Teachers' Inquiry in
Mathematics Education

TIME²

TIME² course guide - design for inquiry-based mathematics education

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A guide for a course on designing for inquiry-based mathematics education

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Teachers' Inquiry in Mathematics Education

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1. Introduction

This is a guide for a course on designing for inquiry-based mathematics education. In this guide we present activities to support teachers in developing a systematic approach to the process of designing mathematics lessons and resources. Teachers are invited to experience how to move through several phases and perform various types of analysis, using the language of "design principles" to discuss their progress and insights.

This course was developed as part of the TIME project, and, as such, has been piloted in the Netherlands and online with international partners. Parts of the course have also featured in workshops in Croatia, Denmark, and Slovenia. The course was taught in between design cycles of Lesson Studies, preparing and designing for the next lesson. However, the course can also be taught independent of Lesson Studies. Some previous experience with lesson design is beneficial, but not prerequisite. As designing is part of the course, participants should have an occasion in mind to pilot the outcome of the course.

Theoretical background for this course is provided by the TIME^2 Compendium for Designing Inquiry-based Mathematics Education (from now on called the "Compendium") and the TIMEless: A short introduction to Lesson Study - TIMEless ideas for professional development (from now on called the "LS guide").

2. Introductory activities

2.1 Workshop activity: Formulating goals

Aim	The main aim of the activity is to raise awareness of the difference between the goals of the lesson (to be achieved by the students) and the goals of the lesson study (to be achieved by the team of teachers), as well as to provide opportunity for the participants to discuss various ways of formulating and evaluating goals appropriate to be pursued in lesson study. Through the activity, the participants are encouraged to consider educational goals beyond delivery of a nice lesson, hence to see themselves not only as teachers, but as researchers.
Prerequisite	Familiarity with the phases and basic tenets of Lesson Study
Time	45 minutes
Required material	
Main issue: Lesson Study (LS) is an organized activity that is based on the intention of a group of teachers to improve their teaching practice by experiments and through reflections based on these experiments. Certainly, each teacher tends to achieve ideal conditions and results in the classroom, but there is a gap between idealism and reality caused by everyday constraints. This activity aims to support teachers in improving the formulation of their intentions, it brings perspective to their LS, that they are invited to perform experiments and that each such experiment should be organized around a concrete goal. In the end, all these goals could be broadly connected under the umbrella of 'closing the gaps', but it is a subtle art to formulate goals which could be reflected on and achieved in one study lesson. A clear articulation of goals is a prerequisite before starting to (re)design activities for the LS.	
Task description	The activity starts with the following instructor's question to the participants: *Could you recall yourself at the beginning of your teaching career, what were your ideals and ambitions?* Next, the instructor asks the participants to quickly reflect on their everyday teaching and how it compares with their initial ambitions. A short open discussion follows in which participants share their reflections. For example, participants first write on a paper "ideals and ambitions" and hang them on one side of a wall marked "ideals"; next they write on a different paper their teaching "reality" and hang them on another side of a wall marked "reality". This way everyone shares and people connect. Possible

	upgrade: hanging papers on two curtains - closing curtains symbolizes closing the gap. After a short recollection (5 minutes) about the main idea of Lesson Study, the instructor raises the question about the difference between the aims of the study lesson for the students and for the teachers. Do the participants see the difference? The discussion lasts around 5 more minutes. The participants are asked to think about their practice and formulate three goals that they find would be suitable for a Lesson Study. After 5 minutes, participants are organized into groups in which they share and discuss the written goals for 15 minutes. The groups are given a further task to write a few pairs of one student goal and one teacher goal. Each pair should correspond to the same lesson. Conclusions are presented plenary with comments from the instructor.
Instructor's actions	In the first part of the activity, the instructor's aim is to raise the participants' attention to the 'educational gaps'. The participants have certainly experienced these gaps which should serve as a source of further development of the goals for lesson study. The instructor recalls that the Lesson Study emphasizes the collaboration of teachers and the inquiry role in which the teachers position themselves. This should ensure that the participants are prepared to differentiate the learning goals of the lesson from the goals of the lesson study. *We advise to use page 14 and 15 of the LS guide for the terminology of research theme, mathematical context, and learning goals.* In the final part, the instructor can motivate or support the discussion by providing some examples: *Goal of the lesson: an understanding of the differential quotient as one way to quantify change.* *Goal of the Lesson Study: identification of the characteristics of the strategy to end a lesson by summarizing the solution of the (specifically designed) problem based on students' results (e.g. time needed, ways for selecting or referring to students' results, conditions for the initial problem)*

	Also see pages 34 and 35 of the LS guide. During the discussion the instructor follows the presentation and provides comments with the particular emphasis on feasibility of the goals, and the difference between goals being set for the students and formulate for the teachers' lesson study.
Further study	TIMEless: A short introduction to Lesson Study – TIMEless ideas for professional development

2.2 Introduction to design: three examples

Aim	To raise awareness of various possibilities and choices when designing for a specific topic. To discuss how design choices could be topic of investigation in a lesson study.
Prerequisite	Suggestions for reading or studying in advance: MERIA: Practical Guide to IBMT & MERIA Teaching Scenarios.
Time	45-60 minutes
Required material	The required material is: three different examples for introducing a topic. These can be - the examples in the Compendium (page 4 to 7), with a short analysis, or - examples prepared by the instructor, or - examples about introducing sequences provided below in the appendix - examples prepared by participants (e.g. for a specific topic from different textbooks they have at hand or that they like to address in a Lesson Study). The latter is probably best. In any case, the examples should vary for instance in use of context, or level of being theoretical or visual.
Main issue: Each topic in mathematics can be addressed in different ways (and for various reasons)	
Task description	Plenary 1: Introduction to the workshop [5 min.] Participants compare the three approaches and discuss in school teams which example is the best, or most appropriate for their context/school/classroom (and try to formulate why), for what reasons and under what kind of conditions. [15 min.] Plenary 2: When using each one of the tasks, what could be the learning goal of a lesson study with that task/example? [10 min.] Participants try to describe what choices the designers have made. Is it clear why they made those choices? [10 min.] Participants present their findings and summarize. [10min.]
Instructor's actions	The instructor shares the three different examples and discusses them with the participants during plenary 2. The instructor invites the participants to

	address similarities and differences, pros and cons, and think of how students might work with these materials. Try to go beyond personal preferences with follow up questions like: - What are your experiences with the teaching/learning difficulties with this topic? - In which approach do you think the learner learns the most in relation to a learning goal? Explain. Compare to how your students are used to work. - To what extent is the topic addressed in a meaningful or relevant way for the students? How is the content motivated? - Do you see any elements in these approaches that stimulate IBMT? As a conclusion, during the summary, the instructor tries to connect arguments for or against certain approaches to underlying design choices related to theories for teaching and learning (general and/or domain specific). This is expected to elicit the need/importance of more theoretical explorations, and analyses of the topic (mathematical and didactical) as an introduction to the next activities.
Further study	Compendium

Appendix to 2.2: alternative examples

These are three textbook designs for the introduction of sequences (recursions, or difference equations) in upper secondary or higher education.

Example 1. A traditional textbook might start by introducing new mathematical language for the new topic (Figure 1). In general, such a choice might be inspired by the attempt to provide a reference book for students.

> 1.1 Introduction
> Difference equations usually describe the evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(n+1)$st generation $x(n+1)$ is a function of the nth $x(n)$. This relation expresses itself in the *difference equation*
> $$x(n+1) = f(x(n)).$$
> We may look at this problem from another point of view. Starting from a point x_0, one may generate the sequence
> $$x_0, f(x_0), f(f(x_0)), f\left(f(f(x_0))\right), \ldots$$
> For convenience we adopt the notation
> $$f^2(x_0) = f(f(x_0)), f^3(x_0) = f\left(f(f(x_0))\right), \text{etc}$$

Figure 1: A structured introduction to difference equations (Elaydi, 2005)

The first activity offered in the textbook (not depicted here) is to explore the sequence $\{f^n(x_0)\}$ with $f(x) = x^2$ and $x_0 = 0.6$. According to the text, it can easily be found (with a calculator) that the iterates $f^n(0.6)$ tend to 0. Most notable about this text is how it uses more formal mathematical language from the outset. It develops from within the "world of mathematics". This entails the risk of isolating the topic from problems to which it provides solutions. This illustrates a choice for designers: does the text provide a resource for inquiry or a mathematically structured overview?

Example 2. The second approach to sequences begins with the following task for students (Figure 2).

Drug level
A patient is ill. A doctor prescribes a medicine for this patient and advises to take a daily dose of 1500 mg. After taking the dose an average of 25% of the drug leaves the body by secretion during a day. The rest of the drug stays in the blood of the patient.
Investigate
- Use calculations to investigate how the amount of the drug (in mg) changes when someone starts taking the drug in a daily dose of 1500 mg with for instance three times 500 mg.
- Are the consequences of skipping a day and/or of taking a double dose so dramatic?
- Can each amount of drug in the blood be reached? Explain your answer.

Product
Design a flyer for patients with answers to the above questions. Include graphs and/or tables to illustrate the progress of the drug level over several days.

Figure 2: Promoting students' inquiry in a task on recursive reasoning (Winsløw, 2017)

The task has the potential to involve students in reasoning about sequences in context, and in developing – or at least feeling the need to develop – symbolisations for sequences. The choice for the drug level context is an attempt to provide students with a meaningful and relevant motive to start performing repeated and recursive calculations. The recursive pattern offers opportunities for students to discover a characteristic like convergence. Moreover, students' results can provide a starting point for introducing symbols for describing calculations with difference equations, and for further mathematical explorations into the converging patterns. This approach tries to provide a meaningful context for the development of the concept of sequence and convergence.

Example 3. A third approach to sequences provides students with context to inquire into sequences. In this approach students are provided with data in the shape of a table and a graph (Figure 3).

Humans have introduced many populations in new habitats, accidentally or intentionally, to study their dynamics. Some of them have become very interesting for ecological research. For example, in 1937, eight pheasants (*Phasianus colchicus torquatus*) were introduced in a protected island in front of Washington. There were enough resources in the island and no predators. Also, the island was far enough from the coast for the pheasants to fly out of it.

In the following table and its corresponding graph representation, there are some collected data on the evolution of the size of that population between 1937 and 1942, taken each year and at the same stage of the cycle life.

Year	Size of population
1937	8
1938	26
1939	85
1940	274
1941	800
1942	1800

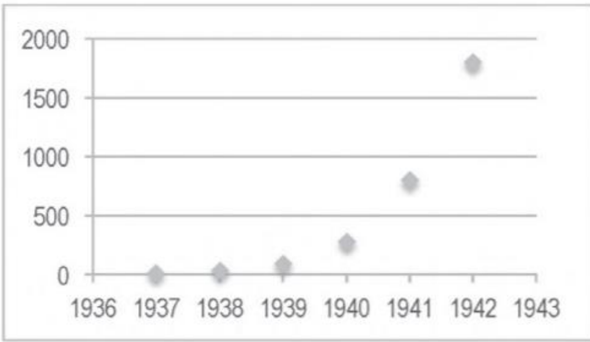


Figure 3: Providing a context for a study research path for dynamic systems (Barquero et al., 2013)

The lesson is expected to start with a discussion of the situation leading to an emerging overarching question (e.g., how to describe and predict population dynamics?) and several related issues might elicit, like: What sort of assumptions on the population and its growth and surroundings should be made? How can one create predictions and test them? By sharing these issues, the group of students is supposed to be able to relate to each other's contributions and to connect intermediate questions and findings to the overarching question. The challenge for the teacher is to connect the contextual questions to generalized mathematical problems and solutions.[1] Like in the previous example, these questions relate the mathematical topic of sequences to a concrete real-world situation and support students in developing coherent concepts and skills that are not isolated in the world of mathematics. Such an approach requires a careful balancing act of student activities and plenum discussions initiated by the teacher. The use of a textbook as a structured source of reference for the topic can be a valuable resource needed during the inquiry process.

These examples show that various choices can be made, depending on your target audience, the learning goal and conditions like time, space and resources available. Consequently, design requires inquiry into your own practice, into the domain of the topic and into potential starting points or sources of inspiration for your design.

[1] The situation in higher education with lectures and workshops (often with teaching assistants) will require a different strategy for this challenge when compared to the situation in secondary schools.

2.3 Workshop activity: Introducing the design aspects, phases, and scheme

Aim	This activity addresses aspects and phases of lesson planning and design. Participants learn how a design process can be structured.
Prerequisite	Experience in lesson design and teaching mathematics
Time	45 min
Required material	On-line session: Jamboard, Padlet or alike Face-to-face session: A2-paper, post-its
Main issue: Designing for (inquiry-based) mathematics education is a challenging endeavor. In order to structure this process, in this task a scheme and essential steps are introduced. In this way participants learn to reflect on the process itself.	
Task description	- The participants discuss in small groups and write down important aspects of lesson design on post-its or in Jamboard. Here is a sample Jamboard with aspects. - The aspects are presented by the "snowball method": The Snowball Method is a way for participants of the workshop to teach each other important concepts and information. Participants begin by working alone. Next, they collaborate with a partner. Partners form groups of four. Groups of four join together to form groups of eight. This snowballing effect continues until the entire group is working together. - In plenary discussion participants order the collected lesson design aspects order on a value scale according to importance. - The instructor introduces the design scheme from the compendium (see also the appendix below). - In plenary discussion participants compare the design scheme to the aspects they had introduced themselves.
Instructor's actions	The instructor - prepares to use either post-it/A2 paper or Jamboard - gives instructions for the several parts of the activity - moderates the plenary discussions

	- introduces the general scheme (use the PowerPoint slides, if desired) - Takes a leading role in comparing the participants aspects to the aspects in the design scheme
Further study	Read the compendium on the design scheme, page 10 and 11.

Figure 4: Sample Jamboard with design aspects.

Appendix to 2.3 Design scheme

See the Compendium for a larger version.

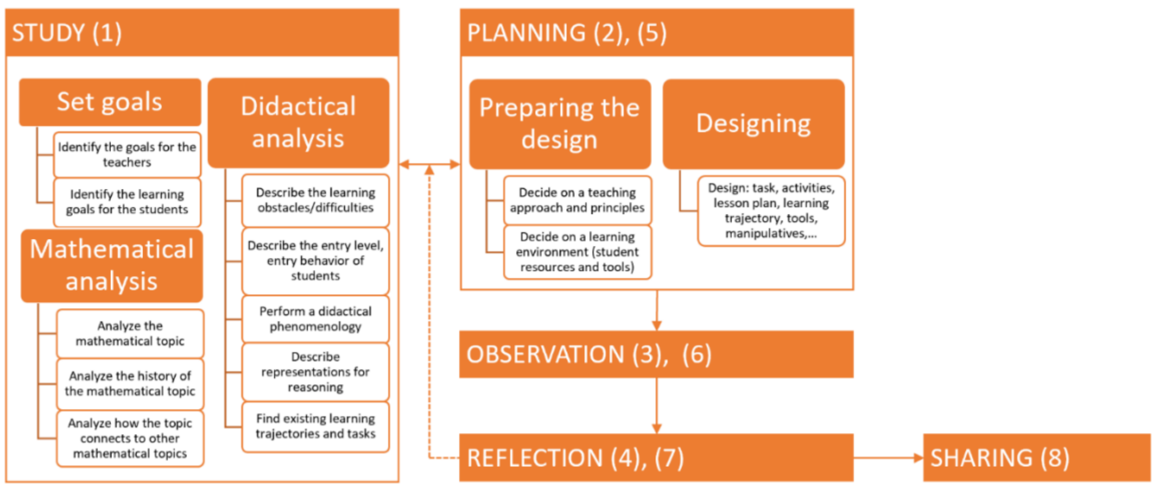


Figure 5: A scheme for the process of designing a mathematics lesson.

3. Analysis activities

3.1 Workshop activity: Mathematical and didactical analysis of a problem situation

<table>
<tr><td>Aim</td><td>Teachers should:
- revisit mathematical knowledge at stake in the open lesson in order not to assume the same approach as textbooks, but to draw on their mathematical background when designing and analysing lessons.
- become able to identify other strategies for students to address the problem at stake in the lesson by revisiting former approaches
- identify potential gaps in knowledge to be taught, which might represent an obstacle for the learning of the knowledge at stake
- see the value of working mathematically when addressing a teaching problem and designing new activities to overcome the problem
- potentially see the value of drawing on the historic origin of the knowledge at stake, when 're-contextualising' the knowledge</td></tr>
<tr><td>Prerequisite</td><td>The mathematical background of upper secondary teachers. If the knowledge at stake goes beyond the core of curriculum, further prerequisites might be relevant.</td></tr>
<tr><td>Time</td><td>90 minutes</td></tr>
<tr><td>Required material</td><td>- Teachers bring teaching materials relevant for the knowledge at stake (though NOT knowing the problem at stake).
- Mega post-its or other poster-like materials suited for sharing group work during the formulation phase and validation phase of the participants work. If the pandemic forces us online a Padlet or Jamboard can be used, but it is less flexible.
- A printed version of the lesson plan in TIMEplate for each participant</td></tr>
<tr><td colspan="2">Main issue: To activate the course participants in the mathematical analysis as part of the study and planning phases of lesson study. In the course it can function as the way to prepare the participants for the open lesson.</td></tr>
<tr><td>Task description</td><td>First part (30 minutes)
Problem: When we are to teach [knowledge at stake in the open lesson], what does it mean? What</td></tr>
</table>

	mathematical notions are needed? What does it consist of? Milieu: Resources (teaching materials for upper secondary and for university courses for preservice teachers) brought to the course by the teachers, materials to be found online. *Second part (30 minutes)* Problem: How do you think your students would solve this problem [of the open lesson]? How would you solve the problem? Try to sketch solutions. Milieu: Same as before, now including the knowledge constructed during the first activity. Teachers also draw on their experiences from own classrooms with current classes. *Third part (30 minutes)* Problem: How to observe the expected and none-expected strategies? Milieu: as previously, though with more information about the specific class (grade, level of mathematics, previous taught knowledge, tools usually available etc.), organisation of groups, classroom and other design choices.
Instructor's actions	*First part* Devolution (3 minutes: Remind the teachers how the particular piece of knowledge is part of curriculum, how it is well known to challenge students. Therefore, it is relevant to revisit this piece of knowledge if we are to address the teaching of it from a different perspective, which might include its historic origin, but could also simply be reminding ourselves of what we know. Maybe, there are subtle details we discover we need to revisit or unfold. Action (12 min.): Participants work in groups and discuss how to present this in upper secondary by revisiting e.g.: curriculum, guidelines, exam exercises, textbooks, university teaching materials and other resources…. Formulation (5 min.): initial a-didactic formulation takes place in the groups and during action. During the last groups present mega post-its. Validation (5 min.): the posters are put on the walls and highlights are shared based on observations

	done by the instructor during the action & formulation phases. Institutionalisation (5 min.): The teacher organise the knowledge represented at the posters as a mind map to identify notions, methods, terms or contexts relevant for addressing and understanding [knowledge at stake in the lesson – instructors most complete this beforehand to guide their own orchestration of the knowledge relevant for their course]. The Idea of icebergs (see the next activity) are used to explain how the subtle details might be crucial for the knowledge at stake to become floating. The mathematical notions drawn upon, potential contexts they stem from etc. from participants' posters are named as key notions and those creating floating capacity. *Second part* Devolution (2 min.): Present the problem of the lesson and the design of milieu for the participants. Pose the problem of this activity Action (10 min.): The participants will start wondering how would high achieving students address the problem? How would the low achieving students? How would you [as a mathematician]? What strategies and mathematics is involved in pen and paper solutions? What strategies are involved if students are allowed to use CAS? What instrumented techniques are required? If not, the instructor can pose those questions to the participants. Or the instructor can ask the participants to revisit the work done during the previous activity, see if some of the mind map can be useful in this context. Formulation (5 min.): Groups share again on mega post-its after formulating ideas in their groups. Validation (7 min.): The post-its are compared. Why do some groups expect more or less of their students? Address the role of CAS or DGS. Institutionalisation (5 min.): This is done by the proposed approaches formulated by the [team who runs the open lesson] teachers inviting all to develop this further if needed. The team might develop a large matrix to facilitate this.

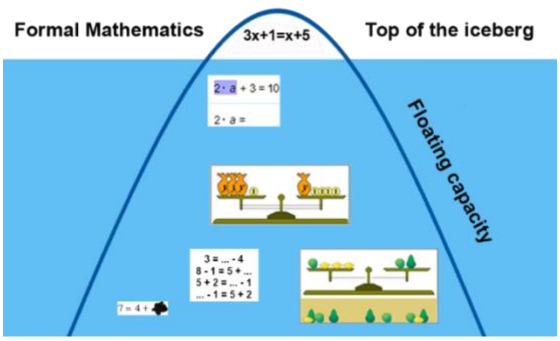
	Third part Devolution (5 min.): Handing over the problem including the setting of the lesson (hand out the lesson plan). Action (10 min.): Participants address questions as: How to observe the strategies are there writings to be noticed? Formulations in the groups we are hoping to hear? How to notice this for further discussion? (pictures, notes,...). What do you want to notice from teachers' actions supporting the goal of the lesson (orchestration of validation, linking different strategies, how to act during students' actions)? Again, if the participants do not consider these questions, they might be posed by the instructor. Formulation (5 min.): The above discussed ideas are written in the lesson plans handed out and shared verbally in plenum. Validation (5 min.): this is run as a guided dialogue by the instructor. This is prepared by considering the question yourselves and based on observations done when the lesson was experimented previously. Institutionalisation (5 min.): The main points, from the discussion is repeated and the 'house rules' of observation is shared with the participants. This should link and reason the idea of expected strategies not being influenced by observers.
Further study	The section in the course material should be introduced as further study offering the participants additional didactic terminology for structuring and using mathematical analysis in their work planning own lessons.
Notice	If you have more time, it would be valuable to ask the participants to run through the activity twice, where the second run is the participants (in the local groups) address a teaching problem including a mathematical topic themselves. Thus, to revisit the content knowledge from both a mathematical and a teacher perspective, inviting historical origin of the knowledge at stake. Then try to formulate at problem, where you again revisit the knowledge involved in solving the problem. In such a second run, the teachers certainly need more time to develop their thinking.

3.2 Workshop activity: Icebergs

Aim	- Reflect on representations of a mathematical object/concept in the curriculum and evaluate how these representations could facilitate learning processes. - Develop an overview of pre-knowledge with respect to a topic, by making and using an inventory of iceberg models.
Prerequisite	The participants should have some overview of the students' learning process leading up to the topic that they would like to analyse as part of this activity.
Time	60-90 minutes
Required material	Examples of models of an iceberg. This task can be done online, in a shared/cloud presentation software, such as Google slides. In face-to-face meetings large sheets of paper and markers will suffice. If possible, as ready-made shape of an iceberg on the slides/sheets could help structure the drawings.
Main issue: Within a network of related mathematical concepts, it is important to identify the ones the students should already be acquainted with: their pre-knowledge. Any gap in pre-knowledge has consequences for the learning process. A design should, if possible, include tasks that activate or assess necessary pre-knowledge of the students. Pre-knowledge needs to be formulated not exclusively in term of formal mathematical knowledge (language and representations). The learning tasks are more likely to be effective when they also connect to students' informal knowledge and real-life experiences, associated to various representations. Formal mathematics becomes meaningful by grounding it in meaningful previous experience and pre-knowledge. The iceberg is a model to depict how pre-knowledge as understood from several representations can be ordered from informal to informal. The informal at the bottom of the iceberg creates floating capacity for the formal at the top (see the compendium, page 14, for more on icebergs. The iceberg helps to identify potential pre-knowledge that might be revisited or need to be (re)emphasized.	
Task description	Participants study some sample icebergs. Then they design an iceberg for the topic they have in mind for a design. Finally, participants' icebergs are discussed.
Instructor's actions	- Introduction of the iceberg metaphor. The iceberg visualizes the following idea: students' more formal knowledge of mathematical manipulations, as, for example, expressed in their notebooks – the visible top of the iceberg – should be supported by previous experiences and pre-knowledge – invisibly, under the water surface. So, the visible more formal mathematical

	manipulations are kept afloat (meaningful) by underwater floating capacity formed by (meaningful) informal experiences and knowledge in various representations (see examples). The designer needs to look under water for those experiences and find tasks that reactivates them. Discuss the examples and the connected iceberg. [15 min] - Participants formulate a topic which they want to work on. This can be the topic concerning the lesson to be designed. They get a blank iceberg and try to make an inventory of prerequisites of students; the models used, and the mathematical concepts concerned. [20 min] - Let participants present their iceberg - In relation to the presented iceberg start a discussion on the level of abstraction used in the different models. 1) To what extent is the floating capacity enough for the design of a lesson around your topic? 2) What is missing? (need for broadening deepen concepts?) 3) Is this sufficient to reach a next step in formalizing the mathematics concerning your topic? (vertical mathematization) [(number of icebergs) *10 minutes]
Further study	Suggestions for reading: Webb, D. C. (2017). The Iceberg model: Rethinking mathematics instruction from a student perspective. In L.West & M. Boston (Eds.), *Annual perspectives in mathematics education: Reflective and collaborative processes to improve mathematics teaching* (pp. 201–209). Reston, VA: NCTM.

Examples (also see the Compendium)



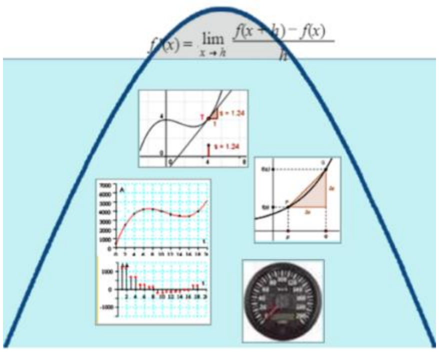


Figure 6: An Iceberg model for equations (left) and for the derivative (right)

Furthermore, one needs to be aware that icebergs for topics can be connected. This awareness, and also the activity of making the floating capacity and these connections explicit is an important activity for teachers and for students.

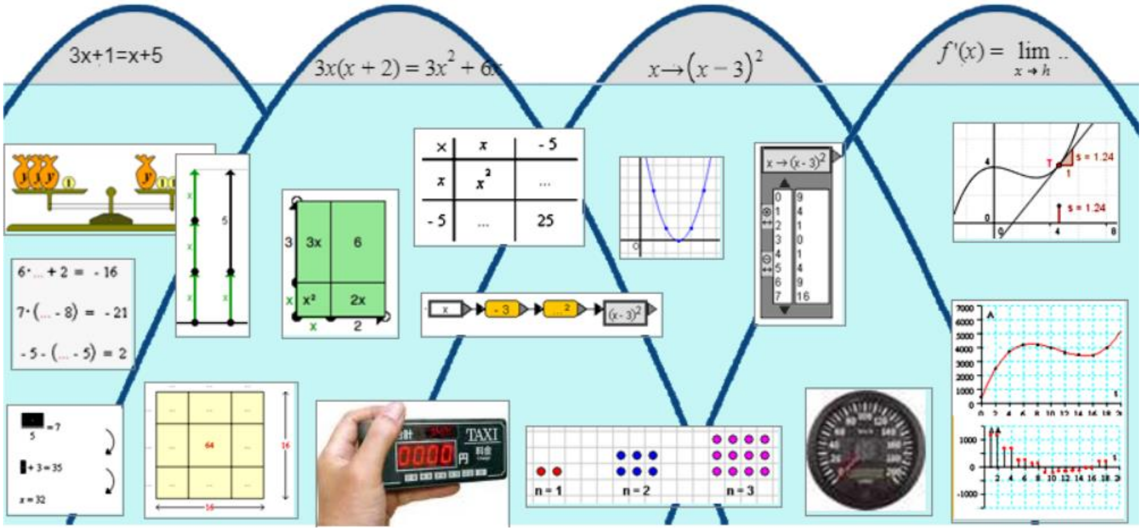


Figure 7: A cascade of connected Icebergs surrounding the derivative

3.3 Workshop activity: Didactical analysis - obstacles

<table>
<tr><td>Aim</td><td>The aim is to discuss the term “students’ difficulties” and use the discussion as a starting point for setting goals for the Study Lesson and its didactical analysis. How do we identify difficulties, are they sometimes advantages and who should help us in the team when designing lessons for students with difficulties?

One more aim is to introduce the notion of obstacle from TDS as a piece of previous knowledge that stands in a way of learning, but also in teaching, of a new mathematical knowledge which are targeted by given mathematical tasks.</td></tr>
<tr><td>Prerequisite</td><td></td></tr>
<tr><td>Time</td><td>45 minutes</td></tr>
<tr><td>Required material</td><td>Post-it notes or sheets of papers on which participants can write their answers.</td></tr>
<tr><td colspan="2">Main issue: We may often hear that “math is hard” and “the students are struggling”. In the didactical analysis the designer identifies a certain “gap” and uses the insights from the analysis to design a lesson that hopefully narrows that gap. In the analysis, when we discuss didactical aspects of teaching and learning, we often use the terms such as “learning difficulty” or “students’ difficulty” that might have many different meanings to different teachers. This may cause confusion among the designers and make it hard to decide what kind of difficulties should be addressed. By making aware that there are different “difficulties”, trying to organize or classify them and discuss ways to deal with them, the participants will develop a stronger sense for didactical issues and improve their designing skills.</td></tr>
<tr><td>Task description</td><td>Discussion on the term “difficulties”, participants write examples on post-it, instructor leads the discussion about the organization of these examples and directs it to the classification (see the text below, 15 minutes).

After the discussion, the instructor gives a small lecture about the Brousseau’s examples of obstacles. (see the text below, 5 minutes)

The participants are asked to discuss one example of generalization or analogy from the perspective of obstacles. The activity ends with presentations of each group. (see the text below, 15+10 minutes)</td></tr>
<tr><td>Instructor’s actions</td><td>Instructor organizes the initial discussion and leads the participants to the classification described in the the text below.</td></tr>
</table>

	The instructor introduces the notion of an obstacle and provides a few examples. In the main activity instructor devolves the task and observes the work of the participants. In the discussion the instructor makes sure that the concepts are described in mathematical terms clearly and that the analogy is explicit. Also, the instructor directs the discussion towards using the term obstacle in this context.
Further study	• Idea of an obstacle from the perspective of TDS • Mathematical learning difficulties subtypes classification, https://www.frontiersin.org/articles/10.3389/fnhum.2014.00057/full • Students struggling with math, https://www.readandspell.com/struggling-with-math

Appendix to activity 3.3: types of challenges that hinder learning

From the literature, but also drawing from teachers' own experience, we are aware that there are different types of challenges that could hinder learning. Some challenges could be classified as **cognitive**, neurological or motorical (e.g. dyscalculia, dyslexia, ADHD or lack of visual-spatial processing) difficulties. Students with such difficulties usually have special needs and require adaptation of the lesson. So, this is the most common meaning of the term "difficulty". Other students might find the lesson challenging for **sociopsychological** reasons – they may have personal or family problems, deal with health issues that hinders their disposition, or lack motivation. These reasons certainly influence student's engagement but can be very unpredictable and can be handled during extra hours.

Next group of difficulties could be jointly called lack of **prerequisite competences**: some students may lack knowledge, skills or attitudes that are necessary for the construction of new knowledge, but which had to be acquired earlier in schooling. The attitudes can vary from lack of interest to inhibited or impulsive engagement ("hasty answer", "speaking without thinking"), while skills may also be connected to cognitive difficulties and surface as slow recall of facts, struggle to keep information in the working memory, poor number sense, or difficulties with mental representations of mathematical concepts. Difficulties might come also from the imperfections of the **resources and teaching style**: students will find the lesson challenging because the complexity has not been gauged appropriately or the instructions for the activities might be unclear. These issues require high awareness of the teacher and might be reduced by the group style of work of teachers promoted in project TIME.

Finally, the lesson may be challenging for the reasons stemming from the nature of mathematics or the order in which the curriculum introduces concepts to students. Brousseau has studied such phenomena in mathematics education and described them under the name "**obstacle**". For him, obstacle is a piece of knowledge that applies correctly in one context but shows to be wrong once the context is generalized (expanded, brought to a higher level of complexity). For example, it is true that every integer has an immediate successor, but this claim is not true (we could even say it does not make sense) once we consider a bigger set of rational numbers. Similarly, multiplying a positive number with a positive integer will yield a bigger number, but this is not true anymore if we multiply a positive number with a positive number (a fraction or a real number) less than 1. A little bit different example comes from the French educational system: there is a strong emphasis to use decimal numbers and approximate values from a very early age and students are used to results with two decimal digits, so when they encounter irrational numbers many misconceptions may surface (such as tendency to think and use that pi is *equal* to 3.14). In this activity the participants will be asked to discuss the example from high school mathematics that a single linear (implicit) equation represents a line in space, since it represents a line in plane.

The following two examples may be used by high-school teachers to discuss the different obstacles that students encounter in geometry and algebra. The first example is given in the passage from 2D to 3D analytical geometry. Both the line and the plane in 3D can be seen as analogues of the line in 2D. The teachers can shape their discussion based on the questions:

Consider the notions of the line in the plane, line in the space and plane in the space, and their corresponding algebraic descriptions in terms of equations. Which difficulties and obstacles can you find as the students move to study of space?

The second example is the generalization of the concept of the tangent. Students first encounter tangents to circles. Next, they might learn about the tangents to ellipsis. In this case, the tangent is a line that has exactly one common point with the curve. Once we consider the tangent to a parabola, this may not be the case as the symmetry axis of the parabola has exactly one common point with it, but it is not its tangent. Furthermore, when dealing with more complicated curves, e.g. graphs of functions, tangents can intersect the curve many times and the property is of "having only one common point with the curve" has to be described locally using a vocabulary of limiting processes.

3.4 Workshop activity: Didactical analysis - contexts

<table>
<tr><td>Aim</td><td>A context of a mathematical task is meant to help students make meaningful interaction with the mathematical content. The main aim of this activity is to provide opportunities for the participants to discuss potentials and drawbacks of contexts of chosen mathematical tasks.</td></tr>
<tr><td>Prerequisite</td><td>...</td></tr>
<tr><td>Time</td><td>45 minutes</td></tr>
<tr><td>Required material</td><td>Examples of mathematical tasks in the supplement</td></tr>
<tr><td colspan="2">Main issue:
Lesson study aims to “close or narrow down gaps” between teaching and students' learning. A particular lesson study activity addresses a gap considering a chosen mathematical content. What makes it hard to learn? What makes it hard to teach? What is known about the learning and teaching of the subject? Didactical analysis focusses on these issues. For instance, it focuses on analysing the context of a chosen mathematical task, its relevance, meaningfulness and further features.</td></tr>
<tr><td>Task description</td><td>Instructor leads a small plenary discussion (5 minutes) about the meaning and relevancy of the context based on the example:

The picture shows a roller coaster in an amusement park. Calculate the area of the region enclosed by the roller coaster track and the ground.

Participants form groups of two or three. Each group is given three mathematical tasks (below). Instructor asks the participants to read and discuss given examples by considering how the context of a task supports or hinders student's learning of mathematical target knowledge. At the end of this activity, groups present their findings (10 minutes).

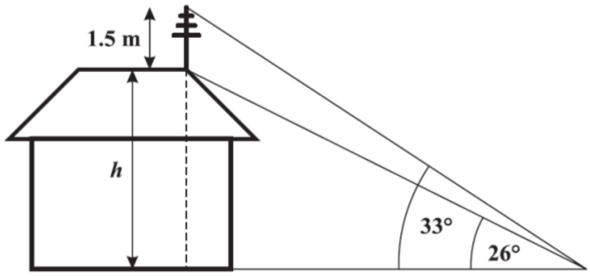
In the second part of this activity, participants analyse the features of the contexts in sample tasks “Slide” and “Stage” in elaborated scenarios and present their findings (30 minutes). The activity ends with presentations of the groups.</td></tr>
<tr><td>Instructor's actions</td><td>Instructor reminds participants to discuss first the mathematical target knowledge of a task and then the role of the context regarding the following questions:

• Is the context important and/or relevant? Can it be (partly) left out? Does it appear as a “noise” to the task? Does it support reasoning about the</td></tr>
</table>

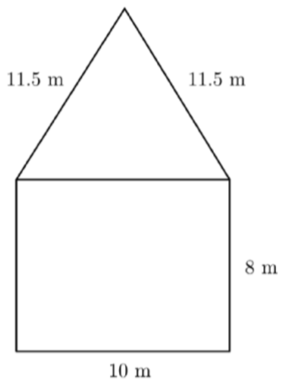
	problem? Does it provide meaningful and accessible mathematics? Does it provide an opportunity to check the solution? • Generally, can the given context be considered as meaningful, realistic, (un)necessary? Can any other context feature be added? • Finally, with the given target knowledge in mind, can the context be changed to be more appropriate one? Instructor further motives the participants to look for these context features in the tasks "Slide" (from project MERIA) and "Stage" (see below). The discussion can be followed by a discussion on possible students' strategies and the choice of the pedagogy pursued: what is the goal of the lesson with this task? After the participants think about this question and comment, the instructor can invite them to comment on the following questions: Is the goal the discussion on the context and hypothesis that should be made before using mathematics? Is it understanding the problem and finding a few numerical examples of possible sides? Is it the application of technology and using the method of regression? Is it the formulation of the function and finding its extreme using algebra? This discussion is wrapped by the concluding remarks of the instructor on the importance of making deliberate didactical choices in planning the lesson.
Further study	• Idea of mathematization in RME, in particular horizontal mathematization (in the booklet "MERIA - Practical guide"). • MERIA Scenario Slide • "A short introduction to Lesson Study – TIMEless ideas for professional development".

Appendix to activity 3.4: examples

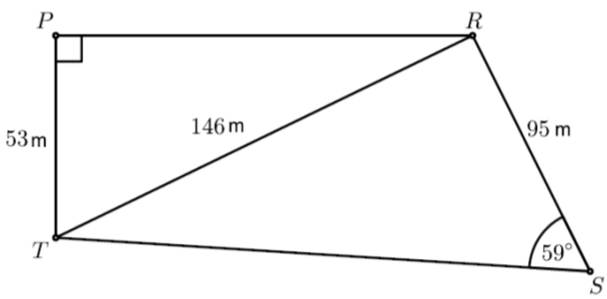
Example 1. Determine the height h of the house shown in the picture.



Example 2. The picture shows the front of the house. What is the height of the house from its base to the top of the roof?



Example 3. The ground plan of popular attractions in an amusement park is shown in the picture.



a) How far are the attractions „Pirates" (P) and „Rafting" (R)?

b) How far are the attractions „Space" (S) and „Tornado" (T)?

Sample problem. "Stage"

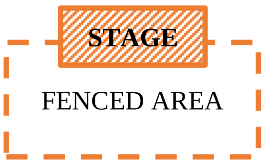
The problem is:

A group of students is organizing a concert after the pandemic. They have bought 240 m of fence and plan to make three sides of the visitors' area with it. The fourth side will be the stage. If one person needs 1 m², what is the maximum number of people that can visit the concert?

In this task there a few decisions deliberately made that will provide possible points for discussion:

1) Is the context of the concert and pandemic supporting the mathematical development of solving strategies? Is there more support, motivation or "noise" in this context?
2) Is the demand of 1 m² per person realistic and does it help or confuse the students? Should the shape of this area of 1 m² be discussed (are we packing squares or disks in a rectangle)? Could we just ask for the maximal possible area?
3) Is it necessary that the visitors' area is a rectangle?
4) Is the width of the stage fixed? Is it better to have the wall instead of the stage?

These considerations will lead to different approaches to modelling the problem and perhaps different mathematical issues.



4. Design principles activities

4.1 Workshop activity: Design principles

Aim	The aims are - For participants to ask about their designs more often: Why do I think this will work? - to introduce participants to the notion of design principles - to motivate participants to use design principles to base their educational designs on and to communicate about their educational designs - for teachers to realize that design principles are a way to learn from testing a scenario in a way that transfers to the times you design other scenarios.
Prerequisite	Participants should bring an old scenario (lesson plan) they designed and a plan for a new scenario (lesson plan). Preferably they have done some of the mathematical and didactical analysis for the new scenario in a previous workshop activity. Having some pre-knowledge of RME or TDS is an advantage, but not necessary.
Time	90 - 150 minutes
Required material	
Main issue: two central issues in Lesson Study are to communicate with colleagues about a lesson design and to learn from testing it. The main questions they should ask themselves are “Why do I think this will work?” and after the lesson “Did it work (as intended)?”. Design principles offer a means to formulate and communicate the ideas and assumptions that underlie a scenario design and answer the first question. A design principle consists of a few central features of an argument to support a design. By talking in terms of design principles participants are invited to formulate ideas that transcend the scenario at hand and apply more generally, thereby allowing them to learn for future designs and educational situations.	
Task description	The task consists of several subtask of which the instructor could select - **Introduction**: introduction of the notion of design principle and motivation for their use (interactive lecture and one exercise) – 60 minutes - **Old scenario**: Looking at an old scenario and studying which design principle were unconsciously applied

<table>
<tr><td></td><td>
New scenario: Looking at a new scenario participants are working on, discussing which design principles might inspire the design.
If there is limited time, one could skip the “Old scenario” part.
</td></tr>
<tr><td>Instructor's actions</td><td>
Introduction (60 minutes)
The instructor gives a short lecture (15 – 20 minutes)
Topics lecture:

The origin of design principle from academic Design Research (slide 3).
The relation (overlap and complementarity) of teacher's knowledge and researcher's knowledge. Both researchers and teacher want to develop knowledge about teaching that applies more generally. Both want to know why lessons are successful. (slide 3)
Introduce the notion of design principle, first form an example, that next is illustrated as applied in a MERIA scenario (slide 4 and 5). Then a formal definition illustrated in this example – What? When? How? Why? (slide 6), followed by the formal definition (slide 7), and then another example (slide 8).

Exercise (30 minutes)
Participants are presented with two design principles:
<table>
<tr><td>The intertwinement principle. Topics should not be taught in isolation. In contrary, many mathematical topics are heavily intertwined and should be taught that way.</td></tr>
<tr><td>The contiguity principle. Align words to corresponding graphics: place printed words near corresponding graphics. Synchronize spoken words to corresponding graphics</td></tr>
</table>
Participants are then invited in small groups to describe to answer the What? When? How? And Why? of these principles (five characteristics).

In short plenary participants' answers are discussed
Next participants are invited to describe one of their own design principles (which they consider a good piece of advice).
In another short plenary (a selection of) these design principles are presented and discussed by the participants.

</td></tr>
</table>

	Questions for the discussion: o Are these principles formulated according to the five characteristics? o Are these principles general, in the sense that they could be applied to all sorts of teaching situations? *Old scenario (30 – 45 minutes)* Prerequisite: participants have brought a scenario that they designed together. - Participants are invited in their groups to discuss which principles underlie this design, even if they were not outspoken during the design. Participants take care to formulate these principles according to the five characteristics. - In short plenary (a selection of) these design principles are presented and discussed by the participants. The same question as before can be posed: o Are these principles formulated according to the five characteristics? o Are these principles general, in the sense that they could be applied to all sorts of teaching situations? *New scenario (30 – 45 minutes)* Prerequisite: participants are planning to develop scenarios in small groups, and have done some preparatory analysis, and have stated some learning goals. The task develops as the task above, but now the challenge is to develop or select principles that could help shaping the scenario to be developed.
Further study	A. Bakker (2019), Design Research in Education: A Practical Guide for Early Career Researchers. Routledge.