Teachers' Inquiry in
Mathematics Education

TIME²

A compendium for designing inquiry-based mathematics education

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EDITOR IN CHIEF
Rogier Bos

TEXT WRITTEN BY
Rogier Bos, Michiel Doorman, Amy Mol, Carl Winsløw

REVIEWS, EDITING AND PROOF READING
Matija Bašić, Rogier Bos, Britta Eyrich Jessen, Željka Milin Šipuš, Carl Winsløw

DESIGN AND VISUALS
Irina Rinkovec

FIGURES
Rogier Bos, Matija Bašić, Kristijan Cafuta

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Introduction

Why would a teacher be interested in designing tasks (for IBMT)? Many resources to shape a lesson—textbooks, tools, lesson plans, etcetera—are already available. However, trying to meet the needs of particular groups of students, one might sometimes feel the need to design one's own tasks, series of task, and sometimes whole lesson series. Moreover, who wouldn't like to be creative and try out innovative ways of teaching? It is good practice to question your current teaching practices at some point, and a way to address such questions might be to try teaching in a novel way. Finally, the ability to design for education is in particular useful when preparing for inquiry-based mathematics teaching, since tasks for IBMT are underrepresented in textbooks.

To support such a design adventure, in this compendium we would like to share some ideas on educational design. The most suitable time to read this compendium as a teacher is when, after building some experience with educational design, you feel the desire to approach this in a more structured and reflexive way.

Many design principles have been formulated by teachers and in (academic) resources that help to design or redesign tasks: to make design choices explicit, and to base design on successes—and not on pitfalls—from the past. An essential goal of this compendium is to support teachers/designers to discuss their design choices.

This compendium offers reading material to support the course on design that is part of the TIME-project. In particular, it offers

- a supporting structure for the designing process (section 1.3),
- many details of the study or analysis that precedes the actual design (sections 2.1 and 2.2),
- an introduction to design principles (chapter 3),
- many aspects to consider during the actual design (chapter 4).

Bear in mind: tasks themselves don't teach (nor do designed lesson plans, or learning trajectories). Teaching is the responsibility of a teacher, and depends on the teacher's choices while teaching. Lesson plans help to embed tasks in a teaching session, but should not be designed or interpreted as a straitjacket. As a teacher, it is important to be able to improvise, and to act and react on students' – always surprising – contributions.

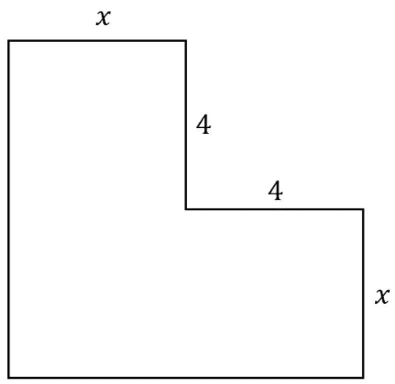
After reading this compendium you still might feel some doubt and difficulties while designing your activities and lessons, but you will have gained a language to discuss educational designs and the process of designing.

The challenges and choices of design: three example

To illustrate the freedom of choice of approach educational designers have, below we present three examples of a task design for completing the square.

Example 1. The first example illustrates the principle of guided reinvention. Moreover, it shows how a geometric model can be used to develop algebraic understanding. Step by step students work their way through an example. After (g) the method is set out more definitively through a worked example.

Solve the following exercise in pairs. Check your answers with the answering model.
An L-shape is obtained by removing a square with side 4 from a square with unknown side.

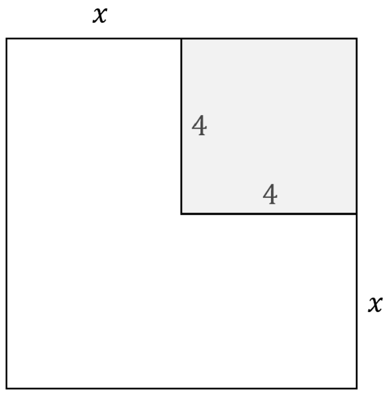


a) Express the area of the L-shape in x.
b) Compute x, if the L-shape has area 20.
c) Compute x, if the L-shape has area 33.

There is also a value for x such that the area is 25. This value you cannot find by solving $x^2 + 8x = 25$, reducing to $x^2 + 8x - 25 = 0$, and decomposing.

d) Check that $x = -4 + \sqrt{41}$ is a solution

How did we find this solution? To do this we complete the L-shape to a square:



e) What is the area of the added square?
f) Express the side of the big square in x. Square this expression to find a formula for the area of the big square.

The area of the L-shape is the difference of the area of the two squares

g) Fill in: $x^2 + 8x = (x + 4)^2 - \cdots$

We solve $x^2 + 8x = 25$ as follows

$x^2 + 8x = 25$ $(x + 4)^2 - 16 = 25$ $(x + 4)^2 = 41$ $x + 4 = \sqrt{41}$ or $x + 4 = -\sqrt{41}$ $x = -4 + \sqrt{41}$ or $x = -4 - \sqrt{41}$	Use g) add +16 both sides square root on both sides -4 on both sides

h) Solve $x^2 + 6x = 12$ in the same way.

Example 2. The second example illustrates how emphasis can be shifted from the algebraic proficiency of the student to the logical order of a solving process through "completing the square". The cards add an element of play to the task and promote discussion in pairs. To some extent the cards allow students to check their own answers. Task (b) and (c) give more challenge, depth, and an opportunity to solidify the procedure.

In pairs you'll receive a set of cards as in the table below. The cards show steps in the solution processes of the equations $x^2 + 6x = 12$ and $x^2 + 8x = 25$.

$x = -4 + \sqrt{41}$ or $x = -4 - \sqrt{41}$	$(x + 4)^2 - 16 = 25$
$x^2 + 6x = 12$	$(x + 3)^2 = 21$

$x+3=\sqrt{21}$ or $x+3=-\sqrt{21}$	$x^2+8x=25$
$(x+3)^2-9=12$	$x=-3+\sqrt{21}$ or $x=-3-\sqrt{21}$
$(x+4)^2=41$	$x+4=\sqrt{41}$ or $x+4=-\sqrt{41}$

a) Put the cards in the right order.
b) Explain what happens in every step.
c) Solve in a similar way $x^2+10x=22$.

Example 3. The third example illustrates the principle of worked examples. The student is invited to follow the calculation and reasoning of a worked example, which is presented in combination with the general method. In the third column the student mimics the examples. This task is more suitable to be done individually, without teacher's guidance. It might only support more superficial understanding.

In this task we teach you a new way to solve quadratic equations like $x^2+8x=25$.

a) Fill in the empty spaces in the right column.

Step	Example	Fill in
Begin with an equation of the form $x^2+p\,x=q$	$x^2+8x=25$	$x^2+6x=12$
Take half of p and introduce brackets: $\left(x+\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2=q$ Where you subtract $\left(\frac{p}{2}\right)^2$ to have an equivalent equation.	$(x+4)^2-16=25$	$(x+\ldots)^2-\ldots=12$
Reduce to brackets left and numbers right of the equality sign.	$(x+4)^2=41$	$(x+\ldots)^2=\ldots$
Take positive and negative the square root	$x+4=\sqrt{41}$ or $x+4=-\sqrt{41}$	$x+\ldots=\ldots$ or $x+\cdots=\ldots$
Isolate x	$x=-4+\sqrt{41}$ or $x=-4+-\sqrt{41}$	$x=\ldots$ or $x=\ldots$

b) Now apply the same method to the equation $x^2 + 10x = 22$

Designing for mathematics education requires a careful consideration of what you want students to do and learn and how this can be supported by tasks and teaching methods. Many aspects may play a role. What do your students need? How do I accommodate for individual differences? How do I vary activities? How do you adapt for the time of day, week, or year? How much inquiry, reinvention, and higher order thinking do I want to invite to? Do I want to stimulate cooperation? Do I want elements of play?

In the TIME project we develop tasks and lessons that promote inquiry-based learning. In this guide design features that support inquiry will therefore be quite prominent. Nevertheless, most aspects of design that we discuss will apply to other educational designs.

1. Shaping the design process

1.1 Design as a process of inquiry

As is mentioned in the document *TIMEless: A short introduction to Lesson Study – TIMEless ideas for professional development* (Bašić, 2020):

> There is a striking similarity between Lesson Study as an activity for teachers, and the experiences aimed at for students in inquiry based education: namely, the principle that you learn from studying a problem through experimenting hypothetical solutions. For teachers, problems are related to students' learning (with specific and more general goals), and you keep finetuning your experiment until you are ready to share your findings with others (p.12)

This process of inquiry is not limited to the context of Lesson Study. Every process of design should be a process of inquiry, in the sense that you go through stages like the ones in Figure 1.1.

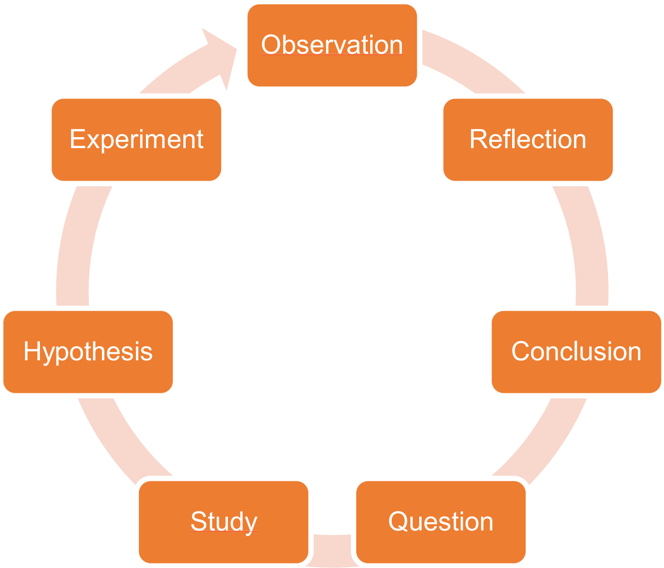


Figure 1.1. A cyclic model for phases of a process of inquiry

Let's describe how these stages fit into this cycle. The need for new design comes from a *conclusion*, based on the *reflection* on an *observation* of students' or teachers' behaviour: obstacles they meet, difficulties they have in the process of learning and teaching, the learning goal you aim at and conditions like time, space and resources available. But sometimes, it also comes from observed changes in the national curriculum, or new goals and directions chosen at the school level.

The observed situation gives rise to concrete *questions* with respect to learning and teaching. The *hypothesis* would be a task, lesson plan, or hypothetical learning trajectory, designed on the basis of a study. The *experiment* is the actual lesson; which is then *observed* and *reflected upon*, from which *conclusions* are drawn. These may, in turn, leave certain issues unresolved or give rise to new questions, and hence a second cycle could begin. This compendium focusses on how the questions lead to a hypothesis on basis of a study.

Concrete design advice
- See your design as an inquiry, an experiment, a continuous, cyclic process
- Test your hypotheses about how your design contributes to learning, and use your insights to improve the design of your teaching

1.2 Design as part of Lesson Study

Lesson Study provides a perfect framework to implement a continuous, possibly cyclic, design process. Design happens on many scales ranging from the individual teachers – working to improve lessons and lesson materials – to the national level where experts develop a curriculum and curriculum resources. In Japan Lesson Studies take place on three levels: at schools, on district level, and on a national level. Sometimes, if time permits, lesson designs are improved and implemented another time. Moreover, in all these situations (new) insight and knowledge into teaching can be acquired.

In Figure 1.2 you find the phases of a Lesson Study as described in TIME Lesson Study Guide.

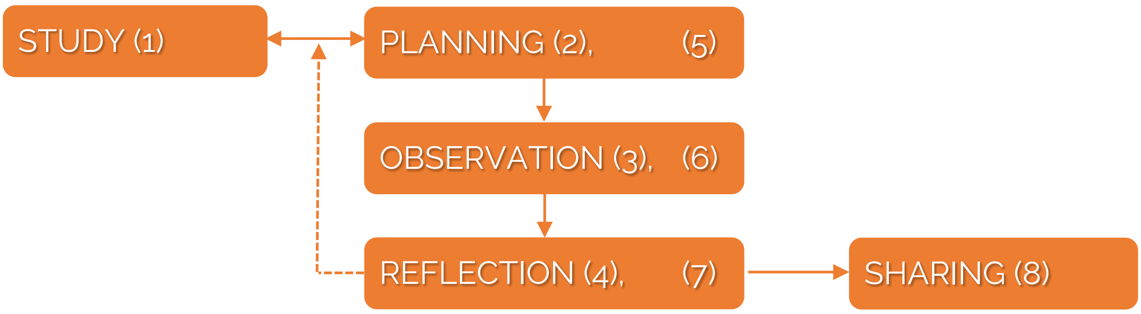


Figure 1.2. Phases of Lesson Study, taken from (Bašić, 2020), an adaptation of a diagram by Stigler and Hiebert (1999)

See how Figure 1.2 fits in with Figure 1.3: "Reflection" in 1.3 includes "Reflection and Conclusion" from Figure 1.2, and similarly "Study" includes "Study and Question", "Observation" includes "Experiment and Observation", whereas the Planning in 1.3 is the forming of a plan or "Hypothesis" in 1.2. The model in Figure 1.3 emphasizes how the period of "Study" and "Planning" may overlap.

The study phase that precedes the design of a lesson plan is called *kyozaikenkyu* in Japanese Lesson Study terminology. One can distinguish four steps in this study (Watanabe, Takahashi, and Yoshida, 2008): understand the scope of the topic and how it fits into a larger teaching sequence; understand students' pre-knowledge; understand the mathematics of the topic; and explore potential problems, activities, and manipulatives. It is interesting to see how these steps match the phases of western traditions of analysis for educational design. These steps of study – or analysis – will be discussed in more detail later in this guide. The result of this study leads to the production of a *gakushushido-an* (lesson proposal). According to Lewis (2002) this could comprise many things: the name of the unit, the unit objectives, the research theme, current characteristics of students, the learning plan for the unit, which includes the sequence of lessons in the unit and the tasks for each lesson, plan for the research lesson, which includes (aims of the lesson, teacher activities, anticipated student thinking and activities, points to notice and evaluate, materials, strategies, major points to be evaluated, and copies of lesson materials, e.g., blackboard plan, student handouts, visual aids), and background information and data collection forms for observers (e.g., a seating chart). This can be compared with the "modules" that were produced in the MERIA-project[1]. In the TIME-project most of these aspects are part of the TIMEplate[2].

1.3 Outline of a design process

The phases Study and Planning of Figure 1.2 encompass the design process. In the scheme below the aspects of this design phases are visualized, starting from the educational problem to be inquired towards the actual design process (Figure 1.3).

We acknowledge that educational design could last a long time when all these aspects are addressed in full detail. Moreover, the boundaries between the aspects are fluent: for example ideas on what to teach (mathematical analysis) and how to teach (didactical analysis) sometimes follow from the same study or reflections. Nevertheless, awareness of these aspects is useful. For some designs the emphasis might be on a mathematical analysis, while others might need more a didactical exploration. In the following sections the successive stages are briefly addressed in connection with potential design tools. In section 2 these stages are further elaborated.

Concrete design advice
Educational design consists of various phases: setting goals, mathematical analysis, didactical analysis, preparation of design, and actual design. Plan to spend time on each phase.

[1] https://meria-project.eu/activities-results/meria-teaching-and-learning-modules
[2] https://time-project.eu/en/intellectual-outputs/template-lesson-plans-and-practice-report

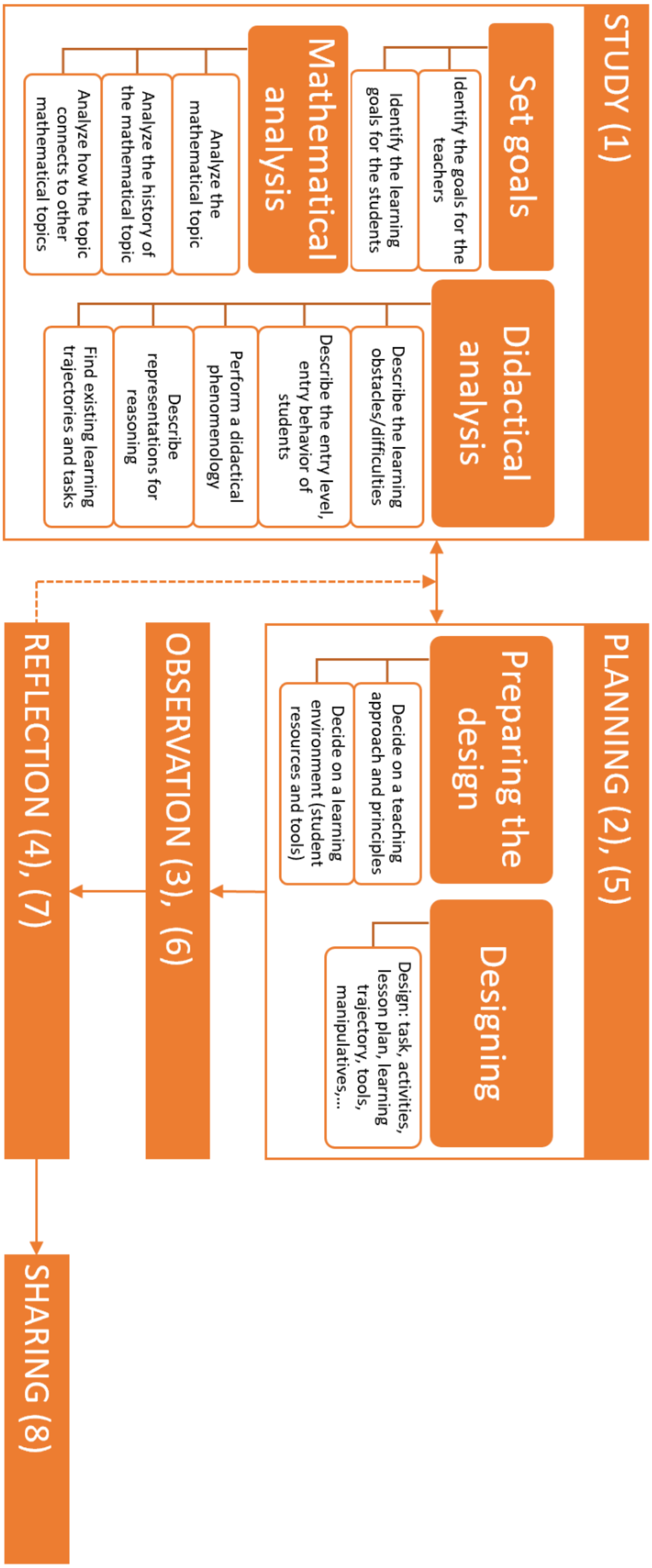


Figure 1.3: Stages of a design process. These are outlined in the next section.

2. Analysis for design

In this chapter we discuss the scheme in Figure 1.3 in more detail. Designing a task or activity can be motivated in many ways, for example initiated by a question or a need for an alternative approach to a certain topic, or a desire to foster a specific skill – possibly not covered by the textbook. The Lesson Study Guide (Bašić, 2020) contains an extensive discussion on the process of arriving at goals for a design. Before you start with designing it is helpful to perform an analysis from a mathematical perspective and from a didactical perspective. These forms of analysis are addressed below.

2.1 Mathematical analysis

A mathematical analysis focuses on a description and analysis of the mathematics of the learning goals. New ways of understanding the mathematics by the designer may inspire designs for new ways of teaching: meanings that teachers attribute to mathematical topics have a strong influence on meanings their students will develop (Thompson, 2013). Such an analysis may also inspire to new situations from which the mathematics arises, and it helps to understand how the topic is related to other mathematics.

Levels of understanding

Mathematical concepts often have several levels of understanding (e.g. Tall, 2013). For example, look at the notion of function. A basic understanding of functions is as an input-output machine, for example a [times 4]-machine: put in number 3, out comes number 12. A next level could be a description of such a machine by an equation using an input-variable x and an output-variable y: $y = 4x$. A next level could be that you name functions (e.g. f) and see them properly as objects with properties: $f(x) = 4x$. Next could be the foundation of functions in set theory: a function $f: X \to Y$ is a set of ordered pairs in $X \times Y$, so that for every $x \in X$ there is exactly one $y \in Y$ that together form such a pair (x, y). There may even be more levels. Learning often is supported by a progression along such levels from situational and concrete to general and abstract.

The point is that the designer benefits from being informed by such a wider picture of the concept. This picture, with connections between representations, language and ways of working on each level, informs the designer for making a choice in a mathematical approach of the concept; to find a level that is most suitable for the student. It allows the designer to consider how the task at stake may be part of a more encompassing learning trajectory in which the concept is developed.

Relation to other mathematical concepts

Mathematical concepts never live on an island: The understanding of a concept depends on the understanding of many other concepts. As a designer it can help to visualize the concept in a concept map (se Figure 2.1), in which you chart the relation of the concept to other concepts. This can, for example, be done using software from https://cmap.ihmc.us/. A conceptual analysis identifies the concept(s) at which the design focuses and possible choices to limit the content (Novak & Canas, 2008). So, concept maps, as design tools, help to visualize the targeted conceptual structure as a step in the design process. They can also be used as a teaching tool to demonstrate or generate the conceptual relations and provide a framework for learning, or to monitor and assess the students' conceptual structure and to help them to become aware of it.

For example, the notion of a function is closely tied to the notion of a graph. The set theoretic definition of the graph of a function $f: X \to Y$ actually leads to exactly the same set as defines the function:

$$\text{graph}(f) = \{(x, f(x)) \in X \times Y |\, x \in X\}$$

This leads some to the viewpoint that the visual graph (e.g. parabola) and an explicit symbolic description ($y = x^2$) are two representations of the same thing. An (incomplete) table, like

x	-2	-1	0	1	2
y	4	1	0	1	4

is seen as another (partial) representation of this same underlying object. Mathematical concept can be closely related, even seen as representations of the same underlying object, and should therefore not be taught in isolation. On contrary, their juxtaposition in a design supports learning.

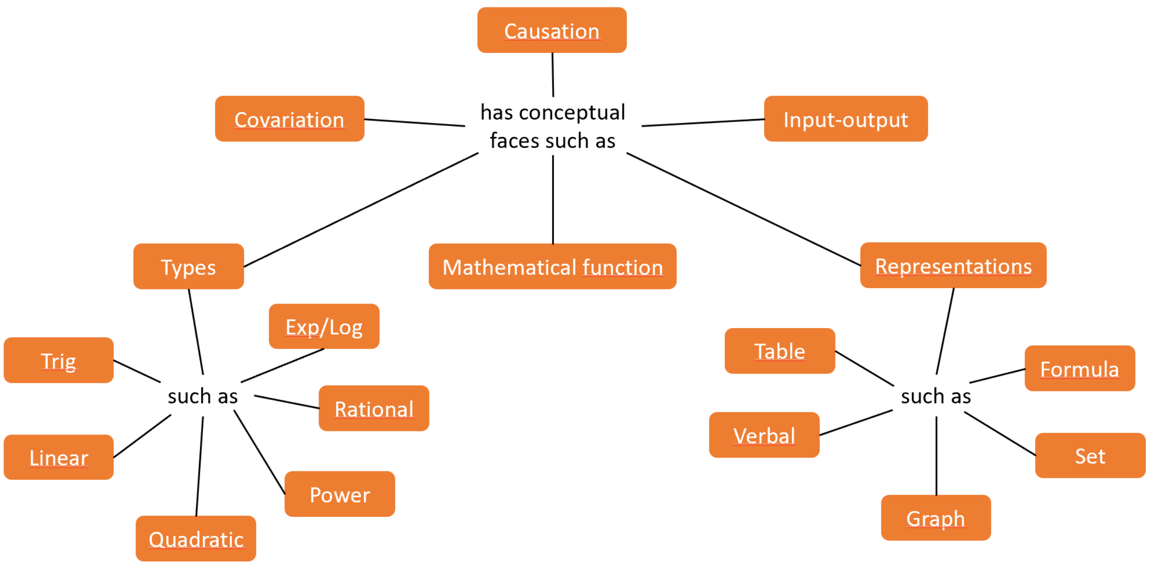


Figure 2.1. a concept map of the notion of function

Concept maps can also be used in teaching. For instance, in a classroom discussion, by collecting and grouping words that emerge when talking about a concept, or as a (formative) assessment task when the teacher provides parts of a concept map together with empty spaces and some words to be used, and asks students to fill in the gaps.

Pre-knowledge

Within a network of related mathematical concepts, it is important to identify the ones the students should already be acquainted with: their pre-knowledge. Any gap in pre-knowledge has consequences for the learning process. A design should, if possible, include tasks that activate or assess necessary pre-knowledge of the students. Pre-knowledge needs to be formulated not exclusively in term of formal mathematical knowledge (language and representations). The designed tasks are more likely to be effective when they also connect to students' informal knowledge and real-life experiences. Formal mathematics becomes meaningful by grounding it in meaningful previous experience and pre-knowledge.

The iceberg metaphor visualizes the following idea: students' more formal knowledge of mathematical manipulations, as, for example, expressed in their notebooks—the visible top of the iceberg—should be supported by previous experiences and pre-knowledge—invisibly, under the water surface (Webb, 2017). So, the visible more formal mathematical manipulations are kept afloat (i.e. meaningful) by underwater floating capacity formed by informal experiences and knowledge (see Figure 2.2). The designer needs to look under water for those experiences, and find tasks that reactivates them.

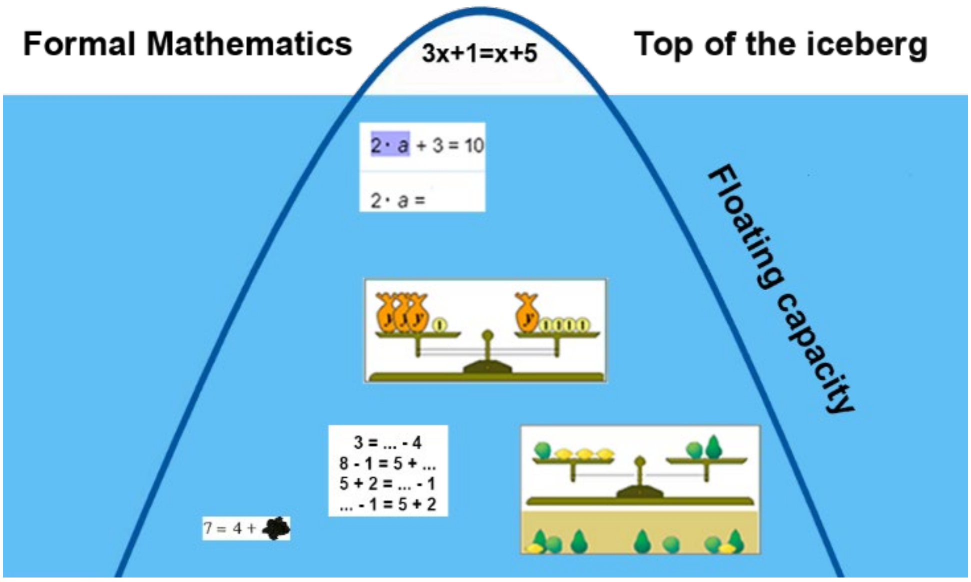


Figure 2.2: An Iceberg model for equations

For example, in Figure 2.2, the ability to formally solve linear equations is supported by more informal tasks with scales, beginning with the relative weights of pears, lemons, and bananas, and then more abstractly, with bags of an unknown number of balls. Additionally, this is supported by solving sums with one number blanked out, and later the method where part of a sum is blocked, to solve for what is underneath.

The iceberg helps to identify potential pre-knowledge that might be revisited or needs to be (re)emphasized. Furthermore, one needs to be aware that icebergs for topics can be connected (e.g. Figure 2.3). This awareness, and also the activity of making the floating capacity and these connections explicit is an important activity for teachers and for students.

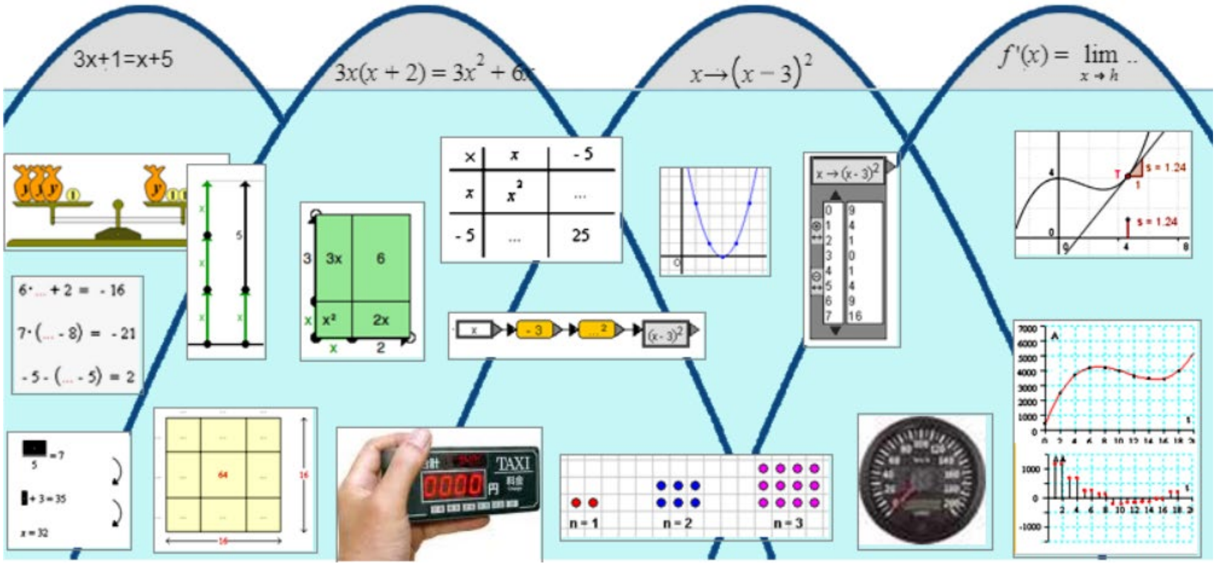


Figure 2.3: A cascade of connected Icebergs surrounding the derivative

Historical analysis

In preparation of a design one could study the origins of the mathematical topic. Knowing where a topic comes from, to what purpose it was developed and what problem it solved, can inspire task design. It can also support a didactical phenomenology (see next section). For example, Galileo developed mathematics of change in the context of investigating free fall, and Euler and Johann Bernoulli thought of a function as an expression in variables and constants. This may also be a good entrance to the concept for students, even though it is not consistent with modern notions and experiences. Clairaut and Euler introduced the notation $f(x)$, which was an important step as it introduced the idea of naming functions. This could similarly be emphasized as an important step in the students' development. When a historical sequence of events guides your design, this is also referred to as a genetic approach.

Concrete design advice

- Analyse the mathematical topic, learn more about it
- Think about the levels at which the topic can be taught
- Make a concepts map for the involved concepts, including relation to other topics
- Chart the pre-knowledge, including an iceberg model for the involved underlying informal knowledge
- Look into the history of the topic

2.2 Didactical analysis

A didactical analysis focusses on the learning and teaching of the mathematical topic. What makes it hard to learn? What makes it hard to teach? What is known about the learning and teaching of the subject? Where lie the opportunities to explore in the design?

The entry level of students also is part of a mathematical analysis: finding out how concepts and propositions depend on other concepts and procedures. In addition, when students have been taught topics that are fundamental, we should also take into account what the level of 'familiarity' is: do they (still) really understand the topic, do they feel comfortable with the topic or is it still challenging? This affective dimension is also relevant when building on prior knowledge. Furthermore, seen from a didactical perspective one should consider whether students have experiences in real life (i.e. with phenomena) or in other subjects like physics or biology that provide starting points for learning. In particular for mathematics education, the role of representations is fundamental for expressing patterns and structures in space or number, and for communicating about new generalizing concepts or procedures. Thus finally, we will address the analysis of representations and possible progressions of representations from informal to formal.

Analysing the teaching-learning situation several aspects that you have also seen in the description of the *gakushushido-an* (lesson plan) appear. We come back to these aspects in the sections on the preparing the design and design phase, in particular with respect to IBMT.

Learning obstacles

Learning obstacles are the aspects of a topic that cause students problems to learn it. It is important to identify the learning obstacles of the topic that the design addresses. The obstacles may have already come up when formulating the challenge and determining the goal for the design. The design team could produce a list with obstacles based both on teachers' personal experience and findings from educational literature. At the surface are the mistakes that students often make, and below the surface there might be the underlying obstacle that is the cause.

A well-known example of a learning obstacle is the *illusion of linearity* (De Bock, Verschaffel, & Janssens, 1999). Students have the tendency to assign the property of linearity to functions that don't have it. For example: $(x+y)^2 = x^2 + y^2$, $\frac{1}{x+y} = \frac{1}{x} + \frac{1}{y}$, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, and this tortuous list goes on and on. More generally, students tend to overgeneralize rules, and many other explanations can be given.

The challenge might stem from the nature of mathematics or the order in which the curriculum introduces concepts to students. Brousseau has studied such phenomena in mathematics education and described them under the name "obstacle" (2002). For him, an obstacle is a piece of knowledge that applies correctly in one context but shows to be wrong once the context is generalized (expanded, brought to a higher level of complexity). For example, it is true that every integer has an immediate successor, but this claim is not true (we could even say it does not make sense) once we consider the bigger set of rational numbers. Similarly, multiplying a positive number with a positive integer will yield a bigger number, but this is not true anymore if we multiply a positive number with a positive number (a fraction or a real number) less than 1. A little bit different example comes from the French educational system: there is a strong emphasis to use decimal numbers and approximate values from a very early age and students are used to results with two decimal digits, so when they encounter irrational numbers many misconceptions may surface (such as tendency to think and use that pi is *equal* to 3.14).

Knowledge of these obstacles should impact the design: the obstacles should be addressed. One way to address obstacles in a design is to ask a question that puzzles the students because of the obstacle, and challenges them to adopt their previous conceptions A first draft of the design should be checked with the list of potential obstacles.

Didactical phenomenology

A didactical phenomenology is an investigation into situations that "beg to be mathematized" by the target knowledge. This means that the natural mathematical approaches to the situation are towards the mathematical topic to be addressed. These situations can be found outside or within mathematics. Let's give two examples.

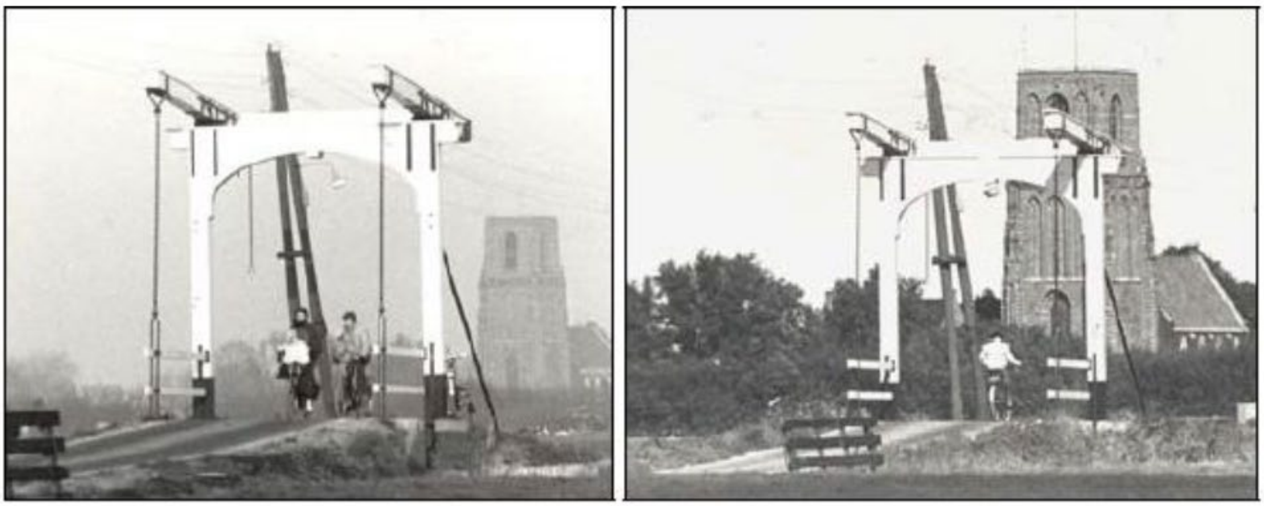


Figure 2.4. Which is higher, the bridge or the tower? Why?

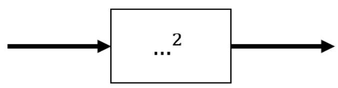
Example 1. In the MERIA-practical guide the following task based on the two photos in Figure 2.4 is presented. The question is: which s higher the bridge or the tower. This situation begs to be mathematized by the notions of sight lines and scaling. There is simply no way of addressing the issue without developing these notions, even if students don't name them that way, or reinvent the full mathematical concept.

Example 2. To illustrate that a context can be purely mathematical, consider the world of integer sequences. A familiar game or challenge is to be presented with the beginning of a sequence 1, 3, 7, 15, 31,... and then guess the next one. Students can try to solve or create such puzzles. A next step could be to classify sequences into various kinds, arithmetic or geometric – but at first in their own terminology. This context begs to be mathematized by recursive and/or direct formulas.

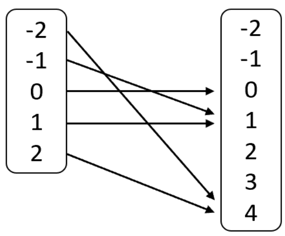
As is clear from these examples, contexts can be used in a fruitful way, offering students meaningful situations as a starting point for mathematizing. The context has features that support informal and more formal mathematical reasoning. So the context is not just dress up, or just showing that the mathematics is useful, it invites and supports mathematical reasoning by being familiar and meaningful to students.

Representations

Another aspects to analyse as educational designers is the representations that are associated to a mathematical topic. For functions this may be: the functional notation $f(x) = x^2$, or some notation involving an arrow $f: x \to x^2$, graphs (the picture of a parabola), tables, pictures like



or



The understanding of an aspect of a topic may be based on a certain representation. So being able to reason about injectivity using the last diagrams does not imply that students are able to reason about injectivity using a graph or a formula. Educational design should aim to address multiple representations and support students to connect them. Note how representation usually play an important role in the iceberg model.

How has the educational challenge been addressed before?

Even though it can be interesting and joyful to reinvent the wheel, and reinvention can be a good learning experience, it is often not necessary, and slows teachers' development of their practice. Knowledge about how to address educational

challenges can come from (academic) literature and from experienced teachers—as we shall also discuss when introducing the notion of design principle. It can be beneficial to discuss the design issues with experienced teachers, within or outside the design team. Additionally, on many educational topics there is a wealth of academic literature. Fortunately, more and more publications are open access these days. Additional resources could be for example teachers' magazines and websites.
In the example below we illustrate how various parts of the analysis influenced the design choice for the slide scenario of the MERIA-project.

Example: Slide task (from the MERIA-project).
This task that aims at introducing the derivative of function. We describe how aspects of study informed the design.

Mathematical analysis
The concept of derivative builds on many subjects previously attended to at school: among others, function-concept (see Figure 2.1), linear functions and their slope, sight lines and tangent lines, growth, proportion. Figure 2.2 shows an iceberg analysis of the derivative.

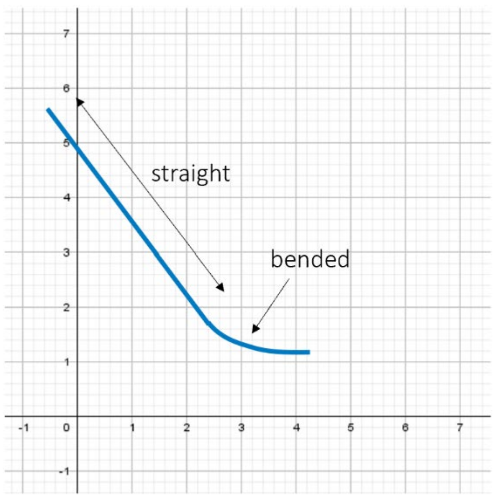
A basic level of understanding of the derivative in a point from a geometric point of view is that it is the slope of a tangent line at that point. From a non-geometric point of view this is the instantaneous growth of a quantity. In textbooks both points of view are usually addressed. A next level is to view the derivative as a function associating slope to an input, or value of a quantity. A next, higher level is to understand the derivative in a formal definition using limits. The slide task addresses the geometric point of view on a basic level.

An interesting finding of a historical analysis is the following. The modern definition of derivative by Newton and Leibniz using limits was preceded by (1) Euclid, who already defined tangent lines in a geometric way (2) mathematicians like Descates, Fermat, and Hudde who computed the slope of polynomial curves in an algebraic way. This opened the designers' eyes to the possibility that such an algebraic approach might be more intuitive for students. Let us briefly illustrate this approach: The line described by $y - 1 = m(x - 1)$ is tangent to $y = x^2$, exactly when the intersection at $(1,1)$ is a double point, that is, when the equation $x^2 - 1 = m(x - 1)$ has $x = 1$ as a solution with multiplicity 2. Rewriting the equation as $(x - 1)(x + 1 - m) = 0$, reveals that this is the case iff $m = 2$. This indeed is the slope of the parabola at $(1,1)$. Actually, this approach has been transformed into a lesson series by Michael Range (2018), which is something to discover looking into prior educational approaches to the subject – which is part of the didactical analysis.

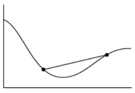
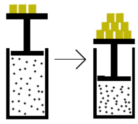
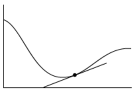
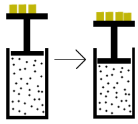
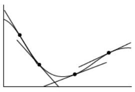
Didactical analysis

There are plenty of obstacles and difficulties for students with the concept of derivative. A major issue is for students to grasp and remember the meaning of the derivative (e.g. as the slope of the tangent line) after they have learnt the rules and procedures for calculation. Another issue is the limiting procedure involved in the definition: Often the derivative is introduced before limits could be properly defined. A well-known obstacle from the geometric perspective is that students consider a tangent line as a bounding line (a line interesting the curve just once), since the first tangent lines the encounter (for e.g. circles and parabola) are all bounding lines. The slide scenario aims very much at meaning making. Moreover, it invites students to develop the notion of slope of a curve in informal ways, postponing the issue of limits. The understanding of a tangent line as a bounding line is considered a vice: It is one of those notions under the water in the iceberg model.

The slide task was an immediate consequence of a didactical phenomenology. What could be a context that begs to be mathematized by "the slope of a curve"? What problem is solved by constructing a tangent line? What reason can we give to mathematize and, in particular, quantify this situation with a tangent line. This gave rise to the task of designing a slide with a bended bit and a straight bit, in such a way that the point where they meet is smooth, no bump. The request for a equations for the curve and line invites students to express the slope as a number.

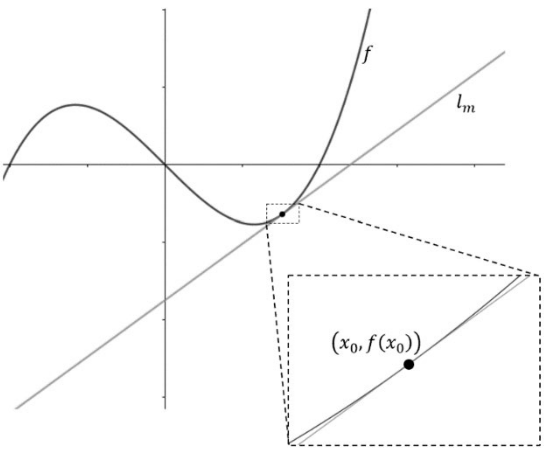


For the derivative the following scheme of representations can be found in academic literature (Round et al., 2015)

	Graphical	Verbal	Symbolic	Numerical	Physical
	Slope	Rate of Change	Difference Quotient	Ratio of Changes	Measurement
Ratio		"average rate of change"	$\frac{f(x+\Delta x)-f(x)}{\Delta x}$	$\frac{y_2 - y_1}{x_2 - x_1}$ numerically	
Limit		"instantaneous ..."	$\lim_{\Delta x \to 0} \cdots$	...with Δx small	
Function		"... at any point/time"	$f'(x) = \cdots$	... depends on x	tedious repetition

Note how here the representations are ordered and unravelled with respect to the limit-definition of the derivative. The scheme could support the teacher in addressing each step in this definition from these various perspectives allowing students to form a rich concept. The slide scenario precedes these considerations and is concerned with forming informal building blocks to which the more formal definition can be connected. However, we expect the representations from this scheme to occur in students' informal approaches to the task.

Finally, various non-standard approaches to teaching the derivative can be found in textbooks and academic literature. We already mentioned the algebraic approach by Michael Range. Additionally, Tall elaborated on an approach emphasizing that zooming in on a graph reveals the local linear character of a differentiable function (Tall, 1985).



By studying the various didactical approaches, the teacher and the designer anticipate a response to the various informal approaches.

Concrete design advice

- Determine the entry situation of the students and the goals from a cognitive and affective point of view
- Find the learning obstacles in the topic from experience and literature
- Perform a didactical phenomenology
- List the involved representations of the involved concepts
- Find out how the educational challenge has been addressed before. Consult experienced teachers and academic and/or other literature, websites,...

3. Design principles

3.1 An introduction to design principles

A design principle is a way to summarize an advice for design characteristics and procedures. In mathematics education the notion finds its origin in design research (Bakker, 2018). Based on such principles one could design (hypothetical) learning trajectories and develop theories on how to teach a subject (local instruction theories). A design principle does not specify the details but takes the form of a general heuristic of what to do in certain educational situations. It is always based on certain norms and values of what *good* education is. For this reason, design principles could contradict each other.

As a teacher, it is important to be able to articulate what is the justification for your decisions in particular when you design a lesson or task. In the same way, it is important for educational researchers to articulate their findings. For educational researchers it is common practice to express their hypotheses and findings in terms of design principles. In the TIME project we investigate whether teachers could express themselves using design principles as well. As such, design principles could form building blocks for communication amongst teachers, and between researcher and teachers; they serve as a means to structure thoughts and discussions on educational design.

So, design principle express researchers' or teachers' (as we propose) knowledge about education. This knowledge develops from research and experience, in particular from Lesson studies (for teachers) and from design-based research for researchers. A design principle not only expresses what we should design, but also how, when and why; that is, it expresses a rationale for the design. In this way, it brings researchers and teachers together facilitating not only on the practicalities of an educational design, but also on the possible underlying theoretical considerations. This is summarized in the diagram of Figure 3.1.

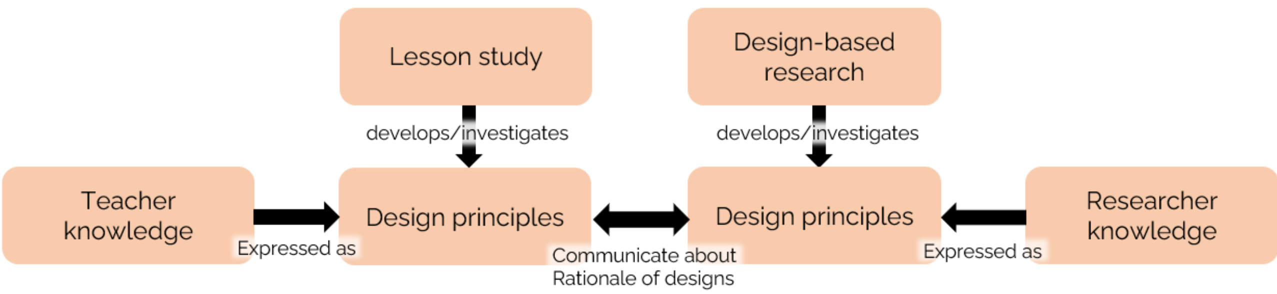


Figure 3.1. How teachers' and researchers' knowledge could be shared and compared using design principles

Example of a design principle:

The activity principle. Teach students to mathematize (to do mathematics) – instead of teaching them ready-made mathematical results. To mathematize means both to translate real-world situations into mathematical representations and to develop those representations further into more abstract and formal mathematics (to prevent mathematics as an isolated discipline, an isolated set of concepts and procedures). So, the task or lesson should not be based on a ready-made mathematical result presented to students. Instead, the task should allow the students to develop this result themselves.

Van den Akker (2013) proposed a general (and rather mathematical) form for a design principle. We present it a bit less formally.

A **design principle** is an advice that

A. is concerned with the *characteristics* of tasks, lessons, lesson series, or curriculums (**what?**)
B. may include *procedures* to arrive at such characteristics (**how?**)
C. may include *conditions* about when it applies, in particular it may relate to the goals of the intervention (**when?**)
D. is based on *theoretical arguments* for those characteristics and procedures (**why?**), and/or
E. is based on *empirical arguments* for those characteristics and procedures (**why?**)

Let's apply this to the above example, the activity principle. The activity principle is formulated in terms of the general characteristics of a task or lesson, and not with the specifics—it applies to more than just a current design. It gives some clues on procedures to develop this principle: students develop results; teachers do not present them ready-made. The activity principle, as phrased above, does not give any specific conditions to when it applies – and actually this is an interesting point of discussion. Since developing results can be time-consuming, some have practical objections against applying it too often. The counter argument is that mathematizing realizes deeper learning, so you learn more through fewer activities. Moreover, the activity principle need not be invoked for every task, but maybe in particular for tasks where meaning-making is a challenge. This example, in turn, illustrates how formulating a design principle can be the starting point of a discussion on choices in design.

The activity principle, in overview.

- Characteristic: students mathematize, develop a result themselves.
- Procedure: finding a suitable situation for mathematizing; setting the problem in the right way.

- Conditions: you need enough time! Mathematization involves developing new mathematical ideas.
- Theoretical argument: TDS and RME both provide theoretical support for the activity principle.
- Empirical argument: our test implementations in the MERIA-project showed in various scenarios that students can mathematize and jointly arrive at a result.

Below is a list of questions that relate to design principles that a team could ask themselves during a design process.

As you work with colleagues and you make decisions on tasks and lesson design, you could ask yourselves the following questions: **What** are we doing, described generally? Are we following a principle or a rule? **How** are we doing it? **Why** are we doing it this way? **Why** do we think this will work? Would it matter if we did it differently? Does this align with any theoretical knowledge from mathematics education? Does it align with the practical knowledge of one of us? Are we basing ourselves on a principle that is new to us?

Ask these questions, for example, for the use of tools, contexts, the problem itself, work in groups or not, etc.

In particular, it is important to coin the question: why do we think this will work? In a lesson study this allows to formulate more precisely which underlying ideas are at stake in the design. During the observation of a study lesson, observers could try to see whether the formulated principles work out in the anticipated way. In this way, the participants not only learn about their specific designs, but also about the underlying principles.

Design principles are based on two knowledge bases regarding teaching practices. Knowledge or advice is gained from theoretical arguments, for example taken from educational literature, and from empirical arguments. The latter could refer to the experimental data in literature, but also to the personal experience of the teachers or designers. If the principle is discussed as part of a theory, then there is no need to add the theoretical arguments to each principle. Additionally, for example, sometimes the procedures need not be specified, because they are obvious from the characteristics. There are many theoretical and empirical arguments for the activity principle; to mention two: actively developing a mathematical result allows students to develop it based on results that are meaningful to them, thereby leading to the new result being meaningful. Students that develop a result themselves experience ownership, which, in turn, results in a positive attitude towards the result, towards their own mathematical abilities, and towards mathematics in general. In literature on RME one can find many examples of the activity principle implemented, e.g. (Gravemeijer, 2020), and within the TIME project team many members have personal experience with this.

The example below highlights how the emergent modelling principle from RME shaped the slide scenario that was discussed before.

Example: Slide task (from the MERIA-project).
The intervention is a reinvention task aiming at the concept of tangent line and slope of a curve.

Task: design a slide by creating equations for a line and a bended curve that meet smoothly.

Design principle. One of the principles for this task design is the emergent models principle. This principle focusses on the purpose of reinvention of mathematical concepts: learning trajectories in which a concept is developed. The principle describes the characteristics of the problem situation and how the teacher should go from there. The problem situation should invite students to develop informal models based on what is meaningful to them. The teacher should then facilitate the process in which the students develop these models into more formal mathematical models.

The designers should thus invest time in finding situations that provide starting points for students to develop those informal models. The designer should try and predict what those informal models would be and how a process could be facilitated to develop those models. These steps could be seen as the advised procedures. The slide task has been shown to offer enough opportunity to students to develop informal models on tangent lines and slope.

The theoretical arguments for the emergent models principle include a detailed description of the various stages of development of a model: from informal to formal. These stages have been observed in teaching experiments and reported in literature, e.g. in (Gravemeijer, 2020).

Concrete design advice

- Base the design not just on ad-hoc decisions, but on design principles. For almost any aspect of the design ask the questions: why do we do it this way? Why do we think this will work?
- Formulate those principles in terms of desired general characteristics and describe procedures to arrive at such characteristics
- Make sure you can justify those principles by including supporting theoretical or empirical arguments
- While observing a lesson and discussing it afterwards, not only learn about the lesson, but also about the underlying principles.

3.2 Preparation for design: choice of design principles

Before concrete tasks are designed, the design team could settle on a teaching approach. A teaching approach can be supported by theoretical frameworks. In the TIME project participants promote and investigate Inquiry-Based Mathematics Teaching (IBMT): a "teaching approach that allows students to be engaged in an activity which leads them to adapt their existing, or construct new, mathematical knowledge" (Winsløw, 2017). According to Marshall et al. (2017), inquiry challenges students to explore concepts, ideas and phenomena before formal explanations are provided. Inquiry-based learning fosters students' understanding of the meaning and foundations of science (Furtak et al., 2012) and supports students' development of research skills (Gormally et al., 2009). For the MERIA and TIME projects we adopted Realistic Mathematics Education and the Theory of Didactic Situations as theoretical frameworks for designing for IBMT. In the next two section we rephrase those theories in terms of some design principles.

Design principles from the theory of didactical situations

As explained in the MERIA handbook, TDS models didactical situations as built from certain *phases* (devolution, action, formulation, validation, institutionalisation) which all require specific attention when designing the situation. Below the most important attention points are formulated as design principles for IBMT.

Target knowledge principle. An inquiry-based didactical situation is always designed for the students to achieve some specific mathematical knowledge, the *target knowledge*. The design process begins by a careful *preliminary analysis* of the target knowledge: How does the target knowledge relate to what students already know? How is it usually taught, with what results? What limitations and obstacles of the usual teaching methods should the inquiry-based situation help students to overcome? (If no serious limitations or obstacles can be identified, IBMT may simply not be needed in this case!)
Milieu principle. Inquiry based didactical situations are designed to let students build the target knowledge while interacting with a *didactical milieu*, consisting of a problem to solve, and some resources (interaction with other students, the knowledge the student group already has, and possibly also material artefacts, texts, computer technology) which they can use to solve the problem and thereby construct the target knowledge. The milieu can vary in different phases of the situation (action, formulation, validation) and is regulated by the teacher according to the designed lesson plan. An explicit analysis of how the students could interact with the milieu (hypotheses about their action, formulation and validation) is a crucial part of the lesson design.
Adidactic potential principle. In IBMT it is essential to devise situations that have *adidactic potential* which means that the milieus students interact with can provide *feedback* (to students' actions, formulations and validations) – so that they construct solutions to the problem by interacting with the milieu, *not only with the teacher*. Adidactic potential in validation situations are of particular importance, so that

formulations are validated against the milieu and not by force of the teachers' authority (in which case the validity of the target knowledge may rely only on the didactical contract – i.e. on trust in the teacher or, worse, in simple compliance with authority).
Devolution principle. To establish a truly adidactic situation, the students must accept that the teacher devolves responsibility for constructing knowledge to them – in contrast with the "normal" type of didactical contract where teachers communicate knowledge directly, and the students' responsibility is limited to applying it in action and formulation phases.
Game principle. For the adidactic potential to be realised, the situation must stimulate students to act as if they are trying to win a game with specific and clear-cut rules – not with the teacher, but with the milieu devolved by her. Winning the game means achieving (part of) the target knowledge. In adidactic phases, the teacher could at most recall the rules given in devolution phases. The explicit analysis of the milieu is supposed to ensure this is plausible, but it can of course only be discovered through experiments whether the analysis was correct.
Institutionalisation principle. When students have achieved (including validated) some piece of the target knowledge, the teacher must still connect students' results to "official formulations" of the target knowledge, which may differ at least in form and frequently also in generality from what the students constructed, but still relates to it in the point of view of the students. This is fully acceptable as long as the validation remains rational to students; it is necessary to ensure the students' new knowledge can be used in other situations.

Example: Slide task (from the MERIA-project).
Let us return once more to the slide task to illustrate some design principles from TDS.

One way to enrich the milieu of the slide task is to allow students to use GeoGebra (or another dynamical geometry tool). If students graph their solutions, they have a way to validate their results without help of the teacher (adidactic potential principle). Applying their everyday knowledge of smoothness, students can judge whether their solution is good or can be improved.

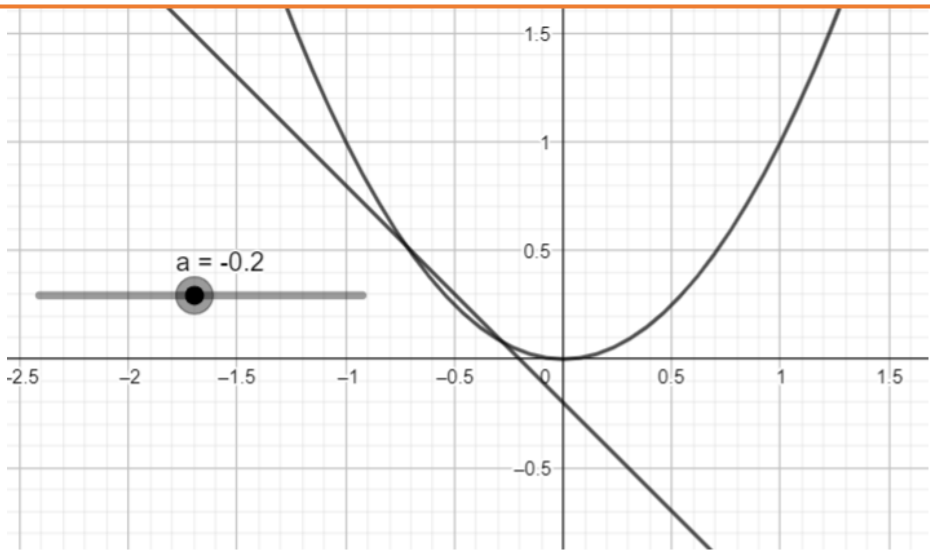


Figure 3.2. by dragging and zooming students can evaluate whether the line and curve meet smoothly and form a suitable slide design

The scenario for the slide task was designed according to the phases as advocated by TDS. The milieu is devolved to the students (devolution principle), that is, after sharing the problem, teachers refrain from scaffolding or helping students with respect to their problem approach. After students have worked on their approaches, their results are presented to the rest of the group, followed by a phase of classroom validation. In this phase the students evaluate the quality of the solution and of the chosen approaches. Which solution are good/better? Were these good solutions constructed in a sound way?

For the slide problem the winning strategy is to construct a tangent line to a curve in a point (or two tangent curves). The crucial ingredient for this is the slope of the curve at this point. Since understanding the slope of a curve is the learning goal the task, in this way the winning strategy relates to the learning goal of the task (the game principle).

To summarize, IBMT focuses, according to TDS, on realising *adidactic situations* of *action*, *formulation* and *validation*, that involve rich and carefully prepared (and devolved) milieus for students to interact with. Common teaching often begins with institutionalisation, followed by limited adidactic action and formulation phases, and finally didactic validation (by the teacher). More responsibility is thus to be assumed by students, with less apparent control for the teacher. However, a careful design – not forgetting institutionalisation in conclusion – can make up for that, even if realizing such situations is more demanding for *both* students and teachers, especially if most of their previous experience come from what we called "common teaching". The benefit of making such efforts is not minor: more solid and autonomous knowledge gained by students, as it is based on reasoning and mathematical necessity - rather than on memory, compliance and authority.

Design principles from realistic mathematics education

Below we present six design principles from realistic mathematics education (van den Heuvel-Panhuizen, Drijvers, 2020). The presentation here focusses on the characteristics and not so much on the procedures for implementation or theoretical and empirical arguments. More on the theoretical background can be found in the MERIA practical guide (Winsløw, 2017). More details on the implementation procedures, as well as empirical studies can be found in the many resources on RME, c.f. the references in (van den Heuvel-Panhuizen, Drijvers, 2020).

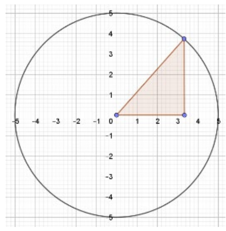
The activity principle. Teach students to mathematize (to do mathematics) – instead of teaching them ready-made mathematical results. To mathematize means both to translate real-world situations into mathematical models and to develop those models further into more abstract and formal models. So, the task/lesson should not be based on a ready-made mathematical result presented to students. Instead, the task should allow the students to develop this result themselves. (to prevent mathematics as an isolated discipline, an isolated set of concepts and procedures).
The reality principle. The starting point of teaching should be what is meaningful and relevant to students. A didactical phenomenology leads designers to find rich situations that allow for mathematical organization using informal version of the targeted learning goals.
The emergent models principle. Teaching a concept should progress from situational models – models that refer to the context in which they are learned – to more general models that are independent of those situations and can be applied in more general situations.
The intertwinement principle. Topics should not be taught in isolation. In contrary, many mathematical topics are heavily intertwined and should be taught that way.
The interactivity principle. Create opportunity for students to share and reflect jointly on the outcomes of mathematization.
The guidance principle. Provide the right amount of guidance. Task should invite and enable students to develop a multitude of strategies. The task should promote students' independent exploration and experimentation with the problem. During the inquiry process the teacher should guide the students - not by providing them with answers, but as an experienced co-researcher who poses questions and thus drives the inquiry process. To provide the right amount of guidance is a balancing act: with too much direction, students' inquiry is limited or even absent, thereby losing the learning potential. However, with too little direction students get stuck or lost, which eventually disengages them from the inquiry process. Let's note that RME and TDS (next section) take different positions on the aspect of guidance.

The next example illustrates how RME-principles can shape a task.

Example: Guided reinvention of the circle equation
This task was developed in preparation for the first Dutch national conference on Lesson Study in 2017.

Task: Find the equation for a circle with centre in the origin.

The guidance is designed in advance in the shape of cards conveying hints and heuristics. Students could come to the teachers table to receive a suitable hint card in case they got stuck. For example:

Draw a concrete circle with radius 5, and find points on it	Look at rectangular triangles within a circle, with one point in the centre, one on the circle and one on the diameter 
A general point (x, y) on the circle has horizontal distance $\|x\|$ and vertical distance $\|y\|$ to the origin	use the Pythagorean theorem

Instead of teaching the circle equation as a ready-made result, students are invited to develop the equation and underlying ideas themselves (activity principle). This affords students to understand why the equation works (instead of just knowing how it works – being able to produce an equation for a circle): they make the connections *circle – equal distance to the centre – distance computed using Pythagorean theorem*.

This task shows a certain interpretation of guided reinvention. Maybe, the teacher does not act as a co-researcher, but instead the teacher shares a crucial nudge, if students are stuck. It is interesting to note that this idea is in conflict with the devolution principle from TDS; more about this below.

Another principle at work here does not come from TDS or RME: the principle of heuristic support. Students are supported in a different way from simply providing the next step. Instead, the cards suggest students to explore concrete examples (left top), or instead look to apply a general technique (left and right bottom).

Design principles from other theories

So far, this guide has emphasized design principles for inquiry-based teaching, originating from RME and TDS. However, design principles could concern many aspects and types of teaching. Let us present one design principle from a completely different perspective.

In Clark & Mayer (2016) the authors present a theory of e-learning. This theory is explicitly built around multiple design principles, grounded in classical theories on human cognition. As an example, we mention two principles from this theory

The multimedia principle. Use words and graphics rather than words alone.
The contiguity principle. Align words to corresponding graphics.

Teachers' design principles

As pointed out in the introduction to design principles, teachers' experiences can form the empirical basis of design principles. The TIME project is a collaboration and exchange between teachers and researchers. Design principles from RME and TDS are "tested" in classroom during study lessons, but as well are teachers own design principles. Researchers that acted as commentators were given the opportunity to reflect on teachers' design principles and try to relate them to principles known from theory. Below we present a teachers' design principle as formulated during a TIME project course.

The dare to make mistakes principle. When students share their work, create a "safe space" in which they feel comfortable to present mistakes and experience how they can learn from them. How to do this? The teacher needs to recognize mistakes and explore them with the students. One way to stimulate this is to have a "Best mistake of the day"-prize. The teachers present TDS as theoretical background for this principle, since it emphasizes the importance of formulation and separates this from validation. From personal experience teachers motivate this principle by pointing out how this activates students, in particular also those with low self-esteem with respect to mathematics.

Another teachers' principle was: *The probability-through-repeated-experiments-principle.* Introduce probability and statistical notion by letting students perform repeated experiment, like throwing dice. One could argue that this is a more concrete implementation of the *reality principle* of RME. In fact, recent design research has elaborated this design principle into a learning trajectory, which has been studied in detail (Droogers, 2019). This is the type of connection that could be made (by e.g. the external commentator in a Lesson Study) between a teachers' principle and a principle from educational research to further enrich the former, in particular to shed light on the aspect of a design principle that is not always provided: why would this work?

Conflicting principles

Design principles, whether coming from theory or teachers, may be conflicting. Different norms and values— possibly expressed in different theoretical frameworks— may lead to conflicting design principles. Formulating design principles actually helps pinpointing the differences in a sharp way and attribute them to underlying values, and theoretical and empirical arguments. In this way, the formulation of design principles facilitates the designers in discussing their convictions and expectation with respect to a design.

For example, the *adidactic potential principle* from TDS seems to be in conflict with the *guidance principle* from RME. In TDS, it is essential that the student interacts with the milieu in an adidactic phase, without intervention of the teacher, whereas in RME the teacher is allowed to guide the inquiry process by posing questions and providing heuristic suggestions. From TDS it can be argued that providing hints is a form of continuous re-devolution of the problem, destroying the adidactic potential: by adapting the problem on students' demand the didactic contract is not challenged anymore. There is even a name for this phenomenon in TDS: the Topaz effect (Brousseau, 2002). However, recent research shows that students can be supported through heuristic/hint cards to improve their metacognitive skills and focus on general mathematical techniques as they try to solve problems (Bos, 2022). What can be learned from this is that a theoretical framework often forms a coherent whole, that does not allow simply adding in principles from another theoretical framework.

Concrete design advice

- For IBMT, implement design principles from TDS and RME.
- For other issues of design consult theory to find many other design principles.
- As a designer/teacher, formulate your own principles; use these principles to discuss a design in a group of designers; invite researchers to relate your own principles to existing theory.
- Beware that design principles can be in conflict. Try to investigate what underlying arguments and values cause the conflict.
- A coherent set of design principles from one theory does not always allow combination with any other design principle.

4. General design issues and choices

4.1. Choosing a problem for IBMT

Design for IBMT is often based on a *problem:*

> *In IBMT a problem is more than a certain task, exercise or an activity. A problem is open in the sense that it requires students to engage in experimenting, hypothesizing about possible solutions, communicating hypotheses and possible solution strategies, and maybe pose further questions to be studied as a part of a process of its solving.* (Winsløw, 2017)

Choosing or designing a problem for IBMT is a major challenge. Both RME and TDS have principles addressing this issue: in particular, the reality principle, the emergent models principle, and the game or winner strategy principle.

Additionally, we formulate some properties the problem should have:

- Open formulation: it should invite to a variety of strategies and answers based on students' prerequisites. The openness of such a problem prompts students' curiosity and imagination, which drives the inquiry process.
- A low floor and a high ceiling. It needs to be accessible to both the weaker and the stronger students so that both can be engaged in the inquiry process. If a problem is real and meaningful to students, this creates a low floor.

4.2. Choosing a context

Problems can be presented in a more or less elaborate context. As a designer, you can have various reasons for your choice. A didactical phenomenology may lead to a context rich in structures that may be mathematized. Such a design will appeal to students' horizontal mathematization or modelling skills. A context may be chosen such that it is meaningful to students: They recognize and are familiar with the structures that are to be developed mathematically. Meaningful contexts can also be fully mathematical, without direct reference to the real world. A context may also be chosen to prove relevance of the topic to students. It can be powerful if contexts are rich and broad enough to support a whole subject or a range of topics, like mechanics as a context for differential calculus.

There are many examples of ill-chosen contexts. A famous one is the taxi context, with start fare and a rate per distance. Why is this less strong? Most students don't use taxis. If they use taxis, their parents are more likely to pay. Hence, taximeters are not part of their everyday life, and even if they are, they do not excite or spark the imagination. Moreover, taximeters actually don't work this way, because they include more parameters, like time—although one could argue that often context need to be somewhat simplified to be accessible to students.

Vocational contexts could be interesting to study: how do modern professionals use a mathematical topic? The use of vocational contexts requires contextual knowledge and skills as well as connecting mathematical content-context knowledge and skills. It might be useful to interview a professional about these issues. One could investigate into four dimensions for designing tasks connected to the world of work: Context, Role, Activity and Product [3]. Designers could try to find authentic problems and reveal how mathematics is needed to solve them. The goal is to identify activities carried out by workers in the workplace (possibly with use of authentic tools) that translate to the classroom situation. The designer could analyse the professional role the workers have, and see if students could fulfil similar roles in classroom, and finally, try to see if the professional fabricates a product, that could be produced in classroom.

Example. The work of the architect contains many mathematical aspects. The most accessible activities for secondary school students would be the designing and technical drawing. The role of communicator to the client can also be translated to the classroom situation. So, a suitable product might be a drawing of a design, including explanation of the choices made. This may lead to the task we sketch below.

Task: The owner of an apartment building wants to build a new parking lot. You are the architect who is given this assignment. Your task is to design a parking lot meeting the requirement… The product you need to deliver is a technical drawing of your as well as a letter to the owner of the building explaining your design and the decision you made. The full task with a floor plan as a workplace artefact can be found here: https://www.fisme.science.uu.nl/toepassingen/22015/.

Concrete design advice

- For IBMT, choose problems with an open formulation, and a low floor/high ceiling.
- Choose a context carefully, or leave it out. Ask yourself: what is the added value of this context? Is it rich? Is it meaningful? Does it add relevance?
- Find out how the content is used nowadays to create a relevant and meaningful context for the students (at a workplace, in daily level or in other subjects).

4.3. Choosing a tool

Both teachers and students usually make use of tools as part of the teaching-learning process. This can be: a pen, notebook, ruler, compass, (graphical) calculator, software/apps, blackboard/whiteboard/smartboard/screen/projector and all sort of manipulatives, like blocks, pieces of string, disks, sticks, balls. A tool can have several purposes in education. In some cases, tools can play an essential role in the students' learning process. A compass can be used not only to facilitate drawing circles, but instead also supporting the insight that a circle consists of all points of equal distance to

[3] Based upon findings from the Mascil project: https://mascil-project.ph-freiburg.de/index.html

the centre. Tools and objects to support learning are sometimes called manipulatives: see for example https://en.wikipedia.org/wiki/Manipulative_(mathematics_education)

The diagram in Figure 4.1 specifies the functionality of IT-tools in education.

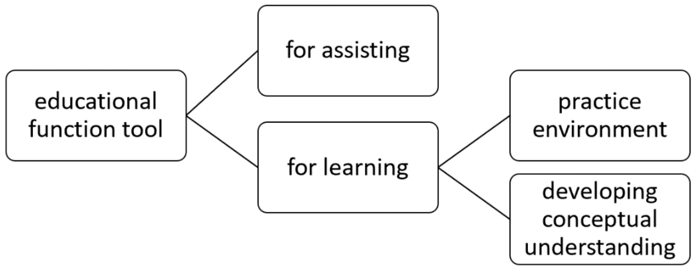


Figure 4.1. A scheme for the functionality of IT-tools, translated from Drijvers et al., 2012

A calculator is in many cases simply there to assist the teacher and students in calculations. Software is sometimes used to practice solving equations, giving instant feedback. In other cases software like GeoGebra is used to acquire a better conceptual understanding of, for example, the derivative of a function.

Every tool a teacher brings to the classroom requires careful thought. What is the added value? How can it be used? Does it contribute or distract, or both? How difficult is it to use? Using tools optimally in classroom is a challenge for teachers. In Lesson Study there is a name for the art of using the blackboard: *bansho*. Similarly, using graphing software (like GeoGebra) for classroom instruction is a skill for teachers to develop.

Bansho. In Lesson Study so much is importance is given to good use of the blackboard that there is a name for it: bansho, the art of blackboard use.

Picture taken from Takahashi (2006)

In Yoshida (2005) summarizes to roles of the blackboard:

- To keep a record of the lesson
- To help students remember what they need to do and to think about

- To help students see the connection between different parts of the lesson and the progression of the lesson
- To compare, contrast, and discuss ideas that students present
- To help to organize student thinking and discovery of new ideas
- To foster organized student note-taking skills by modelling good organization

For students the challenge can be expressed by the following slogan

Use to learn versus learn to use.

A teacher decides to use computer algebra software (say Mathematica) to let students experiment with the expansion of brackets in algebraic formulas—use to learn. Instead of experimenting, students spend all time learning the syntax of software—learn to use. On a positive note, the struggles with learning to use new tools can also offer opportunity to learn about the involved mathematics. In many cases learning to use a mathematical tool and learning the involved mathematics can and should go hand in hand.

Concrete design advice

- Every tool that is brought to the classroom for learning needs careful thought. What are the benefits and what are the challenges?
- Become a *bansho* master!
- Think about the balance between use to learn and learn to use

Epilogue

This compendium contains advice on how to shape an educational design process, in particular for inquiry-based mathematics teaching and as part of Lesson Study. There is an overall outline of several aspect or phases of the process in chapter 1, details of analyses one could perform in chapter 2, a concept of design principles to facilitate discussion on design, and further interesting aspects, like problem, context, and tools, in chapter 3,

This compendium has been developed as part of the TIME project and the ideas presented were tested as part of the process. The compendium is background reading for a course on educational design that was also developed in the project; and we advise you to try participate in the activities that this compendium supports.

Educational design is a craft. According to common knowledge it takes 10.000 hours to master a craft; so, more than anything, we invite you to build this practice. Lesson Study is a very suitable form to perform this collaboratively and enjoy your colleagues support, feedback and expertise. Make it an adventurous joint inquiry!

Final (concrete) design advice

- Try to learn how to think beyond the traditional teaching approaches.
- Learn to take risks.
- It is okay to fail and learn from mistakes.

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