



Canonical vertex formalism in DT theory of toric Calabi-Yau 4-folds

Sergej Monavari

Mathematical Institute, Utrecht University, P.O. Box 80010, 3508 TA Utrecht, the Netherlands

ARTICLE INFO

Article history:

Received 17 November 2021

Received in revised form 11 January 2022

Accepted 20 January 2022

Available online 31 January 2022

Keywords:

DT theory of Calabi-Yau 4-folds

Virtual localization

Hilbert schemes

Orientations

K-theoretic invariants

ABSTRACT

Motivated by previous computations of Y. Cao, M. Kool and the author, we propose square roots and sign rules for the vertex and edge terms that compute Donaldson-Thomas invariants of a toric Calabi-Yau 4-fold, and prove that they are canonical, exploiting the combinatorics of plane and solid partitions.

© 2022 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

1.1. DT theory of Calabi-Yau 4-folds

Donaldson-Thomas invariants were firstly defined in Thomas' thesis [31] as a way to count stable sheaves on projective Calabi-Yau 3-folds,¹ and were recently extended to Calabi-Yau 4-folds in the seminal work of Cao-Leung [14].

Let X be a quasi-projective variety, $\text{Hilb}^n(X, \beta)$ the Hilbert scheme of closed subschemes $Z \subset X$ with proper support in homology class $[Z] = \beta \in H_2(X, \mathbb{Z})$ and $\chi(\mathcal{O}_Z) = n$ and denote by \mathcal{I} the universal ideal sheaf on $X \times \text{Hilb}^n(X, \beta)$. By the work of Huybrechts-Thomas [19], the Atiyah class gives an *obstruction theory* on $\text{Hilb}^n(X, \beta)$

$$\mathbf{R}\mathcal{H}om_{\pi}(\mathcal{I}, \mathcal{I})_0^{\vee}[-1] \rightarrow \mathbb{L}_{\text{Hilb}^n(X, \beta)}, \quad (1.1)$$

where $(\cdot)_0$ denotes the trace-free part, $\mathbf{R}\mathcal{H}om_{\pi} = \mathbf{R}\pi_* \circ \mathbf{R}\mathcal{H}om$, $\pi : X \times \text{Hilb}^n(X, \beta) \rightarrow \text{Hilb}^n(X, \beta)$ and $\mathbb{L}_{\text{Hilb}^n(X, \beta)}$ is the truncated cotangent complex.

If X is a smooth projective 3-fold, the obstruction theory (1.1) is *perfect*, so M carries a virtual fundamental class; Donaldson-Thomas invariants are defined by integrating insertions against it. Unfortunately, if X is a smooth projective 4-fold, the obstruction theory fails to be perfect, so the machineries of Behrend-Fantechi [1] and Li-Tian [21] do not produce a virtual class. Nonetheless, if X is a projective Calabi-Yau 4-fold, Borisov-Joyce [6] constructed a virtual fundamental class

$$[\text{Hilb}^n(X, \beta)]_{o(\mathcal{L})}^{\text{vir}} \in H_{2n}(\text{Hilb}^n(X, \beta), \mathbb{Z})$$

¹ E-mail address: s.monavari@uu.nl.

¹ Here we denote by *Calabi-Yau variety* X a smooth complex quasi-projective variety with $K_X \cong \mathcal{O}_X$ and $b_1(X) = 0$.

which depends on a choice of orientation of $\mathrm{Hilb}^n(X, \beta)$, i.e. a choice of square root of the isomorphism

$$Q : \mathcal{L} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_{\mathrm{Hilb}^n(X, \beta)},$$

induced by Serre duality pairing, where $\mathcal{L} := \det(\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}, \mathcal{I}))$ is the determinant line bundle. The existence of orientations was proved for arbitrary compact Calabi-Yau 4-folds by Cao-Gross-Joyce [8] and in the non-compact setting by Bojko [2]. Recently Oh-Thomas [27] proposed an alternative (algebraic) construction of the virtual cycle

$$[\mathrm{Hilb}^n(X, \beta)]_{o(\mathcal{L})}^{\mathrm{vir}} \in A_n \left(\mathrm{Hilb}^n(X, \beta), \mathbb{Z} \left[\frac{1}{2} \right] \right),$$

which coincides with Borisov-Joyce virtual cycle under the cycle map and proved a virtual localization formula. Morally, the virtual class is obtained (at least locally) as the zero locus of an *isotropic* section of an $SO(r, \mathbb{C})$ -bundle over a smooth ambient space.

1.2. Hilbert schemes of toric CY 4-folds

It is in general very difficult to compute Donaldson-Thomas type invariants of a Calabi-Yau 4-fold. Here we assume that X is a toric Calabi-Yau 4-fold and denote by $\mathbf{T} = \{t_1 t_2 t_3 t_4 = 1\} \subset (\mathbb{C}^*)^4$ the subtorus preserving the Calabi-Yau form of X . The \mathbf{T} -action naturally lifts to $\mathrm{Hilb}^n(X, \beta)$, making the obstruction theory (1.1) \mathbf{T} -equivariant by [30, Thm. 53]. Despite being almost never the Hilbert scheme $\mathrm{Hilb}^n(X, \beta)$ proper, its fixed locus $\mathrm{Hilb}^n(X, \beta)^{\mathbf{T}}$ is reduced, 0-dimensional (Lemma 2.1) and the induced obstruction theory is trivial (Proposition 2.3), so we can *define* invariants \mathbf{T} -equivariantly by means of Oh-Thomas virtual localization theorem (cf. [27, Thm. 7.1, Rem. 7.4]).

Definition 1.1. Let $\gamma \in H_{\mathbf{T}}^*(\mathrm{Hilb}^n(X, \beta))$. The \mathbf{T} -equivariant Donaldson-Thomas invariants of X are

$$\mathrm{DT}_n(X, \beta; \gamma) = \sum_{Z \in \mathrm{Hilb}^n(X, \beta)^{\mathbf{T}}} \sqrt{e^{\mathbf{T}}(-T_Z^{\mathrm{vir}})} \cdot \gamma|_Z \in \frac{\mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}, \quad (1.2)$$

where $T^{\mathrm{vir}} \in K_{\mathbf{T}}^0(\mathrm{Hilb}^n(X, \beta))$ is the class of the dual of the obstruction theory (1.1) and $\lambda_i = c_1^{\mathbf{T}}(t_i)$.

These invariants had been intensively studied (e.g. [11, 9, 15, 12]), where γ is respectively a primary, descendant, tautological or K -theoretic insertion.

Here we denoted by $\sqrt{e^{\mathbf{T}}}(\cdot)$ the (\mathbf{T} -equivariant) square root Euler class defined by Edidin-Graham in [16] (cf. also [27, Sec. 7]). The definition of this class is far from being explicit since relies on the chosen orientation of $\mathrm{Hilb}^n(X, \beta)$. To ease our life, we define the *square root* of a \mathbf{T} -representation.

Definition 1.2. Let $V \in K_{\mathbf{T}}^0(\mathrm{pt})$ be a virtual \mathbf{T} -representation. We say that $T \in K_{\mathbf{T}}^0(\mathrm{pt})$ is a *square root* of V if

$$V = T + \overline{T} \in K_{\mathbf{T}}^0(\mathrm{pt}),$$

where $\overline{(\cdot)}$ denotes the dual \mathbf{T} -representation.

Let V admit a square root T in $K_{\mathbf{T}}^0(\mathrm{pt})$. Its square root Euler class satisfies

$$\sqrt{e^{\mathbf{T}}}(V) = \pm e^{\mathbf{T}}(T),$$

where the sign depends on the chosen orientation. Therefore, we can compute the invariants (1.2) by explicitly finding a square root of the virtual tangent bundle, at the price of introducing a (non-explicit!) sign at every \mathbf{T} -fixed point of $\mathrm{Hilb}^n(X, \beta)^{\mathbf{T}}$.

Finding the correct signs given a square root of the virtual tangent space is one of the major difficulties of the theory, as the number of possible choices of signs rapidly increases with the number of fixed points, which makes the application of the virtual localization formula not feasible — we recall that the virtual localization formula is one of the rare tools at our disposal to perform computations in DT theory. The main goal of this paper is to propose various square roots of the virtual tangent bundle with *sign rules*, which would make the invariants (1.2) effectively computable. As evidence to our proposals, we show that they

- are canonical (Theorem 2.8, 2.14, 2.17),
- play well with the dimensional reduction studied in [12, Sec. 2.1] (Remark 2.7, 2.13, 2.18),
- are consistent with previous computations [9–12, 24, 26], both in the math and physics literature.

To each fixed point in $\text{Hilb}^n(X, \beta)^{\mathbf{T}}$, we associate some combinatorial data using *solid partitions*, which can be displayed as arrangements of boxes in $\mathbb{Z}_{\geq 0}^4$ (cf. Section 2.1). We exploit the combinatorics of solid partitions to propose our sign rules and to prove they are canonical. It is important to notice that these signs carry independent interesting information on the combinatorics of solid partitions — a field where not much is currently known (see Section 1.4).

For $X = \mathbb{C}^4$, our proposals generalize the ones already discussed by Nekrasov-Piazzalunga [26, Sec. 2.4] for $\beta = 0$ and by Cao-Kool and the author [12, Rem. 1.18] when the \mathbf{T} -invariant subscheme $Z \subset X$ is supported in at most two \mathbf{T} -invariant lines. Here we deal with the full theory, where Z can be supported in four \mathbf{T} -invariant lines.

1.3. Relation to string theory

Donaldson-Thomas invariants of Calabi-Yau 4-folds — and their K -theoretic refinement — appear in String Theory as the result of supersymmetric localization of $U(1)$ super-Yang-Mills theory with matter on a Calabi-Yau 4-fold, studied by Nekrasov-Piazzalunga [24, 26] and by Bonelli-Fasola-Tanzini-Zenkevich [5] with an ADHM-type construction. In these cases, the authors study the quantum mechanics of a system of $D0$ - $D8$ branes. Our results would allow to easily perform computations in the presence of points and curves, that is for a system of $D0$ - $D2$ - $D8$ branes.

1.4. Magnificent four

MacMahon [22] classically solved the combinatorial problem of enumerating *partitions* and *plane partitions*. Remarkably, his conjectural formula for *solid partitions* fails and no closed expression is expected to exist. From a modern geometric point of view, the enumeration of partitions and plane partitions appears as a suitable limit of K -theoretic invariants of $\text{Hilb}^n(\mathbb{C}^2)$ and $\text{Hilb}^n(\mathbb{C}^3)$, exploiting respectively the smooth structure and the symmetric perfect obstruction theory — cf. [28]. In the case of $\text{Hilb}^n(\mathbb{C}^4)$ the Borisov-Joyce/Oh-Thomas virtual structure crucially depends on a choice of *orientation*, which is responsible for the ambiguity of sign at each fixed point — a phenomenon that does not arise in lower dimension — and partially explains the failure of MacMahon's conjecture, where the signs are not playing any rôle. For $d \geq 5$ there are no known virtual structures on $\text{Hilb}^n(\mathbb{C}^d)$ and no direct analogues of the combinatorial formulas in dimensions 2, 3, 4 seem to hold — see [10]. This may indicate that dimension 4 is special and we believe that our sign conjectures could play an important rôle in Combinatorics as well; quoting Nekrasov's *Magnificent four* [24]

"The adjective 'Magnificent' reflects this author's conviction that the dimension four is the maximal dimension where the natural albeit complex-valued probability distribution exists."

Acknowledgments. I am grateful to Martijn Kool for asking a question that led to this paper and for the many suggestions. I wish to thank Arkadij Bojko and Yalong Cao for many conversations on orientations in DT theory of Calabi-Yau fourfolds. I also thank the anonymous referee for several helpful comments, which improved the exposition of the paper. S.M. is supported by NWO grant TOP2.17.004.

2. Hilbert schemes of toric Calabi-Yau 4-folds

2.1. Toric varieties

Let X a toric Calabi-Yau 4-fold and consider the Hilbert scheme $\text{Hilb}^n(X, \beta)$, parametrizing closed subschemes $Z \subset X$ with proper support in the homology class $[Z] = \beta \in H_2(X, \mathbb{Z})$ and $\chi(\mathcal{O}_Z) = n$. Denote by $\Delta(X)$ the Newton polytope of X , by $V(X)$ its set of vertices and by $E(X)$ its set of edges. Vertices $\alpha \in V(X)$ correspond to $(\mathbb{C}^*)^4$ -fixed point of X , each contained in a maximal $(\mathbb{C}^*)^4$ -invariant open subset $U_\alpha \subset X$. Edges $\alpha\beta \in E(X)$ correspond to $(\mathbb{C}^*)^4$ -invariant lines $L_{\alpha\beta} \cong \mathbb{P}^1$, whose normal bundle is

$$N_{L_{\alpha\beta}/X} \cong \mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta}) \oplus \mathcal{O}(m''_{\alpha\beta})$$

and satisfies $m_{\alpha\beta} + m'_{\alpha\beta} + m''_{\alpha\beta} = -2$, being X a Calabi-Yau variety. We may choose coordinates t_i on $(\mathbb{C}^*)^4$ and x_i on U_α such that the $(\mathbb{C}^*)^4$ -action on U_α is determined by

$$(t_1, t_2, t_3, t_4) \cdot x_i = t_i x_i.$$

If the line $L_{\alpha\beta}$ is defined in these coordinates by $\{x_2 = x_3 = x_4 = 0\}$, the transition function between the charts U_α and U_β are of the form

$$(x_1, x_2, x_3, x_4) \mapsto (x_1^{-1}, x_2 x_1^{-m_{\alpha\beta}}, x_3 x_1^{-m'_{\alpha\beta}}, x_4 x_1^{-m''_{\alpha\beta}}).$$

Denote by $\mathbf{T} = \{t_1 t_2 t_3 t_4 = 1\} \subset (\mathbb{C}^*)^4$ the subtorus preserving the Calabi-Yau form of X . The $(\mathbb{C}^*)^4$ -action and the \mathbf{T} -action naturally lift on $\text{Hilb}^n(X, \beta)$, whose fixed locus is reduced and 0-dimensional.

Lemma 2.1 ([11, Lemma 2.1, 2.2]). *We have an isomorphism of schemes*

$$\mathrm{Hilb}^n(X, \beta)^{\mathbf{T}} = \mathrm{Hilb}^n(X, \beta)^{(\mathbb{C}^*)^4},$$

which consists of finitely many reduced points.

We recap the description of the $(\mathbb{C}^*)^4$ -fixed locus, which is completely analogous to [23, Sec. 4.2] in the setting of toric 3-folds. For an extensive treatment in the case of toric 4-folds, look at [11, Sec. 2.1].

Let $Z \in \mathrm{Hilb}^n(X, \beta)^{(\mathbb{C}^*)^4}$ and I be its ideal sheaf; $Z \subset X$ is preserved by the torus action, hence it must be supported on the $(\mathbb{C}^*)^4$ -fixed points (corresponding to vertices $\alpha \in V(X)$) and $(\mathbb{C}^*)^4$ -invariant lines of X (corresponding to edges $\alpha\beta \in E(X)$). Since I is $(\mathbb{C}^*)^4$ -fixed on each open subset, I must be defined on U_α by a monomial ideal

$$I_\alpha = I|_{U_\alpha} \subset \mathbb{C}[x_1, x_2, x_3, x_4],$$

and may also be viewed as a solid partition π_α ,

$$\pi_\alpha = \left\{ (k_1, k_2, k_3, k_4), \prod_{i=1}^4 x_i^{k_i} \notin I_\alpha \right\} \subset \mathbb{Z}_{\geq 0}^4. \quad (2.1)$$

The associated subscheme of I_α is (at most) 1-dimensional. The corresponding partition π_α may be infinite in the direction of the coordinate axes. If the solid partition is viewed as a box diagram in \mathbb{Z}^4 , the vertices in (2.1) are determined by the interior corners of the boxes, the corners closest to the origin.

The asymptotics of π_α in the coordinate directions are described by four ordinary finite-size plane partitions. In particular, in the direction of the $(\mathbb{C}^*)^4$ -invariant curve $L_{\alpha\beta}$ given by $\{x_2 = x_3 = x_4 = 0\}$, we have the plane partition $\lambda_{\alpha\beta}$ with the following diagram

$$\begin{aligned} \lambda_{\alpha\beta} &= \{ (k_2, k_3, k_4) : x_1^{k_1} x_2^{k_2} x_3^{k_3} x_4^{k_4} \notin I_\alpha, \forall k_1 \geq 0 \} \\ &= \{ (k_2, k_3, k_4) : x_2^{k_2} x_3^{k_3} x_4^{k_4} \notin I_{\alpha\beta} \} \subset \mathbb{Z}_{\geq 0}^3, \end{aligned}$$

where

$$I_{\alpha\beta} = I|_{U_\alpha \cap U_\beta} \subset \mathbb{C}[x_1^{\pm 1}, x_2, x_3, x_4].$$

The vertices of $\lambda_{\alpha\beta}$ defined above are the interior corners of the squares of the associated plane partition.

In summary, a $(\mathbb{C}^*)^4$ -fixed ideal sheaf can be described in terms of the following data:

- (i) a finite-size plane partition $\lambda_{\alpha\beta}$ assigned to each edge $\alpha\beta \in E(X)$;
- (ii) a (possibly infinite) solid partition π_α assigned to each vertex $\alpha \in V(X)$, such that the asymptotics of π_α in the three coordinate directions is given by the plane partitions $\lambda_{\alpha\beta}$ assigned to the corresponding edges.

Let $Z \in \mathrm{Hilb}^n(X, \beta)^{\mathbf{T}}$ correspond to the partition data $\{\pi_\alpha, \lambda_{\alpha\beta}\}_{\alpha, \beta}$. We see

$$\beta = \sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}| [L_{\alpha\beta}] \in H_2(X, \mathbb{Z}),$$

where $|\lambda_{\alpha\beta}|$ denotes the size of the plane partition λ , the number of the boxes in the diagram.

For a (possibly infinite) solid partition π such that its asymptotic plane partitions λ_i , $i = 1, 2, 3, 4$, are of finite size $|\lambda_i| < \infty$, we define the renormalized volume $|\pi|$ as follows. We set

$$|\pi| := \# \{ \pi \cap [0, \dots, N]^4 \} - (N+1) \sum_{i=1}^4 |\lambda_i|, \quad N \gg 0.$$

The renormalized volume is independent of the cut-off N as long as N is sufficiently large. We will say that a solid partition is *point-like* if all the asymptotics $\lambda_i = 0$, $i = 1, \dots, 4$ and *curve-like* if at least one $\lambda_i \neq 0$. See [29] for a similar discussion of the renormalization of infinite plane partitions and its interpretation in terms of melting crystals.

To conclude, let $\mathbf{m} = (m_2, m_3, m_4)$, λ a finite-size plane partition and set

$$f_{\mathbf{m}}(\lambda) = \sum_{(i,j,k) \in \lambda} (1 - m_2 \cdot i - m_3 \cdot j - m_4 \cdot k).$$

By [11, Lemma 2.4], if a \mathbf{T} -invariant closed subscheme $Z \subset X$ corresponds to a partition data $\{\pi_\alpha, \lambda_{\alpha\beta}\}_{\alpha, \beta}$, then

$$\chi(\mathcal{O}_Z) = \sum_{\alpha \in V(X)} |\pi_\alpha| + \sum_{\alpha\beta \in E(X)} f_{\mathbf{m}_{\alpha\beta}}(\lambda_{\alpha\beta}), \quad (2.2)$$

where $\mathbf{m}_{\alpha\beta}$ is the multidegree of the normal bundle of the \mathbf{T} -invariant line $L_{\alpha\beta}$.

2.2. The vertex formalism

Denote by \mathcal{I} the universal ideal sheaf of the universal subscheme $\mathcal{Z} \subset X \times \text{Hilb}^n(X, \beta)$. The obstruction theory $\mathbf{R}\mathcal{H}om_{\pi}(\mathcal{I}, \mathcal{I})_0^\vee[-1] \rightarrow \mathbb{L}_{\text{Hilb}^n(X, \beta)}$ is naturally \mathbf{T} -equivariant by [30, Cor. 4.4] and endowed by the \mathbf{T} -equivariant Serre quadratic pairing. Over a point $Z \in \text{Hilb}^n(X, \beta)$ the fiber of the virtual tangent space is

$$T_Z^{\text{vir}} = \mathbf{R}\mathcal{H}om_{\pi}(\mathcal{I}, \mathcal{I})_0[1]|_Z = \mathbf{R}\text{Hom}(I_Z, I_Z)_0[1],$$

where I_Z is the ideal sheaf of Z . To a \mathbf{T} -fixed point Z corresponds a partition data $\{\pi_{\alpha}, \lambda_{\alpha\beta}\}$ with $\alpha \in V(X), \alpha\beta \in E(X)$; denote by

$$Z_{\alpha} = H^0(Z|_{U_{\alpha}}, \mathcal{O}_{Z|_{U_{\alpha}}}) = \sum_{(i,j,k,l) \in \pi} t_1^i t_2^j t_3^k t_4^l, \quad (2.3)$$

$$Z_{\alpha\beta} = \sum_{(a,b,c) \in \lambda_i} t_j^a t_k^b t_l^c, \quad (2.4)$$

where the line $L_{\alpha\beta}$ is defined by $\{x_j = x_k = x_l = 0\}$. Recall the vertex formalism² developed in [11, Sec. 2.4]

$$V_{\alpha} = Z_{\alpha} + \overline{Z_{\alpha}} - \overline{P_{1234}} Z_{\alpha} \overline{Z_{\alpha}} + \sum_{i=1}^4 \frac{F_{\alpha\beta_i}(t_{i'}, t_{i''}, t_{i'''})}{1 - t_i}, \quad (2.5)$$

$$F_{\alpha\beta} = -Z_{\alpha\beta} + \frac{\overline{Z_{\alpha\beta}}}{t_2 t_3 t_4} + \overline{P_{234}} Z_{\alpha\beta} \overline{Z_{\alpha\beta}}, \quad (2.6)$$

$$E_{\alpha\beta} = t_1^{-1} \frac{F_{\alpha\beta}(t_2, t_3, t_4)}{1 - t_1^{-1}} - \frac{F_{\alpha\beta}(t_2 t_1^{-m_{\alpha\beta}}, t_3 t_1^{-m'_{\alpha\beta}}, t_4 t_1^{-m''_{\alpha\beta}})}{1 - t_1^{-1}} \quad (2.7)$$

where $\{t_i, t_{i'}, t_{i''}, t_{i'''}\} = \{1, 2, 3, 4\}$ and for a set of indices I , we set $P_I = \prod_{a \in I} (1 - t_a)$. This vertex formalism allows us to compute the virtual tangent space at a \mathbf{T} -fixed point.

Proposition 2.2 ([11, Prop. 2.11]). *Let X be a toric Calabi-Yau 4-fold, $\beta \in H_2(X, \mathbb{Z})$ and $Z \in \text{Hilb}^n(X, \beta)^{\mathbf{T}}$. Then*

$$T_Z^{\text{vir}} = \sum_{\alpha \in V(X)} V_{\alpha} + \sum_{\alpha\beta \in E(X)} E_{\alpha\beta}.$$

Anticipating some results from Section 2.4, 2.5 we prove that the obstruction theory induced on the \mathbf{T} -fixed locus is trivial.

Proposition 2.3. *Let X be a toric Calabi-Yau 4-fold and $\beta \in H_2(X, \mathbb{Z})$. Then the induced obstruction theory on $\text{Hilb}^n(X, \beta)^{\mathbf{T}}$ is trivial. In particular, for $Z \in \text{Hilb}^n(X, \beta)^{\mathbf{T}}$, the virtual tangent space T_Z^{vir} is \mathbf{T} -movable.*

Proof. By Lemma 2.10, 2.15 there exist \mathbf{T} -movable square roots $v_{\alpha}, e_{\alpha\beta}$ of $V_{\alpha}, E_{\alpha\beta}$ for any $\alpha \in V(X), \alpha\beta \in E(X)$, which implies that also $V_{\alpha}, E_{\alpha\beta}$ and T_Z^{vir} are \mathbf{T} -movable by Proposition 2.2. We have an identity in \mathbf{T} -equivariant K -theory

$$T_Z^{\text{vir}} = \text{Ext}^1(I_Z, I_Z) - \text{Ext}^2(I_Z, I_Z) + \text{Ext}^3(I_Z, I_Z) \in K_{\mathbf{T}}^0(\text{pt}).$$

It follows by Lemma 2.1 that $\text{Ext}^1(I_Z, I_Z)^{\mathbf{T}} = \text{Ext}^3(I_Z, I_Z)^{\mathbf{T}} = 0$, by which we conclude that $\text{Ext}^2(I_Z, I_Z)^{\mathbf{T}} = 0$ as well. \square

In the remainder of the paper, we will exhibit (several) explicit square roots $v_{\alpha}, e_{\alpha\beta}$ of $V_{\alpha}, E_{\alpha\beta}$ reducing the DT invariants in (1.2) to

$$\text{DT}_n(X, \beta; \gamma) = \sum_{Z \in \text{Hilb}^n(X, \beta)^{\mathbf{T}}} \prod_{\alpha \in V(X)} (-1)^{\sigma(Z, v_{\alpha})} e^{\mathbf{T}}(-v_{\alpha}) \prod_{\alpha\beta \in E(X)} (-1)^{\sigma(Z, e_{\alpha\beta})} e^{\mathbf{T}}(-e_{\alpha\beta}) \cdot \gamma|_Z, \quad (2.8)$$

and propose explicit canonical signs $(-1)^{\sigma(Z, v_{\alpha})}, (-1)^{\sigma(Z, e_{\alpha\beta})}$.

² We only write down $F_{\alpha\beta}$ and $E_{\alpha\beta}$ when $L_{\alpha\beta} \cong \mathbb{P}^1$ is given by $\{x_2 = x_3 = x_4 = 0\}$, i.e. the leg along the x_1 -axis. The other cases follow by symmetry.

2.3. The vertex term: points

To each fixed point $Z \in \text{Hilb}^n(\mathbb{C}^4)^T$ corresponds a solid partition π of size n . Denote by Z_π, V_π the vertex terms Z_α, V_α in (2.3), (2.5). By \mathbf{T} -equivariant Serre duality, we know that V_π admits a square root. We set

$$v_\pi^i = Z_\pi - \overline{P_{jkl}} Z_\pi \overline{Z_\pi},$$

which enjoys

$$V_\pi = v_\pi^i + \overline{v_\pi^i},$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. For $i = 4$, it recovers the square root already found in [26, 12]. The next lemma was already proven by Nekrasov-Piazzalunga [26], whose proof we sketch for completeness and to introduce useful notation.

Lemma 2.4 ([26, Sec. 2.4.1]). *Let π be a point-like solid partition and $i = 1, \dots, 4$. Then v_π^i is \mathbf{T} -movable.*

Proof. Without loss of generality, suppose that $i = 4$. We prove the statement by induction on the size of π . If $|\pi| = 1$, then v_π^4 has no constant terms. Suppose now that the claim holds for all solid partitions π of size $|\pi| \leq n$. Consider a solid partition $\tilde{\pi}$ of size $|\tilde{\pi}| = n + 1$; this can be seen as a solid partition π of size n with an extra box over it, corresponding to a \mathbb{Z}^4 -lattice point $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$. We have

$$Z_{\tilde{\pi}} = Z_\pi + t^\mu$$

and

$$\begin{aligned} v_{\tilde{\pi}}^4 &= v_\pi^4 + t^\mu - \overline{P_{123}}(t^\mu \overline{Z_\pi} + t^{-\mu} Z_\pi + 1) \\ &= v_\pi^4 + t^\mu - \overline{P_{123}}(t^\mu (\overline{Z_\pi} + t^{-\mu}) + t^{-\mu} (Z_\pi + t^\mu) - 1). \end{aligned}$$

By induction, we know that v_π^4 is \mathbf{T} -movable. Consider the subdivision $\tilde{\pi} = \pi' \sqcup \pi''$ of the boxes of $\tilde{\pi}$, where π' corresponds to the lattice points ν such that $\nu \leq \mu$ and π'' corresponds to the lattice points ν such that $\nu \not\leq \mu$. Here, given $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$, we say that $\nu \leq \mu$ if $\nu_i \leq \mu_i$ for all $i = 1, \dots, 4$. Denote by

$$\begin{aligned} Z_{\pi'} &:= \sum_{\nu \in \pi'} t^\nu = \sum_{i \leq \mu_1, j \leq \mu_2, k \leq \mu_3, l \leq \mu_4} t_1^i t_2^j t_3^k t_4^l, \\ Z_{\pi''} &:= \sum_{\nu \in \pi''} t^\nu. \end{aligned}$$

By construction, $Z_{\tilde{\pi}} = Z_{\pi'} + Z_{\pi''}$. We want to prove that

$$(t^\mu + \overline{P_{123}} - \overline{P_{123}} t^\mu \overline{Z_{\pi'}} - \overline{P_{123}} t^\mu \overline{Z_{\pi''}} - \overline{P_{123}} t^{-\mu} Z_{\pi'} - \overline{P_{123}} t^{-\mu} Z_{\pi''})^{\text{fix}} = 0,$$

where by $(\cdot)^{\text{fix}}$ we denote the invariant factors under the \mathbf{T} -action.³ For a set of indices I , denote by δ_I the function which is 1 if only if all the indices are equal. The contribution of each summand is

$$\begin{aligned} (t^\mu + \overline{P_{123}})^{\text{fix}} &= 1 + \delta_{\mu_1, \mu_2, \mu_3, \mu_4}, \\ (\overline{P_{123}} t^\mu \overline{Z_{\pi''}})^{\text{fix}} &= 0, \\ (\overline{P_{123}} t^{-\mu} Z_{\pi''})^{\text{fix}} &= 0, \\ (\overline{P_{123}} t^\mu \overline{Z_{\pi'}})^{\text{fix}} &= \sum_{i=0}^{\mu_4} \delta_{\mu_1, \mu_2, \mu_3, i}, \\ (\overline{P_{123}} t^{-\mu} Z_{\pi'})^{\text{fix}} &= 1 - \sum_{i=0}^{\mu_4-1} \delta_{\mu_1, \mu_2, \mu_3, i}, \end{aligned}$$

by which we conclude the induction step. \square

We propose a sign rule⁴ for the sign in (2.8), relative to the square root v_π^i .

³ In other words, for a virtual \mathbf{T} -representation $F \in K_T^0(\text{pt})$, $(F)^{\text{fix}}$ is its constant term when viewing F as a Laurent polynomial in the torus coordinates.

⁴ After a first draft of the paper was written, Kool-Rennemo [20] announced a proof for this sign rule.

Conjecture 2.5. Let π be a solid partition corresponding to a \mathbf{T} -fixed point in $\text{Hilb}^n(\mathbb{C}^4)$. Then the sign relative to the square root v_π^i is $(-1)^{\sigma_i(\pi)}$, where

$$\sigma_i(\pi) = |\pi| + \# \{ (a_1, a_2, a_3, a_4) \in \pi : a_j = a_k = a_l < a_i \} \quad (2.9)$$

and $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Remark 2.6. For $i = 4$, the sign rule (2.9) was proposed by Nekrasov-Piazzalunga in the physics literature in [26], as the result of supersymmetric localization in string theory.⁵ It was also proposed in [12], based on explicit low order computations. This sign rule is consistent with the previous computations of [9–12, 24, 26].

Remark 2.7. Let π corresponds to a \mathbf{T} -invariant closed subscheme $Z \subset \mathbb{C}^4$ supported in the hyperplane $\{x_i = 0\} \subset \mathbb{C}^4$, for $i = 1, \dots, 4$. Then $\sigma_i(\pi) = |\pi|$, which is consistent with the dimensional reduction studied in [12, Sec. 2.1].

We prove now that the sign rule (2.9) is canonical, meaning that it does not really depend on choosing a preferred x_i -axis, as one should expect from the correct sign rule.

Theorem 2.8. Let π a point-like solid partition. For every $i, j = 1, \dots, 4$ we have

$$(-1)^{\sigma_i(\pi)} e^{\mathbf{T}}(-v_\pi^i) = (-1)^{\sigma_j(\pi)} e^{\mathbf{T}}(-v_\pi^j).$$

Proof. v_π^i and v_π^j are both square roots of V_π and are \mathbf{T} -movable by Lemma 2.4. To prove the claim, we need to write

$$v_\pi^i - v_\pi^j = U_\pi - \overline{U_\pi},$$

for a $U_\pi \in K_{\mathbf{T}}^0(\text{pt})$ and compute the parity of $\text{rk } U_\pi^{\text{mov}}$; in fact

$$\frac{e^{\mathbf{T}}(v_\pi^i)}{e^{\mathbf{T}}(v_\pi^j)} = \frac{e^{\mathbf{T}}(U_\pi^{\text{mov}})}{e^{\mathbf{T}}(\overline{U_\pi^{\text{mov}}})} = (-1)^{\text{rk } U_\pi^{\text{mov}}}.$$

Without loss of generality, suppose $i = 4$ and $j = 3$; we prove the claim by induction on the size of π . If $|\pi| = 1$, we have $Z_\pi = 1$ and

$$\begin{aligned} v_\pi^4 - v_\pi^3 &= \overline{P_{12}}(t_3^{-1} - t_4^{-1}) \\ &= \overline{P_{12}}t_3^{-1} - \overline{P_{12}t_3^{-1}}, \end{aligned}$$

where used that $t_1 t_2 t_3 t_4 = 1$; clearly $\text{rk}(\overline{P_{12}t_3^{-1}})^{\text{mov}} = 0$. Suppose now the claim holds for any partition of size $|\pi| \leq n$ and consider a solid partition $\tilde{\pi}$, obtained by a solid partition π of size n by adding a box whose lattice coordinates are $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$. We have

$$\begin{aligned} v_{\tilde{\pi}}^4 - v_{\tilde{\pi}}^3 &= v_\pi^4 - v_\pi^3 + \overline{P_{12}}(t_3^{-1} - t_4^{-1}) + \overline{P_{12}}(t_3^{-1} - t_4^{-1})(\overline{Z_\pi}t^\mu + Z_\pi t^{-\mu}) \\ &= v_\pi^4 - v_\pi^3 - \overline{P_{12}}(t_3^{-1} - t_4^{-1}) + \overline{P_{12}}(t_3^{-1} - t_4^{-1})(\overline{Z_{\tilde{\pi}}}t^\mu + Z_{\tilde{\pi}}t^{-\mu}), \end{aligned}$$

and by the induction step

$$\begin{aligned} e^{\mathbf{T}}(v_{\tilde{\pi}}^4 - v_{\tilde{\pi}}^3) &= (-1)^{\sigma_4(\tilde{\pi}) - \sigma_3(\tilde{\pi})}, \\ e^{\mathbf{T}}(\overline{P_{12}}(t_3^{-1} - t_4^{-1})) &= 1. \end{aligned}$$

The final piece to compute is of the form

$$\begin{aligned} \overline{P_{12}}(t_3^{-1} - t_4^{-1})(\overline{Z_{\tilde{\pi}}}t^\mu + Z_{\tilde{\pi}}t^{-\mu}) &= (W_1 + \overline{W_1})(W_2 - \overline{W_2}) \\ &= (W_1 W_2 - \overline{W_1} \overline{W_2}) + (\overline{W_1} W_2 - W_1 \overline{W_2}), \end{aligned}$$

with

$$\begin{aligned} W_1 &= t^{-\mu} Z_{\tilde{\pi}}, \\ W_2 &= t_3^{-1} \overline{P_{12}}. \end{aligned}$$

⁵ In [26] the sign rule is slightly different due to a slightly different choice of square root.

Since $\text{rk } W_1 W_2 = \text{rk } \overline{W}_1 W_2 = 0$, we just need to compute the parity of $\text{rk}(W_1 W_2)^{\text{fix}} + \text{rk}(\overline{W}_1 W_2)^{\text{fix}}$. As in the proof of Lemma 2.4, consider the subdivision $\tilde{\pi} = \pi' \sqcup \pi''$, with $Z_{\tilde{\pi}} = Z_{\pi'} + Z_{\pi''}$, where

$$Z_{\pi'} = \sum_{i \leq \mu_1, j \leq \mu_2, k \leq \mu_3, l \leq \mu_4} t_1^i t_2^j t_3^k t_4^l,$$

$$Z_{\pi''} = \sum_{v \in \pi''} t^v.$$

We have

$$(W_1 W_2)^{\text{fix}} = \left(t_3^{-1} \overline{P}_{12} t^{-\mu} Z_{\pi'} + t_3^{-1} \overline{P}_{12} t^{-\mu} Z_{\pi''} \right)^{\text{fix}}.$$

As in the proof Lemma 2.4 we compute

$$\left(t_3^{-1} \overline{P}_{12} t^{-\mu} Z_{\pi''} \right)^{\text{fix}} = 0,$$

$$\left(t_3^{-1} \overline{P}_{12} t^{-\mu} Z_{\pi'} \right)^{\text{fix}} = \sum_{k=0}^{\mu_3} \sum_{l=0}^{\mu_4-1} \delta_{\mu_1, \mu_2, k, l}.$$

Notice now that

$$\overline{W}_2 = t_4^{-1} \overline{P}_{12},$$

thus, by symmetry, we have

$$(\overline{W}_1 W_2)^{\text{fix}} = \sum_{k=0}^{\mu_3-1} \sum_{l=0}^{\mu_4} \delta_{\mu_1, \mu_2, k, l}.$$

We compute the parity

$$\begin{aligned} \text{rk}(W_1 W_2)^{\text{fix}} + \text{rk}(\overline{W}_1 W_2)^{\text{fix}} &= \sum_{k=0}^{\mu_3} \sum_{l=0}^{\mu_4-1} \delta_{\mu_1, \mu_2, k, l} + \sum_{k=0}^{\mu_3-1} \sum_{l=0}^{\mu_4} \delta_{\mu_1, \mu_2, k, l} \\ &= \sum_{l=0}^{\mu_4-1} \delta_{\mu_1, \mu_2, \mu_3, l} + \sum_{k=0}^{\mu_3-1} \delta_{\mu_1, \mu_2, k, \mu_4} \pmod{2}. \end{aligned}$$

Notice that

$$\sum_{l=0}^{\mu_4-1} \delta_{\mu_1, \mu_2, \mu_3, l} = \begin{cases} 1 & \mu_4 > \mu_1 = \mu_2 = \mu_3 \\ 0 & \text{else} \end{cases}$$

therefore $\sigma_4(\tilde{\pi}) = \sigma_4(\pi) + \sum_{l=0}^{\mu_4-1} \delta_{\mu_1, \mu_2, \mu_3, l}$ and $\sigma_3(\tilde{\pi}) = \sigma_3(\pi) + \sum_{k=0}^{\mu_3-1} \delta_{\mu_1, \mu_2, k, \mu_4}$, by which we conclude the proof. \square

2.4. The vertex term: curves

Set $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_i are finite plane partitions. Denote by $P_{\underline{\lambda}}$ the collection of (possibly infinite) solid partitions, whose asymptotic profile is given by $\underline{\lambda}$. To any solid partition $\pi \in P_{\underline{\lambda}}$ corresponds a \mathbf{T} -invariant closed subscheme $Z \subset \mathbb{C}^4$; denote by $Z_{\pi}, Z_{\lambda_i}, V_{\pi}$ the vertex terms $Z_{\alpha}, Z_{\alpha\beta_i}, V_{\alpha}$ in (2.3), (2.4), (2.5). \mathbf{T} -equivariant Serre duality implies that V_{π} admits a square root; set

$$v_{\pi}^i = Z_{\pi} - \overline{P}_{jkl} Z_{\pi} \overline{Z_{\pi}} + \sum_{j \neq i, j=1}^4 \frac{f_{\lambda_j}^i}{1-t_j} + \frac{1}{(1-t_i)} \left(-Z_{\lambda_i} + \overline{P}_{jkl} (\overline{Z_{\pi}} Z_{\lambda_i} - Z_{\pi} \overline{Z_{\lambda_i}}) + \frac{\overline{P}_{jkl}}{1-t_i} Z_{\lambda_i} \overline{Z_{\lambda_i}} \right),$$

$$f_{\lambda_j}^i = -Z_{\lambda_j} + \overline{P}_{kl} Z_{\lambda_j} \overline{Z_{\lambda_j}},$$

which enjoy

$$V_{\pi} = v_{\pi}^i + \overline{v_{\pi}^i},$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Setting $i = 4$, we recover the explicit square root choice studied in [12]. We redistribute now the vertex terms in a different way. Write, for $N \gg 0$

$$Z_\pi = Z_{\pi^{\text{nor}}} + \sum_{i=1}^4 \frac{Z_{\lambda_i}}{1-t_i} t_i^{N+1},$$

$$Z_{\pi_i} = Z_{\lambda_i} \sum_{n=0}^N t_i^n,$$

$$\frac{Z_{\lambda_i}}{1-t_i} = Z_{\pi_i} + \frac{Z_{\lambda_i}}{1-t_i} t_i^{N+1}.$$

Here the (point-like) solid partition π^{nor} is the cut-off for $N \gg 0$ of the (possibly curve-like) solid partition π . If π is a point-like solid partition, then simply $\pi^{\text{nor}} = \pi$, while π_i is simply the cut-off for $N \gg 0$ of the curve-like solid partition corresponding to the infinite leg along the x_i -axis containing π . Using the above expressions, we can express the vertex terms as

$$v_\pi^i = v_{\pi^{\text{nor}}}^i - \sum_{a=1}^4 v_{\pi_a}^i + A_\pi^i + B_\pi^i + C_\pi^i - \overline{C}_\pi^i, \quad (2.10)$$

where

$$A_\pi^i = -\overline{P}_{jkl} \sum_{\substack{a \neq b \\ a, b \neq i}} \frac{Z_{\lambda_a}}{1-t_a} \frac{\overline{Z}_{\lambda_b}}{1-t_b^{-1}} (t_a t_b^{-1})^{N+1} - \overline{P}_{1234} \sum_{a \neq i} \frac{Z_{\lambda_a}}{1-t_a} \frac{\overline{Z}_{\lambda_i}}{1-t_i^{-1}} (t_a t_i^{-1})^{N+1},$$

$$B_\pi^i = -\overline{P}_{jkl} \sum_{a \neq i} \left(\frac{Z_{\lambda_a}}{1-t_a} t_a^{N+1} (\overline{Z}_{\pi^{\text{nor}}} - \overline{Z}_{\pi_a}) + \frac{\overline{Z}_{\lambda_a}}{1-t_a^{-1}} t_a^{-(N+1)} (Z_{\pi^{\text{nor}}} - Z_{\pi_a}) \right) \\ - \overline{P}_{1234} \frac{\overline{Z}_{\lambda_i}}{1-t_i^{-1}} t_i^{-(N+1)} (Z_{\pi^{\text{nor}}} - Z_{\pi_i}),$$

$$C_\pi^i = \overline{P}_{123} \left(Z_{\pi_i} (\overline{Z}_{\pi^{\text{nor}}} - \overline{Z}_{\pi_i}) + \sum_{a \neq i} \frac{\overline{Z}_{\lambda_a}}{1-t_a^{-1}} t_a^{-(N+1)} Z_{\pi_i} \right),$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Motivated by the above expression, we define a new square root of V_π

$$v_\pi^{i'} = v_\pi^i - C_\pi^i + \overline{C}_\pi^i.$$

Remark 2.9. It is an equivalent problem to find the sign rule for $v_\pi^{i'}$ or for v_π^i . In fact, such two sign rules will differ just by $(-1)^{\text{rk}(C_\pi^i)^{\text{mov}}}$, which is completely determined by the solid partition π . Moreover, if the \mathbf{T} -invariant closed subscheme $Z \subset \mathbb{C}^4$ corresponding to a solid partition π is not supported in the \mathbf{T} -invariant line $\{x_j = x_k = x_l = 0\}$, we have that $v_\pi^{i'} = v_\pi^i$.

Lemma 2.10. Let π be a curve-like solid partition and $i = 1, \dots, 4$. Then $v_\pi^i, v_\pi^{i'}$ are \mathbf{T} -movable.

Proof. By Lemma 2.4 $v_{\pi_{\text{red}}}^i, v_{\pi_a}^i$ are \mathbf{T} -movable, for $a = 1, \dots, 4$. For $N \gg 0$, we clearly have $(A_\pi^i)^{\text{fix}} = (B_\pi^i)^{\text{fix}} = 0$. \square

We propose a sign rule for the sign in (2.8), relative to the square root $v_\pi^{i'}$.

Conjecture 2.11. Let π be a curve-like solid partition. Then the sign relative to the square root $v_\pi^{i'}$ is $(-1)^{\sigma_i(\pi)}$, where

$$\sigma_i(\pi) = |\pi| + \# \{ (a_1, a_2, a_3, a_4) \in \pi : a_j = a_k = a_l < a_i \} \\ - \sum_{\text{leg}} \# \{ (a_1, a_2, a_3, a_4) \in \text{leg} : a_j = a_k = a_l < a_i \} \quad (2.11)$$

and $\{i, j, k, l\} = \{1, 2, 3, 4\}$, where leg denote the curve-like solid partitions obtained by translating the plane partitions λ_i along the x_i -axis.

Remark 2.12. For $i = 4$ the sign rule (2.11) was proposed in [12] for at most two non-empty legs, based on explicit low order computations. This sign rule is consistent with the computations of [11, 12].

Remark 2.13. Let π corresponds to a \mathbf{T} -invariant closed subscheme $Z \subset \mathbb{C}^4$ supported in the hyperplane $\{x_i = 0\} \subset \mathbb{C}^4$, for $i = 1, \dots, 4$. Then $\sigma_i(\pi) = |\pi|$, which is consistent with the dimensional reduction studied in [12, Sec. 2.1].

We prove now that the sign rule (2.11) is canonical, meaning that it does not really depend on choosing a preferred x_i -axis.

Theorem 2.14. Let π a curve-like solid partition. For every $i, j = 1, \dots, 4$ we have

$$(-1)^{\sigma_i(\pi)} e^{\mathbf{T}}(-v_{\pi}^{i'}) = (-1)^{\sigma_j(\pi)} e^{\mathbf{T}}(-v_{\pi}^{j'}).$$

Proof. $v_{\pi}^{i'}$ and $v_{\pi}^{j'}$ are both square roots of V_{π} and are \mathbf{T} -movable by Lemma 2.10. The difference of the two sections is

$$v_{\pi}^{i'} - v_{\pi}^{j'} = (v_{\pi}^{i'} - v_{\pi}^{j'}) - \sum_{a=1}^4 (v_{\pi_a}^{i'} - v_{\pi_a}^{j'}) + A_{\pi}^i - A_{\pi}^j + B_{\pi}^i - B_{\pi}^j.$$

By Theorem 2.8, we know that

$$\begin{aligned} e^{\mathbf{T}}(v_{\pi}^{i'} - v_{\pi}^{j'}) &= (-1)^{\sigma_i(\pi^{\text{nor}}) - \sigma_j(\pi^{\text{nor}})}, \\ e^{\mathbf{T}}(v_{\pi_a}^{i'} - v_{\pi_a}^{j'}) &= (-1)^{\sigma_i(\pi_a) - \sigma_j(\pi_a)}, \end{aligned}$$

which satisfies

$$\sigma_i(\pi) - \sigma_j(\pi) = \sigma_i(\pi^{\text{nor}}) - \sigma_j(\pi^{\text{nor}}) - \sum_{a=1}^4 (\sigma_i(\pi_a) - \sigma_j(\pi_a)).$$

To conclude, with a simple computation it is possible to show that

$$\begin{aligned} A_{\pi}^i - A_{\pi}^j &= U_{A,i,j,\pi} - \overline{U_{A,i,j,\pi}}, \\ B_{\pi}^i - B_{\pi}^j &= U_{B,i,j,\pi} - \overline{U_{B,i,j,\pi}}, \end{aligned}$$

where, for $N \gg 0$, we have $\text{rk}(U_{A,i,j,\pi})^{\text{mov}} = \text{rk}(U_{A,i,j,\pi})^{\text{fix}} = 0$ and $\text{rk}(U_{B,i,j,\pi})^{\text{mov}} = \text{rk}(U_{B,i,j,\pi})^{\text{fix}} = 0$. \square

2.5. The edge term

In this section we study square roots and propose a sign rule for the edge term (2.7). For simplicity, let's assume that the edge $\alpha\beta \in E(X)$ corresponds to the \mathbb{P}^1 given, in the local coordinates of U_{α} , by $\{x_2 = x_3 = x_4 = 0\}$, with normal bundle

$$N_{\mathbb{P}^1/X} \cong \mathcal{O}(m_2) \oplus \mathcal{O}(m_3) \oplus \mathcal{O}(m_4),$$

satisfying $m_2 + m_3 + m_4 = -2$; we set $\mathbf{m} = (m_2, m_3, m_4)$. We also fix a finite-size plane partition λ , corresponding to the profile of the non-reduced \mathbf{T} -fixed leg. The discussion for the legs along the other directions will be completely analogous. Denote here by Z_{λ}, E_{λ} the edge terms $Z_{\alpha\beta}, E_{\alpha\beta}$ in (2.4), (2.7), where

$$Z_{\lambda} = \sum_{(j,k,l) \in \lambda} t_2^j t_3^k t_4^l.$$

Denote by $(\tilde{\cdot}) : K_{\mathbf{T}}^0(\text{pt}) \rightarrow K_{\mathbf{T}}^0(\text{pt})$ the map sending

$$V \mapsto V(t_1^{-1}, t_2 t_1^{-m_2}, t_3 t_1^{-m_3}, t_4 t_1^{-m_4}).$$

The edge term admits a square root; set

$$\begin{aligned} e_{\lambda}^j &= t_1^{-1} \frac{f_{\lambda}^j}{1 - t_1^{-1}} - \frac{\tilde{f}_{\lambda}^j}{1 - t_1^{-1}}, \\ f_{\lambda}^j &= -Z_{\lambda} + \overline{P_{kl} Z_{\lambda} Z_{\lambda}}, \end{aligned}$$

which enjoys

$$E_{\lambda} = e_{\lambda}^j + \overline{e_{\lambda}^j},$$

where $\{j, k, l\} = \{2, 3, 4\}$. Setting $j = 4$, we recover the explicit square root choice studied in [12].

Lemma 2.15. Let λ be a finite plane partition and $j = 2, 3, 4$. Then e_λ^j is \mathbf{T} -movable.

Proof. Without loss of generality, suppose that $j = 4$. Write $f_\lambda^4 = \sum_\nu t^\nu$, where the sum is over $\nu = (\nu_2, \nu_3, \nu_4)$ and we set $t^\nu = t_2^{\nu_2} t_3^{\nu_3} t_4^{\nu_4}$. We have

$$\frac{1}{1 - t_1^{-1}} (t_1^{-1} t^\nu - t^\nu t_1^{-\mathbf{m}\nu}) = \begin{cases} -t^\nu \sum_{i=0}^{-\mathbf{m}\nu} t_1^i & \mathbf{m}\nu \leq 0, \\ 0 & \mathbf{m}\nu = 1, \\ t^\nu t_1^{-1} \sum_{i=0}^{\mathbf{m}\nu-2} t_1^{-i} & \mathbf{m}\nu \geq 2 \end{cases} \quad (2.12)$$

where $\mathbf{m}\nu$ denotes the standard scalar product in \mathbb{Z}^3 . Therefore the contribution to the \mathbf{T} -fixed part of each t^ν is

$$\text{rk} \left(\frac{1}{1 - t_1^{-1}} (t_1^{-1} t^\nu - t^\nu t_1^{-\mathbf{m}\nu}) \right)^{\text{fix}} = \begin{cases} -\sum_{i=0}^{-\mathbf{m}\nu} \delta_{i, \nu_2, \nu_3, \nu_4} & \mathbf{m}\nu \leq 0, \\ 0 & \mathbf{m}\nu = 1, \\ \sum_{i=1-\mathbf{m}\nu}^{-1} \delta_{i, \nu_2, \nu_3, \nu_4} & \mathbf{m}\nu \geq 2 \end{cases} \quad (2.13)$$

$$= \begin{cases} -1 & \nu_2 = \nu_3 = \nu_4 \geq 0, \\ 1 & \nu_2 = \nu_3 = \nu_4 \leq -1, \\ 0 & \text{else.} \end{cases} \quad (2.14)$$

Denote by W_l the sub-representation of f_λ^4 corresponding to the irreducible \mathbf{T} -representation $(t_2 t_3 t_4)^l$, for $l \in \mathbb{Z}$. Equation (2.13) translates into

$$\text{rk} \sum_\nu \left(\frac{1}{1 - t_1^{-1}} (t_1^{-1} t^\nu - t^\nu t_1^{-\mathbf{m}\nu}) \right)^{\text{fix}} = \sum_{l \geq 0} (\text{rk } W_{-l-1} - \text{rk } W_l).$$

Notice that $f_\lambda^4 - \overline{f}_\lambda^4 (t_2 t_3 t_4)^{-1}$ is the 3-fold vertex of [23, Eqn. (12)] in the variables t_2, t_3, t_4 , which is \mathbf{T}_0 -movable for $\mathbf{T}_0 = \{t_2 t_3 t_4 = 1\} \subset (\mathbb{C}^*)^3$ (cf. [23, pag. 1279]). This implies that for any $l \in \mathbb{Z}$

$$\text{rk } W_l = \text{rk } W_{-l-1},$$

by which we conclude the proof. \square

We propose a sign rule for the sign in (2.8), relative to the square root e_λ^i .

Conjecture 2.16. Let λ be a finite plane partition. Then the sign relative to the square root e_λ^i is $(-1)^{\sigma_i(\lambda)}$, where

$$\sigma_i(\lambda) = f_{\mathbf{m}}(\lambda) + |\lambda| m_i + \# \{ (a_2, a_3, a_4) \in \lambda : a_j = a_k < a_i \}, \quad (2.15)$$

and $\{i, j, k\} = \{2, 3, 4\}$.

We prove now that the sign rule (2.15) is canonical, meaning that it does not really depend on choosing a preferred x_i -axis.

Theorem 2.17. Let λ be a finite plane partition. For every $i, j = 2, 3, 4$ we have

$$(-1)^{\sigma_i(\lambda)} e^{\mathbf{T}}(-e_\lambda^i) = (-1)^{\sigma_j(\lambda)} e^{\mathbf{T}}(-e_\lambda^j).$$

Proof. Without loss of generality assume $i = 4, j = 3$. Say, for $k \in \mathbb{Z}$,

$$A(k) = \begin{cases} -\sum_{i=0}^{-k} t_1^i & k \leq 0, \\ 0 & k = 1, \\ t_1^{-1} \sum_{i=0}^{k-2} t_1^{-i} & k \geq 2 \end{cases}$$

and, for a \mathbf{T} -representation V ,

$$B(V) = \sum_{\nu \in V} t^\nu A(\mathbf{m}\nu) \in K_{\mathbf{T}}^0(\text{pt}),$$

where the sum is over the weight spaces of V . We extend the definition of $B(V)$ by linearity to $K_{\mathbf{T}}^0(\text{pt})$. By (2.12), we have

$$e_{\lambda}^4 - e_{\lambda}^3 = B(f_{\lambda}^4 - f_{\lambda}^3).$$

Notice the decomposition

$$\begin{aligned} f_{\lambda}^4 - f_{\lambda}^3 &= W_{\lambda} + \overline{W_{\lambda}}(t_2 t_3 t_4)^{-1}, \\ W_{\lambda} &= Z_{\lambda} \overline{Z_{\lambda}}(t_4^{-1} - t_3^{-1}). \end{aligned}$$

Then

$$\begin{aligned} e_{\lambda}^4 - e_{\lambda}^3 &= B(W_{\lambda}) + B(\overline{W_{\lambda}}(t_2 t_3 t_4)^{-1}) \\ &= B(W_{\lambda}) - \overline{B(W_{\lambda})}, \end{aligned}$$

by which we conclude that

$$e^{\mathbf{T}}(e_{\lambda}^4 - e_{\lambda}^3) = (-1)^{\text{rk } B(W_{\lambda})^{\text{mov}}}.$$

We compute the parity of $\text{rk } B(W_{\lambda})^{\text{mov}}$ by induction on the size of λ . If $|\lambda| = 1$, we clearly have

$$\text{rk } B(W_{\lambda})^{\text{mov}} = m_4 + m_3 \pmod{2}.$$

Suppose now that the claim holds for all plane partition of size $|\lambda| \leq n$ and consider a plane partition $\tilde{\lambda}$ of size $|\tilde{\lambda}| = n + 1$; this can be seen as a plane partition λ of size n with an extra box over it, corresponding to a \mathbb{Z}^3 -lattice point $\mu = (\mu_2, \mu_3, \mu_4)$. We have

$$\begin{aligned} B(W_{\tilde{\lambda}}) &= B(W_{\lambda}) + B(Y_4) - B(Y_3) + B(t_4^{-1} - t_3^{-1}), \\ Y_i &= t_i^{-1}(Z_{\lambda} t^{-\mu} + \overline{Z_{\lambda}} t^{\mu}) \quad i = 3, 4, \end{aligned}$$

and by the inductive step

$$\begin{aligned} \text{rk } B(W_{\lambda})^{\text{mov}} &= \sigma_4(\lambda) - \sigma_3(\lambda) \pmod{2}, \\ \text{rk } B(t_4^{-1} - t_3^{-1})^{\text{mov}} &= m_4 - m_3 \pmod{2}. \end{aligned}$$

Clearly, $\text{rk } B(Y_4)^{\text{mov}} = \text{rk } B(Y_4)^{\text{fix}} \pmod{2}$. In fact,

$$\begin{aligned} \text{rk } B(Y_4) &= \sum_{v \in \mathbb{Z}_{\lambda}} (\mathbf{m}(\mu - v + (0, 0, -1)) + \mathbf{m}(v - \mu + (0, 0, -1))) \\ &= -2m_4 |\lambda|. \end{aligned}$$

A simple analysis of $B(Y_4)^{\text{fix}}$ as in (2.13) yields

$$\begin{aligned} \text{rk}(B(Y_4))^{\text{fix}} &= \#\{(\nu \in \lambda: \mu_2 - \nu_2 = \mu_3 - \nu_3 = \mu_4 - \nu_4 + 1)\} \\ &\quad - \#\{\nu \in \lambda: \mu_2 - \nu_2 = \mu_3 - \nu_3 = \mu_4 - \nu_4 - 1\}, \end{aligned} \quad (2.16)$$

where $\nu = (\nu_2, \nu_3, \nu_4)$; in particular, it has to satisfy $\nu \leq \mu$. Therefore we can write it as

$$\begin{aligned} \text{rk}(B(Y_4))^{\text{fix}} &= \sum_{i=0}^{\mu_2} \sum_{j=0}^{\mu_3} \sum_{k=0}^{\mu_4} (\delta_{\mu_2-i, \mu_3-j, \mu_4-k+1} - \delta_{\mu_2-i, \mu_3-j, \mu_4-k-1}) \\ &= \sum_{i=0}^{\mu_2} \sum_{j=0}^{\mu_3} \left(\sum_{k=-1}^{\mu_4-1} \delta_{\mu_2-i, \mu_3-j, \mu_4-k} - \sum_{k=1}^{\mu_4+1} \delta_{\mu_2-i, \mu_3-j, \mu_4-k} \right) \end{aligned}$$

By symmetry we may compute the difference

$$\begin{aligned} \text{rk}(B(Y_4) - B(Y_3))^{\text{fix}} &= \sum_{i=0}^{\mu_2} \left(\sum_{k=-1}^{\mu_4-1} \delta_{\mu_2-i, 0, \mu_4-k} + \sum_{j=0}^{\mu_3-1} \delta_{\mu_2-i, \mu_3-j, \mu_4+1} - \sum_{j=-1}^{\mu_3-1} \delta_{\mu_2-i, \mu_3-j, 0} \right. \\ &\quad \left. - \sum_{k=0}^{\mu_4-1} \delta_{\mu_2-i, \mu_3+1, \mu_4-k} - \sum_{k=1}^{\mu_4+1} \delta_{\mu_2-i, \mu_3, \mu_4-k} - \sum_{j=1}^{\mu_3} \delta_{\mu_2-i, \mu_3-j, -1} + \sum_{j=1}^{\mu_3+1} \delta_{\mu_2-i, \mu_3-j, \mu_4} + \sum_{k=1}^{\mu_4} \delta_{\mu_2-i, -1, \mu_4-k} \right) \end{aligned}$$

Further analyzing which of these sums actually contribute to the rank, we finally get that

$$\mathrm{rk}(B(Y_4) - B(Y_3))^{\mathrm{fix}} = \begin{cases} 1 & \mu_2 = \mu_3 < \mu_4, \\ 1 & \mu_2 = \mu_4 < \mu_3, \\ 0 & \text{else.} \end{cases} \quad \text{mod } 2$$

Therefore we conclude that

$$\mathrm{rk} B(W_{\tilde{\lambda}})^{\mathrm{mov}} = \sigma_4(\tilde{\lambda}) - \sigma_3(\tilde{\lambda}) \quad \text{mod } 2,$$

which finishes the inductive step. \square

Remark 2.18. Let $X = K_Y$ be the canonical bundle of a smooth projective toric 3-fold Y and consider $\mathrm{Hilb}^n(X, \beta)$, where $\beta \in H_2(X, \mathbb{Z})$ is a class pulled-back from Y . Consider a \mathbf{T} -fixed point $Z \in \mathrm{Hilb}^n(X, \beta)^{\mathbf{T}}$ (corresponding to a partition data $\{\pi_\alpha, \lambda_{\alpha\beta}\}_{\alpha, \beta}$) scheme-theoretically supported on the zero section of $X \rightarrow Y$. Locally on the toric charts, label the fiber direction by x_4 and denote by $m''_{\alpha\beta}$ the degree of the normal bundle of $L_{\alpha\beta}$ in the x_4 -direction. Consider the square root of T_Z^{vir} given by

$$v_Z = \sum_{\alpha \in V(X)} v_\alpha^4 + \sum_{\alpha\beta \in E(X)} e_{\alpha\beta}^4.$$

By Remark 2.12, the sign rules proposed for vertex and edge terms imply that the correct sign would be

$$\begin{aligned} (-1)^{\sigma(Z, v_Z)} &= \prod_{\alpha \in V(X)} (-1)^{\sigma_4(\pi_\alpha)} \cdot \prod_{\alpha\beta \in E(X)} (-1)^{\sigma_4(\lambda_{\alpha\beta})} \\ &= \prod_{\alpha \in V(X)} (-1)^{|\pi_\alpha|} \cdot \prod_{\alpha\beta \in E(X)} (-1)^{|\lambda_{\alpha\beta}| m''_{\alpha\beta} + f_{\mathbf{m}_{\alpha\beta}}(\lambda_{\alpha\beta})} \\ &= (-1)^{n + c_1(Y) \cdot \beta}, \end{aligned}$$

where the last equality follows from (2.2) and

$$\begin{aligned} \sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}| m''_{\alpha\beta} &= - \sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}| \deg N_{Y/X}|_{L_{\alpha\beta}} \\ &= - \sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}| \deg K_Y|_{L_{\alpha\beta}} \\ &= \sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}| c_1(T_Y) \cdot [L_{\alpha\beta}] \\ &= c_1(Y) \cdot \beta. \end{aligned}$$

The same sign was proposed in a similar setting for stable pair invariants [13, Prop. 4.2, Rmk. A.2], where such local geometries are studied, motivated by a choice of preferred orientation as in [7].

3. Refinements

3.1. K -theoretic invariants

Let X be a smooth Calabi-Yau 4-fold and M a moduli space of compactly supported sheaves on X . Oh-Thomas defined [27, Def. 5.9] a (twisted) virtual structure sheaf

$$\widehat{\mathcal{O}}_M^{\mathrm{vir}} \in K_0(M),$$

which depends on a chosen orientation and whose properties mimic the virtual structure sheaf with the Nekrasov-Okounkov twist in the classical 3-fold theory [25]. While in the 3-fold theory the twist is introduced to make DT invariants more symmetric, here it is actually necessary to be defined. We define the K -theoretic version of the invariants (1.2) by means of Oh-Thomas K -theoretic virtual localization theorem (cf. [27, Thm. 7.3]); such invariants are studied in [24, 26, 12, 4, 3, 5].

Definition 3.1. Let $V \in K_0^{\mathbf{T}}(\mathrm{Hilb}^n(X, \beta))$. The K -theoretic \mathbf{T} -equivariant Donaldson-Thomas invariants of X are

$$\mathrm{DT}_n^K(X, \beta; V) = \sum_{Z \in \mathrm{Hilb}^n(X, \beta)^{\mathbf{T}}} \sqrt{\epsilon^{\mathbf{T}}} (-T_Z^{\mathrm{vir}}) \cdot V|_Z \in \frac{\mathbb{Q}(t_1, t_2, t_3, t_4)}{(t_1 t_2 t_3 t_4 - 1)}.$$

Here $\epsilon^{\mathbf{T}}$ is the (\mathbf{T} -equivariant) K -theoretic Euler class, which is defined as follows. Let X be a scheme and V a \mathbf{T} -equivariant locally free sheaf on X . We define

$$\epsilon^{\mathbf{T}}(V) := \Lambda^{\bullet} V^{\vee} = \sum_{i \geq 0} (-1)^i \Lambda^i V^{\vee} \in K_{\mathbf{T}}^0(X),$$

and extend it by linearity to any class $V \in K_{\mathbf{T}}^0(X)$. Finally, $\sqrt{\epsilon^{\mathbf{T}}}(\cdot)$ is the (\mathbf{T} -equivariant) K -theoretic square-root Euler class; the complete description and construction of this class is in [27, Sec. 5.1]. Let V be a \mathbf{T} -representation with a square root T in $K_{\mathbf{T}}^0(\text{pt})$. Its K -theoretic square root Euler class satisfies

$$\sqrt{\epsilon^{\mathbf{T}}}(V) = \pm \epsilon^{\mathbf{T}}(T) \otimes (\det T)^{\frac{1}{2}} \in K_{\mathbf{T}}^0\left(\text{pt}, \mathbb{Z}\left[\frac{1}{2}\right]\right).$$

For an irreducible \mathbf{T} -representation⁶ t^{μ} , define

$$[t^{\mu}] = t^{\frac{\mu}{2}} - t^{-\frac{\mu}{2}} \in K_{\mathbf{T}}^0(\text{pt})$$

and extend it by linearity to any $V \in K_{\mathbf{T}}^0(\text{pt})$. It is shown in [18, Sec. 6.1] that

$$\epsilon^{\mathbf{T}}(V) \otimes (\det V)^{\frac{1}{2}} = [V],$$

for any virtual \mathbf{T} -representation $V \in K_{\mathbf{T}}^0(\text{pt})$. Therefore, given square roots $v_{\alpha}, e_{\alpha\beta}$ of $V_{\alpha}, E_{\alpha\beta}$, we have

$$\text{DT}_n^K(X, \beta; V) = \sum_{Z \in \text{Hilb}^n(X, \beta)^{\mathbf{T}}} \prod_{\alpha \in V(X)} (-1)^{\sigma(Z, v_{\alpha})} [-v_{\alpha}] \prod_{\alpha\beta \in E(X)} (-1)^{\sigma(Z, e_{\alpha\beta})} [-e_{\alpha\beta}] \cdot V|_Z.$$

The operator $[\cdot]$ satisfies $[t^{-\mu}] = -[t^{\mu}]$, the same multiplicative property of $e^{\mathbf{T}}(\cdot)$; therefore, the results of Theorem 2.8, 2.14 and 2.17 hold the same replacing $e^{\mathbf{T}}(\cdot)$ by $[\cdot]$.

3.2. Elliptic invariants

An elliptic refinement of DT invariants was proposed in [18, Sec. 8.2] and studied in [4]. Set

$$\theta(p; y) = -ip^{1/8}(y^{1/2} - y^{-1/2}) \prod_{n=1}^{\infty} (1 - p^n)(1 - yp^n)(1 - y^{-1}p^n),$$

$$\eta(p) = p^{\frac{1}{24}} \prod_{n \geq 1} (1 - p^n).$$

Set $p = e^{2\pi i \tau}$, with $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$. Denoting $\theta(\tau|z) := \theta(e^{2\pi i \tau}; e^{2\pi i z})$, θ enjoys the modular behavior

$$\theta(\tau|z + a + b\tau) = (-1)^{a+b} e^{-2\pi i bz} e^{-i\pi b^2 \tau} \theta(\tau|z), \quad a, b \in \mathbb{Z}.$$

See [18, Sec. 8.1] and [17, Sec. 6] for related discussion on the modularity of these functions. For an irreducible \mathbf{T} -representation t^{μ} , define

$$\theta[t^{\mu}] = (i \cdot \eta(p))^{-1} \theta(p; t^{\mu}) \in K_{\mathbf{T}}^0(\text{pt})[[p]][p^{\pm \frac{1}{12}}]$$

and extend it by linearity to any $V \in K_{\mathbf{T}}^0(\text{pt})$. Let $V \in K_0^{\mathbf{T}}(\text{Hilb}^n(X, \beta))$. The *elliptic* \mathbf{T} -equivariant Donaldson-Thomas invariants of X are

$$\text{DT}_n^{\text{ell}}(X, \beta; V) = \sum_{Z \in \text{Hilb}^n(X, \beta)^{\mathbf{T}}} \prod_{\alpha \in V(X)} (-1)^{\sigma(Z, v_{\alpha})} \theta[-v_{\alpha}] \prod_{\alpha\beta \in E(X)} (-1)^{\sigma(Z, e_{\alpha\beta})} \theta[-e_{\alpha\beta}] \cdot V|_Z.$$

Again, as $\theta[t^{-\mu}] = -\theta[t^{\mu}]$, the results of Theorem 2.8, 2.14 and 2.17 hold the same replacing $e^{\mathbf{T}}(\cdot)$ by $\theta[\cdot]$.

⁶ To be precise, we should replace the torus \mathbf{T} with its double cover where the character $t^{\frac{\mu}{2}}$ is well-defined (cf. [25, Sec. 7.1.2]).

References

- [1] K. Behrend, B. Fantechi, The intrinsic normal cone, *Invent. Math.* 128 (1) (1997) 45–88.
- [2] A. Bojko, Orientations for DT invariants on quasi-projective Calabi-Yau 4-folds, *Adv. Math.* 388 (2021) 107859.
- [3] A. Bojko, Wall-crossing and orientations for invariants counting coherent sheaves on CY fourfolds, PhD Thesis, University of Oxford, 2021.
- [4] A. Bojko, Wall-crossing for zero-dimensional sheaves and Hilbert schemes of points on Calabi-Yau 4-folds, arXiv:2102.01056, 2021.
- [5] G. Bonelli, N. Fasola, A. Tanzini, Y. Zenkevich, ADHM in 8d, coloured solid partitions and Donaldson-Thomas invariants on orbifolds, arXiv:2011.02366, 2020.
- [6] D. Borisov, D. Joyce, Virtual fundamental classes for moduli spaces of sheaves on Calabi-Yau four-folds, *Geom. Topol.* 21 (6) (2017) 3231–3311.
- [7] Y. Cao, Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-folds II: Fano 3-folds, *Commun. Contemp. Math.* 22 (07) (2020) 1950060.
- [8] Y. Cao, J. Gross, D. Joyce, Orientability of moduli spaces of Spin(7)-instantons and coherent sheaves on Calabi-Yau 4-folds, *Adv. Math.* 368 (2020) 107134.
- [9] Y. Cao, M. Kool, Zero-dimensional Donaldson-Thomas invariants of Calabi-Yau 4-folds, *Adv. Math.* 338 (2018) 601–648.
- [10] Y. Cao, M. Kool, Counting zero-dimensional subschemes in higher dimensions, *J. Geom. Phys.* 136 (2019) 119–137.
- [11] Y. Cao, M. Kool, Curve counting and DT/PT correspondence for Calabi-Yau 4-folds, *Adv. Math.* 375 (2020) 107371.
- [12] Y. Cao, M. Kool, S. Monavari, K -theoretic DT/PT correspondence for toric Calabi-Yau 4-folds, arXiv:1906.07856, 2019.
- [13] Y. Cao, M. Kool, S. Monavari, Stable pair invariants of local Calabi-Yau 4-folds, *Int. Math. Res. Not. rna061* (2021) 1–46.
- [14] Y. Cao, N.C. Leung, Donaldson-Thomas theory for Calabi-Yau 4-folds, arXiv:1407.7659, 2014.
- [15] Y. Cao, Y. Toda, Gopakumar-Vafa type invariants on Calabi-Yau 4-folds via descendent insertions, *Commun. Math. Phys.* 383 (1) (2021) 281–310.
- [16] D. Edidin, W. Graham, Characteristic classes and quadric bundles, *Duke Math. J.* 78 (2) (1995) 277–299.
- [17] B. Fantechi, L. Göttsche, Riemann-Roch theorems and elliptic genus for virtually smooth schemes, *Geom. Topol.* 14 (1) (2010) 83–115.
- [18] N. Fasola, S. Monavari, A.T. Ricolfi, Higher rank K -theoretic Donaldson-Thomas theory of points, *Forum Math. Sigma* 9 (2021) 51.
- [19] D. Huybrechts, R.P. Thomas, Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes, *Math. Ann.* 346 (3) (2010) 545–569.
- [20] M. Kool, J.V. Rennemo, Proof of a magnificent conjecture, in preparation.
- [21] J. Li, G. Tian, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties, *J. Am. Math. Soc.* 11 (1) (1998) 119–174.
- [22] P.A. MacMahon, *Combinatory Analysis*. Vol. I, II, Dover Publications, Inc., Mineola, NY, 2004. Reprint of “An Introduction to Combinatory Analysis” (1920) and “Combinatory Analysis. Vol. I, II” (1915, 1916).
- [23] D. Maulik, N. Nekrasov, A. Okounkov, R. Pandharipande, Gromov-Witten theory and Donaldson-Thomas theory. I, *Compos. Math.* 142 (5) (2006) 1263–1285.
- [24] N. Nekrasov, Magnificent four, *Ann. Inst. Henri Poincaré D* 7 (4) (2020) 505–534.
- [25] N. Nekrasov, A. Okounkov, Membranes and sheaves, *Algebr. Geom.* 3 (3) (2016) 320–369.
- [26] N. Nekrasov, N. Piazzalunga, Magnificent four with colors, *Commun. Math. Phys.* 372 (2) (2019) 573–597.
- [27] J. Oh, R.P. Thomas, Counting sheaves on Calabi-Yau 4-folds, I, arXiv:2009.05542, 2020.
- [28] A. Okounkov, Lectures on K -theoretic computations in enumerative geometry, in: *Geometry of Moduli Spaces and Representation Theory*, in: IAS/Park City Math. Ser., vol. 24, Amer. Math. Soc., Providence, RI, 2017, pp. 251–380.
- [29] A. Okounkov, N. Reshetikhin, C. Vafa, Quantum Calabi-Yau and classical crystals, in: *The Unity of Mathematics*, in: *Progr. Math.*, vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 597–618.
- [30] A.T. Ricolfi, The equivariant Atiyah class, *C. R. Math. Acad. Sci. Paris* 359 (3) (2021) 257–282.
- [31] R.P. Thomas, A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations, *J. Differ. Geom.* 54 (2) (2000) 367–438.