THE VALUES OF SIMPLICITY AND GENERALITY IN CHASLES'S GEOMETRICAL THEORY OF ATTRACTION [AUTHOR'S VERSION]

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Abstract. French mathematician Michel Chasles (1793-1880), a staunch defender of pure geometrical methods, is now mostly remembered as the author of the *Aperçu Historique* (1837). In this book, he retraced the history of geometry in order to expound epistemological theses on what constitutes a virtuous practice of geometry. Amongst these stands out the assertion that the values of generality and simplicity in mathematics are intimately connected. In this paper, we flesh out this claim by analysing Chasles' geometrical solutions to the century-old problem of the attraction of the ellipsoids. We show how these solutions echo Chasles' evaluation of the relative strengths of geometrical and analytical methods, and how they embody a set of normative rules for the geometer's practice whose observance Chasles deemed necessary and sufficient for the development of general methods and theories. Geometry and Mechanics and Simplicity and Generality and Chasles

1. Introduction

In the twelfth section of the first book of the *Principia*, Newton dealt with the attraction of spherical bodies. Most notably, he stated what later became known as his 'Shell theorem', which states that the attraction of a spherical surface is equal to that of a fictional point located at its center, and where all of the mass of the surface is concentrated. However, Newton did not attempt to study the attractive forces of bodies or surfaces of other, more elaborate shapes. A few decades later, MacLaurin would write his *Treatise of Fluxions* as a response to some of the criticisms that Newton's concept of fluxions had elicited, and especially to Berkeley's. MacLaurin's treatise showcased the power and fruitfulness of Newton's fluxions, while grounding them in what he considered to be a rigorous framework. The rigor of his work, he claimed, was made undeniable by the fact that he proceeded *'after the manner of the Antients*[1]*'*.

To do so in a manner that would meet the criterias of rigour laid out by Newton's critics, MacLaurin elected to use in this treatise a reworked Euclidean framework, while still incorporating fluxions and algebraic calculations. As a result of this adaptation of the language of Ancient Greek geometry, his propositions were mostly stated in terms of equality of ratios. In chapter XIV of book I of the *Treatise*, MacLaurin expanded on Newton's work on the attractive forces of bodies. In particular, he stated and proved important results on the attraction of ellipsoids. When computing attractions, rather than writing formulas explicitly giving a direct expression for the attraction of a body, MacLaurin would compare the attractions of two similar bodies. One of MacLaurin's results, which drew a great deal of attention, can be thus restated: the attractions exerted by two confocal ellipsoids of

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[1][Mac42], Book I, Preface, p.ix

revolution on an exterior point S are in the same ratio as their volumes[2], provided that S be on the axis of revolution of these ellipsoids or in the plane of their equator. MacLaurin was also able to show that the same result holds for two general confocal ellipsoids if the point is on their main axis[3].

Such results, which state that the attractions of two confocal ellipsoids on an exterior point are in the same proportion as their volumes, provided that certain conditions be satisfied, would all shortly thereafter be gathered under the label 'MacLaurin's theorem'. To prove that this theorem holds for any two confocal ellipsoids on any exterior point (which, in what follows, we will call 'MacLaurin's generalized theorem'), and to give an expression for this attraction, was a most important problem for mathematicians to try their hands (and new methods) at throughout the 18^{th} century. This was especially so in France, where MacLaurin's work on tides and the shape of the Earth had been awarded a prize by the Académie de Paris[4]. Thus, questions pertaining to the attractions of ellipsoids became topics of interest for mathematicians and physicists all throughout the development of Newtonian mechanics. However, what the actors of this development deemed to be a general and satisfactory answer to these questions changed, and so did the mathematical tools used to that purpose[5].

One of the mathematicians involved in this scientific endeavour is French geometer Michel Chasles, who published several articles and memoirs on the attraction of ellipsoids between 1837 and 1842. An important actor in the renewal of pure geometry in the wake of the works of Monge, Carnot, and Poncelet, he used this problem as a case-in-point to showcase the newly-found power of modern, pure geometry, and its various epistemic advantages over analytical methods, whose hegemony he had staunchly criticized[6]. In this paper, we set out to examine his work on the attraction of ellipsoids, and its connections with some of the broader epistemological theses he had previously expressed in his *Aperçu Historique*[7].

[2]Or masses, if these ellipsoids are taken to be of homogeneous density. To abbreviate and simplify expressions, in this paper, we will only consider homogeneous ellipsoids of density $\rho = 1$, and omit this term when reproducing equations and formulas.

[3]These results and MacLaurin's proofs can be found in [Mac42], Book I, Chapter XIV, §649-654. To what extent did MacLaurin prove these results was, as we will see in section 3, a point of contention between Chasles and previous commentators.

[4]See [Gra97] for more on the context of MacLaurin's reception in continental Europe.

[5]A longitudinal study of the history of mechanics following this particular problem would probably give rise to a finer picture of the fluctuation of disciplinary and epistemological boundaries between geometry, algebra, and mechanics. Indeed, the computational difficulties this problem entails, and the rich geometrical interpretations it allows, make it a robust and abundant source for reflections and discussions. Therefore, it is no surprise that such a picture would include the works of most of the mathematicians usually associated with the development of mechanics: D'Alembert, Legendre, Poisson.. all worked at some point in their career on the attraction of ellipsoids. Even far into the 20th century, mathematical physicists continued to go back to this problem for renewed insights (see, for instance, [Arn85]).

[6]In first approximation, analytical geometry here refers to geometry done with the help of Cartesian coordinates, and algebraic or infinitesimal calculus. Pure geometry, on the other hand, is geometry done without such tools. We will go back to this distinction later in the paper, and explain how such crude distinctions do not really capture the disciplinary and epistemological boundaries that actors such as Chasles identify with these terms.

[7]See [Cha37a]. The complete title of this book translates into *'General historical survey of the origin and development of methods in geometry, in particular of those that relate to modern geometry, followed by a memoir of geometry on two general principles of that science, that is, duality and homography'*. To a large extent, this text will constitute our main source to study Chasles' early epistemology and practice of geometry, which substantially evolves in his later works, as he obtained a chair at the Sorbonne where he developed and taught what he called *Higher Geometry* (see [Cha52]).

Despite Chasles' fame and institutional importance in his time[8], his mathematical works have been the object of considerably fewer studies than some of his direct contemporaries, such as for instance Poncelet for the history of projective geometry , or Poinsot for the history of geometrical mechanics[9]. And yet, his influence on generations of students and scientists is undeniable. In the decades following the publications of his work on ellipsoids, many students from various French universities would refer to and comment upon it, especially in the context of doctoral dissertations[10]. These mentions all bear witness to the large and rapid circulation of the ideas we will be discussing in the following sections. This circulation would be a fruitful one. Chasles' papers on the attraction of ellipsoids stem from his theory of second-degree surfaces, which would later prove to play an important role for his student Darboux, as the latter studied orthogonal systems of surfaces for his doctoral dissertation[11], but also for the development of Hamilton's theory of quaternions, in which Chasles' geometry of the ellipsoid (via a translation into English by Reverend Charles Graves) is mentioned[12]. In this context, a thorough analysis of Chasles' proofs of MacLaurin's generalized theorems provides us with a vantage point atop which a finer picture of French geometrical mechanics, and its influence on several episodes in 19^{th} century geometry, can be gained.

Furthermore, Chasles' proofs are particularly interesting inasmuch as their primary aim is not the establishment of new results, but rather that of a strong case for pure geometrical methods. Indeed, a few years prior to Chasles' writings on this subject, a memoir by Poisson[13] had definitively solved the problem with regards to the standard expectations for such a proof shared by the Parisian mathematical community[14]. What Chasles produced, however, is a set of alternative proofs whose value resides not in the epistemic assurance they provide, but rather in their epistemological quality[15]. While previous results concerning the attraction of ellipsoids answered the problem in a satisfactory manner, their proof remained insufficiently simple, to Chasles' eyes. He did not pursue simplicity for its own sake, however.

[8] Let us here remember that, when he died, Chasles was a member of most European academies, the first foreign recipient of the Copley medal, and had been teaching at the Sorbonne for several decades. In a obituary published in the New York Times shortly after his death, he is even said to be *'the most distinguished mathematician in France'*. However, his fame quickly declined over the decades following his death. The reasons for this decline are still unclear. Possible explanations include the infamous Vrain-Lucas affair, by which he was publicly ridiculed and which left a lasting mark on the collective memory of his life, but also the celebration of other 19^{th} century geometers such as Poncelet or Von Staudt as the main protagonists of the development of projective geometry in subsequent historical narratives.

[9] In his general history of 19^{th} geometry, [Gra07] acknowledges that *'research needs to be done on Chasles' presentation of projective geometry, and the way his work eclipsed that of Poncelet'* (Introduction, p.vii). Recent attempts to do just so include [Nab06], [Che16]. Despite the fact that several of Chasles' first successful scientific contributions dealt with mechanics (whether it be with kinematics or, in our case, the theory of attraction), and that Chasles taught mechanics at the Ecole Polytechnique between 1841 and 1851, his work is rarely mentioned in more than passing footnotes in general studies in the history of mechanics. See for instance [GG90].

[10] See for instance [Bor40], [Cat41], [Hop63], [Pes43] among many others. [Ber92] (p.xiii), in a eulogy pronounced a dozen years after Chasles' death, asserts that these proofs of MacLaurin's theorem have become *'classics'*, taught by most professors who desire to teach this subject. To what extent this claim is truthful, however, remains unclear.

[11] [Dar66]. For a comprehensive analysis of said dissertation, as well as of its relation with Chasles' early geometrical works, see [Cro16], particularly chapters 2 & 3.

[12] See [Ham63], p.280-288.

[13] [Poi33]

[14] Chasles himself acknowledges it, see [Cha46], p.640.

[15] In that respect, these proofs yield rich insights into the practice of re-proving, a practice which has been studied in detail in [Daw15].

What motivated the production of his alternative proofs was the notion that simplicity and generality, both in mathematical methods, proofs, and theorems, are intimately linked[16]. Chasles conceded that analytical proofs such as Poisson's were swift, ingenious, and efficient; but their convoluted calculations could not lead to a satisfactory explanation of the truth of MacLaurin's generalized theorem. To his eyes, the complexities of artificial, computational machines such as Algebra may have led to quick and powerful solutions, but at a certain cost: these proofs make us blind to what is really happening to the figures at play, behind the cogwheels of calculus. These proofs do not permit any insight into the mathematical truths they produce, and as such, are unfit for further generalizations. However, Chasles claimed, pure geometry was finally able to provide epistemologically virtuous solutions. Modern methods, inspired by the likes of Monge and Poncelet, made it possible to find proofs that resort only to the resources provided by human reasoning, and the figures themselves. Such proofs, he asserted, would necessarily be simple and general. To understand these claims, and the way they inform Chasles' mathematical practice, is what we set out to do in this paper.

2. A geometer's ethos

First, we must sketch Chasles' early epistemology of geometry in greater detail[17], which we will then connect to both his historiographical and his mathematical practice, through the study of his work on the problem of ellipsoids. We will show that Chasles' epistemology of geometry was no mere theoretical musing. His philosophy of geometry, and the epistemological values it enlists, actively guided and structured his scientific work. In that sense, these values played a strong normative role, both in Chasles' selection of problems, and in the tools he elected to use in order to prove and communicate them. These epistemological values take a moral turn, as from them derive a set of normative rules for the practice of the geometer that Chasles not only preaches, but closely observes.

2.1. **Tracking the means for a greater generality in past geometrical methods.**

In 1829, the Académie Royale des Sciences de Belgique proposed a prize for the best essay on the topic of *'the philosophical examination of various geometrical methods used in recent geometry, and, in particular, of the method of reciprocal polars'*. Chasles' winning entry was to be immediately published, but some political turmoil caused by the Belgian Revolution put this project to a temporary halt[18]. As he was finally allowed to send a manuscript to press in 1837, Chasles had more than doubled the size of his dissertation. On top of the two memoirs on the principles of homography and duality which the Belgian Academy had rewarded, he had added a detailed historical and philosophical study of the development of Geometry, from the classical works of Thales and Pythagoras to the recent discoveries of Dupin and Poncelet; as well as 34 notes which go in depth into some technical, historical, or philosophical details evoked during the historical account itself.

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[16]Such a claim is far from being unique in the history of mathematics. For instance, in his autobiography, *Récoltes & Semailles*, French mathematician Alexandre Grothendieck makes constant use of the notion of *'childish simplicity'* ('simplicité enfantine'), which he uses to describe extremely complex theorems, precisely because of their perceived generality. See [McL03] for more on that issue.

[17]We will mainly focus on Chasles' *Aperçu Historique*; a more detailed description thereof, with special emphasis on the theme of generality, can be found in [Che16], from which some of the examples discussed below are borrowed. Several shifts occur after a chair of Higher Geometry was created at the Sorbonne for Chasles, which we do not discuss here.

[18]See [Que72], p.36-37.

Despite its title, this book should not be regarded as a purely historical account of the development of a certain science. History is here used as a means to an epistemological end : to understand what enabled geometers to develop ever more general methods and results, and to further these investigations on that very same path. As Chasles puts it,

> 'While recounting the march of Geometry, and while presenting the state of its discoveries and recent doctrines, we mainly set our sights on showing, by a few examples, that the hallmark of these doctrines is to bring, in all parts of the science of extension, a new simplicity and the means for a generalization, until then unknown, of all geometrical truths[19].'

Such a project very much bears the mark of a certain *milieu*, namely that of the École Polytechnique. An important tradition, perhaps best embodied by the influential teaching of Lagrange, had been shaped there, in which historical considerations were central to the very understanding of what constituted the strengths and advantages of the methods and discipline being taught[20]. For instance, in the very first pages of his *Traité de Mécanique Analytique*, Lagrange sets out to *'reduce this science to general formulas'*, and to *'unite the different principles found until now under a single viewpoint'.* To do so, in both parts of his book (respectively dealing with statics and dynamics), the first section is devoted to a historical overview of these previously found principles, while the second section unites them within so-called general formulas. History thus plays an instrumental as well as a pedagogical role, and serves to put on display the perfection reached by analytical methods, which reduce to two principles, and two formulas, the sum of all mechanical knowledge acquired throughout the centuries. While Chasles and Lagrange reach very different conclusions as to what tools are best suited for the development of mechanics, the parallelism in their argumentations, and their shared way of producing and reflecting on mathematical knowledge must be emphasized here[21].

Chasles' history of geometry takes the form of a succession in five stages (*'époques'*), each marking a decisive turn in the sort of geometrical methods available to mathematicians, and characterized by an ever-greater generality. However, unlike the sort of narratives for the development of sciences suggested by Comte's well-known theory of the *'three stages'*, Chasles allows for parallel developments to take place. For instance, he claims that both Descartes' and Desargues' geometries display levels of generality which, in their respective fields of application, were previously unattainable, while being contemporary. And yet, these geometries appear in different *'époques'*. Despite this fact, Chasles' historical account can be regarded as narrating the progress of geometry towards generality, and stops at a point where various powerful methods have been found, which still remain to be unified under a minimal set of principles. Chasles' initial project when expanding on his prize-winning dissertation was to include a *exposition dogmatique* of some of these modern geometrical theories[22], including a theory of second-degree surfaces which

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[19]'Nous avons eu en vue surtout, en retraçant la marche de la Géométrie, et en présentant l'état de ses découvertes et de ses doctrines récentes, de montrer, par quelques exemples, que le caractère de ces doctrines est d'apporter, dans toutes les parties de la science de l'étendue, une facilité nouvelle et les moyens d'arriver à une généralisation, jusqu'ici inconnue, de toutes les vérités géométriques.', [Cha37a], p.2.

[20]See [Wan17].

[21]This mathematical *habitus* can be detected in many other works from that period and that *milieu*, such as Lacroix's *Traité du Calcul Différentiel* (1797/8), or Fourier's *Théorie Analytique de la Chaleur* (1822).

[22][Cha37a], p.254. This project was eventually dropped, but gave Chasles the initial content for the redaction of several notes.

we will briefly discuss in what follows. This abandoned project betrays the influence of another fellow Polytechnicien, on whom this perceived interest of returning to the historical development of sciences as a means to track what allowed for their perfecting also left a mark. Indeed, in his *Leçons de Philosophie Positive*, briefly quoted in Chasles' *Aperçu*[23], French philosopher Auguste Comte resorted to a similar instrumental use of the history of science. Comte famously claimed that *'one can only completely know a science once one knows its history*[24]*'*, as only history can reveal how scientific knowledge was formed, how generality was reached, how founding principles for a given science were discovered. This historical study, however, can and must be supplemented in Comte's view by an exposition of the *'dogmatic march'* of said science[25], so that the historical account becomes intelligible.

2.2. **Striving for generality.** The *Aperçu Historique* opens with what historian Karine Chemla calls a *'diagnosis about the limits of ancient geometry'*[26]. These limits consist mainly in a lack of generality that can be found in both the mathematical statements and the geometrical methods used by ancient Greek geometers. While they can certainly be linked, these two limits must first be separated and explained in their own rights[27].

The first of these limits is that certain theorems, or propositions, are redundant for the modern geometer. For instance, in the seventh book of Pappus' *Collection*, 43 lemmas are given which, according to Chasles, *'express a single theorem'*[28]. The multiplicity of these lemmas follows from the fact that ancient geometers had to give different proofs for different configurations of the points involved in the theorem. Several of these propositions pertain to a point on a line, which happens to be either within or without a certain segment. In each of these cases, a different proposition and a different proof are needed, which modern geometry can unite within a single statement (and, thus, a single proof). In this manner, for instance, Chasles explains that *'through the consideration of negative and positive quantities, under a single statement, one theorem can display diverse cases'*[29]. By allowing the symbol ab, which refers to the segment bounded by points a and b, to be either a positive or a negative quantity (depending on the relative position of these points with respect to an arbitrary direction), Chasles would show in his lectures at the Sorbonne how one can achieve generality through the choice of adequate notations[30]. Therefore, the importance for Chasles of the development of an adequate language for geometrical studies must be stressed. Not only is generality linked with the capacity of grouping *statements* (as opposed to truths, results, or formulas), but it is through the development of a certain symbolic or linguistic grasp on geometrical statements, that the latter have been simplified. The development of geometry requires the shaping of certain discursive tools which allow for the description of various related

[23][Cha37a], p.415. It is also likely that the entire Note V (p.288-290) is directed against Comte's 10^{th} *Leçon*, where Geometry is defined a the science of the measurement of extension, which Chasles very much refused.

[24][Com30], Deuxième Leçon, p.82

[25]ibid.

[26][Che16], p.50.

[27]Note that this criticism does not apply equally to the whole of Ancient geometry. For instance, Apollonius seems somewhat immune to it, as Chasles reads his *Conics* as bearing the mark of what would become the foundation of Descartes' analysis; namely that a single propriety between two magnitudes on a conic serves as a unifying notion on which the whole theory is built. See [Cha37a], p.17-18.

[28][Cha37a], p.41.

[29]ibid.

[30]See [Cha52], ch.1. These lectures were given a few years after the texts on the attraction of ellipsoids, and Chasles' reflections on generality expressed therein display some subtle variations, but these are out of the scope of this paper.

configurations to be purveyed at once[31]. The benefits of such general statements are manifold. Not only does a single theorem expressing many other ones allow for a better understanding of the *'general modes of transformation'* at play behind the actual theorems[32], but also from a truly general statement, a large amount of lesser results can be easily deduced. Thus, Chasles claims, from his mystical hexagram alone, Pascal was able to deduce 400 corollaries, precisely because these corollaries all *'**express** a certain property of six points on a conic'*[33].

A second criticism expressed by Chasles towards Ancient geometry is the lack of systematicity in the application of its methods[34]. For instance, Archimedes' method of exhaustion, while quite general with regards to the variety of figures to which it can be, or has been, applied, is no match for modern integral methods as far as the systematicity of the method is concerned. Chasles writes:

> 'The method of exhaustion, which rested on a completely general main idea, did not deprive Geometry of its character of narrow-mindedness and specialization, since this conception, lacking general means of application, became, in each particular case, a wholly new question, which found resources only in the individual properties of the figure to which it was applied[35]'.

This method requires a renewed effort every time it is applied to a new figure, and thus does not follow systematically from the application of a certain set of rules. Let us compare this to the optimistic description of modern geometry Chasles gave at the very end of his history of geometry:

> 'Nowadays, anyone can step up, pick any known truth, and submit it to the various general principes of transformation. In so doing, they will gather other truths, either different or more general, and on these truths similar operations can be carried out-so that one can multiply, almost to infinity, the number of new truths deduced from the first. [..] Genius is no longer required to contribute[36]'.

The generality of a method is here linked to its systematicity, even to the effortlessness of its application. The very need for proof seems to disappear: the study of modern geometry, to Chasles' eyes, resembles an easy, spontaneous passage from one truth to another. This stands in stark contrast to his assessment of the sheer mental effort that Ancient geometry requires from its practicioner.

The form of pure geometry here envisioned and promoted cannot rely on diagrammatic reasoning. While no concern for the rigour of such a form of reasoning is expressed here, a different argument is produced. Diagrams, Chasles claims, are

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[31]This call for a renewal of the language of geometry can also be found in Poncelet's well-known *Traité des Propriétés Projectives*, see [Pon22], p.xxii.

[32]ibid., but also [Cha37a], Note VIII, p.297-301.

[33][Cha37a], p.73. Our emphasis.

[34]This notion of generality can be compared to Steiner's *'systematicity'*. See for instance [Lor16], in particular p.429.

[35]'La méthode d'exhaustion, qui reposait sur une idée mère tout à fait générale, n'ôta point à la Géométrie son caractère d'étroitesse et de spécialité, parce que cette conception y manquant de moyens généraux d'application, devenait, dans chaque cas particulier, une question toute nouvelle, qui ne trouvait de ressources que dans les propriétés individuelles de la figure à laquelle on l'appliquait.', [Cha37a] p.52.

[36]'Aujourd'hui, chacun peut se présenter, prendre une vérité quelconque connue, et la soumettre aux divers principes généraux de transformation; il en retirera d'autres vérités, différentes ou plus générales; et celles-ci seront susceptibles de pareilles opérations; de sorte qu'on pourra multiplier, presque à l'infini, le nombre des vérités nouvelles déduites de la première. [..] le génie n'est plus indispensable pour ajouter une pierre à l'édifice', [Cha37a] p.268-269.

an impediment for truly general methods, to this free derivation of truths we just described. Such a claim is expressed when describing Monge's descriptive geometry:

> 'This useful influence of descriptive Geometry extended naturally just as well to our style and language in mathematics, which it made easier and clearer, by freeing the complications brought about by figures, whose use distracts from the attention we owe essentially to ideas, and which hinders imagination and speech. In a nutshell, descriptive Geometry was adequate for fortifying and developing our powers of conception; to give our faculty of judgement more sharpness and certainty; our language more precision and clarity[37]'.

More so than merely an epistemology of mathematical generality, Chasles' writings are shaping a moral economy of mathematical practice, that is to say a *'web of affect-saturated values that stand and function in well-defined relationship to one another[38]'*. Indeed, as we have seen, the values of generality, simplicity, or systematicity, when mobilized by Chasles, are very much saturated with affects, as the hindrance caused by a diagram contrasts with the easiness through which geometrical truths are combined in one's mind, the blindness of analytical calculations with the clarity provided by proper geometrical reasoning. Furthermore, a theory of their mutual dependence is given, as we will see in what follows. Most important, however, is the fact that from these values derive a set of normative rules, by which the mathematician must abide in order to gain knowledge that is deemed epistemologically good. Indeed, in the midst of historical narrative, Chasles expresses two rules, whose observance he deems to be still required of geometers who wish to advance geometry:

> - 'Generalize more and more particular propositions, in order to attain, step by step, what is most general; which will always be, at the same time, the most natural and the simplest.
> - Within the proof of a theorem or the solution of a problem, never be satisfied by an initial result which would be enough in a particular case viewed independently of its place within a general system in science; but be satisfied by a proof or solution only when its simplicity, or its intuitive deduction from some known theory, will prove that you have attached the question to the very doctrine it naturally depends on.
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> To indicate a way to recognize whether the practice of these two rules has led to the desired goal, that is to say whether we have marched on the true roads of definitive truth, and reached its source, we believe that, in each theory, there must always be, and we must always be able to recognize, some principal truth from which all others easily follow, as simple transformations or natural corollaries; and that this fulfilled condition only will be the mark of the true perfection of a science[39]'.

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[37]'Cette influence utile de la Géométrie descriptive s'étendit naturellement aussi sur notre style et notre langage en mathématiques, qu'elle rendit plus aisés et plus lucides, en les affranchissant de cette complication de figures dont l'usage distrait de l'attention qu'on doit au fond des idées, et entrave l'imagination et la parole. La Géométrie descriptive, en un mot, fut propre à fortifier et à développer notre puissance de conception; à donner plus de netteté et de süreté à notre jugement; de précision et de clarté à notre langage', [Cha37a], p.190.

[38][Das95], p.4.

[39][Cha37a], p.115.

- 'Généraliser de plus en plus les propositions particulières, pour arriver de proche en proche à ce qu'il y a de plus général; ce qui sera toujours, en même temps, le plus simple, le plus naturel et le plus facile;

Generality is thus seen as the benefit of a virtuous mathematical practice. Such a practice, Chasles contends, should strive to root each proposition it establishes into its natural theoretical setting. Such an endeavour, in return, is said to reward the virtuous geometer with the most general proposition they could have hoped to establish. Chasles' successive solutions of the problem of the attraction of ellipsoids attempt to do just so. As we will see in the next sections, selection of this problem as a worthy challenge for further geometrical investigations, of the tools used to solve it, and of the narrative in which his success would be cast, all abide by this normative ideal. But this is no mere rhetoric: as he reproves MacLaurin's generalized theorem, Chasles progressively does away with the specific properties of ellipsoids, and ends up with a proof that can be applied to more general surfaces. What's more, he is able to extract from these very proofs the seeds for what he thought would become a general, geometrical theory of attraction.

The intimate connection between the values of generality and simplicity, at this stage, remains somewhat unclear. In the quote given above, it also seems to be something of a creed, a conviction that virtuous mathematical practice pays off. However, in a footnote immediately following it, Chasles expands on this claim[40]: not only is it an empirical and experimental truth, that generality and simplicity come hand in hand, but it also follows from a theoretical argument *a priori*:

> 'The most general principles, that is to say those that extend their domain to the largest number of particular facts, are necessarily free from the various circumstances which seemed to to give a distinct and different character to each of these particular facts when conceived in isolation, prior to the discovery of their common link and origin: if they were complicated by all these particular circumstances and properties, they would bear the mark of these particularities in all of their corollaries, and would, in general, only give rise to truths which excessively embarassed and complicated themselves. These most general principles are therefore, by nature, necessarily the simplest[41]'.

However convincing or not this argument may seem to us, from its conclusions sprung several epistemic norms and rules which Chasles attempted to abide by, as best he could. Moreover, these norms would guide and inform his reading and understanding of other mathematicians' works, as will be shown in the next section. This circulation of epistemic norms from mathematical to historiographical practice

- Ne point se contenter, dans la démonstration d'un théorème ou la solution d'un problème, d'un premier résultat, qui suffirait s'il s'agissait d'une recherche particulière, indépendante du système général d'une partie de la science; mais ne se satisfaire d'une démonstration ou d'une solution, que quand leur simplicité, ou leur déduction intuitive de quelque théorie connue, prouvera qu'on a rattaché la question à la véritable doctrine dont elle dépend naturellement.

Pour indiquer un moyen de reconnaître si la pratique de ces deux règles a conduit au but désiré, c'est-à-dire si l'on a rencontré les vraies routes de la vérité définitive, et pénétré jusqu'à son origine, nous croyons pouvoir dire que, dans chaque théorie, il doit toujours exister, et que l'on doit reconnaître, quelque vérité principale dont toutes les autres se déduisent aisément, comme simples transformations ou corollaires naturels; et que cette condition accomplie sera seule le cachet de la véritable perfection de la science'.

[40]ibid., p.115-116.

[41]'Les principes les plus généraux, c'est-à-dire qui s'étendent sur le plus grand nombre de faits particuliers, doivent être dégagés des diverses circonstances qui semblaient donner un caractère distinctif et différent à chacun de ces faits particuliers, considéré isolément, avant qu'on eüt découvert leur lien et leur origine commune: s'ils étaient compliqués de toutes ces circonstances ou propriétés particulières, ils en porteraient l'empreinte dans tous leurs corollaires, et ne donneraient lieu, généralement, qu'à des vérités excessivement embarrassées et compliquées elles-mëmes. Ces principes les plus généraux sont donc nécessairement, par leur nature, les plus simples', [Cha37a], p.116.

is usually carried out by the transfer of what we could call a 'stability criterion'. When Chasles assesses positively the generality of a certain mathematical method or proof (past or present), it is rather common for him to base his judgment on the fact that a certain term in the proof (or in the method) can be replaced by a more general one without altering the validity of the proof. For instance, if every occurrence of the word 'circle' can replaced by the word 'conic' in a proof (and if all related terms are replaced accordingly), in a way that does not falsify the proof, then the latter is usually deemed general by Chasles. While such a criterion is not explicitly stated, it matches many generality judgements that Chasles makes both in his historical accounts and in his arguments for the worth of certain notations or principles[42], up until late in his career where this view is more clearly outlined[43]. This stability, for Chasles, bears witness to the fact that a generalization of a result can be achieved without *'genius'*, to the fact that an adequate geometrical language has been achieved, as well as to the deep connection between generality and simplicity.

2.3. **Analysis versus Geometry.** We now turn to Chasles' argument for pure geometry against so-called analytic methods, which he felt dominated the institutional and scientific landscape in French mathematics at the time. Chasles rarely defines the terms Analysis and Synthesis[44] and even seems reluctant to use them, as he agrees with Poinsot, Poncelet, and others that they do not provide adequate descriptions of modern geometry[45]. When he does, however, he resorts to characterizing analytical methods by the use of coordinates and algebraic calculus. Conversely, Synthesis then refers to geometry without coordinates, without algebra, built through 'natural reasoning alone'[46]. However, the term 'synthetic geometry' is almost never used by Chasles (a very special exception being precisely the memoirs on the attraction of ellipsoids), who much prefers to talk of pure geometry, or sometimes of modern geometry[47].

It must be pointed out that Chasles is not opposed to the practice, nor to the teaching of analytical methods. He never mentions doubting the validity of an analytical proof, or expresses distrust towards the metaphysics of the calculus, and he even acknowledges its sheer efficiency. Furthermore, he concedes that Analysis benefits from a certain kind of generality which the very nature of its instruments, such as Algebra, bestows upon it. Indeed, when working with Cartesian equations, the analyst is able to prove a property of all curves of a certain degree via a single proof or computation. More precisely, Chasles claims:

> 'Descartes' Geometry distingues itself from Ancient geometry in another particular regard, which ought to be noted; it establishes, through a single formula, general properties of entire families of curves; so that one could not in this way discover some property of a curve without it immediately yielding similar or analogous properties in an infinity of other lines.[48]'

[42]See for instance [Cha52], Préface, especially p.v-xii.

[43]See [Cha74], p.579.

[44]The classical definitions of these two terms in Ancient Greek geometry notwithstanding ([Cha37a], p.5), as these are of little relevance to our case. Note that in [Cha52], however, an interesting link is drawn between the ancient use of these terms and their more recent acceptations. See the *Discours Inaugural*, p.550-576.

[45]See for instance the quote by Poinsot given in [Cha37a], p.252.

[46][Cha52], p.551.

[47]The term 'rational geometry' also appears sometimes in Chasles' writings.

[48]'La Géométrie de Descartes, [..], se distingue encore de la Géométrie ancienne sous un rapport particulier, qui mérite d'être remarqué; c'est qu'elle établit, par une seule formule, des propriétés générales de familles entières de courbes; de sorte que l'on ne saurait découvrir par cette voie

Thus, the achievements of Analysis are not only valid, but also laudable, as they enabled important generalisations in geometry. However, Chasles found much to be faulted in the epistemological quality of the knowledge provided by Analysis. These flaws have to do mainly with the fact that analytical calculations obfuscate the *intermediary propositions* which form the chain of truths linking hypotheses and results[49]. This obscurity is both a strength, as it allows for a steady march onto the path of resolution without requiring the formation of these intermediary propositions that pure Geometry must construct step-by-step; but it also is the source of a major weakness, in that it prevents the mathematician from finding out what principles, what primary truths are causing the theorem to be true. This is why a certain lack of generality and simplicity can be found in the methods of analytical geometers: as they steadily compute and prove theorems, they are sometimes unable to see that their work could be subject to geometrical interpretation, and then placed in a more natural setting. In particular, Chasles' criticism of Legendre's proofs of MacLaurin's theorem, which will be studied in what follows, exemplifies and strongly substantiates this thesis on the compared strengths of analytical and geometrical methods. As analytical computations swiftly reach the targeted theorems, without the help of geometrical interpretations, they do not provide any information on why these theorems are true, what their causes are. Hence, they cannot form the basis of a generalisation in the sense described previously. Geometrical methods, on the other hand, do not suffer the same fate. To display this difference between both sciences *in vivo* is precisely what Chasles intended to do by giving alternative solutions to the problem of the attraction of ellipsoids and commenting both on its historical development and on the limitations of the existing proofs.

3. Chasles' historical account of the problem of ellipsoids

Chasles' first writings on this problem occur in the context of the historical section of the *Aperçu Historique*, as he described the achievements of MacLaurin, and in particular his *Treatise on Fluxions*[50]. In this text, Chasles found many a reason for admiration and hope for the future of pure geometry. However, he observed that the attraction of ellipsoids, throughout the 18$^{\text{th}}$ century, had mostly remained a *chasse gardée* for analysts. Consequently, after describing the improvements that had been built upon MacLaurin's initial results, and the relative success of these analysts who tried to generalize MacLaurin's theorem, Chasles expressed the desire to find a *'more synthetic proof'* (p.166) of these new, extended results. He thought such a feat had been made possible by the development of a modern, geometrical theory of second-degree surfaces. Later in the *Aperçu Historique*, in Note XXXI[51], such a theory would be sketched[52], after which Chasles claimed to have found such a proof (p.396), thus giving us an idea of the chronology of his works. Furthermore, the first memoir sent by Chasles to the Académie des Sciences on this very subject contains a long historical introduction[53]. These historical accounts are largely structured by the Analysis/Synthesis distinction: Chasles frames his narrative as a competition between two sides, one of which has largely pulled ahead over the past decades. Such an emphasis on this distinction serves to reinforce the strategical

quelque propriété d'une courbe, qu'elle ne fasse aussitôt connaître des propriétés semblables ou analogues dans une infinité d'autres lignes', [Cha37a], p.95.

[49] [Cha37a], p.114

[50] [Cha37a], p.162-170.

[51] [Cha37a], p.384-399.

[52] Let us remember that this very theory was first supposed to be part of an *'exposition dogmatique'* of modern geometry previously mentioned.

[53] [Cha46], p.629-640.

and rhetorical point of his new solutions to this century-old problem: Chasles does not aim to bring to the fore new results, but to show what can be gained by the pursuit of purely geometrical methods. We will not render Chasles' entire narrative here, nor will we attempt to assess its historical accuracy. Rather, we wish to delve into a couple of episodes which reveal how the epistemological theses explored in the previous section inform his historiographical practice, but also his motivation for attempting a new resolution of said problem.

3.1. **MacLaurin's undervalued achievements.** As we've already mentioned, MacLaurin was (in Chasles' view) the first to properly tackle the problem of finding the attraction of ellipsoids on a particule, whether it be inside or outside the body. In particular, he had stated and proved that, in certain configurations, two confocal ellipsoids exerted attractions on an exterior point in the same direction, and with intensities in the same proportion as their volumes. This result, to which we refer as MacLaurin's theorem in what follows, was obtained in a manner that Chasles describes as *'synthetic'*, but also *'elegant'* and *'simple'*. In particular, Chasles explains:

> 'MacLaurin was able to draw, from a few properties of conics, all of the required resources for the solution of this question, which had always passed, in the eyes of the most renowned analysts, as one of the most difficult[54]'.

One striking feature of Chasles' historical account is the extension of the theorems he credits MacLaurin with proving. Previous commentators such as D'Alembert, Lagrange, or Legendre had all asserted that MacLaurin had proved his theorem in one particular configuration (two confocal ellipsoids of revolution attracting a point on the plane of their equator) but only stated a more general case (two general confocal ellipsoids, attracting a point on one of their axes), without giving a proof. To that interpretation, Chasles objects that MacLaurin's turn of phrase when stating this more general case, *'and it will appear in the same manner'*[55] shows that the Scottish geometer was keenly aware that the same proof held in this new configuration, *'without adding or subtracting a single word'*[56]. We do not wish to assess the validity of Chasles' claim here, but simply to point out that it deeply connects with his view of generality, described in the previous section, as he defends the generality of MacLaurin's geometrical method in terms of stability of a proof (and even of a proof-text). This stability, in Chasles' reading, derives from MacLaurin's ability to find a few fundamental properties of conics, and to tie his proofs to these very properties, so that no new adjustement was required when moving from a particular case to a more general one. Chasles as a reader of past mathematical texts does not do away with his epistemological theses. On the contrary, they inform and guide his understanding of MacLaurin's proofs, and conversely, he uses MacLaurin as a case in point to show that geometry has the resources to rival analytical methods on the field of generality.

3.2. **The criticism of Legendre's proofs.** Even in light of Chasles' generous reading, however, there remained room for genereralisation in MacLaurin's theorems. While many analysts attempted to extend these results during the 18^{th} century, little to no progress was achieved prior to Legendre's series of memoirs

[54]'Mac-Laurin sut tirer, de quelques propriétés des coniques, toutes les ressources suffisantes pour la solution de cette question, qui a toujours passé, auprès des plus célèbres analystes, pour l'une des plus difficiles', [Cha37a], p.163

[55][Mac42],p.131. Chasles' emphasis is striking in [Cha46], p.633: '*MacLaurin a formellement démontré son théorème*'.

[56][Cha37a],p.168-169

published between 1788 and 1812. In these texts, and in particular in [Leg88], he was able to prove that MacLaurin's theorem holds generally for the attraction of any two confocal ellipsoids on an exterior point. However, Chasles diagnosed several limitations hindering Legendre's analytical proofs.

The first of these limitations is that Legendre's memoirs were deemed too obscure and complex, so much that even fellow analysts would declare them unsatisfactory, merely on that ground. For instance, Poisson declared Legendre's 1788 solution to be full of *'inextricable computations'* in his own memoir[57]. Secondly, a more technical criticism had also been levelled by Poisson against Legendre's but also other analytical solutions: the integral formulas they ended up with were not direct enough, and involved auxiliary quantities, which it was not easy to determine or eliminate[58]. However, a more crucial limitation to Chasles' eyes is the fact that Legendre's proofs suffer from a certain blindness, which is characteristic of purely analytical methods. After complimenting Legendre's 1788 memoir for proving for the first time the total generality of MacLaurin's theorem, Chasles describes it as a:

> 'Very beautiful and very deep memoir, which would be even richer in interesting results, had Legendre given the geometrical meaning of some of the many formulas through which he had to pass, in order to arrive at the conclusion of the theorem at hand[59]'.

Indeed, as Chasles will keenly point out, some of the algebraic quantities Legendre handled in his memoir could have benefited from a geometrical interpretation. Had he known what some of his infinitesimal elements represented, he could have used some geometrical properties to simplify and improve his solution. But there's more: in so doing, he would have been led to understand what the real cause behind MacLaurin's generalised theorem was, and the connections between the problem of the attraction of ellipsoids, and the theory of second-degree surfaces. Discussing more generally the application of Analysis to such problems inspired by physics, Chasles asserted that:

> 'Already, in the most skilful investigations in mathematical physics, Analysis has uncovered the presence of surfaces [of the second degree]; but most of the times, this ever so happy circumstance has been viewed as fortuitous and secondary, no one has thought that, on the contrary, it may be directly linked to the first cause of the phenomena, and even be conceived as the real origin, and not an accidental one, of all the circumstances it can offer[60]'.

Analysis may have quickly led Legendre to a proof, where geometers were at their wit's ends. In so doing, however, Legendre did not uncover the wealth of riches that a proper understanding of this problem would have disclosed. Therefore, Analysis did now allow him, nor any other analyst for that matter, to generalize his proof properly. Furthermore, Chasles reiterates his opposition between the analyst's reliance on shrewd, artificial computational devices, where the geometer's keen eye observes the natural causes at play. Virtuous geometrical practice, by uncovering

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[57][Poi33], p.499.

[58]ibid., p.500.

[59]'Mémoire fort beau et très-profond, et qui serait plus riche encore en résultats intéressants, si M.Legendre avait donné la signification géométrique de plusieurs des nombreuses formules par lesquelles il lui faut passer, pour arriver à la conclusion du théorème en question', [Cha37a], p.165

[60]'Déjà, dans les plus savantes recherches physico-mathématiques, l'Analyse a dévoilé la présence de ces surfaces; mais le plus souvent on a regardé une si heureuse circonstance comme fortuite et secondaire, sans songer qu'au contraire elle pouvait se rattacher directement à la cause première du phénomène, et mëme ëtre prise pour l'origine réelle, et non pas accidentelle, de toutes les circonstances qu'il peut offrir', [Cha37a], p.251.

such natural causes, is wont to achieve more general and simpler proofs: such is the claim that Chasles' memoirs attempt to show.

Chasles' historical account of this problem serves as a counterpoint to his proofs: it emphasises the hidden, underestimated strengths of geometry, while putting on display some glaring weaknesses of analytical methods. Chasles' alternative proofs, to which we now turn our attention, then completes the argument by not only displaying a rather simple access to MacLaurin's theorem, but also by leading naturally to a more general theory of attraction based on notions which emerged from the specialised proofs themselves.

4. From generality to simplicity : the proof by homography

Around 1837, Chasles wrote and published four texts on the subject of the attraction of ellipsoids, two of which were sent to the Journal de l'École Polytechnique[61], and two of which to the Académie des Sciences[62]. As these memoirs were directed towards different audiences, they display different rhetorics, but also different sets of mathematical tools. In particular, the papers for the Journal were to be read by students of the École Polytechnique, and Chasles did not deem these to be the appropriate place for a full-blown defense of synthetic methods[63]. Therefore, we will focus here solely on the memoirs sent to the Académie des Sciences.

The first proof we examine was first sent to the Académie des Sciences at the end of the year 1837, but was only published in 1846 in the *Mémoires des Savants Etrangers*[64]. Chasles describes it as a *'synthetic proof'*, both within the title and at several occasions in the proof itself. As we already mentioned, this term rarely occurs under Chasles' pen. In the context of these proofs, however, Chasles dons the banner of synthesis for strategical purposes. Indeed, one of the central (and explicit) purposes of this memoir is to convince a broad audience of mathematicians that pursuing modern geometry is a worthy endeavour, and to correct the misguided judgement of past mathematicians who deemed Synthesis incapable of solving this problem[65].

Despite the strategical status of this term of 'synthesis', and the fact that one shouldn't look in Chasles' writings for a precise definition of what Synthesis is (in the context of modern geometry), one may be surprised when looking at the proof itself. Indeed, it contains both algebraic and infinitesimal calculations, Cartesian coordinates, integrals etc., which seemed to be the preserve of Analysis. Therefore, we must understand how some calculations, even when based on algebraic and infinitesimal notions, can be mobilized in a so-called synthetic proof while maintaining the simplicity and intuitiveness that characterizes geometrical insights. In this section, we sketch Chasles' first proof of MacLaurin's generalized theorem, which we divide into three parts. First, we describe how Chasles sets up a geometrical transformation (which he borrows from Poncelet), and establishes two equations through which he is able to control what this transformation does to geometrical figures. Second, we follow Chasles as he shows how fruitful this transformation can be when applied to second-degree surfaces. Third and last, Chasles produces what he calls a 'synthetic calculation' of the attraction of ellipsoids, in ways that were naturally suggested by the geometrical investigations carried out previously. Readers not familiar with projective geometry may skip the technical details on first reading.

[61] [Cha37b], [Cha37c].

[62] [Cha46], [Cha38].

[63] For more on the readership of this journal, see [Mas14].

[64] All references are made to this 1846 edition.

[65] Chasles gives quotes by Legendre and Poisson (among others) to that effect, see for instance [Cha46], p.640.

4.1. **Setting up a geometrical transformation.** Chasles' proof starts off with a series of results pertaining to the geometry of second-degree surfaces[66]. Most of these results were already stated and proved in Note XXXI of the *Aperçu Historique*[67], where they were presented as consequences of a general theory of second-degree surfaces, centered around what Chasles called the *excentric* or *focal conic* of a quadric. These conics are to quadrics what the focal points are to a conic, and were taken to be the basis of a modern, geometrical (read : non-analytical) theory of surfaces. However, in his memoir to the Académie des Sciences, Chasles does not employ this mode of exposition, and instead follows a narrower path into the theory of second-degree surfaces. He claims this shouldn't be seen as a hurdle and a difficulty specific to geometrical methods, for had geometry been studied as much as it should have, therefore as much as Analysis, this geometrical introduction wouldn't have been necessary[68]. What must be shown, then, is how a proof of MacLaurin's theorem derives easily from this geometrical knowledge that is sorely lacking in the training of most young mathematicians[69]. To do so, Chasles will only mobilize theories which were developed some decades prior to his own works, namely the theory of transversal of Carnot[70] and the theory of polar transformations of Poncelet[71]. We now sketch these geometrical preliminaries.

First, considering two homothetic second-degree surfaces[72] U, V, of ratio λ, with distinct centers G and S, Chasles lets a transversal 'turn about the point S'[73]. Readers unfamiliar with the language of 19$^{\text{th}}$ century geometry may think of a transversal line either as a mobile line turning about a point S, or as a collection of lines (mathematicians nowadays would speak of a *'pencil of lines'*) which all pass through a common point S. The aim of the theory of transversals, as devised by Carnot, was to express properties of a system of figures through equations which involve the intersection points of this transversal and the figures of the system. Remarkably, this notion enables Chasles, after Carnot, to use a single letter to refer to the transversal (or to its intersection with a fixed line, for instance), despite the transversal being mobile (or one element of a collection). Thus, for Carnot, 'the theory of transversal is, in the end, the same as that of coordinates[74]'. In Chasles' case, these transversals cross U in two points noted Π, Π', and V in a point π[75]. A relation holds identically during the motion of this transversal:

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[66][Cha46], p.645-669.

[67][Cha37a], p.384-399.

[68][Cha46], p.644

[69]Note that a similar concern is expressed in the *Aperçu*, Ch.VI, p.253 : *'Monge's descriptive geometry is being taught. [..] But the other methods we have talked about are still scattered in the Memoirs of the geometers who used them, Memoirs which may seem lengthy and painful to read, because of the very large amount of new results they include. This is, I believe, the real cause for the detachment to rational geometry, where one mistakenly perceives, and this mistake is to be deplored, a mere chaos of new propositions found by chance, with no connection between them, and no future for a noteworthy improvement of the science of extension'*. Here, rational geometry can be roughly understood to refer to pure geometry.

[70][Car06]

[71][Pon22]

[72]Homotheties (a term introduced by Chasles himself) refer to figures which derive from one another by a homogeneous dilation. Here, the second ellipsoid is obtained by enlarging the first ellipsoid (and shifting its center from G to S), with a scale factor of λ.

[73]This sort of expression would disappear in his later works, where talks of homographic correspondences replace this cinematic viewpoint, in particular in the wake of [Cha52].

[74][Car06], p.65. For more on Carnot, see [Che98], in particular p.172.

[75]A transversal should cross V in two points. However, here, it seems that Chasles is looking at transversals as rays or half-lines, which is unusual, although he does not comment it. See fig.1 & fig.2 below for the geometrical configuration that is being constructed.

(a) $$S\Pi \cdot S\Pi' = \lambda^2(\frac{SG^2}{GH^2} - 1) \cdot S\pi^2$$

Chasles obtains this relation from what could be called an intersecting chord theorem for conics: with the notations above, $\frac{S\Pi \cdot S\Pi'}{G\rho^2}$ (where $G\rho$ is the radius of U, parallel to $S\Pi$) remains constant for all transversal lines passing through S[76]; in particular, it is equal to $\frac{SH' \cdot SH}{GH'^2}$. Since the two ellipsoids are λ-homothetic, $G\rho = \lambda S\pi$, hence the formula (a).

Furthermore, there are exactly two planes perpendicular to one given transversal line of this pencil, which are also tangent to U. Similarly, there is one plane perpendicular to this line and tangent to V. If Γ, Γ', γ denote the points on which the line and the planes cut each other, Chasles notices that $(S\Gamma - S\Gamma')$ is the distance between two planes tangent to U, while $S\gamma$ is half the distance between the two corresponding planes tangent to V. Because of the λ-homothety, the following equation always holds[77] :

(b) $$S\Gamma - S\Gamma' = 2\lambda \cdot S\gamma$$

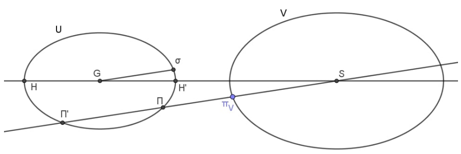

Figure 1

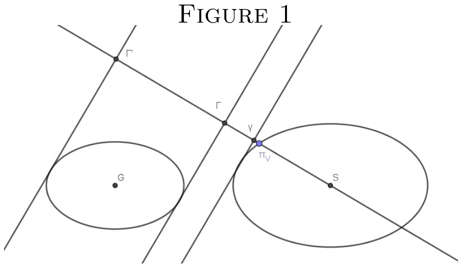

Figure 2. My figures.

Chasles then makes use of the polar transformation of these surfaces U, V with regards to the sphere of center S, and radius 1, which Poncelet had devised a few years earlier in [Pon22]. This transformation was first conceived as a correlation of the points and lines of a plane through geometrical constructions which involve a given, fixed conic. To construct the polar line of a point, with respect to a given conic, one simply draws the two tangent lines to the conic passing through the point, and we join the points where the tangents intersect the conic. Conversely, to construct the pole of a line, one draws the tangent to the conic passing through the points where the line intersects the conic. The pole is the intersection of these two lines.

[76]This quantity can be thought of as the power of point S with respect to a conic. One can find a similar result already in Apollonius' *Conics*, III, 27. Chasles does not mention any particular source for this theorem.

[77]See figures below. These figures do not appear in Chasles' texts, in which very few figures are present, and whose role is mostly to clarify some notations.

In the three-dimensional case, things are more complicated. However, a similar transformation can be constructed in three-dimensional space[78], which correlates points and planes. The key property of this transformation is that coplanar points correspond to planes which all intersect, and conversely. This makes the transformation reciprocal (taking the polar plane of a point, then the pole of this plane, means returning to the original point), and yields several properties for the locus of the poles of the planes enveloping a given surface, or for the surface envelopped by the polar planes of the points of a certain surface. Moreover, here, the polar plane of a point m is perpendicular to the radius of the given sphere, and the distance between the polar plane and S is equal to $\frac{1}{Sm}$.

This transformation yields two second-degree surfaces U', V', corresponding to U, V respectively; but also two planes tangent to U' corresponding to Π, Π', as well as a plane tangent to V' corresponding to π, whose distances to S are respectively

$$SP = \frac{1}{S\Pi}\ ,\ SP' = \frac{1}{S\Pi'}\ ,\ \text{and}\ Sp = \frac{1}{S\pi}$$

Replacing $S\Pi, S\Pi', S\pi$ in (a) and (b), Chasles obtains two polar equations :

(a') $$\frac{1}{SP} \cdot \frac{1}{SP'} = \lambda^2 (\frac{SG^2}{GH^2} - 1) \cdot \frac{1}{Sp}$$

(b') $$\frac{1}{SM} - \frac{1}{SM'} = 2\lambda \cdot \frac{1}{Sm}$$

where M, M', m are the points corresponding to the three planes perpendicular to the transversal line described above. Through these polar equations, Chasles can control the surfaces generated by transversal lines (or specific points thereon). In fact, the first relation helps control the points of this surface, whilst the second helps control its tangent planes. In particular, for any second-degree surface A, if S is a fixed point in space (be it within or without the surface), a transversal line turning around S will cross A in points M, M'. Let $O\mu$ denote the radius of U parallel to the direction of the transversal line. The locus described by the point m on a line of the pencil can now be controlled, and Chasles states the following theorem:

> Let m be a point on each transversal line defined by the equation $Sm = C\frac{O\mu^2}{MM'}$ (where C is an arbitrarily chosen, constant number), then m generates a second-degree surface A', of center S.

This sort of result is central to Chasles' early geometrical practice : a general configuration is set up, in which certain figures (here, a transversal line) can move in certain fashions, while certain relations are maintained between the elements of the configuration. Hence, transversal lines, and the intersections they form with certain fixed elements of the configuration play a role akin to those of the analysts' variables. However, these 'geometrical variables' are imbued with geometrical meaning, as their role in the configuration is not arbitrary, but rather appears naturally from the motion of the transversal lines. Furthermore, by specifying the configuration, and observing the transformations such specifications impose on the *control-relations* (my term), Chasles is able to obtain information on the surface-locus generated by these transversal lines. Note that the notion of transformation plays here a key role: it is by recentering geometrical methods around it that Chasles hopes to elevate geometry to a status more befitting of its capacities. While transformations are here still largely thought of, and used as, transformations of extension and of

[78]See the footnote p.646 in [Cha46] for the details of this transformation.

figures, in later texts they will rather play the role of transformation of *statements*, through the development of the notion of (homographic) correspondences[79].

The theorem Chasles has obtained at this point is of a peculiar kind: it is both a geometrical construction, a relation between certain magnitudes, and a theorem about second-degree surfaces. A good deal of computations went into the derivation of this theorem, but these calculations can be correlated to successive transformations (or specifications) of a certain geometrical configuration. That is to say, they play a role quite different from that of a blind series of analytical calculations, whose only goal is to obtain a formula for a certain magnitude.

Specifying the conditions of this transformation even further with regard to the problem of the ellipsoids, Chasles takes the point attracted, S, to be external to the attracting ellipsoid, A. Then, some of the lines within the pencil of transversals will be tangent to A, crossing it in a single double point $M = M'$. It follows from the equations above that Sm will go to infinity when m correspond to this double point M. Hence, A' is an hyperboloid, and its principal diameters can be shown to be none others than the principal axes of the cone of apex S and circumscribed to A.

This stems partly from the fact that the sets of transversal lines for which $\frac{O\mu^2}{MM'}$ is constant form cones, which all share the same principal axes. These cones, Chasles explains, are precisely the cones involved in Legendre's clumsy and inextricable analytic computation of the attraction of an ellipsoid. What appeared to Legendre as a *'lucky and fortuitous circumstance'*[80] has now been generated through a transformation of a configuration, and a certain relation which controls said transformation. Hence, these cones appear organically, from within the geometrical reasoning itself, and from the study of similar second-degree surfaces. This transformation has plenty of properties which will shed light and clarity upon the proof itself. In particular, they will provide Chasles with a natural, hence simple, choice for the infinitesimal volumes to be integrated. But more can be said of these geometrical objects in the context of confocal ellipsoids, to which we now turn our attention.

4.2. **Rooting the proof in the theory of second-degree surfaces.** The polar transformation of surfaces can now be put to use on pairs of confocal second-degree surfaces to great effect. First, Chasles notices that, for two second-degree surfaces A, B of same center C,

> A and B are confocal if and only if for any plane T_M tangent to A, if T'_M denotes the plane parallel to T_M and tangent to B, the quantity $|d(C, T_M)^2 - d(C, T'_M)^2|$ is constant (see Fig.3).

Chasles' proof of this result goes as follows: if A and B are confocal, suppose their principal axes are CX, CY, CZ. Then, considering two tangent planes T_M and T'_M as described above, let P_A, P_B denote the points where these planes respectively cross A and B. Then, one can see that $CP_A^2 = a^2\cos^2(\angle P_ACX) + b^2\cos^2(\angle P_ACY) + a^2\cos^2(\angle P_ACZ)$, where a, b, c are the principal radii of A. In a similar fashion, $CP_B^2 = a'^2\cos^2(\angle P_BCX) + b'^2\cos^2(\angle P_BCY) + a'^2\cos^2(\angle P_BCZ)$. But due to the parallelism between both planes, $\angle P_ACX = \angle P_BCX$, and so on for Y, Z, and the sum of these squared cosinus equals 1. Therefore, $CP_A^2 - CP_B^2 = a'^2 - a^2$. Conversely, suppose A and B are concentric, and for each pair of parallel, tangent planes T_M, T'_M, $|d(C, T_M)^2 - d(C, T'_M)^2| = \lambda$. Then consider the surface B', confocal with A, such that its principal radii α, β, γ satisfy $a^2 - \alpha^2 = \lambda$. The

[79]In particular, in [Cha52], correspondences are substituted to transformations, which elicited some bitter remarks by Poncelet.

[80][Cha37a], p.251

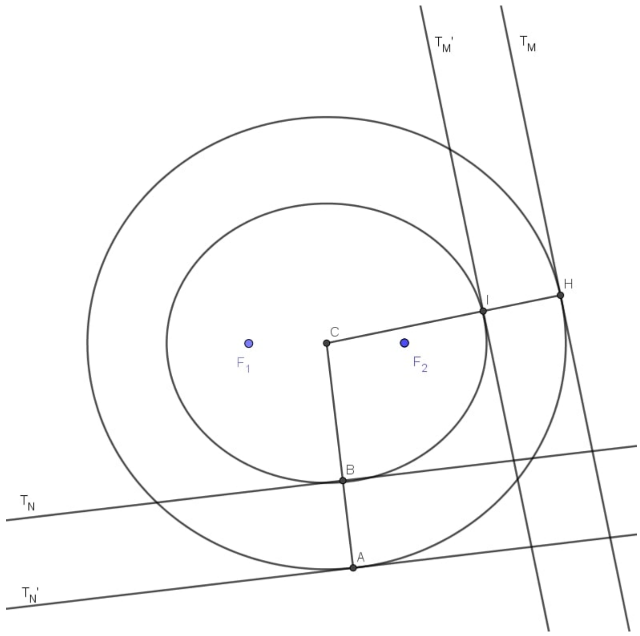

FIGURE 3. Here, $|CA^2 - CB^2| = |CH^2 - CI^2|$ (my figure)

reasoning above shows here that the planes tangent to B' are exactly those tangent to B, hence $B = B'$, thus allowing Chasles to conclude.

Chasles then considers two confocal, same-centered surfaces A, B, and S a fixed point. The polar transformation discussed in the previous subsection yields two polar surfaces A', B', of same center S. Using the equations $(a'), (b')$ given above, Chasles shows that the conditions of the theorem stated previously are satisfied, and thus that A', B' are confocal. This implies that their principal axes are the same. But these axes are precisely the axes of the cones circumscribed to A, B of apex S. Hence, these cones have the same axes as well.

Taking S to be on one of these surfaces, for instance B, then one of the cones will be the plane tangent to B at S. One of its principal axes will be the normal to B at S. The two other axes are also the normals to two other surfaces that go through S, and which are confocal to A. These three surfaces will play an crucial role in what follows[81]. For any second-degree surface A, and any external point S, there will consequently be three confocal, second-degree surfaces $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ passing through S, whose normals at S are the principal axes of the cone of apex S, circumscribed to A. From that follows as well that the principal axes of A' are these normals as well, hence A' and A are confocal.

With these geometrical properties in mind, Chasles is able to determine the principal diameters of A', through a series of (in part, algebraic) computations. From these computations[82] emerge a specific polar transformation of special interest. If $\varepsilon, \varepsilon'$ are two confocal ellipsoids, S a fixed point in space, for any transversal line turning about S, Chasles can construct a specific corresponding transversal line so

[81]Let us note here that they form an exemple of a triply orthogonal system of surfaces, which would go on to form the basis of important works by Gaston Darboux, whose doctoral thesis on this subject was supervised by Chasles himself in 1866.

[82]While we can't give the details here, the reader is referred to [Cha46], see p.655-663.

that both transversal respectively cross $\varepsilon, \varepsilon'$ at E, F and E', F', and so that the following equation always holds[83]:

$$(\star) \qquad \frac{Oe^2}{EF} : \frac{Oe'}{E'F'} = \frac{abc}{a'b'c'} : \frac{\sqrt{a_1^2 - a^2}\sqrt{a_3 - a^2}}{\sqrt{a_2^2 - a'^2}\sqrt{a_3^2 - a'^2}}$$

where O is the center of ε and ε', Oe, Oe' their radiuses, a, a' half their major principal axes, a_i half the major principal axis of $\mathcal{S}_i$.

4.3. **A 'synthetic calculation'.** From this moment onwards, Chasles is ready to tackle the mechanical part of this problem. First he shows MacLaurin's theorem for an infinitely thin, ellipsoidal layer, comprised between two infinitely close, confocal ellipsoidal surfaces ϵ_1, ϵ_2. Denoting the external point attracted S, Chasles considers the infinitely small element of volume intercepted by an infinitely small cone whose apex is S. Let dv denote its volume, r its distance to S, then the attraction it exerts on S is, as Newton's mechanics state:

$$\frac{dv}{r^2}$$

Now, Chasles can decompose this attraction onto three axes SA, SB, SC, using a transformation into spherical coordinates that was well-known of any student of the Ecole Polytechnique. This yields formulas which only involve the angles between the axes and r. Let θ denote the angle formed by SA and r, and ω the angle formed by SA and the plane generated by SB, SC. Then

$$dv = dr \times rd\theta \times r\sin\theta d\omega = r^2 \sin\theta \cdot drd\theta d\omega$$

Hence, the attraction of the ellipsoid along each axis is respectively

$$\sin\theta\cos\theta \cdot d\theta d\omega dr \ , \ \sin^2\theta\cos\omega \cdot d\theta d\omega dr \ , \ \sin^2\theta\sin\omega \cdot d\theta d\omega dr$$

Now, denoting E, F the points where r crosses ϵ_2, G their center, D, D' the points where OS crosses ϵ_2, Oe the radius parallel to r (see Fig.4), the quantities $\frac{SE \cdot SF}{Oe^2}$ and $\frac{SD \cdot SD'}{OD^2}$ are equal[84]. The same points can be considered for ε_1, and since ϵ_1 and ϵ_2 are confocal, G stays the same, as well as $\frac{Oe}{OD}$. Hence, the infinitesimal dGE between each surface of the thin layer, which is equal to dr, can be computed. Indeed, one simply rewrites $SE \cdot SF = SG^2 - GE^2 = \frac{Oe^2}{OD^2}(SO^2 - OD^2)$; differentiating this equation to take into account the thinness of the layer, since we know which quantities are constant, we obtain

$$dr = \frac{Oe^2}{GE}\frac{d \cdot OD}{OD} = 2\frac{Oe^2}{EF} \cdot \frac{da}{a}$$

Therefore, the attraction of the thin layer on the point S, along the axis SA, is

$$2\frac{da}{a}\frac{Oe}{EF}\sin\theta d\theta d\omega$$

This derivation of the attraction of the thin ellipsoidal layer is what Chasles calls a *calcul synthéthique*. Within the context of Chasles' theory of second-degree surfaces, there is now a natural choice for the axes SA, SB, SC. Indeed, by taking SA, SB, SC to be the three normals to the surfaces aforementioned, and if considering a second confocal, infinitely thin ellipsoidal layer, attracting the same particle

[83] Giving the details of this construction is well without the scope of this paper. We hope, however, that the details given above will enable the curious reader to follow Chasles' proofs, which is to be found in [Cha46], p.645-663.

[84] This is the very same '*intersecting chords theorem for conics*' used at the very beginning of this proof.

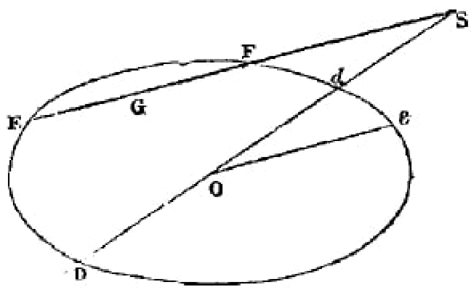

FIGURE 4. One of Chasles' few figures in this memoir. It mainly serves to fix the notations.

S, the same computations hold and Chasles obtains an expression for the attraction of this second layer (along SA) analogous to the one above. Now, using equation ($\star$) to determine the ratio of the attractions of these two layers, Chasles easily finds that this ratio is

$$\frac{bcda}{b'c'da'}$$

Since $dV = 4\pi bcda$, and $dV' = 4\pi b'c'da'$, $\frac{bcda}{b'c'da'} = \frac{dV}{dV'}$, the attractions of two thin, ellipsoidal layers have the same ratio as their weights, as well as the same direction. Decomposing two confocal ellipsoids $\mathcal{E}_1, \mathcal{E}_2$ into series of confocal thin layers, one can establish a correspondance between these layers : let A, B, C denote the principal diameters of $\mathcal{E}_1$, A', B', C' those of $\mathcal{E}_2$, the layers of each ellipsoid will have, as principal diameters, respectively nA, nB, nC, and $n'A', n'B', n'C'$. Two layers are said to be corresponding when $n = n'$. It is then clear that corresponding layers are confocal, and therefore the formulas can be summed. Thus concludes Chasles' first proof of MacLaurin's generalized theorem.

Furthermore, at this point, Chasles is able to determine the direction of the attraction of an ellipsoid on an exterior point. Considering MacLaurin's generalized theorem, it suffices to determine this attraction for a thin ellipsoidal layer, for which we have obtained an expression earlier. Among the three surfaces we have considered earlier, one is the confocal ellipsoid passing through S. Suppose that SA is the normal of this ellipsoid at S, it can easily be seen that the attractions exerted by two elements of volume of the ellipsoidal layer, located on transversal lines passing through S and forming equal angles with SA, are equal. Hence, the attraction is entirely directed along the axis SA. Therefore, we can define level-surfaces as the surfaces which are, at every point, normal to the direction of the attraction of the ellipsoid on this point. It appears then that the level-surfaces of an ellipsoidal layer are its confocal ellipsoids[85].

Chasles' proof displays close to no figures, and proceeds via a large amount of calculations and formulas, even resorting to infinitesimal, integral, and algebraic computations. One could be tempted to rather tip the scales in favor of Analysis when describing it, despite the fact Chasles himself regarded it as a *faire-valoir* of the synthetic approach. To understand this apparent contradiction, one must

[85]While we do not tackle it in this paper, the computation of the intensity of this attraction is carried out in several different ways by Chasles across his memoirs, which also highlight different practices of computations, with various levels of geometrical interpretation involved. In [Cha37c], Chasles uses the Laplacian equation $\Delta V = 0$ (V denoting here the potential), and an integration method first devised in [Lam37]. In other texts, he tries to limit the need for differential calculus.

remember that Chasles' criticism of Analysis did not bear on calculations *qua* calculations, but only on blind calculations, that is calculations based on a thoughtless use of arbitrary cartesian coordinates, and on the idea of reducing all of geometry to a matter of mechanical computations[86]. The case of Legendre is but one striking example of the blindness analytical calculations sometimes provoke; to which bears witness his inability to see the specific nature and role of the cones involved in his integration[87]. As such, Chasles' *'synthetic'* approach cannot purely be characterized by the mathematical theories and tools it relies on, but rather by a set of interconnected epistemological values and a specific practice of the proof[88]. The virtuous practice derived from this set of values and rules is not necessarily one without computations, but it is one where each and every computation has to be correlated to a geometrical interpretation, so as to be clearly situated in this chain of truths.

What this proof first shows, to Chasles, is that modern geometry has the resources to attain truths which were previously thought to be the preserve of Analysis. In that regard, new methods, such as the theory of transversals or polar transformations play a crucial role. Secondly, these alternative, geometrical proofs cast a new light on a subject already well-studied by analysis. By virtue of these advantages of pure geometry, which we described abstractly in the previous sections, Chasles has been able to root MacLaurin's theorem in its natural setting, namely the theory of second-degree surfaces. In so doing, he gained a clearer understanding of the causes of this theorem, and enlightened the central role of some auxiliary surfaces: the confocal ellipsoids, which happen to be the level-surfaces to the attraction of the initial ellipsoid. From the generality of modern geometry, a simpler, more intuitive proof has been obtained[89].

5. From simplicity to generality : the proof by correspondence

The second proof we wish to examine now was communicated to the Académie des Sciences a few months later than the first one, during one of its weekly session, on June 25$^{\text{th}}$ 1838[90]. This proof does not require prior knowledge of Chasles' theory of second-degree surfaces, and does not proceed by rooting the problem of the attraction of ellipsoids within a more general geometrical theory. Rather, Chasles carved a new geometrical object from some properties which played a key role in the proof we outlined in the previous section, as well as from an 1809 memoir written by Scottish mathematician James Ivory[91]. In this section, we first show how Ivory's notion of a correspondence between points on two confocal ellipsoids enabled Chasles to drastically shorten and simplify his proof of MacLaurin's generalized theorem. Then, we describe how from this simpler proof, he extracted a more general notion of geometrical correspondence, on which he set out to build a general

[86]As did Comte, in his *Leçons de philosophie positive*, in particular in his tenth lesson, where he states : 'One can form a very clear idea of the geometrical science, conceived in its totality, if we assign as its general goal the reduction of comparisons of all sorts of extended volumes, surfaces, lines etc. to simple comparisons of straight lines..'

[87][Cha37a], p.395

[88]Do note, however, that this argument is limited to the case of Chasles' specific conception of geometry, and wouldn't describe the synthetic geometry of, say, Von Staudt.

[89]Of course, this simplicity judgment by a 19$^{\text{th}}$ century actor is wont to be at odds with our contemporary assessments. The fact that Chasles' proof may seem more difficult to us than intricate calculations, whose technical basis is more likely to have been taught to us, should not distract us from the fact that our aim here is to understand how an actor's epistemic values structure and guide his mathematical practice, and not to search for an hypothetical conceptual content of simplicity.

[90][Cha38].

[91][Ivo09].

geometrical theory of attraction. In so doing, we will show another facet of the connection between simplicity and generality, as in this case, simplicity paves the way for a more geometrical theory to emerge.

5.1. **Ivory's memoir revisited.** Ivory's 1809 memoir was of great value and importance for both Chasles and analysts such as Poisson[92]. As part of an effort to incorporate French Analysis and mechanics into British science, Ivory wrote his memoir in part to simplify and make more accessible new results obtained in France in both of these scientific fields[93]. To that effect, his strategy was to work on the analytic expression for the attraction of an ellipsoid $\mathcal{E}$ on an exterior point S, and to transform it so that it can be read as the attraction of another ellipsoid $\mathcal{E}'$, on an another point S', interior to $\mathcal{E}'$. Effectively, this would reduce the problem of the attraction of an ellipsoid on an exterior point, to that of an ellipsoid on an interior point. To do so, Ivory constructed $\mathcal{E}'$ as the ellipsoid confocal with $\mathcal{E}$, and whose external surface passes through S[94]. After some analytic calculations, S' was determined as well, and it was shown that these attractions were in the same proportion. Ivory then proceeded to call these pairs of points and surfaces *'correspondent'*. This was enough to show that MacLaurin's theorem held for any two confocal ellipsoids attracting points on the surfaces of one another.

Poisson put this memoir to great use, as he used the analytical transformation at the basis of Ivory's correspondence to compute the attraction of an ellipsoidal body as the sum of infinitely-thin layers of confocal ellipsoidal layers. It is noteworthy that Chasles still classified Ivory's proof as a synthetic one[95], despite it being largely composed of calculations we would easily regard as analytic, especially with Poisson's proof in mind. In a way, Chasles' reading is basically an inversion of that of Poisson : while the latter sees the notion of correspondence as a mere tool to gain a tractable decomposition of ellipsoid, Chasles thematizes it and centers his second proof around it. To do so, he devises the following definition[96]: *'two points of two confocal ellipsoids are said to be* ***correspondent*** *if and only if their coordinates along each principal axis are in the same proportion as the diameters of the ellipsoids along these axes'*. This is the same notion as the one present in Ivory's memoir. However, it does not result from a calculation anymore: the correspondence is taken to be a basic notion from which the study of confocal pairs of ellipsoids will proceed, and will then be a potential object for generalization. This shows how diverse the readings of a single mathematical text can be. While the analyst's computational mastery allows him to read in Ivory's memoir an infinitesimal decomposition of ellipsoids which allows for a tractable integration of their attraction, the very same text becomes a study in the geometrical correspondence between confocal ellipsoids, while under scrutiny from a trained geometer.

92[Poi33], p.499, mentions his early and continued interest in Ivory's work.

93'It will not be altogether unworthy of the notice of the Royal Society, if [my method contributes] to simplify a branch of physical astronomy of great difficulty, and which has so much engaged the attention of the most eminent mathematicians', [Ivo09], p.347. These *'eminent mathematicians'*, according to what precedes this quotation, are mainly Legendre and Laplace. For more on the circulation of French Analysis in Great Britain (especially in context of astronomical studies), see [Cra16].

94[Ivo09],p.351

95[Cha37a], p.165

96[Cha38], p.903

Chasles' rewriting and reinterpretation of Ivory's correspondences had already led to an important result in his *Aperçu Historique*, which would later be known as Ivory's theorem[97]. In Chasles' terms, this theorem states that[98]:

> For any two arbitrary points S, m on an ellipsoid, denoting their correspondent points S', m', we have $Sm' = S'm$.

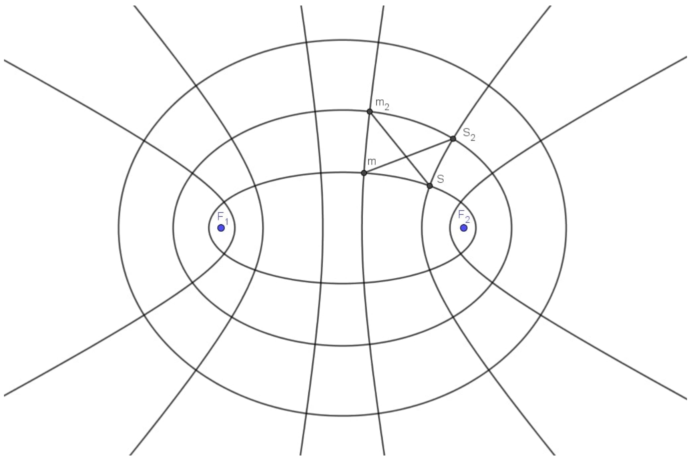

Figure 5. Ivory's theorem (my figure)

In his 1838 memoir on the attraction of ellipsoids, Chasles was able to use this theorem to derive a much shorter solution, which we now reproduce. Chasles first considered the following ellipsoids:

$$\text{(A)} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$(A_n) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = n^2$$

Furthermore, defining the correspondence between two points in space (x, y, z) and (x', y', z') by the equations

$$\frac{x}{a} = \frac{x'}{a'}\ ,\ \frac{y}{b} = \frac{y'}{b'}\ ,\ \frac{z}{c} = \frac{z'}{c'}$$

where a', b', c' are arbitrary numbers, Chasles formed two ellipsoids A', A'_n, corresponding respectively to A and A_n. Choosing a', b', c' so that

$$\begin{aligned} a^2 - b^2 &= a'^2 - b'^2 \\ a^2 - c^2 &= a'^2 - c'^2 \end{aligned}$$

ensures that A, A' (resp. A_n, A'_n) are confocal[99]. Chasles then takes n smaller than but infinitely close to 1, so that A and A_n become the internal and external

[97] This result has applications in the theory of billards, as well as in hyperbolic geometry, see for instance [SW04]. Despite its name, most likely inherited after the publication of Dingeldey's article on the geometry of conics in Klein's and Meyer's *Encyklopädie der mathematischen Wissenschaften*, the theorem itself is not explicitly stated in Ivory's memoir.

[98] [Cha37a], p.393-394. See fig.5.

[99] On a sheaf of confocal ellipsoids, correspondent points form confocal hyperboloids, which are orthogonal to the ellipsoids (see fig.5). These hyperboloids are among the orthogonal surfaces we described in the previous proof

surfaces of an infinitely thin layer C (the same holds for A', A'_n, C'). The width of each layer along the x-axis is, respectively:

$$da = (1-n)a \ , \ da' = (1-n)a'$$

hence

$$\frac{da}{da'} = \frac{a}{a'}$$

For two points S, m on the surface A, and $d\nu$ a volume element around m, and their corresponding elements on A',

$$\frac{d\nu}{d\nu'} = \frac{abc}{a'b'c'}$$

Now, Chasles' theorem for corresponding points states that $mS' = m'S$. Hence, the previous equation can be rewritten:

$$\frac{d\nu}{mS'} : \frac{d\nu'}{m'S} = \frac{abc}{a'b'c'}$$

Summing that expression with regards to each and every *'molecule*[100]*'* of each layer, Chasles obtains the following equation:

$$\Sigma\frac{d\nu}{mS'} : \Sigma\frac{d\nu'}{m'S} = \frac{abc}{a'b'c'}$$

which is also the ratio of the weights of each layer.

5.2. **A second proof of MacLaurin's generalized theorem.** This last equation expresses what Chasles describes as a *'property that is sufficient to solve the whole question*[101]*'*. First, he notices that the differential coefficients of the function $\Sigma\frac{d\nu}{mS'}$ are equal to the coordinates of the attraction the layer C exerts on S'. Taking C to be the external layer of the ellipsoidal layer, then S' belongs to the internal surface of the layer. Then, a generalized form of Newton's *'Shell theorem'* for ellipsoids shows that the layer exerts no attraction on S' (see fig. below).

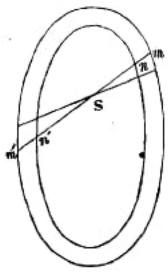

FIGURE 6. A shell theorem for ellipsoids: the attraction of a thin ellipsoidal layer on an internal point is null

Therefore, the differential coefficients of $\Sigma\frac{d\nu}{mS'}$ are null on the internal surface of C, and $\Sigma\frac{d\nu'}{m'S}$ is constant on A, and equal to $\frac{a'b'c'}{abc}\Sigma\frac{d\nu}{mS'}$. It follows that A is normal to the attraction C' exerts on S. Chasles then considers another layer C'', defined in an analogous manner to C', for which

$$\frac{\Sigma\frac{d\nu'}{mS'}}{\Sigma\frac{d\nu''}{mS''}} = \frac{a'b'c'}{a''b''c''}$$

[100]This is Chasles' term for elements of volume.

[101][Cha38], p.905

Differentiating this equation along the x-axis for instance, Chasles shows that the attractions of both layers along this axis have the same ratio as their weights, and states the following result[102]:

> Two confocal, ellipsoidal layers exert on an external point an attraction of similar direction, and whose magnitudes are in the same proportion as the masses of the layers. This direction is that of the normal of the ellipsoid passing through the attracted point and confocal with these layers.

From there onwards, MacLaurin's generalized theorem can be established following the same line of thought as in Chasles' memoir. However, along the way, new information has been gained, as the series of ellipsoids confocal to a given ellipsoid can now be regarded as level-surfaces for the attraction of this given ellipsoid. Chasles regarded this alternative proof as much simpler than the previous one. Not only is it considerably shorter, it also does not require an advanced knowledge of modern geometry. More important, however, is the fact that it is centered around one geometrical property that is very simple, therefore subject to further generalizations, through the study of the geometry of level-surfaces.

5.3. **Towards a geometrical theory of attraction.** As early as 1837, within his second memoir for the Journal de l'Ecole Polytechnique, Chasles had worked out general theorems on the attraction of bodies, which allowed for analogous statements in the theory of electricity, of heat, or hydrodynamics. However, in 1839, in a public communication to the Académie des Sciences, he was able to go much further, on the basis of a generalization of his work on ellipsoids. After stating a theorem on level-surfaces of a body of any shape, he explained

> 'This theorem enables us, once we know the level-surfaces related to the attraction of a body, to reduce the calculation of this attraction to that of the attraction of an infinitely thin layer. [..] And so this problem, considered from a general point of view, is stripped of the great difficulties it had presented when we attacked it through considerations both narrow and specific to the special shape of this body. This case seems to offer a new example of the advantages of *generalization* in geometry, to simplify theories and shed an intuitive light on them[103]'.

Two years later, he published a short mémoire in the Connaissance des Temps[104], in which this striking claim is made more explicit. In this memoir, Chasles does not start with level-surfaces, but rather with isotherms. Indeed, as he had already noted in the specific case of ellipsoids, the isotherms can be obtained by fixing the value of Laplace's potential function V. On the importance of the introduction of this function, Chasles makes the following observation:

> 'Although the function considered by Laplace has not ceased to play a major role in [all research related to attraction, magnetism, or electricity], it has been exclusively studied under an analytic point of view and within a second order differential equation; and

[102][Cha38], p.906.

[103]'Ce théorème permet, quand on connaît les surfaces de niveau relatives à l'attraction d'un corps, de ramener le calcul de cette attraction à celui de l'attraction d'un corps infiniment mince. [..] De sorte que ce problème, envisagé ainsi d'un point de vue général, se dépouille des grandes difficultés qu'il avait présentées quand on l'attaquait par des considérations restreintes et toutes spéciales à la forme particulière du corps. Ce cas paraît offrir un nouvel exemple des avantages de la *généralisation* en géométrie, pour simplifier les théories et y répandre une clarté intuitive', [Cha39]; p.209-210.

[104][Cha42]

> it has not yet been thought of to consider some surfaces which this function gives rise to; surfaces that are analogous to those we call, in hydrodynamics, *level surfaces*, and which we may also call *level surfaces relative to the attraction of bodies* because the attractions exerted by the body on the different points of each of these surfaces, are directed along their normals[105]'.

What Chasles would then show, is that one can obtain these objects geometrically, thus preserving their meaning and developing a geometrical theory of attraction. For that purpose, he considered a finite, closed body of any shape, and the level-surfaces with regards to the attraction it exerts, given by the equations

$$V = \lambda$$

where λ is a constant number. In what follows, only surfaces that are entirely external to the body are considered by Chasles. Let A be one of these surfaces, $m \in A$, and dn the normal to A at m comprised between A and another level-surface infinitely close to A. Chasles defines a more general concept of correspondence on these level-surfaces: two points m, m' on two level-surfaces A, A' are said to be correspondent if and only if they are on a line that is orthogonal to every level-surface. Corresponding volume elements are similarly defined.

The main property of these correspondences, around which Chasles' memoir is structured, is that the body exerts an equal attraction on corresponding volume elements. This translates into the following equation[106]:

$$\frac{dV}{dn} d\omega = \frac{dV'}{dn'} d\omega'$$

Chasles doesn't prove this formula in his 1840 memoir; instead, he claims to have already done so in his second memoir for the Journal de l'Ecole Polytechnique. Tracing back his original statement, one cannot find any proof, but rather the assertion that the result for the case of ellipsoids *'se prête à la généralisation'*[107]. By this expression, it is to be understood that the reasoning adopted within the case of ellipsoids can be applied *mutatis mutandis* to that of any sort of body, in line with the stability criteria we described previously. The generalization from the attraction of ellipsoids to the attraction of general surfaces is an effortless one: because Chasles has found a simple, natural solution of the particular case, it suffices to replace every particular term by its more general equivalent for the solution of the general problem to follow.

This generalization, shows Chasles, turns out to be a most fruitful one. For instance, his equation shows that the sum of the attractions exerted by the body on the elements of a single level-surface is constant, and Chasles is able to compute this sum, thus independently finding a result already proved by Gauss and Green[108]:

$$\int\int \frac{dV}{dn} d\omega = 4\pi M$$

---

[105]'Mais, bien que la fonction considérée par Laplace n'ait pas cessé depuis de jouer un röle principal dans toutes les recherches de ce genre, c'est toujours sous un point de vue exclusivement analytique et dans l'équation différentielle du second ordre, qu'on l'a étudiée ; et l'on n'a pas songé à considérer certaines surfaces auxquelles donne lieu cette fonction ; surfaces analogues à celles qu'on appelle, dans la théorie des fluides, *surfaces de niveau*, et qu'on peut appeler aussi *surfaces de niveau relatives à l'attraction du corps*, parce que les attractions exercées par le corps sur les différents points de chacune de ces surfaces, sont dirigées suivant les normales.', [Cha42], p.19.

[106]Notice the similarity with the *geometrical property* at the center of the proof by correspondence above.

[107][Cha37c], p.284-286

[108][Cha42], p.25.

where M is the mass of the body, and the integral is extended to all of the elements of the surface A. Perhaps more importantly, Chasles can rewrite the equality of the attractions of the body on corresponding elements of volume in the following manner[109]:

$$\frac{KdV'}{K'dV} = \frac{Kd\omega}{dn} : \frac{K'd\omega'}{dn'}$$

where K, K' are two infinitely-small coefficients of the second order. Since dV and dV' are constant for points on level-surfaces A, A', so is the right-hand term, which is also the ratio between corresponding elements of volume on A and A'. Summing on the whole surfaces, this ratio is shown to be also the ratio between the whole layers on A and A'. Hence[110],

> Any canal orthogonal to every level-surface intercepts in two layers volumes which are in the same proportion as the total volumes of the layers, which can be expressed by the equation:

$$\frac{d\mu}{\mu} = \frac{d\mu'}{\mu'}$$

This equation is shown by Chasles to have many consequences, most notably a general shell theorem for a wide category of bodies, which shows how powerful this geometrical perspective on mechanics can be[111]. The seeds for a geometrical theory of attraction were planted. Chasles, however, did not pursue these matters any further, leaving these promises mostly unfulfilled[112]. Whatever the merits of this theory, what we assuredly gained from the study of this memoir, is an insight into how simplicity in geometrical practices can lead to generality, and how these values actively structured Chasles' quest for simpler, more general proofs.

6. Conclusion

Chasles' sequence of publications on the problem of the attraction of ellipsoids illustrates and substantiates his theses on the connection between the values of simplicity and generality in mathematical practice. This connection is manifold but also dynamic: through the search for a simpler proof, more general settings and results are found, and vice-versa. Chasles' alternative proofs are motivated by an active search for a general theoretical setting in which the solution to the problem of the attraction of ellipsoids becomes much more simple. Conversely, as Chasles demonstrates, such simple proofs necessarily reveal and indicate a clear and open path towards generalization. Thusly, these proofs adduce evidence to the claim that virtuous geometrical practices do end up yielding a wealth of riches that the analyst, only concerned with the speedy derivation of a formula, will fail to apprehend.

In this sense, the connection Chasles establishes between the values of simplicity and generality is not only constitutive of a certain epistemology of geometry, but also of an ethos, of a way of acting as a geometer. Such an outlook explains why Chasles' epistemological discussions echo so strongly his mathematical practice. Rather than viewing the latter as an application of a general theory of mathematical knowledge, whose internal coherence and stability may be subject to further

[109]ibid., p.26.

[110]ibid., p.27.

[111]Although some restrictions ought to be put on Chasles' statements for the theorems to satisfy modern criterias of exactness.

[112]Although these ideas were not lost to everyone; see, for example, Benjamin Peirce, whose *System of Analytic Mechanics* develops a concept of *'Chaslesian shells'*.

discussion and criticism, we suggest regarding both as produced by a single agent bounded by the same set of epistemic imperatives. Investigating this ethos demands to read closely both Chasles' mathematics, with their own autonomous rationality, and the expression of the rules which determine how this autonomous mathematics comes into being, that is to say is actually produced and written down. Through the case of the attraction of ellipsoids we come to see how both epistemological discussions and self-imposed epistemic rules are necessary to understand an actor's practice. The history of Newtonian mechanics, therefore, is not merely that of a succession of results: it is in fact shaped by the fluctuation in disciplinary boundaries and identities, and one can do without neither the discussion of these results, nor of these disciplinary changes.

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