

Problems hard for treewidth but easy for stable gonality

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Abstract

We show that some natural problems that are XNLP-hard (which implies $W[t]$ -hardness for all t) when parameterized by pathwidth or treewidth, become FPT when parameterized by stable gonality, a novel graph parameter based on optimal maps from graphs to trees. The problems we consider are classical flow and orientation problems, such as UNDIRECTED FLOW WITH LOWER BOUNDS (which is strongly NP-complete, as shown by Itai), MINIMUM MAXIMUM OUTDEGREE (for which $W[1]$ -hardness for treewidth was proven by Szeider), and capacitated optimization problems such as CAPACITATED (RED-BLUE) DOMINATING SET (for which $W[1]$ -hardness was proven by Dom, Lokshtanov, Saurabh and Villanger). Our hardness proofs (that beat existing results) use reduction to a recent XNLP-complete problem (ACCEPTING NON-DETERMINISTIC CHECKING COUNTER MACHINE). The new, “easy”, parameterized algorithms use a novel notion of weighted tree partition with an associated parameter that we call “treebreadth”, inspired by Seese’s notion of tree-partite graphs, as well as techniques from dynamical programming and integer linear programming.

1 Introduction

The parameterization paradigm Problems on finite (multi-)graphs that are NP-hard may become polynomial by restricting a specific graph parameter k . If furthermore the degree of the resulting polynomial bound is constant, we say that the problem becomes *fixed parameter tractable* (FPT) for the parameter k [15, 1.1]. More precisely, there should exist an algorithm that solves the given problem in time bounded by a computable function of the parameter k times a power of the input size of the problem. Despite the fact that computing the parameter itself can often be shown to be NP-hard or NP-complete, the FPT-paradigm, originating in the work of Downey and Fellows [18], has shown to be very fruitful both in theory and practice.

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One successful approach is to consider graph parameters that measure how far a given graph is from being acyclic; more specifically, how the graph may be decomposed into “small” pieces, such that the interrelation of the pieces is described by a tree-like structure. A prime example of such a parameter is the *treewidth* $\text{tw}(G)$ of a graph G ([15, Ch. 7]; reviewed briefly below). Some reasons for the success are that graphs of bounded treewidth are amenable to algorithms based on dynamic programming [15, 7.3], and Courcelle’s theorem [14] [15, 7.4], that states that graph problems described in second order monadic logic are FPT, linear in the number of vertices, provided a tree decomposition realising the treewidth is given. Examples of problems that are FPT for treewidth are DOMINATING SET and VERTEX COVER [15, 7.3].

Other parameters have been considered; for example, in [20], it is shown that colouring problems such as EQUITABLE COLOURING (existence of a vertex colouring with at most k colours, so that the sizes of any two colour classes differ by at most one) is hard for treewidth (technically, $W[1]$ -hard), but FPT for vertex cover number. In [25] and [15, 7.9] one finds more problems and parameters that shift their complexity. Still, some famous graph orientation and graph flow problems, as well as capacitated version of classical problems such as DOMINATING SET have so far not succumbed to the FPT paradigm for any reasonable parameter, despite being of practical importance in logistics and resource allocation. Below, we will consider UNDIRECTED FLOW WITH LOWER BOUNDS, Problem [ND37] in [22], known to be strongly NP-complete since the 1970’s. We will show that, when parameterized by treewidth, the problem is hard (in fact, XNLP-complete with pathwidth as parameter, cf. *infra*). The new paradigm of this paper is that such problems become FPT for a novel, natural graph parameter based on *mapping* the graph to a tree, rather than decomposing the graph.

A novel parameter: stable gonality This novel multigraph parameter, based on “tree-likeness”, is the so-called *stable gonality* $\text{sgon}(G)$ of a multigraph G , introduced in [13, §3]. As we will briefly describe in Remark 2.5, the parameter originates in algebraic geometry – more precisely, the theory of Riemann surfaces – where a similar construction has been used since the 19th century; the analogy between Riemann surfaces and graphs stretches much further, as seen, for example, in [3] and [4]. The basic idea is to replace the use of tree *decompositions* of a graph G by graph *morphisms* from G to trees, and to replace the “width” of the decomposition by the “degree” of the morphism, where lower degree maps correspond to less complex graphs. For example, graphs of stable gonality 1 are trees [7, Example 2.13], those of stable gonality 2 are so-called hyperelliptic graphs, i.e., graphs that admit, after refinement, a graph automorphism of order two such that the quotient graph is a tree; this is decidable in quasilinear time [7, Thm. 6.1]. There are two technicalities that we will explain in the body of the text, but ignore for now: (a) in order to be able to define a notion of “degree” of a graph morphism, one needs to assume that the map is *harmonic* for a certain choice of edge weights; (b) it turns out that the most useful parameter occurs by further allowing refinements of the graph (this explains the terminology “stable”, a loanword from algebraic geometry). A detailed definition with intuition and examples is given in Section 2.2.

It has been shown that $\text{tw}(G) \leq \text{sgon}(G)$ [16, §6] and $\text{sgon}(G) \leq (c(G) + 3)/2$, where $c(G)$ is the cyclomatic number of G [13, Thm. 5.7], that $\text{sgon}(G)$ is computable, and NP-

complete [23, 24]. One attractive point of stable gonality as parameter for weighted problems stems from the fact that it is sensitive to multigraph properties, whereas the treewidth of a multigraph equals that of the underlying simple graph. Just like for treewidth [12], lower bounds are known, depending on the Laplace spectrum and maximal degree of the graph, leading, for example, to lower bounds for the stable gonality of expanders linear in the number of vertices [13, Cor. 6.10].

Three sample problems As stated above, the goal of this paper is to show that certain problems that are hard in bounded treewidth become easy (in fact, FPT) for stable gonality. In this introduction, we discuss three examples (one about orientation, one about flow, and one on capacitated domination), but in the body of the paper we consider many more related problems. We always assume that integers are given in unary.

An orientation problem A typical orientation problem is the following.

MINIMUM MAXIMUM OUTDEGREE (cf. Szeider [29])

Given: Undirected weighted graph $G = (V, E, w)$ with a weight function $w: E \rightarrow \mathbb{Z}_{>0}$; integer r

Question: Is there an orientation of G such that for each $v \in V$, the total weight of all edges directed out of v is at most r ?

This and related problems naturally concern weighted graphs, and we define their stable gonality in terms of an associated multigraph: given an undirected weighted graph $G = (V, E, w)$, we have an associated (unweighted) multigraph \tilde{G} , with the same vertex set, but where each simple edge $e = uv$ in G is replaced by $w(e)$ parallel edges between the vertices u and v . We call the stable gonality of the associated multigraph \tilde{G} the stable gonality of the weighted graph G , and denote it by $\text{sgon}(G) := \text{sgon}(\tilde{G})$.

A flow problem To describe a typical flow problem, we first recall some notions from the theory of network flow. A *flow network* is a *directed* graph $D = (N, A)$ with for each arc $e \in A$ a capacity $c(e) \in \mathbb{Z}_{>0}$, and two nodes s (source) and t (target) in N . Given a function $f: A \rightarrow \mathbb{Z}_{\geq 0}$ and a node v , we call $\sum_{uv \in A} f(uv)$ the *flow to* v and $\sum_{vw \in A} f(vw)$ the *flow out of* v . We say f is an *s-t-flow* if for each arc $a \in A$, the flow over the arc is nonnegative and at most its capacity (i.e., $0 \leq f(a) \leq c(a)$), and for each node $v \in N \setminus \{s, t\}$, the flow conservation law holds: the flow to v equals the flow out of v . The *value* $\text{val}(f)$ of a flow is the flow out of s minus the flow to s . A flow f is a *circulation* if it has value $\text{val}(f) = 0$, which is equivalent to flow conservation also holding at s and t .

UNDIRECTED FLOW WITH LOWER BOUNDS

Given: Undirected graph $G = (V, E)$, for each edge $e \in E$ a positive integer capacity $c(e) \in \mathbb{Z}_{>0}$ and a non-negative integer lower bound $\ell(e) \in \mathbb{Z}_{\geq 0}$, vertices s (source) and t (target), a non-negative integer $R \in \mathbb{Z}_{>0}$ (value)

Question: Is there an orientation of G such that the resulting directed graph D allows an *s-t-flow* f that meets capacities and lower bounds (i.e., $\ell(a) \leq f(a) \leq c(a)$ for all arcs in D), with value R ?

This is Problem [ND37] in the classical reference Garey and Johnson [22] (in that reference it is required that $\text{val}(f) \geq R$ rather than $\text{val}(f) = R$, but the problems are seen to be of the same complexity by adding a new target vertex t' with edge tt' . For the transformation in one direction, set $\ell(tt') = c(tt') = R$; in the other direction, set $\ell(tt') = R$ and choose $c(tt')$ sufficiently large, e.g., equal to the sum of the weights of all edges incident to t in G , cf. [26].)

The corresponding problem on a *directed* graph is solvable in polynomial time.

A capacitated problem Capacitated version of classical graph problems concern imposing a limitation on the available “resources”, placing them closer to real-world situations. Our final type of problem is a capacitated version of DOMINATING SET (to find out whether there is a set of vertices of size $\leq k$ that is connected to all vertices in the graph), that can be viewed as an abstract form of facility location questions.

CAPACITATED DOMINATING SET

Given: Undirected graph $G = (V, E)$, for each vertex $v \in V$ a positive integer capacity $c(v) \in \mathbb{Z}_{>0}$, integer k

Question: Is there a set $D \subset V$ of size $|D| \leq k$ and a function $f: V \setminus D \rightarrow D$ such that $vf(v) \in E$ for all $v \in V \setminus D$ and $|f^{-1}(v)| \leq c(v)$ for all $v \in D$?

Complexity classes To make the (parameterized) hardness of these problems more precise, we use the complexity class XNLP parameterized by a graph parameter k , first considered by Elberfeld, Stockhusen and Tantau in [19]; this is the class of problems that can be solved non-deterministically in time $O(f(k)n^c)$ ($c \geq 0$) and space $O(f(k) \log(n))$ where n is the input size and f is a computable function. More familiar is the W-hierarchy of Downey and Fellows up to parameterised reduction [15, 13.3], where $W[0]$ is FPT, and $W[1]$ is a class for which INDEPENDENT SET is complete (parameterized by the size of the set). We presently note that XNLP-hardness implies $W[t]$ -hardness for all t (for a given parameter) [9, Lemma 2.2].

Main result: hard problems for treewidth become easy for stable gonality

Theorem 1.1. MINIMUM MAXIMUM OUTDEGREE, UNDIRECTED FLOW WITH LOWER BOUNDS and CAPACITATED DOMINATING SET are XNLP-complete for pathwidth, and XNLP-hard for treewidth (given a path or tree decomposition realising the path- or treewidth), but are FPT for stable gonality (given a refinement and graph morphism from the associated multigraph to a tree realising the stable gonality).

Unparameterized UNDIRECTED FLOW WITH LOWER BOUNDS has been known since 1977 to be NP-complete in the strong sense, cf. Itai [26, Theorem 4.1], [22, p. 216]). Unparameterized DOMINATING SET is $W[2]$ -complete for the size of the dominating set [15, Theorem 13.28], and FPT for treewidth [15, Theorem 7.7]. The $W[1]$ -hardness of MINIMUM MAXIMUM OUTDEGREE for treewidth was proven by Szeider [29] and of CAPACITATED DOMINATING SET by Dom, Lokshtanov, Saurabh and Villanger [17].

XNLP-completeness of CAPACITATED DOMINATING SET for pathwidth in the main theorem is not due to us, but was proven very recently by Bodlaender, Groenland and Jacob building upon our results, see [8, Theorem 8].

As far as we know, our proof is the first parameterized-hardness result for **UNDIRECTED FLOW WITH LOWER BOUNDS**. We prove the **XNLP**-hardness for pathwidth (and hence, for treewidth) by reduction from **ACCEPTING NON-DETERMINISTIC CHECKING COUNTER MACHINE** from [9]. see Section 6.

For proving FPT under stable gonality, we revive an older idea of Seese on tree-partite graphs and their widths [28]; in contrast to the tree decompositions used in defining treewidth, we partition the original graph vertices into *disjoint* sets (‘bags’) labelled by vertices of a tree, such that adjacent vertices are in the same bag or in bags labelled by adjacent vertices in the tree. Seese introduced *tree partition width* to be the maximal size of a bag in such any partition. We consider weighted graphs and define a new parameter, *breadth*, given as the maximum of the bag size and the sum of the weights of edges between adjacent bags; cf. Subsection 2.2 below, in particular, Figure 2 for a schematic illustration. Taking the minimal breadth over all tree partitions gives a new graph parameter, that we call *treebreadth*. This allows us to divide the proof in two parts: (a) show that, given a graph morphism from the associated multigraph to a tree, one can compute in polynomial time a tree partition of the weighted graph of breadth upper bounded by the stable gonality of the associated multigraph — see Theorem 2.9; (b) provide an FPT-algorithm, given a tree partition of bounded breadth. The very general intuition, to be made precise in the detailed proofs, is that the tree that we map the graph to (or that labels the bags in the tree partition) “structures” the algorithm by consecutively running bounded algorithms over the pre-images of individual vertices and edges in the tree, using dynamical programming and integer linear programming to control extension of partial solutions to the entire graph. We mix this with classical tools for equivalence of various flow problems, and Edmonds’ algorithm for matching. By reductions, the two algorithms we specify are described in the following theorem for the indicated parameters.

Theorem 1.2. *MINIMUM MAXIMUM OUTDEGREE is FPT for treebreadth (given a partition tree realising the treebreadth), and CAPACITATED DOMINATING SET is FPT for tree partition width (given a tree partition with bounded width).*

We underline that Seese’s original tree partition width suffices for the capacitated problem, by an argument described at the start of Subsection 5.2.

The algorithm for **MINIMUM MAXIMUM OUTDEGREE** is given in Section 4 (in fact, for the closely related problem **OUTDEGREE RESTRICTED ORIENTATION**, in which outdegree is required to belong to a given interval instead of not exceeding a given value), and the algorithm for **CAPACITATED DOMINATING SET** is given in Section 5.

In a related direction, we may weaken the complexity class but increase the strength of the parameter. Here, we prove the following, by reduction from **BIN PACKING** [27].

Theorem 1.3. *MINIMUM MAXIMUM OUTDEGREE and UNDIRECTED FLOW WITH LOWER BOUNDS are $W[1]$ -hard for vertex cover number.*

Further problems In Section 2.3, we define some more related circulation problems (sometimes used as intermediaries in our arguments) and show that they have the same properties; these concern finding orientations on weighted graphs that make the weights define a circulation; or with the outdegree belonging to a given set, having a given value,

or not exceeding a given value; and finding an s - t -flow on a given directed graph for which all non-zero values on arcs match the capacity exactly; and, finally, a coloured version of CAPACITATED DOMINATING SET.

2 Preliminaries

2.1 Conventions and notations

We will consider *multigraphs*, where we allow for parallel edges and self-loops. Said otherwise, a multigraph $G = (V, E)$ consists of a finite set V of vertices, as well as a finite multiset E of unoriented (unweighted) edges, i.e., a set of pairs of (possibly equal) vertices, with finite multiplicity on each such pair. We denote such an edge between vertices $u, v \in V$ as uv . For $v \in V$, E_v denotes the edges incident with v , and for two subsets $X, Y \subset V$, $E(X, Y)$ is the collection of edges from any vertex in X to any vertex in Y .

We also consider *weighted simple graphs*, where edges have positive integer weights. We will make repeated use of the correspondence between integer weighted simple graphs and multigraphs given by replacing every edge with weight k by k parallel edges.

All graphs we consider are connected. For convenience, we use the terminology “vertex” and “edge” for undirected graphs, and “arc” and “node” for either directed graphs, or for trees that occur in graph morphisms or tree partitions.

We write \mathbb{Z} for all integers, with unique subscripts indicating ranges (so $\mathbb{Z}_{>0}$ is the positive integers and $\mathbb{Z}_{\geq 0}$ the non-negative integers). We use interval notation for sets of integers, e.g., $[2, 5] = \{2, 3, 4, 5\}$.

2.2 Stable gonality and treebreadth

Stable gonality The notion of stable gonality of a multigraph is a measure of tree-likeness of a multigraph defined using the minimal “degree” of a *map* to a tree, rather than the more conventional “decomposition” in terms of trees. As usual, a *graph homomorphism* between two loopless multigraphs G and H , denoted $\phi: G \rightarrow H$, is a map of vertices that respects the incidences given by the edges, i.e., it consists of two (not necessarily surjective) maps $\phi: V(G) \rightarrow V(H)$ and $\phi: E(G) \rightarrow E(H)$ such that $\phi(uv) = \phi(u)\phi(v) \in E(H)$ for all $uv \in E(G)$. One would like to define the “degree” of such a graph homomorphism as the number of pre-images of any vertex or edge, but in general, this obviously depends on the chosen vertex or edge. However, by introducing certain weights on the edges via an additional index function, we get a large collection of “indexed” maps for which the degree can be defined as the sum of the indices of the pre-image of a given edge, as long as the indices satisfy a certain condition of “harmonicity” above every vertex in the target. We make this precise in the following definition.

Definition 2.1. A *finite morphism* ϕ between two loopless multigraphs G and H consists of a graph homomorphism $\phi: G \rightarrow H$ (denoted by the same letter), and an index function $r: E(G) \rightarrow \mathbb{Z}_{>0}$ (hidden from notation). The *index* of $v \in V(G)$ in the direction of

$e \in E(H)$, where e is incident to $\phi(v)$, is defined by

$$m_e(v) := \sum_{\substack{e' \text{ incident to } v, \\ \phi(e')=e}} r(e').$$

We call ϕ *harmonic* if this index is independent of the direction $e \in E(H)$ for any given vertex $v \in V(G)$. We call this simply the *index of v* , and denote it by $m(v)$. The *degree* of a finite harmonic morphism ϕ is

$$\deg(\phi) = \sum_{\substack{e' \in E(G), \\ \phi(e')=e}} r(e') = \sum_{\substack{v' \in V(G), \\ \phi(v')=v}} m(v').$$

where $e \in E(H)$ is any edge and $v \in V(H)$ is any vertex. Since ϕ is harmonic, this number does not depend on the choice of e or v , and both expressions are indeed equal.

The second ingredient in the definition of stable gonality is that of a refinement.

Definition 2.2. Let G be a multigraph. A *refinement* of G is a graph obtained using the following two operations iteratively finitely often:

- add a leaf (i.e., a vertex of degree one),
- subdivide an edge.

Definition 2.3. Let G be a multigraph. The *stable gonality* of G is

$$\text{sgon}(G) = \min\{\deg(\phi) \mid \phi: H \rightarrow T, \text{ a finite harmonic morphism, where} \\ H \text{ is a loopless refinement of } G \text{ and} \\ T \text{ is a tree}\}.$$

Example 2.4. Two examples are found in Figure 1. The left-hand side illustrates the need for an index function (the middle edge needs label 2), and the right hand side shows the effect of subdivision (without subdivision, the minimal degree to a tree is the total number of edges). Since the top multigraphs are not trees, and trees are precisely the multigraphs having stable gonality 1 ([7, Example 2.13]), we conclude that they both have stable gonality two.

Remark 2.5. The notion is similar to that of the gonality of a compact Riemann surface (or smooth projective algebraic curve), which is defined as the minimal degree of a non-constant holomorphic map to the Riemann sphere (or the projective line; the unique Riemann surface of first Betti number 0, so the analogue of a tree). The need for refinements in the definition of stable gonality of graphs reflects the fact that there are infinitely many trees, whereas there is only one compact Riemann surface of first Betti number 0.

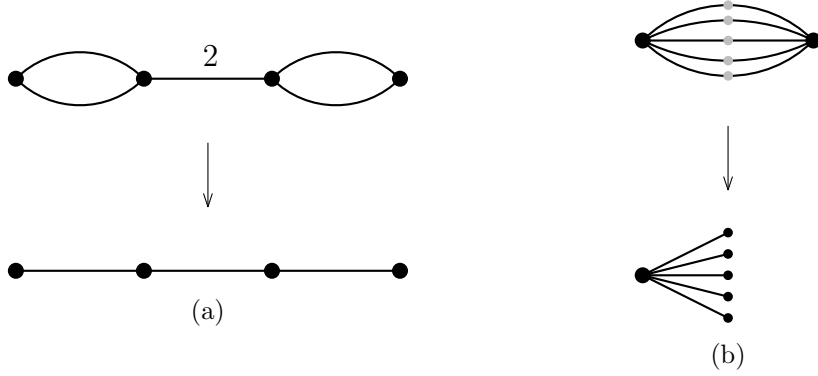


Figure 1: Two examples of a finite harmonic morphisms of degree 2. The edges without label have index 1. The small grey vertices represent refinements of the graph.

Tree partitions and their breadth The existence of a harmonic morphism to a tree imposes a special structure on the graph that we can exploit in designing algorithms. To capture this structure, we define the “breadth” of *tree partitions* of weighted graphs. The notion resembles that of “tree-partite graphs”, introduced by Seese [28]. The idea is to *partition* a (weighted) graph according to an index set given by the vertices of a tree, and use the incidence relations on the tree to define an associated measure for the graph.

Definition 2.6. A *tree partition* \mathcal{T} of a weighted graph $G = (V, E, w)$ is a pair

$$\mathcal{T} = (\{X_i \mid i \in I\}, T = (I, F))$$

where X_i are subsets of the vertex set V and $T = (I, F)$ is a tree, such that $\{X_i \mid i \in I\}$ forms a partition of V (i.e., for each $v \in V$, there is exactly one $i \in I$ with $v \in X_i$); and adjacent vertices are in the same set X_i or in sets corresponding to adjacent nodes (i.e., for each $uv \in E$, there exists an $i \in I$ such that $\{u, v\} \subseteq X_i$ or there exists $ij \in F$ with $\{u, v\} \subseteq X_i \cup X_j$).

The *breadth* of a partition tree \mathcal{T} of G is defined as

$$b(\mathcal{T}) := \max\{|X_i|, |E(X_j, X_k)| \mid i \in I, jk \in F\},$$

with

$$|E(X_j, X_k)| = \sum_{e \in E(X_j, X_k)} w(e)$$

the weighted number of edges connecting vertices in X_j to vertices in X_k .

We refer to Figure 2 for a schematic view of a tree partition with weights and bounded breadth.

If we have a tree partition of a weighted graph G using a tree T , for convenience we will call the vertices of T *nodes* and the edges of T *arcs*. We call the sets X_i *bags*. Observe that in a tree partition of breadth k , if there are more than k parallel edges between two vertices u and v , then u and v will be in the same bag.

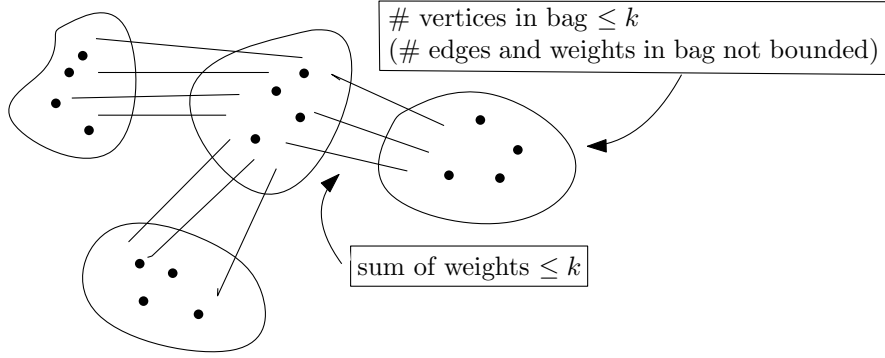


Figure 2: Schematic representation of a tree partition of a graph of breadth $\leq k$

Definition 2.7. The *treebreadth* $\text{tb}(G)$ of a weighted graph G is the minimum breadth of a tree partition of G . The *stable treebreadth* $\text{stb}(G)$ of a graph G is the minimum treebreadth of any refinement of G .

Remark 2.8. By contrast to tree partitions, a *tree decomposition* of a simple graph G is a pair $\mathcal{T} = (\{X_i \mid i \in I\}, T = (I, F))$ such that $\bigcup X_i = V$ but the X_i are not necessarily disjoint, where for every $uv \in E$, $\{u, v\} \subseteq X_i$ for some $i \in I$, and for any $v \in V$ the set of vertices $i \in I$ such that $v \in X_i$ induces a connected subtree of T . The width of \mathcal{T} is $\max\{|X_i| - 1 \mid i \in I\}$, and *treewidth* is the minimal width of all tree decompositions. For *pathwidth*, one furthermore insists that T is a path graph.

In Seese's work [28], the structure/weights of edges between bags does not contribute to the total width; Seese's tree-partition-width $\text{tpw}(G)$ of a simple graph G , defined as the minimum over all tree partitions of G of the maximum bag size in the tree partition, is thus a lower bound for the treebreadth $\text{tb}(G)$ (in particular, for sgon , cf. infra). For any G , $\text{tpw}(G)$ is lower bounded in terms of $\text{tw}(G)$, but also upper bounded in terms of $\text{tw}(G)$ and the maximal degree in G , cf. [30].

From morphisms to tree partitions There is a direct relation between the existence of a finite harmonic morphism ϕ of some degree k from a multigraph to a tree, and the existence of a partition tree of breadth k for the associated weighted simple graph. The basic idea is to use the pre-images of vertices in T as partitioning sets, but this needs some elaboration.

Theorem 2.9. Suppose G is a weighted simple graph, and $\phi: H \rightarrow T$ is a finite harmonic morphism of degree $\deg(\phi) = k$, where H is a loopless refinement of the multigraph corresponding to G and T is a tree. Then one can construct in time $O(k \cdot |V(T)|)$ a tree partition $\mathcal{T} = (X, T')$ for a subdivision of G such that

1. $\text{b}(\mathcal{T}) \leq k$, and
2. $|V(T')| \leq 2|V(G)|$.

In particular, $\text{stb}(G) \leq \text{sgon}(G)$.

Proof. Construct a tree partition $\mathcal{T} = (X, T)$ as follows. For every node $t \in V(T)$, define $X_t = \phi^{-1}(t) \cap V(G)$. For every edge $uv \in E(G)$, do the following. Let $i \in V(T)$ be such that $u \in X_i$ and let $j \in V(T)$ be such that $v \in X_j$. Let $i, t_1, t_2, \dots, t_l, j$ be the path between i and j in T . Subdivide the edge uv into a path $u, s_1, s_2, \dots, s_l, v$ and add the vertex s_r to the set X_{t_r} for each r . Notice that this indeed yields a tree partition.

We claim that this tree partition has breadth at most k . Let $t \in V(T)$, and suppose that $s_i \in X_t$ is a subdivision vertex of an edge $uv \in E(G)$. The edge uv is subdivided in H as well (if $w(uv) > 1$, there are parallel edges uv in the corresponding multigraph, so all such edges are subdivided; pick any), and the path from u to v is mapped by ϕ to a walk from $\phi(u)$ to $\phi(v)$. By definition t is in the unique path from $\phi(u)$ to $\phi(v)$, so every walk from $\phi(u)$ to $\phi(v)$ contains t . It follows that there is some subdivision vertex in H that is mapped to t . We conclude that $|X_t| \leq |\phi^{-1}(t)| \leq k$. The argument for the edges is analogous. We conclude that the breadth of this tree partition is at most k .

Now we will change the tree partition slightly to ensure that the number of nodes is at most $2|V(G)|$. First remove all vertices from T for which $X_t = \emptyset$. Notice that the resulting graph T' will still be a tree. Moreover, for every leaf t of T' the set X_t will contain a vertex of G . For every degree 2 vertex t of T' for which X_t does not contain a vertex of $V(G)$, contract t with one of its neighbours, and contract all vertices in X_t with a neighbour as well. The number of nodes in the resulting tree is at most $2|V(G)|$ since every node either contains a vertex of G or has degree at least 3.

The runtime analysis is as follows. For every edge $uv \in E(G)$, we can find the (unique) shortest path between $\phi(u)$ and $\phi(v)$ in T in time the length of the path. We can subdivide the edge uv and add the subdivisions to the corresponding sets X_t in time linear in the length of this path. We conclude that the total construction can be done in the sum of the lengths of the paths, which is upper bounded by $k \cdot |V(T)|$ time. \square

Remark 2.10. Theorem 2.9 can also be shown using the results of [24, Lemma 6.5]. The properties of the morphism considered in that paper are stronger than the ones we need here, which allows one to obtain a tree partition with at most $|V(G)|$ nodes.

Example 2.11. For the multigraphs in Figure 1, the constructed tree partitions have breadth equal to the stable gonality: for (a), each vertex forms an individual bag, and bags are connected by edges of weight 2, leading to a total breadth of 2; for (b), the constructed tree partition consists of one bag containing both non-subdivision vertices, and no edges, again leading to a breadth of 2.

The main application of the above result is the following reduction for the proof of one part of the main theorem.

Corollary 2.12. *To prove that a weighted graph problem is FPT for sgon (given a morphism of a refinement of the corresponding multigraph to a tree whose degree realizes the stable gonality), it suffices to prove that it is FPT for the breadth of a given tree-partition of a subdivision of the weighted graph.*

2.3 Problem definitions

In this subsection, we give the formal definitions of the flow, orientation, and capacitated problems studied in this paper. We always assume that integers are given in unary.

We consider five (strongly related) problems that ask for orientations of a weighted undirected graph.

MINIMUM MAXIMUM OUTDEGREE (MMO)

Given: Undirected weighted graph $G = (V, E, w)$ with a weight function $w : E \rightarrow \mathbb{Z}_{>0}$; integer r

Question: Is there an orientation of G such that for each $v \in V$, the total weight of all edges directed out of v is at most r ?

MMO was shown to be W[1]-hard for treewidth by Szeider [29].

CIRCULATING ORIENTATION (CO)

Given: Undirected weighted graph $G = (V, E, w)$ with a weight function $w : E \rightarrow \mathbb{Z}_{>0}$

Question: Is there an orientation of the edges such that for all $v \in V$, the total weight of all edges directed to v equals the total weight of all edges directed from v ?

OUTDEGREE RESTRICTED ORIENTATION (ORO)

Given: Undirected weighted graph $G = (V, E, w)$ with a weight function $w : E \rightarrow \mathbb{Z}_{>0}$; for each vertex $v \in V$, an interval $D_v \subseteq \mathbb{Z}_{\geq 0}$

Question: Is there an orientation of G such that for each $v \in V$, the total weight of all edges directed out of v is an integer in D_v ?

TARGET OUTDEGREE ORIENTATION (TOO)

Given: Undirected weighted graph $G = (V, E, w)$ with a weight function $w : E \rightarrow \mathbb{Z}_{>0}$; for each vertex $v \in V$, an integer d_v

Question: Is there an orientation of G such that for each $v \in V$, the total weight of all edges directed out of v equals d_v ?

CHOSEN MAXIMUM OUTDEGREE (CMO)

Given: Undirected weighted graph $G = (V, E, w)$ with a weight function $w : E \rightarrow \mathbb{Z}_{>0}$; for each vertex $v \in V$, an integer m_v

Question: Is there an orientation of G such that for each $v \in V$, the total weight of all edges directed out of v is at most m_v ?

We also consider two (variants of) classical graph flow problems, the first of which is couched using orientations.

UNDIRECTED FLOW WITH LOWER BOUNDS (UFLB)

Given: Undirected graph $G = (V, E)$, for each edge $e \in E$ a positive integer capacity $c(e) \in \mathbb{Z}_{>0}$ and a non-negative integer lower bound $\ell(e) \in \mathbb{Z}_{\geq 0}$, vertices s (source) and t (target), a non-negative integer $R \in \mathbb{Z}_{\geq 0}$ (value)

Question: Is there an orientation of G such that the resulting directed graph D allows an s - t -flow f that meets capacities and lower bounds (i.e., $\ell(a) \leq f(a) \leq c(a)$ for all arcs in D), with value R ?

ULFB was shown to be strongly NP-complete by Itai [26, Theorem 4.1].

ALL-OR-NOTHING FLOW (AoNF)

Given: Directed graph $G = (V, E)$, for each arc a positive capacity $c(e)$, vertices s, t , positive integer R

Question: Is there a flow f from s to t with value R such that for each arc $e \in E$, $f(e) = 0$ or $f(e) = c(e)$?

An NP-completeness proof of ALL OR NOTHING FLOW is given in [2].

Finally, we consider a coloured and uncoloured capacitated version of DOMINATING SET.

CAPACITATED DOMINATING SET (CDS)

Given: Undirected graph $G = (V, E)$, for each vertex $v \in V$ a positive integer capacity $c(v) \in \mathbb{Z}_{>0}$, integer k

Question: Is there a set $D \subset V$ of size $|D| \leq k$ and a function $f: V \setminus D \rightarrow D$ such that $vf(v) \in E$ for all $v \in V \setminus D$ and $|f^{-1}(v)| \leq c(v)$ for all $v \in D$?

CDS was shown to be $W[1]$ -hard for treewidth in [17], even when restricted to planar graphs [10]. Recently, it was shown that the problem is XNLP-complete for pathwidth [8, Theorem 8]. The usefulness of the following auxiliary coloured version of the problem was first pointed out in [21].

CAPACITATED RED-BLUE DOMINATING SET (CRBDS)

Given: Undirected bipartite graph $G = (V = R \sqcup B, E)$, for each “red” vertex $v \in R$ a positive integer capacity $c(v) \in \mathbb{Z}_{>0}$, integer k

Question: Is there a set $D \subset R$ of size $|D| \leq k$ and a function $f: B \rightarrow D$ such that $bf(b) \in E$ and $|f^{-1}(v)| \leq c(v)$ for all $v \in D$?

3 Transformations between orientation and flow problems

In this section, we give a number of relatively simple transformations between seven of the main problems that we study in this paper. A summary of the transformations can be found in Figure 3. In Section 4, we show that ORO is fixed parameter tractable when parameterized by the stable gonality. In Section 6, we show that AoNF is XNLP-hard when parametrized by pathwidth and that TOO is $W[1]$ -hard when parameterized by the vertex cover number. The algorithms and hardness results for the other problems follow from these reductions.

First, a number of these problems can be seen as special cases of others:

- **TARGET OUTDEGREE ORIENTATION** is the special case of **OUTDEGREE RESTRICTED ORIENTATION** by taking each interval $D_v = \{d_v\} = [d_v, d_v]$ a singleton.
- **CHOSEN MAXIMUM OUTDEGREE** is the special case of **OUTDEGREE RESTRICTED ORIENTATION** by taking each interval $D_v = [0, m_v]$ starting at 0.
- **MINIMUM MAXIMUM OUTDEGREE** is the special case of **CHOSEN MAXIMUM OUTDEGREE** where all values m_v are equal to r .

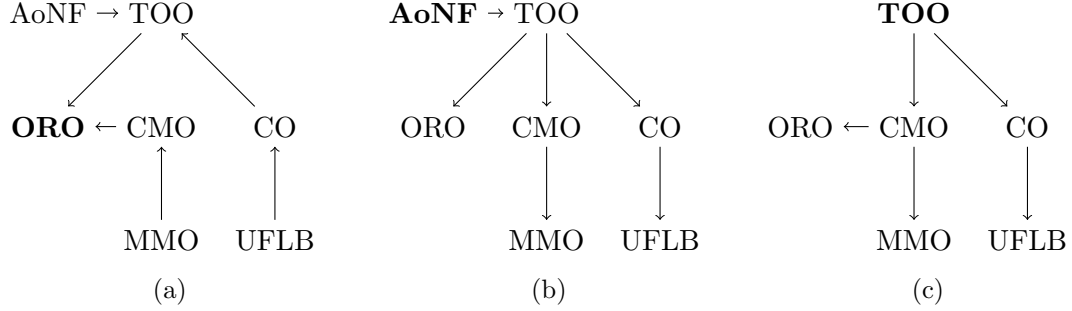


Figure 3: Transformation between different problems with respect to parameter (a) tree-breadth or stable gonality, for which ORO is FPT, (b) pathwidth, for which AoNF is XNLP-complete, and (c) vertex cover number, for which TOO is W[1]-hard.

- **CIRCULATING ORIENTATION** is the special case of **TARGET OUTDEGREE ORIENTATION** where for each vertex v , we set $d_v = \frac{1}{2} \deg(v)$, with $\deg(v)$ equal to the sum of all weights of edges incident to v .
- **CIRCULATING ORIENTATION** is the special case of **UNDIRECTED FLOW WITH LOWER BOUNDS** where we set for each edge the lower bound equal to its capacity, and the target value of the flow to 0.

3.1 From ALL-OR-NOTHING FLOW to TARGET OUTDEGREE ORIENTATION

We now show how to transform instances of **ALL-OR-NOTHING FLOW** to equivalent instances of **TARGET OUTDEGREE ORIENTATION**, see Lemma 3.2. The transformation increases the pathwidth of the graph by at most one, and does not change the stable gonality of the graph.

This transformation has two consequences. On the positive side, when we have bounded stable gonality, we can use the algorithm for **OUTDEGREE RESTRICTED ORIENTATION** from Section 4; on the negative side, combined with Theorem 6.1, this shows that **TARGET OUTDEGREE ORIENTATION** with pathwidth as parameter is XNLP-hard.

Suppose we are given an instance (G, s, t, c, R) of **ALL-OR-NOTHING FLOW**. The transformation uses a number of steps.

First, we subdivide all arcs, with both subdivisions having the same capacity. Note that this step increases the pathwidth by at most one, and does not increase the stable gonality. (However, the step can increase the size of a minimum vertex cover.) Also note that this creates an equivalent instance of **ALL-OR-NOTHING FLOW**.

We call the resulting graph G^1 , and denote the capacity function again by c .

In the second step, we create an equivalent instance of **TARGET OUTDEGREE ORIENTATION** except that we allow weights of edges to be a multiple of $1/2$.

Let $G^2 = (V^2, E^2)$ be the underlying undirected graph of G^1 ; i.e., G^2 is obtained by dropping all directions of arcs in G^1 . Note that G^2 is a simple graph, as a result of the step where we subdivided all arcs.

If uv is an arc in G^1 with capacity $c(uv)$, then the weight of the undirected edge uv in G^2 is $w(uv) = c(uv)/2$.

Let $\text{IN}_{G^2}(v)$ be the set of edges in G^2 whose originating arc was directed towards v , i.e., all edges uv in G^2 with uv an arc in G^1 . Let $\text{OUT}_{G^2}(v)$ be the set of edges in G^2 whose originating arc was directed out of v , i.e., all edges uv in G^2 with vu an arc in G^1 .

For each vertex $v \in V^2$, we compute the target outdegree value d_v , as follows.

- Set β_v to be the sum of the capacities of all arcs that are directed towards v in G^1 .
- Now, if $v \in V \setminus \{s, t\}$, then set $d_v = \beta_v/2$. Set $d_s = R/2 + \beta_s/2$, and $d_t = -R/2 + \beta_t/2$.

Lemma 3.1. *There is an all-or-nothing flow in G^1 from s to t with value R , if and only if there is an orientation of G^2 such that for each vertex v , the total weight of all outgoing edges of v is equal to d_v .*

Proof. Suppose that we have an all-or-nothing flow f in G^1 from s to t with value R . Consider an arc xy in G^1 . If f sends a positive amount of flow over this arc (and thus, flow equal to the capacity), then orient the edge xy from x to y . Otherwise, 0 flow is sent over the arc, and we orient the edge xy from y to x , i.e., edges are oriented in the same direction as their corresponding arc when they have non-zero flow, and in the opposite direction as their corresponding arc when the flow over the arc is 0. (The step is inspired by the principle of cancelling flow.)

We now verify that this orientation meets the targets. First, consider a vertex $v \in V \setminus \{s, t\}$. Suppose the total inflow to v by f is δ_v . Then, the total outflow from v by f is also δ_v , due to flow conservation. We start by looking at the edges from $\text{IN}_{G^2}(v)$. The total capacity of all arcs that send flow to v is δ_v , so the total weight of all edges from $\text{IN}_{G^2}(v)$ that are directed towards v equals $\delta_v/2$. The total weight of all edges from $\text{IN}_{G^2}(v)$ equals $\beta_v/2$. So, the total weight of all edges from $\text{IN}_{G^2}(v)$ that are directed out of v equals $(\beta_v - \delta_v)/2$. Now, look at the edges in $\text{OUT}_{G^2}(v)$. The total capacity of all arcs that send flow out from v is again δ_v , so the total weight of all edges in $\text{OUT}_{G^2}(v)$ that are directed out of v equals $\delta_v/2$. Thus, the total weight of the edges that are directed out of v in the orientation equals $(\beta_v - \delta_v)/2 + \delta_v/2 = \beta_v/2$.

Now, consider s . Suppose f has inflow δ_s at s , and thus outflow $R + \delta_s$. Again, the total weight of edges from $\text{IN}_{G^2}(s)$ that are directed out of s equals $(\beta_s - \delta_s)/2$. The total weight of the edges from $\text{OUT}_{G^2}(s)$ equals $(R + \delta_s)/2$. So, the total weight of all outgoing arcs in the orientation is $R/2 + \beta_s/2$. The argument for t is similar.

Suppose that we have an orientation of G^2 such that each vertex has total weight of all edges directed out of v equal to d_v . If an edge is oriented in the same direction as its originating arc, then send flow over the arc equal to its capacity, otherwise, send 0 flow over that arc. Let f be the corresponding function. We claim that f is a flow in G^1 from s to t with value $R/2$.

Suppose the inflow by f to a vertex $v \in V$ equals ϵ_v . The total weight of edges in $\text{IN}_{G^2}(v)$ that are directed towards v thus is $\epsilon_v/2$. The total weight of all edges in $\text{IN}_{G^2}(v)$ equals $\beta_v/2$. Hence, the total weight of all edges in $\text{IN}_{G^2}(v)$ that are oriented out of v is $\beta_v/2 - \epsilon_v/2$. Hence, the total weight of all edges in $\text{OUT}_{G^2}(v)$ that are oriented out of v is $d_v - (\beta_v/2 - \epsilon_v/2)$. This value equals $\epsilon_v/2$ for $v \notin \{s, t\}$, $R/2 + \epsilon_v/2$ for $v = s$, and $-R/2 + \epsilon_v/2$ for $v = t$. The total amount of flow sent out of v by f is twice this number, so for $v \notin \{s, t\}$, the inflow equals the outflow, namely ϵ_v ; for $v = s$, the outflow is R larger than its inflow, and for $v = t$, the outflow is R smaller than its inflow. The result follows. \square

We now have shown that we can transform an instance of ALL-OR-NOTHING FLOW to an equivalent instance of TARGET OUTDEGREE ORIENTATION, except that in the latter problem, values are allowed to be a multiple of $\frac{1}{2}$. To obtain an instance of TARGET OUTDEGREE ORIENTATION with all values integral, we multiply all weights and outdegree targets by two.

Note that the undirected graph G^2 is obtained from the directed graph G by subdividing each edge, and then dropping directions of edges. As above, the operation increases the pathwidth by at most one, and does not increase the (stable) treebreadth and stable gonality, i.e. sgon of the undirected graph underlying G equals sgon of G^2 . One easily observes that the transformations can be carried out in polynomial time and logarithmic space. The latter is needed for XNLP-hardness proofs.

Lemma 3.2. *There is a parameterized log-space reduction from ALL-OR-NOTHING FLOW to TARGET OUTDEGREE ORIENTATION with respect to parameters pathwidth, (stable) treebreadth and stable gonality, which also transforms the associated given finite harmonic morphism from a refinement of the input of degree $\text{sgon}(G)$ to one of the new graph with the same degree or transforms the given tree partition of (a refinement of) G of breadth $(\text{stb})(G)$ to one of the new graph with the same breadth.*

3.2 From CHOSEN MAXIMUM OUTDEGREE to MINIMUM MAXIMUM OUTDEGREE

Szeider [29] gives a transformation from CHOSEN MAXIMUM OUTDEGREE to MINIMUM MAXIMUM OUTDEGREE. This reduction changes the graph in the following way: two additional vertices are added, as well as a number of edges — each new edge has at least one of the two new vertices as endpoint. Thus, the vertex cover number and the pathwidth of the graph is increased by at most two by this reduction.

Lemma 3.3 (Szeider [29]). *There is a parameterized log-space reduction from CHOSEN MAXIMUM OUTDEGREE to MINIMUM MAXIMUM OUTDEGREE for the parameterizations by pathwidth and by vertex cover number.*

3.3 From TARGET OUTDEGREE ORIENTATION to CIRCULATION ORIENTATION

Again, the following reduction is inspired by standard insights from network flow theory.

Lemma 3.4. *There is a parameterized log-space reduction from TARGET OUTDEGREE ORIENTATION to CIRCULATION ORIENTATION when parameterized by pathwidth or by vertex cover number.*

Proof. Let (G, w, d_v) be an instance of TARGET OUTDEGREE ORIENTATION. We turn the instance into a circulation, and do this with a construction that is classical in flow theory, see e.g. [1]. First we determine the demand of each vertex. If a vertex v has outdegree bound d_v , and total weight of incident edges r_v , then the indegree will be $r_v - d_v$, and this gives in a flow a demand of $r_v - 2d_v$. Let α be the sum of all positive demands, that is, $\alpha = \frac{1}{2} \sum_{v \in V(G)} |r_v - 2d_v|$. The construction is to add a supersource s , a supersink t , and an edge from t to s with weight α . Moreover, we add an edge with weight $r_v - 2d_v$ from s to

each vertex v with negative demand, and an edge with weight $r_v - 2d_v$ from each vertex v with positive demand to t . Call H the thus constructed graph.

Claim 3.5. *If G is a yes-instance of TARGET OUTDEGREE ORIENTATION, then H is a yes-instance of CIRCULATING ORIENTATION.*

Proof. Suppose that there is an orientation of G such that each vertex $v \in V(G)$ has weighted outdegree d_v . Then we can extend this orientation to a circulation of H by directing all edges vt from v to t , all edges sv from s to v and the edge st from t to s .

To show that this is indeed an orientation, we distinguish cases. For each vertex v with $r_v - 2d_v = 0$, the outdegree is d_v and the indegree is $r_v - d_v = d_v$. For each vertex v with $r_v - 2d_v > 0$, the outdegree is $d_v + r_v - 2d_v = r_v - d_v$, since the edge vt is oriented out of v , and the indegree is $r_v - d_v$, which indeed equals the outdegree. The case when $r_v - 2d_v < 0$ is similar. The in- and outdegree of s and t is α . \lrcorner

Claim 3.6. *If H is a yes-instance of CIRCULATING ORIENTATION, then G is a yes-instance of TARGET OUTDEGREE ORIENTATION.*

Proof. Suppose that there is an orientation of H such that each vertex $v \in V(H)$ has equal weighted in- and outdegree. Suppose that the edge st is oriented from t to s . Then t has outdegree α , so all edges vt are oriented towards t . Now consider a vertex v with $r_v - 2d_v > 0$. It follows that $\text{indeg}_G(v) = \text{outdeg}_G(v) + r_v - 2d_v$, where $\text{indeg}_G(v)$ and $\text{outdeg}_G(v)$ are the in- and outdegree of v restricted to the edges of G . Since $\text{indeg}_G(v) = r_v - \text{outdeg}_G(v)$, it follows that $\text{outdeg}_G(v) = d_v$. The case of vertices with $r_v - 2d_v \leq 0$ is similar. We conclude that restricting the orientation to G will give the desired outdegree for all vertices.

If the edge st is oriented from s to t , then flipping the orientation of all edges gives another circulating orientation, and the result follows as above. \lrcorner

Notice that this is a polynomial time and logarithmic space construction. Moreover, the pathwidth of H is at most $\text{tw}(G) + 2$, which can be seen by adding s and t to all bags of a path decomposition of G . Also, the vertex cover number of H is at most two more than the vertex cover number of G : if S is a vertex cover of G , then $S \cup \{s, t\}$ is a vertex cover of H . \square

3.4 From TARGET OUTDEGREE ORIENTATION to CHOSEN MAXIMUM OUTDEGREE

With a pigeonhole argument, we obtain a simple reduction from TARGET OUTDEGREE ORIENTATION to CHOSEN MAXIMUM OUTDEGREE.

Consider an instance of TARGET OUTDEGREE ORIENTATION, i.e., we are given an undirected graph $G = (V, E)$, for each edge $e \in E$ a positive integer weight $w(e)$, and for each vertex $v \in V$ a positive integer target value d_v . We have the following two simple observations.

Lemma 3.7. *If G has an orientation such that for each vertex $v \in V$ the sum of the weights of edges directed out of v equal to d_v , then $\sum_{e \in E} w(e) = \sum_{v \in V} d_v$.*

Proof. For a given orientation, each edge is directed out of exactly one vertex. \square

Lemma 3.8. *Suppose $\sum_{e \in E} w(e) = \sum_{v \in V} d_v$. For each orientation of G , we have that for each vertex $v \in V$ the total weight of edges directed out of v equals d_v , if and only if for each vertex $v \in V$ the total weight of edges directed out of v is at most d_v .*

Proof. Consider an orientation of G , and suppose that for each vertex $v \in V$ the total weight of edges directed out of v is at most d_v . If there is a vertex u for which the total weight of edges directed out of u is less than d_u , then the sum over all vertices $v \in V$ of the total weight of the edges directed out of v is less than $\sum_{v \in V} d_v$. But each edge $e \in E$ is counted once in this sum, hence $\sum_{v \in V} d_v < \sum_{e \in E} w(e)$, a contradiction.

The other direction is trivial. \square

Lemma 3.9. *There is a parameterized log-space reduction from TARGET OUTDEGREE ORIENTATION to CHOSEN MAXIMUM OUTDEGREE parameterized by pathwidth, vertex cover number, or stable gonality.*

Proof. The lemmas above show that we can use the following reduction: first, check whether $\sum_{e \in E} w(e) = \sum_{v \in V} d_v$. If not, then reject (or transform to a trivial no-instance); otherwise, set $m_v = d_v$ for each v . As G is not changed, pathwidth, vertex cover number and stable gonality are the same. \square

3.5 From UNDIRECTED FLOW WITH LOWER BOUNDS to CIRCULATING ORIENTATION

Lemma 3.10. *Suppose we have an instance of UNDIRECTED FLOW WITH LOWER BOUNDS with a tree partition of breadth at most k . Then we can build, in polynomial time and logarithmic space, an equivalent instance of CIRCULATING ORIENTATION with a tree partition of breadth $O(k^2)$.*

Proof. The transformation is done in four steps.

Suppose we are given an instance of UNDIRECTED FLOW WITH LOWER BOUNDS, i.e., an undirected graph $G = (V, E)$, capacity function $c : E \rightarrow \mathbb{Z}_{>0}$, lower bound function $\ell : E \rightarrow \mathbb{Z}_{\geq 0}$, vertices $s, t \in V$, and target flow value $R \in \mathbb{Z}_{\geq 0}$.

First, we turn the instance into an orientation problem, or, equivalently, an instance of UNDIRECTED FLOW WITH LOWER BOUNDS with flow value 0 (a circulation). This is done by adding an edge st with lower bound and capacity equal to R . Let G^1 be the resulting graph. (This step is skipped when $R = 0$.)

Claim 3.11. *There is a flow in an orientation of G from s to t with value R fulfilling lower bounds, if and only if there is a flow in an orientation of G^1 from s to t with value 0 fulfilling lower bounds.*

Proof. Suppose there is a flow f in an orientation of G from s to t with value R fulfilling lower bounds. Orient the new edge from t to s and send R flow over this edge. All other edges have their flow dictated by f . This gives the desired solution for G^1 .

Suppose there is a flow in an orientation of G^1 from s to t with value 0 fulfilling lower bounds. If the edge st is oriented from s to t , then reverse the direction of all arcs in the

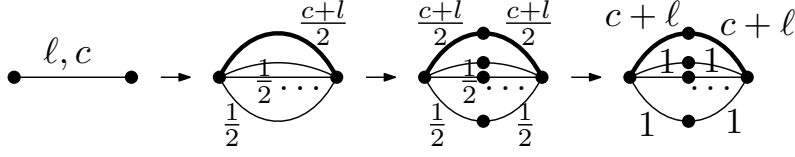


Figure 4: The second, third and fourth step of the transformation for a single edge

orientation, but send the same amount of flow over each edge (but now in the opposite direction). This gives an equivalent solution. We can now assume that the edge st is oriented from t to s . Deleting this edge and its flow gives the desired orientation and flow for G . \lrcorner

We now assume that $R = 0$, and thus, the flow we look for is a circulation. The vertices s and t no longer play a special role as flow conservation also holds for s and t .

The second, third and fourth step are illustrated in Figure 4, where the steps that are applied to a single edge are shown.

In the second step, we create an intermediate undirected graph with parallel edges. These edges have weights that are a multiple of $1/2$. This is done as follows. Suppose we have an edge e with capacity $c(e) > 0$ and lower bound $\ell(e) \geq 0$; $0 \leq \ell(e) \leq c(e)$. Now, replace e by the following parallel edges: one *heavy* edge of weight $(c(e) + \ell(e))/2$, and $c(e) - \ell(e)$ *light* edges of weight $1/2$. Let G^2 be the resulting multigraph.

Claim 3.12. *There is an integer flow with value 0 respecting lower bounds and capacities in an orientation of G^1 , if and only if there is a circulating orientation in G^2 .*

Proof. Suppose f is an integer flow with value 0, or equivalently, a circulation, that respects lower bounds and capacities in an orientation of G^1 .

For each edge $uv \in E$, suppose f send α units of flow from u to v in G^1 . Now, in G^2 , we orient the heavy edge (with weight $(c(e) + \ell(e))/2$) from u to v . Orient $(\alpha - \ell(e))$ of the light parallel edges (of weight $1/2$ between u and v) from u to v and all other of these light parallel edges from v to u . Thus, we have $c(e) - \ell(e) - (\alpha - \ell(e)) = c(e) - \alpha$ light edges directed from v to u .

Thus, the heavy edge sends $(c(e) + \ell(e))/2$ flow from u to v ; the light edges send $(\alpha - \ell(e))/2$ flow from u to v and $(c(e) - \alpha)/2$ flow from v to u . The net flow contribution that goes from u to v of all these edges adds up to

$$\frac{c(e) + \ell(e)}{2} + \frac{\alpha - \ell(e)}{2} - \frac{c(e) - \alpha}{2} = \alpha.$$

Thus, the flow directed by the constructed orientation of the multigraph is for each pair of adjacent vertices the same as the flow in f . As the latter is a circulation, the constructed orientation of the multigraph is also an orientation.

Suppose we have an orientation that gives a circulation in G^2 . Consider an edge $uv \in E$. Suppose the ‘heavy’ edge between u and v (with weight $(c(e) + \ell(e))/2$) is oriented from u to v . Then, in G^1 , we send flow in the direction from u to v . Suppose

γ light edges of weight $1/2$ are oriented from u to v ; thus $c(e) - \ell(e) - \gamma$ light edges (of weight $1/2$) are oriented from v to u . Now, send

$$\frac{c(e) + \ell(e)}{2} + \gamma \cdot \frac{1}{2} - (c(e) - \ell(e) - \gamma) \cdot \frac{1}{2} = \ell(e) + \gamma$$

flow from u to v over the edge uv in G^1 . Thus, the flow from u to v in G^1 equals the net flow from u to v over all parallel edges in G^2 (where flow from v to u cancels the same amount of flow from u to v , as usual in flow theory). Note that the flow sent from u to v is an integer in $[\ell(e), c(e)]$. As the difference of what is sent from u to v and what is sent from v to u in G^2 equals the amount sent in G^1 , flow conservation also holds for the flow in G^1 , and we again have a circulation. \lrcorner

The last two steps are relatively simple. In the third step, we turn the graph into a simple graph (without parallel edges), by subdividing each edge. When subdividing an edge, the two resulting edges get the same lower bound and capacity as the original edge. One easily sees that this step gives equivalent instances.

In the fourth step, we obtain an equivalent instance with only integral values by multiplying all capacities and lower bounds by two. Let G^4 be the resulting graph.

Using these four steps, we transformed an instance of UNDIRECTED FLOW WITH LOWER BOUNDS into an equivalent instance of CIRCULATING ORIENTATION.

Now, if we have a tree partition of G of breadth k , we can build a tree partition of a subdivision of G^4 of breadth at most $O(k)$ as follows. We first build a tree partition of G^1 . When s and t are in the same or adjacent bags, then we do not need to change the graph or tree partition. When s and t are not in the same or adjacent bags, then suppose the path in \mathcal{T} from the bag containing s to the bag containing t has q intermediate nodes. Subdivide the edge st q times, and place in each intermediate node of this path between the bags one of the subdivision nodes, in order. This increases the breadth of the tree partition by at most R . Now, note that if s and t are not in the same bag, then take the edges between the bag containing s and the neighbouring bag on the path in \mathcal{T} towards the bag containing t . These form an s - t cut of size at most the breadth of the tree partition. Hence, if $R > k$, we can reject (by the minimum cut maximum flow theorem.) So, the step increases the breadth by at most k .

We need to change the tree partition again when we subdivide the graph in the third step. First, consider edges between vertices in different bags. To accommodate the subdivisions of these edges, we subdivide each arc ij of \mathcal{T} and place the subdivisions of edges between a vertex in X_i and a vertex in X_j in this new bag. Each vertex in this new bag can be associated with an edge of weight at least one with one endpoints in X_i and X_j , thus these new bags have $O(k)$ vertices. Second, we may subdivide the edge st one or more additional times, such that each bag on the path from s to t contains exactly one subdivision vertex of the edge st . Third, for each adjacent pair of vertices $v, w \in X_i$ in the same bag, we add the subdivision vertex of the heavy edge between v and w in X_i , and for each light edge, we add an additional bag that is made incident to i . As there are $O(k^2)$ pairs of vertices in a bag, this step increases the breadth by $O(k^2)$. The result is a tree partition of the graph obtained in the third step, of breadth $O(k^2)$.

The fourth step does not change the graph. Doubling the weight of the edges can

double the breadth of the tree partition. Thus the result is a tree partition of breadth $O(k^2)$. \square

Remark 3.13. The above reduction is also a reduction with respect to the parameter sgon . Let (G, c, l, R) be an instance of **UNDIRECTED FLOW WITH LOWER BOUNDS** and $\phi: G' \rightarrow T$ a finite harmonic morphism of degree k . Recall that G' is a refinement of the multigraph corresponding to G . We obtain a morphism $\phi': G^{4l} \rightarrow T'$ with degree $O(4k)$ as follows. If $\phi(s) = \phi(t)$ subdivide the new edges from s to t once, and map the new vertices to unique new leaves. Otherwise, subdivide the edges from s to t into l edges, where l is the length of the $\phi(s), \phi(t)$ -path, and map those new vertices to the $\phi(s), \phi(t)$ -path. Refine the graph such that the morphism becomes harmonic again. This results in a morphism of degree $O(2k)$, as above. In step 3, when subdividing all edges, subdivide all edges of T as well. This does not change the degree of the morphism. Finally, multiplying all capacities by two means doubling all edges in the corresponding multigraph, and this results in a morphism of degree $O(4k)$.

4 An algorithm for OUTDEGREE RESTRICTED ORIENTATION for graphs with small (stable) treebreadth

In this section, we give our main result, and show that **OUTDEGREE RESTRICTED ORIENTATION** is fixed parameter tractable for graphs with small stable treebreadth, given a tree partition of a subdivision graph G' realizing the breadth.

Structure of the algorithm We give each edge in G' the same weight as it has in G ; if an edge e' resulted from subdividing edge e , then its weight is set to $w(e)$. If vertex x_e resulted from subdividing edge e , then we set $D_{x_e} = [w(e), w(e)]$.

With this setting of weights and targeted intervals, the problem on G' is equivalent to the problem on G . From this, it follows that we can assume that we have a tree partition of the input graph itself, of given breadth.

Claim 4.1. *There is an orientation of the edges in G with for each $v \in V$, the total weight of all edges directed out of v in D_v , if and only if there is such an orientation in G' .*

Proof. If we have the desired orientation in G , then orient each edge created by a subdivision in the same way as the original one. If we have the desired orientation in G' , then note that for each vertex created by a subdivision, one of its incident edges is incoming and one is outgoing. Thus, we can orient the original edge in the direction that all its subdivisions use. \lrcorner

Next, we add a new root vertex r to the tree partition, and set $X_r = \emptyset$.

After these preliminary steps, we perform a dynamic programming algorithm on the resulting tree partition \mathcal{T} , as follows.

For each arc a from a node to its parent, we compute a table A_a . We do this bottom-up in \mathcal{T} . If the table of the arc to the root has a positive entry, then accept, otherwise reject. Correctness of this follows from the definition of the information in the tables, as will be discussed below.

Notation Note that we assume we are given a graph G' with a tree partition \mathcal{T} of breadth at most k . We denote the vertices in G' by V , the edges by E , the weight function by w , and for each vertex v its target interval by D_v .

An orientation of the edges in G' is said to be *good*, if for each vertex $v \in V$, the total weight of edges directed out of v is an element of D_v .

For a node $i \in I$, we denote the union of all vertex sets X_j with $j = i$ or j a descendant of i as V_i .

For an arc $a = ii' \in F$ with i the parent of i' , we write E_a for the set of all edges of G' with one endpoint in V_i and one endpoint in $V_{i'}$. I.e., we take all edges with both endpoints in V_i except the edges with both endpoints in X_i : $E_a = (V_i \times V_{i'} \cap E) \setminus X_i \times X_{i'}$.

A *partial solution* for arc a is an orientation of E_a such that for each vertex $v \in V_{i'}$, the total weight of all edges directed out of v is an integer in D_v . Note that for partial solutions, the condition is not enforced for vertices in X_i .

Let ρ be a partial solution for a . The *fingerprint* of ρ is the function $f : X_i \rightarrow \mathbb{Z}_{\geq 0}$ where for each $v \in X_i$, $f(v)$ equals the total weight of all edges directed out of v for the orientation ρ .

We say that a partial solution ρ for a is *extendable*, if there is a good orientation ρ' of G' with all edges in E_a oriented in the same way in ρ and ρ' .

Some observations

Claim 4.2. *G' has a good orientation if and only if there is a partial solution for the arc between the root and its child.*

Proof. Let i be the unique child of root r . Observe that $E_{ri} = E$ and $V_i = V$, and thus, a partial solution for ri is a good orientation, and vice versa. \lrcorner

Claim 4.3. *Let f be the fingerprint of a partial solution for $a = ii'$. For all $v \in X_i$, $0 \leq f(v) \leq k$.*

Proof. All edges in E_a with $v \in X_i$ as endpoint have their other endpoint in $X_{i'}$, so use the arc a , and thus the total weight of all such edges is at most k . \lrcorner

Claim 4.4. *Let ρ_1 and ρ_2 be partial solutions for a with the same fingerprint. Then ρ_1 is extendable if and only if ρ_2 is extendable.*

Proof. Suppose ρ extends ρ_1 . Consider the orientation that orients all edges in E_a as in ρ_2 and all edges in $E \setminus E_a$ as in ρ . One easily checks that this is an extension of ρ_2 . \lrcorner

In the algorithm, we compute for each arc a in \mathcal{T} the set of all fingerprints of partial solutions for a .

Computing sets of fingerprints for leaf arcs We have a separate, simple algorithm for arcs with one endpoint a leaf of \mathcal{T} . Let $a = ii'$ be an arc in \mathcal{T} , with i the parent of leaf i' . Note that V_i has at most $2k$ vertices, so E_a has $O(k^2)$ edges. To compute all fingerprints for ii' , we can simply enumerate all $2^{O(k^2)}$ possible orientations of E_a , and then check for each if the outdegree weight condition is fulfilled for all $w \in X_{i'}$, and if so, compute the fingerprint and store it in a table.

Computing sets of fingerprints for other arcs Now, suppose $a = ii'$ is an arc with i the parent of i' , and i' has at least one child. Let the children of i' be j_1, j_2, \dots, j_q . Write $a_p = i'j_p$ for the arc from i' to its p th child; $1 \leq p \leq q$.

We assume that we already computed (in bottom-up order) tables A_{a_p} that contain the set of all fingerprints of partial solutions of a_p , for $p \in [1, q]$.

We now consider the equivalence relation on the arcs a_1, \dots, a_q given by $a_p \sim a_{p'}$ if and only if A_{a_p} and $A_{a_{p'}}$ are the same, i.e., each fingerprint that belongs to A_p also belongs to $A_{p'}$ and vice versa.

Claim 4.5. *The number of equivalence classes of \sim is bounded by $2^{(k+1)^k}$.*

Proof. Each fingerprint maps at most k vertices to an integer in $[0, k]$, so there are at most $(k+1)^k$ fingerprints. In a table, each of these can be present or not, which gives the bound. \square

(A sharper bound is possible by using that the sum of the values is bounded by k .)

We denote the set of all equivalence classes of \sim by Γ , and denote the set of all possible fingerprints for arcs between i' and a child by Δ , i.e., Δ is a subset of the set of all functions $f: X_{i'} \rightarrow [0, k]$.

For an equivalence class $\gamma \in \Gamma$, and fingerprint $f \in \Delta$, we write $f \in \gamma$ if there exists an arc a_p which belongs to equivalence class γ and $f \in A_{a_p}$. Note that this implies that $f \in A_{a_{p'}}$ for all $a_{p'}$ equivalent to a_p .

Let ρ be a partial solution of a . The *blueprint* of ρ is the function $g: \Gamma \times \Delta \rightarrow [0, q]$, such that $g(\gamma, f)$ is the number of arcs a_p in equivalence class γ for which the restriction of ρ to E_{a_p} has fingerprint f .

Overall procedure The procedure to compute the set of fingerprints for a has a main loop. Here, we enumerate all orientations ρ of the edges between a vertex in X_i and a vertex in $X_{i'}$ and the edges with both endpoints in $X_{i'}$. In a subroutine, which will be given later, we check if this orientation can be extended to a partial solution for ii' . If so, we store the fingerprint of this orientation in the table $A_{ii'}$. If not, the orientation is ignored and we continue with the next orientation in the ordering.

Note that ρ gives all information to compute the fingerprint, as all edges in E_a that have an endpoint in X_i receive an orientation in ρ .

Assuming the correctness of the subroutine, this gives the complete set of all fingerprints of ii' . Note that the number of edges we orient in this step is bounded by a function of k : all these edges are in $(X_i \times X_{i'} \cup X_{i'} \times X_{i'}) \cap E$, and thus, its number is bounded by $2k^2$, and we call the subroutine for at most 2^{2k^2} orientations.

The main subroutine We now finally come to the heart of the algorithm. In this subroutine, we check whether an orientation coming from the enumeration as described above can be extended to a partial solution.

To be more precise, we have an arc $a = ii'$ with i the parent of i' . Say i' has $q > 0$ children, j_1, \dots, j_q . We are given an orientation ρ of the edges with either one endpoint in X_i and one endpoint in $X_{i'}$ or with both endpoints in $X_{i'}$. For each child j_p , we are given the table $A_{i'j_p}$ of fingerprints of partial solutions of $i'j_p$. The procedure returns a

Boolean, that is true if ρ has an extension that is a partial solution of $a = ii'$, and false otherwise.

To do this, we search for the blueprint of such an extension. Let Γ and ρ be as above.

First, compute for all equivalence classes $\gamma \in \Gamma$, the number $n(\gamma)$ of arcs to children of i' that belong to the equivalence class γ .

Second, we build an integer linear program (ILP). The ILP has a variable $g_{\gamma,f}$ for each equivalence class $\gamma \in \Gamma$ and $f: X_{i'} \rightarrow [0, k]$ one of the fingerprints stored in tables in class γ . This variable denotes the value in the blueprint of the extension that we are searching for. Furthermore, we have a number of constraints.

1. For all $\gamma \in \Gamma$, $f \in \Delta$: $g_{\gamma,f} \geq 0$. (When the variable exists.)
2. For all $\gamma \in \Gamma$: $\sum_f g_{\gamma,f} = n(\gamma)$. Indeed, the extension has a partial solution for each $i'j_p$ whose fingerprint belongs to the equivalence class of $i'j_p$. The total number of times such a fingerprint is taken from this equivalence class must be equal to the number of child arcs which belong to the class.
3. For all $v \in X_{i'}$, we have a condition that checks that the outdegree of v in the orientation belongs to D_v . Let $D_v = [d_{min,v}, d_{max,v}]$. Let α be the total of the weight of all edges in ρ that have v as endpoint and are directed out of v . Now, add the inequalities:

$$d_{min,v} \leq \alpha + \sum_{\gamma,f} f(v) \cdot g_{\gamma,f} \leq d_{max,v}$$

where we sum over all $\gamma \in \Gamma$, and $f \in \gamma$.

Claim 4.6. *The set of inequalities has an integer solution if and only if there exists a partial solution for a that extends ρ .*

Proof. First, suppose the set of inequalities has an integer solution with values $g_{\gamma,f}$. Start by assigning to each child arc a_p a fingerprint from A_{a_p} in such a way that for each equivalence class $\gamma \in \Gamma$, exactly $g_{\gamma,f}$ members of the class have f assigned to it. We can do this because of the second set of inequalities of the ILP. Then, for each child j_p , orient the edges in A_{a_p} as in a partial solution that has f as fingerprint — we can do this, since $f \in A_{a_p}$ by construction. Combine these with the orientation ρ . We claim that this gives a partial solution for ii' . For every vertex v belonging to a bag that is a descendant of i' , its outdegree lies in D_v , since all its incident edges belong to a partial solution for an arc between i' and a child. For a vertex $v \in X_i$, the total weight of outgoing edges equals $\alpha + \sum_{\gamma,f} f(v) \cdot g_{\gamma,f}$: α for the edges in ρ , and for each γ, f , there are $g_{\gamma,f}$ children of i which are in equivalence class γ and are oriented with a fingerprint f , which implies that the total weight of outgoing edges from v to vertices in that subtree equals $f(v)$. The third set of conditions of the ILP thus enforces that the total weight of outgoing edges for v is in D_v .

In the other direction, suppose we have a partial solution ρ' of a that extends ρ . Take the blueprint of ρ' . One can easily verify that setting the variables according to this blueprint gives a solution of the ILP. \square

We can now finish the argument. Note that the number of variables and inequalities is a function of k . Thus, we can solve the ILP with an FPT algorithm, see e.g., the discussion

in [15, Section 6.2]. In fact, the number of variables of the ILP is at most $2^{(k+1)^k} \cdot (k+1)^k$. The number of inequalities is $O(2^{(k+1)^k})$: we have $O(1)$ inequalities per equivalence class and one for each vertex in X_i . Each integer in the description of the ILP is bounded by either k , or the number of children of i .

Solving ILP's is fixed parameter tractable when we take the number of variables as parameter, see [15, Theorem 6.4]). We note that the number of variables is double exponential in k ; and applying Theorem 6.4 from [15] gives an algorithm which is quadruple exponential in k .

Wrap-up All elements of the algorithm have been given. We compute the table A_a bottom-up for each arc a in the partition tree \mathcal{T} (e.g., in postorder). A simple procedure suffices when the arc has a leaf as one of its endpoints. Otherwise, we enumerate over orientations of edges in $X_{i'}$ and edges between X_i and $X_{i'}$ as described above, and for each orientation, we use an ILP to answer the question whether the orientation can be extended to a partial solution. When the test succeeds, we store the corresponding fingerprint.

When we have finally obtained the table for the arc to the root node, we just check whether this table is non-empty; if so, we answer positively, otherwise, we answer negatively.

The running time is dominated by the total time of solving all ILP's. The number of arcs in \mathcal{T} is linear in the number of vertices; for each, we consider 2^{2k^2} orientations in the enumeration, and for each of these, the time for solving the ILP can be done with an algorithm that is fixed parameter tractable in k .

We obtain the following result.

Theorem 4.7. *OUTDEGREE RESTRICTED ORIENTATION is fixed parameter tractable when parameterized by stable gonality, assuming that a finite harmonic morphism of a refinement of the input graph to a tree of degree $\text{sgon}(G)$ is given as part of the input; and when parameterized by (stable) treebreadth, assuming a tree partition (of a subdivision) is given realizing the given breadth.*

With help of reductions, we obtain fixed parameter tractability for the six other problems parameterized by stable gonality or treebreadth.

Corollary 4.8. *The following problems are fixed parameter tractable with stable gonality as parameter, assuming that a finite harmonic morphism of a refinement of the input graph to a tree of degree $\text{sgon}(G)$ is given as part of the input; ; and when parameterized by (stable) treebreadth, assuming a tree partition (of a subdivision) is given realizing the given breadth.*

1. TARGET OUTDEGREE ORIENTATION
2. CHOSEN MAXIMUM OUTDEGREE
3. MINIMUM MAXIMUM OUTDEGREE
4. CIRCULATING ORIENTATION
5. UNDIRECTED FLOW WITH LOWER BOUNDS

6. ALL-OR-NOTHING FLOW

Proof. TARGET OUTDEGREE ORIENTATION, CHOSEN MAXIMUM OUTDEGREE and MINIMUM MAXIMUM OUTDEGREE are special cases of OUTDEGREE RESTRICTED ORIENTATION, so we can apply the algorithm of Theorem 4.7. We can also use this algorithm for CIRCULATING ORIENTATION, as this problem is the special case of TARGET OUTDEGREE ORIENTATION, where for each vertex v , d_v is set to half the sum of the weights of all edges incident to v .

To solve UNDIRECTED FLOW WITH LOWER BOUNDS, we transform an input of that problem to an equivalent input of CIRCULATING ORIENTATION, as in Lemma 3.10. The breadth of the tree partition is still bounded by 4 times the original value, which yields fixed parameter tractability.

Finally, to solve ALL-OR-NOTHING FLOW, we follow the transformation described by Lemma 3.2 — we obtain an equivalent instance of CIRCULATING ORIENTATION with an associated finite harmonic morphism from a refinement to a tree of bounded degree, and thus can apply the FPT algorithm for CIRCULATING ORIENTATION on that equivalent input. \square \square

5 CAPACITATED DOMINATING SET for graphs with small tree partition width

In this section, we give the algorithm for CAPACITATED RED-BLUE DOMINATING SET for graphs with a given tree partition of bounded tree partition width, and discuss at the end of this section how to adapt this to CAPACITATED DOMINATING SET. The description of the algorithm has a large number of technical details. Before giving these details, we first give a high level overview of the main ideas.

5.1 A high level overview

Let \mathcal{T} be the given tree partition of the input graph G . The algorithm employs dynamic programming on the tree for \mathcal{T} . For each node, we compute a table with the minimum sizes for equivalence classes of partial solutions.

To a node i , we associate the subgraph G_i , consisting of all vertices in the bags equal to or a descendant of i , and the edges in G between these vertices. A partial solution knows which red vertices in this subgraph belong to the dominating set, and how blue vertices in this subgraph are dominated. Possibly the blue vertices in X_i are not yet dominated, as they can be dominated by vertices from the bag of the parent of i .

We have a technical lemma claiming that the differences of the minimum values for the equivalence classes of a node i is bounded by $2k$. This lemma is similar to and inspired by a notion called *finite integer index* [11]; the techniques used to prove the lemma are taken from the classical theory of algorithms to find matchings in graphs (with one step basically employing the principle of an alternating path).

If we want to build a partial solution for a node i , we decide ‘what happens around X_i ’, and for each child, we select a partial solution.¹

¹A technicality is that we actually take a slight extension of a partial solution, but we leave that

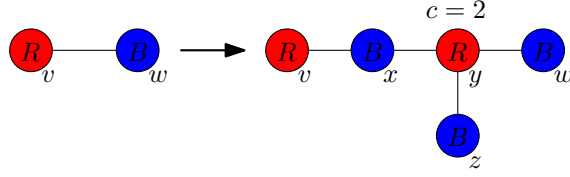


Figure 5: A transformation to handle subdivisions

Instead of looking at all children separately, we note that if we take the table of minimum sizes of a child, and subtract from each the smallest value, then all remaining values are in the interval $[0, 2k]$. We sum these smallest values over all children — as we need to always ‘pay’ this amount, we store this value separately.

We now define an equivalence relation on the children of the node — two children are equivalent if after subtracting the minima, the table of values of minimum sizes are the same. The table size is a function of k , and the values after subtraction are in $[0, 2k]$, so this equivalence relation has a number of classes that is bounded by a function of k .

Now, instead of deciding for each child separately what type of partial solution we take in its subgraph, we decide for each equivalence class and each type of partial solution how many children in that equivalence class have a partial solution of that type. Which of these decisions are possible and combine to a partial solution, and which gives the minimum total size for the type of partial solution at i we want to build can be formulated as an integer linear program (ILP). This ILP has a number of variables that is a function of k , and thus can be solved with an FPT algorithm.

Finally, if we have the table for the root node, it is easy to read off from it the minimum size of a capacitated red-blue dominating set.

5.2 Detailed explanation

Suppose that we have a graph $G = (V, E)$, with $V = R \cup B$; R is the set of the red vertices, and B the set of blue vertices. We have a capacity function $c: R \rightarrow \mathbb{Z}_{>0}$, that gives each red vertex a positive capacity.

Dealing with subdivisions If we have a tree partition of a subdivision of G , then we can build an equivalent instance where we have a tree partition of the graph itself. For each bag that contains a subdivision vertex, we replace this subdivision vertex by three new vertices, as in Figure 5. The three vertices are placed in the subdivision bag instead of the subdivision vertex. This gives an equivalent instance: The new red vertex y has capacity 2, and must be in the dominating set in order to dominate its private neighbour z ; if v dominates w in the original graph, then in the transformed graph, v dominates x and y dominates w ; if v does not dominate w in the original graph, then another vertex dominates w , and y can be used to dominate z and x . The width of the tree partition at most triples.

From now on, we assume that we are given a tree partition \mathcal{T} of $G = (R \cup B, E)$ of width k , and a weight function $w: R \rightarrow \mathbb{Z}_{>0}$.

discussion for Section 5.2.

View the tree of \mathcal{T} as a rooted tree. For a bag X_i , let V_i be the set of vertices in the bags X_j with $j = i$ or j a descendant of i , E_i be the set of edges in E with both endpoints in V_i , and $G_i = (V_i, E_i)$.

Solutions A *solution* is a pair (S, f) , with $S \subseteq R$ a set of red vertices in G , and $f: B \rightarrow S$ a mapping that assigns each blue vertex to a neighbour, such that the capacity constraints are satisfied: for all $v \in S$, $|f^{-1}(v)| \leq c(v)$.

One can check in polynomial time whether at least one solution exists: if there is a solution, then we can take $S = R$, so what is needed is to see whether we can assign all blue vertices to red neighbours satisfying the capacities; this step can be done by a standard generalized matching algorithm.

Partial solutions We will use three different, but slightly similar, notions that describe a part of a solution restricted to a subgraph: partial solutions, partially extended partial solutions (peps), and extended partial solutions (eps). While it may seem somewhat confusing to have three such notions, this helps for a clear explanation of the algorithm and the proof of its correctness.

For each of these three notions, we define an equivalence relation by identification of a ‘characteristic’, and we define the minimum ‘size’ of an element in an equivalence class (for convenience, assigning size ∞ when no solution with the given characteristic exists).

We start with giving these definitions for partial solutions.

Suppose i is a node from \mathcal{T} .

A *partial solution* for i is a triple (S, D, f) , where $S \subseteq R \cap V_i$ is a set of red vertices in V_i , $D \subseteq B \cap X_i$ is a set of blue vertices from X_i , and $f: B \cap (V_i \setminus X_i) \cup D \rightarrow S$ is a function, that maps each blue vertex in $V_i \setminus X_i$ or in D to a vertex in S , such that for all $v \in B \cap (V_i \setminus X_i) \cup D$, $f(v)$ is a neighbour of v , and for all $w \in S$, $|f^{-1}(w)| \leq c(w)$.

In other words, in the partial solution, we need to dominate all blue vertices in sets $X_{i'}$ with i' a descendant of i , and from X_i , we need to dominate all vertices in D but not those in $X_i \setminus D$. This domination is done by the vertices in S without exceeding the capacities.

Note that a partial solution $(S, X_r \cap B, f)$ for root bag r of \mathcal{T} is a solution to the CAPACITATED RED-BLUE DOMINATING SET problem, and we thus want to determine the minimum size of a set S such that there is a partial solution $(S, X_r \cap B, f)$ for the root bag.

The *characteristic* of a partial solution (S, D, f) is the pair (D, h) , with $h: R \cap X_i \rightarrow [0, k]$ the function such that for $v \in S$ we have $h(v) = \min\{k, c_f(v)\}$ with $c_f(v) = c(f) - |f^{-1}(v)|$, and for $v \notin S$, we set $h(v) = 0$. The value $c_f(v)$ is what remains from the capacity of v after all vertices are mapped to v by f . When we extend a partial solution, the only additional vertices that are mapped to vertices in X_i are those from the parent bag of i , and thus, we will not need more than k additional capacity — hence, we can take the maximum of the remaining capacity and k . Vertices not in S cannot have additional vertices mapped to it, so their remaining capacity is set to 0.

We say that two partial solutions for i are equivalent if and only if they have the same characteristic.

The *size* of a partial solution (S, D, f) is $|S|$. The *minimum size* of an equivalence class is the minimum size of a partial solution in the class; we denote by $A_i(D, h)$ the minimum size of the equivalence class of partial solutions at i with characteristic (D, h) .

We say that a solution (S, f) *extends* a partial solution (S', D, f') when $S' = S \cap V_i$ and f' is obtained from f by restricting the domain to $((V_i \setminus X_i) \cap B) \cup D$, with the additional properties that blue vertices in $X_i \setminus D$ are mapped to the parent bag: if $v \in (X_i \setminus D) \cap B$, then $f(v) \notin V_i$.

Exchange arguments show that a partial solution that is not of minimum size in its equivalence will never extend to a solution of minimum size. Thus, in the dynamic programming algorithm, we need to compute for all the equivalence classes of partial solutions their minimum size.

Partial extended partial solutions For nodes that are not the root of \mathcal{T} , we define the notion of partial extended partial solution. We use these to prove a property of values of minimum sizes which is needed in the dynamic programming algorithm.

Suppose i is not the root of \mathcal{T} , and let j be the parent of i . A *partial extended partial solution* or *peps* for i is a triple (S, D, f) , where $S \subseteq R \cap V_i$ is a set of red vertices in V_i , $D \subseteq B \cap (X_i \cup X_j)$, and $f: B \cap (V_i \setminus X_i) \cup D \rightarrow S$ is a function that maps each blue vertex in $V_i \setminus X_i$ or in D to a vertex in S , such that for all $v \in B \cap (V_i \setminus X_i) \cup D$, $f(v)$ is a neighbour of v , and for all $w \in S$, $|f^{-1}(w)| \leq c(w)$.

The only difference between partial solutions and partial extended partial solutions is that a peps knows which blue vertices in X_j are dominated by red vertices in X_i .

The *characteristic* of a peps (S, D, f) is D . Two peps are equivalent when they have the same characteristic. The *size* of a peps (S, D, f) is $|S|$, and the *minimum size* of an equivalence class of peps is the minimum size of a peps in the class. We denote this by $B_i(D)$ for the class with characteristic D .

A solution (S, f) extends a peps (S', D, f') , when $S' = S \cap V_i$ and f' is the restriction of f to $((V_i \setminus X_i) \cap B) \cup D$ and every blue vertex in $X_i \setminus D$ is mapped to a vertex in X_j and every blue vertex in $X_j \setminus D$ is not mapped to a vertex in X_i . (I.e., the peps tells for all red vertices in V_i which blue vertices they dominate.) Again, a peps that is not of minimum size in its equivalence class cannot be extended to a minimum size solution, and thus, the dynamic programming algorithm needs only store the minimum size for each equivalence class of peps.

Extended partial solutions Again, let i be a node that is not the root, with parent j .

An *extended partial solution* or *eps* for i is a triple (S, D, f) , where $S \subseteq R \cap V_j$ is a set of red vertices in V_j , $D \subseteq B \cap X_j$, and $f: (B \cap V_i) \cup D \rightarrow S$ is a function that maps each blue vertex in V_i or in D to a vertex in S , such that for all $v \in (B \cap V_i) \cup D$, $f(v)$ is a neighbour of v , and for all $w \in S$, $|f^{-1}(w)| \leq c(w)$, and for all $x \in D$, $f(x) \in V_i$.

The difference between a partial extended partial solution and an extended partial solution is that in the latter, all blue vertices in X_i are dominated, possibly by red vertices from X_j . Blue vertices in X_j can but do not have to be dominated, and we only look at domination of these vertices by red vertices from X_i . In the characteristic, we record how much capacity of the red vertices in X_j is used to dominate the blue vertices in X_i .

Note that at this point, the eps only tells us which blue vertices in X_i are dominated by red vertices in X_j ; in an extension, these red vertices can dominate other vertices not in V_i .

The *characteristic* of an eps (S, D, f) is the pair (D, g) with $g : X_j \cap R \rightarrow [0, k]$ is given by $g(v) = |f^{-1}(v)|$. Two eps are equivalent if they have the same characteristic.

In the size of an eps, we do not yet count the red vertices in $X_j \cap S$ — this helps to prevent to count these multiple times, and makes later steps slightly simpler. The *size* of an eps (S, D, f) at i is $|S \setminus X_j|$. The *minimum size* of an equivalence class of eps is the minimum size of an eps in the class. We denote the minimum size of an eps at i with characteristic (D, g) by $C_i(D, g)$.

A solution (S, f) is an extension of an eps (S', D, f') when $S' \cap V_i = S \cap V_i$? and f' is obtained by restricting the domain of f to the union of the blue vertices in V_i and the blue vertices in X_j that are dominated by vertices in X_i . Again, an eps that is not of minimum size in its equivalence class has no extension of minimum size, and thus, in the dynamic programming algorithm, we need to tabulate only the minimum sizes of equivalence classes of eps.

The finite integer index property Before we proceed with the algorithm, we need a technical lemma, Lemma 5.2. This lemma is used to show that a certain equivalence relation has a finite number of equivalence classes. This shows that the problem is finite integer index in the terminology of [11]; we do not use the terminology from that source any further, but we note that our methods are similar to the ones in that reference. The proof of the lemma is heavily inspired by well known techniques from matching, in particular, the notion of an alternating path, and the proof that such exist.

Lemma 5.1. *Let i be a node. If there is no peps (S, \emptyset, f) for any S and f for i , then no solution exists.*

Proof. Suppose we have a solution for G , say with dominating set S' , and f' assigns each blue vertex to a neighbour in S without violating capacities. Then set $S = S' \cap V_i$, and f the restriction of f' to domain $(V_i \setminus X_i) \cap B$. \square

From now on, we assume that a peps (S, \emptyset, f) for some S and f for i exists.

Lemma 5.2. *Let i be a node with parent j . Suppose $\alpha = B_i(\emptyset)$. Let $D \subseteq (X_i \cup X_j) \cap B$. If there is a peps with characteristic D , then*

$$\alpha \leq B_i(D) \leq \alpha + |D|.$$

Proof. We have that $\alpha \leq B_i(D)$, as we can take a solution for D , and restrict the domination function f by removing D from the domain.

We prove that $B_i(D) \leq \alpha + |D|$ by induction with respect to the size $|D|$ of D . The result trivially holds when $|D| = 0$.

Consider a $D \subseteq X_i \cup X_j$ with $|D| > 0$, and suppose the lemma holds for smaller sized sets. Suppose there exists a peps (S, D, f) for some S and f ; if not, we are done. Take a vertex $v \in D$. Let $f \setminus v$ be the restriction of f where we remove v from the domain of f . Now, $(S, D \setminus \{v\}, f \setminus v)$ is a peps.

Thus, the minimum size of a set S' for which there is a peps of the form $(S', D \setminus \{v\}, f')$ exists for some f' , and from the induction hypothesis, it follows that $|S'| \leq |D \setminus \{v\}| = |D| - 1$.

We now define an auxiliary graph G^+ , which is formed from G by replacing each red vertex v by $c(v)$ copies of v , each incident to all neighbours of v . To a peps (S'', D'', f'') , we associate a matching in G^+ as follows: for each vertex x in the domain of f'' , we add to the matching an edge from x to a copy of $f(x)$. As the size of a preimage of a red vertex is at most its capacity, we can add these edges such that they form a matching, i.e., have no common endpoints.

Let M_1 be the matching associated with the peps (S, D, f) and let M_2 be the matching associated with the peps $(S', D \setminus \{v\}, f')$. Consider the symmetric difference $M_1 \oplus M_2 = (M_1 \cup M_2) \setminus (M_1 \cap M_2)$.

Vertices in the symmetric difference of two matchings have degree at most two, so this symmetric difference consists of a number of cycles and paths. The vertex v is incident to an edge in M_1 but not in M_2 , so is an endpoint of a path in $M_1 \oplus M_2$. All other blue vertices are incident to 0 or to 2 edges in $M_1 \cup M_2$, so have degree 0 or 2 in the symmetric difference, hence cannot be endpoint of a path. We find that in $M_1 \oplus M_2$ there is a path from v to a red vertex. Say this last red vertex is z , and let M'_1 and M'_2 be the edges from M_1 and M_2 that belong to this path. We thus have a path P that starts at v , and then alternately has an edge in M_1 from a blue vertex to a red vertex, and an edge in M_2 from a red vertex to a blue vertex, ending with an edge in M_1 .

We now change the peps $(S', D \setminus \{v\}, f')$ as follows. Set $S'' = S' \cup \{z\}$. The domain of f'' is obtained by adding v to the domain of f' . All blue vertices that are not on P are mapped in the same way by f'' as by f' . For each edge in M'_1 , say from blue vertex x to a copy of red vertex y , we map x to y . This effectively cancels the mappings modelled by the edges in M'_2 .

We claim that $(S' \cup \{z\}, D, f'')$ is a peps.² Indeed, consider a red vertex on P , unequal to z ; say it is a copy of the red vertex y . The vertex has an incident edge in M'_1 (which causes one additional mapping to this vertex) and an incident edge in M'_2 (which cancels one mapping to this vertex), and thus, the total number of blue vertices dominated by y stays the same, in particular, at most $c(y)$. Since z has no incident edge in M'_2 , it is a copy of a vertex that has at least one capacity left in f' .

As $(S' \cup \{z\}, D, f'')$ is a peps, $B_i(D) \leq B_i(D - \{v\}) + 1 \leq \alpha + |D|$, and the induction step is proved. \square

Lemma 5.3. *Let (D_1, g_1) and (D_2, g_2) be characteristics of an eps at i . Then*

$$|C_i(D_1, g_1) - C_i(D_2, g_2)| \leq 2k.$$

Proof. Each eps of minimum size at i can be obtained from a peps of minimum size at i by extending it by dominating the not yet dominated vertices from X_i by red vertices in X_j . Note that we do not count the red vertices in X_j in the sizes of eps at i . Thus, the largest difference in sizes of eps at i is bounded by the largest difference in sizes of peps at i , which, by Lemma 5.2, is at most $2k$. \square

²This step is very similar to the use of an alternating path to augment a flow, as in classical network flow theory.

Suppose the problem has at least one solution for G . By Lemma 5.2, for each node i and $D \subseteq (X_i \cup X_j) \cap B$, we have $B_i(D) \in [B_i(\emptyset), B_i(\emptyset) + 2k]$.

Main shape of algorithm We now describe the algorithm.

We start with a generalized matching algorithm to determine whether the set of all red vertices is a capacitated dominating set. If not, there is no solution, and we halt. Otherwise, all values $B_i(\emptyset)$ are integers (and at most $|R|$).

We then process all nodes in postorder. Each computation step below is done by a subroutine whose details are discussed further below. For a leaf node, we do a direct computation of the table with the minimum sizes of equivalence classes of partial solutions (A_i). For a node with children, we use the tables with minimum sizes of equivalence classes of eps of its children (C_{j_α} for all children j_α) to compute the table with minimum sizes of partial solutions (A_i). If the node is the root, we decide the problem. If the node is not the root, we next compute a table with minimum sizes of equivalence classes of peps (B_i) and then a table with minimum sizes of equivalence classes of partial solutions (C_i). We are now done processing this node, and continue with the next.

Partial solutions of leaf nodes A simple exhaustive search gives all partial solutions for a leaf node, since when i is a leaf of \mathcal{T} , then G_i has at most k vertices, and $O(k^2)$ edges.

Partial solutions for nodes with children Suppose now i is a node in \mathcal{T} with at least one child. Let j_1, \dots, j_q be the children of i . We describe how to obtain a table with minimum sizes for equivalence classes of partial solution for i , given tables with minimum sizes for equivalence classes of extended partial solutions for the children of i . This is the most involved (and most time consuming) step of the algorithm; the step uses as a further subroutine solving an integer linear program with the number of variables a function of k .

We assume we have computed, for each child j_α and each equivalence class of extended partial solutions for j_α , the minimum size of the set $|S|$ in such an eps; i.e., for all equivalence classes, represented by (D, g) , we know the value $C_{j_\alpha}(D, g)$.

For each child j_α , let $m_\alpha = \min_{D, g} C_{j_\alpha}(D, g)$, and let $C'_{j_\alpha}(D, g) = C_{j_\alpha}(D, g) - m_\alpha$, if $C_{j_\alpha}(D, g) \neq \infty$; if $C_{j_\alpha}(D, g) = \infty$, then we set $C'_{j_\alpha}(D, g) = \infty$.

By Lemma 5.3, we have $C'_{j_\alpha}(D, g) \in [0, 2k] \cup \{\infty\}$, for all children j_α and all characteristics of eps at j_α (D, g).

We now define another equivalence relation, this time on the children of i , and say that $j_\alpha \sim j_\beta$ if for all (D, g) , we have $C'_{j_\alpha}(D, g) = C'_{j_\beta}(D, g)$.

The index of this equivalence relation is bounded by a (double exponential) function of k . The domain size of functions C'_{j_α} is bounded by $2^k \cdot (k+1)^k$, and the image size is bounded by $2k+2$.

Each equivalence class is identified by a function C' that maps pairs of the form (D, g) to a value in $[0, 2k] \cup \{\infty\}$.

The first step is to compute the number of children in each equivalence class of the equivalence relation \sim described above. If C' is a function mapping pairs (D, g) to values in $[0, 2k] \cup \{\infty\}$, we write $N[C']$ for the number of children j_α with $C'_{j_\alpha} = C'$.

We also compute the integer $m_{\text{tot}} = \sum_{\alpha=1}^q m_\alpha$.

Note that for a child j_α , the function C_{j_α} captures all essential information pertaining to the subgraph G_{j_α} . Thus, the function N together with the integer m_{tot} gives at this stage all essential information for all children of i . At this point, we can forget further information and tables for the children of i , and keep only the function N and the value m_{tot} in memory.

We now iterate over all characteristics of partial solutions (D, h) at i , and compute the minimum size of the characteristic, as described below.

We then assume (D, h) to be fixed and iterate over all guesses which red vertices in X_i belong to S ; say this set is Q . (Each iteration yields the minimum size for the characteristic, which is a non-negative integer, or ∞ . We keep the best value over all iterations.)

For simplicity, in the remainder of this iteration, we assume that vertices in $(X_i \cap R) \setminus Q$ (i.e., red vertices that are not used in the dominating set) have capacity 0.

The (blue) vertices in D can be dominated by vertices in X_i or in a child of X_i . We iterate over all possible options. If we guess that $v \in D$ is dominated by a vertex $w \in X_i$, then subtract one from the capacity of w (and reject this case when it leads to a negative capacity), and remove v from D . All vertices that remain in D must be mapped to vertices in child bags.

An ILP for characteristic selection We now want to determine if we can choose from each child bag a characteristic of an eps such that the combination of these characteristics and the guesses done so far combine to a partial solution with characteristic (D, h) ; and if so, what is its minimum size.

We can do this in FPT time by formulating the question as a linear program, with the number of variables a function of k .

For each equivalence class of \sim and each eps at children of i , we have a variable. The variable $x_{C', (D, g)}$ denotes the number of children in the equivalence class with function C' for which the restriction of the partial solution to an eps for that child has characteristic (D, g) .

We have a number of constraints.

1. For each $x_{C', (D, g)}$, $0 \leq x_{C', (D, g)}$.
2. For each $x_{C', (D, g)}$, if $C'(D, g) = \infty$, then $x_{C', (D, g)} = 0$. In this case, we have subtrees where (D, g) is not the characteristic of an eps, so we cannot choose such eps in those subtrees.
3. For each C' , $\sum_{(D, g)} x_{C', (D, g)} = N[C']$. This enforces that we choose for each child in the equivalence class for C' precisely one extended partial solution.
4. For each $v \in D$, $\sum_{C', (D, g), v \in D} x_{C', (D, g)} = 1$. This enforces that each vertex in D is dominated exactly once by a vertex from a child node.
5. For each $v \in X_i \cap R$ with $h(v) < k$: $c(v) - \sum_{C', (D, g)} g(v) \cdot x_{C', (D, g)} = h(v)$. The sum tells how much capacity of v is used by vertices in child bags of v ; $h(v)$ is the exact remaining capacity. This case and the next enforce that h is the minimum of the remaining capacity and k .

6. For each $v \in X_i \cap R$ with $h(v) = k$: $c(v) - \sum_{C', (D, g)} g(v) \cdot x_{C', (D, g)} \geq k$. This case is similar as the previous one. We have a different case here because we took the minimum of k and the remaining capacity for functions h in the characteristics of partial solutions.

The total size of the partial solutions equals

$$|Q| + m_{\text{tot}} + \sum_{C', (D, g)} C'(D, g) \cdot x_{C', (D, g)}.$$

Indeed, we have $x_{C', (D, g)}$ many children in the class of C' where we choose an eps with characteristic (D, g) . Consider one of these, say j_α . The number of vertices in the dominating set in V_{j_α} is the sum of m_{j_α} and $C'(D, g)$. We have m_{j_α} counted within m_{tot} , and $C'(D, g)$ is counted in the summation, once for each time we choose this eps for this equivalence class.

Thus, in the ILP, we minimize $\sum_{C', (D, g)} C'(D, g) \cdot x_{C', (D, g)}$, and report for this iteration, if there is at least one solution of the ILP, the sum of the minimum value of the ILP and $m_{\text{tot}} + |Q|$.

From partial solutions to peps Suppose we have a table of values $A_i(D, h)$ for node i with parent j . To compute the table of values $B_i(D)$, we do the following. For each set $D \subseteq (X_i \cup X_j) \cap B$ of blue vertices in $X_i \cup X_j$ (i.e., characteristics of peps at i) consider all characteristics of partial solutions at i of the form $(D \cap X_i, h)$. Check all mappings of the vertices in $D \cap X_j$ to red neighbours in X_i . For each such mapping, we check whether we obtain a peps (necessarily with characteristic D) if we extend a partial solution with characteristic $(D \cap X_i, h)$ using this mapping; and if so, we check whether the size $A_i(D, h)$ is the current best for the characteristic D . (The check amounts to verifying that red vertices $v \in X_i \cap R$ have at most $h(v)$ vertices from X_j mapped to it. Note that red vertices in X_i that are not in S have remaining capacity $h(v) = 0$ so will not dominate blue vertices in X_j .)

From peps to eps Suppose we have a table of values $B_i(D)$ for node i with parent j . To compute the table of values $C_i(D, g)$, we do the following.

For each characteristic (D, g) of an eps at i , look at all sets $D' \subseteq X_i \cap B$ of blue vertices in X_i . Check if there exists a mapping of the vertices in $(X_i \cap B) \setminus D'$ to red neighbours in X_j , such that for each vertex $v \in X_j \cap R$ exactly $g(v)$ vertices are mapped to it. If this is so, we can extend a peps with characteristic D' to an eps with characteristic (D, g) ; the size stays the same, since in an eps, vertices in X_j do not yet contribute to the size. The minimum size for the characteristic (D, g) is the minimum over all the $B_i(D')$ for all D' which have such a mapping.

Finding the answer A partial solution (S, D, f) at the root node r gives a solution (S, f) if $D = X_r \cap B$ — in that case, all vertices, including all blue vertices in the root bag are dominated. Thus, the minimum size of a solution (S, f) is the minimum size over all equivalence classes of partial solutions at r with a characteristic of the form $(X_r \cap B, f)$. We thus answer the problem by computing $\min_f A_r(X_r \cap B, f)$, after we computed the table A_r for the root r .

Time analysis We note that for each of the equivalence relations (on partial solutions, peps, and eps), the index is a function of k . Also, the equivalence relation on children of a node is of finite index, and thus, we solve an ILP with the number of variables a function of k . Again, we use that solving ILP's with the number of variables as parameter is fixed parameter tractable; see e.g., [15, Theorem 6.4]. This all amounts to an FPT algorithm.

We can now summarize our findings in the following result.

Theorem 5.4. *CAPACITATED RED-BLUE DOMINATING SET is fixed parameter tractable when parameterized by tree partition width, assuming that a tree partition realizing the width is given as part of the input (in particular, it is also FPT when parameterized by (stable) tree breadth and stable gonality, assuming a corresponding refinement of the graph with a tree partition or morphisms to a tree is given as part of the input).*

5.3 Capacitated Dominating Set

It is not hard to deduce the same result for CAPACITATED DOMINATING SET, either by adapting the previous algorithm, or — somewhat easier — by using a transformation of the instance as follows.

Lemma 5.5. *Suppose we have an instance (G, c) of CAPACITATED DOMINATING SET and a tree partition of a refinement of G of breadth k . Then one can build in polynomial time an equivalent instance of CAPACITATED RED-BLUE DOMINATING SET (G', c') and a tree partition of a refinement of G' of breadth $2k$.*

Proof. Replace each vertex v by a red vertex of capacity $c(v) + 1$ and a blue vertex; the red vertex is incident to its blue copy and all blue copies of neighbours of v . The breadth thus precisely doubles. \square

Corollary 5.6. *CAPACITATED DOMINATING SET is fixed parameter tractable when parameterized by tree partition width, assuming that a tree partition realizing the width is given as part of the input (in particular, it is also FPT when parameterized by (stable) tree breadth and stable gonality, assuming a corresponding refinement of the graph with a tree partition or morphisms to a tree is given as part of the input).*

6 Hardness results

6.1 XNLP-completeness for bounded pathwidth

In this section, we first consider the ALL-OR-NOTHING FLOW problem with the pathwidth as parameter, and show that this problem is complete for the class XNLP. After that, we use the transformations given in Section 3 to obtain XNLP-hardness for the the six other problems in Figure 3(b).

XNLP is the class of parameterized problems that can be solved with a non-deterministic algorithm in $O(f(k)n^c)$ time and $O(f(k) \log n)$ space, with f a computable function, c a constant, and n the input size. This class was studied in 2015 by Elberfeld et al. [19] under the name $N[f \text{ poly}, f \log]$. Recently, Bodlaender et al. [9] showed that a large number of problems is complete for this class.

We now describe one of these problems, that we subsequently use to show hardness of ALL-OR-NOTHING FLOW.

A NON-DETERMINISTIC NON-DECREASING CHECKING COUNTER MACHINE (NNCCM) is a machine model where we have k counters, an upper bound $B \in \mathbb{Z}_{\geq 0}$, and a sequence of n tests. Each test is described by a 4-tuple of integers in $[1, k] \times [0, B] \times [1, k] \times [0, B]$. The machine works as follows. Initially, all counters are 0. We can increase some (or all, or none) of the counters, to integers that are at most B by alternating the following steps:

- Each counter is possibly increased, i.e., for each i , if the current value of counter i equals c_i , then the counter is non-deterministically given a value in $[c_i, B]$.
- The next check is executed. If the next check is the 4-tuple (i, a, j, b) , then the machine halts and rejects if counter i has value a and counter j has value b .

If the machine has completed all tests without rejecting, then the machine accepts.

For a test (i, a, j, b) , we say that a counter i' *participates* in the test if $i' \in \{i, j\}$, and we say it *fires*, if $i = i'$ and the value of counter i equals a , or $j = i'$ and the value of counter j equals b . We consider the following problem.

ACCEPTING NNCCM

Given: A Non-deterministic Checking Counter Machine, with k counters, upper bound B , and a sequence of n tests in $[1, k] \times [0, B] \times [1, k] \times [0, B]$, with all values given in unary.

Question: Is there an accepting run of this machine?

Theorem 6.1. ALL-OR-NOTHING FLOW for graphs given with a path decomposition, with the width of the given path decomposition as parameter, is complete for XNLP.

Proof. Membership in XNLP is easy to see: go through the path decomposition from left to right; for each edge between two vertices in the current bag with at least one endpoint not in the previous bag, non-deterministically choose the flow over the edge. For all vertices in the bag, keep track of the difference between the inflow and outflow. For a vertex v in the current bag that is not in the next bag, check whether the difference between the inflow and outflow is 0 if $v \notin \{s, t\}$, $-R$ if $v = s$, and R if $v = t$, and if not, reject. Accept when all bags are handled.

We have in memory the current value of the k counters, and a pointer to the current check to be performed; these all can be written with $O(\log n)$ bits.

We next show hardness, with a reduction from ACCEPTING NNCCM, which, we recall, was shown to be XNLP-complete in [9].

Suppose we are given the NNCCM, with k counters, an upper bound B , and a sequence S of n checks, with the t -th check whether counter $c_{t,1} \in [1, k]$ equals $v_{t,1} \in [0, B]$ and counter $c_{t,2}$ equals $v_{t,2}$.

We construct the following all-or-nothing flow network $G = (V, E)$ with capacity function c , see also Figure 6.

For each counter $j \in [1, k]$ and each time $t \in [0, n]$, we take a vertex $v_{j,t}$ and a vertex $w_{j,t}$. We have two additional vertices s, t , and for each i -th check, we have two vertices $x_{i,1}, x_{i,2}$.

were smaller than B to B . Fix such an accepting run. In the remainder of this first part of the proof, the values of counters are as in this accepting run.

First, use in full all edges from s with capacity L , and all edges to t with capacity $L + 2B$.

It follows from the construction that if at the t -th check, counter j has value α , then the outflow of $w_{j,t-1}$ and the inflow of $v_{j,t}$ equal $B + 2\alpha$. Consider such t , j , and α , and suppose that between the t -th and $(t + 1)$ -st check we increased the counter to β (and $\beta = B$ if $t = n$); if the counter was not changed, set $\beta = \alpha$. Thus, β is the value of counter j during the $(t + 1)$ -st check. We now use the following arcs:

- α arcs with value 2 from s to $w_{j,0}$ if $t = 1$.
- $\beta - \alpha$ arcs with value 2 from s to $w_{j,t}$.
- The arc with value $L + 2\alpha$ from $v_{j,t}$ to $w_{j,t}$.
- If the j th counter is not involved in the t th check, or if the firing value of the j th counter in the t th check unequals α , then use the arc from $w_{j,t-1}$ to $v_{j,t}$ with capacity $L + 2\alpha$.
- If the j -th counter is involved in the t -th check, and the firing value of the j -th counter in the t -th check equals α , then use the arc from $w_{j,t-1}$ to $v_{j,t}$ with capacity $L + 2\alpha - 1$, and the following arcs with capacity 1: from $w_{j,t-1}$ to $x_{t,1}$, from $x_{t,1}$ to $x_{t,2}$, and from $x_{t,2}$ to $v_{j,t}$. Note that, as we have an accepting run, the other counter involved in the t -th check has the firing value, and thus the edge from $x_{t,1}$ to $x_{t,2}$ is used at most once.

One verifies that the resulting function is indeed an all-or-nothing flow. The value equals the inflow at t : we have k arcs that send $L + 2B$ flow to t , to a total value of $R = k \cdot (L + 2B)$.

Now, suppose we have an all-or-nothing flow from s to t with value R .

First, look at the arcs out of s : we have k arcs with capacity L , and $(n + 1)kB < L/2$ arcs with capacity 2. We must use each of these k arcs with capacity L , otherwise the outflow from s and thus the flow value would be at most $(k - 1)L + 2(n + 1)kB < kL < R$.

We claim that for each $j \in [1, k]$, the inflow of $w_{j,t}$ is an even number of the form $L + 2\alpha$, with $\alpha \in [0, B]$: we show this by induction w.r.t. t , the claim being easy for $t = 0$. Suppose it holds for $t - 1$. The node $w_{j,t-1}$ has at most one outgoing edge of capacity 1, and all other outgoing edges have capacities between L and $L + 2B$. As $2B < L$, we thus must use exactly one of these ‘heavy’ edges. Now consider the inflow of $v_{j,t}$. It receives the flow of the used heavy edge and has, in addition to the heavy edges, at most one other incoming edge of capacity 1. Hence the inflow to $v_{j,t}$ is in $[L, L + 2B + 1]$. As $2B + 1 < L$, we see that we use exactly one of the outgoing edges from $v_{j,t}$ — all have weights in $[L, L + 2B]$. The node $w_{j,t}$ has further B incoming edges of weight 2, so the inflow of $w_{j,t}$ is in $[L, L + 4B]$. It cannot be in $[L + 2B + 2, L + 4B]$, as there is no combination of edges out of $w_{j,t}$ with total capacity in that interval (using that $4B < L$). Since all arcs into $w_{j,t}$ have even capacity, the inflow of $w_{j,t}$ is even.

From the above proof, it also follows that the inflow of $w_{j,t+1}$ is at least the inflow of $w_{j,t}$. Now set the j -th counter to α before check t when the inflow of $w_{j,t}$ equals $L + 2\alpha$. We see that these counters are non-decreasing.

Finally, we verify that the machine does not halt at a check. Suppose that both halves of the t -th check fire, say, for counters j and j' , having values α and α' . Then $w_{j,t-1}$ has an outgoing arc of weight $L + 2\alpha - 1$, and thus $w_{j,t-1}$ must send flow 1 to $x_{t,1}$. Similarly, $w_{j',t-1}$ has an outgoing arc of weight $L + 2\alpha' - 1$, and thus $w_{j',t-1}$ must send flow 1 to $x_{t,1}$. Now $x_{t,1}$ receives 2 inflow, but has only one outgoing arc of capacity 1, which contradicts the law of flow conservation. Thus, we cannot have a check where both halves fire, and hence the machine accepts. \square

A path decomposition of G of width $O(k)$ can be constructed as follows. For $0 \leq i < n$, we take a bag

$$X_i = \{s, t\} \cup \{v_{j,i} \mid j \in [1, k]\} \cup \{v_{j,i+1} \mid j \in [1, k]\} \cup \{w_{j,i} \mid j \in [1, k]\} \\ \cup \{w_{j,i+1} \mid j \in [1, k]\} \cup \{x_{i+1,1}, x_{i+1,2}\}.$$

The size of G is polynomial in k and n , and we can construct G with weight function c and the path decomposition of width $O(k)$ in $O(k^c \log n)$ space and time, polynomial in k and n . Thus, the hardness result follows. \square

Corollary 6.3. *The following problems are XNLP-complete with pathwidth as parameter:*

1. TARGET OUTDEGREE ORIENTATION
2. OUTDEGREE RESTRICTED ORIENTATION
3. CHOSEN MAXIMUM OUTDEGREE
4. MINIMUM MAXIMUM OUTDEGREE
5. CIRCULATING ORIENTATION
6. UNDIRECTED FLOW WITH LOWER BOUNDS

Proof. Membership in XNLP follows easily, with similar arguments as for ALL-OR-NOTHING FLOW in the proof of Theorem 6.1.

XNLP-hardness for TARGET OUTDEGREE ORIENTATION follows from the hardness for ALL-OR-NOTHING FLOW (Theorem 6.1 and the reduction given by Lemma 3.2). XNLP-hardness of OUTDEGREE RESTRICTED ORIENTATION follows directly as this problem contains TARGET OUTDEGREE ORIENTATION as a special case.

XNLP-hardness for CHOSEN MAXIMUM OUTDEGREE follows from the hardness of TARGET OUTDEGREE ORIENTATION by the reduction given by Lemma 3.9. From the hardness of CHOSEN MAXIMUM OUTDEGREE, we obtain by Szeider's transformation ([29], see Lemma 3.3), XNLP-hardness for MINIMUM MAXIMUM OUTDEGREE.

To prove that CIRCULATING ORIENTATION is XNLP-hard, we use the hardness of TARGET OUTDEGREE ORIENTATION and the reduction given by Lemma 3.4. Finally, as CIRCULATING ORIENTATION is a special case of UNDIRECTED FLOW WITH LOWER BOUNDS, XNLP-hardness of UNDIRECTED FLOW WITH LOWER BOUNDS follows. \square

6.2 $W[1]$ -hardness for bounded vertex cover number

In this final section, we consider the complexity of our seven basic orientation and flow problems for a stronger parameter, namely, vertex cover number. We can still prove $W[1]$ -hardness by reduction to the following problem.

BIN PACKING

Given: A set $A = \{a_1, \dots, a_n\}$ of n positive integers, and integers B and k such that $\sum_{i=1}^n a_i = B \cdot k$

Question: Partition A into k sets such that each has sum exactly B .

Jansen et al. [27] showed that BIN PACKING parameterized by the number of bins is $W[1]$ -hard.

Theorem 6.4. TARGET OUTDEGREE ORIENTATION *parameterized by the size of a vertex cover is $W[1]$ -hard.*

Proof. We give a transformation from BIN PACKING parameterized by the number of bins.

Suppose $A = \{a_1, \dots, a_n\}$, B and k are given. We take a complete bipartite graph $K_{k,n}$. Denote the k vertices at the left colour class by v_1, \dots, v_k and the n vertices at the right colour class by w_1, \dots, w_n .

We assign the following weights and target outdegrees. All edges incident to w_i have weight a_i . Each w_i has as target outdegree $d_{w_i} = a_i$. Each v_i has as target outdegree $d_{v_i} = Bk - B$. Let G be the resulting weighted graph.

Lemma 6.5. *A can be partitioned into k sets A_1, \dots, A_k such that each has sum exactly B , if and only if edges in G can be oriented such that the given target outdegrees are fulfilled.*

Proof. Suppose we can partition A into k sets A_1, \dots, A_k such that each has sum exactly B . Now construct the following orientation: for each $i \in [1, n]$, $j \in [1, k]$, if $a_i \in A_j$ then orient the edge between v_j and w_i from w_i to v_j ; otherwise, orient it from v_j to w_i . Each w_i has one outgoing edge of weight a_i . For each vertex v_i , the total weight of all incident edges equals $\sum_{i=1}^n a_i = Bk$. Precisely the edges with weights in A_i are directed to v_i — as we have a solution of the BIN PACKING problem, these have total weight exactly B for each v_i . All other edges are directed out of v_i and have weight $Bk - B$.

Suppose we have an orientation where each vertex has outdegree at most d_v . For each w_i , exactly one incident edge must be directed out of w_i , as each edge incident to w_i has the same weight, equal to the target outdegree. If the edge between v_j and w_i is directed from w_i to v_j , then assign a_i to A_j . Thus, we have that each a_i is assigned to exactly one set A_j . For each v_j , the total weight of all edges directed to v_j must be B , namely, the total weight of all incident edges minus the target outdegree. There is a one-to-one correspondence between the weights of these edges and the numbers assigned to A_j , and thus the sum of each A_j equals B . \square

The result follows from this claim and the quoted result of Jansen et al. [27], noting that $K_{k,m}$ has a vertex cover of size k . \square

Corollary 6.6. *The following problems, parameterized by the size of a vertex cover are $W[1]$ -hard:*

1. CHOSEN MAXIMUM OUTDEGREE
2. OUTDEGREE RESTRICTED ORIENTATION
3. MINIMUM MAXIMUM OUTDEGREE
4. CIRCULATING ORIENTATION
5. UNDIRECTED FLOW WITH LOWER BOUNDS

Proof. Throughout this proof, we look at parameterizations by vertex cover number.

$W[1]$ -hardness of CHOSEN MAXIMUM OUTDEGREE follows from Theorem 6.4 and the reduction given by Lemma 3.9. As OUTDEGREE RESTRICTED ORIENTATION contains CHOSEN MAXIMUM OUTDEGREE, it is also $W[1]$ -hard.

Lemma 3.3 gives a reduction from CHOSEN MAXIMUM OUTDEGREE to MINIMUM MAXIMUM OUTDEGREE, which gives $W[1]$ -hardness of MINIMUM MAXIMUM OUTDEGREE.

The reduction from TARGET OUTDEGREE ORIENTATION to CIRCULATING ORIENTATION from Lemma 3.4 gives $W[1]$ -hardness of CIRCULATING ORIENTATION. UNDIRECTED FLOW WITH LOWER BOUNDS contains CIRCULATING ORIENTATION as a special case, and thus is also $W[1]$ -hard. \square

We have not yet dealt with ALL-OR-NOTHING FLOW. It turns out that the NP-completeness proof of ALL-OR-NOTHING FLOW from Alexandersson [2] also provides a $W[1]$ -hardness proof for ALL-OR-NOTHING FLOW with vertex cover as parameter. For completeness, we present the argument.

Theorem 6.7. *ALL-OR-NOTHING FLOW with vertex cover number as parameter is $W[1]$ -hard.*

Proof. Suppose we have an instance of BIN PACKING with k bins of size B , and positive integers a_1, \dots, a_n . Take a graph with vertices $s, t, v_1, \dots, v_n, w_1, \dots, w_k$, with arcs (s, v_i) with capacity a_i ($1 \leq i \leq n$), (v_i, w_j) with capacity a_i ($1 \leq i \leq n, 1 \leq j \leq k$), and (w_j, t) with capacity B ($1 \leq j \leq k$). Observe that there is an all-or-nothing flow of value kB in this network if and only if the BIN PACKING instance has a solution. Indeed, arguing as in [2], each vertex v_i receives a_i flow, and must send this across exactly one outgoing arc to a ‘bin vertex’ w_j . Each w_j must receive exactly B flow and send that to t .

Note that $\{s, w_1, w_2, \dots, w_k\}$ is a vertex cover of size $k + 1$ of the network. \square

7 Conclusion

We showed that various classical instances of flow, orientation and capacitated graph problems are XNLP-hard when parameterized by treewidth (and even pathwidth), but FPT for a novel graph parameter, stable gonality. Following Goethe’s motto “Das Schwierige leicht behandelt zu sehen gibt uns das Anschauen des Unmöglichen”, we venture into stating some open problems.

1. Is stable gonality fixed parameter tractable? Can multigraphs of fixed stable gonality be recognized efficiently (this holds for treewidth; for $\text{sgon} = 2$ this can be done in quasilinear time [7])? Given the stable gonality of a graph, can a refinement and morphism of that degree to a tree be constructed in reasonable time (the analogous problem for treewidth can be done in linear time)? Can we find a tree partition of a subdivision with bounded treebreadth? The same question can be asked in the approximate sense.
2. Find a multigraph version of Courcelle’s theorem (that provides a logical characterisation of problems that are FPT for treewidth, see [14]), using stable gonality instead of treewidth: give a logical description of the class of multigraph problems that are FPT for stable gonality.
3. Stable gonality and (stable) treebreadth seems a useful parameter for more edge-weighted or multigraph problems that are hard for treewidth. Find other problems that become FPT for such a parameter. Here, our proof technique of combining tree partitions with ILP with a bounded number of variables becomes relevant.
4. Conversely, find problems that are hard for treewidth and remain hard for stable gonality or (stable) treebreadth. We believe candidates to consider are in the realm of problems concerning “many” neighbours of given vertices (where our use of ILP seems to break down), such as DEFENSIVE ALIANCE and SECURE SET, proven to be $W[1]$ -hard for treewidth (but FPT for solution size) [5], [6]. For such problems, it is also interesting to upgrade known $W[1]$ -hardness to XNLP.
5. Other flavours of graph gonality (untied to stable gonality) exist, based on the theory of divisors on graphs (cf. [3], [4]). Investigate whether such ‘divisorial’ gonality is useful parameter for hard graph problems.

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