LECTURE NOTES: TOPOLOGICAL ASPECTS IN THE STUDY OF TANGENT DISTRIBUTIONS

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Abstract. These are the lecture notes for the minicourse on tangent distributions given during the 13th International Young Researchers Workshop on Geometry, Mechanics and Control, which took place at the University of Coimbra, Portugal, during the days 6th-8th December 2018.

Distributions (and their cousins the differential forms) first appeared in the works of E. Cartan, G. Darboux, S. Lie, J.F. Pfaff and others as the intrinsic language underlying the theory of Partial Differential Equations. For some of the readers, they might be most relevant because they provide geometric footing in Control Theory.

We will begin these notes by reviewing their local theory, which is rather *geometric* in nature. Once we have some understanding of their local invariants, we will look instead at global, and therefore *topological*, properties. We will quickly focus on the following question: On a given manifold, can we describe the homotopy type of the space of tangent distributions with some given invariants?

1. Introduction

The following will be the object of interest in these notes:

Definition 1.1. Let M be a smooth manifold. A **distribution** ξ is a subbundle of the tangent bundle TM. The dimension of its fibres is called the **rank**.

We will devote this brief introduction to a few examples showcasing the many settings in which distributions appear.

1.1. The rolling coin.

Suppose we want to describe the rolling movement of a coin on a table. To this end we first choose our state space to be 3-dimensional: we use two variables $(x, y) \in \mathbb{R}^2$ to describe the position of the coin and an angle $\theta \in \mathbb{S}^1$ to describe where the coin points.

Remark 1.2. The variable θ takes values in $\mathbb{S}^1$ because we remember what "heads" and "tails" are. If we forget this information, we would write $\theta \in \mathbb{RP}^1$ instead. Another possible variation: We may use a second angular variable to record the position of a marked point sitting at the edge of the coin; such a system would be 4-dimensional. ♦

Let $\gamma(t) = (x(t), y(t), \theta(t))$ be a curve describing a rolling motion. Any such curve must satisfy the following property: The variation in position $(x'(t), y'(t))$ should be proportional to the direction $(\cos(\theta(t)), \sin(\theta(t)))$ in which the coin points. Proportionality of these two vectors is equivalent to any of the following conditions:

- The following determinant is zero:
$$\begin{vmatrix} x'(t) & \cos(\theta(t)) \\ y'(t) & \sin(\theta(t)) \end{vmatrix} = \sin(\theta(t))x'(t) - \cos(\theta(t))y'(t) = 0.$$
- The 1-form $\alpha = \sin(\theta)dx - \cos(\theta)dy$ is zero when restricted to the curve $\gamma(t)$. I.e our motion system is described by the linear constraint $\gamma^*\alpha = 0$.

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- The curve $\gamma(t)$ is tangent to the plane field
$$\xi = \ker(\alpha) = \langle \cos(\theta)\partial_x + \sin(\theta)\partial_y, \partial_\theta \rangle$$
which α annihilates.
- The vector $\gamma'(t)$ is a linear combination of $\cos(\theta)\partial_x + \sin(\theta)\partial_y$ (which is the vector field responsible for "pure rolling", i.e. rolling without changing the angle θ) and ∂_θ (which allows the direction θ to change as we roll).

Any of these equivalent conditions tells us that movement in our system is *constrained*. Despite of this:

Lemma 1.3. *This system is* fully controllable. *That is, any two points (x_0, y_0, θ_0) and (x_1, y_1, θ_1) in our state space can be joined by a path γ tangent to ξ.*

Proof. We first turn from (x_0, y_0, θ_0) to (x_0, y_0, θ_2), where θ_2 is the angle of the straight segment connecting (x_0, y_0) with (x_1, y_1). Then we roll along this segment, eventually reaching (x_1, y_1, θ_2). Finally, we turn again to set the angle to θ_1. This is a piecewise smooth path tangent to ξ that connects the two points. □

This controllability phenomenon is the global counterpart of the following infinitesimal fact:

Lemma 1.4. *All directions of motion (in the state space) are linear combinations of Lie brackets of vector fields in ξ.*

Proof. Indeed, the vector field
$$[\cos(\theta)\partial_x + \sin(\theta)\partial_y, \partial_\theta] = \sin(\theta)\partial_x - \cos(\theta)\partial_y$$
completes, along with the framing $\{\cos(\theta)\partial_x + \sin(\theta)\partial_y, \partial_\theta\}$ of ξ, a global framing of the tangent bundle $T(\mathbb{R}^2 \times \mathbb{S}^1)$. □

Lemma 1.4 states that, infinitesimally, we are able to move in all directions, so it is within reason that the same is true globally, as in Lemma 1.3.

1.2. **Other control systems.** You can probably already see that many similar systems can be described using this language: cars towing some number of trailers, robot arms, satellites... They are all systems where movement is constrained (e.g. by the direction in which the wheels or a joint point) but we can nonetheless act on it (e.g. by turning the wheel, or introducing movement with an engine). Systems like these are the object of study in **Control Theory**; we will look at a simple control problem in Subsection 2.4.

The general setup is as follows: We think of a manifold M as the space of states for a particle and of ξ, a subbundle of TM, as the set of *admissible* directions of motion for the particle. A trajectory/path that is everywhere tangent to ξ is said to be *admissible*.

There are then two immediate questions:

- Starting from a point $p \in M$, can we reach another given point $q \in M$ using only admissible directions?
- If so, can we find an admissible path from p to q that is *optimal*?

For some systems, like the coin, the first question has always a positive answer; such systems are said to be *controllable*. To answer the second question we first need to define what "optimal" means. For this we need to introduce some cost functional in the space of all admissible paths[1] and look for minimisers.

A natural choice would be to fix a metric along ξ and study the standard energy functional:
$$E(\gamma) = \frac{1}{2} \int |\gamma'(t)|^2 dt.$$
Even though this resembles the usual variational setup for geodesics in the Riemannian case, there is a substantial difference: now our curves are constrained to be tangent to ξ, so we are not allowed to

[1]Note: One can consider spaces of paths with different regularities (in terms of their differentiability), depending on what we are aiming to do.

vary γ freely. That is: we have a functional E, which is defined on the space $\Omega_{p,q}(M)$ of all paths in M with endpoints $p, q \in M$. We want to find its minima over a certain subset, the set $\Omega_{p,q}(M,\xi)$ of paths which are additionally tangent to ξ. The issue is that this subset is not a submanifold in the Frechet sense, so we cannot really understand this through a lagrangian multiplier approach. It is not a submanifold because it has singularities: certain curves $\gamma \in \Omega_{p,q}(M,\xi)$ have infinitesimal variations that do not arise from deformations of γ relative to its endpoints. Understanding these issues is a central question in **Subriemannian Geometry** [34].

1.3. **Functions and jet spaces.** Let us move away from Control Theory now. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth real-valued function. To any such f, we can assign the graph:

$$j^1 f : \mathbb{R} \to \mathbb{R}^3$$

$$j^1 f(x) = (x, y(x) = f(x), z(x) = f'(x)).$$

Tautologically, $j^1 f$ satisfies all of the following equivalent conditions:

- $\frac{dy}{dx}(x) = z(t)$. I.e. the slope of y with respect to x is precisely z.
- The 1-form $\alpha = dy - zdx$ vanishes when restricted to $j^1 f$. I.e. $(j^1 f)^*\alpha = 0$.
- $j^1 f$ is tangent to the subbundle $\xi_{\text{can}} = \ker(\alpha) = \langle \partial_z, \partial_x + z\partial_y \rangle$ of $T\mathbb{R}^3$.

This particular case can be generalised to smooth maps $f : \mathbb{R}^n \to \mathbb{R}^m$, associating to them (for some given integer r) the graph:

$$j^r f : \mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \times \mathbb{R}^{(n(n-1)m)/2} \times \cdots$$
$$j^r f(x) = (x, f(x), f'(x), f''(x), \cdots, f^{(r)}(x)).$$

where $f^{(j)}$ denotes the collection of all derivatives of order j. Then, there is a subbundle $\xi_{\text{can}} \subset TJ^r(\mathbb{R}^n, \mathbb{R}^m)$ (called the **Cartan distribution**, **canonical distribution**, or **tautological distribution**), which is defined by the following universal property: A section of $J^r(\mathbb{R}^n, \mathbb{R}^m)$ is of the form $j^r f$ if and only if it is tangent to ξ_{can}. The Cartan distribution can be written explicitly as the kernel of a collection of 1-forms just like in the first example.

Suppose we are interested in functions $f : \mathbb{R}^n \to \mathbb{R}^m$ satisfying a partial differential equation (PDE) of order r. Any such equation can be written in the form:

$$R(x, f(x), f'(x), \cdots, f^{(r)}(x)) = 0,$$

where R is some algebraic expression on x, f, and its derivatives, that we can immediately interpret as a function $R : J^r(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^k$ (for some k). In many cases the set $\mathcal{R} = \{R = 0\}$ is a submanifold which in fact fibres $\mathcal{R} \subset J^r(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^n$ over $\mathbb{R}^n$. It can be endowed with the subbundle $\xi_R = T\mathcal{R} \cap \xi_{\text{can}}$ and, thus, solutions f of R correspond to sections of $\mathcal{R}$ tangent to ξ_R.

This is the setup for the **Geometric Theory of PDEs** initiated by J.F. Pfaff and E. Cartan. The following questions are quite natural:

- Given a manifold (e.g. $\mathcal{R}$) with a subbundle (e.g. ξ_R) how do we construct submanifolds tangent to the subbundle (e.g. $j^r f$ with f a solution of $R = 0$)? This can be regarded as a generalisation of: how do we solve a PDE?
- Given two manifolds M_i with subbundles $\xi_i \subset TM_i$, can we find a diffeomorphism $f : M_0 \to M_1$ such that $f^*\xi_1 = \xi_0$? This particularises to: when are two seemingly different PDEs actually the same after diffeomorphism?

A key observation here is the following: a solution of R corresponds to a section $j^r f : \mathbb{R}^n \to \mathcal{R}$ tangent to ξ_R. However, there may be (and indeed, there are) submanifolds of $\mathcal{R}$ tangent to ξ_R which are not of the form $j^r f$ because they are not graphical over $\mathbb{R}^n$. These are sometimes called **generalised solutions** of the PDE, and play an important role in the Singularity Theory of PDEs. We may think of them (under some regularity assumptions) as graphs $j^r f$ where f is not a function but a *multiple valued function solving R.*

1.4. **Complex/Algebraic Geometry.** Let us look at the real sphere $\mathbb{S}^{2n-1}$ sitting in complex Euclidean space $\mathbb{C}^n$. Fix a point $p \in \mathbb{S}^{2n-1}$. The tangent space $T_p\mathbb{S}^{2n-1}$ is a real vector subspace of $T_p\mathbb{C}^n \cong \mathbb{C}^n$. Since the latter is a complex vector space we ask ourselves: What is the largest complex subspace of $\mathbb{C}^n$ contained in $T_p\mathbb{S}^{2n-1}$? The answer is $\xi_p = T_p\mathbb{S}^{2n-1} \cap i(T_p\mathbb{S}^{2n-1})$: by definition, it is invariant under multiplication by i and, being the intersection of two distinct hyperplanes, is $(2n-2)$-dimensional. We can do this at every point, and the collection of all these complex subspaces is a subbundle $\xi_{\mathrm{std}} \subset T\mathbb{S}^{2n-1}$ of dimension $2n-2$. We call it the **standard contact structure**; we will look at contact structures (which are a particular class of distributions) in Subsubsection 2.11.2 and Subsection 3.2 onwards.

More generally, consider a complex manifold (X, J), where J denotes the (integrable) almost complex structure. Given a real submanifold $M \subset X$, we can look at the space of complex lines $\xi = TM \cap J(TM)$ contained in TM. The interaction of M with the holomorphic/meromorphic functions on X is very much related to the properties of (M, ξ) as an abstract manifold; as such, it is studied in **Complex Geometry**.

A particular case of interest arises in **Complex Singularity Theory**: One looks at complex polynomials $F : \mathbb{C}^n \to \mathbb{C}$ having an isolated critical point at the origin. Thus, the set $F^{-1}(0)$ is a complex codimension-1 algebraic subvariety of $\mathbb{C}^n$ passing through the origin and failing to be smooth there. The fundamental classification question one would like to address is:

- Given two polynomials F_i, are there biholomorphisms ϕ (locally defined close to the origin $0 \in \mathbb{C}^n$) and ψ (in $\mathbb{C}$) such that $F_0 = \psi \circ F_1 \circ \phi$? I.e. when are the corresponding singularities the same up to holomorphic change of coordinates?

Given a polynomial F, we can look at the intersection $L_F = F^{-1}(0) \cap \mathbb{S}^{2n-1}$, which is usually called the *link*. It is a submanifold of $\mathbb{S}^{2n-1}$ transverse to ξ_{std}. It turns out that many of the properties of F can be detected using L_F and the way it interacts with ξ_{std}.

1.5. **What can you find in these notes?** These examples should give you a flavour of some of the areas in Mathematics in which distributions appear. This list is far from complete. To name some more: they also play an important role in Thermodynamics, the theory of hypoelliptic operators, and Smooth Topology.

A disclaimer is in order: These notes are very much biased by my own taste. I thought it best to cover some of the results that first got me interested in distributions with the hopes of getting you, the reader, interested too.

These notes have two parts: Section 2 deals with the *Geometry* of distributions (i.e. with their *local* classification up to diffeomorphism) and Section 3 with their *Topology* (i.e. with their *global* classification up to homotopy once some of their local invariants have been fixed).

In Section 2 we will introduce the most important local invariants of distributions: the growth vector and the curvatures. This will allow us to classify all (regular) distributions up to (but not including) dimension 5.

In Section 3 we will rapidly focus on *contact structures*, since they are the family of distributions that is best understood from a topological viewpoint. The main two results we will cover are due to M. Gromov (Theorem 3.17) and Y. Eliashberg (Theorem 3.21). The first one classifies all contact structures in open manifolds up to homotopy and the second one classifies the *overtwisted* subclass of contact structures in closed manifolds up to isotopy.

2. Geometry of distributions

The fundamental question we want to study in this Section is the following:

- When are two germs[2] of distributions ξ_0 and ξ_1 equivalent under a germ of diffeomorphism?

This issue was first addressed by Cartan, Engel, Goursat, Lie, Pfaff and others. A complete answer is in general intractable, but one can nonetheless define invariants.

The reader should be reminded of the case of metrics: to show that two metrics are not diffeomorphic, it is sufficient to show that their curvatures are not diffeomorphic. Often, this can be seen by checking simpler tensors, like the sectional or scalar curvatures. For distributions we will, analogously, construct certain differential objects (the Lie flag and the curvatures) which in turn will allow us to define numerical invariants.

Remark 2.1. Classical authors like Cartan [5] and Pfaff [40] phrased their work not in the language of distributions, but in the language of *Pfaffian systems* (i.e. collections of 1-forms). The correspondence between the two is given by duality: given a distribution $\xi \subset TM$, its annihilator $\xi^\perp \subset T^*M$ is a Pfaffian system. We provide a few more details in Subsection 2.10. ♦

2.1. **The associated flag.**

When we study distributions, it is fruitful to look at the map:

$$\{\text{Distributions}\} \leftrightarrow \left\{\begin{matrix} C^\infty\text{-modules of vector fields} \\ \text{of pointwise constant rank} \end{matrix}\right\}.$$

Which assigns to ξ the C^∞-module $\Gamma(\xi) \subset \mathfrak{X}(M)$ of those vector fields that are tangent to it.

Remark 2.2. When M is closed, this map is in fact a 1-to-1 correspondence. Indeed, ξ can be recovered from the corresponding module by evaluating at every point.

If M is open some subtleties arise. For instance, we may assign to ξ the C^∞-submodule $\Gamma_c(\xi) \subset \Gamma(\xi)$ of compactly supported vector fields instead. Evaluating $\Gamma_c(\xi)$ pointwise gives us ξ back, so the correspondence fails. There are in fact other possible choices of C^∞-submodule Γ: they all amount to taking vector fields that are compactly supported only in certain directions. As such, $\Gamma_c(\xi)$ and $\Gamma(\xi)$ are the two extreme cases and all other sensible choices lie in-between, i.e. $\Gamma_c(\xi) \subset \Gamma \subset \Gamma(\xi)$. We can therefore single out $\Gamma(\xi)$ as our preferred choice by observing that it is the unique module satisfying the sheaf condition. These issues have been studied in depth in the literature concerning singular foliations; see [2] and Remark 2.17 below. ♦

Since the space of all vector fields $\mathfrak{X}(M)$ is an (infinite dimensional) Lie algebra, we are naturally lead to the question: "how does the linear subspace $\Gamma(\xi)$ behave with respect to the Lie bracket?" This motivates us to define:

Definition 2.3. A string of the form "a", whose input is the formal variable a, is said to be a **bracket expression** of length 0. Inductively, we define a bracket expression of length $i+j+1$ to be a string of the form

$$\text{“}[A(a_0,\cdots,a_i), B(a_{i+1},\cdots,a_{i+j+1})]\text{”},$$

where $A(-)$ and $B(-)$ are bracket expressions of lengths i and j. ♦

Definition 2.4. Given a distribution ξ, we define the **associated Lie flag** to be the sequence of C^∞-modules

$$\xi^{(0)} \subset \xi^{(1)} \subset \cdots \xi^{(i)} \subset \xi^{(i+1)} \subset \cdots$$

given by:

$$\xi^{(i)} = \langle A(v_0,\cdots,v_i) \,|\, v_0,\cdots,v_i \in \Gamma(\xi);\ A \text{ bracket expression of length } \le i.\rangle_{C^\infty}.$$

Here the braces indicate taking the C^∞-span. ♦

[2]We recall that a germ of distribution at a point p is a equivalence class of pairs (U,ξ), where U is an open set containing p and ξ is a distribution defined over U. Two pairs (U_0,ξ_0) and (U_1,ξ_1) are equivalent if there is a smaller open set $U_2 \ni p$ contained in both U_0 and U_1 such that $\xi_0|_{U_2} = \xi_1|_{U_2}$. In fact, this notion makes sense not just for distributions, but for sections of any bundle.
In this Section we will look at properties and invariants that are mostly local in nature. Even though we will state them for distributions defined over a manifold, we invite the reader to check that they depend only on the germ at each point. In some cases, these local properties will have global consequences. Examples of this are the theorems of Frobenius and Chow; see Subsection 2.3.

In particular, $\xi^{(0)} = \Gamma(\xi)$.

Example 2.5. In general, the elements $\xi^{(i)}$ in the associated flag do not correspond to distributions, because their pointwise ranks $\dim(\xi^{(i)}(p))$ may vary with $p \in M$. Indeed, consider the following distribution:

$$\xi = \ker(dy - z^2 dx) = \langle \partial_x + z^2\partial_y, \partial_z \rangle$$

which is called the *Martinet distribution* [31]. We can readily compute Lie brackets:

$$[\partial_z, \partial_x + z^2\partial_y] = 2z\partial_y$$
$$[\partial_z, [\partial_z, \partial_x + z^2\partial_y]] = 2\partial_y.$$

This shows that $\xi^{(1)}$ has rank 3 everywhere except along the hypersurface $\{z = 0\}$. $\xi^{(2)}$ is the module $\mathfrak{X}(M)$ of all vector fields in M. ♦

We would like to avoid cases like this one, so we introduce the following definition:

Definition 2.6. A distribution is **weakly regular** if its associated Lie flag is comprised of distributions. ♦

Under this regularity assumption, the associated flag stabilises:

Lemma 2.7. *If ξ is weakly regular there exists some integer m such that $\xi^{(i)} = \xi^{(m)}$ for every $i \geq m$.*

Proof. If $\xi^{(i+1)}$ is different from $\xi^{(i)}$, it must have larger rank because both are distributions. However, rank is bounded above by the rank of TM, so the rank eventually stabilises. □

Finally, we define:

Definition 2.8. The **growth vector** of a weakly regular distribution ξ is defined as:

$$(\mathrm{rank}(\xi^{(0)}), \mathrm{rank}(\xi^{(1)}), \cdots, \mathrm{rank}(\xi^{(m-1)}), \mathrm{rank}(\xi^{(m)})),$$

where m is the step in which the associated flag stabilises. ♦

This notion can still be defined for general distributions, but then the growth vector is a lower semi-continuous function which varies with the point.

2.2. **Intermezzo: some remarks about differential systems.** Even in the non-weakly regular case, the modules in the associated flag are reasonably well-behaved:

Lemma 2.9. *Let ξ be a distribution on a smooth manifold M. The entries $\xi^{(i)}$ in the Lie flag are locally finitely generated C^∞-modules.*

Proof. Over any ball in M, ξ is trivial as a bundle and therefore a local framing exists. Since any vector field tangent to ξ can be expressed as a C^∞-combination of the elements of the framing, we deduce that $\xi^{(0)}$ is locally finitely generated.

By construction, $\xi^{(i)}$ is locally spanned by all the bracket expressions of order at most i evaluated on arbitrary vector fields tangent to ξ. Given such a bracket expression, we expand each entry in terms of the framing and we apply the Leibniz rule. This shows that $\xi^{(i)}$ is in fact locally spanned by bracket expressions whose entries are just elements of the chosen framing of ξ. The number of such expressions is finite, which proves the claim. □

This motivates us to introduce the following definition:

Definition 2.10. Let M be a smooth manifold. A **differential system** on M is a locally finitely generated C^∞-submodule of vector fields. ♦

We cannot ignore differential systems when we develop the theory of distributions (as seen in Example 2.5). Nonetheless, to keep these notes lean, we will pay little attention to them. We invite the reader to study the statements we provide (which deal just with distributions) and see whether they can be adapted to this more general setting.

Before doing so, let us show that the associated flag does not stabilise, in general, without the weak regularity assumption:

Example 2.11. Here is a variation on the Martinet distribution:

$$\xi = \ker(dy - f(z)dx) = \langle X = \partial_x + f(z)\partial_y, \partial_z \rangle,$$

where $f : \mathbb{R} \to \mathbb{R}$ is some arbitrary function. We may readily compute:

$$X' = [\partial_z, X] = f'(z)\partial_y, \qquad X^{(n+1)} = [\partial_z, X^{(n)}] = f^{(n+1)}(z)\partial_y,$$
$$\xi^{(n)} = \langle \partial_z, X, X', \cdots, X^{(n)} \rangle.$$

We can choose $f(z)$ to have a zero of order exactly j at 0, and this will imply that the associated flag stabilises in the term $\xi^{(j)}$. However, we can take this a step further and make f vanish to infinite order at 0, but have non-zero derivatives everywhere else; a possible choice is $f(z) = e^{-1/z^2}$. What is happening is the following: even though the C^∞-modules associated to $\xi^{(i)}$ keep growing with i, their pointwise spans remain the same.

By considering functions f vanishing to different orders at 0, we have that the corresponding (germs of) distributions ξ are not diffeomorphic to one another at the origin, because the associated flags cannot possibly be diffeomorphic. ♦

2.3. **A couple of relevant cases.** Before the Intermezzo, we introduced the associated Lie flag and the growth vector. These are differential invariants of the distribution. Within the spectrum of all distributions, these invariants may fall between two extreme situations: foliations and bracket-generating distributions.

2.3.1. *Foliations.* Before defining what a foliation is, we need some preliminary concepts:

Definition 2.12. Let M be a smooth n-manifold and let ξ be a rank-k distribution. A **foliation chart** for ξ at a point $p \in M$ is an embedding

$$\psi : [0,1]^n \to M$$

whose image contains p and which satisfies $\psi^*\xi = \ker(dx_{k+1}, \cdots, dx_n)$. Here $(x_1, \cdots, x_n)$ are the coordinates in $[0,1]^n$.

Then, we say that ξ is:

- **involutive** if $\xi^{(j)} = \xi^{(0)}$ for all j.
- **integrable** if it admits a foliation chart at every point in M. ♦

Identically, ξ is involutive if and only if $\Gamma(\xi)$ is a Lie subalgebra of $\mathfrak{X}(M)$. It is easy to see that integrability implies involutivity by simply computing the Lie flag in a foliation chart. A celebrated theorem of Frobenius provides the converse:

Theorem 2.13 (Frobenius)**.** *A distribution ξ is integrable if and only if it is involutive.*

This is an instance in which the differential invariants of the distribution (in this case the associated flag being trivial) determine its local form.

Remark 2.14. When ξ is a **line field** (i.e. of rank 1), it is always involutive and therefore always integrable. This tells us that there are always local coordinates in which ξ corresponds to a coordinate direction. The Frobenius theorem in this setting is also known as the **flowbox theorem**. ♦

In the domain of a foliation chart ψ, the common level sets of the functions $x_{k+1}, \cdots, x_n$ are submanifolds of dimension k tangent to ξ. Their images provide a partition of $\psi([0,1]^n)$. If ξ is a integrable distribution on a manifold M, these local partitions are compatible with one another in the following sense:

Definition 2.15. Let M be a smooth manifold. A partition $\mathcal{F}$ of M into k-dimensional manifolds $\{N_i\}_{i\in I}$ is called a **foliation** if:

- the collection of tangent spaces $T\mathcal{F} = \{T_pN_i \mid p \in M, i \in I\}$ is a smooth, integrable, k-dimensional distribution,
- if N_i and $N_{i'}$ intersect the same level set in a foliation chart, then $i = i'$.

The submanifolds in $\mathcal{F}$ are called **leaves**. ♦

The second condition tells us that the leaves are the *maximal* submanifolds tangent to $T\mathcal{F}$. Alternatively, one can define leaves as follows:

Lemma 2.16. *Let $\mathcal{F}$ be a foliation. Leaves correspond to equivalence classes of the following relation: $p, q \in M$ are related if and only if they can be connected by a path tangent to $T\mathcal{F}$.*

PROOF. Two points in a foliation chart can be joined by an admissible path if and only if they lie in a common level set of $x_{k+1}, \cdots, x_n$, which is the condition defining what a leaf is. □

Remark 2.17. A notion explored first by Stefan [45] and Sussmann [47] and later by Androulidakis and Skandalis [2] is that of a **singular foliation**. Following our reasoning so far, it is clear what it should be: an involutive differential system. This concept appears naturally in the study of general distributions due to the following result: If a distribution is not weakly regular and its associated flag stabilises, the final term $\xi^{(m)}$ is a singular foliation; see Lemma 2.9.

Given an involutive differential system, we can still provide a partition of M into leaves by following the procedure above using admissible paths. If the differential system does not have constant rank, the resulting leaves will have different dimensions. ♦

2.3.2. *Bracket-generating distributions.* The case that will be more interesting for us is the following:

Definition 2.18. A distribution ξ is **bracket-generating** if there exists some m such that the m^{th} element of the associated flag satisfies $\xi^{(m)} = \mathfrak{X}(M)$. ♦

That is, $\Gamma(\xi)$ generates $\mathfrak{X}(M)$ as an algebra. Our example of the coin was precisely of this form. We remarked then that any two points in the state space of the coin can be connected by a path tangent to the distribution. This is in fact a general phenomenon:

Theorem 2.19 (Chow)**.** *Let (M, ξ) be a connected manifold endowed with a bracket-generating distribution. Then any two points $p, q \in M$ can be connected by a path tangent to ξ.*

PROOF. We start with a general remark about vector fields: Given vector fields X and Y, we can take their flows ϕ_t^X and ϕ_t^Y. The definition of Lie bracket tells us that the flow $[\phi_t^X, \phi_t^Y]$ (which is non-autonomous, unlike ϕ_t^X and ϕ_t^Y) approximates $\phi_{t^2}^{[X,Y]}$ with error $o(t^2)$.

Locally around p we can find a framing $\{X_1, \cdots, X_k\}$ of ξ. The bracket-generating hypothesis tells us that these vector fields and some of their Lie brackets $\{X_{k+1}, \cdots, X_{\dim(M)}\}$ span the whole of TM. For each $j > k$, we can write $X_j = A_j(X_{i_0^j}, \cdots, X_{i_{a_j}^j})$, where A_j is some bracket expression of length a_j.

For $j \leq k$ we define ϕ_j^t to be the flow of X_j, and for $j > k$ we set

$$\phi_j^t = A_j(\phi_{i_1^j}^{t^{1/a_j}}, \cdots, \phi_{i_{a_j}^j}^{t^{1/a_j}})$$

which approximates the flow of X_j; this is a generalisation of the statement above we gave about commutators of two flows.

A first observation is that p and $\phi_j^t(p)$ can be joined by a path tangent to ξ, for all j. If $j \leq k$, this is clear because flowlines of ϕ_j^t are tangent to ξ. Suppose then that $j > k$. For simplicity we can look first at the case $X_j = [X_{i_1^j}, X_{i_2^j}]$ so $\phi_j^t = [\phi_{i_1^j}^t, \phi_{i_2^j}^t]$. Then $\phi_j^t(p)$ can be reached from p by first flowing a time $t^{1/2}$ along the flow of $X_{i_1^j}$, then a time $t^{1/2}$ along $X_{i_2^j}$, then a time $t^{1/2}$ in the direction of $-X_{i_1^j}$ and then a time $t^{1/2}$ along $-X_{i_2^j}$. The general case follows by flowing iteratively along each element in the bracket-expression that defines ϕ_j^t.

Now we define a map (which we call the **endpoint map** based at p):

$$\psi : \mathbb{R}^{\dim(M)} \to M$$

$$\psi(t_1, \cdots, t_{\dim(M)}) = \phi_{\dim(M)}^{t_{\dim(M)}} \circ \cdots \circ \phi_1^{t_1}(p).$$

The bracket-generating hypothesis tells us that this map is a local submersion at p. By construction, p can be joined with each point in the image of ψ by a path tangent to ξ. This argument is true for every $p \in M$, so any two points in M can be connected by a piecewise linear path tangent to ξ. □

Corollary 2.20. *Let (M,ξ) be a manifold endowed with a bracket-generating distribution. Let γ be a path connecting $p,q \in M$. Then there exists $\widetilde{\gamma}$ tangent to ξ, connecting p with q, and C^0-close to γ.*

Proof. Given $\gamma : [0,1] \to M$, divide $[0,1]$ in intervals of length $1/N$, for N large enough. Fix a finite covering of γ by little balls U_i such that U_i is neighbourhood of the interval $\gamma([i/N,(i+1)/N])$. Since each $(U_i, \xi|_{U_i})$ is a manifold endowed with a bracket generating distribution, we have that there is a path tangent to ξ and contained in U_i connecting $\gamma(i/N)$ with $\gamma((i+1)/N)$. Concatenating these paths yields $\widetilde{\gamma}$. □

In [25] M. Gromov explains how to modify this proof to yield a smooth immersed path (essentially by providing an argument to "round the corners" of the path we construct).

Example 2.21. Let us compare Chow's theorem with the following example: In $\mathbb{R}^3$ with coordinates (x,y,z), we look at the distribution $\xi = \ker(dz)$. This distribution is the tangent space of the foliation $\mathcal{F}$ of $\mathbb{R}^3$ whose leaves are level sets of z. Any admissible path must be tangent to $\mathcal{F}$.

We can now tweak it: Consider the distribution $\tilde{\xi} = \ker(dz + f(y)dx)$, where the function $f : \mathbb{R} \to \mathbb{R}$ is identically 0 if $y \leq 0$ and satisfies $f'(y) > 0$ everywhere else. This implies that $\tilde{\xi}$ is a foliation in the region $\{y \leq 0\}$ and bracket-generating everywhere else. Suppose we want to connect two points (x,y,z) and (x',y',z') lying in $\{y \leq 0\}$. Then, we can first connect to $(x,0,z)$ by using a straight path (which is contained in a leaf), then travel within the $\{y \geq 0\}$ region to $(x',0,z')$ using Chow's theorem, and finally go to (x',y',z') (again, staying within a leaf). Here the key fact we use is that all the leaves of the foliated region connect to the bracket-generating region.

This tells us that Chow's theorem can hold even without the bracket-generating assumption. ♦

If ξ is not bracket-generating, what can we say about the points q that can be reached from p by using an admissible path? As hinted in Remark 2.17, we may define a **leaf** of ξ to be a equivalence class of the relation: $p,q \in M$ are related if and only if they can be connected by a path tangent to ξ. Then, the following result holds:

Corollary 2.22 (Orbit theorem). *Let ξ be a distribution which stabilises but that is possibly not bracket-generating. Let $\xi^{(m)}$ be the involutive differential system to which it stabilises. Then, the leaves of ξ are precisely the leaves of $\xi^{(m)}$.*

Proof. The idea is very similar to Chow's theorem but the details are much more involved. We will provide a very rough sketch and we refer the reader to the original sources [45, 47].

We fix a point p. We construct an endpoint map (based at p) by using commutators of flows of vector fields tangent to ξ. However, this endpoint map does not take values in the whole of M, but only in the leaf of $\xi^{(m)}$ passing through p (because flows tangent to ξ are in particular tangent to $\xi^{(m)}$). It is possible to show (but we will not do it here) that this leaf is in fact a smooth submanifold, and the endpoint map provides a local chart of the leaf around p, proving the claim. □

2.4. **Intermezzo: A linear control problem.**
Having introduced some basic notions, we are ready to look at one of the problems that P. Lissy proposed and solved in his lectures (appearing in this same volume) using the *fictitious control method*. The contents of this Subsection are not needed later on, so the reader may safely skip it.

2.4.1. *The setup.* We consider a linear ODE $y'(t) = Ay(t)$ with two equations and two unknowns to which we add a control u:

$$y = (y_1, y_2) : \quad \mathbb{R} \to \mathbb{R}^2$$
$$u = (0, u_2) : \quad \mathbb{R} \to \{0\} \times \mathbb{R} \subset \mathbb{R}^2$$
$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ u_2(t) \end{pmatrix}.$$

That is, without our input, the system naturally follows a solution of the ODE $y'(t) = Ay(t)$. By making u non-zero, we are able to modify the behaviour of the trajectories.

We want to prove **complete controllability**. That is, given a starting point $(y_1(0), y_2(0))$, a time T, and an endpoint $(y_1(T), y_2(T))$, we want to find a control $u_2 : [0,T] \to \mathbb{R}$ such that $(y_1(t), y_2(t))_{t \in [0,T]}$, solving the control system above, has the desired boundary values.

2.4.2. *The solution.* Let us start by putting this in a more geometric language. Adding the time variable, we can regard our system as living in $\mathbb{R}^3$ with coordinates (y_1, y_2, t). The ODE can be expressed as the vector field

$$Y(y_1, y_2, t) = (a_{11}y_1 + a_{12}y_2)\partial_{y_1} + (a_{21}y_1 + a_{22}y_2)\partial_{y_2} + \partial_t,$$

i.e. we move at unit speed in time. Our control is another vector field:

$$U(y_1, y_2, t) = \partial_{y_2}.$$

And the two of them span a plane field $\xi = \langle Y, U\rangle$. We can now check that:

Lemma 2.23. *The distribution ξ is bracket-generating if and only if the* coupling constant a_{12} *is non-zero.*

Proof. The computation $[U, Y] = Y' = a_{12}\partial_{y_1} + a_{22}\partial_{y_2}$ shows that the collection $\{U, Y, Y'\}$ is a framing of $\mathbb{R}^3$ if and only if $a_{12} \neq 0$, as claimed.

Equivalently, we write $\xi = \ker(\alpha)$ where $\alpha = dy_1 - (a_{11}y_1 + a_{12}y_2)dt$. The bracket-generating condition is equivalent to

$$\alpha \wedge d\alpha = -a_{12}dy_1dy_2dt$$

being a volume form. This is true if and only if a_{12} is non zero. □

In particular, if $a_{12} = 0$, the distribution

$$\xi = \langle \partial_t + a_{11}y_1\partial_{y_1}, \partial_{y_2}\rangle = \ker(dy_1 - a_{11}y_1dt)$$

defines a foliation whose leaves are the level sets of the function $H(y_1, y_2, t) = e^{a_{11}t} - y_1$. Thus, under the hypothesis $a_{12} = 0$, the movement in y_1 cannot be affected at all by the control.

Proposition 2.24. *Complete controllability holds if and only if $a_{12} \neq 0$.*

Proof. We have just discussed the only if direction. The if statement can be rephrased geometrically as follows: Can we find a curve

$$\gamma : [0, T] \to \mathbb{R}^3$$

tangent to ξ, starting at the given point $(y_1(0), y_2(0), 0)$, finishing at the desired destination $(y_1(T), y_2(T), T)$, and of the form

$$\gamma'(t) = Y + u_2(t)U$$

for some control function $u_2 : [0, T] \to \mathbb{R}$? Note that, according to Chow's theorem, we can find a curve tangent to ξ connecting $(y_1(0), y_2(0), 0)$ with $(y_1(T), y_2(T), T)$. However, such a curve will not be of the form we want in general: it will sometimes make a reversal and go negatively in the time direction.

Nonetheless, we can find a curve of the form claimed. Consider the projection $\pi \circ \gamma : [0, T] \to \mathbb{R}^2$ to the (y_1, t)-coordinates. The expression

$$\xi = \ker(dy_1 - (a_{11}y_1 + a_{12}y_2)dt)$$

tells us that we can (uniquely) recover the y_2 coordinate of γ from its projection $\pi \circ \gamma$ using the closed formula:

$$y_2(t) = \frac{1}{a_{12}}(y_1'(t) - a_{11}y_1(t)).$$

The boundary conditions for y_1 and y_2, together with this expression, uniquely give us boundary conditions for the derivative y_1'. Now we can pick $y_1(t)$ to be an arbitrary smooth function satisfying these given boundary conditions (for itself and its derivative), and recover $y_2(t)$ from it.

Lastly, we recover the desired control $u_2(t)$ from $y_1(t)$ and $y_2(t)$. It is given by the expression:

$$u_2(t) = y_2'(t) - (a_{21}y_1(t) + a_{22}y_2(t))$$

i.e. the discrepancy between the evolution of y_2 given by the matrix A (i.e. if we were not acting on the system) and its actual value. □

2.4.3. *A slightly more general case.* We can in fact improve the statement, studying general ODEs of two equations and two unknowns with one control in the second equation. That is, consider the control system

$$y'(t) = A(y(t), t) + u(t) \tag{2.1}$$

$$A = (a_1, a_2) : \mathbb{R}^3 \to \mathbb{R}^2$$

$$u = (0, u_2) : \mathbb{R} \to \mathbb{R}^2,$$

where A is an arbitrary smooth function and u is the control.

Proposition 2.25. *Consider the control system presented in Equation 2.1. Assume that for each $(y_1, t) \in \mathbb{R}^2$ the map $y_2 \to a_1(y_1, y_2, t)$ is a diffeomorphism of $\mathbb{R}$. Then, the system is completely controllable.*

PROOF. The function A can be interpreted as a vector field $Y = \partial_t + a_1(y,t)\partial_{y_1} + a_2(y,t)\partial_{y_2}$ in $\mathbb{R}^3$ and the control as a second vector field $U = \partial_{y_2}$. They span a plane field ξ whose defining equation is $\alpha = dy_1 - a_1(y,t)dt$; you can check that it is bracket-generating if and only if $\partial_{y_2} a_1 \neq 0$.

We look for a curve $\gamma : [0, T] \to \mathbb{R}^3$ satisfying $\gamma'(t) = Y + u_2(t)U$ and some given boundary conditions at $t = 0, T$. We look at its projection $\pi \circ \gamma(t) = (y_1(t), t)$. Looking at α we see that $a_1(y,t) = y_1'(t)$ should hold. Evaluating the left-hand side at $t = 0, T$ we obtain boundary conditions for y_1'. We choose y_1 arbitrarily and satisfying the boundary conditions for itself and its derivative. Under the hypothesis, we can apply the implicit function theorem, parametrically in t, to $y_2 \to a_1(t)(y_1, y_2, t)$, i.e. we solve for the unique $y_2(t)$ satisfying $y_1'(t) = a_1(y,t)$. Lastly, we recover the control by setting $u_2(t) = y_2'(t) - a_2(y,t)$. □

You should think of the hypothesis as a quantitative version of the bracket-generating condition. It tells us that ξ turns very aggressively in the y_2 direction (equivalently, the control acts very strongly in the y_1 direction, thanks to the coupling). Observe that the linear case is a particular instance of this statement, by letting A be a fixed matrix.

2.4.4. *Localisation in time.* The key fact we used in the proofs of Propositions 2.24 and 2.25 is that finding a solution of the control problem amounts to picking a curve γ which is only constrained at the endpoints. This followed from the fact that the slope of γ (of y_1 with respect to t) could be taken to be arbitrary because the control allowed us to do so.

One can then prove a stronger result by noticing that this freedom for the slope is only necessary in a small time interval:

Corollary 2.26. *Consider the control system presented in Equation 2.1. Assume that there are constants t_0, δ such that: For every $y_1 \in \mathbb{R}$ and every $t \in [t_0 - \delta, t_0 + \delta] \subset [0, T]$ the map $y_2 \to a_1(y_1, y_2, t)$ is a diffeomorphism of $\mathbb{R}$. Then the system is completely controllable.*

PROOF. Consider the (unique!) solution s_0 of the (uncontrolled) ODE starting at $(y_1(0), y_2(0))$ at time 0 and finishing at time $t_0 - \delta$. Similarly, we consider s_1, the unique solution of the uncontrolled ODE starting at time $t_0 + \delta$ and finishing at $(y_1(T), y_2(T))$ at time T. Now the apply the previous Proposition to the control problem restricted to the time interval $[t_0 - \delta, t_0 + \delta]$ and with boundary conditions $s_0(t_0 - \delta)$ and $s_1(t_0 + \delta)$. □

2.4.5. *Local controllability.* Using the same approach, one can prove **local controllability**. I.e. given a trajectory of our system and a slight change of the endpoint, we can perturb the control slightly to land at the new endpoint. We leave this result as an exercise for the reader:

Lemma 2.27. *The system is locally controllable under the bracket-generating assumption $\partial_{y_2} a_1 \neq 0$.*

Observe that $\partial_{y_2} a_1 \neq 0$ implies that each function $y_2 \to a_1(y_1, y_2, t)$ is an immersion, but it may not be a diffeomorphism, as we required previously.

2.5. **The curvatures.** Before the Intermezzo we defined the associated Lie flag, which measures the non-integrability of the distribution ξ. We also defined weak regularity, which for simplicity we will henceforth assume:

Assumption 2.28. All distributions, unless stated otherwise, will be **weakly regular**. ♦

Prior results (unless explicitly stated) did not need to assume this. From now on (and thanks to this assumption) we will use the following notation: $\Gamma(\xi^{(j)})$ will be the j^{th} submodule of vector fields in the Lie flag associated to ξ. The corresponding distribution will be denoted by $\xi^{(j)}$.

Under the weak regularity hypothesis, we may look at the following morphism induced by the Lie bracket:

$$\Gamma(\xi^{(j)}) \times \Gamma(\xi^{(i)}) \to \Gamma(\xi^{(i+j+1)}) \subset \mathfrak{X}(M).$$

Observe that the elements in $\Gamma(\xi^{(j)})$ and $\Gamma(\xi^{(i)})$ are spanned by bracket expressions of lengths i and j, respectively. As such, their Lie bracket indeed lands in $\Gamma(\xi^{(i+j+1)})$. Since the Lie bracket is a derivation, we obtain a C^∞-linear map as follows:

Lemma 2.29. *The Lie bracket yields a well-defined bundle morphism:*

$$\Omega_{i,j}(\xi) = \xi^{(i)}/\xi^{(i-1)} \times \xi^{(j)}/\xi^{(j-1)} \to \xi^{(i+j+1)}/\xi^{(i+j)}$$

called the (i,j)*-**curvature**.*

Proof. It suffices to show that the composition

$$\Gamma(\xi^{(j)}) \times \Gamma(\xi^{(i)}) \to \Gamma(\xi^{(i+j+1)}) \to \Gamma(\xi^{(i+j+1)}/\xi^{(i+j)}),$$

where the second map is simply the quotient, is C^∞-linear. Indeed, given $u \in \Gamma(\xi^{(j)})$, $v \in \Gamma(\xi^{(i)})$, and a smooth function f, we have:

$$[u, fv] = f[u,v] + df(u)v$$

where the second term belongs to $\Gamma(\xi^{(i)}) \subset \Gamma(\xi^{(i+j)})$. The rest of the proof follows analogously. □

The curvature tells us much more about the non-integrability of the distribution, since it completely captures the behaviour of the distribution with respect to the Lie bracket.

Remark 2.30. Due to the antisymmetry of the Lie bracket, the first curvature $\Omega_{0,0}(\xi) : \xi \times \xi \to \xi^{(1)}/\xi$ factors through $\wedge^2(\xi)$. That is, it is a 2-form on ξ with values in $\xi^{(1)}/\xi$. ♦

Probably many of the readers have already encountered the curvature in the following incarnation:

Example 2.31. Let $\pi : X \to M$ be a fibre bundle. X comes tautologically endowed with a vertical distribution $V = \ker(d\pi)$; i.e. V is the collection of tangent spaces of the fibres of X. A (Ehresmann) **connection** on X is a distribution $H \subset TX$ that is complementary to V, i.e. $V \oplus H = TX$. We can think of H as a recipe to lift $T_{\pi(p)}M$ to a subspace of T_pX: Given a point $p \in X$ and a vector $u \in T_{\pi(p)}M$, there is a unique vector $\tilde{u} \in H_p$ satisfying $d\pi(\tilde{u}) = u$.

In this context we can look at the first curvature $\Omega_{0,0}(H)$. Given $p \in X$ and a pair $u, v \in T_{\pi(p)}M$, we lift them to a pair $\tilde{u}, \tilde{v} \in H_p$ and we apply $\Omega_{0,0}(H)$. The resulting vector $\Omega_{0,0}(H)(u,v)$ can be regarded (in a unique manner) as a vector in V_p. I.e. the curvature tells us how the (infinitesimal) commutation of two vector fields in the base produces a displacement in the fibre.

When X is a vector bundle, it is possible to recover the standard notion of (linear) connection and curvature as a particular case. ♦

Let us prove a simple lemma:

Lemma 2.32. *All the curvatures are determined by the ones of the form*

$$\Omega_{0,i}(\xi) : \xi^{(0)} \times \xi^{(i)}/\xi^{(i-1)} \to \xi^{(i+1)}/\xi^{(i)}.$$

Proof. This follows from the fact that any bracket expression A of length $i+1$ can be written, using the Jacobi identity, as a linear combination of expressions of the form $[-, B()]$, with B of length i. □

In the weakly regular setting, the rank of the curvatures of ξ may vary with the point, as the following example shows:

Example 2.33. In $\mathbb{R}^5$ with coordinates (x_1, x_2, y_1, y_2, z) we can define a distribution

$$\xi = \ker(dz - y_1 dx_1 - f(y_2) dx_2) = \langle \partial_{y_1}, \partial_{y_2}, \partial_{x_1} + y_1 \partial_z, \partial_{x_2} + f(y_2)\partial_z \rangle$$

which is in fact a connection over the $\mathbb{R}^4$ spanned by (x_1, x_2, y_1, y_2). We choose $f : \mathbb{R} \to \mathbb{R}$ to be identically zero if $y_2 \geq 0$ and strictly increasing otherwise. The only non-trivial Lie brackets among elements of this framing are:

$$[\partial_{y_1}, \partial_{x_1} + y_1\partial_z] = \partial_z, \qquad [\partial_{y_2}, \partial_{x_2} + f(y_2)\partial_z] = f'(y_2)\partial_z.$$

From this we see that the distribution is bracket-generating and weakly regular. However, the curvature $\Omega_{0,0}(\xi)$ has maximal rank (i.e. 4) if $y_2 < 0$ but rank 2 if $y_2 \geq 0$. ♦

2.6. **Notation: graded Lie algebras.** Our next goal is to put all the curvatures together into an algebraic object that is easier to deal with. In order to do this, we must introduce some new notation:

Definition 2.34. A **graded Lie algebra** is a Lie algebra $(V; [,])$ such that:

- Its underlying vector space V is graded:
$$V = \oplus_{i=0}^m V_i \oplus V_\infty,$$
where m is some non-negative integer.
- The Lie bracket is compatible with the grading in the following sense: all brackets involving V_∞ are zero and, for i and j finite, $[V_i, V_j] \subset V_{i+j+1}$.

An element v of V_i is said to have **grading** or **degree** i. We denote this by $\mathrm{gr}(v) = i$. ♦

A Lie algebra is **nilpotent** if any sufficiently long bracket-expression is zero. Graded Lie algebras are a particular case, since bracket-expressions of length $m + 1$ are zero by construction.

Remark 2.35. The reader may refer to [39] for a more in-depth account of graded Lie algebras and their deformation theory. We note that our grading convention differs from the one used in the literature by a -1 shift. That is, the graded Lie algebras we consider would start at degree 1 (instead of 0) using this alternate convention.

Example 2.36. Let us look at examples in low dimensions. The first non-trivial example appears when $\dim(V) = 3$. We can take the bracket to be $[e_1, e_2] = e_3$. This is called the **Heisenberg** algebra. Several gradings are compatible with this; they are all of the form $\mathrm{gr}(e_1) = i$, $\mathrm{gr}(e_2) = j$, and $\mathrm{gr}(e_3) = i + j + 1$. Any other non-trivial example in dimension 3 is isomorphic to these ones.

Suppose now $\dim(V) = 4$. We take brackets $[e_1, e_2] = e_3$ and $[e_1, e_3] = e_4$. This is called the **Engel algebra**. We may again choose several gradings, but the most natural one is simply $\mathrm{gr}(e_1) = 0$, $\mathrm{gr}(e_2) = 0$, $\mathrm{gr}(e_3) = 1$, and $\mathrm{gr}(e_4) = 2$. All examples in dimension 4 are either trivial, decomposable into a trivial piece plus the Heisenberg algebra, or isomorphic to the Engel algebra.

If $\dim(V) = 5$, we still get a finite list of non-isomorphic algebras. From dimension 6 onwards, graded Lie algebras might have moduli: that is, there are families of Lie algebras parametrised by real parameters whose elements are pairwise non-isomorphic. ♦

Now we take this concept to the bundle-theoretical framework:

Definition 2.37. A **weak bundle of graded Lie algebras** is a pair $(L \to M; [,])$ satisfying:

- $L \to M$ is a graded vector bundle
$$L = \oplus_{i=0}^m L_i \oplus L_\infty.$$
- $[,]$ is a fibrewise Lie bracket turning the fibres of L into graded Lie algebras. The pairing $[,]$ varies smoothly with the base point in M.

If $(L \to M; [,])$ is locally trivial (i.e. modelled on a single Lie algebra), we say that it is a **bundle of graded Lie algebras**. ♦

We will often abuse notation and simply say that L is a bundle of graded Lie algebras; the pairing $[,]$ will be implicit.

2.7. **The nilpotentisation.** Our goal in the next Section will be to study certain classes of distributions having interesting properties. What we do is we impose a certain set of local invariants and we try to classify all distributions having those invariants. At this point we have looked at some of them: the growth vector and the curvatures. We can package all this information using the following algebraic gadget:

Definition 2.38. Let (M, ξ) be a weakly regular distribution with m the degree in which its associated flag stabilises. The **nilpotentisation** associated to ξ is the weak bundle of graded Lie algebras:

$$L(\xi) = \oplus_{i=0}^{m} L_i(\xi) \oplus L_\infty(\xi)$$

$$L_0(\xi) = \xi^{(0)}, \qquad L_i(\xi) = \xi^{(i)}/\xi^{(i-1)}, \qquad L_\infty(\xi) = TM/\xi^{(m)},$$

where the Lie bracket is $\oplus_{i,j}\Omega_{i,j}(\xi)$. ♦

The nilpotentisation $L(\xi)$ associated to ξ is, in general, just a weak bundle. A first issue appeared already in Example 2.33, where the rank of the curvature varied with the point; this yields very different Lie brackets over different points. More subtle issues exist, as we pointed out in Example 2.36: there are smooth families of graded Lie algebras where the ranks of the curvatures remain the same but the algebras are not isomorphic. Thus:

Definition 2.39. We say that ξ is **regular** if and only if $L(\xi)$ is a *bundle of graded Lie algebras*. I.e. if and only if there are local trivialisations identifying all the fibrewise graded Lie algebras with a fixed graded Lie algebra. ♦

And from now on we will work under the hypothesis:

Assumption 2.40. ξ is **regular**. ♦

Example 2.41. A plane field ξ in a 3-manifold is said to be a *contact structure* if $L(\xi)(p)$ is isomorphic to the Heisenberg algebra at each p; you may check that the coin example from Subsection 1.1 is of this form. A plane field in a 4-manifold is *Engel* if $L(\xi)(p)$ is isomorphic to the Engel algebra. We will look at these examples (and many others) in Subsection 2.11. ♦

2.8. **Aside: filtered manifolds.** The attentive reader has probably observed that a weak bundle $L(\xi)$ arising from a distribution ξ has a particular form: By construction, bracket expressions of length i with entries in $L_0(\xi) = \xi^{(0)} = \xi$ generate $L_i(\xi) = \xi^{(i)}/\xi^{(i-1)}$. Not every bundle of graded Lie algebras behaves like this because, for a general graded Lie algebra $(V; [,])$, the inclusion $[V_i, V_j] \subset V_{i+j+1}$ might not be an equality.

What this is telling us is that (weak) bundles of graded Lie algebras do not just package invariants of distributions, but of more general objects:

Definition 2.42. Let M be a smooth manifold. A **filtered structure** on M is a flag of differential systems:

$$\xi_0 \subset \xi_1 \subset \cdots$$

satisfying $[\xi_i, \xi_j] \subset \xi_{i+j+1}$. Its **growth vector** is

$$(\mathrm{rank}(\xi_0), \mathrm{rank}(\xi_1), \cdots).$$

♦

We will say that the filtered structure $\{\xi_i\}_i$ is **weakly regular** if all the ξ_i correspond to distributions. A weakly regular filtered structure has an associated **nilpotentisation** $L(\xi_i)$ which is constructed, just like in the distribution case, using the curvatures. We then say that a filtered structure is **regular** if the Lie algebras $L(\xi_i)(p)$ are all isomorphic for varying p.

Example 2.43. Let M be a 3-dimensional manifold. Consider a filtered structure $\xi_0 \subset \xi_1 \subset \xi_2$ with ξ_1 a contact structure, ξ_0 a line field, and ξ_2 the space of all vector fields. Its nilpotentisation is fibrewise isomorphic to the Heisenberg Lie algebra $[e_1, e_2] = e_3$ with grading $\mathrm{gr}(e_1) = 0$, $\mathrm{gr}(e_2) = 1$, and $\mathrm{gr}(e_3) = 2$. ♦

2.9. **Submanifolds.** Given a manifold M endowed with a distribution ξ, there are two natural constraints one can impose on a submanifold $N \subset M$. We can require it to be **tangent** (sometimes called **integral**), which means that $TN \subset \xi|_N$, or we can require it to be **transverse**, which means that $TN \oplus \xi|_N$ is as large as possible (i.e. we require the usual transversality condition for linear subspaces of T_pM at each point $p \in N$).

These two cases are the most studied ones in the literature, but one can in fact look at the following more general situation:

Definition 2.44. We say that N is **weakly regular** if $TN \cap \xi^{(i)}$ has constant rank, for all i. ♦

Under this hypothesis, the sequence of distributions

$$TN \cap \xi^{(0)} \subset TN \cap \xi^{(1)} \subset \cdots \subset TN \cap \xi^{(m)} \subset TN$$

is a filtered structure on N. In general, this filtered structure is *not* the flag associated to $TN \cap \xi$. It is easy to see that the growth vector of the Lie flag associated to $TN \cap \xi$ is bounded above by the growth of the filtered structure.

The key observation is that weakly regular submanifolds yield Lie subalgebras of $L(\xi)$ at each point.

Proposition 2.45. *Let N be weakly regular. At every $p \in N$ we have a Lie subalgebra:*

$$L(\xi, N)(p) = (T_pN \cap \xi^{(0)}) \oplus \frac{T_pN \cap \xi^{(1)}}{T_pN \cap \xi^{(0)}} \oplus \cdots \oplus \frac{T_pN \cap \xi^{(m)}}{T_pN \cap \xi^{(m-1)}} \oplus \frac{T_pN}{T_pN \cap \xi^{(m)}} \subset L(\xi)(p).$$

PROOF. Given vectors $v, w \in L(\xi, N)(p)$, we can find local extensions $\tilde{v}$ and $\tilde{w}$ tangent to ξ and N. Then their Lie bracket $[\tilde{v}, \tilde{w}]$ is tangent to N at p. This tells us that $[v, w]$ is contained in $L(\xi, N)(p)$. □

All of these subalgebras glue together to yield a subbundle of graded Lie algebras $L(\xi, N) \subset L(\xi)|_N$ over N.

Definition 2.46. Fix a distribution (M, ξ), a weakly regular submanifold N, and a family $\mathfrak{G}$ of graded Lie algebras.

- N is a **$\mathfrak{G}$-submanifold** if, for all $p \in N$, $L(\xi, N)(p)$ is isomorphic to an element of $\mathfrak{G}$.
- N is **regular** if $L(\xi, N)$ is fibrewise isomorphic to a fixed graded Lie algebra. ♦

2.10. **Intermezzo: Pfaffian systems.** Before moving on to examples, let us go over an important notion for completeness. Instead of studying distributions, one may look at their duals:

Definition 2.47. Let M be a smooth manifold. A **Pfaffian system** in M is a subbundle of T^*M. ♦

Given a distribution $\xi \subset TM$, one may consider its *annihilator* $\xi^\perp \subset T^*M$, which is a Pfaffian system. Given a Pfaffian system I, we can look at $\Gamma(I)$, its space of sections, which is a C^∞-submodule of $\Omega^1(M)$.

Definition 2.48. Let M be a smooth manifold and I a Pfaffian system. The **dual flag** of the distribution $I^\perp$ is the flag defined inductively by:

$$\begin{aligned} I^{(0)} &= \Gamma(I) \\ I^{(j)} &= \{\alpha \in I^{(j-1)} \mid d\alpha(v, -) \in I^{(j-1)}\, \forall v \in I^\perp\} \subset I^{(j-1)}. \end{aligned}$$

♦

The following lemma provides the desired duality between distributions and Pfaffian systems:

Lemma 2.49. *Let ξ be a weakly regular distribution. Then $(\xi^{(j)})^\perp = (\xi^\perp)^{(j)}$.*

PROOF. We prove it by induction on j; the base case $j = 0$ holds by definition. A 1-form α in $(\xi^\perp)^{(j-1)}$ belongs to $(\xi^\perp)^{(j)}$ if and only if $d\alpha(v, -)$ is in $(\xi^\perp)^{(j-1)}$ for every vector field v tangent to ξ. I.e. if and only if $d\alpha(v, w) = 0$ for every vector field w tangent to $\xi^{(j-1)}$, by induction hypothesis. We can then apply Cartan's formula:

$$0 = d\alpha(v, w) = -\alpha([v, w]).$$

Since $\xi^{(j)}$ is spanned by all elements of the form $[v,w]$, we have shown that the previous conditions are equivalent to α belonging to $(\xi^{(j)})^\perp$, concluding the claim. □

However, there are many subtleties when the regularity hypothesis is dropped and, in fact, the duality breaks down:

Example 2.50. The annihilator of the Martinet distribution

$$\xi = \ker(dy - z^2dx) = \langle X = \partial_x + z^2\partial_y, \partial_z\rangle.$$

is simply $I = \langle\alpha = dy - z^2dx\rangle$. We claim that $I^{(1)} = 0$; to prove this we must show that $d(g\alpha)(v,w) = 0$ for every $v, w \in \Gamma(\xi)$ implies that g is the identically zero function. Since ξ is bracket-generating, g must be zero in $\{z \neq 0\}$, and therefore also along $\{z = 0\}$ by continuity. Thus we see that $I^{(1)}$ is different from $(\xi^{(1)})^\perp$ (which is supported purely along $\{z = 0\}$ and, as such, it is not a submodule of 1-forms).

We can vary this theme a little and consider instead:

$$I = \langle\alpha = dy - f(z)dx\rangle$$

If $f(z) = z^k$, for some $k > 1$, we can reason as above and show that $I^{(1)} = 0$. However, if f vanishes in the half-space $\{z \leq 0\}$ and it is strictly increasing in $\{z > 0\}$, the result is a bit more interesting. Indeed, $g\alpha$ belongs to $I^{(1)}$ if and only if g vanishes identically in $\{z > 0\}$. However, the space of such functions (and thus $I^{(1)}$) is not locally finitely generated over $C^\infty(\mathbb{R}^3)$! Additionally, you can check that, just as before, $I^{(1)}$ is not $((I^\perp)^{(1)})^\perp$, since the latter is non-zero along $\{z = 0\}$. ♦

Remark 2.51. One can define the **dual growth vector** to be the sequence of numbers

$$(\mathrm{corank}(I^{(0)}), \mathrm{corank}(I^{(1)}), \cdots, \mathrm{corank}(I^{(m-1)}), \mathrm{corank}(I^{(m)})),$$

where m is the step in which the dual flag stabilises. In the weakly regular case, as we have seen, this is just the usual growth vector. In the general case the two might be quite different: in the Martinet example the dual growth vector is $(2,3)$ and does not depend on the point, despite the fact that the growth vector has a discontinuity at $\{z = 0\}$ (where it grows more slowly). ♦

We will not look at Pfaffian systems in depth in these notes, but for many purposes this dual picture is quite useful. Indeed, we will often present local models for distributions using 1-forms (for instance, in the upcoming Subsection 2.11).

2.11. Examples of distributions.

2.11.1. *Foliations.* Let M be an n-dimensional manifold endowed with an involutive distribution ξ of rank k. By definition, the associated flag is constant, i.e. $\xi^{(i)} = \xi$, and the curvatures are necessarily zero. The nilpotentisation is simply $L(\xi) = \xi \oplus TM/\xi$. The first term lies in degree 0 and the second one in degree ∞. The bracket is trivial.

As we saw already, Frobenius' theorem states that ξ is involutive if and only if it is tangent to a foliation of rank k. This tells us that there are local coordinates $(x_1, \cdots, x_n)$ in which $\xi = \ker(dx_{k+1}, \cdots, dx_n)$, so in this particular case the differential invariants determine completely the local model of the distribution.

2.11.2. *Contact structures.* Let M be a $2n+1$-dimensional manifold with ξ a distribution of rank $2n$. We may look at the first curvature:

$$\Omega_{0,0} : \xi \times \xi \to TM/\xi.$$

Being a 2-form with values in the line bundle TM/ξ, we can study its rank. We say that ξ is a **contact structure** if the rank is maximal (i.e. $2n$). This means that, given a vector v in ξ, we can always find another vector w such that $\Omega_{0,0}(v,w) \neq 0$.

Just like in the foliation case, contact structures possess a *unique local model*: there are local coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, z)$ in which $\xi = \ker(dz + \sum_i (y_i dx_i - x_i dy_i))$. We will prove this claim for $2n + 1 = 3$ in Proposition 3.11.

If ξ is a contact structure, then all the fibres of $L(\xi)$ are isomorphic to the *Heisenberg Lie algebra* H_n. We already encountered it in Example 2.36 when $2n+1=3$. In the general case, it is defined by the following relations:

$$H_n = (\mathbb{R}^{2n+1}(x_1, \cdots, x_n, y_1, \cdots, y_n, z); [x_i, y_i] = z)$$

where x and y lie in degree zero and z lies in degree 1. You should compare this with the local model.

The first topological result about contact structures was proven by Gray [23]:

Theorem 2.52 (Gray)**.** *Let $(\xi_t)_{t\in[0,1]}$ be a 1-parametric family of contact structures on a closed smooth manifold M. Then, there exists a 1-parametric family of diffeomorphisms $(\phi_t)_{t\in[0,1]} : M \to M$ such that $\phi_t^*\xi_t = \xi_0$.*

This is a remarkable fact: any two contact structures that are homotopic to one another are in fact the same (they simply differ by the diffeomorphism ϕ_1)! This makes contact structures extremely nice to work with: any invariant we may define remains unchanged when we homotope the structure. This global stability is in fact a very rare phenomenon in the world of distributions (for instance, it does not hold for foliations, as Example 3.1 below shows).

2.11.3. *Engel structures.* We have already covered all the regular distributions when M is a 3-manifold: they are either line fields (rank 1), contact structures (rank 2), foliations by surfaces (rank 2), or the whole of TM (rank 3).

Let us look at 4-manifolds. Let ξ be a distribution of rank 2. First we inspect its curvature

$$\Omega_{0,0}(\xi) : \xi \times \xi \to \wedge^2(\xi) \cong \det(\xi) \to TM/\xi.$$

This map has image at most 1-dimensional, since it factors through the line bundle $\det(\xi)$. That is, unless ξ is a foliation, it induces an isomorphism between ξ_1/ξ and $\det(\xi)$. Under this assumption we may then look at the second curvature:

$$\Omega_{0,1}(\xi) : \xi \times \xi^{(1)}/\xi \to TM/\xi^{(1)}.$$

Since the target is a line bundle, this map must necessarily have a non-trivial kernel. Now there are two possible cases:

- $\Omega_{0,1}(\xi)$ is surjective, and therefore the kernel is a line bundle $\mathcal{W}$ contained in ξ.
- $\Omega_{0,1}(\xi)$ is the zero map.

In the first case we say that ξ is maximally non-integrable or **Engel**. In the second case $\xi^{(1)}$ is a foliation by 3-dimensional leaves and ξ is a leafwise contact structure.

The **Engel algebra**, as introduced in Example 2.36, is given by:

$$\mathfrak{E} = (\mathbb{R}^4(x, y, z, w); [x, w] = z, [x, z] = y),$$

where x and w lie in the degree 0 part, z in degree 1, and y in degree 2.

The nilpotentisation associated to an Engel structure is fibrewise isomorphic to $\mathfrak{E}$. F. Engel proved that Engel structures admit a unique local model; that is, there are local coordinates (x, y, z, w) in which

$$\begin{aligned} \xi^{(1)} =& \ker(dy - zdx) = \langle \partial_w, \partial_z, \partial_x + z\partial_y \rangle \\ \xi =& \ker(dy - zdx, dz - wdx) = \langle \partial_w, (\partial_x + z\partial_y) + w\partial_z \rangle \\ \mathcal{W} =& \langle \partial_w \rangle \end{aligned}$$

The 3-distribution $\xi^{(1)}$ corresponds, at the Lie algebra level, to the 3-dimensional subspace of elements of degree at most 1 (spanned by the x, w, and z axes). The line field $\mathcal{W}$ corresponds to the w-axis.

Engel structures present (from a topological perspective) many parallels with contact structures. We will mention some of them at the end of these notes in Subsection 3.6.

2.11.4. *1-forms of constant rank.* Generalising the contact case we can look at distributions (M^{2k+j+1}, ξ^{2k+j}) whose nilpotentisation $L(\xi)$ is pointwise isomorphic to:

$$H_k \oplus I_j = (\mathbb{R}^{2k+j+1}(x_1, \cdots, x_k, y_1, \cdots, y_k, z, e_1, \cdots, e_j); [x_i, y_i] = z)$$

where x, y, and e lie in degree zero and z lies in degree 1. This is a *stabilisation* of the Heisenberg algebra by adding the trivial algebra with j generators. When $j = 1$, such distributions are called **even-contact structures** and in the general case they are called 1-forms of constant rank $2k$.

Any such ξ contains the smaller distribution $\mathcal{F}_\xi = \ker(\Omega_{0,0}(\xi))$ and it is not difficult to see that $\mathcal{F}_\xi$ is in fact involutive. This allows us to think of ξ as a foliation $\mathcal{F}_\xi$ whose transverse structure is contact. This means that the holonomy of $\mathcal{F}_\xi$ is by contactomorphisms (i.e. transformations that preserve the transverse contact structure).

Using the local model for contact structures it follows that any such ξ is locally isomorphic to:

$$(\mathbb{R}^{2k+j+1}[x_1, \cdots, x_k, y_1, \cdots, y_k, z, e_1, \cdots e_j], \xi = \ker(dz + \sum_i (y_i dx_i - x_i dy_i))),$$

that is, the contact model times $\mathbb{R}^j$.

Remark 2.53. A distribution $\mathcal{D}$ is Engel (as defined before) if and only if $\mathcal{D}^{(1)}$ is even-contact. ♦

2.11.5. *Goursat structures.* At this point we have classified all possible local models for regular distributions in dimension 4. In rank 1 we have line fields. In rank 2, we have foliations by surfaces, Engel structures, and contact structures tangent to a foliation of rank 3. In rank 3, we have foliations by hypersurfaces and even-contact structures. All of these have a unique local model.

We go on then to dimension 5. Let ξ be a rank-2 distribution. The following cases are variations of the ones we have already seen:

- Growth vector (2) is the foliation by surfaces case.
- Growth vector $(2, 3)$ is a foliation $\xi^{(1)}$ by 3-dimensional leaves with ξ a leafwise contact structure.
- Growth vector $(2, 3, 4)$ is a foliation $\xi^{(2)}$ by 4-dimensional leaves with ξ a leafwise Engel structure.

This leaves two cases: growth $(2, 3, 4, 5)$ and $(2, 3, 5)$. Structures with growth vector $(2, 3, 4, 5)$ are a particular case of:

Definition 2.54. Let M be a smooth manifold of dimension $k+2$. A bracket-generating distribution of rank 2 and growth vector $(2, 3, 4, \cdots, k+2)$ is said to be a **Goursat structure**. ♦

In particular, observe that 3-dimensional contact structures correspond to the case $k = 1$, and Engel structures to the case $k = 2$.

In the next Subsubsection we will discuss jet spaces and their canonical distribution ξ_{can}. The following local model for Goursat structures holds:

Lemma 2.55. *The Goursat structure with growth* $(2, 3, 4, \cdots, k+2)$ *is locally diffeomorphic to the canonical structure* ξ_{can} *in the jet space* $J^k(\mathbb{R}, \mathbb{R})$.

Remark 2.56. In Remark 2.51 we pointed out that one can define the notion of growth vector using Pfaffian systems and that this might not agree with the usual definition. Goursat structures are sometimes defined in the literature by requiring the dual growth vector to be constant and equal to $(2, 3, 4, \cdots, k+2)$ (or, alternatively, using the big growth vector, which we have not defined here). This yields a larger class of distributions, some of which have "exotic local models", different from the one claimed in Lemma 2.55. This is hardly surprising, given the fact that the usual growth vector grows more slowly at the points with exotic local model [36]. ♦

2.11.6. *(2,3,5)-structures.* Lastly, we look at rank 2-distributions with growth $(2, 3, 5)$. Such a structure is often said to be of **Cartan type**, since they were studied by Cartan in his 5-variables paper [5]. We can readily see that both curvatures are isomorphisms:

$$\Omega_{0,0}(\xi) : \xi \times \xi \cong \det(\xi) \to \xi^{(1)}/\xi$$

$$\Omega_{0,1}(\xi) : \xi \times \xi^{(1)}/\xi \to TM/\xi^{(1)}.$$

The nilpotentisation associated to a $(2,3,5)$-structure is fibrewise modelled on the following Lie algebra:

$$(\mathbb{R}^5(x,y,z,a,b); [x,y] = z, [x,z] = a, [y,z] = b).$$

A reasonable choice of degrees is $\mathrm{gr}(x) = \mathrm{gr}(y) = 0$, $\mathrm{gr}(z) = 1$, and $\mathrm{gr}(a) = \mathrm{gr}(b) = 2$. These structures lack a unique local model, and instead there is a whole moduli of them, as studied by Cartan himself.

2.11.7. *Jet bundles.* Lastly, we can construct a large class of distributions in varying dimensions and ranks by looking at **jet spaces**. As we advanced in the Introduction, these are fundamental in the geometric formulation of PDEs. Let us explain how that goes by starting with the simplest case.

Given a map $f : \mathbb{R}^n \to \mathbb{R}^m$, we can consider the graph of its differential information up to order 1, that is:

$$j^1 f : \mathbb{R}^n \to J^1(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{mn}$$
$$x \to j^1 f(x) = (x, y(x) = f(x), z(x) = \partial_x f(x))$$

Here $\partial_x f$ denotes the Jacobian of f. The target $J^1(\mathbb{R}^n, \mathbb{R}^m)$ is called the **space of 1-jets** of maps $\mathbb{R}^n \to \mathbb{R}^m$; it fits into the tower of bundles:

$$J^1(\mathbb{R}^n, \mathbb{R}^m) \to J^0(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n.$$

When we talk about sections of $J^1(\mathbb{R}^n, \mathbb{R}^m)$, we always mean sections over $\mathbb{R}^n$, which we call the **base**. The forgetful map $J^1(\mathbb{R}^n, \mathbb{R}^m) \to J^0(\mathbb{R}^n, \mathbb{R}^m)$ is called the **front projection**; it discards the information about the derivative.

Definition 2.57. A section $s(x) = (x, y(x), z(x))$ of $J^1(\mathbb{R}^n, \mathbb{R}^m)$ of the form $s(x) = j^1 f(x)$ is said to be **holonomic**.

By construction, every holonomic section $(x, y(x) = f(x), z(x) = \partial_x f(x))$ of $J^1(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^n$ satisfies

$$\frac{\partial y_i}{\partial x_j}(x) = z_{ij}(x).$$

Identically, if we look at the 1-forms:

$$\alpha_i = dy_i - \sum_{j=1}^{n} z_{ij} dx_j$$

we have that $j^1 f$ lies in the kernel of all them:

$$(j^1 f)^* \alpha_i = df_i - \sum_{j=1}^{n} (\partial_{x_j} f_i) dx_j = 0.$$

We have shown:

Lemma 2.58. *A section $s(x)$ of $J^1(\mathbb{R}^n, \mathbb{R}^m)$ is holonomic if and only if it is tangent to the **tautological distribution***

$$\xi_{\mathrm{can}} = \cap_i \ker(\alpha_i).$$

The distribution ξ_{can} is sometimes also called the **Cartan distribution**. If $m = 1$, this is simply the standard contact structure in $\mathbb{R}^{2n+1}$!

Remark 2.59. Holonomic sections are in particular integral submanifolds of ξ_{can}. Conversely, any integral submanifold which is graphical is the image of a holonomic section. However, not all integral submanifolds are necessarily graphical. Indeed, consider the embedding

$$\begin{aligned}(t, x_2, \cdots, x_n) \quad \mapsto \quad & (t^2/2, x_2, \cdots, x_n, \\ & y_1 = t^3/3, y_2 = 0, \cdots, y_m = 0, \\ & z_{1,1} = t, z_{1,2} = 0, \cdots, z_{m,n} = 0).\end{aligned}$$

At $t = 0$, the image is tangent to the fibre of $J^1(\mathbb{R}^n, \mathbb{R}^m) \to J^0(\mathbb{R}^n, \mathbb{R}^m)$, so its front projection has a singularity. This singularity is the usual *cusp*, and it is the simplest type of singularity of tangency an integral submanifold might have. Studying these singularities is an important area of research in the intersection of real Singularity Theory and Geometry of PDEs [1, 22, 42]. ♦

This can be generalised to higher orders, yielding the **space of r-jets**:

$$j^r f : \mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{mn} \times \cdots$$
$$x \to j^r f(x) = (x, y(x) = f(x), z(x) = \partial_x f(x), w(x) = \partial_x^2 f(x), \cdots ...)$$

Where $\partial_x f$ denotes the Jacobian of f, $\partial_x^2 f$ the Hessian, and so on. Similarly, instead of considering base $\mathbb{R}^n$ and fibre $\mathbb{R}^m$, one can consider arbitrary bundles over a smooth manifold. In all these cases there is a Cartan distribution ξ_{can} characterised by the fact that sections are holonomic if and only if they are tangent to ξ_{can}.

As we have seen, contact structures appear for $m = 1$ and $r = 1$. If $n = 1$ and $m = 1$ and $r = 2$, the Cartan distribution is Engel. In all other cases (for instance, the Goursat case $n, m = 1$, $r > 2$) the Cartan distribution is far from being *generic*, that is, for its given dimension and rank, its growth vector is smaller than it should be (even though it remains bracket-generating) and its curvatures have a large kernel. Generic distributions (particularly their growth) were studied in detail by Gershkovich and Vershik in [21].

3. Topology of distributions

We will now leave the realm of Geometry and instead put on our Topology glasses, asking ourselves:

- Can we classify distributions not *locally* but *globally* (i.e. not germs but distributions in a whole manifold)?

There are at least two ways of approaching this classification question. First, we could try to classify distributions up to *diffeomorphism* of the ambient manifold. Consider the following example:

Example 3.1. Let $\mathbb{T}^2$ be the 2-torus with coordinates x, y. Endow it with the foliations $\mathcal{F}_a = \langle \partial_x + a\partial_y \rangle$, where $a \in \mathbb{R}$. If a is rational, the leaves of $\mathcal{F}_a$ are circles. Otherwise, they are diffeomorphic to $\mathbb{R}$. As such, foliations that are arbitrarily close to one another in the space of distributions might display completely different dynamical behaviours and thus not be diffeomorphic. This tells us that the classification up to diffeomorphism is extremely fragile. ♦

This example tells us that the classification of distributions up to diffeomorphism is a rather geometric problem. As topologists, we will instead focus on the classification up to homotopy (once certain differential invariants have been fixed).

3.1. **The space of all distributions.**

Before we impose any differential condition (say, fixing the growth vector or the behaviour of the curvatures), we should understand a bit better the space of all distributions $\mathrm{Dist}(M, k)$ of rank k on a given manifold M. This is clearly a set and we can furthermore endow it with a topology using the following correspondence:

Lemma 3.2. *There is a bijective correspondence:*

$$\mathrm{Dist}(M, k) \cong \Gamma(\mathrm{Gr}(TM, k))$$

between distributions of rank k and smooth sections of the Grassmann bundle $\mathrm{Gr}(TM, k) \to M$.

Proof. This follows from the definition of what a distribution is, but let us clarify a bit. First recall: Given a vector space V, the Grassmanian $\mathrm{Gr}(V, k)$ is the space of all k-dimensional linear subspaces of V. A particular example is $\mathrm{Gr}(V, 1)$, projective space. A k-dimensional vector subspace $W \subset V$ determines tautologically an element in $\mathrm{Gr}(V, k)$.

Analogously, we define $\mathrm{Gr}(TM, k)$ to be the bundle over M whose fibre at a point $p \in M$ is $\mathrm{Gr}(T_pM, k)$. The correspondence $\mathrm{Dist}(M, k) \cong \Gamma(\mathrm{Gr}(TM, k))$ is given by the fibrewise tautological map. □

As a space of sections, $\mathrm{Dist}(M, k)$ admits several distinct topologies. For our purposes we will use the **Whitney C^r-topologies**, $r \in \{0, 1, \cdots, \infty\}$. We will make particular choices of r depending on what we aim to do.

Example 3.3. Let us study the space $\mathrm{Dist}(M, 1)$ when M is a closed 3-dimensional manifold. Any such manifold M is parallelisable, i.e. its tangent bundle is trivial $TM \cong M \times \mathbb{R}^3$. This automatically provides a trivialisation of the bundle $\mathrm{Gr}(TM, 1) \cong M \times \mathbb{RP}^2$.

I.e. providing a rank-1 distribution in M is the same as providing a map $M \to \mathbb{RP}^2$. As such, the space of all rank-1 distributions is homotopy equivalent to the space of maps of M into $\mathbb{RP}^2$. Suppose that $M = \mathbb{S}^3$. Then, the connected components $\pi_0(\mathrm{Dist}(\mathbb{S}^3, 1))$ are in 1-to-1 correspondence with homotopy classes of maps $\mathbb{S}^3 \to \mathbb{RP}^2$; these are, by definition, the elements of the third homotopy group $\pi_3(\mathbb{RP}^2) \cong \pi_3(\mathbb{S}^2) \cong \mathbb{Z}$; in the first isomorphism we use that $\mathbb{S}^2$ is the universal cover of $\mathbb{RP}^2$. ♦

Example 3.4. We could instead look at $\mathrm{Dist}(M, 2)$, with M still a closed 3-dimensional manifold. Note that $\mathrm{Gr}(\mathbb{R}^3, 2) \cong \mathrm{Gr}(\mathbb{R}^3, 1)$ (because choosing a plane is the same as choosing the orthogonal line), so $\mathrm{Dist}(M, 2)$ is also isomorphic to the space of sections of $M \times \mathbb{RP}^2 \to M$. In particular, $\pi_0(\mathrm{Dist}(\mathbb{S}^3, 2)) \cong \mathbb{Z}$.

For line fields, there are no interesting local invariants to look at (because they have a unique local model thanks to the flowbox theorem, Remark 2.14). However, we know that regular plane distributions in dimension 3 might be foliations or contact structures. We may then ask: Given a connected component in $\pi_0(\mathrm{Dist}(\mathbb{S}^3, 2)) \cong \mathbb{Z}$, does it contain a foliation? Does it contain a contact structure? The answer is yes to both. We will dedicate the rest of these notes to proving the contact case. ♦

Remark 3.5. What the previous examples tell us is that tools from Algebraic Topology (in particular Homotopy Theory) allow us to compute the homotopy type of $\mathrm{Dist}(M, k)$. These computations reduce to so-called *obstruction theory*, which essentially phrases the homotopy type of $\mathrm{Dist}(M, k)$ in terms of cohomology classes of M with values in homotopy groups of Grassmannians. However, providing a explicit answer can be terribly complicated! ♦

3.2. **An overview of Contact Topology.** We present now a panoramic view of Contact Topology (i.e. the area of Differential Topology that studies contact structures from a global perspective). We will be rather terse at times and we will ignore many important aspects and techniques. We refer the reader to [20] for an in-depth treatment of the topic.

3.2.1. *The definition.* We introduced contact structures already in Subsubsection 2.11.2:

Proposition 3.6. *Let M be a $(2n+1)$-dimensional manifold. A hyperplane field $\xi \subset TM$ is a* ***contact structure*** *if and only if one of the following equivalent conditions holds:*

- *the first curvature $\Omega : \xi \times \xi \to TM/\xi$ is a non-degenerate two-form,*
- *locally there exists a 1-form α such that $\ker(\alpha) = \xi$ and $\alpha \wedge d\alpha^n \neq 0$,*
- *there are local coordinates (x, y, z) in which ξ is the kernel of $\alpha_{\mathrm{std}} = dz - \sum_i y_i dx_i$.*

Proof. The first condition is just the definition. The second one can be deduced as follows: locally any hyperplane can be writen as the kernel of a 1-form α. Now we compute, using the explicit formula for the exterior differential:

$$\alpha \circ \Omega(u, v) = \alpha([u, v]) = -d\alpha(u, v)$$

which means that $d\alpha$ is non-degenerate (as a two-form on ξ) if and only if Ω is non-degenerate. Non-degeneracy of $d\alpha$ means that $d\alpha^n$ is a volume form on ξ, which is equivalent to the second condition.

It is easy to see that $\alpha_{\mathrm{std}} \wedge d\alpha_{\mathrm{std}}^n \neq 0$, so $\ker(\alpha_{\mathrm{std}})$ is contact. We will prove the other implication in Proposition 3.11 (but only for the case $2n + 1 = 3$ for simplicity). □

Remark 3.7. In contrast to Riemannian structures, whose isometry groups are finite dimensional, the group of automorphisms of a contact structure is infinite dimensional. The infinitesimal automorphisms (i.e. vector fields preserving the contact structure), are in 1-to-1 correspondence with the smooth functions on the manifold. This makes contact structures extremely topological in nature (for instance, any two points in a contact manifold can be mapped to one another using a global contactomorphism). ♦

3.2.2. *Formal contact structures.*

Definition 3.8. Let M be a $(2n+1)$-dimensional manifold. A **formal contact structure** is a pair (ξ, Ω) where:

- ξ is a hyperplane field,
- $\Omega : \xi \times \xi \to TM/\xi$ is a non-degenerate 2-form (playing the formal role of the curvature). ♦

Any contact structure ξ can be regarded as a formal contact structure by taking the pair $(\xi, \Omega_{0,0}(\xi))$. Doing this we are effectively *decoupling* ξ from its curvature. Since the curvature is constructed from ξ using the Lie bracket (which is a linear differential operator of first order), you should think of this as decoupling ξ from its first derivative.

Remark 3.9. Given a formal contact structure, we can produce a bundle of Lie algebras L by taking $L = \xi \oplus TM/\xi$ and Ω as Lie bracket. The non-degeneracy condition for Ω tells us that L is fibrewise isomorphic to the Heisenberg algebra H_n. As a vector bundle, L is clearly isomorphic to TM (but not canonically).

In general, one could define a **formal distribution** as an identification of TM with a (weak) bundle of Lie algebras. By taking its nilpotentisation, any (weakly regular) distribution induces a formal distribution (for this, one must show that the identification between TM and the nilpotentisation is unique up to homotopy). Any general classification statement (up to homotopy) for distributions must be phrased using this language. However, as far as the author is aware, such a setup has not been worked out in the literature yet. We leave it as an interesting open question. ♦

We write $\mathrm{Cont}(M)$ for the space of contact structures in M. Similarly, we write $\mathrm{Cont}^f(M)$ for the space of formal contact structures. It is suitable to take the C^0-topology in $\mathrm{Cont}^f(M)$, which induces the C^1-topology in $\mathrm{Cont}(M)$. This provides us with a continuous inclusion:

$$\mathrm{Cont}(M) \to \mathrm{Cont}^f(M).$$

3.2.3. *Orientations.* We now prove a little remark about orientations. Only the 3-dimensional case will be relevant for us later on; as such, I invite the reader to think about the proof in that particular setting.

Lemma 3.10. *Let M be a manifold admitting a formal contact structure (ξ, Ω). Then*

- *If n is even: ξ is canonically oriented. M is orientable if and only if ξ is coorientable.*
- *If n is odd: M is canonically oriented. ξ is orientable if and only if it itself is coorientable.*

Proof. Since $\Omega : \xi \times \xi \to TM/\xi$ is non-degenerate, we have that Ω provides an isomorphism between $\det(\xi) = \wedge^{2n}\xi$ and $\otimes^n(TM/\xi)$. If n is even, there is an orientation-preserving isomorphism between $\det(\xi) \cong \otimes^n(TM/\xi)$ and the oriented trivial line bundle that is unique up to homotopy. This provides an orientation of ξ. If n is odd, we deduce similarly $\det(\xi) \cong TM/\xi$.

In general we can write $TM = \xi \oplus TM/\xi$ and compute the determinant bundle of TM:

$$\det(TM) \cong \det(\xi) \otimes TM/\xi \cong \otimes^{n+1}(TM/\xi).$$

That is: If n is odd, M is naturally oriented. Otherwise, if n is even, $\det(TM)$ is naturally isomorphic to TM/ξ. □

In particular: A 3-dimensional contact manifold has a canonical orientation. Therefore, given an oriented 3-manifold M, we may then talk about **positive** contact structures (those whose orientation is compatible with the given one) and **negative** contact structures.

3.2.4. *Local model.* As promised in Proposition 3.6, let us prove:

Proposition 3.11 (G. Darboux)**.** *Let (M, ξ) be a 3-dimensional contact manifold. Let $p \in M$. Then there exist local coordinates (x, y, z) around p in which $\xi = \ker(dz - ydx)$.*

Proof. Over a ball, ξ is trivial as a vector bundle. This implies that, in a neighbourhood of p, we can find a non-vanishing vector field Y tangent to ξ. Let us pick now a surface S containing p and transverse to Y. Being transverse to ξ, we have that $TS \cap \xi$ is a line field: we choose a non-vanishing vector field $Z \in \mathfrak{X}(S)$ spanning it.

The flowbox theorem tells us that, in a neighbourhood of $p \in S$, there are coordinates (x, z) such that $Z = \partial_z$. Using the flow of Y we can extend these to coordinates (x, y, z) in which $Y = \partial_y$ (possibly by further shrinking the neighbourhood of p). In this local model, we can pick a 1-form α satisfying $\ker(\alpha) = \xi$. By construction, α must be of the form $dz + f(x, y, z)dx$.

Now we observe that $\alpha \wedge d\alpha = (\partial_y f)dx \wedge dy \wedge dz$. This implies that ξ is contact if and only if $\partial_y f \neq 0$. The implicit function theorem implies that we can reparametrise in the y–coordinate so that $f(x, y, z) = -y$, as claimed. □

A similar reasoning yields:

Lemma 3.12. *Let S be disc, a 2-torus, or a cylinder; fix coordinates (x, y) in S. Suppose $S \times [-1, 1]$ is endowed with a plane field $\xi = \ker(\alpha)$ with $\alpha = fdx + gdy$ and $f, g : S \times [-1, 1] \to \mathbb{R}$.*

The plane field ξ is contact at the point (x_0, y_0, t_0) if and only if the curve

$$t \to \frac{(f(x_0, y_0, t), g(x_0, y_0, t))}{|f, g|} \in \mathbb{S}^1$$

is an immersion at time t_0.

PROOF. Indeed, we check that

$$\alpha \wedge d\alpha = [f(\partial_t g) - g(\partial_t f)]dx \wedge dy \wedge dt.$$

The condition $f(\partial_t g) - g(\partial_t f) \neq 0$ means that (f, g) and $(\partial_t f, \partial_t g)$ are linearly independent vectors in $\mathbb{R}^2$, which is precisely the immersion condition for $(f, g)/|f, g|$ (I invite the reader to draw this claim, as it is then easy to see). □

This is usually phrased, in more informal terms, as follows: ξ is contact if and only if it "turns" with respect to any line field tangent to it (if this does not make sense to you, please try to follow the proof and make a picture of it!). This lemma will be extremely useful, because it will allow us to construct contact structures by taking any plane field and "adding to it a bit of turning".

3.2.5. *Examples.* Let us provide some explicit examples of globally defined contact structures on a manifold. They all can be shown to be contact using Lemma 3.12 (or using the characterisations of contactness from Proposition 3.6).

Example 3.13. The structure

$$(\mathbb{R}^3, \xi_{\mathrm{std}'} = \ker(\cos(z)dx + \sin(z)dy))$$

turns infinitely many times with respect to the line field $\langle \partial_z \rangle$. It is (globally) diffeomorphic to

$$\xi_{\mathrm{std}} = \ker(dz - ydx), \quad \text{and} \quad \xi_{\mathrm{std}''} = \ker(dz - ydx + xdy).$$

We may then take cylindrical coordinates (r, θ, z) in $\mathbb{R}^3$ and define:

$$\xi_{\mathrm{std}'''} = \ker(dz - r^2 d\theta),$$

which turns with respect to the radial vector field and makes almost a turn of $\pi/2$ from the origin to infinity. It is yet again diffeomorphic to the previous ones. We simply say that all of them are the **standard contact structure** in $\mathbb{R}^3$. We invite the reader to provide explicit diffeomorphisms between all of them. ♦

Example 3.14. Consider again cylindrical coordinates (r, θ, z) in $\mathbb{R}^3$. The structure

$$(\mathbb{R}^3, \xi_{\mathrm{OT}} = \ker(\cos(r)dz + \sin(r)rd\theta))$$

turns infinitely many times with respect to the line field $\langle \partial_r \rangle$. It is **not** diffeomorphic to the standard contact structure, and it is called the **contact structure overtwisted at infinity**; the subscript OT denotes "overtwisted". See Theorem 3.20 below and the subsequent discussion. ♦

Example 3.15. The structures

$$(\mathbb{T}^3, \xi_k = \ker(\cos(\pi kz)dx + \sin(\pi kz)dy)) \qquad k \in \mathbb{Z}^+,$$

turn $k/2$ times with respect to the line field $\langle \partial_z \rangle$. They are not diffeomorphic to one another. ♦

Example 3.16. Consider $\mathbb{S}^3 \subset \mathbb{C}^2$. It is defined as the level set $f^{-1}(1)$ of

$$f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2.$$

As such, its tangent space is the kernel of the 1-form

$$df = 2(x_1 dx_1 + y_1 dy_1 + x_2 dx_2 + y_2 dy_2).$$

The complex tangencies (i.e. the vectors v such that both v and iv are in $T\mathbb{S}^3$) are simply the complex lines:

$$\begin{aligned}\xi_{\mathrm{std}} = T\mathbb{S}^3 \cap i(T\mathbb{S}^3) &= \ker(df) \cap \ker(df \circ i) \\ &= \ker(-x_1 dy_1 + y_1 dx_1 - x_2 dy_2 + y_2 dx_2) \subset T\mathbb{S}^3,\end{aligned}$$

which we encountered already in the Introduction. $(\mathbb{S}^3, \xi_{\mathrm{std}})$ is the compactification of $(\mathbb{R}^3, \xi_{\mathrm{std}})$. ♦

3.2.6. *Classification.* We will now outline, very superficially, some central results in Contact Topology, so that the reader gets an idea of the types of results one may care about (very biased by my own taste, of course). The first one, due to M. Gromov, will be proven in Subsection 3.4:

Theorem 3.17 (Gromov)**.** *If M is open, the following inclusion is a homotopy equivalence:*

$$\mathrm{Cont}(M) \to \mathrm{Cont}^f(M).$$

Remark 3.18. Usually this theorem is stated by saying: "the ***h*-principle holds** for contact structures in open manifolds". Let us explain what this means.

As we pointed out before, when we look at the inclusion

$$\mathrm{Cont}(M) \to \mathrm{Cont}^f(M)$$

we are essentially decoupling the contact condition. That is, we forget that the hyperplane field ξ is contact and instead we think of its curvature $\Omega_{0,0}(\xi)$ as a separate piece of data. This is an extremely natural thing to do: if we want to construct a contact structure, a necessary condition is that, first, we must be able to construct a formal one. One may then hope that one can deform some given formal contact structure until it becomes truly contact.

One can proceed similarly when constructing any other geometric structure: we decouple the geometric object from its derivatives and we denote the result as a "formal geometric structure". We certainly need to be capable of constructing the latter to construct the former. Constructing formal structures is usually a matter of Algebraic Topology (as seen in Examples 3.3 and 3.4) so we leave that to the topologists. As geometers, we focus on the next step: how to turn a formal structure into a genuine one. When the spaces of geometric structures and formal structures are homotopy equivalent, we say that "*the problem satisfies the h-principle*"; here h stands for homotopy. ♦

Remark 3.19. There is no single recipe for proving that a problem satisfies the h-principle. Gromov's result above is based on the **method of flexible sheaves** [25], which is in turn a generalisation of earlier work of M.W. Hirsch and S. Smale [28, 46] proving the h-principle for (subcritical) immersions. This idea was later streamlined by Y. Eliashberg and N. Mishachev [13] in the form of the **holonomic approximation lemma**.

Gromov also devised another approach called **convex integration** which, for instance, can be used to prove the h-principle for even-contact structures; see [32]. The idea for convex integration appeared first in the works of J. Nash [37] regarding C^1-isometric embeddings and has seen an explosion in popularity recently in the world of Fluid Dynamics [9].

There are many other flavours of h-principle. In his proof of the h-principle for foliations [50, 51], W. Thurston uses two central ideas: First, that triangulations can be made transverse to distributions (which is the first example of an *h-principle without homotopical assumptions*); see Proposition 3.56 below. Secondly, that **surgery techniques** can be used to remove the singularities of a foliation which is singular in the sense of A. Haefliger [27]. Surgery techniques had already appeared in the context of simplification of singularities of maps in Y. Eliashberg's work [10], which was later further developed, jointly with N. Mishachev, in the **wrinkling** saga [14, 15, 16]. These tools also play a central role in homological stability h-principles in Homotopy Theory [30]. ♦

After Theorem 3.17, we are naturally lead to wonder: what about closed manifolds? In the early 80's, D. Bennequin [3] showed that the h-principle in fact fails in the closed setting:

Theorem 3.20 (Bennequin)**.** *There is a connected component in the space of plane fields in $\mathbb{S}^3$ that contains two different connected components of positive contact structures.*

That is, the map $\pi_0(\mathrm{Cont}(\mathbb{S}^3)) \to \pi_0(\mathrm{Cont}^f(\mathbb{S}^3))$ at the level of connected components is not injective: in one of the components of $\pi_0(\mathrm{Cont}^f(\mathbb{S}^3))$ there are two contact structures which are not homotopic to one another. D. Bennequin proved that $(\mathbb{R}^3, \xi_{\mathrm{std}})$ and $(\mathbb{R}^3, \xi_{\mathrm{OT}})$ are not diffeomorphic to each other by looking at their spaces of integral curves. The two structures in $\mathbb{S}^3$ are suitable compactifications of ξ_{std} and ξ_{OT} and are also not diffeomorphic. Then one recalls Gray's Theorem 2.52: if they are not diffeomorphic, they cannot be homotopic.

Nonetheless, the h-principle still holds in a partial sense:

Theorem 3.21 (Eliashberg in dimension 3 [11]. Borman, Eliashberg and Murphy in all dimensions [4])**.** *Let M be closed. There exist subclasses*

$$\mathrm{Cont}^{\mathrm{OT}}(M,\Delta) \subset \mathrm{Cont}(M), \qquad \mathrm{Cont}^f(M,\Delta) \subset \mathrm{Cont}^f(M)$$

such that the inclusion

$$\mathrm{Cont}^{\mathrm{OT}}(M,\Delta) \to \mathrm{Cont}^f(M,\Delta)$$

is a weak homotopy equivalence.

These concepts will be introduced in detail in Subsection 3.5. The main idea to keep in mind is that even though the h-principle in full generality fails (already at the level of π_0 according to Theorem 3.20), there is a particular family of contact structures, which we call **overtwisted**, for which it holds. Overtwistedness amounts to the existence of a ball in the manifold isomorphic to $(\mathbb{R}^3, \xi_{\mathrm{OT}})$. Non-overtwisted contact structures are said to be **tight**.

Remark 3.22. This idea of having particular subclasses of geometric structures for which the h-principle holds is not exclusive to Contact Topology. For instance, a similar behaviour can be observed in the world of foliations: Reeb components (which are a semi-local model closely related to the overtwisted disc) play a central role in W. Thurston's constructions [51] (but the situation is rather subtle because a full h-principle still holds in dimension 3 in π_0, see [19]). This can also be observed in the classification of convex curves in $\mathbb{S}^2$ [44] and in the classification of immersions with prescribed folds [17]. ♦

These results of D. Bennequin, Y. Eliashberg, and M. Gromov constitute the beginning of Contact Topology. A force guiding much of the development in this field has been the relation between contact and symplectic structures. Although this relationship will not be explored in this notes, it is central to the construction of contact invariants. The main tools in this direction are the theory of pseudoholomorphic curves, introduced first by M. Gromov [24], the theory of generating functions, and the theory of microlocal sheaves.

Remark 3.23. Due to its relationship with Symplectic Topology, dynamical aspects play a very important role in Contact Topology. In particular, each contact manifold (M,ξ) has a canonical collection of associated vector fields, which are called the **Reeb fields**. Given a contact form α (i.e. a 1-form such that $\ker(\alpha) = \xi$), the corresponding Reeb field is uniquely determined by the equations $i_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$.

A driving question in the area is whether R_α always has closed orbits, which is the so-called **Weinstein conjecture**. It was proven in dimension 3 by C.H. Taubes [48] using Seiberg-Witten theory, but the general case remains open. However, there is a remarkable fact: overtwisted contact structures always have contractible closed orbits [29, 38]. This relationship between Dynamics (how do orbits behave?) and Topology (e.g. is the structure overtwisted?) is in fact extremely deep. ♦

3.3. **Detour: Submanifolds of contact manifolds.** Bennequin's result (or rather, its proof, based on the study of integral curves tangent to contact structures) indicates that the study of submanifolds of a contact manifold can be extremely fruitful. This, of course, happens in many other geometries: it is often interesting to look for submanifolds interacting with the geometric structure in a meaningful way. Even though not essential for the main theorems we want to prove later on, it is convenient to provide an overview.

3.3.1. *The linear question.* In the case of distributions, we already devoted some time to the general setup in Subsection 2.9. We argued that one should focus on submanifolds intersecting the associated flag in a weakly regular filtered structure. In this manner, TN inherits fibrewise the structure of a Lie subalgebra of the nilpotentisation. Thus, in the contact case, we must understand Lie subalgebras of the Heisenberg algebra.

Remark 3.24. At this point, some symplectic linear algebra is unavoidable. We refer the reader to [33]. ♦

Lemma 3.25. *The graded Lie subalgebras of H_n are isomorphic to:*

- H_j *with* $j \leq n$,
- $\mathbb{R}^j[0]$ *i.e. the trivial Lie algebra in degree* 0*, with* $j \leq n$,

- $\mathbb{R}^k[0] \oplus H_j$ *with* $j + k \leq n$.

PROOF. As a vector space, H_n is isomorphic to $\mathbb{R}^{2n} \oplus \mathbb{R}$, where the first term lies in degree 0 and the second one in degree 1. The bracket is given by $[x_i, y_i] = z$.

Let $V \subset H_n$ be the (graded) vector subspace underlying a Lie subalgebra. Suppose first that $0 \oplus \mathbb{R} \not\subset V$, which is equivalent to $V \subset \mathbb{R}^{2n} \oplus 0$ (recall that we are looking at graded subspaces!). V being a Lie subalgebra implies that the bracket restricted to V should take values in V. However, z is not in V, so we deduce that the bracket over V should be zero; V is then isomorphic to a trivial Lie algebra. Now, the bracket in H_n is a non-degenerate 2-form on $\mathbb{R}^{2n} \oplus 0$ with values in $0 \oplus \mathbb{R}$; this tells us that V is an isotropic subspace and therefore it is at most n dimensional.

Suppose instead that V contains $0 \oplus \mathbb{R}$. Then $V \cap \mathbb{R}^{2n} \oplus 0$ is an arbitrary vector subspace. As such, it can be decomposed as $W \oplus W'$, with W symplectic and W' isotropic. Then, $W \oplus \mathbb{R}$ is isomorphic to the Heisenberg subalgebra H_j, for some j, and $W' \oplus 0$ is the trivial Lie algebra. □

Remark 3.26. This lemma is not the whole story at the linear level. One can now look at the Grassmannian $\mathrm{Gr}(H_n, \mathfrak{h})$ of all Lie subalgebras of H_n isomorphic to a given graded Lie algebra $\mathfrak{h}$. This produces several non-trivial manifolds with interesting topology. For instance, when we look at trivial Lie subalgebras of dimension n in degree 0, we obtain the *Lagrangian Grassmannian.*

One can analogously consider Lie algebras other than H_n: then the Grassmannians might be varieties with singularities instead of smooth manifolds. This happens already when one looks at the Cartan distribution in a higher jet space.

If the distribution we look at is regular (as it is in the contact case), these Grassmannians vary smoothly with the point, so the nilpotentisation $L(\xi)$ has an associated Grassmannian bundle $\mathrm{Gr}(\xi, \mathfrak{h})$. Given a submanifold N with $TN \cap L(\xi)$ everywhere isomorphic to $\mathfrak{h}$, we obtain a corresponding section of $\mathrm{Gr}(\xi, \mathfrak{h})|_N$. This tells us that understanding these Grassmannians (say, their homology) is essential in the study of submanifolds. ♦

3.3.2. *Definitions.* These linear algebra computations lead us to the definition:

Definition 3.27. Let (M^{2n+1}, ξ) be a contact manifold. A submanifold $N \subset M$ is:

- **contact** if $TN \cap L(\xi)$ is fibrewise isomorphic to H_j, for some j,
- **isotropic** or **integral** if $TN \cap L(\xi)$ is fibrewise isomorphic to the trivial Lie algebra in degree zero.
- **legendrian** if it is isotropic and n-dimensional.
- **coisotropic** if $TN \cap L(\xi)$ is fibrewise isomorphic to $\mathbb{R}^k \oplus H_j$ with $j + k = n$.
- **transverse** if $TN \cap L(\xi)$ is fibrewise isomorphic to $V \oplus H_0$, for some V in degree 0. ♦

Remark 3.28. Do note that not all of the linear algebra cases are covered by the definition and some cases fit in more than one category. For instance, contact and coisotropic submanifolds are always transverse. ♦

Remark 3.29. One remarkable instance of the topological/flexible nature of contact structures is the so–called **Moser's trick**: one can use flows to put contact structures in a *normal form.* This can be used to prove Darboux' Theorem 3.11 and Gray's stability Theorem 2.52 in all dimensions. Furthermore, it can be used to construct normal form theorems in the vicinity of a submanifold. We will look at a particular instance of this in Proposition 3.48. ♦

3.3.3. *Gromov's h-principle.* Much like we are trying to classify contact structures, one could try to classify all the submanifolds of a given type. Such results are usually stated in the language we presented above: we define the notion of a "formal submanifold of a given type" and we try to compare the homotopy type of the corresponding spaces of submanifolds.

Remark 3.30. We have been vague about how these submanifolds sit in M. As the reader may guess, one can ask them to be embedded or consider the more general class of immersed ones. The first case is more rigid and (at least in Contact Topology) it is the one that presents interesting behaviours. All of our statements will refer to the embedded case. For more general distributions, the existence of submanifolds (even just locally) is rather involved [26], so the immersed case is still interesting. ♦

It turns out that Gromov's theory of flexible sheaves provides a classification in some of these cases:

Theorem 3.31 (Gromov)**.** *Let (M, ξ) be a contact manifold. Then the following submanifolds satisfy the h-principle:*

- *Contact submanifolds of codimension at least* 4.
- *Subcritical isotropics (i.e. not legendrian).*

A consequence of this theorem is that these submanifolds are therefore not useful for distinguishing contact structures. The reason for this is that the spaces of formal submanifolds of a given type do not depend on the contact structure ξ itself, but only on its underlying formal distribution.

3.3.4. *Legendrians.* Historically, much of the study in Contact Topology has dealt with legendrians. The first fundamental fact is that legendrians do *not* satisfy the h-principle, meaning that their classification is quite complicated and does not correspond to classifying *formal legendrians.* Additionally (and relatedly), their behaviour tells us a lot about the contact structure in which they live.

There is a deep relationship between legendrians and functions, which tells us that, morally, some aspects of Contact Topology are essentially Morse Theory on steroids. We said a bit about this (for integral submanifolds in general jet spaces) in Subsubsection 2.11.7.

Example 3.32. Consider the standard contact structure $\xi_{\text{std}} = \ker(\alpha_{\text{std}}) = \ker(dy - \sum_i z_i dx_i)$. By construction, a function $f : \mathbb{R}^n \to \mathbb{R}$ of the x–coordinates immediately yields a legendrian submanifold by taking the graph of f and df:

$$(x_1, \cdots, x_n) \to \left(x_1, \cdots, x_n, f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

Indeed, if we pullback α_{std} by $(x, f(x), df(x))$, we get zero. Recall that ξ_{std} in this case is just the Cartan distribution in 1-jet space.

Suppose that we now have a legendrian submanifold L of ξ_{std}. We can project it to the zero jet using the front projection $\pi(x, y, z) = (x, y)$. Whenever $\pi(L)$ is immersed, it is graphical over the x–coordinates and we can differentiate y in terms of x to recover the missing z–coordinate. At some points $\pi(L)$ will have singularities (these correspond to the points where L was tangent to the z–coordinates), like in the following example: in the 3-dimensional case, the curve $(t^2/2, t^3/3, t)$ is an embedded legendrian, but its projection to x and y has a cusp singularity at $t = 0$.

The punchline is that legendrian submanifolds L can be manipulated through their projections $\pi(L)$, which can be thought of as *multivalued functions.* This says that the study of legendrians is very close to the study of smooth functions, which is of course a deep subject of research. In fact, tools arising in the world of functions can be used to study legendrians; for instance, Morse theory, under the name of **generating function theory**, is fundamental in Contact Topology. ⧫

Even though we cannot hope to classify *all* legendrians, many results can be proven. The following is a very small sample of the very many results obtained since the mid 80's:

Theorem 3.33 (Bennequin, Chekanov, Eliashberg, Murphy, Vogel, many others)**.** *Let (M, ξ) be a contact manifold. Then:*

- *Legendrians do not satisfy the h-principle.*
- *If ξ is overtwisted, legendrians in the complement of an overtwisted disc satisfy the h-principle. Those intersecting every overtwisted disc display interesting behaviours* [53].
- *In dimension at least* 5*, there exists a family of legendrians, called **loose**, that satisfies the h-principle* [35]*. Non-loose legendrians exist.*
- *If ξ is a tight* 3*-dimensional contact structure, the legendrians it contains satisfy certain bounds* [12]*. Using this one can distinguish tight and overtwisted structures.*

The case that is understood best is when M is 3-dimensional. Then legendrians are knots tangent to the contact structure.

3.4. **Gromov's h-principle for open manifolds.** Before our detour about submanifolds, we stated the following foundational result:

Theorem 3.34 (Gromov)**.** *Let M be a smooth open $(2n+1)$-dimensional manifold. Then the inclusion*

$$\mathrm{Cont}(M) \to \mathrm{Cont}^f(M)$$

is a weak homotopy equivalence.

In order not to clutter the main idea, we will just prove the following particular case:

Proposition 3.35. *Let M be a smooth open $(2n+1)$-dimensional manifold. Then any formal contact structure is homotopic to a contact structure.*

I.e. we are proving that $\pi_0(\mathrm{Cont}(M)) \to \pi_0(\mathrm{Cont}^f(M))$ is a surjection. The approach we use is called *holonomic approximation* [13].

Remark 3.36*.* Let us provide some intuition on why the open condition for M is important. Consider the following geometric problem: *how do we construct a function $f : M \to \mathbb{R}$ with no critical points?* What we may do first is pick some Morse function $g : M \to \mathbb{R}$ (i.e. a function with non-degenerate and therefore isolated critical points). Since M is open, g may have an infinite number of critical points, but on each compact set we find only finitely many. Our goal now will be to push them out to infinity.

We pick an exhaustion of M by compacts

$$M_0 \subset M_1 \subset \cdots \subset M_i \subset \cdots$$

and we proceed inductively on i. Our induction hypothesis is that there is a diffeomorphism $\phi_i : M \to M$ which is the identity outside of M_i and such that $g_i = g \circ \phi_i$ has no critical points in M_{i-1}. This means that $g \circ \phi_i$ is like g but all the critical points in M_{i-1} have been pushed into $M_i \setminus M_{i-1}$ by ϕ_i. This is trivially true in the base case $i = 0$.

Assume the hypothesis for i. The function g_i has finitely many critical points $\{c_j\}$ in M_i (all of them in $M_i \setminus M_{i-1}$). We pick paths γ_j starting at c_j, contained in $M_{i+1} \setminus M_{i-1}$, and finishing somewhere in M_{i+1}. We require all paths to be embedded and disjoint from one another (this is possible because we have a finite collection of points). Now we can find a diffeomorphism $\phi : M_i \to M_i$ that is the identity outside a little neighbourhood of each path; we require that it takes the other endpoint of γ_j to c_j (essentially by pushing along γ_j). The diffeomorphism $\phi_{i+1} = \phi_i \circ \phi$ satisfies the claim: $g_{i+1} = g \circ \phi_{i+1}$ has no critical points in M_i (they were all pushed into M_{i+1}).

Consider now the sequence of functions g_i. Observe that $g_i|_{M_n} = g_{i'}|_{M_n}$ if $i, i' > n$, by construction. This implies that the sequence converges in the C^∞_{loc}-topology to a function f with no critical points.

Another (maybe more useful) way of thinking about it is to consider the C^∞_{loc}-limit ϕ of the sequence ϕ_i. This limit, as a map $M \to M$, indeed exists because $\phi_i|_{M_n}$ does not depend on i as long as $i > n$. It is also a smooth map. However, it is not a diffeomorphism. Indeed, the critical points and the paths that connect them to infinity are not in the image. In particular, we see that f cannot be possibly proper. ♦

The rest of the section is dedicated to the proof of Proposition 3.35. The contact structures resulting from it will be like the function f: their behaviour at infinity will be wild and uncontrolled, which is exactly the kind of flexibility we need.

3.4.1. *Contact structures along the skeleton.* In Remark 3.36 we saw that M being open allows us to "push our problems to infinity". Let us try to mimick this argument in the contact case. To avoid some technical details, we will work with *contact forms* instead of *contact structures.* I.e. we will try to construct 1-forms α such that $\alpha \wedge d\alpha^n \neq 0$. Our input will be a *formal contact form* (λ, ω), where λ is a 1-form and ω is a 2-form such that together they satisfy $\lambda \wedge \omega^n \neq 0$. It is a good exercise for the reader to compare this to the case of contact structures.

Suppose that we are able to prove:

Proposition 3.37. *Given a formal contact form (λ, ω), there exists:*

- *a triangulation* $\mathcal{T}$ *of* M*,*
- *a neighbourhood* $\mathcal{U}$ *of its codimension-*1 *skeleton,*
- *a homotopy of formal contact forms* $(\lambda_t, \omega_t)_{t\in[0,1]} \in \mathrm{Cont}^f(\mathcal{U})$ *with*

$$(\lambda_0, \omega_0) = (\lambda, \omega)|_{\mathcal{U}}, \qquad (\lambda_1, \omega_1) = (\alpha, d\alpha)$$

with α *some contact form on* $\mathcal{U}$*.*

Assuming Proposition 3.37 we can prove Proposition 3.35:

Proof. Consider the collection of barycenters $\{c_j\}$ of the $(2n+1)$-simplices of $\mathcal{T}$. This is a discrete, but possibly infinite, collection of points. Proceeding as in Remark 3.36, we can push these barycenters to infinity: that is, we find a diffeomorphism ϕ_1 between M and a subset $\bar{M} \subset M$ lying in the complement of the barycenters. In fact, this diffeomorphism is part of an isotopy of embeddings $(\phi_s)_{s\in[0,1]} : M \to M$, with ϕ_0 the identity.

Now, the manifold $M \setminus \{c_j\}$ deformation retracts to an arbitrarily small neighbourhood $\bar{\mathcal{U}}$ of the codimension-1 skeleton of $\mathcal{T}$. Indeed, consider a single $(2n+1)$-simplex Δ: If we remove its barycenter, we can just retract the rest of the simplex towards the boundary $\partial\Delta$ by pushing with a radial vector field with origin at the barycenter. Doing this for all top-simplices simultaneously yields the desired deformation retraction, which we denote by $(\psi_s)_{s\in[0,1]} : M \setminus \{c_j\} \to M \setminus \{c_j\}$, with ψ_0 the identity.

The neighbourhood $\bar{\mathcal{U}}$ is arbitrary, so we can take it to be small enough to satisfy $\bar{\mathcal{U}} \subset \mathcal{U}$. Then $\phi_1^*\psi_1^*\alpha$ is a contact form in M. We have to check that it is homotopic to (λ, ω). Now, Proposition 3.37 tells us that $\phi_1^*\psi_1^*(\alpha, d\alpha)$ and $\phi_1^*\psi_1^*((\lambda, \omega)|_{\mathcal{U}})$ are homotopic. In turn, $\phi_1^*\psi_1^*((\lambda, \omega)|_{\mathcal{U}})$ is homotopic to $\phi_0^*\psi_0^*(\lambda, \omega) = (\lambda, \omega)$, where the homotopy is given by $\phi_t^*\psi_t^*(\lambda, \omega)$. □

So now the question is, how do we prove Proposition 3.37? The statement of Proposition 3.37 itself suggests an inductive approach.

3.4.2. *The beginning of the inductive argument.* Since M is a smooth manifold, it can be triangulated, yielding some triangulation $\mathcal{T}$. We write $\mathcal{T}^{(i)}$ for the collection of i-dimensional simplices of $\mathcal{T}$. Our desired output is a homotopy of (λ, ω) in a neighbourhood of $\mathcal{T}^{(2n)}$ that yields a contact form. Thus, by setting as base case

$$(\lambda^{-1}, \omega^{-1}) = (\lambda, \omega),$$

a natural induction hypothesis would be:

- there is a neighbourhood $\mathcal{U}_i$ of the union of all i-simplices $\mathcal{T}^{(i)}$,
- and a homotopy $(\lambda_t, \omega_t)_{t\in[0,1]}$ such that

$$(\lambda^{i-1}, \omega^{i-1}) = (\lambda_0, \omega_0), \qquad (\lambda^i, \omega^i) = (\lambda_1, \omega_1) \text{ is contact in } \mathcal{U}_i.$$

That is, we construct the homotopy inductively on the dimension i of the simplices. During the course of the argument we shall see that this does not quite work. However, the very nature of the proof will tell us how to adapt the hypothesis: instead of achieving the contact condition in a neighbourhood of $\mathcal{T}^{(2n)}$, we will achieve it in a neighbourhood of $\widetilde{\mathcal{T}}^{(2n)}$, where $\widetilde{\mathcal{T}}$ is a triangulation obtained from $\mathcal{T}$ by a C^0-small but C^1-big perturbation.

Let us look at the base case $i = 0$. We must deform our formal contact form around a collection of points (the zero simplices) to be contact. This is relies on the following lemma:

Lemma 3.38. *Let* (λ, ω) *be a formal contact form and let* p *be a point. Then, there exists a* 1*-form* β*, defined on a neighbourhood of* p*, such that* $(\beta(p), d\beta(p)) = (\lambda(p), \omega(p))$*.*

Proof. Since the claim is at a point p, this is a matter of linear algebra. The inequality $\lambda \wedge \omega^n \neq 0$ says that $\omega(p)$ is a two-form of maximal rank (i.e. $2n$) and that this rank is attained over the kernel of $\lambda(p)$. We can therefore find a basis in T_pM in which the kernel of $\lambda(p)$ corresponds to the plane spanned by the first $2n$-coordinates and the kernel of $\omega(p)$ is the last coordinate direction. Further, we may assume (by applying a base change) that in this basis $(\lambda(p), \omega(p))$ is expressed as $(dz, \sum_i dx_i \wedge dy_i)$. This basis at T_pM can be extended to local coordinates around p in which this expression holds at p. Now we simply pick $\beta = dz + \sum_i x_i dy_i$. □

Remark 3.39. Let us provide an alternate and probably more conceptual proof: We pick local coordinates around p and we write down the Taylor expansions for λ and ω at p. The zeroeth order part of ω is a constant 2-form and, as such, it is closed and therefore exact. We pick a primitive ρ. We write down the Taylor expansion of ρ at p as well. We can then define β to be the sum of the zeroeth order part of λ and the first order part of ρ. ♦

Applying Lemma 3.38 to all the vertices in $\mathcal{T}^{(0)}$, we obtain a 1-form β which is defined in a little neighbourhood $\bar{\mathcal{U}}_0$ of $\mathcal{T}^{(0)}$. We choose a bump function $\chi : M \to [0,1]$ whose value outside of $\bar{\mathcal{U}}_0$ is zero and whose value inside a slightly smaller neighbourhood $\mathcal{U}_0$ is one. Then

$$(\lambda_s, \omega_s)(q) = (1 - s\chi(q))(\lambda, \omega)(q) + s\chi(q)(\beta, d\beta)(q)$$

satisfies the induction hypothesis. Indeed: since $(\beta, d\beta)$ and (λ, ω) agree at the points $\mathcal{T}^{(0)}$, the pair (λ_s, ω_s) will be a formal contact form as long as $\bar{\mathcal{U}}_0$ is sufficiently small.

3.4.3. 1-*simplices.* We want to produce a contact form on a neighbourhood of the edges $\mathcal{T}^{(1)}$. Thanks to the previous step, we have some (λ^0, ω^0) which is already of the form $(\beta, d\beta)$ in a neighbourhood $\mathcal{U}_0$ of the zero simplices. Let $\mathcal{U}_0' \subset \mathcal{U}_0$ be a slightly smaller neighbourhood of $\mathcal{T}^{(0)}$.

Let Δ be a component of $\mathcal{T}^{(1)} \setminus \mathcal{U}_0'$ (i.e. an edge from which we have removed a neighbourhood of the endpoints). We can find a neighbourhood U of Δ which is disjoint from all other edges in $\mathcal{T}^{(1)}$ (and, in particular, disjoint from all vertices in $\mathcal{T}^{(0)}$). We can then choose a collection of disjoint neighbourhoods for each component of $\mathcal{T}^{(1)} \setminus \mathcal{U}_0'$, yielding an open set $\tilde{\mathcal{U}}_1$ such that $\tilde{\mathcal{U}}_1 \cup \mathcal{U}_0$ is a covering of the 1-skeleton.

In order to create the desired homotopy of formal contact forms $(\lambda_s, \omega_s)_{s\in[0,1]}$, it is sufficient to work with a particular Δ with neighbourhood U. The first claim is that:

Lemma 3.40. *There exists a family of* 1*-forms* $(\beta_p)_{p\in\Delta}$ *such that:*

- β_p *is defined in a small neighbourhood of* p*,*
- $(\beta_p(p), d\beta_p(p)) = (\lambda^0(p), \omega^0(p))$*,*
- $(\beta_p, d\beta_p) = (\lambda^0, \omega^0)$ *if* p *is close to the endpoints of* Δ*.*

Recall that close to the endpoints (λ^0, ω^0) is already of the form $(\beta, d\beta)$, so all the $(\beta_p, d\beta_p)$ can be assumed to be contact forms in the domain in which they are defined.

Proof. The argument was sketched already in Remark 3.39: We take the Taylor expansions of λ^0 and ω^0 and we pick a primitive ρ_p for the zeroeth order part of ω^0 (all of this parametrically on p). Then we assemble β_p as the sum of the zeroeth order part of λ_0 at p plus ρ_p. Close to the endpoints of Δ, we interpolate, linearly on p, between (λ^0, ω^0) and the resulting $(\beta_p, d\beta_p)$ so that the two agree. Observe that this argument can be done not just for 1-simplices, but for any subset Δ. ☐

The idea at this point is clear: we are thinking of Δ as a 1-parametric family of points, and over each of them we have contact forms β_p. Our goal then is to glue the family $(\beta_p)_{p\in\Delta}$ to produce a single contact form in the vicinity of the whole of Δ. We decompose the neighbourhood U of Δ in a manner suitable for this: We can find local coordinates so that $U = [0,1] \times \mathbb{D}(\varepsilon)$, with $\Delta = [0,1] \times 0$; here $\mathbb{D}(\varepsilon)$ denotes the disc of radius ε, for $\varepsilon > 0$ small. For an integer N to be determined later, we consider the points $x_i = (i/N, 0) \in [0,1] \times 0 = \Delta$. By possibly shrinking U (and thus the radius ε) and taking N larger, we can assume that β_{x_i} is defined over the region $B_i = [i/N, (i+1)/N] \times \mathbb{D}(\varepsilon)$. We can further decompose

$$B_i = B_i' \cup B_i'' = [i/N, (2i+1)/(2N)] \times \mathbb{D}(\varepsilon) \cup [(2i+1)/(2N), (i+1)/N] \times \mathbb{D}(\varepsilon).$$

We can define a contact form β' in the union of all the B_i' by simply setting $\beta'|_{B_i'} = \beta_{x_i}$. The question now is how to define β' over the B_i''. A naive idea would be to interpolate between β_{x_i} and $\beta_{x_{i+1}}$ using as interpolation parameter the first coordinate (i.e. the coordinate corresponding to Δ). This cannot possibly work, because the interpolation function would have to increase from zero to one in an interval of size $1/2N$, so it would have derivative of size roughly $O(N)$, while $|\beta_{x_i} - \beta_{x_{i+1}}| = O(1/N)$. That is, the resulting one form would have uncontrolled derivative and the contact condition would be lost.

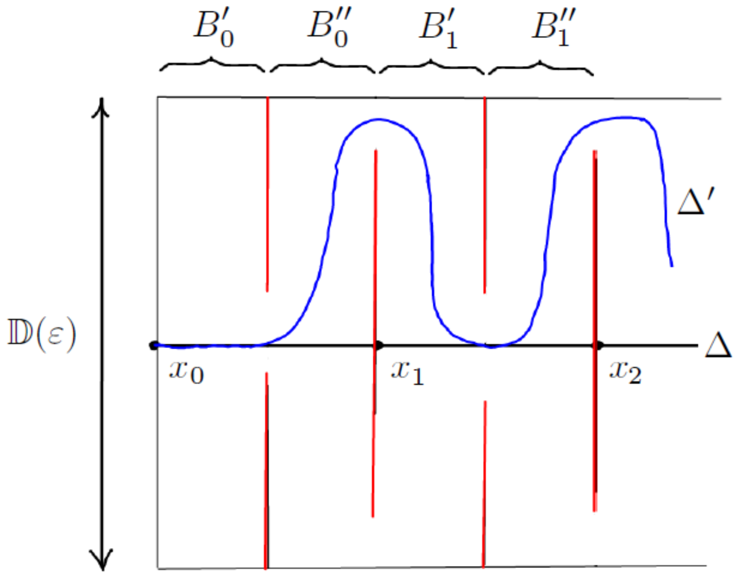

FIGURE 1. Here we see the different regions involved in the wiggling process. The thick horizontal line in the middle is the simplex Δ. Its normal bundle, of size ε, is the vertical direction. We have marked the points $x_i \in \Delta$, which are spaced at distance $1/N$. The region inbetween two points is divided into two strips: B_i' and B_i''. The discontinuities of β' take place in the transition between these strips; they are marked by thick vertical lines. The simplex Δ can be pushed into its normal bundle to avoid these discontinuities, yielding the wiggling Δ'.

A better idea is to use the directions along the normal bundle of Δ to interpolate. We denote by r the radial coordinate along $\mathbb{D}(\varepsilon)$. Then we define β' in B_i'' by the expression:

$$\beta'(q) = (1 - \chi(r(q)))\beta_{x_i}(q) + \chi(r(q))\beta_{x_{i+1}}(q)$$

where $\chi : [0, \varepsilon] \to [0, 1]$ is a function that is identically 1 close ε and identically 0 close to 0. As such, its n-th derivative is of the order of ε^{-n}.

The first observation is that β' has discontinuities. Indeed, it will be continuous in the slice $\{(2i+1)/2N\} \times \mathbb{D}(\varepsilon)$ only for radii close to 0 (because for those radii β' agrees with β_{x_i}). In $\{(i+1)/N\} \times \mathbb{D}(\varepsilon)$ it will be continuous only for radii close to 1 (because there it agrees with $\beta_{x_{i+1}}$).

The second observation is that (apart from its discontinuities), it is contact. Indeed, we may compute:

$$d\beta' = [(1 - \chi)d\beta_{x_i} + \chi d\beta_{x_{i+1}}] + d\chi \wedge (\beta_{x_{i+1}} - \beta_{x_i}).$$

The first term is what we want, since it is the linear interpolation between the derivatives of the contact forms β_{x_i} and $\beta_{x_{i+1}}$. We claim that we can control the behaviour of the second term. Indeed, the expression $d\chi$ is roughly of the order of $1/\varepsilon$ and, in particular, it does not depend on N. Crucially, $(\beta_{x_{i+1}} - \beta_{x_i})$ does depend on N and indeed it behaves as $O(1/N)$. This tells us that if we keep ε fixed and we take N sufficiently large, β' will be contact.

What have we exactly achieved? Due to the discontinuities, β' is not contact along Δ, as we would have wanted. However, Δ can be pushed (pointwise at most a distance ε) to lie in the domain in which β' is defined. The resulting edge Δ' obtained by this perturbation process is said to be the **wiggling** of Δ. Just like in the case of the zeroeth skeleton, we can now interpolate between (λ^0, ω^0) and β' as we leave a small neighbourhood of Δ'. This yields (λ^1, ω^1), which is contact not along $\mathcal{T}^{(1)}$, but along a wiggled version of it. See Figure 1.

Remark 3.41. The key point about this proof is the interplay between N and ε. As we take N larger, the points x_i get closer to one another, so the Taylor expansions of (λ^0, ω^0) at x_i and x_{i+1} (i.e. β_{x_i} and $\beta_{x_{i+1}}$) approach each order with order $O(1/N)$. However, we argued that a potential interpolation function along Δ between the two would have derivative of the order $O(N)$. Do observe that linear interpolation would yield something similar to (λ^0, ω^0), which is certainly not contact.

What we do then is we "gain interpolation length". Instead of interpolating along Δ, we push Δ in the normal direction, so the distance between x_i and x_{i+1} is not of the order of $1/N$, but of the order of ε. Since ε is fixed and independent of N, we now have plenty of space to interpolate slowly. ♦

3.4.4. *A sketch of the argument for higher simplices.* For higher simplices the argument is extremely similar. We think of an i-simplex Δ as a family of $(i-1)$-simplices Δ_p, to which we apply our argument parametrically. As long as i is smaller than the dimension of the manifold, we can wiggle Δ in the normal direction to gain space and therefore be able to interpolate between the contact forms we have obtained along the Δ_p. Since we need the normal direction to be non-empty, we are only able to carry out the argument in the skeleton of codimension 1. This concludes the proof. □

Remark 3.42. The reader can probably imagine that the argument we just presented is not really about contact forms. The key property we used is the following: any C^1–small perturbation of a contact form is still contact; i.e. the contact condition is an open condition in the space of all distributions. As such, Gromov's argument applies to any other geometric structure given by an open condition. For instance: immersions, submersions, symplectic structures (via the little trick of looking at a primitive), even-contact structures, Engel structures... ♦

Remark 3.43. The key idea was to use the normal directions to the simplices to wiggle them to gain additional space to interpolate. As such, it seems hopeless to try and modify this argument for closed manifolds. However, this is far from being the case! The key idea in the wrinkling saga [14] is to wiggle the top dimensional simplices back and forth onto themselves (imagine folding a sheet over and over). This yields geometric structures that have simple singularities called **folds**. The study of these singularities and how to remove them is a vital question in the Topology of PDEs and geometric structures. In fact, the h-principle for overtwisted contact structures in higher dimensions [4], relies on this idea. ♦

3.5. **Eliashberg's classification of overtwisted contact structures.** In the previous Subsection we showed that contact structures in open manifolds satisfy the h-principle. Even though this is not the case for closed manifolds, there is a particular subfamily of contact structures, the overtwisted ones, for which the h-principle holds.

3.5.1. *Overtwistedness.*

Definition 3.44. The **overtwisted disc** is the smooth disc

$$D_{\mathrm{OT}} = \{(r,\theta,0) \in \mathbb{R}^3 \mid |r| \leq \pi\} \subset (\mathbb{R}^3, \xi_{\mathrm{OT}})$$

along with the germ of contact structure ξ_{OT}. ♦

Since the germ of ξ_{OT} along D_{OT} is the crucial piece of data, let us describe how it looks. At the origin we have the tangency $\xi_{\mathrm{OT}}(0) = T_0 D_{\mathrm{OT}}$. Similarly, the contact structure is tangent to the disc along its boundary i.e. $\xi_{\mathrm{OT}}(p) = T_p D_{\mathrm{OT}}$ for every $p \in \partial D_{\mathrm{OT}}$. For every other $q \in D_{\mathrm{OT}}$, $\xi_{\mathrm{OT}}(q)$ and $T_q D_{\mathrm{OT}}$ are transverse to one another and they intersect along the radial direction. You can readily see that ξ_{OT} essentially makes "half a turn" from the centre of D_{OT} to its boundary. This twisting gives it its name.

Remark 3.45. In Example 3.14 we introduced the contact structure overtwisted at infinity $(\mathbb{R}^3, \xi_{\mathrm{OT}})$, which we remarked was not diffeomorphic to the standard contact $(\mathbb{R}^3, \xi_{\mathrm{std}})$. This was proven by Bennequin by looking at the legendrian knots in each of these structures. The boundary of D_{OT} is precisely a exotic legendrian which has no analogue in $(\mathbb{R}^3, \xi_{\mathrm{std}})$. ♦

Now we define:

Definition 3.46. A contact manifold (M^3, ξ) is **overtwisted** if there exists an embedding $\Psi : \mathcal{O}p(D_{\mathrm{OT}}) \to M$ such that $\Psi^*\xi = \xi_{\mathrm{OT}}$. The image of Ψ is called the **overtwisted disc**. ♦

We use $\mathcal{O}p(D_{\mathrm{OT}})$ to denote an arbitrarily small, unspecified open neighbourhood of D_{OT}. An equivalent way of stating this is that ξ is overtwisted if we can find an embedded disc $D = \Psi(D_{\mathrm{OT}}) \subset M$ such that ξ makes half a turn along D in the sense we discussed before.

An important observation is the following:

Lemma 3.47. *Let ξ and ξ' be homotopic as contact structures. Then ξ is overtwisted if and only if ξ' is overtwisted.*

PROOF. Since ξ and ξ' are homotopic, they are isotopic thanks to Gray's theorem. In particular, they are diffeomorphic by some diffeo $\psi : (M, \xi) \to (M, \xi')$. If Ψ is an embedding of an overtwisted disc of ξ, $\psi \circ \Psi$ is an overtwisted disc of ξ'. □

This implies that the connected components of $\mathrm{Cont}(M)$ divide in two groups: the overtwisted ones, which we denote collectively as $\mathrm{Cont}^{\mathrm{OT}}(M)$, and the **tight** ones (i.e. everything else).

3.5.2. *Lutz twist.* Overtwistedness is thus characterised by the presence of a semi-local model in which the contact structure "overtwists" with respect to a disc. It turns out that there is a procedure that allows us to modify a contact structure to introduce such twisting. The input we need is the following:

Proposition 3.48. *Let (M, ξ) be a 3-dimensional contact manifold. Let $\gamma : \mathbb{S}^1 \to M$ be a knot (i.e a smooth embedded curve) transverse to ξ. Then, there are local cylindrical coordinates (r, θ, z) in a tubular neighbourhood $\mathbb{D}^2_\varepsilon \times \mathbb{S}^1$ of $\gamma = \{r = 0\}$ such that $\xi = \xi_{\mathrm{trans}} = \ker(dz - r^2 d\theta)$.*

This statement is an example of a normal form theorem that can be proven using *Moser's trick*, as we advanced in Remark 3.29.

Let us thus consider the manifold $\mathbb{D}^2_\varepsilon \times \mathbb{S}^1$ endowed with the contact structure $\xi_{\mathrm{trans}} = \ker(dz - r^2 d\theta)$. We may then find functions $f, g : [0, \varepsilon] \to \mathbb{R}$ such that:

- $f(r) = 1$, $g(r) = -r^2$ in a neighbourhood of 0 and ε.
- $f(\varepsilon/2) = -1$, $g(\varepsilon/2) = 0$.
- $r \to (f(r), g(r))/|f, g|$ is an immersion into $\mathbb{S}^1$ describing one clockwise turn.

The resulting structure is denoted by $\xi_{\mathrm{Lutz}} = \ker(f(r)dz + g(r)d\theta)$. Observe that it agrees with ξ_{trans} along $0 \times \mathbb{S}^1$ and along the boundary $(\partial\mathbb{D}^2_\varepsilon) \times \mathbb{S}^1$, but in-between it makes one whole additional turn. Using Lemma 3.12 you may check that ξ_{Lutz} is still contact.

Definition 3.49. Let (M, ξ) be a 3-dimensional contact manifold. Let γ be a knot transverse to ξ with neighbourhood isomorphic to $(\mathbb{D}^2_\varepsilon \times \mathbb{S}^1, \xi_{\mathrm{trans}})$. Let $\tilde{\xi}$ be the structure obtained from ξ by replacing the tubular neighbourhood of γ by $(\mathbb{D}^2_\varepsilon \times \mathbb{S}^1, \xi_{\mathrm{Lutz}})$.

We say that $\tilde{\xi}$ is obtained from ξ by adding a **full Lutz twist** along γ. ♦

And now, as we claimed:

Lemma 3.50. *The structure $\tilde{\xi}$ is always overtwisted.*

PROOF. It is sufficient to show that the model $(\mathbb{D}^2_\varepsilon \times \mathbb{S}^1, \xi_{\mathrm{Lutz}})$ is overtwisted. This follows from the fact that the discs $\{(r, \theta, z_0) \,|\, |r| \leq \varepsilon/2\}$ are all overtwisted. □

Remark 3.51. In our construction of ξ_{Lutz} we could have chosen f and g differently. The key properties we needed were that they define an immersion (since that is equivalent to the contact condition) and that at radius $r = \varepsilon$ they define the same plane field as ξ_{trans} (since our model needs to glue with whatever we had outside). I invite you to think how to choose f and g so that the resulting structure now makes n half-turns (instead of a whole turn).

In fact, the case of half a turn (i.e the one where we seemingly introduce an $\mathbb{S}^1$-family of overtwisted discs) will play an important role subsequently. ♦

Remark 3.52. The reader should also ponder the following question: are all these contact structures homotopic as distributions relative to the boundary of the model? When we introduce an even number of half-turns (i.e. full turns), the answer is yes. Otherwise, no. Keeping track of this homotopy class is important, since our goal is classifying distributions up to homotopy. ♦

3.5.3. *Main result and structure of the proof.* Our goal in this last subsection is to prove the following simplified incarnation of Theorem 3.21:

Theorem 3.53 (Eliashberg)**.** *Let M be a closed 3-dimensional manifold. The following inclusion induces a surjection between connected components:*

$$\mathrm{Cont}^{\mathrm{OT}}(M) \to \mathrm{Cont}^{f}(M).$$

That is, we say nothing about higher homotopy groups, but we are not required to fix the overtwisted disc. This will allow us to avoid several technical details. However, do refer to Proposition 3.59 below for a sketch of π_0-injectivity.

In dimension 3, a formal contact structure is simply a plane field ξ and a choice of orientation for the whole manifold. For that given orientation, the statement asks us to deform ξ until it is a positive contact structure. The proof will follow a standard scheme in h-principles: First we modify ξ so it has an overtwisted disc somewhere. We choose a triangulation $\mathcal{T}$ of the manifold M suitably. Then, we perturb ξ in the vicinity of the 2-skeleton $\mathcal{T}^{(2)}$ until it is contact there. Finally, we modify ξ in the interior of every 3-simplex (with the help of the overtwisted disc) so that it becomes contact everywhere.

Remark 3.54. To obtain a contact structure on a neighbourhood of $\mathcal{T}^{(2)}$, we could simply apply Gromov's theorem (since such a neighborhood is an open manifold!). However, the behaviour of the resulting structure would be uncontrolled, and it would be harder to carry out the final step (achieving the contact condition in the 3-simplices). As such, we will describe an alternate approach that yields a better behaved plane field everywhere.

Nonetheless, it is worth remarking that one can apply Gromov's result with a certain finesse to yield contact structures which are "well-behaved at infinity" (which in this case corresponds to the interior of the top simplices). This approach was used in [4] to prove the h-principle for overtwisted contact structures in higher dimensions. A key ingredient in this process is Gray's Theorem, which is contact theoretical in nature. As such, it is unclear to the author whether these ideas can be applied to geometries lacking some form of local stability. ♦

3.5.4. *Preliminaries.* Let ξ be a plane field in the oriented manifold M. We first explain how to modify ξ so it has an overtwisted disc. We choose a random point $p \in M$ and we find local coordinates (x, y, z) around p in which ξ is of the form $\ker(dy + f(x, y, z)dx)$, as in Lemma 3.12. We require the standard orientation in the coordinates (x, y, z) to agree with the orientation of M; this is important to ensure that we construct a *positive* contact structure. We can modify f locally, relative to the boundary of the model, so that $\partial_z f(p) > 0$; we denote the resulting structure still by ξ.

Now we use the following result:

Proposition 3.55. *Let (N, ν) be a contact 3-manifold. Let $\tilde{\gamma}$ be a smooth knot. Then there exists γ transverse to ν and C^0-approximating $\tilde{\gamma}$.*

This is the simplest form of h-principle for transverse knots. When applied to ξ close to p, it yields a smooth unknot γ transverse to ξ and contained in an arbitrarily small neighbourhood of p. We then introduce a full Lutz twist to ξ along γ. According to Remark 3.52 and Lemma 3.50, the resulting plane field (which we still call ξ) is contact close to p and has an overtwisted disc.

The next step is triangulating our manifold. We want to do it in a manner that is suited to our purposes latter on (i.e. being able to introduce turning to ξ using Lemma 3.12). We will need the following statement, which is known as *Thurston's jiggling*, and which is also a form of h-principle:

Proposition 3.56. *Let (N, ν) be a manifold endowed with a distribution. Then, there exists a triangulation $\mathcal{T}$ such that all its simplices are transverse to ν.*

Roughly speaking, the idea behind this proposition is that if a given simplex is not transverse to ν we can subdivide it repeatedly and perturb the smaller simplices so that they become transverse. A key step in this process is observing that, as we subdivide, our simplices become smaller and, by zooming in, ν becomes progressively flatter with respect to them.

We apply the proposition to (M, ξ) to yield a triangulation $\mathcal{T}$. If U is a fixed ball containing the overtwisted disc, we require the triangulation $\mathcal{T}$ to be relative to U (i.e. we triangulate ∂U, which is a sphere, and then we extend this to a triangulation of $M \setminus U$). This is just needed so that our subsequent arguments do not modify the overtwisted disc.

3.5.5. *The reduction process.* We want to construct a contact structure in a neighbourhood of $\mathcal{T}^{(2)}$. We first observe that, due to the transversality between ξ and $\mathcal{T}$, each simplex Δ has a (properly oriented) neighbourhood $\mathbb{D}^2\times[-1,1]$ with coordinates (x,y,z) in which Δ is strictly contained in $\mathbb{D}^2\times 0$ and ξ is tangent to ∂_z (as in Lemma 3.12). In particular, ξ is locally of the form $dy + f_\Delta(x,y,z)dx$. Our goal is to modify the functions f_Δ, inductively on the dimension of the corresponding simplex Δ.

First we choose the neighbourhoods of the zero simplices so that they are pairwise disjoint. In each neighbourhood, and relative to its boundary, we deform f_Δ so that $\partial_z f_\Delta(x,y,z)(\Delta) > 0$. We did the same in the first step of introducing the overtwisted disc.

Instead of working with 1-simplices, we proceed as in Subsection 3.4. We take a 1–simplex Δ and we remove a little neighbourhood of its endpoints in which ξ is already contact. We denote the resulting path by $\tilde{\Delta}$. Now we may choose pairwise disjoint neighbourhoods $\mathbb{D}^2 \times [-1,1]$ for each of these paths. In each of them we can achieve the condition $\partial_z f_\Delta(x,y,z)|_{\tilde{\Delta}} > 0$ relative to the boundary of the model. Close to the boundary of $\tilde{\Delta}$ we can leave f_Δ untouched (because there it is already increasing).

For the 2-simplices we proceed analogously. We remove a neighbourhood of their boundary and then choose disjoint neighbourhoods so our deformations do not interact with one another. Over each neighbourhood we modify f_Δ to be increasing close to $\mathbb{D}^2 \times 0 \supset \Delta$; we do not have to modify the functions close to the boundary of $\mathbb{D}^2 \times 0$ because there they were already increasing.

Remark 3.57. All the deformations we do must be C^0-small, so that ξ remains transverse to the triangulation. This can easily be achieved, since we are just imposing the functions $z \to f_\Delta(x,y,z)$ to be increasing at the single point $z=0$. In particular, we do not care how these functions behave away from $z=0$. ♦

3.5.6. *The extension problem.* At this point we have a plane field ξ which is contact in a neighbourhood of $\mathcal{T}^{(2)}$ and transverse to the triangulation. The latter assumption actually tells us a lot about the structure of ξ in the boundary of each 3-simplex Δ: the intersection $\xi \cap T\Delta$ (whenever it is defined) is a line field that looks almost horizontal. I.e. if we were to smooth Δ suitably, ξ would only be tangent to Δ in two points, which we might as well call the north and south poles. See Figure 2.

Remark 3.58. Here one must be careful with orientations. We must choose a coorientation for ξ at the beginning of the whole argument (i.e. what does it mean to transverse ξ "positively"?), and this will determine what is "north" and what is "south". If ξ is not coorientable, one can still carry out this argument by using a covering. Let us ignore this technicality. ♦

Let $U \subset M$ be a neighbourhood of the overtwisted disc. As a contact manifold, (U,ξ) is isomorphic to $(\mathbb{D}^2_{\pi+\delta} \times [0,\delta], \xi_{\mathrm{OT}})$, for some small δ. That is, it looks like a piece of a half Lutz twist; the curve $\nu_0 = 0 \times [0,\delta] \subset U$ is the transverse curve along which this half Lutz twist has seemingly been introduced. We will now move ν_0 around M; as we do so, U will follow it around. Our goal is to make ν_0 go across each 3-simplex from the south to the north pole, effectively replacing the interior of the simplex by a piece of U. This will achieve the contact condition everywhere.

More formally: Let $\{\Delta_i\}$ be the collection of top simplices. For each Δ_i, we denote by $\tilde{\Delta}_i \subset \Delta_i$ a slightly smaller copy of it such that ξ is contact in $\Delta_i \setminus \tilde{\Delta}_i$. We denote by $\tilde{\xi}$ the contact structure in M obtained from ξ by "undoing" the half Lutz twist along ν. Do note that it has discontinuities along the slices $\mathbb{D}^2_{\pi+\delta} \times \{0,\delta\}$. We pick a homotopy $(\nu_s)_{s\in[0,1]}$ of ν_0 satisfying:

- The homotopy is relative to the endpoints of ν_0.
- The ν_t are embedded transverse curves which are disjoint from the interior of the $\tilde{\Delta}_i$.
- The path ν_1 intersects each $\tilde{\Delta}_i$ in its boundary as a meridional curve connecting the poles.

Constructing such a homotopy is easy if we do not require the ν_t to be transverse curves. To achieve the transversality condition, we must apply a relative and parametric analogue of Proposition 3.55.

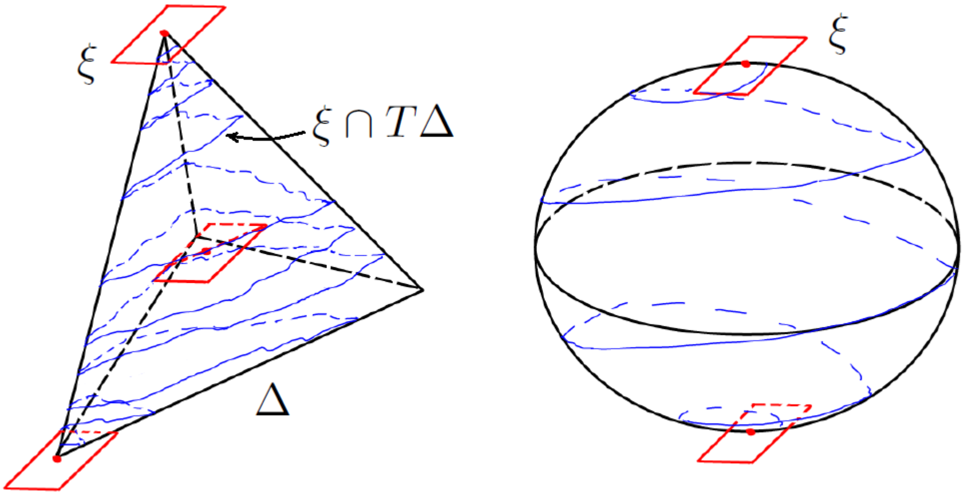

FIGURE 2. On the left hand side we have a 3-simplex Δ. The horizontal planes denote the plane field ξ, which intersects each face in a line field $\xi \cap T\Delta$, due to the transversality hypothesis. On the right hand side we have a smoothing of Δ, which is a smooth ball. The plane field ξ is tangent to it at the poles; when we intersect it with the boundary, it yields a smooth line field that is approximately horizontal. We have chosen to draw the line field without periodic orbits, but these might exist.

I.e. transverse curves can be constructed in 1-parametric families as long as we do not require the process to be relative to both ends of the family; the reader should accept this statement as a blackbox.

The collection $\{\tilde{\Delta}_i\}$ (the beads), together with the path ν_1 (the string), looks like an open necklace. If we thicken ν_1 slightly, the necklace becomes a solid cylinder, which we denote by T. We can divide the boundary in two parts: the horizontal part $\partial_h T$, which corresponds to the slices $\mathbb{D}^2_{\pi+\delta} \times \{0, \delta\}$, and the vertical part $\partial_v T$, which is everything else. See Figure 3.

Since the path ν_1 was transverse to $\tilde{\xi}$, Proposition 3.48 tells us that the line field $T(\partial_v T) \cap \tilde{\xi}$ is almost horizontal. How do we modify $\tilde{\xi}$ in the interior of T, relative to $\partial_v T$, so that it becomes contact? We proceed much like in Subsubsection 3.5.2: $\tilde{\xi}$ is flat along the core ν_1 of T and we may assume that it is tangent to the radial direction. We then replace $\tilde{\xi}|_T$ by a contact structure that makes roughly half a turn as it goes from the centre to $\partial_v T$. Since the line field $T(\partial_v T) \cap \tilde{\xi}$ is almost horizontal, this allows us to exactly match the desired slope at the vertical boundary. See Figure 4.

Close to the endpoints of ν_1, where it agrees with ν_0, this process amounts to reintroducing the half Lutz twist, so the resulting structure matches smoothly with ξ. It is also easy to see that the resulting structure is homotopic as a plane field to ξ, since the process of reintroducing the turning can be done parametrically for all ν_t. In fact, doing so provides us with a nice picture: as we move ν_t, the half Lutz twist comes along with it and eventually attaches to the balls $\tilde{\Delta}_i$. As it attaches, their interior is filled by a contact structure. This concludes the proof of (the π_0-surjectivity version of) Theorem 3.21. □

3.5.7. *Final remarks about the argument.* The argument we have just presented can be adapted to prove that:

Proposition 3.59. *Let M be closed and 3-dimensional. The following map is a bijection:*

$$\pi_0(\mathrm{Cont}^{\mathrm{OT}}(M)) \to \pi_0(\mathrm{Cont}^f(M)).$$

PROOF. We are given two overtwisted contact structures ξ_0 and ξ_1 that are homotopic as plane fields; we denote the homotopy by $(\xi_s)_{s\in[0,1]}$. We must then deform the homotopy, relative to $s = 0, 1$, until the ξ_s are all overtwisted contact structures. Let us provide a sketch of the argument.

The reduction part of the argument is very similar to what we already did:

- By applying a diffeomorphism of M we might assume that ξ_0 and ξ_1 have the same overtwisted disc.

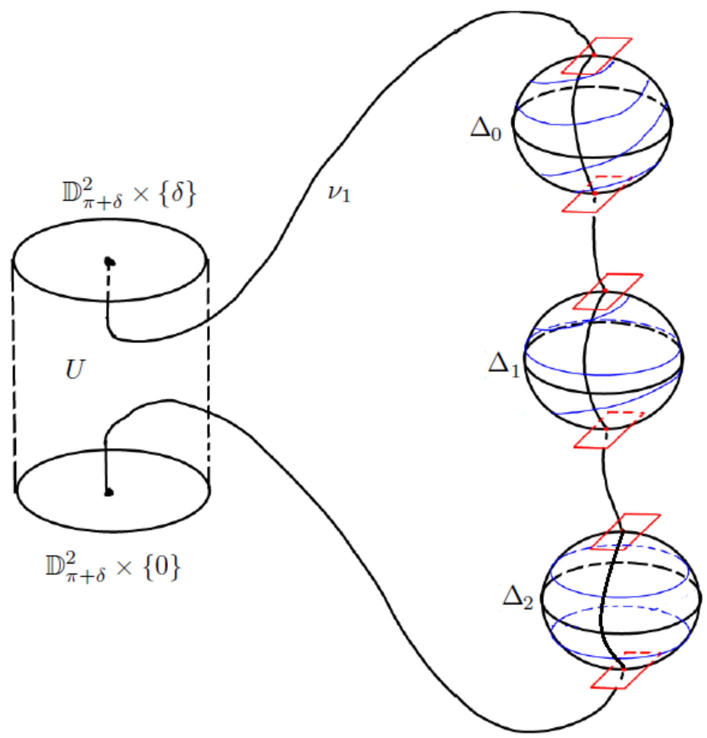

FIGURE 3. The necklace in $(M, \tilde{\xi})$. On the left hand side we see U which, before the untwisting, was the neighbourhood of the overtwisted disc. The structure $\tilde{\xi}$ has discontinuities along its upper and lower boundary $\mathbb{D}^2_{\pi+\delta} \times \{0, \delta\}$. The curve departing from U is ν_1: it attaches to each Δ_i along a meridian; this is depicted on the right.

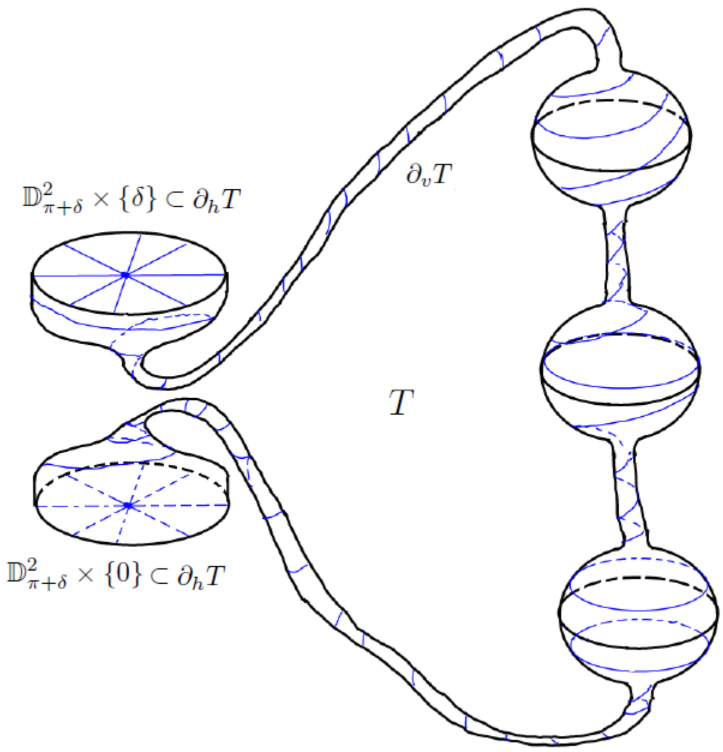

FIGURE 4. The thickened necklace T after the twisting is reintroduced along ν_1. On the left hand side we see U, which now has overtwisted discs close to its upper and lower boundary $\partial_h T = \mathbb{D}^2_{\pi+\delta} \times \{0, \delta\}$ (just like ξ had before).

- Parametrically in s we can modify the ξ_s so that they have overtwisted discs too (which should be the same for all s).
- We can take the manifold $M \times [0,1]$ (where the second term is parametrised by s and denotes the parameter space) and triangulate it as before. The triangulation will be required to be transverse to the fibres $M \times s$ and also to the ξ_s. It should also be relative to the region in which the overtwisted disc lives.
- Simplices of positive codimension are transverse to the ξ_s, so we can achieve the contact condition in a neighbourhood of them.

For the extension argument, we reason as we did in the non-parametric case: as new balls appear (as we move in s), ξ_s is not contact in their interior a priori. To solve this, a piece of the overtwisted disc extends itself to attach to the new ball as soon as it is born, spreading the contact condition to the interior. □

Remark 3.60. This argument is an example of *surgery of singularities*, an important technique in the theory of h-principles. Here we may think of the failure of the contact condition in the interior of the top cells as a "singularity" (this is a proper statement that can be made precise). The overtwisted disc is itself a singularity that has been "resolved" in a nice smooth manner. When we undo the half Lutz twist that defines the overtwisted disc, we are somehow going back to the unresolved singularity. This idea of moving the core of the overtwisted disc so that it attaches to the top simplices amounts to saying that a single singularity can be used to absorb all others. Once they are together we can resolve them again (reintroducing the half Lutz twist), yielding a smooth contact structure.

This reasoning is one of the central ingredients in the theory of wrinkles. For instance, in the classification of *soft* (i.e. overtwisted/flexible) immersions with prescribed folds [17], one also uses a certain model of singularity (called the *zig-zag*), to absorb all other singularities one has to introduce during the argument. In the theory of loose legendrians [35] exactly the same role is played by the *loose chart*. ♦

Overtwisted structures have almost magical properties. For instance:

Corollary 3.61. *Let (M, ξ) be an overtwisted contact manifold. Let n be any positive integer. Then, there exists a solid torus $T \subset M$ such that (T, ξ) is isomorphic to a Lutz twist with n half-turns.*

Proof. Take a ball $B \subset M$ disjoint from the overtwisted disc. Pick a transverse knot contained in B and introduce along it a Lutz twist with N full turns. The resulting structure $\tilde{\xi}$ is contact, still overtwisted, and homotopic to ξ as a plane field, according to Remark 3.52. As such, Proposition 3.59 tells us that ξ and $\tilde{\xi}$ are homotopic as overtwisted contact structures. They are therefore diffeomorphic thanks to Gray stability. Since $\tilde{\xi}$ contains n half-turns for any $n \leq 2N$, so does ξ. □

More generally, using the same argument one can show that:

Corollary 3.62. *Let (M, ξ) be an overtwisted contact manifold. Then:*

- *M contains any semi-local model as long as there are no homotopical obstructions.*
- *Submanifolds of a given type in the complement of a fixed overtwisted disc satisfy the h-principle.*

In particular: checking whether a particular type of submanifold lives inside (M, ξ) amounts to performing an algebraic topology computation using the nilpotentisation of ξ. This requires no actual Geometry!

Remark 3.63. Both corollaries find the desired semi-local model in the complement of the overtwisted disc. Those submanifolds that intersect every overtwisted disc still present interesting behaviours [51]. There is still plenty to be understood in this direction. ♦

3.6. **Further directions.** The reader might be intrigued at this point: Even though these notes are almost over, we have only looked at the topological behaviour of contact structures. What about all other tangent distributions?

The answer is that Contact Topology is possibly the most remarkable example of a class of distributions displaying a nice topological theory. Another great example is 3-dimensional Foliation Theory: it turns out that there is a class of foliations, those having **Reeb components**, that in many ways is analogous to the overtwisted class. In fact, the argument that we presented for overtwisted contact structures can be adapted to construct also such foliations. Similarly, those foliations with no Reeb components resemble tight contact structures. These parallels are best understood through the lens of **Confoliation Theory** [18], in which 3-dimensional contact structures and foliations come together.

It is also known that certain families of distributions, for instance even-contact structures [32], are completely flexible. That is, they satisfy the complete h-principle. This makes them uninteresting from our perspective, but one can still pose many questions about them from a geometric viewpoint. In the case of even-contact structures, we still have to explore their dynamics further.

Now comes some self-promotion: After looking at dimension 3, one would look at dimension 4, and therefore at Engel structures. The first general method for constructing them appeared in the thesis [52] of T. Vogel, relying on techniques from Contact Topology. More recently, there have been developments [6] from an h-principle perspective: We now know that there is a subclass of Engel structures that are *overtwisted* and therefore the h-principle applies to them [43]. In fact, the proof of Theorem 3.21 given in these notes is an adaptation to the contact case of the Engel proof.

Surprisingly, there is a second class of Engel structures, that we call *loose*, that satisfy the h-principle as well [8]. This has no analogue in the world of contact structures, but it is in fact very similar to the manifestation of flexibility for convex curves in $\mathbb{S}^2$ [44]. At this point we do not know what the relationship between Engel looseness and overtwistedness is. We also do not know if there are non-loose and non-overtwisted structures!

Beyond these cases, I would say we know very little. Seeing the parallels between Engel and contact structures, one might hope that the story goes on. Maybe one can provide h-principle statements for (possibly an overtwisted subclass of) the generic distribution in each dimension and rank. I believe this can be proven for distributions of rank 3 in dimension 5 using known techniques, but for rank 2 (i.e. distributions of Cartan type) it seems more involved. A question in this direction is the following: Given a class of distributions whose nilpotentisation is fibrewise isomorphic to some generic graded Lie algebra $\mathfrak{g}$ (or a generic family of them), can we tell whether some classic h-principle technique (say, convex integration) applies to provide a complete classification result, purely in terms of $\mathfrak{g}$? For distributions that are non-generic (and thus given by a closed condition), one would need analytic arguments (à la Nash-Moser [26]) to attack the problem.

One can pose similar questions about submanifolds. In the Engel case, curves are completely flexible [7, 41], but the case of surfaces has to be understood still. Due to dimensional reasons, this is the critical case and thus it might yield interesting behaviours. In general jet spaces, a theory of (embedded) integral manifolds can be developed, which in many ways mimicks the theory of legendrians but displaying greater flexibility [49, 42]. Some results for the general immersed case can be found in [26], but they only apply to subcritical dimensions.

The punchline is: the topological theory of tangent distributions is largely an unexplored land still!

Acknowledgments

I would like the organisers of the "13th International Young Researchers Workshop on Geometry, Mechanics and Control" for giving me the opportunity of discussing this beautiful theory. I would also like to thank you, the reader, for taking the time to look at these notes. I hope you can find something interesting in them!

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