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Towards a Universal Mathematical Braille Notation

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Abstract

Introduction: Across the world, mathematical expressions are represented very differently in braille. The aim of this study was (1) to gain an overall insight in mathematical braille notations and (2) to investigate how mathematical braille notations support braille readers in reading and comprehending mathematical expressions. **Method:** Twenty teachers from sixteen countries (thirteen European Union, EU, and three non-EU) were asked to transform 21 mathematical expressions and equations into the mathematical braille notation currently used by their braille readers. Three mathematical expressions were selected, and the transformed expressions in the different braille notations were qualitatively compared at braille and mathematical structure level. **Results:** The results illustrated that most mathematical braille notations use mathematical structures that either support braille readers in getting an overview of an expression—for example, by announcing the start and end of a fraction—or facilitate communication between braille readers and people who can see. **Discussion:** The method of comparing transformed expressions at structure level can be extended to other types of mathematical expressions and other mathematical braille notations. Agreement on the structure of different mathematical expressions can be a first step towards a universal mathematical braille notation. **Implications for Practitioners:** Mathematics teachers should be aware of and use the strengths of the mathematical braille notation and try to compensate for weaknesses of the notation in the support of braille readers.

Keywords

braille reader, mathematical notation, mathematical braille notation

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It is generally recognized that language plays an important role in teaching and learning mathematics (e.g., Morgan, Craig, Schütte, & Wagner, 2014). Developments in the study of language in mathematics education are closely related to other developments. The shift in thinking about learning as an individual activity to a socially organized one stimulates language-oriented studies to contribute to the understanding of mathematics education (Kress & Selander, 2012). To access mathematics, students must communicate through different languages (Riccomini, Smith, Hughes, & Fries, 2015; Schleppegrell, 2007; Van Eerde, 2009). For example, students need to distinguish the meaning of the word "function" in daily life, school language and formal mathematical language. Moreover, a grasp of symbols is needed to act and communicate mathematically. A mathematical notation like $3/4$ is supposed to evoke images such as  and actions such as dividing 3 among 4. These practices are part of a mathematical culture that has developed over centuries and resulted in a shared symbolism (Nasir & Cobb, 2002). This symbolism, however, has barriers to entry for students who cannot hear or who cannot see (e.g., O'Neill, Cameron, Quinn, O'Neil, & McLean, 2015; Schermer, 2003). These students use rather recently developed alternatives to our spoken and written language that have not developed universal mathematical notations. In the current study, we investigated how mathematical braille notations (from here on also referred to as braille notations) in different countries support braille readers in reading and comprehending mathematical expressions. This can be a first step towards a uniform mathematical braille notation.

Mathematical Notation and Braille

The formal mathematical notation (from here on mathematical notation) uses two-dimensional arrangements of symbols to convey information. The symbols are arranged according to specific rules. For instance, 2^x is has a different meaning than 2x. In mathematical notations, Latin and Greek letters, for example, e, π, and $\sum$, are used as well as specific forms, for example, $\sqrt{\ }$. In general, symbols are used to save time and space. For instance, "the square root of x to the power of three" is denoted by "$\sqrt{x^3}$." This example shows that the mathematical notation can be very compact and offers little redundancy. It is in many instances impossible to guess the identity of a symbol based on context.

A braille cell consists of a pattern of raised dots arranged in a 2 * 3 (⠿) or 2 * 4 (⣿) configuration. Each pattern represents a braille character. The mathematical braille notations that braille readers use have rules for transforming expressions into braille. Worldwide, different mathematical braille notations are used. Every mathematical braille notation has its own rules. In this article, we use the 6-dot or 8-dot Dutch mathematical braille notation. If another braille notation is used, this is explicitly stated.

In 6-dot braille, 64 braille characters are possible. Mathematical text, however, needs more characters (Edwards, McCartney, & Fogarolo, 2006). For instance, extra characters are needed to distinguish between ⠼⠙⠃ (42) and ⠼⠙ ⠠⠃ (4b) or between ⠼⠃ ⠭ (2x) and ⠼⠃ ⠤⠭ (2^x). In 8-dot braille, 256 braille characters are possible. As a consequence, many modifier signs can be removed. For example, 42 and 4b are transformed into ⢙⢃ and ⢙⠃, 2x and 2^x into ⢃⠭ and ⢃⡔⠭.

Braille can be read on paper or on a braille display linked to a computer. Typically, braille readers read or write on paper in 6-dot braille. There are, however, new developments that make it possible to use 8-dot braille on paper (e.g., Four Line 8-Dot Braille Slate, MakerBot Industries, LLC). When braille readers write on paper, they use a slate and stylus, or a braillewriter (Dixon, 2009). With a slate and stylus, you write from right to left, one dot at a time, and reverse the dots because they are embossed on the other side of the paper. This device is still widely used in developing countries. In western countries, most braille readers use a braillewriter or a one-line braille display. A braillewriter is a typewriter with a key corresponding to each of the six dots, a space key, a backspace key, and a line space key. With a braillewriter, you write—in contrast to a slate and stylus—one braille character at a time. The one-line braille display allows braille readers to read the content on a computer screen one text line at a time in the form of a line of braille characters. When using this device, it is difficult to get an overview of a few lines of text. This is less of a problem when using a multi-line braille display—which is, as far as we know, not yet widely used—or braille on paper.

Challenges in Reading Mathematical Expressions in Braille

It is challenging to read and comprehend mathematical expressions in braille (e.g., Van Leendert, Doorman, Drijvers, Pel, & Van der Steen, 2019). These challenges are related to accurate reading, getting an overview of an expression, and mathematical communication. Accurate reading is important, because an error in decoding the braille characters of an expression can change the meaning. Accurate reading is difficult, because braille characters have low redundancy, which means that characters are difficult to distinguish (Millar, 1997; Tobin & Hill, 2015). Getting an overview is challenging, because braille is a linear output modality (Stöger & Miesenberger, 2015). Some braille notations also allow for spatially arranged structures such as matrices and grade school level arithmetic sum, multiplication and division problems. That does not completely solve the challenge of getting an overview, because braille readers still need to build an overview by touching one braille character after the other (Millar, 1997; Van Leendert et al., 2019) Therefore, braille readers cannot take advantage of the layout of a mathematical expression that helps people who can see (from here on print readers) to understand the structure of an expression at a glance (Karshmer & Bledsoe, 2002). Finally, mathematical communication between braille and print readers is difficult due to the differences in perception and notation. This is critical, as communicating mathematically is essential for the overall development of mathematical abilities (Riccomini et al., 2015).

The Transformation from a Mathematical Expression to an Expression in Braille

The transformation from a mathematical expression to an expression in braille can be considered as a two-step process, see Figure 1. Step 1 is the transformation from a mathematical to a linear-print expression. This may result in a change in the mathematical structure of the expression. Step 2 is the conversion from the linear-print to the linear-braille expression. This transformation is called a conversion because of the one-to-one correspondence between the ASCII characters and the braille characters. This conversion depends on the braille table used. It does not change the mathematical structure of the expression. If no distinction between the linear-print and the corresponding linear-braille expression is necessary or desired, we use the term transformed expression. In some cases, when no confusion is possible, we use the term expression instead of transformed, linear-print or linear-braille expression.

Support in Reading Mathematical Expressions in Braille

Braille notations differ from each other in how they transform mathematical expressions into linear-print expressions and/or in the braille table that they use. We will explain this in more detail. In this section, we will describe how notations can support braille readers in reading accurately, in getting an overview and in communication. Accurate reading is supported by using braille tables that are unambiguous and use good mnemonics (Martos, Kouroupetroglou, & Argyropoulus 2015; Nemeth, 2001). An example of good mnemonics is using symmetric braille characters for the "(" and ")" signs. Getting an overview is supported by transformed expressions that 1)

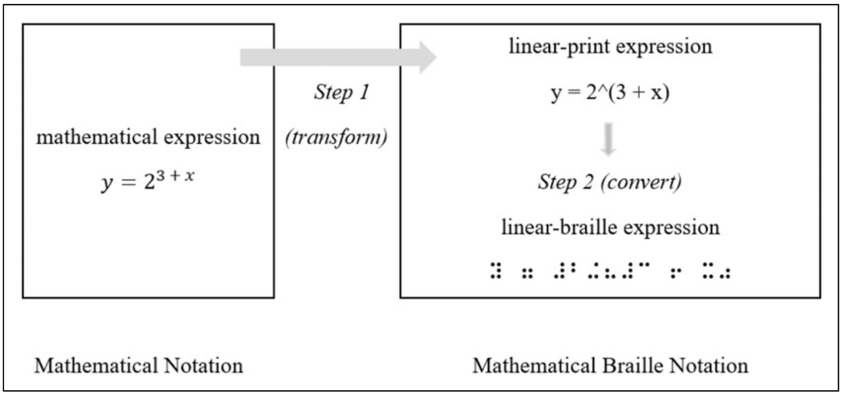

Figure 1. Transformation from a mathematical to a linear-braille expression.

are compact, 2) use structure announcement and/or 3) use context awareness. An expression that is compact helps to provide an overview because such an expression does not include unnecessary characters. An expression that uses structure announcement also supports getting an overview. This will be illustrated with the fraction (x+1)/(x-1). We start with two non-examples of structure announcement. In the Dutch braille notation, this fraction is transformed into (x+1)/(x-1). This expression introduces many brackets that are not present in mathematical notation and are therefore called phantom brackets. The French braille notation uses blocks to avoid the use of phantom brackets. This results in bbx+1eb/bbx–1eb. The abbreviation bb (⠰ dot 56) stands for begin and eb (⠆ dot 23) for end block. The problem with the aforementioned transformed expressions is that braille readers only know that they are reading a fraction when they come across the symbol “/” (Karshmer, Gupta, & Pontelli, 2007). Therefore, some braille notations provide a variety of grouping symbols to announce the start and end of the structure of an expression or sub-expression. For instance, the Nemeth Code, a notation that is mainly used in the United States, transforms the above fraction as ?x+1/x-1#, where ? (⠹) stands for start and # (⠼) for end fraction. This is called structure announcement.

Another important feature that also helps to provide an overview is keeping the braille reader aware of the context he or she is in at all times (Karshmer et al. 2007). This is because braille readers are focused on one braille character and the part to the right of the finger is not known to them at all, while the text to the left is in the braille readers’ memory. For example, the expression y^{x^a+3} is transformed into the Dutch notation as yˆ(xˆa + 3). This does not work so well, as the braille reader will immediately forget the exponent level as he or she moves left to right in the expression and a significant number of backtracking with the finger will be required to comprehend it. Therefore, some notations use context awareness. The Nemeth Code, for example, transforms $x^a + 3$ and y^{x^a+3} into xˆa”+3 and yˆxˆaˆ+ 3, respectively. The superscripted expression yˆxˆˆaˆ+ 3 is terminated by the space. The symbol ˆ indicates superscript and the symbol ” indicates a shift to the baseline. The sub-expression $x^a + 3$ is transformed into xˆˆa+ 3 rather than xˆa”+ 3 because of the context.

Finally, a transformed expression can support mathematical communication between braille and print readers. This is the case for an expression that is true to the print (Nemeth, 2001). This means that a transformed expression is very similar to the expression in the mathematical notation, apart form spacing and format. For example, the aforementioned expression yˆxˆˆaˆ+ 3 is true to the print. Expressions that use Excel or LaTeX conventions also support communciation between braille

Table 1. Support in reading and comprehending mathematical expressions in braille.

Level	Features of expression	Support
Braille level	unambiguity good mnemonics	accurate reading
	compactness	getting an overview
Structure level	compactness structure announcement context awareness	getting an overview
	Excel or LaTeX conventions true to the print	communication

and print readers even when these expressions are not true to the print. This transformed is because many print readers are familiar with these notations. An example is y^(x^a + 3). This expression is not true to the print, but most print readers will recognize and comprehend the structure. Table 1 summarizes the features of expressions that support braille readers when reading and comprehending mathematical expressions.

Research Question

The challenges that braille readers face when reading and understanding mathematical expressions relate to accurate reading, getting an overview of an expression, and communication with print readers. In this study, we investigate whether and how braille notations from different countries support braille readers with reading and comprehending mathematical expression with the following research question: What are similarities and differences in the support that braille notations from different countries offer braille readers in reading and comprehending mathematical expressions?

We assume that most braille notations use good mnemonics to support accurate reading. This seems a very natural thing to do. In addition, we expect that braille notations differ in how they support braille readers. They have to make choices between the features described in the last section.

Methods

Design of the Study

An English-language questionnaire was made. In the first part, the participants were required to give demographic information. In the second part, the participants had to transform mathematical expressions into the mathematical braille notation used by their braille reader or braille readers.

Participants

In 2019, a conference on mathematics for braille readers took place in France. There were 22 teachers—in addition to other professionals—from fourteen European countries, who worked in mathematical education for braille readers. After the conference, we approached them to participate in the current study. None of them used the Nemeth Code or UEB (Unified English Braille) notation, which are common in English-speaking countries. For this reason, we used our personal contacts and approached two teachers who used the Nemeth Code (both from the United States) and two teachers who used the UEB notation (one from Ireland and one from New Zealand). In addition, we approached a teacher from Mexico, who is a friend of one of the authors. After agreeing to participate in the current study, each teacher received an email and was asked to complete the questionnaire. Table 2 shows the participants' demographics.

Procedure

Each teacher was requested to complete the questionnaire within six weeks. 50% of the teachers responded within this period. After a reminder, all remaining teachers responded within three months after the first contact.

Pilot Study

We conducted a pilot study and asked four teachers, two from the Czech Republic and two from Flanders (northern Belgium), to complete the questionnaire. They identified some issues in readability, understanding, and phrasing. We discussed these issues and adapted the text accordingly. They mentioned that the selected expressions contribute directly to the factors being evaluated for comparison.

Data Collection and Analysis

The questionnaire consisted of 21 items involving expressions and equations. The teachers transformed these expressions and equations into the braille notation that their braille readers use. To address the research question, we first analyzed the representations of numbers and the "+" and "-" symbols in braille for the presence of mnemonics and compactness. Three mathematical expressions have been selected for further analysis:

$$\frac{1}{4} \tag{1}$$

$$\frac{2a+3b}{n} \tag{2}$$

$$y^{x^{a}+b} \tag{3}$$

Expression (1) was selected to investigate the extent to which braille notations differ from each other. Expression (2) and (3) were selected to investigate how different braille notations support braille readers in reading and comprehending expressions. For expression (2), we investigated whether braille notations use structure announcement or other ways to group symbols. For expression (3), we investigated whether braille notations use context awareness or other ways to transform the mathematical expression.

Results

The response rate was 80%. Twenty teachers from sixteen countries completed the questionnaire. We checked each completed questionnaire for inconsistencies in the transformed expressions and equations. In five cases, we discovered some inconsistencies and these teachers corrected their answers. The results show that most countries have their own braille notation. In some countries, 6-dot braille is not—or hardly—used in secondary education. In that case, we only gave the representation in 8-dot braille. In some cases, we referred to a braille notation only using the country's name. In other cases, we needed to give some additional information. This is necessary when a

Table 2. Participants' demographics.

Number of participants	Experience in Mathematical Education of Braille readers (Years)	Braille or Print Reader	European or Non-European
1	<1	Print	European
4	5–10	Print	European
10	>10	Print	European
1	>10	Braille	European
4	>10	Print	Non-European

country uses different braille notations or when a braille notation is used in different countries. For example, Czech Republic uses three different notations: Czech Republic (6-dot), Czech Republic (BlindMoose), and Czech Republic (Lambda). BlindMoose is a Microsoft add-in that provides access through braille and visual display (Wiazowski, 2018). Lambda is a mathematical editor that provides access through braille, synthetic speech, and visual display (Edwards et al., 2006). Flanders uses two braille notations: Flanders Mathematical Notation (FMN) and Spermalie. A plug-in for MS Word enables on-the-fly conversion between expressions in the mathematical notation and FMN. Both Ireland and New Zealand use the UEB notation. This is referred to as UEB (Ireland & New Zealand}. In the United States, the UEB and the Nemeth Code are used. Our teachers from the United States used the Nemeth Code. This notation is named USA (Nemeth Code). Finally, we refer to the Swedish 8-dot notation as Sweden (AsciiMath). This notation is very similar to AsciiMath, a well-known notation for mathematics teachers.

Table 3 summarizes how numbers are transformed in different braille notations. This table shows that most 6-dot braille notations use number signs. The French notation uses "letter a..j + dot 6." However, the number zero is transformed into ⠼ instead of ⠺ to avoid a conflict with the letter w. The Nemeth Code

Table 3. Transformation of numbers in braille.

Table	Numbers in braille	Country
6-dot	number sign + letter a, ..., j example number 3: ⠼⠉	Czech Republic (6-dot), Estonia (6-dot), Ireland & New Zealand (UEB), Latvia, Lithuania, Mexico, the Netherlands (6-dot), Poland, Slovenia (6-dot), Sweden (6-dot)
	letter a, ..., j + dot 6[a] exception number 0: ⠼ dot 3456 example number 3: ⠩ dot 146	France
	"dropped" letter a, ..., j example number 3: ⠒ dot 25	USA (Nemeth Code)
8-dot	letter a, ..., j + dot 6[b] exception number 0: ⠬ dot 346 example number 3: ⠩ dot 146	Flanders (FMN), Germany (pseudo-LaTeX, LaTeX), the Netherlands (8-dot)
	letter a, ..., j + dot 8 example number 3: ⢉ dot 148	Czech Republic (BlindMoose, Lambda), Estonia (8-dot), Flanders (Spermalie), Norway, Slovenia (8-dot), Sweden (AsciiMath)

[a]The number zero is an exception and is transformed into ⠼.
[b]The number zero is a special case and is transformed into ⠬ (dot 346).

Table 4. Transformation of the plus sign in braille.

Table	Plus sign	Braille notation
6-dot	⠲ dot 256	Czech Republic (6-dot), Sweden (6-dot)
	⠖ dot 235	Estonia (6-dot), France, Latvia, Lithuania, Mexico, the Netherlands (6-dot), Poland
	⠐⠖ dot 5 235	Ireland & New Zealand (UEB)
	⠳ dot 1256	Slovenia (6-dot)
	⠬ dot 346	USA (Nemeth Code)
8-dot	⠲ dot 256	Czech Republic (BlindMoose, Lambda), Sweden (AsciiMath)
	⠖ dot 235	Estonia (8-dot), Flanders (Spermalie), Germany (pseudo-LaTex, LaTeX), the Netherlands (8-dot)
	⡖ dot 2357	Norway
	⢖ dot 2358	Flanders (FMN), Slovenia (8-dot)

uses "dropped" letters. This notation requires that ⠼, the numeric indicator, is used before numbers that would otherwise be preceded by a space. That helps determine the braille character alignment. In this study, the 8-dot notations use "letter a..j + dot 6"—except for the number zero—or "letter a..j + dot 8." The number zero is transformed into ⠬ (dot 346) to avoid a conflict with the letter w (⠺). Table 4 shows how the "+" sign is transformed in braille. The "-" sign is transformed into ⠤ (dot 36) in all notations, except for the UEB notation that uses ⠐⠤ (dot 5 36).

Table 5 shows how different braille notations transform $\frac{1}{4}$. The braille notations are divided into four categories based on the structure of the transformed expressions. In the first category, the notations use "dropped" numbers for the

Table 5. Different ways to represent $\frac{1}{4}$ in braille.

Category 1 Mathematical structure: *numerator - denominator*
Feature: compactness

Table	Linear-print expression	Linear-braille expression	Braille notation
6-dot	#a	⠼⠁⠲	Estonia (6-dot), Latvia, Lithuania
	#a/	⠼⠁⠲	Poland
	#,d	⠼⠂⠙	Mexico

Category 2 Mathematical structure: start fraction - *numerator* - fraction line - *denominator* - end fraction
Feature: structure announcement

Table	Linear-print expression	Linear-braille expression	Braille notation
6-dot	;#a/#d[caps lock][1]	⠰⠼⠁⠻⠼⠙⠰	Czech Republic (6-dot)
	?1/4#	⠹⠂⠌⠲⠼	USA (Nemeth Code)
	;#a:#d[letter prefix][2]	⠰⠼⠁⠒⠼⠙⠰	Slovenia (6-dot)
8-dot	;1/4¨	⠰⢁⠻⢙⠰	Czech Republic (BlindMoose)
	//1Ø4\\	⠰⢁⢶⢙⠰	Czech Republic (Lambda)

Category 3 Mathematical structure: *numerator* - fraction line - *denominator*
Feature: Excel conventions

Table	Linear-print expression	Linear-braille expression	Braille notation
6-dot	#a/#d	⠼⠁⠌⠼⠙	Estonia (6-dot), the Netherlands (6-dot)
	1/4	⠡⠌⠹	France
	#aü#d	⠼⠁⠳⠼⠙	Sweden (6-dot)
	#a/d	⠼⠁⠌⠙	Ireland & New Zealand (UEB)
	1/4	⢁⠌⢙	Estonia (8-dot), Sweden, (AsciiMath)
	1/4	⠡⠌⠹	Flanders (FMN), the Netherlands (8-dot)
8-dot	1/4	⠡⠲⠹	Germany (pseudo-LaTeX)
	1/4	⢁⡐⢙	Norway, Flanders (Spermalie)

Category 4 Mathematical structure: backslash fraction - opening accolade - *numerator*- closing accolade - opening accolade - *denominator* - closing accolade
Feature: LaTeX conventions

Table	Linear-print expression	Linear-braille expression	Country
8-dot	\ul{1}{4}	⡳⠥⠇⠣⢁⠜⠣⢙⠜	Slovenia (8-dot)
	\frac{1}{4}	⡌⠋⠗⠁⠉⠶⠡⠾⠶⠹⠾	Germany (LaTeX)

[1]The braille character ⠰ (dot 56) represents Caps Lock.
[2]The braille character ⠰ (dot 56) represents letter prefix.

numerator or denominator. This results in a very compact expression. Estonian (6-dot) uses two different structures: numerator—denominator (category 1) and numerator—fraction line—denominator (category 2).

The notations for expression (2) are divided into five categories based on how grouping symbols are used (Table 6). The Polish notation transforms the expression in two different ways. One transformed expression uses structure announcement, the other is true to the print. Usually, in Polish notation, a space is placed before the plus sign. However, in the representation in category 2, this space is replaced by the braille character ⠈ (dot 4). This character is needed to group the numerator "2a plus 3b." If you were to write 2a +3b/n, the numerator would be 3b. The French notation uses blocks. This notation is also true to the print. Sweden (6-dot), which is typically read on paper, uses structure announcement. In contrast, Sweden (AsciiMath), which is read on the braille display, uses Excel conventions.

The braille notations for expression (3) are divided into six categories (Table 7). In the first category, the notations announce "that the exponent of y is an exponential expression." This is an example of structure announcement. The Nemeth Code, in the second category, uses context awareness and is true to the print. The notations in the third and fourth category are also true to the print. In the third category, the notations give the location of each exponent relative to the neighbor. The notations in category four use blocks—in the UEB notation named braille grouping symbols—to avoid the use of phantom brackets. Most notations use Excel or LaTeX conventions (category five and six).

Table 6. Different ways to support reading $\frac{2a+3b}{n}$ in braille.

Mathematical structure	Table	Braille notation
Category 1 start fraction - *numerator*-fraction line - *denominator* - end fraction Example: ?2a+3b/*n*# (Nemeth) Feature: structure announcement	6-dot	Czech Republic (6-dot), Estonia (6-dot), Lithuania, USA (Nemeth), Ireland and New Zealand (UEB), Poland, Sweden (6-dot)
	8-dot	Czech Republic (BlindMoose, Lambda), Estonia (8-dot), Flanders (FMN), Norway
Category 2 *numerator* - fraction line - *denominator* Example: 2a ⠈+3b/n [1] Feature: true to the print	6-dot	Poland
Category 3 begin block - *numerator* - end block - fraction line - *denominator* Example: ⠰2a+3b⠆ /n [2] Feature: true to the print	6-dot	France
Category 4 opening bracket - *numerator* - closing bracket - fraction line - *denominator* Example: (2a+3b)/*n* Feature: Excel conventions	6-dot	Latvia, Mexico, the Netherlands (6-dot)
	8-dot	Flanders (Spermalie), Germany (pseudo-LaTeX), the Netherlands (8-dot), Sweden (AsciiMath)
Category 5 backslash - "fraction" - opening accolade - *numerator* - closing accolade - opening accolade - *denominator*- closing accolade Example: \fraction{2a+3b}{*n*} Feature: LaTeX conventions	8-dot	Germany (LaTeX), Slovenia (8-dot)

[1]The braille character ⠈(dot 4) is used as a grouping symbol.
[2]The braille characters ⠰ (dot 56) and ⠆ (dot 23) denote begin and end block.

Table 7. Different ways to support reading y^{x^a+b} in braille.

Mathematical Structure	Table	Braille notation
Category 1	6-dot	Poland, Sweden (6-dot)
y - shift up for exponential expression - x - shift up for expression - a - plus b (eventually: - shift down)[1] Example: y ⠘⠬x⠬a+bû (Sweden 6-dot)[2] Feature: structure announcement	8-dot	Czech Republic (Lambda)
Category 2 y - shift up - x - shift up shift up (two times) - a - shift up - plus b - space Example: yˆxˆˆaˆ+b Feature: context awareness, true to the print	6-dot	USA (Nemeth Code)
Category 3	6-dot	Czech Republic (6-dot), Latvia, Lithuania
y - shift up - x - shift up - a - shift down - plus b - shift down Example: yíxíaš +bš (Czech Republic BlindMoose) Feature: true to the print	8-dot	Czech Republic (BlindMoose), Flanders (FMN)
Category 4 y - shift up - begin block - x - shift up - a - plus b - end block Example: yˆ⠰ xˆa+b⠆ [3] Feature: true to the print	6-dot	France, Ireland & New Zealand (UEB)
Category 5	6-dot	Estonia (6-dot), Mexico, the Netherlands (6-dot)
y - shift up - open bracket - x - shift up - a - plus b - closed bracket Example: yˆ(xˆa + b) (the Netherlands) Feature: Excel conventions	8-dot	Estonia (8-dot), Flanders (Spermalie), Germany (pseudo-LaTeX), the Netherlands (8-dot), Norway, Sweden (AsciiMath)
Category 6 y - shift up - opening accolade - x - shift up - opening accolade - a - closing accolade - plus b - closing accolade Example: yˆ{xˆ{a}+b} Feature: LaTeX conventions	8-dot	Germany (LaTeX), Slovenia (8-dot)

[1]Shift up means shift to a higher level. Shift down means shift to a lower level.
[2]The combination of braille characters ⠘⠬ (dot 45 346) denotes shift up for exponential expression, the braille character ⠬ (dot 346) denotes shift up for expression.
[3]The braille character ⠰ (dot 56) denotes begin block and the braille character ⠆ (dot 23) denotes end block.

Conclusions and Discussion

We investigated how braille notations of different countries support braille readers while reading and comprehending mathematical expressions. The results of the transformations of numbers and the “+”and “-” signs show that braille notations differ in compactness. All notations, except the UEB notation transform the “-” sign into ⠤ (dot 36) which is very similar to the representation in print. UEB uses two braille characters ⠐⠤ (dot 5 36). In the Czech Republic, Estonia, the Netherlands and Sweden, the braille characters for the “+” sign in 6-dot and 8-dot braille are the same. These are all examples of good mnemonics.

For expression (1), the transformed expressions were compared at structure and braille level. This resulted in eighteen different linear-braille expressions. These expressions were grouped into four categories based on mathematical structure. For expression (2) and (3), the transformed expressions were only compared at structure level. For expression (2), the notation in the first category supports getting an overview of an expression. The notations in the other categories support communication between print and braille readers. As to expression (3), the notations in the first category support getting an overview. The Nemeth Code (category two) supports getting an overview and communication. That is because this structure uses context awareness and is true to the print. The notations in the last four categories support communication. The results are in line with what we expected. Most notations, except the Nemeth Code for expression (3), do not support getting an overview ánd communication. Other findings are that the categories are not stable. For example, Latvia and Mexico fall in the same category for expression (1) and (2) but in different categories for expression (3). Another finding is, related to the one we just mentioned, that a notation can support getting an overview for one transformed expression and support communication for another transformed expression.

A limitation of this study is the low number of mathematical expressions, as well as the low number of mathematical braille notations. However, the method of comparing expressions at structure level can be easily scaled up to other types of mathematical expressions and other mathematical braille notations. A second limitation is that we investigated the notations in isolation. We did not take into account the context of the braille reader and/or teacher. For example, the assistive devices that braille readers use and how they use them also play a role in reading and comprehending mathematical expressions (e.g., Van Leendert et al., 2019). Future studies should investigate the notations in relation to different contexts.

Our study sheds light on how braille notations support braille readers in reading and comprehending mathematical expressions. For expressions (2) and (3), we compared the transformed expressions only at structure level. That resulted in manageable differences and similarities. Therefore, we suggest that mathematics teachers of braille readers from different countries come together and try to agree on (features of) the structure of different kind of expressions and equations. That could be a first step towards a universal mathematical braille notation.

As a next step, we might opt for a more comprehensive universal mathematical approach to supporting braille readers in doing mathematics. Such an approach should be developed in close collaboration with braille readers. A universal mathematical braille notation can be part of it. Speech synthesis can also play an important role and may compensate for the weaknesses of the mathematical braille notation. The practical implications are that mathematics teachers of braille readers should get opportunities to study the mathematical braille notations that their braille reader(s) use at braille and structural level. They should use the strengths of the braille notation and compensate for its weaknesses.

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