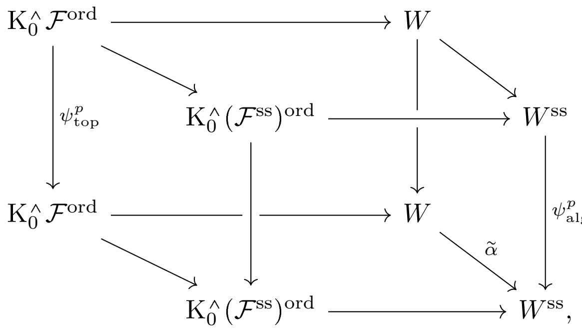
$$\alpha_{\mathrm{chrom}}\colon \mathcal{F}^{\mathrm{ord}} = \mathscr{O}_{K(1)}(A) \to (\mathscr{O}_{K(2)}(A))_{K(1)} = (\mathcal{F}^{\mathrm{ss}})^{\mathrm{ord}}$$

is isomorphic to $\widetilde{\alpha}$.

Proof of Clm.B.3.6. We have already seen that $\mathcal{F}^{\mathrm{ord}} = \mathcal{F}_{K(1)}$ is an elliptic cohomology theory for the map $\operatorname{Spf} A^{\mathrm{ord}} \to \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}$, and similarly by [Beh14, Lm.8.8], we see that $(\mathcal{F}^{\mathrm{ss}})^{\mathrm{ord}}$ is an elliptic cohomology theory for the map

$\mathrm{Spf}(A^{\mathrm{ss}})^{\mathrm{ord}} \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}$. The same is true for $\mathcal{A} = \mathscr{O}_{K(1)}^{\mathrm{top}}(A)$ and $(\mathscr{O}_{K(2)}^{\mathrm{top}}(A))_{K(1)} = \mathcal{A}'$, and in this case we know that taking π_0 of $\alpha_{\mathrm{chrom}}^{\mathrm{top}} \colon \mathcal{A} \to \mathcal{A}'$ is isomorphic to $\alpha\colon A^{\mathrm{ord}} \to (A^{\mathrm{ss}})^{\mathrm{ord}}$ by construction; see [Beh14, p.43-4]. The naturality of $\mathrm{h}\phi\colon \mathscr{O}^{\mathrm{top}} \to \mathscr{O}$ and the chromatic fracture square imply that π_0 of the natural map of $\mathbf{E}_\infty$-rings $\alpha_{\mathrm{chrom}}\colon \mathcal{F}^{\mathrm{ord}} \to (\mathcal{F}^{\mathrm{ss}})^{\mathrm{ord}}$ also realises α, and hence taking zeroth p-adic K-theory realises $\widetilde{\alpha}$. This proves Clm.B.3.6. △

Recall that $\mathcal{F}^{\mathrm{ord}} = \mathscr{O}_{K(1)}(A)$ for our choice of A above. Consider the diagram of $\mathbf{Z}_p$-algebras



(B.3.7)

where the maps are the obvious ones used above, and all the vertical morphisms are the unstable Adams operations; ψ^p_{top} on the left, and ψ^p_{alg} on the right. Note that the top and bottom faces commute by Clm.B.3.6, the right face commutes as $\widetilde{\alpha}\colon W \to W^{\mathrm{ss}}$ is a morphism of θ-algebras, and the left face commutes as α_{chrom} is a morphism of $\mathbf{E}_\infty$-rings. Most importantly, the front face also commutes. Indeed, from the arguments in the $K(2)$-local case of the proof of Th.B.1.1 we see $\mathcal{F}^{\mathrm{ss}}$ is naturally equivalent to a product of $K(2)$-local Lubin–Tate spectra recognising the given elliptic curve over Spf A^{ss}, and we can then apply [Beh14, Th.6.10]; the hypotheses and proof of this theorem are dispersed between pages 21 and 24 of *idem*. The back face of (B.3.7) is precisely (B.3.3) for $R = A$.

Claim B.3.8. The morphism $\widetilde{\alpha}\colon W \to W^{\mathrm{ss}}$ is injective.

Using this claim, to show that the back face of (B.3.7) commutes, it suffices to do so after post-composing with $\widetilde{\alpha}$. This follows from the above considerations by a diagram chase. Hence the back face of (B.3.7) commutes, which yields the commutativity of (B.3.3) for this particular choice of étale map Spec $A \to \mathcal{M}_{\mathrm{Ell}}$.

Proof of Clm.B.3.8. As $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}(p^\infty) \to \mathcal{M}_{\mathrm{Ell}}^{\mathrm{ord}}$ is ind-étale (see [Beh14, Lm.5.1]), it is flat. By base-change, we see that $A^{\mathrm{ord}} \to W$ is also flat, hence $\widetilde{\alpha}$ is injective if we can show α is injective. To do this, we will show $\alpha_*\colon A_*^{\mathrm{ord}} \to (A_*^{\mathrm{ss}})^{\mathrm{ord}}$ is injective. Using the notation above, we find ourselves with the following commutative diagram of graded rings, where all maps are the indicated localisations or completions:

$$\begin{array}{ccccc} A_* & \longrightarrow & A_*[v_1^{-1}] & \longrightarrow & A_*[v_1^{-1}]_p^\wedge = A_*^{\mathrm{ord}} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha_*} \\ A_*^{\mathrm{ss}} & \longrightarrow & A_*^{\mathrm{ss}}[v_1^{-1}] & \longrightarrow & A_*^{\mathrm{ss}}[v_1^{-1}]_p^\wedge = (A_*^{\mathrm{ss}})^{\mathrm{ord}}. \end{array}$$

Let us now make the following remarks from this diagram:

1. From our choice of A, we have $A = A_1 \times A_2$, where A_1 and A_2 both integral domains; see [BO16, Th.1.1.1] for the $\mathcal{M}_1^{\mathrm{sm}}(5)$-case, and the $\mathcal{M}_1^{\mathrm{sm}}(4)$-case is similar.[55] It follows that γ can be written in the following commutative diagram of graded rings

$$\begin{array}{ccc} A_* & \xrightarrow{\gamma} & A_*^{\mathrm{ss}} = (A_*)^\wedge_{(v_1)} \\ \downarrow & & \downarrow \\ A_{*,1} \times A_{*,2} & \xrightarrow{\gamma_1 \times \gamma_2} & A_{*,1}^{\mathrm{ss}} \times A_{*,2}^{\mathrm{ss}}. \end{array}$$

   The ring A is Noetherian as it is finitely presented over $\operatorname{Spec} \mathbf{Z}$, so both A_1 and A_2 are Noetherian integral domains. In particular, the completion maps γ_i are flat for $i = 1, 2$. If we know these maps γ_i are nonzero, then it immediately follows that they are injective. To see that they are nonzero, it suffices to show that v_1 is not a unit inside both $A_{*,1}$ and $A_{*,2}$. This is where our choice of A comes in. If our fixed prime $p \neq 2, 5$, then for both $i = 1, 2$ the image of the map

$$\operatorname{Spf} \widehat{A}_i \to \widehat{\mathcal{M}}_{\mathrm{Ell}}^{\mathrm{sm}} \to \widehat{\mathcal{M}}_{\mathrm{Ell}}$$

   contains a supersingular elliptic curve, as all supersingular elliptic curves are contained in the smooth locus of $\widehat{\mathcal{M}}_{\mathrm{Ell}}$. This implies that v_1 cannot be a unit, else $\operatorname{Spf} \widehat{A}_i \to \widehat{\mathcal{M}}_{\mathrm{Ell}}$ would define only ordinary elliptic curves of height one. Similarly, if $p = 2$, then the p-completion of A is $\widehat{A}_2$, and we again see v_1 is not a unit so $\gamma_2 = \gamma$ is injective. The same holds for $\widehat{A}_1$ when $p = 5$. This implies that $\gamma_1 \times \gamma_2$ is always injective, hence γ is injective.

2. As β is the v_1-localisation of γ, and localisation is exact, we see that β is also injective.

3. Standard arguments show that the p-completion of β, also known as α_*, is also injective. Indeed, limits are left exact, so it suffices to show each

[55] Indeed, following the proof of [BO16, Th.1.1.1], which in turn uses [Hus04, §4.4], each elliptic curve E in Weierstraß form over a $\mathbf{Z}[\frac{1}{2}]$-algebra R with $(0,0)$ a point of order 4 can be moved into (non homogeneous) *Tate normal form* $y^2 + (1-c)xy - by = x^3 - bx^2$. As $(0,0)$ has order 4 we have $[2](0,0) = [-2](0,0)$, which by [Hus04, Ex.4.4] yields $c = 0$. As in the proof of [BO16, Th.1.1], we see $\mathcal{M}_1(4)$ is equivalent to spectrum of $\mathbf{Z}[\frac{1}{2}, b, \Delta^{-1}]$ where $\Delta = b^4(1 + 16b)$.

α_*^k in the following commutative diagram of rings is injective, for every $k \geqslant 1$:

$$\begin{array}{ccc} A_*[v_1^{-1}] & \longrightarrow & A_*[v_1^{-1}]/p^k \\ \big\downarrow \beta & & \big\downarrow \alpha_*^k \\ A_*^{\mathrm{ss}}[v_1^{-1}] & \longrightarrow & A_*^{\mathrm{ss}}[v_1^{-1}]/p^k. \end{array}$$

Given an element $\overline{x}$ such that $\alpha_*^k(\overline{x}) = 0$, then we first note that any lift x over $\overline{x}$ is sent to a $\beta(x)$ such that $p^k\beta(x) = 0$. However, $A_*^{\mathrm{ss}}[v_1^{-1}]$ is flat over $\mathbf{Z}$, as we have the following composite of flat maps:

$$\mathbf{Z} \to \mathbf{Z}_p \to \widehat{A} \to A_* \xrightarrow{\gamma} A_*^{\mathrm{ss}} \to A_*^{\mathrm{ss}}[v_1^{-1}];$$

the second map is flat as $\operatorname{Spec} A \to \mathcal{M}_{\mathrm{Ell}}$ is étale and $\mathcal{M}_{\mathrm{Ell}}$ is smooth over $\mathbf{Z}$, and the third map is flat as each $\omega_E^{\otimes k}(\operatorname{Spf} A)$ is a line bundle and hence projective of rank 1. This implies that $A_*^{\mathrm{ss}}[v_1^{-1}]$ is torsion-free, hence $\beta(x) = 0$. As β is injective, this implies $x = 0$ and $\overline{x} = 0$, hence α_*^k is injective.

It follows that $\widetilde{\alpha}$ is injective. △

Reduction to a general étale morphism Let $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$ be an arbitrary étale morphism now, and consider the Cartesian diagram of stacks

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} A \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} R & \longrightarrow & \mathcal{M}_{\mathrm{Ell}}, \end{array}$$

where $\operatorname{Spec} A = \mathcal{M}_1^{\mathrm{sm}}(4) \sqcup \mathcal{M}_1^{\mathrm{sm}}(5)$ is that of the previous paragraph; the stack $\operatorname{Spec} B$ is affine as $\mathcal{M}_{\mathrm{Ell}}$ is separated. All of the morphisms above are étale by base-change, so we can consider the morphism of $\mathbf{E}_\infty$-rings $\mathscr{O}(A) \to \mathscr{O}(B)$.

Claim B.3.9. The morphism of $\mathbf{E}_\infty$-rings $\mathscr{O}(A) \to \mathscr{O}(B)$ is étale.

Proof. Recall from [Lur17, §7.5] that a morphism $\mathcal{A} \to \mathcal{B}$ of $\mathbf{E}_\infty$-rings is *étale* if the morphism $\pi_0\mathcal{A} \to \pi_0\mathcal{B}$ of discrete commutative rings is étale and the natural map of $\pi_0\mathcal{B}$-modules

$$\pi_0\mathcal{B} \underset{\pi_0\mathcal{A}}{\otimes} \pi_*\mathcal{A} \to \pi_*\mathcal{B}$$

is an isomorphism. The fact that $\pi_0\mathscr{O}(A) \to \pi_0\mathscr{O}(B)$ is étale follows from the facts that $A \to B$ is étale and $\mathscr{O}$ defines a natural elliptic cohomology theory. The condition on the higher homotopy groups also follows as $\mathscr{O}$ defines a natural elliptic cohomology theory; see Rmk.B.1.4. △

By [Lur17, Th.7.5.0.6], the π_0-functor induces an equivalence of ∞-categories

$$\mathrm{CAlg}^{\text{ét}}_{\mathscr{O}(A)} \xrightarrow{\pi_0} \mathrm{CAlg}^{\text{ét}}_{A},$$

where the superscript indicates subcategories of étale algebras. By Clm.B.3.9, for any étale $\mathbf{E}_\infty$-$\mathscr{O}(A)$-algebra $\mathcal{B}$ such that $\pi_0\mathcal{B}$ is isomorphic to B as an A-algebra, there is a equivalence of $\mathbf{E}_\infty$-$\mathscr{O}(A)$-algebras $\mathscr{O}(B) \simeq \mathcal{B}$, which is unique up to contractible choice. As we have proven Lm.B.3.1 for $\operatorname{Spec} A$, it follows from the proof of Th.B.1.1 above that the equivalence of homotopy commutative ring spectra $\mathrm{h}\phi(A)\colon \mathscr{O}^{\mathrm{top}}(A) \simeq \mathscr{O}(A)$ can be lifted to a morphism of $\mathbf{E}_\infty$-rings. The composite $\mathscr{O}(A) \simeq \mathscr{O}^{\mathrm{top}}(A) \to \mathscr{O}^{\mathrm{top}}(B)$ is also an étale $\mathbf{E}_\infty$-$\mathscr{O}(A)$-algebra recognising B, hence we obtain a natural equivalence of $\mathbf{E}_\infty$-$\mathscr{O}(A)$-algebras $\mathscr{O}^{\mathrm{top}}(B) \simeq \mathscr{O}(B)$. As $\mathscr{O}^{\mathrm{top}}(B)$ is θ-compatible (see [Beh14, Rmk.6.3]), we see $\mathscr{O}(B)$ is also θ-compatible, meaning that (B.3.3) commutes for $R = B$. Finally, let us turn our attention to $\mathscr{O}(R) \to \mathscr{O}(B)$.

Claim B.3.10. The morphism induced by $\mathscr{O}(R) \to \mathscr{O}(B)$ on zeroth p-adic K-theory is injective.

Assuming the above claim, it immediately follows that (B.3.3) for our arbitrary R. Indeed, Clm.B.3.10 provides us with an injection of θ-algebras induced by $\mathscr{O}(R) \to \mathscr{O}(B)$, which allows us to check the commutativity of (B.3.3) in the same diagram for $R = B$, which we know commutes by the above paragraph.

Proof of Clm.B.3.10. The morphism $\operatorname{Spec} B \to \operatorname{Spec} R$ can be factored into the following diagram of formal stacks:

$$\begin{array}{ccccccc}
\operatorname{Spf} W_B & \longrightarrow & \operatorname{Spf} W_R^{\mathrm{sm}} & \longrightarrow & \operatorname{Spf} W_R & \longrightarrow & \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}}(p^\infty) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spf} \widehat{B}^{\mathrm{ord}} & \longrightarrow & \operatorname{Spf} \widehat{R}^{\mathrm{ord,sm}} & \longrightarrow & \widehat{R}^{\mathrm{ord}} & \longrightarrow & \mathcal{M}^{\mathrm{ord}}_{\mathrm{Ell}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spf} \widehat{B} & \longrightarrow & \operatorname{Spf} \widehat{R}^{\mathrm{sm}} & \longrightarrow & \operatorname{Spf} \widehat{R} & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{Ell}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} R^{\mathrm{sm}} & \longrightarrow & \operatorname{Spec} R & \longrightarrow & \mathcal{M}_{\mathrm{Ell}} \\
\downarrow & & \downarrow & & \downarrow & & \\
\operatorname{Spec} A & \longrightarrow & \mathcal{M}^{\mathrm{sm}}_{\mathrm{Ell}} & \longrightarrow & \mathcal{M}_{\mathrm{Ell}} & &
\end{array} \tag{B.3.11}$$

Every square above is Cartesian, and the $\widehat{(-)}$ indicates base-change with $\operatorname{Spf} \mathbf{Z}_p$. By [Beh14, Lm.6.1], the morphism $W_R \to W_B$ above is isomorphic to the morphism induced by $\mathscr{O}(R) \to \mathscr{O}(B)$ on p-adic K-theory, hence it suffices to see the composite map

$$W_R \to W_R^{\mathrm{sm}} \to W_B, \tag{B.3.12}$$

featured in the top-left corner of (B.3.11), is injective. As $\operatorname{Spec} A \to \mathcal{M}^{\mathrm{sm}}_{\mathrm{Ell}}$ is an étale cover, then by base-change we see $W_R^{\mathrm{sm}} \to W_B$ is also faithfully flat, and hence injective. Observe that $W_R \to W_R^{\mathrm{sm}}$ is an open immersion of formal affine schemes by base-change as $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}} \to \mathcal{M}_{\mathrm{Ell}}$ is an open immersion of stacks. Moreover, we claim the open immersion $R \to R^{\mathrm{sm}}$ has scheme theoretically dense image as Δ is a nonzero divisor in R; see [Sta, Tag 01RE]. Indeed, to see Δ is not a zero divisor, it suffices to show that the image of $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{Ell}}$ has nontrivial intersection with the image of $\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}$. This is clear on the level of underlying topological spaces, as the inclusion $|\mathcal{M}_{\mathrm{Ell}}^{\mathrm{sm}}| \to |\mathcal{M}_{\mathrm{Ell}}|$ is equivalent to open immersion of coarse moduli spaces $|\mathbf{A}^1_{\mathbf{Z}}| \to |\mathbf{P}^1_{\mathbf{Z}}|$ which adds the point at ∞, and the map $|\operatorname{Spec} R| \to |\mathcal{M}_{\mathrm{Ell}}|$ is open as étale morphisms are in particular flat and locally of finite presentation; see [Sta, Tag 06R7]. As all the right vertical maps in (B.3.11) are flat, and $R \to R^{\mathrm{sm}}$ is quasi-compact (as a map of affine schemes), then [Sta, Tag 0CMK] tells us that $W_R \to W_R^{\mathrm{sm}}$ also have scheme theoretically dense image. Another application of [Sta, Tag 01RE] shows this open immersion $W_R \to W_R^{\mathrm{sm}}$ must be injective. Therefore, the composite (B.3.12) is injective. $\triangle$

This finishes our proof of Lm.B.3.1. $\square$

Appendix C

Summaries

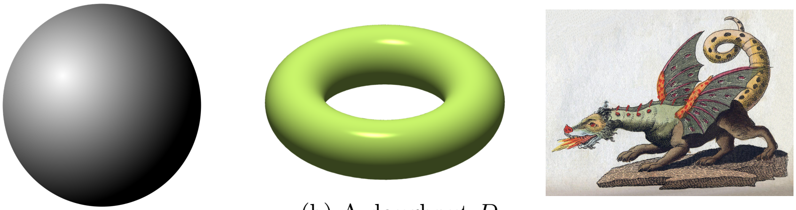
On a distant beach, lonely and wild,
At a later time, see a man and a child,
And the man takes the child up into his arms,
Takes her over the breakers to where the water is calm.

Paul Kelly, Deeper Water

C.1 Summary for a general audience

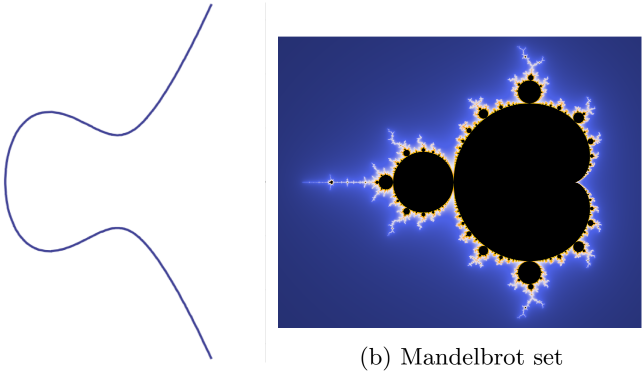
The goal of this thesis is to further demonstrate that the philosophy and language of *homotopy theory*, when applied to *number theory*, produces interesting results in both disciplines. Homotopy theory was born to aid in the study of geometry, but since the 1960s it has been viewed as a new philosophy to study mathematics. Number theory, on the other hand, has its roots in ancient Babylon and Egypt and seeks to understand numbers and the relationships between them. In this summary, I discuss some of the motivations behind homotopy theory and highlight the point-of-view that this theory is built to understand the *whys* and the *hows* (rather than just the *whats*) before we see this philosophy in the world of number theory.

The concept of a *space*—a set with a notion of "closeness"—frequently occurs in mathematics. Some examples quickly come to mind, like a sphere and a doughnut, as well as more complex pictures, like a dragon; see Fig.C.1. Quite often, you might also find yourself with something you want to understand, like the equation $y^2 = x^3 - 2x + 2$ or the iterative equation $f_c(z) = z^2 + c$, and produces a *space of solutions*; see Fig.C.2. A thorough study of these spaces of solutions can help discover new and interesting things about the equations you started with. In more general terms, many things that you want to study in mathematics can be understood through a space whose elements are what one



(a) A sphere S (b) A doughnut D (c) A dragon

Figure C.1: Some expected examples of spaces



(a) $y^2 = x^3 - 2x + 2$ (b) Mandelbrot set

Figure C.2: Some less expected examples of spaces

is truly interested in. With this in mind, one then wants a collection of tools to study spaces in general, before undergoing a study to learn what these tools say about our favourite spaces.

For example, you might want to prove that the plane $\mathbf{R}^2$ is not the same as space $\mathbf{R}^3$ (two dimensions vs three dimensions). We might hope this is true, but how could you prove these spaces are different? One way is to use a tool that counts the holes in a space X, written as $h_1(X)$, and the "removing a point trick". In more detail, notice that if you remove a point p from $\mathbf{R}^2$, you create a hole, meaning there is a looped path around p which cannot be deformed so that it does not go around p without this deformation crossing through p. This means there is a hole in $\mathbf{R}^2 - p$, and it turns out this is the only hole, so $h_1(\mathbf{R}^2 - p) = 1$. When you remove a point q from $\mathbf{R}^3$, any looped path you draw around q can be deformed so that it doesn't go around q anymore—there is an extra dimension of wiggle room to do this compared to the case in two dimensions! This means that $h_1(\mathbf{R}^3 - q) = 0$, and so $\mathbf{R}^2 - p$ and $\mathbf{R}^3 - q$ cannot be the same, and hence $\mathbf{R}^2$ and $\mathbf{R}^3$ are also not the same.[56] For an example used in this thesis, a question about the uniqueness of a certain object $\mathscr{O}^{\mathrm{top}}$ is rephrased in terms of a space $\mathscr{Z}$ having the property that any two points on that space can be connected with a line; this is Th.B.1.1 and the focus of §B.

What does this all have to do with homotopy theory? Superficially, it is the fact that the deformations of the paths in the above example are called *homotopies*. More importantly, these homotopies are individual pieces of data that *witness* that a loop can be deformed in a certain way, and the lack of such a homotopy confirms that such a deformation does not exist! A deeper exploration into the mathematics behind loops will further expose why these homotopies are important.

We all know the equation $x + (y + z) = (x + y) + z$, or at least we find its implications obvious—picking up 2 apples when I already have 3 in my left hand and 1 in my right totals the same number as picking up 1 apple when I already have 5 in my hands. In these simple cases, we do not need to remember the reason *why* an equation is true, we just need to remember the *fact* that it is true. This can lead us into trouble further down the mathematical path, for example, if we want to build a theory of adding loops together. To be more precise, a *loop* α on a space X is a drawing of a circle on X where the pen does not leave the space, and let us assume it always takes 1 minute to draw a loop. We do not just want to remember the picture of this circle on X, we want to remember how we drew it—otherwise, we cannot tell if someone drew one circle in 60 seconds, or drew that same circle in 1 second 60 times in a row. Using all of this information, we can add two loops α and β together as long as they start at the same point. Indeed, we can define the loop $\alpha+\beta$ by first drawing α and then immediately drawing β—as they start at the same point the pen does not leave X—making sure to draw each loop twice as fast as they were defined to assure that together it still only takes 1 minute to draw $\alpha + \beta$. Let us now consider what happens when we add three loops α, β, and γ together in a row. There are (at least) two ways to do this:

- $\alpha + (\beta + \gamma)$, which travels twice as fast around α for 30 seconds, and then four times as fast around β and as well as γ for 15 seconds each, or
- $(\alpha + \beta) + \gamma$, which travels four times as fast around α and as well as β for 15 seconds each, followed by going around γ twice as fast for 30 seconds.

Both of these options have the same picture (see Fig.C.3), but these loops are just **not** the same! At 15 seconds, on the first loop, we are halfway through α, but on the second loop, we have just finished α. However, you can deform between these two options by drawing α for $30-t$ seconds, then β for 15 seconds, and finally γ for $15 + t$ seconds, as t varies between 0 and 15. This is the very definition of a *homotopy* $H_{\alpha,\beta,\gamma}$ between $\alpha+(\beta+\gamma)$ and $(\alpha+\beta)+\gamma$. As t varies,

[56] As an exercise to the reader: can you calculate $h_1(S)$ or $h_1(D)$, ie, can you count the holes on the sphere and the doughnut? What does the answer say about S and D?

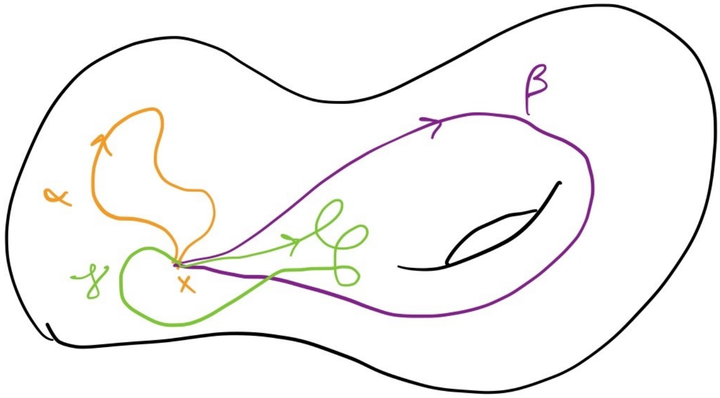


Figure C.3: Three loops α, β, and γ all starting at a common point x.

this homotopy gives us a little path from one loop to the other. We write this as:

$$H_{\alpha,\beta,\gamma} \colon \alpha + (\beta + \gamma) \to (\alpha + \beta) + \gamma$$

More geometrically, in the *space of loops on* X, suggestively denoted by ΩX, the loops $\alpha + (\beta + \gamma)$ and $(\alpha + \beta) + \gamma$ are two different points in this space ΩX, and $H_{\alpha,\beta,\gamma}$ is a path between these two points; see Fig.C.4. It seems this single homotopy has saved the day, as it witnesses that $\alpha + (\beta + \gamma)$ can be deformed into $(\alpha + \beta) + \gamma$, in other words, this homotopy witnesses these two different loops are equivalent. However, now we can investigate the sum of four loops α, β, γ, and δ. It turns out there are five different ways to add four loops together, but, even worse, there are two different ways to deform between these choices using the homotopy H above:

$$\begin{array}{ccccc} & & (\alpha+\beta)+(\gamma+\delta) & & \\ & \overset{H_{\alpha,\beta,\gamma+\delta}}{\nearrow} & & \overset{H_{\alpha+\beta,\gamma,\delta}}{\searrow} & \\ \alpha+(\beta+(\gamma+\delta)) & & & & ((\alpha+\beta)+\gamma)+\delta \\ {\scriptstyle \alpha+H_{\beta,\gamma,\delta}}\downarrow & & & & \uparrow{\scriptstyle H_{\alpha,\beta,\gamma}+\delta} \\ \alpha+((\beta+\gamma)+\delta) & & \xrightarrow[H_{\alpha,\beta+\gamma,\delta}]{} & & (\alpha+(\beta+\gamma))+\delta \end{array} \tag{C.1.1}$$

It is not clear which direction from $\alpha+(\beta+(\gamma+\delta))$ to $((\alpha+\beta)+\gamma)+\delta$ is better—to go up and over, or down and under—just as it was not clear if $\alpha + (\beta + \gamma)$ was better than $(\alpha+\beta)+\gamma$. Similar to how the homotopy H was a deformation between two loops, there is a *higher homotopy* between the top two arrows and the bottom three—another kind of deformation (which you could write down with enough patience). Just as H is a path in the space of loops on X, this

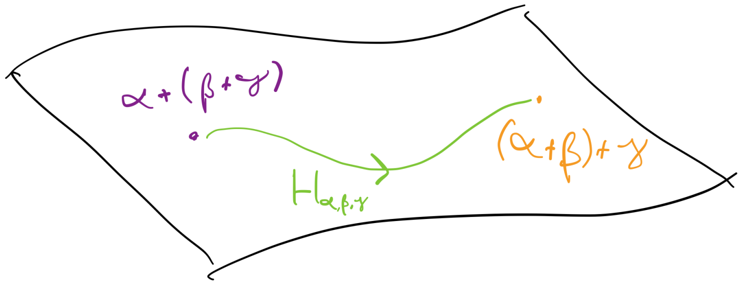


Figure C.4: A part of the space of loops ΩX of X, and a path $H_{\alpha,\beta,\gamma}$ from the point $\alpha + (\beta + \gamma)$ to the point $(\alpha + \beta) + \gamma$.

higher homotopy is a surface in the space of loops: the space in the middle of (C.1.1). You may wonder about adding together five loops, or six, or as many as you can count, and you may ask if it is possible to write down all of the higher homotopies as well. This particular problem was solved by Jim Stasheff in [Sta63], where he shows that all of this structure, the homotopies and all of the higher homotopies, can be captured using polygons called *associahedra*.[57]

There are many things you might want to do in homotopy theory other than add loops together, but whatever you do, you need to make sure to remember all of the homotopies and higher homotopies—to remember all of the *whys*, and the *whys between the whys*, etc. This seems like an unwieldy thing to do, and indeed, it took homotopy theorists many years and many different systems to try and make this intuition rigorous. Recently, this has culminated in the theory of ∞-*categories*, where the ∞ indicates that we are interested in an infinite amount of homotopical data (all of the higher homotopies). This is not the only system used to control all of these homotopies, but it is the system used in this thesis, and it allows us to easily work with all of this data at once without becoming (too) confused. Let us now discuss how this philosophy of homotopy theory can be applied to study certain equations.

There is a mathematical concept known as a *ring* which is a place where many of our favourite equations make sense—for any two elements x, y in a ring R, you can ask if $x^2 + 2x = y$ holds inside the ring R. In more detail, a ring R is a set with an addition and a multiplication operation that satisfy the usual algebraic conditions from high school, such as $x + (y + z) = (x + y) + z$, for example. To introduce homotopy theory into the study of rings, you want

[57] The curious reader might find the Wikipedia page for *associahedron* quite entertaining.

to replace the equalities in the algebraic conditions for a ring with something of a homotopical flavour. A *derived ring* R is a space with an addition and a multiplication operation satisfying a list of algebraic conditions not up to equality, but up to a particular choice of homotopy (and higher homotopies between these homotopies, and even higher homotopies ...).[58] For example, given a triple of points x, y, z in a derived ring R, there is a homotopy (so a path in R) from $x + (y + z)$ to $(x + y) + z$ which witnesses the fact that these two points are "the same", and higher homotopies like those between the different ways to add loops together above. The space of loops on X discussed above gives this space of loops half of the structure of a derived ring (either the addition or multiplication). The homotopies (and all higher homotopies) that remember that the space R is a derived ring is **a lot** of extra information to carry around, but if we are going to be honest homotopy theorists we need to keep track of this!

A natural question in the readers' mind might be:

What do these derived rings have over the classical notions of rings?

Well, there are some reasons, a few of which you can find with a quick google search of "applications of derived algebraic geometry". I want to focus on one particular reason for now though. Given two rings R and S, a *morphism* from R to S is a way of assigning to each element in R an element in S, such that for any pair of elements x, y inside R, the image of $x+y$ in S is the same as the sum in S of the image of x with the image of y, and likewise for multiplication. For example, if an equation like $x^2 + 2x = y$ holds in R, then writing $f(x)$ and $f(y)$ for the image of x and y inside S, respectively, we have the equation $f(x)^2 + 2f(x) = f(y)$ also in S. For a non-example, the morphism from the integers $\mathbf{Z}$ to itself which sends an integer n to $f(n) = 2n$ preserves addition

$$f(n + m) = 2(n + m) = 2n + 2m = f(n) + f(m)$$

but it does not preserve multiplication:

$$f(2 \cdot 3) = 2 \cdot 2 \cdot 3 = 12 \neq 24 = 2 \cdot 2 \cdot 2 \cdot 3 = f(2) \cdot f(3)$$

This means f is not a morphism of rings.[59] In other words, morphisms of rings have to preserve equations—why would we be interested in such morphisms otherwise! The converse is not generally true though—if an equation holds in S and f is a morphism from R to S, then we might not be able to find that same equation in R. For example, consider the integers $R = \mathbf{Z}$ and let $S = \mathbf{Z}/12\mathbf{Z}$—this is the ring that behaves like the hours on a clock. The elements of $\mathbf{Z}/12\mathbf{Z}$ are $0, 1, 2, 3, \ldots, 10, 11$ and to add or multiply, we first do so pretending these elements are integers, and then we ignore any factors of 12. For example,

$$10 + 5 = 3 + 12 = 3 \qquad 4 \cdot 5 = 20 = 8 + 12 = 8$$

[58] On any random page of this thesis you will most likely find the phrase $\mathbf{E}_\infty$-*ring*, which is very similar to a derived ring.

[59] The curious reader might want to show that there is precisely **one** morphism of rings from $\mathbf{Z}$ to $\mathbf{Z}$.

This reflects what you know about time—if your 5-hour shift at work starts at 10 o'clock, it ends at 3 o'clock (rather than 15 o'clock). There is a morphism of rings from $\mathbf{Z}$ to $\mathbf{Z}/12\mathbf{Z}$ which sends an integer n to its remainder after dividing by 12.[60] Notice that some equations hold in $\mathbf{Z}/12\mathbf{Z}$ but do not have a counterpart in $\mathbf{Z}$, like $5 + 7 = 12 = 0$, which holds in $\mathbf{Z}/12\mathbf{Z}$ but does **not** hold in $\mathbf{Z}$—the sum $5 + 7 = 12$ is not zero in $\mathbf{Z}$!

The mathematical word used to describe morphisms of rings which not only preserve equations but where equations in the target can always find a counterpart in the source, are called *flat* morphisms.[61] If a morphism of rings from R to S is not flat, then there is a collection of data that remembers which equations in S are not true in R, although quantifying this data is very abstract and can frequently be difficult in practice. If you have a morphism of rings that is not flat, it is a lot of extra effort to carry around this abstract collection of data, and it is not always clear how to deal with it. What is that? Carrying around extra abstract data? Sounds like homotopy theory can provide a solution! Indeed, this is what I see as one of the main benefits of derived rings: some of the extra data a derived ring carries around in all of its homotopies and higher homotopies comprises exactly these collections of data that occur when a morphism of derived rings is not flat. For example, the morphism of rings $\mathbf{Z}$ to $\mathbf{Z}/12\mathbf{Z}$ described above is not flat as there are equations in $\mathbf{Z}/12\mathbf{Z}$ with no parallel in $\mathbf{Z}$. The classical ring $\mathbf{Z}/12\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/12\mathbf{Z}$, which is supposed to remember that the morphism from $\mathbf{Z}$ to $\mathbf{Z}/12\mathbf{Z}$ is not flat, is simply $\mathbf{Z}/12\mathbf{Z}$, and no extra data is floating around. However, when considered as a derived ring, we see that $\mathbf{Z}/12\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/12\mathbf{Z}$ has a path from 0 to $5 + 7$ which remembers that this is true in $\mathbf{Z}/12\mathbf{Z}$, even though it was not in $\mathbf{Z}$. I am stretching this analogy pretty far now, but this is one reason why derived rings occur naturally in the wild—regardless, this motivation for studying derived rings has many benefits.[62]

There is another reason for studying derived rings over classical rings, and this brings us back to the first paragraph and the construction of tools to study spaces. For each ring R, you can create an invariant $H_i(X; R)$ similar to $h_1(X)$ we saw before. For example, in $H_1(X; \mathbf{Z})$ you can add holes together in a very similar way to how we added loops together above, and in $H_1(X; \mathbf{Z}/12\mathbf{Z})$ you simply demand that going around a loop twelves times is zero just as $12 = 0$ inside $\mathbf{Z}/12\mathbf{Z}$. The objects $H_i(X; R)$ for i larger than 1 measure how many i-dimensional holes there are in X and adds these holes together according to R—mathematics is wonderful at these kinds of generalisation. These invariants $H_i(X; R)$ are fantastic, but they do have their weaknesses. For example, there are many examples of spaces X and Y such that $H_i(X; R)$ and $H_i(Y; R)$ are the same for all i and all classical rings R, but the spaces X and Y are quite

[60] Check this is a morphism of rings yourself if you want to practice!

[61] Such a characterisation of flat morphisms can be found in [Sta, 00HK].

[62] There is a further (more advanced) discussion in this direction in the introduction of [SAG]. In particular, [SAG, Rmk.0.0.0.5] also highlights that derived rings should capture not just the facts, but the *why* behind those facts.

different. For a derived ring R, there is also an invariant $H_i(X; R)$, and this captures much more information about X than the case where R is a classical ring. This intuitively makes sense, as derived rings contain a lot more information than classical rings, so its interaction with spaces X might also contain a lot more information. If we consider the collection of invariants $H_i(X; R)$ for all natural numbers i and all derived rings R, we obtain a pretty exhaustive list of modern mathematics' favourite ways to study spaces.[63]

The purpose of this thesis is to study derived rings coming from *elliptic curves*,[64] meaning rings related to solutions of equations like $y^2 = x^3 - 2x + 2$ as in Fig. C.2a. These particular derived rings are called *elliptic cohomology theories*. The universal elliptic cohomology theory is written as TMF, which we call *topological modular forms*—wait a second, that is in the title of this thesis! This thesis can then be summarised as follows:

- Part I proves a complicated theorem of Jacob Lurie which constructs many natural examples of derived rings and elliptic cohomology theories from number theory, and, in particular, shows that TMF has a lot of extra symmetry compared to how it was originally defined.
- Part II uses these extra symmetries on TMF, and some inspiration from number theory, to construct operations on TMF and the powerful invariant $H_i(X; \mathrm{TMF})$.
- Part III then shows that these operations on $H_i(X; \mathrm{TMF})$, even in the simplest case when X is a single point, give number-theoretic insight into homotopy theory and also homotopy theoretic insight into number theory. I hope that the more we know about $H_i(X; \mathrm{TMF})$ for more interesting spaces X, the more these operations can say about number theory and vice versa.

In essence, the operations constructed in this thesis are further evidence that using homotopy theoretic techniques in number theory is beneficial to both subjects.

C.2 Samenvatting

Het doel van dit proefschrift is om aan te tonen dat de filosofie van *homotopietheorie*, toegepast op *getaltheorie*, interessante resultaten opleverde in beide ge-

[63]For those with some more background, these invariants $H_i(X; R)$ contain information such as isomorphism classes of vector bundles over X (topological K-theories), cobordism classes of bundles over X whose fibres are various structured manifolds ((co)bordism theories), linear combinations of n-dimensional triangles in your space X (singular (co)homology), and many, many more.

[64]The interested reader should have a read through the Wikipedia page for *Fermat's last theorem*, which states that there are no whole number solutions to the equation $x^n + y^n = z^n$ for $n > 2$. This was proved over 350 years after it was originally claimed to be proven, and the proof heavily relies on the theory of elliptic curves, which also has its own interesting Wikipedia page!

bieden van de wiskunde. In deze samenvatting bespreek ik het perspectief dat homotopietheorie de *waarom* (en niet alleen de *wat*) onthoudt, en bespreek ik hoe dit kan worden gebruikt in de getaltheorie.[65]

We kennen allemaal de vergelijking $x+(y+z) = (x+y)+z$, of in ieder geval vinden we de implicaties ervan voor de hand liggend—2 appels oppakken als ik er al 3 in mijn linkerhand en 1 in mijn rechterhand heb, geeft in totaal hetzelfde aantal als het oppakken van 1 appel als ik er al 5 in mijn handen heb. In deze eenvoudige gevallen hoeven we de reden *waarom* een vergelijking waar is niet te onthouden, maar enkel dat de vergelijking waar is. Dit kan ons verderop op het wiskundige pad in de problemen brengen, bijvoorbeeld als we een theorie willen bouwen om lussen aan elkaar te plakken. Om preciezer te zijn, een *lus* α op een ruimte X is een tekening van een cirkel op X waarbij de pen X niet verlaat; laten we hierbij aannemen dat het altijd één minuut duurt om een lus te tekenen. We willen niet alleen het plaatje van deze cirkel op X onthouden, maar ook hoe we hem hebben getekend—anders kunnen we niet zeggen of iemand één cirkel in 60 seconden heeft getekend, of 60 keer diezelfde cirkel in één seconde heeft getekend. Met al deze informatie kunnen we twee lussen α en β aan elkaar plakken, zolang ze maar op hetzelfde punt beginnen. Hier denken we over na als een optelling: we definiëren de lus $\alpha + \beta$ door eerst α te tekenen en dan onmiddellijk β te tekenen—aangezien ze op hetzelfde punt beginnen, verlaat de pen X niet. Hierbij moeten we erop letten dat we elke lus twee keer zo snel tekenen als voorheen, om ervoor te zorgen dat het in totaal nog steeds maar één minuut duurt om $\alpha + \beta$ te tekenen. Laten we nu eens kijken wat er gebeurt als we drie lussen α, β en γ bij elkaar optellen. Er zijn twee manieren om dit te doen:

- $\alpha + (\beta + \gamma)$, die twee keer zo snel α doorloopt in 30 seconden, en dan vier keer zo snel β en γ doorloopt in 15 seconden elk, of
- $(\alpha + \beta) + \gamma$, die vier keer zo snel α en β doorloopt in 15 seconden elk, en daarna twee keer zo snel door γ heen loopt in 30 seconden.

Beide opties hebben weliswaar hetzelfde plaatje (zie Fig. C.3), maar de twee lussen zijn gewoon **niet** hetzelfde! Na 15 seconden in de eerste lus zijn we halverwege α, terwijl we in de tweede lus dan net klaar zijn met α. We kunnen echter deze twee opties in elkaar omvormen, door α te tekenen voor $30 - t$ seconden, vervolgens β voor 15 seconden en tenslotte γ voor $15 + t$ seconden, waar t varieert tussen 0 en 15. Dit is de definitie van een *homotopie* $H_{\alpha,\beta,\gamma}$ tussen $\alpha + (\beta + \gamma)$ en $(\alpha + \beta) + \gamma$. Als t varieert, geeft deze homotopie ons een klein pad van de ene lus naar de andere:

$$H_{\alpha,\beta,\gamma} \colon \alpha + (\beta + \gamma) \to (\alpha + \beta) + \gamma$$

Meetkundig gezegd: in de *ruimte van lussen op X* vormen de lussen $\alpha+(\beta+\gamma)$ en $(\alpha + \beta) + \gamma$ twee verschillende punten, en vormt $H_{\alpha,\beta,\gamma}$ een pad tussen die

[65] De Engelse samenvatting heeft nog een paar extra details.

twee punten. Het lijkt erop dat deze homotopie ons heeft gered, aangezien het getuigt dat $\alpha + (\beta + \gamma)$ kan worden vervormd tot $(\alpha + \beta) + \gamma$—hierdoor kunnen we toch nog doen alsof de twee lussen 'hetzelfde' zijn. Maar we kunnen ook de som van *vier* lussen α, β, γ en δ onderzoeken. Het blijkt dat er vijf verschillende manieren zijn om vier lussen bij elkaar op te tellen, maar erger nog, er zijn twee verschillende manieren om deze keuzes in elkaar om te vervormen door middel van de homotopie H hierboven—zie Equation (C.1.1). Het is niet duidelijk welke richting van $\alpha+(\beta+(\gamma+\delta))$ naar $((\alpha+\beta)+\gamma)+\delta$ beter is—langs boven of langs onderen. Net zoals de homotopie H een vervorming is tussen twee lussen, is er een *hogere homotopie* tussen de bovenste twee pijlen en de onderste drie—een ander soort vervorming die men met voldoende geduld kan opschrijven. Net zoals H een pad is in de ruimte van lussen op X, is deze hogere homotopie een oppervlak in de ruimte van lussen. Je kunt je nu afvragen hoe dit werkt met vijf lussen, of zes lussen, of zoveel lussen als je kunt tellen, en je kunt je afvragen of je ook alle hogere homotopieën kunt opschrijven. Dit specifieke probleem is opgelost door Jim Stasheff in [Sta63], waar hij laat zien dat al deze structuur, de homotopieën en alle hogere homotopieën, kunnen worden vastgelegd met behulp van polygonen genaamd *associahedra*.[66]

Er zijn veel dingen die je zou willen doen in de homotopietheorie naast het optellen van lussen, maar wat je ook doet, je moet ervoor zorgen dat je alle homotopieën en hogere homotopieën onthoudt—om alle *waaroms* te onthouden, en de *waaroms tussen de waaroms*, enz. Dit lijkt onpraktisch om te doen, en inderdaad, het kostte homotopietheoretici vele jaren en veel verschillende systemen om te proberen deze intuïtie rigoureus te maken. Onlangs heeft dit zijn hoogtepunt bereikt in de theorie van *∞-categorieën*, waarbij de ∞ aangeeft dat we geïnteresseerd zijn in een oneindige hoeveelheid homotopische gegevens (alle hogere homotopieën). Dit is niet het enige systeem dat wordt gebruikt om al deze homotopieën te beschrijven, maar het is wel het systeem dat in dit proefschrift wordt gebruikt. Laten we nu bespreken hoe deze filosofie van homotopietheorie kan worden toegepast om bepaalde vergelijkingen te bestuderen.

Er is een wiskundig concept genaamd een *ring*, een plaats waar veel van onze favoriete vergelijkingen zinvol zijn—voor elke twee elementen x, y in een ring R kan men vragen of $x^2 + 2x = y$ geldt in de ring R. Meer in detail is een ring R een verzameling met een optel- en een vermenigvuldigingsbewerking die voldoet aan de gebruikelijke algebraïsche voorwaarden van de middelbare school, zoals $x + (y + z) = (x + y) + z$, bijvoorbeeld. Om homotopietheorie in de studie van ringen te introduceren, moet je de gelijkheden in de algebraïsche voorwaarden voor een ring vervangen door iets met een homotopische smaak. Een *afgeleide ring* R is een ruimte met een optel- en een vermenigvuldigingsbewerking die voldoet aan een lijst van algebraïsche voorwaarden, niet als gelijkheidheden, maar tot op een bepaalde keuze van homotopieën (en hogere homotopieën tussen

[66] De nieuwsgierige lezer kan de Wikipedia-pagina over de *associahedron* lezen.

deze homotopieën, en zelfs hogere homotopieën ...).[67] Bijvoorbeeld, gegeven een drietal punten x, y, z in een afgeleide ring R, is er een homotopie (dus een pad in R) van $x + (y + z)$ naar $(x + y) + z$ dat getuigt van het feit dat deze twee punten "hetzelfde" zijn, en hogere homotopieën, die lijken op die tussen de verschillende manieren om lussen bij elkaar op te tellen zoals eerder genoemd. De eerder besproken ruimte van lussen op X geeft de helft van de structuur van de afgeleide ring (ofwel de optelling of vermenigvuldiging). Deze homotopieën (en alle hogere homotopieën) die onthouden dat de ruimte R een afgeleide ring is, is **veel** extra informatie om mee te nemen, maar als we echte homotopietheoretici willen zijn, mogen we die niet vergeten!

Wat is het voordeel van deze afgeleide ringen ten opzichte van klassieke ringen? Er zijn een aantal redenen, waarvan je er een paar kunt vinden met een snelle Google-zoekopdracht naar "toepassingen van afgeleide algebraïsche meetkunde". Ik wil me nu echter op één specifieke reden concentreren. Gegeven twee ringen R en S, is een *morfisme* van R naar S een manier om aan elk element in R een element in S toe te kennen (dat we zijn *beeld* noemen), zodanig dat voor elke twee elementen x, y in R, het beeld van $x+y$ in S hetzelfde is als de som in S van het beeld van x met het beeld van y, en evenzo voor vermenigvuldiging. Als bijvoorbeeld een vergelijking als $x^2 + 2x = y$ in R geldt, dan schrijven we $f(x)$ (resp. $f(y)$) voor het beeld van x (resp. y). In S geldt dan de vergelijking $f(x)^2 + 2f(x) = f(y)$. Voor een niet-voorbeeld, het morfisme van de gehele getallen $\mathbf{Z}$ naar zichzelf dat een geheel getal n naar $f(n) = 2n$ stuurt, behoudt weliswaar de optelling:

$$f(n + m) = 2(n + m) = 2n + 2m = f(n) + f(m)$$

maar **niet** de vermenigvuldiging:

$$f(2 \cdot 3) = 2 \cdot 2 \cdot 3 = 12 \neq 24 = 2 \cdot 2 \cdot 2 \cdot 3 = f(2) \cdot f(3)$$

Dit betekent dat f geen morfisme van ringen is.[68] Met andere woorden, morfismen van ringen moeten vergelijkingen behouden—waarom zouden we anders geïnteresseerd zijn in dergelijke morfismen! Het omgekeerde is echter over het algemeen niet waar—als een vergelijking geldt in S en f een morfisme is van R naar S, dan kunnen we diezelfde vergelijking misschien niet terugvinden in R. Beschouw bijvoorbeeld de gehele getallen $R = \mathbf{Z}$ en laat $S = \mathbf{Z}/12\mathbf{Z}$—dit is de ring die zich gedraagt als de uren op een klok. De elementen van $\mathbf{Z}/12\mathbf{Z}$ zijn $0, 1, 2, 3, \ldots, 10, 11$. Om op te tellen of te vermenigvuldigen in $\mathbf{Z}/12\mathbf{Z}$ doen we eerst alsof deze elementen gehele getallen zijn, en dan we negeren alle factoren van 12. Bijvoorbeeld,

$$10 + 5 = 3 + 12 = 3 \qquad 4 \cdot 5 = 20 = 8 + 12 = 8$$

[67] Op elke willekeurige bladzijde van dit proefschrift zal men hoogstwaarschijnlijk de term $\mathbf{E}_\infty$-*ring* vinden, wat erg lijkt op een afgeleide ring.

[68] De nieuwsgierige lezer wil misschien bewijzen dat er precies **één** morfisme is van ringen van $\mathbf{Z}$ naar $\mathbf{Z}$.

Dit gedraagt zich als een klok—als mijn 5 uur durende vergadering op het werk om 10 uur begint, eindigt deze om 3 uur in plaats van om 15 uur. Er is een morfisme van ringen van **Z** naar **Z**/12**Z** die een geheel getal n stuurt naar de rest na deling door 12.[69] Merk op dat er vergelijkingen zijn die gelden in **Z**/12**Z**, maar die geen tegenhanger hebben in **Z**. Een voorbeeld is $5 + 7 = 12 = 0$, wat wel geldt in **Z**/12**Z**, maar **niet** in **Z**—de som $5 + 7 = 12$ is zeker niet nul in **Z**!

Als een morfisme niet alleen vergelijkingen behoudt, maar ook de eigenschap heeft dat vergelijkingen in het doel altijd een tegenhanger in de bron hebben, wordt het morfisme een *plat* morfisme genoemd.[70] Als een morfisme van ringen van R naar S niet plat is, dan is er een verzameling gegevens die onthoudt welke vergelijkingen in S niet waar zijn in R, hoewel het kwantificeren van deze gegevens erg abstract is en in de praktijk vaak moeilijk kan zijn. Als men een morfisme van ringen heeft dat niet plat is, kost het veel extra moeite om deze abstracte verzameling gegevens mee te nemen en is het niet altijd duidelijk hoe ermee om te gaan. Wat is dat? Extra abstracte gegevens meesjouwen? Dat klinkt alsof homotopietheorie een oplossing kan bieden! Dit is inderdaad wat ik zie als een van de belangrijkste voordelen van afgeleide ringen: sommige van de extra gegevens die een afgeleide ring in al zijn homotopieën en hogere homotopieën met zich meedraagt, omvatten precies deze verzamelingen gegevens die optreden wanneer een morfisme van afgeleide ringen niet plat is. Het hierboven beschreven morfisme van de ring **Z** naar **Z**/12**Z** is bijvoorbeeld niet plat omdat er vergelijkingen zijn in **Z**/12**Z** zonder parallel in **Z**. De klassieke ring $\mathbf{Z}/12\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/12\mathbf{Z}$ die moet onthouden dat het morfisme van **Z** naar **Z**/12**Z** niet vlak is, is gewoon **Z**/12**Z**, en deze ring onthoudt geen extra data. Als we het echter als een afgeleide ring beschouwen, dan heeft $\mathbf{Z}/12\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/12\mathbf{Z}$ een pad van 0 tot $5 + 7$. Dit pad onthoudt dat de vergelijking $5 + 7 = 0$ waar is in **Z**/12**Z**, ook al is het niet waar in **Z**. Ik rek deze analogie nu vrij ver uit, maar dit is een reden waarom afgeleide ringen van nature in het wild voorkomen—hoe dan ook, deze motivatie om afgeleide ringen te bestuderen heeft vele voordelen.[71]

Het doel van dit proefschrift is om bepaalde afgeleide ringen te bestuderen, namelijk afgeleide ringen die afkomstig zijn van *elliptische krommen*,[72] d.w.z., ringen die gerelateerd zijn aan oplossingen van vergelijkingen zoals $y^2 = x^3 - 2x + 2$ uit in Fig.C.2a. Deze specifieke afgeleide ringen worden *elliptische cohomologietheorieën* genoemd. De universele elliptische cohomologietheorie wordt aangeduid met TMF, wat staat voor *topologische modulaire vormen*—

[69]Controleer zelf of dit een morfisme van ringen is, als je wilt oefenen!

[70]Een precieze karakterisering van platte morfismen staat bijvoorbeeld in [Sta, 00HK].

[71]Er is een verdere (meer geavanceerde) discussie in deze richting in de inleiding van [SAG]. In het bijzonder benadrukt [SAG, Rmk.0.0.0.5] ook dat afgeleide ringen niet alleen de feiten moeten vastleggen, maar ook de *waarom* achter die feiten.

[72]De geïnteresseerde lezer zou de Wikipedia-pagina moeten lezen voor *de laatste stelling van Fermat*, die zegt dat er zijn geen gehele oplossingen zijn van de vergelijking $x^n + y^n = z^n$ als $n > 2$. Pas 350 na de oorspronkelijke claim werd deze uitspraak bewezen, en het bewijs is sterk afhankelijk van de theorie van elliptische krommen, die ook zijn eigen interessante Wikipedia-pagina heeft!

wacht even, dat staat in de titel van dit proefschrift! Dit proefschrift kan als volgt worden samengevat:

- Deel I bewijst een gecompliceerde stelling van Jacob Lurie die veel natuurlijke voorbeelden van afgeleide ringen en elliptische cohomologietheorieën construeert uit de getaltheorie, en die in het bijzonder laat zien dat TMF veel extra symmetrie heeft vergeleken met de oorspronkelijke definitie.

- Deel II gebruikt deze extra symmetrieën op TMF, en enige inspiratie uit de getaltheorie, om operaties op TMF te construeren.

- Deel III laat vervolgens zien dat deze operaties op TMF getaltheoretische inzichten geven in homotopietheorie en ook homotopietheoretische inzichten in de getaltheorie. Ik hoop dat hoe meer we weten over TMF voor interessantere ruimtes X, hoe meer deze bewerkingen kunnen zeggen over getaltheorie en vice versa.

In wezen laten de operaties uit dit proefschrift zien dat het gebruik van homotopietheoretische technieken in de getaltheorie gunstig is voor beide vakgebieden.

C.3 Zusammenfassung

Vergiss die Techniken in dieser Doktorarbeit, und lass uns über ihren Kern diskutieren. Das Ziel dieser Arbeit ist es zu demonstrieren, dass die Philosophie der *Homotopietheorie*, angewendet auf die *Zahlentheorie*, interessante Ergebnisse in beiden Bereichen hervorbringt. In dieser Zusammenfassung betone ich die Perspektive, dass Homotopietheorie das *Warum* und das *Wie* kennt, und zeige wie diese Philosophie in der Welt der Zahlentheorie funktionieren kann.[73]

Wir kennen die Gleichung $x + (y + z) = (x + y) + z$ für drei natürliche Zahlen x, y, z. Zumindest finden wir ihre Konsequenzen klar: wenn du 2 Äpfel bekommst und du noch 3 in deiner rechten Hand und 1 in deiner linken hast, ist dies dasselbe wie wenn du 1 Apfel bekommst und du noch 5 hast. In diesen einfachen Szenarien wissen wir sofort die *Tatsache*, dass die Gleichung gilt, bevor wir wissen *Warum* sie gilt. Wenn wir nur die Tatsachen wissen, können wir uns im späteren mathematischen Pfad irren. Nehmen wir zum Beispiel eine Theorie des Zusammenfügen von *Schleifen.* Eine Schleife α in einem Raum X ist eine Zeichnung einer Kurve in X, bei der der Stift auf X bleibt. Nehmen wir auch an, dass es eine Minute dauert um eine Schleife zu zeichnen. Wir wollen uns nicht nur an das Bild der Schleife in X erinnern, sondern auch daran wie es gezeichnet wird. Andernfalls können wir nicht unterscheiden, ob jemand in 60 Sekunden *einen* Kreis oder 60 mal denselben Kreis in 1 Sekunde gezeichnet hat.

[73] Die englische Zusammenfassung enthält noch ein paar Details.

Man kann zwei verschiedene Schleifen hinzufügen solange beide am gleichen Punkt beginnen. Bei zwei Schleifen α, β in X die am selben Punkt beginnen, definieren wir die Schleifen $\alpha + \beta$ indem wir zuerst α zeichnen und dann, ohne den Stift hochzuheben, die Schleife β weiterzeichnen. Wir zeichnen beide bei doppeler Geschwindigkeit, also 30 Sekunden für α und 30 Sekunden für β, sodass es nur 1 Minute dauert, um $\alpha + \beta$ zu zeichnen. Jetzt können wir betrachten, wie man drei Schleifen hinzufügt. Es gibt zwei offensichtliche Möglichkeiten:

- $\alpha + (\beta + \gamma)$, wo man α doppelt so schnell (30 Sekunden), dann β und γ viermal so schnell (jeweils 15 Sekunden) zeichnet.
- $(\alpha + \beta) + \gamma$, wo man α und β viermal so schnell (jeweils 15 Sekunden), und γ nur doppelt so schnell (30 Sekunden) zeichnet.

Diese zwei Schleifen haben das gleiche Bild in X (Fig. C.3), aber sie sind nicht die gleiche Schleife! Es ist jedoch möglich, diese beiden Schleifen zu verformen. Zum Beispiel, indem man zuerst α für $30 - t$ Sekunden, dann β für 15 Sekunden, und zuletzt γ für $15 + t$ Sekunden zeichnet, solange t zwischen 0 und 15 variiert. Diese Verformung nennen wir eine *Homotopie* von $\alpha + (\beta + \gamma)$ nach $(\alpha + \beta) + \gamma$. Topologisch gesehen können wir einen *Raum der Schleifen in X* bilden, bezeichnet als ΩX, wo die Punkte Schleifen in X und die Pfade Homotopien zwischen Schleifen sind. Zum Beispiel, die Homotopie $H_{\alpha,\beta,\gamma}$ ist ein Pfade von $\alpha + (\beta + \gamma)$ nach $(\alpha + \beta) + \gamma$ in ΩX; sieh Fig.C.4. Wir betrachten diese Homotopie als Grund dafür, dass diese beiden Schleifen gleichwertig sind. Eine Homotopie ist ein *Warum* zwischen zwei Punkten (oder zwei Schleifen), die äquivalent sind.

Als nächstes können wir fragen, ob wir vier Schleifen α, β, γ, δ zusammenfügen können. Mit anderen Worten, wie viele Punkte in ΩX durch die Schleifen α, β, γ, δ entstehen können, und was die Pfade dazwischen sind. Es stellt sich heraus, dass es fünf verschiedene Möglichkeiten gibt, vier Schleifen hinzufügen. Es kommt noch dazu, dass es zwei verschiedene Möglichkeiten zwischen diesen fünf Schleifen gibt; siehe Diagram C.1.1. Es ist nicht klar, welche Richtung wir von $\alpha+(\beta+(\gamma+\delta))$ nach $((\alpha+\beta)+\gamma)+\delta$ nehmen sollen—genauso wie es nicht klar ist, ob wir $\alpha + (\beta + \gamma)$ oder $(\alpha + \beta) + \gamma$ nehmen sollen! Sollen wir von $\alpha + (\beta + (\gamma + \delta))$ nach $((\alpha + \beta) + \gamma) + \delta$ mit den beiden obigen Wegen oder mit den drei unteren? So wie die Homotopie H eine Verformung zwischen zwei Schleifen war, gibt es eine *höhere Homotopie* zwischen den oberen beiden Pfaden und den unteren drei Pfaden in Diagram C.1.1—eine andere Art der Verformung, die man mit genügend Geduld aufschreiben kann. So wie eine Homotopie ein *Pfad* im Raum der Schleifen ΩX ist, ist diese höhere Homotopie eine *Fläche* in ΩX—der leere Raum in der Mitte des Diagram C.1.1. Philosophisch gesehen: ein *höreres Warum zwischen anderen Warums.* Man kann sich fragen, ob man fünf oder sechs (oder mehr) Schleifen hinzufügen kann, und ob man auch alle höheren (und noch höheren) Homotopien aufschreiben kann. Dieses spezielle Problem wurde von Jim Stasheff in [Sta63] gelöst. Er nutzt Polygonen, die sogenannten *associahedra*, um alle diese Strukturen, die Homotopien und alle höheren Homotopien zu erfassen.[74]

Es gib viele Dinge, die man in der Homotopietheorie tun kann, neben Schleifen hinzuzufügen. Egal was man in Homotopietheorie tut, man muss sich immer an alle Homotopien und höheren Homotopien erinnern—an alle *Warums* und die *Warums zwischen den anderen Warums*. Das ist eine schwerfällige Aufgabe. Homotopietheoretiker*innen haben in der Tat viele Jahre und viele verschiedene Theorien gebraucht, um diese Intuition rigoros zu machen. In jüngster Zeit kulminierte dies in der Theorie der ∞-*Kategorien*. Das ∞ Symbol bedeutet, dass wir an einer unendlichen Menge homotopischer Daten (alle Homotopien und höheren Homotopien) interessiert sind. Wir verwenden die Sprache der ∞-Kategorien in dieser Arbeit, da es uns erlaubt, gleichzeitig mit all diesen Daten zu arbeiten, ohne (zu) verwirrt zu werden. Lasst uns nun erörtern, wie die Philosophie der Homotopietheorie auf die Untersuchung bestimmter Gleichungen angewendet werden kann.

Es gibt ein mathematisches Konzept, das als *Ring* bekannt ist und in dem viele unserer Lieblingsgleichungen einen Sinn ergeben—für zwei beliebige Elemente x, y in einem Ring R kann man fragen, ob $x^2 + 2x = y$ innerhalb des Rings R gilt. Ein Ring R ist eine Menge mit einer Additions- und einer Multiplikationsoperation, die die üblichen algebraischen Bedingungen aus der Schule erfüllt, z.B. $x + (y + z) = (x + y) + z$. Beispielsweise sind ganze Zahlen $\mathbf{Z}$ ein Ring. Um die Homotopietheorie in das Studium der Ringe einzuführen, möchten wir die Gleichheiten in den algebraischen Bedingungen für einen Ring durch etwas Homotopisches ersetzen. Ein *abgeleiteter Ring R* ist ein **Raum** (nicht nur eine Menge) mit einer Additions- und einer Multiplikationsoperation, der eine Liste von algebraischen Bedingungen nicht bis zur Gleichheit, sondern bis zu einer bestimmten Wahl der Homotopien (und höheren Homotopien zwischen diesen Homotopien, und noch höheren Homotopien ...) erfüllt. [75] Zum Beispiel gibt es bei drei Punkten x, y, z in einem abgeleiteten Ring R eine Homotopie (einen Pfad in R) von $x + (y + z)$ nach $(x + y) + z$, die bezeugt, dass diese beiden Punkte "äquivalent" sind. Zwischen vier Punkten w, x, y, z in R gibt es zwei Pfade zwischen $w+(x+(y+z))$ und $((w+x)+y)+z$ in R, und eine Fläche zwischen diese Pfaden; wie in Diagram C.1.1 in ΩX. Diese Pfade und Flächen (Homotopien und höhere Homotopien) kennen die "Gleichungen", die wir in einem Ring erwarten. Diese Homotopien (und alle höheren Homotopien) sind eine Menge zusätzlicher Informationen, die man mit sich herumtragen muss. Diese Informationen können schwer sein, aber wenn wir ehrliche Homotopietheoretiker*innen sein wollen, müssen wir dies im Auge behalten!

Was haben diese abgeleiteten Ringe gegenüber den klassischen Ringen? Es gibt einige Gründe, die du mit einer schnellen Google-Suche nach "Applications of derived algebraic geometry" finden kannst. Wir werden uns jetzt aber auf

[74] Die neugierige Leserin könnte die Wikipedia-Seite für *associahedron* interessant finden.

[75] Auf einer beliebigen Seite dieser Arbeit wird man höchstwahrscheinlich die Formulierung $\mathbf{E}_\infty$-*ring* finden, die einem abgeleiteten Ring sehr ähnlich ist.

einen bestimmten Grund konzentrieren. Ein *Morphismus* von einem Ring R nach einem Ring S ordnet jedem Element x in R *genau ein* Element $f(x)$ in S zu, sodass für jedes Paar x, y in R das Bild von $x + y$ in S dasselbe ist, wie die Summe des Bildes von x mit dem Bild von y in S. Das Gleiche gilt auch für die Multiplikation. Wenn eine Gleichung wie $x^2 + 2x = y$ in R gilt, dann haben wir die Gleichung $f(x)^2 + 2f(x) = f(y)$ auch in S. Ein Gegenbeispiel: Der Morphismus von den ganzen Zahlen $\mathbf{Z}$ zu $\mathbf{Z}$, der eine ganze Zahl n die Zahl $2n$ zuordnet. Die Addition bleibt

$$f(n+m) = 2(n+m) = 2n + 2m = f(n) + f(m)$$

aber die Multiplikation bleibt nicht erhalten:

$$f(2 \cdot 3) = 2 \cdot 2 \cdot 3 = 12 \neq 24 = 2 \cdot 2 \cdot 2 \cdot 3 = f(2) \cdot f(3)$$

Das bedeutet, dass dieses f kein Morphismus von Ringen ist.[76] Mit anderen Worten: Morphismen von Ringen müssen Gleichungen erhalten—warum sollten wir uns sonst für solche Morphismen interessieren! Das Umgekehrte gilt jedoch nicht generell: Wenn eine Gleichung in S gilt und f ein Morphismus von R nach S ist, dann kann es sein, dass wir die gleiche Gleichung in R nicht finden können. Betrachten wir zum Beispiel die ganzen Zahlen $R = \mathbf{Z}$ und die Ringe $S = \mathbf{Z}/12\mathbf{Z}$—das ist der Ring, der sich wie die Stunden auf einer Uhr verhält. Die Elemente von $\mathbf{Z}/12\mathbf{Z}$ sind $0, 1, 2, 3, \ldots, 10, 11$, und um zu addieren oder zu multiplizieren, tun wir zunächst so, als seien diese Elemente ganze Zahlen, und ignorieren, dass alle Faktoren von 12 sind. Zum Beispiel:

$$10 + 5 = 12 + 3 = 3 \qquad 4 \cdot 5 = 20 = 8 + 12 = 8$$

Das ist das, was du über Zeit weißt—wenn deine 5-Stunden-Schicht auf der Arbeit um 10 Uhr beginnt, endet sie um 3 Uhr. Es gibt einen Morphismus der Ringe q von $\mathbf{Z}$ nach $\mathbf{Z}/12\mathbf{Z}$, der eine ganze Zahl n nach der Division durch 12 in ihren Rest überführt. Beobachte, dass einige Gleichungen in $\mathbf{Z}/12\mathbf{Z}$ gelten, aber kein Gegenstück in $\mathbf{Z}$ haben. Zum Beispiel: $5 + 7 = 12 = 0$ gilt in $\mathbf{Z}/12\mathbf{Z}$, aber nicht in $\mathbf{Z}$—die Summe $5 + 7 = 12$ ist in $\mathbf{Z}$ definitiv nicht Null!

Das mathematische Wort zur Beschreibung von Morphismen von Ringen, die nicht nur Gleichungen erhalten, sondern bei denen Gleichungen im Ziel immer ein Gegenstück in der Quelle finden, nennt man *flache* Morphismen.[77] Wenn ein Morphismus von Ringen von R nach S nicht flach ist, dann gibt es eine Sammlung von Daten, die sich daran erinnern, welche Gleichungen in S in R nicht wahr sind. Die Quantifizierung dieser Daten ist sehr abstrakt und kann in der Praxis oft schwierig sein. Wenn man einen Morphismus von Ringen hat, der nicht flach ist, ist es ein großer zusätzlicher Aufwand, diese abstrakte Datensammlung mit sich herumzutragen. Es ist jedoch nicht immer klar, wie man

[76] Die neugierige Leserin möchte vielleicht zeigen, dass es genau **einen** Morphismus von Ringen von $\mathbf{Z}$ nach $\mathbf{Z}$ gibt. Diese Leserin muss wissen, dass ein Morphismus zwischen Ringen 0 nach 0 und 1 nach 1 nehmen muss.

[77] Eine solche Charakterisierung von flachen Morphismen befindet sich in [Sta, 00HK].

damit umgehen kann. Wie bitte? Zusätzliche abstrakte Daten mit sich herumtragen? Das klingt, als könnte die Homotopietheorie eine Lösung bieten! In der Tat sehe ich darin einen der Hauptvorteile abgeleiteter Ringe. Einige der zusätzlichen Daten, die ein abgeleiteter Ring in (höheren) Homotopien mit sich herumträgt, sind genau diese Datensammlungen, die daran errinern, *Warum* ein Morphismus von abgeleiteten Ringen nicht flach ist. Zum Beispiel ist der oben beschriebene Morphismus der Ringe q von $\mathbf{Z}$ nach $\mathbf{Z}/12\mathbf{Z}$ nicht flach, da es Gleichungen in $\mathbf{Z}/12\mathbf{Z}$ gibt, die keine Parallele in $\mathbf{Z}$ haben. Der klassische Ring $\mathbf{Z}/12\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/12\mathbf{Z}$, der daran erinnern soll, dass der Morphismus von $\mathbf{Z}$ nach $\mathbf{Z}/12\mathbf{Z}$ nicht flach ist, ist einfach $\mathbf{Z}/12\mathbf{Z}$, und es gibt keine zusätzlichen Daten. Betrachtet man jedoch $\mathbf{Z}/12\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/12\mathbf{Z}$ als abgeleiteten Ring, so sieht man, dass $\mathbf{Z}/12\mathbf{Z}$ einen Pfad von 0 nach $5 + 7$ hat, der daran erinnert, dass dies in $\mathbf{Z}/12\mathbf{Z}$ wahr ist, auch wenn es in $\mathbf{Z}$ nicht der Fall ist. Wir ziehen diese Analogie jetzt ziemlich weit, aber das ist mitunter ein Grund, warum abgeleitete Ringe in der Natur vorkommen. Unabhängig davon hat diese Motivation für das Studium abgeleiteter Ringe viele Vorteile.[78]

Der Zweck dieser Arbeit ist die moderne Studie von abgeleiteten Ringen die aus *elliptischen Kurven* stammen.[79] In anderen Worten: Ringe, die sich auf Lösungen von Gleichungen wie $z^2 = x^3 - 2x + 2$, wie in Fig. C.2a dargestellt, beziehen. Diese besonderen abgeleiteten Ring werden *elliptische Kohomologietheorien* genannt. Die universelle elliptische Kohomologietheorie heißt *topologische modulare Formen* und wird als TMF geschrieben—Moment mal, das steht doch im Titel dieser Arbeit! Diese Arbeit lässt sich also wir folgt zusammenfassen:

- Teil I beweist ein kompliziertes Theorem von Jacob Lurie, das viele natürliche Beispiele von abgeleiteten Ringen und elliptischen Kohomologietheorien aus der Zahlentheorie konstruiert. Dieser Teil zeigt insbesondere, dass TMF eine Menge zusätzlicher Symmetrien im Vergleich zu seiner ursprünglichen Definition hat.

- Teil II verwendet diese zusätzlichen Symmetrien auf TMF, und einige Anregungen aus der Zahlentheorie, um Operationen auf TMF zu konstruieren.

- Teil III zeigt dann, dass diese Operationen auf TMF einen zahlentheoretischen Einblick in die Homotopietheorie, und einen homotopietheoretischen Einblick in die Zahlentheorie geben. Ich hoffe, dass je mehr wir über TMF

[78] Eine weitere (fortgeschrittenere) Diskussion in dieser Richtung gibt es in der Einführung von [SAG]. Insbesondere hebt [SAG, Rmk.0.0.0.5] auch hervor, dass abgeleitete Ringe nicht nur die Tatsachen, sondern auch das *Warum* hinter diesen Tatsachen erfassen sollten.

[79] Die interessierte Leserin sollte sich die Wikipedia-Seite für *der Großen Fermatscher Satz* durchlesen, die besagt, dass es keine ganzzahligen Lösungen der Gleichung $x^n + y^n = z^n$ für $n > 2$ gibt. Dieser Satz wurde über 350 Jahre nach seiner ursprünglichen Behauptung bewiesen, und der Beweis stützt sich stark auf die Theorie der elliptischen Kuren, die auch eine eigene interessante Wikipedia-Seite hat!

wissen, diese Operationen mehr über Zahlentheorie aussagen können—und umgekert.

Im Wesentlichen sind die in dieser Arbeit konstruierten Operationen ein weiterer Beweis dafür, dass die Anwendung homotopietheoretischer Techniken in der Zahlentheorie für beide Fächer von Nutzen ist.

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C.5 Curriculum Vitae

Jack Morgan Davies was born on 29th May 1993 in Canberra, Australia. After finishing high school, he enrolled in a B. Sc. with honours at the Australian National University in 2012, and was supervised by Vigleik Angeltveit for a bachelor thesis on some of the basic elements of equivariant stable homotopy theory. He then moved to Bonn, Germany in 2016 to study for an M. Sc. with a thesis in global homotopy theory supervised by Stefan Schwede. In 2018, he started a Ph.D. under the supervision of Lennart Meier at Utrecht University, of which this thesis is a product. He will begin a postdoctoral position in Bonn in October 2022.

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