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SUPPORTING STUDENTS TO COMPRESS MATHEMATICAL KNOWLEDGE WHILE PROBLEM SOLVING

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Compression of mathematical objects, procedures and statements could play a major role in successful problem solving. We report on a study in which we aim to design an instrument, named heuristic tree, to stimulate the process of compression in students while working in a digital problem-solving environment. In particular, we report on the evidence a second pilot study provided that improvements in the design helped to overcome issues with help-seeking that presented themselves in a first pilot.

INTRODUCTION

Proficient problem solvers express their solution strategy in compressed, abstract language. For instance, the sentence: "By constructing a perpendicular line, I found a perpendicular triangle to which I applied the Pythagorean theorem" is easily understood by a more advanced students, but is too compressed for a novice. For a novice the process of constructing perpendiculars, the concept of a perpendicular triangle and the statement and application of the Pythagorean Theorem are intricate, extensive, not easily applied, nor understood in one sentence, because they are the result of compression.

Compression is a central organizational feature of mathematical knowledge. Mathematician William Thurston conjectured that compression, that is easily observed in mathematical discourse, is mirrored by a cognitive compression process:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics (Thurston, 1990, p.847)

Compression is not unique to mathematical cognition, but unique to mathematics is the way compressed content can be part of many new rounds of compression creating deeply nested abstract cognitive structures. Moreover, mathematical compression consists not merely of filing away details of processes or ideas in a reliable way to long-term memory, but is complemented by a shift of attention from a multitude of phenomena to *common properties* of those phenomena.

An issue in mathematics education is that it does not prepare students enough for dealing with mathematical problem situations. Our hypothesis is that a major cause for this is that compression processes are not attended to, facilitated or stimulated in mathematics education. The aim of our research is to develop a tool that does just that. The tool, named *heuristic tree*, offers students support implemented in a digital problem-solving environment, allowing students to decompress the heuristic statements into constituents, by clicking through a tree-like structure (see Figure 1).

Other tools and approaches have been developed to address this or a very similar problem, among others, the ACE-approach and genetic decomposition linked to APOS-theory (Arnon et al., 2014), and the Teaching for abstraction-model (White & Mitchelmore, 2010). New about our approach is the foundation on self-regulated learning in a digital problem-solving environment.

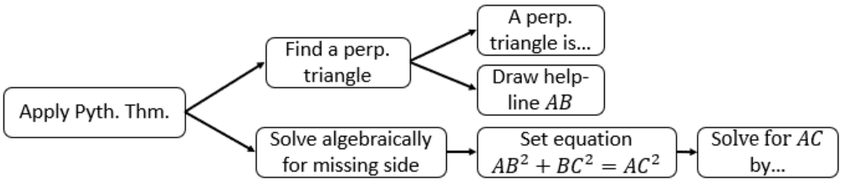

Figure 1. A sketch of a heuristic tree to support an undescribed problem about computing a length

This paper reports on the second pilot as part of a broader design study, consisting of several cycles of (re)design, classroom tests and analysis. A pilot study (Bos, 2017) revealed that students had difficulty with self-regulation, in particular adopting the right help-seeking behavior using heuristic trees. In the method section we describe how we redesigned the project's setup for a second pilot study to overcome this issue. In the last section we present a brief overview demonstrating how these changes were successful in dealing with these difficulties.

THEORETICAL FRAMEWORK

Compression applies to three aspects of mathematical organization: objects, procedures and statements. Each form of compression is characterized by a shift of attention – represented both in the individual (cognitive perspective) and in the mathematical discourse (socio-cultural perspective) – from a multitude of phenomena to common properties of those phenomena together with a filing away of details in a reliable way to long-term memory or reference books.

We distinguish two forms of compression: compression on cases – cases of objects, cases where procedures apply, and cases where statements apply – and compression on steps/details of a process/technique. Tall calls compression on cases of mathematical objects *categorization* (Tall, 2013): a multitude of objects is compressed to a category. Similarly, reorganizing knowledge on procedures and statements, such that separate cases are treated uniformly is a form of compression on cases. Literature provides various vocabularies to discuss the phases, stages or levels of abstraction processes that apply to compression on cases (Hershkowitz, Schwarz, & Dreyfus, 2001; Tall, 2013; van Hiele, 1986). In this paper we use the stages of the model of White and Mitchelmore (White & Mitchelmore, 2010): familiarity, similarity, reification and application. In short, in a first stage one familiarizes oneself with the separate cases. Then, one shifts one's attention to the similarities between these cases. Next, one sees the totality (called *condensation*) of cases as a category, a new object, described by its properties – this is called the *reification* phase. In this phase the shift of attention to properties, mentioned before, takes shape. Finally, the new object (object, procedure or statement) is seen and used in applications, like problem solving, dealing with all underlying cases at once.

Compression on the steps of a process or details of a technique have also been described within several frameworks: for example operational-structural by Sfard (Sfard, 1991), APOS-theory by Dubinsky and collaborators (Arnon et al., 2014), and procept-theory by Tall (Tall, 2013). Our description builds on these frameworks, while trying to stay close to the description of compression on objects. We again discern four stages/levels/phases. In a first stage, *recognition*, individually performed actions (steps) are recognized as parts of overarching procedures; in a second stage, *interiorization*, similarities between the performed procedures in different cases are observed, and therefore more efficiently stored in long-term memory (or a book) as a reliable process or technique. In the following reification phase, one sees the process as a totality (*condensation*), a new object (or *procept* (Tall, 2013)), and shifts attention to the properties of the process or technique; properties like the theoretical underpinning/justification of the mechanisms at work or the domain to which the technique applies. In the final stage, *application*, the process can be recalled more easily in cases where it applies and can be applied more flexibly, adapted to the problem situation, if necessary. If students are stuck between the familiarity and similarity stage, they will not have the flexibility to use the concept, technique or statement (like theorems) to solve a problem.

A proficient problem solver develops ideas into strategies by combining, adapting and elaborating techniques. To this purpose, the employed mathematical discourse needs to be abstract, formulated in terms of compressed objects, to make insurmountable series of steps, that form the obstacles that compose the problem, surmountable. Heuristic trees provide support for students in compressed language at the root, allowing students to click down the branches to decompress the these heuristics into more concrete hints (Bos, 2017).

DESIGN IMPROVEMENTS

To deal with the issue of help-seeking we implemented two improvements. A central insight, inspired by instrumentation theory (Artigue, 2002), was that compression – as a process of cognitive reorganization filing details away to long-term memory and shifting attention to properties – should develop in parallel with the use of the heuristic tree tool that allows students to experience the compression/decompression dynamic. The instrumental genesis of the heuristic tree instrument involves developing a cognitive scheme about this dynamic and how it is implemented in the tree structure. In the first pilot study students had difficulty navigating the structure because such scheme had not developed. In the second pilot this was overcome by a process of incrementation: The complexity of the problems and the supporting heuristic tree were gently increased, beginning with

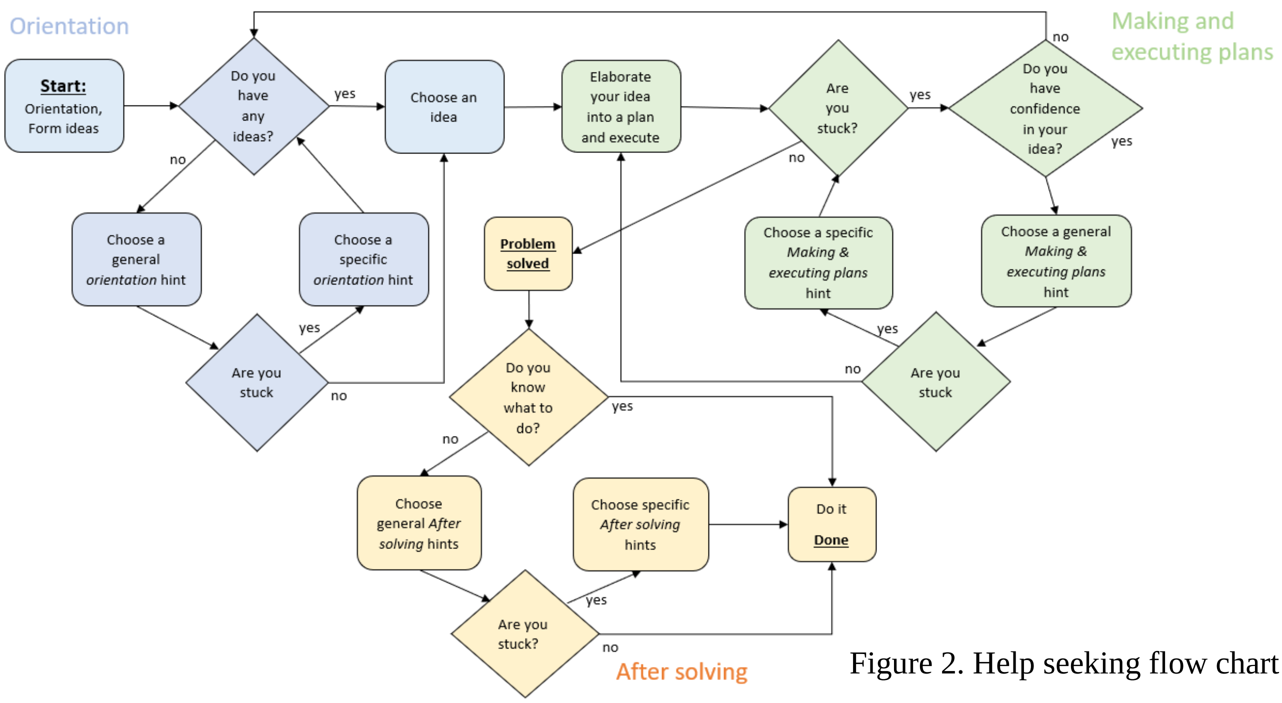

Figure 2. Help seeking flow chart

simple problems supported by trees of depth two growing to more complex problems supported by trees of depth four.

The second improvement to support students help-seeking behavior was to provide students with a help-seeking flow-chart (see Figure 2). This is an major extension of the chart in (Roll, Aleven, McLaren, & Koedinger, 2011), to include the various phases of problem solving and facilitate the dynamic between compressed heuristics and more decompressed hints. The use of the chart is introduced to students by a demonstration video.

RESULTS FROM A SECOND PILOT STUDY, DISCUSSION AND OUTLOOK

Two classes (grade 9 and 11) each had two 90 minute lessons working in the online problem environment. For each problem students tracked their work flow in a flow chart. We registered their use of the heuristic trees and their results on the problems. Students filled in questionnaires before and after the experiment.

The data supported the conclusion that help-abuse – like clicking through all hints or not asking for help when needed – was reduced to *nearly not occurring*, while success in problem solving went up multiple times compared to the first pilot. Moreover, approximately 45% of students reported in the questionnaires that they improved their approach to problem solving.

Up next is a main experiment in which we will collect data on the use of language by students thinking out loud while problem solving, to detect if working in our digital problem solving environment with support provided by heuristic trees facilitates the compression process, as described above.

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