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Embodied design using augmented reality: the case of the gradient

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We study the augmented reality sandbox (ARSB) as an embodied learning environment to foster meaning making in the context of bivariable calculus. We present the case of Tiago, a first-year bachelor chemistry student, performing a series of tasks based on embodied design, including perception-based, action-based and incorporation-based tasks. Tiago's work demonstrates the affordances of the ARSB, e.g. to trace a height line and to manipulate plastic planes either with or without feedback from projected height lines. Tiago's reasoning about mathematical concepts, e.g. the parameters in a plane equation and the gradient vector, is supported by perceptual structures that he discovers during these embodied tasks. We distinguished two ways in which ARSB affordances were used in the learning sequence. In perception-based and action-based tasks, the affordances of the ARSB were immediately available and intensively involved in the interaction. In incorporation tasks, on the contrary, a critical affordance was deliberately removed and the student was able to reproduce its functionality without technology.

1. Introduction

Students' meaning making is a central challenge in mathematics education (Thompson 2013). Meaning propagates in a recursive way, because (mathematical) concepts gain meaning by being grounded and integrated in previous meaningful experiences. In this study we focus on a powerful design heuristic for meaningful education: ground new concepts on embodied experiences (Abrahamson 2009; Nemirovsky *et al.* 2020; Radford 2014). As foundation for meaning making, theories of embodied design promote goal-oriented actions (Abrahamson *et al.* 2020). In embodied design, goal-oriented action is often mediated by digital artefacts, e.g. for providing feedback. As the goals are achieved, the artefact becomes incorporated in the constitution of new instrumented actions (Drijvers 2019; Shvarts *et al.* 2021). In this paper we study conceptually oriented tasks that foster embodied interaction with suitable tools designed to support the constitution and grounding of the gradient of a bivariable function. The tasks are enacted in a so-called *embodied learning environment* (Duijzer *et al.* 2019).

Fig. 1 Tiago at work in the ARSB.

The main component of our embodied learning environment is the augmented reality sandbox (ARSB), as developed at UC Davis[1] (Fig. 1). This interactive artefact consists of a box of sand on which a projector visualizes height lines, altitude colours and flowing water caused by imaginary local showers in real time. Augmented reality can be deployed in various ways in an educational context (Ibáñez and Delgado-Kloos 2018; Levy *et al.* 2020). We are interested in the ARSB's affordances for students' exploration based on the height lines projected in real time on the sand and on plastic planes handled inside the sandbox.

The mathematical topic at stake is bivariable calculus. It is known that students try to make sense of concepts in bivariable calculus in terms of their one-variable equivalents (Weber and Thompson 2014). Other recent work confirms how students struggle to make sense of gradients and other notions in bivariable calculus (Gaisman *et al.* 2018; Martínez-Planell *et al.* 2015; Nagle *et al.* 2019). Our research focuses on planar equation $z = ax+by+c$, in particular on constituting the meaning of (1) the parameters a, b and c in relation to transformations of the plane; and of (2) the gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial y}{\partial y}\right) = (a, b)$ as a vector in the contour plot and in relation to the steepest direction in a given point. These are rather specific topics, but they demonstrate a general issue in calculus: both the plane equation and the gradient are computed by straightforward procedures, which makes it tempting for students and teachers to disregard their meanings as grounded in spatial visual experiences related to geometry (Tall 1992). The recent survey by Martínez-Planell and Trigueros (2021) confirms the need to introduce the geometric point of view in bivariable calculus education.

Our aim is to study the ARSB as an embodied learning environment to foster meaning making in the context of bivariable calculus. We study how the concepts of plane equation and gradient can be grounded in sensorimotor experience and instrumented actions emerging from embodied designs in the ARSB. To this purpose we designed a task sequence. In this paper we discuss a case study of a student

1 https://arsandbox.ucdavis.edu/

performing the tasks, to illustrate the potential and challenges of this approach involving embodiment and augmented reality.

2. Theoretical framework

2.1. Embodied design: action-based and perception-based tasks

Embodied design, as developed by Abrahamson, can be classified in action-based and perception-based designs (Abrahamson 2009, 2014; Abrahamson *et al.* 2020). The action-based design framework aims to ground mathematical concepts in students' natural capacity to adaptively solve sensorimotor problems (Bernstein 1996).

> Action-based embodied designs are predicated on the research-based general hypothesis that, in the course of attempting to perform complex movements, such as simultaneous orthogonal bimanual manipulations, people spontaneously discern new sensorimotor perceptual structures that facilitate and regulate effective motor control; with appropriate intervention, these new structures, in turn, can become signified as mathematical objects. (Abrahamson *et al.* 2020, p. 5)

The embodied action-based design framework is summarized in three consecutive phases: moving in new ways—signifying—reconciling (Abrahamson *et al.* 2020). Within this design framework sensorimotor problems are designed so that the adaptively obtained sensorimotor solutions carry mathematical meaning as observed by teachers and, of course, the designer, but is initially not necessarily articulated as such by the student. In the next stage of the task, through a guided process of reflection, students are invited to develop their sensorimotor experiences into mathematical conceptualizations through verbalizations and gestural expressions (Flood 2018; Flood *et al.* 2020). In our study, we also facilitated this transition by offering the students an opportunity for expressing their experiences through diagrams and symbols.

A paradigmatic example of action-based embodied design is the *Mathematics Imagery Training (MIT)*. Movement is mediated by an artefact, like a touch screen, to provide continuous feedback. This feedback is used to promote a certain type of goal-oriented movement.

> This is the dynamical conservation principle of action-based design: enacting continuous motion that varies positional/quantitative properties of topical elements yet sustains an overall target feedback. (Abrahamson 2014, p. 8)

An example of an MIT task is to maintain two vertical bars on a touch screen in a fixed proportional height (Abrahamson and Bakker 2016). The feedback is provided in the form of the bars or the screen gradually turning green as the proportional height is achieved. The task is to reach green and then keep green while moving. Eye-tracking data and verbal reports revealed that solving this motor task coincides with the emergence of a new perceptual structure: an attentional anchor, where in instead of controlling the two hands separately, students coordinate the movement by focussing attention on an imaginary diagonal interval between the two hands. Later, students use this imaginary object to qualitatively describe their solution strategies. Overlaying a grid and then number axes on the screen supports students developing their initially sensorimotor solution into statements such as 'the height of one bar is a multiple of (e.g. twice) the height of the other' or symbolically '$h_1 = 2\,h_2$'.

Perception-based designs aim to ground the meaning of mathematical concepts in students' natural perceptual ability, in their naïve views with respect to a situation. The student recognizes familiar structures in a new mathematically meaningful context, thus expanding natural perceptual ability in directions intended by the designer. Similarly to the action-based genre this is followed by a phase of reflection in which these views are developed and reconciled with a more formal view. In an example of a perception-based task, students perceive a ladle that scoops four balls from a container with equal amount of blue and green balls. The initial judgement that two blue: two green is the most likely situation is developed into a probability model for the relative occurrence of outcomes (Abrahamson 2014).

2.2. Incorporation-based tasks

We introduce another type of embodied design: incorporation-based tasks. For an incorporation-based task, students are first invited to solve a sensorimotor task with feedback of some artefacts (e.g. an action-based task) or observe perceptual qualities enabled by an artefact (e.g. perception-based task), and then invited to perform the same task without the artefact, just with their body. Such a sensorimotor problem can be adaptively solved, accompanied by spontaneous transformation of instrumented actions to new bodily actions based on the anticipation of the feedback from the environment, which is not perceived, but only imagined in this case. This practice was appropriated by the tutors as they work within action-based embodied designs (e.g. Flood *et al.* 2020). We thematized it as a type of activity, highlighting the potential for environments enriched by augmented reality. By removing an artefact and actual feedback from the environment, we stimulate the students to imagine this feedback while enacting the task. This way, a mathematical component that was performed by the tool, becomes now incorporated into students' bodily system and can be later reflected upon, thus grounding mathematical meaning making. Incorporation is in a sense the opposite of outsourcing a task to an artefact instead of a person. Thus, inviting a student to incorporate the function of the artefact, we provide an opportunity for embodied understanding of the artefact's functionality (Shvarts *et al.* 2021).

2.3. Research question

How can the affordances of the ARSB in embodied design tasks foster meaningful reasoning about contour plots, planar equations and the gradient of linear functions in two variables?

By an affordance of an artefact we mean an opportunity the artefact offers for a student to act (Gibson 1986).

3. Method

The general methodological approach of this design study is a micro-ethnographical analysis (Streeck and Mehus 2005) of a case study. With respect to the design study methodology (Bakker, 2018), we built our design sequence based on theoretical assumptions of embodied cognition theories that stress enactment as a source of meaning and particular design principles of diverse genres of embodied designs (Abrahamson 2014). We adapted the design principles to the context of augmented reality and advanced mathematical concepts, which led us to introducing a new type of embodied design. Based on these theoretical assumptions and design principles, we developed the intervention and formulated hypothetical student behaviour. Here we report on the result of an empirical try-out of the developed design with an eye on the ARSB affordances and meaning-making processes. The first author of the paper acted as a tutor facilitating the learning.

3.1. Case study participant

The student in this case study, Tiago, participated in a basic mathematics course on calculus and linear algebra for first year bachelor chemistry students. The intervention took place in October 2020, just before the part on bivariable calculus began, after one-variable calculus had been revised. Tiago, 18 years old, responded to a call to volunteer as a participant. At the mid-way test of the course Tiago scored 80%, where the average was 72% (SD 22%).

3.2. Description of intervention

The design consists of a sequence of tasks as presented in Table 1. We briefly introduce and motivate each task. We also indicate whether the task is based on perception (**P**), action (**A**) or incorporation (**I**).

3.3. Data collection and analysis

The session was recorded on video from two viewpoints in the Teaching & Learning Lab Studio of Utrecht University—an overview viewpoint and a view on the sandbox. The viewpoints were edited into one video. With respect to micro-ethnographic methodology (Streeck and Mehus 2005), the video was watched independently by three of the authors with attention for hypothesized student behaviour as described in Table 1, including motor action, gestures and verbal utterances. The researchers marked and annotated the fragments of the behaviours where the use of the ARSB's affordances and their role in eliciting mathematical meaning could be seen. One of the authors was focussing in particular on Tiago's gestures and motor actions. From these fragments they selected vignettes that provided the most vivid data with respect to the research question. These vignettes were analysed in depth.

4. Results

Below we present key features of Tiago's observed behaviour task by task and how behaviours at different tasks are related. The goal is to highlight the moments when Tiago speaks or gestures in a way that suggest a process of meaning making and emphasize the relation to the features of the task (action-, perception- or incorporation-based) he is performing. We suggest the reader performs the described movements and gestures by Tiago while reading.

4.1. Tasks 1–3: height lines and contour plots

During Task 1 Tiago talks about the 'angle of a slope'. This happens during the incorporation-based task of following a height line with eyes closed. Hence, it is likely that this was caused by Tiago's shift of attention to touch (or maybe proprioception). It is important to note that the notion of angle was not mentioned by the teacher, and it is a priori not clear what angle he refers to. In Task 6, he returns to the notion of angle.

While working on Task 2, Tiago says 'I expect that the height difference between height lines is always the same' and makes a rhythmical gesture: three vertical markings in the air. Then he says 'I use my thumb as a reference for the distance', as he uses his thumb to produce the regular markings on the

TABLE 1. *The designed task sequence, theoretical support for those tasks and hypothetical student behaviour*

Task	Theoretical support for task and hypothetical student behaviour
1. *Height lines* The student traces a projected height line in the AR Sandbox, first with eyes open (**A**), then with eyes closed (**I**). Then they describe how they are doing it.	After closing eyes, the student is required to incorporate the height sensing functionality of the stereo-camera. While at first, the student relies on the visual information provided by stereo-camera and projector, later the student needs to use touch and/or proprioception[2]. This switch between registers is an extra challenge. This task grounds the students' concept of height lines in a motor experience, supported by deep levels of movement control.
2. *Vertical scale* The student draws the heights of two height lines on a stick in the sandbox (**P**), predicts the heights of the others and checks this (**I**).	This task introduces the concept of vertical axis/scale into the situation and connects it to the ARSB height lines projection functionality. The student transfers the functionality of height measurement to a new, more tangible, instrument: the stick.
3. *Contour plot* The student draws an image of level lines onto a transparent plastic plane (**I**). Also, the student is invited to draw these contour lines holding the marker below the transparent sheet (**I**).	The ARSB projects the height lines on the landscape, not on a horizontal plane; with the augmented reality feedback switched off, the student performs the transition from a three-dimensional landscape to a two-dimensional picture. This transition is a form of projection as well, so it is an incorporation of the initial functionality of ARBS.
5. *Horizontal plane*[3] The student positions and then moves the plane in a way that the height lines on it disappear and then reflects on the mathematical rule that support such movement (**A**). The teacher introduces the axes of a coordinate system. For a fixed plane the teacher asks 'Look, this is a point with coordinates (x, y, z), what is the rule for being on this plane?' The teacher asks: 'Now we move the plane such that the height lines are still gone, how does the rule change?'	This task has the characteristics of an action-based task, beginning with a sensorimotor task of trying to maintain a continuous feedback. We expect the notion of keeping horizontal to emerge. The follow-up task immediately pushes for the algebraic description, assuming some familiarity with analytical geometry in two dimensions, and the three-dimensional coordinate system. The student needs to express the rule that was used to solve the task by symbolic means. The absence of height lines means the height is invariant along the plane. Since the height corresponds to the z-variable, this means the z-variable should not change: z is constant, $z = c$. Keeping the plane horizontal, leaves rotation within the plane and all translational degrees free. This rotation and horizontal translation does not change the equation, but vertical translation does. This vertical movement is expressed mathematically as a variation of the parameter c. In the last task the student is invited to develop this sort of reasoning.
6 & 7. *Tilted planes* Tasks 6 and 7 are similar to 5, but now the planes are tilted. In Task 6, students are invited to position and then move a rectangular plane such that the projected height lines are parallel to the y-axis, and in Task 7 parallel to the x-axis (**A**).	A plane with height lines parallel to the y-axis can be described algebraically by $z = a\,x$ (or, more generally, by $z = a\,x + c$). The motor problem draws the student's attention to height lines being parallel to the y-axis. This may help them to see more easily that on the plane the height z does not depend on y, but only on x. We expect it could be challenging for students to express the rule as a linear relation. Possibly, students are helped by viewing the plane from the side to see it as line. There is one rotational degree of freedom now (tilting around an axis parallel to y-axis and three translational degrees of freedom. These movements are associated in a subtle way with the parameters of the equation (horizontal movement perpendicular to the height lines and vertical movement changes c; rotation round an axis parallel to the y-axis changes a en c), which can be explored in discussion with the teacher.

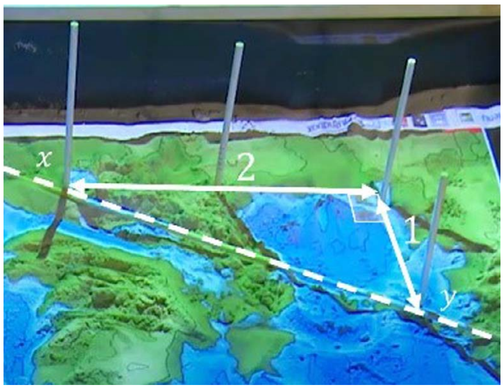
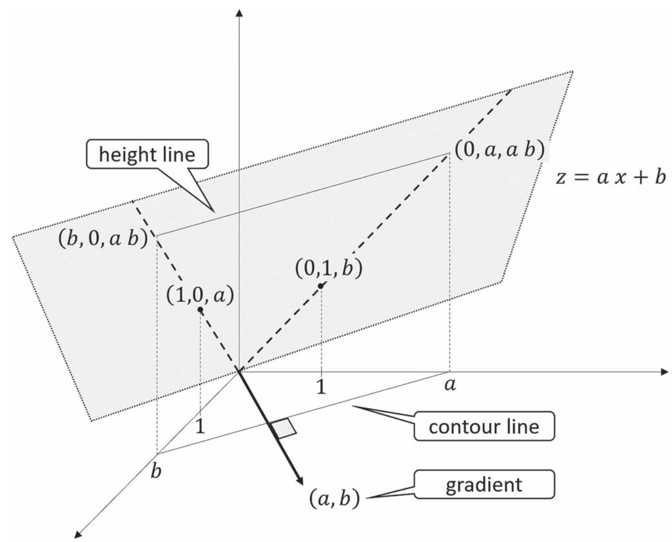
(Continued)

TABLE 1. *Continued*

Task	Theoretical support for task and hypothetical student behaviour
8. *Tilting and lifting: plane equation* The teacher shows how you can combine both forms of tilting and vertical lifting with a plane. They put a rectangular plane in a fixed, tilted position in the sandbox, with height lines not parallel to any of the axes and ask: 'Can you move the horizontal plane to join the plane spanned by the tilted one?' (**A**). This task is repeated for a different position of the tilted plane. Then the teacher asks for the general equation of the plane.	The task is designed to let students experience that a horizontal plane associated with $z = 0$ can be transformed into any other plane by just three transformation: rotations round the y-axis and x-axis and vertical translation. These movements are associated with the three parameters a, b and c. Solving the sensorimotor problem adaptively should support students in understanding the general equation for a plane is a linear combination of the previous terms: $z = a\,x + b\,y + c$. This is not exactly an action-based task. The goal is to reach a goal-state, using feedback, but not to continuously maintain an affirmative feedback.
9. *Gradient of a plane, qualitatively* The student is invited to move a circular plane in a way that the direction of the height lines stays the same, but the distance varies and further reflect on the embodied strategy (**A**). Later, the student is invited to roll a marble down the plane for several positions and to explain the direction of the rolling (**P**). In the second sub-task, they do the same but with the distance between the height lines fixed and the direction variable (**P**). Lastly, for a fixed plane they draw on a horizontal transparent plastic plane some contour lines and the rolling direction of the marble (**I**).	The gradient of a plane (associated with $z = a\,x + b\,y + c$) is the vector (a, b). A vector consists of two properties: length and direction. The direction of the gradient is the (vertically projected) steepest upwards direction of the plane–this is perpendicular to the contour lines. The length equals the tangent of the angle between the plane and the xy-plane. The aim of this task is to distinguish these two properties (length and direction) separately in an embodied, qualitative way, each as a reflection on an invariant of the constraint movement under control. The allowed movements in the first sub-task are translations (that the student now knows only influences the parameter c) and rotation around an axis parallel to the projected height lines. The latter influences the speed of the marble but not its projected direction. In the second sub-task, keeping the distance between height lines invariant is achieved by keeping the steepness invariant and thus, unbeknown for the student, keeping invariant the aforementioned angle and the length of the gradient vector. The allowed movements are horizontal and vertical translation and rotation around a vertical axis. This rotation is a way to set the orientation of the plane very different from the tilting that determines a and b. This is precisely why it is difficult to understand how the gradient is related to the parameters the way it is. Based on the student's reflections on the performance in the two action-based sub-tasks, the teacher establishes the gradient qualitatively as a vector constituted of these to invariants: direction and length (∼steepness∼acceleration of the marble). Moreover, we expect students to discover that the direction in which the marble rolls is perpendicular to the height lines. This is a central insight about the relation between contour lines and gradient direction—although the gradient points opposite to the projected direction the marble goes. The final task invites the student to incorporate the rolling marble as their hand moves the marker along the trajectory of the marble projected onto the two-dimensional contour plot as a 2-vector. This task allows the student and teacher to consolidate the gradient vector as a new concept as described above.

(Continued)

TABLE 1. *Continued*

Task	Theoretical support for task and hypothetical student behaviour
10. *The gradient of a plane quantitatively* The teacher lays a long stick in the sandbox as in the picture (dashed line)  and discusses the direction vector $(2, -1)$ and the normal vector $(1, 2)$ with the student. Then they ask: 'Can you move the plane by the tilting around the *x*- and *y*-axes—the *a* and *b* parameters—to keep height lines parallel to the stick?' **(A)** Next, the teacher asks 'What is the direction of the gradient with respect to the contour lines?'	This task similar to the action-based task in the theoretical framework section. The direction of the height lines, which in the previous task were obtained by rotation, need now be obtained by coordinating the tilting of a plane around the *y*-axis (parameter *a*) and *x*-axis (parameter *b*). The right height line is obtained when the proportion $a : b$ equals $1 : 2$. The teacher could invite the student to test more proportions, like $1 : 3$ and $2 : 3$. This should lead to the hypothesis that the proportion of the gradient vector coordinates (which has the same direction as the normal vector) is the same as the proportion of *a* and *b*. A final step for the teacher could be to explain why the gradient is precisely (a, b), and not a multiple, using the picture below. 

[2]the sense of self-movement and body position

[3]Task 4 was removed from the sequence

stick. The thumb is deployed to incorporate a functionality of the sandbox. In Task 10, he again uses his fingers to measure heights. Vignette A shows Tiago work on Task 3.

Vignette A: Drawing contour lines on a transparent without AR feedback

Tiago moves with his upper body so it is positioned above the sandbox. One hand is used to hold the transparent screen in place, the other to hold the marker. Tiago draws a line by trying to follow the edge of a "hill".

Tiago: "I go and hang above it a little. And then I try….yes…to see with my eyes where the mountain… the hill begins" (traces the same height line on the sand again) "and I put a line there".

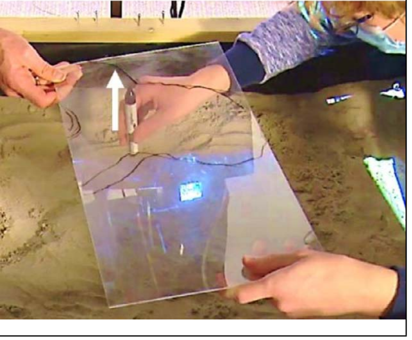
Rogier: "Can you draw a line from below… so that you use the marker from below?".

Tiago moves the marker with the bottom to a point on the sand; then he carefully moves it up vertically until it touches the transparent; then he traces out a contour line onto the transparent.

Tiago: "[I] try to move around… such that the distance between the sand and the bottom of the marker stays a bit the same".

When he has gone round and arrives at the starting point, he makes a similar movement as in the beginning moving the marker vertically, going from bottom on the sand to tip on the transparent.

This procedure he repeats when drawing another line.



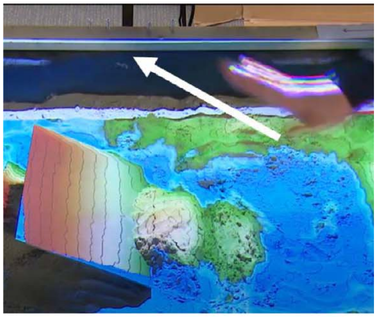
Tiago views the transparent plane and sand below from right above, in a perpendicular angle to the floor, to draw the lines *on top of* imaginary height lines in the sandbox. Positioning the vision line from the sand to the eye this way is an incorporation of the projection. In case the marker is below: the vertical movement between screen and surface incorporates the vertical projection using the marker in a new way: as an artefact to which vertical movement is outsourced. So, the functionality of the ARSB is transferred partly to the movement and partly to the marker. Tiago's behaviour supports the constitution of a contour plot as a vertical projection of the three-dimensional landscape on a sheet. Also noteworthy is Tiago's use of the word 'mountain' to make meaning of the height lines and help him perform the task.

4.2. Tasks 5–8: the plane equation

Tiago has little difficulty performing the motor part of Task 5. He solves it by rotating the plane around one of height lines until the others disappear. The transition to the algebraic description ($z = c$) is unproblematic. He uses a gesture for vertical translation, moving his stretched-out hand up and down. Also, for Task 6 Tiago performs the motor task easily. There is little continuous repositioning. First very quickly, but mistakenly he tilts the plane such that the height lines are parallel to the x-axis. After the teacher points this out, Tiago swiftly adjusts the position. Then he suggests that the angle with the sand needs to be 45 degrees, but using the ARSB feedback Tiago discovers that any angle works. Tiago uses a lot of gestures as he acts, tentatively explores and explains. Vignette B begins when Tiago is asked to find the algebraic description of the tilted plane.

Vignette B: An equation for a tilted plane

Tiago first describes the plane in a qualitative way: How the height does not depend on y, and how it increases as x increases; the latter accompanied by a distinguished gesture, moving a stretched hand away from himself first horizontally, tilting it, and then diagonally up, parallel to the surface of the plane. Thus, he uses the experience of moving the plane up and down and then tilting it in action-based Tasks 5 and 6.



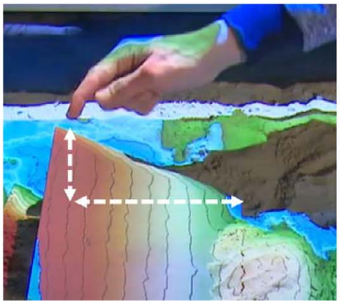
Tiago: "Let me think… this z-value is related to… now I have to think a bit… something to do with …" (Tiago looks again at the markings of axes on the box)… the sine or cosine of the angle" (Tiago points along the tilted edge of the plane) "between the plane and the x-axis and, so, with the place on the x-as… I expect."

Rogier confirms and draws the attention back to finding an equation.

Tiago: "Let me think a bit."

Rogier asks Tiago to think aloud

Tiago: "In these cases I always need to remember that rule. In any case there is the angle" (Tiago points at the point where the edge meets the sand, and then gestures with his index finger along the edges of the plane) "you know this one, so you can provide it, it has to be given or whatever, and…uhm… you know the x-length" (gestures along the sand under the edge parallel to the x-axis) "… if you call this point zero" (indicates where the plane meets the sand) "then you know the position on the x-axis" (again gestures along the sand) "… so then it is… uhm… in the end you want to know the height of this triangle" (Tiago points up and down, and then back and forth along the sand, implying the edge of the plane as the third edge of the triangle) "So then you know… an adjacent and an opposite side [of the triangle], so you arrive at the tangent [note: he means the trig function]"



Rogier: "mm-hmm"

Tiago: "So it is the x-value times the tangent of the triangle, and that plus a constant value to move the plane" (Tiago moves his stretched hand up and down) "in this angle" (puts hand parallel to the plane and continues to move up and down) "with respect to the z-axis".

As early as Task 2 Tiago talks about angles. He sees a relation of angles to steepness. Maybe this explains why he introduces trigonometry to solve the problem. The experience of interacting with the AR of the sand box supports his gestures; most importantly, he gestures how he sees a new structure revealed by the new action of tilting the plane parallel to y-axis in Task 6, the sides of a right triangle. Note also that Tiago introduces a zero on an axis now that he needs it.

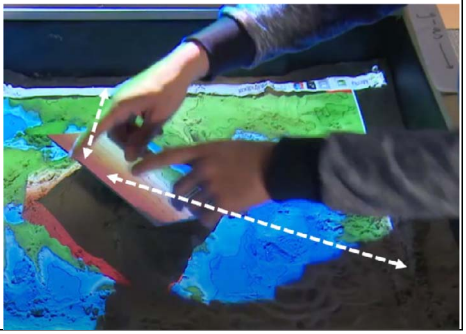
Task 7 of tilting along the x-axis is easily performed, with even less apparent use of the feedback. Many spoken concepts and ideas, like the angle and the translations are supported by gestures. The problem in Task 8 is easily solved for the first position of plane. As also becomes clear later, this appears to be the result of adaptive behaviour, as the students finds a solution through multiple try-outs observing the result of movements and feedback. For the second (and more difficult) position of the plane the feedback of the projected height lines is evidently used.

Tiago agrees that any plane could be 'made' using the three transformations linked to the parameters. When asked about the algebraic description of the plane Tiago comes up with the solution as he looks at the flip chart with the previous equations: $z = a\,x + c$ and $z = b\,y + c$). He seems to just combine these equations linearly, but seems not to know why. Vignette C begins where the teacher asks him to explain why.

Vignette C: Gesturing the plus in $z = ax + by$

Rogier has asked why there is a plus between $a\,x$ and $b\,y$.

Tiago: "I think of it now, that if this is positioned like this" (Tiago moves his hand parallel to the plane) "with here the Origin" (Tiago indicates a corner of the sandbox as the origin) "then the point that has both a high x- and y-value, so that is there" Tiago indicates a point above the opposite corner of the sandbox that is part of the infinite plane through the cardboard plane "So actually this point" (now he points at the mid of the top edge of the plane) that is higher than the point that is not far away… that does not have a high y-value, but does have a high x-value" (Tiago move his hand away from the Origin along the x-axis and up) "or a heigh y-value, but a low x-value" (same gesture but now along y-axis) "these are then lower than if both are high." Tiago makes synchronous movements of the right-hand index finger from the x-axis (from the position where the y-value is small) to the middle of the top edge of the plane, and the left hand the same beginning from the y-axis.



During Task 8 Tiago gestures are very expressive. Again, Tiago puts the Origin where it suits him. He struggles and does not manage to fully explain the equation, but during his last attempt his hand gestures actually explain the main idea: his hands can be interpreted as indicating the vector sum, as a parallelogram, of vectors $(x, 0, a\,x)$ and $(0, y, b\,y)$, for not further specified values of x and y. So the mismatch between gestures and speech seems to indicate that his embodied knowledge is beyond of what he is able to verbally express (Goldin-Meadow 1999).

4.3. *Tasks 9 and 10: steepness and the gradient*

In Task 9, keeping the direction of the height lines the same is a task easily solved by Tiago. Keeping the distance between height lines invariant leads to more serious exploration. Vignette D begins after Tiago has found a few discrete positions that solve the task.

Vignette D: Discovery of steepness as invariant

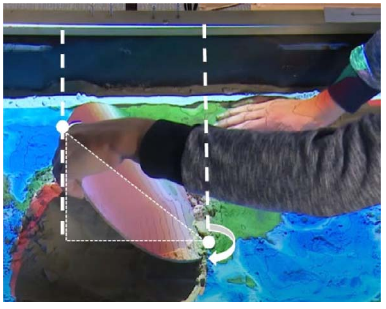
Tiago: "Like this you get [it] too, and then you can make all those movement again" [Tiago refers to horizontal and vertical translations] "but then, in any case, it will remain just as steep".

Then Tiago begins to rotate the plane round the place where it touches the sand

Tiago: "Whether you hold it like this or this"

Rogier: "Ah.. What movement are you making?"

Tiago: "I rotate it. I rotate it round an axis. That is actually what I'm doing."



Tiago keeps rotating the plane. He chooses a correct position, then waits for the feedback to update, thus clearly using the affordance of the ARSB for establishing a new action.

Rogier: "Do you pay attention to anything in particular while rotating?"

Tiago: "No, nothing special. Yes, that I keep it equally steep. Moreover, it doesn't matter whether you rotate around the point at the bottom [where it touches the sand] or at the top [where Tiago holds it]"

As is clear from 'I keep it equally steep', Tiago discovers steepness as the essential invariant to perform the task. The last sentence contains clues on his points of reference: the points at the top of the disc and the point where it touches the sand, and less obviously, the two implied vertical rotational axes. These hint towards the triangle that Tiago gestures in Vignette B. The triangle, in turn, hosts the angle that Tiago has reinvented as a measure of steepness, tracing back to Task 2, thus meaningfully connecting the new experience with the previous one.

Furthermore, at Task 9 Tiago discovers that the marble rolls perpendicular to the height lines. At first he finds this hard to explain, but just after he draws a contour plot with a vector for the direction of the ball, he comes up with an explanation in term of shortest distance travelled.

Tiago clearly uses the ARSB feedback in Task 10. He then forms a hypothesis about the proportion of the angles, which changes into *a* and *b*, which changes into heights, which he tests using his fingers as units of measurement—a nice solution, probably inspired by his approach in Task 2.

Vignette E: Inquiry into gradient-coefficient relation

In the action-based Task 10, using the AR feedback, Tiago easily tilts the plane such that the height lines are parallel to the slat on the sand representing a line in the xy-plane with direction $(-2,1,0)$.

Tiago: "I definitely see that this angle, say,…" (Tiago gestures with his index along the sand and along the edge of the plane, drawing the angle) "the …" (Tiago looks at the x-axis) "a in this case, it is smaller than the b, and I expect it's half, because of this distance" (Tiago gestures along the marking sticks at the x-axis) "is twice as long, so then…. you get…." (Tiago looks at the flip chart with the equations on it) "let me think … $a = 2$, and then $b = 1$" [note: wrong way round: a little slip?]

Rogier: "Can you test this?"

Tiago: "Erm… then one finger like this and two fingers there… and then it seems to be quite right."

(Tiago pulls away his two fingers) "If I do this, then it is not right."

When Rogier asks whether there are more solutions, Tiago immediately says and shows there is also a solution with two and four fingers. Later he explains the ratio has to be right.

5. Discussion

The research question of this study was: How can the affordances of the ARSB in embodied designed tasks foster meaning making and reasoning about contour plots, planar equations and the gradient of linear functions of two variables? Our proposed learning trajectory consisted of three types of embodied tasks: perception-, action- and incorporation-based tasks. Let us discuss how the task characteristics and the affordances of the ARSB contributed to Tiago's meaning making and reasoning.

5.1. Action-based tasks

In action-based tasks (as in Tasks 5–10), new perceptual structures should emerge from adaptive motor control based on continuous feedback from ARBS. These structures can be developed into mathematical notions. While previously embodied action-based designs were based on continuous colour feedback, in this study we explored other possibilities of the feedback in this design genre. In action-based tasks, we used one feature of the ARBS—ability to project height lines—to set up multiple affordances, as we asked the students to maintain different sorts of feedback. Tiago's behaviour in action-based tasks varied with the level of the task and his pre-knowledge. In some tasks his actions were based on the feedback from the system and were explorative, while in other cases the students was confident and knowledgeable. For example in Task 5, he rotated the plane around a height line to make the other height lines disappear. Similarly, in Tasks 6 and 7, Tiago's performance was a mixture of well-planned enactments and spontaneous corrections and discoveries based on the feedback. Feedback sometimes confirmed his initial hypothesis, sometimes provided opportunities to discover new forms of action. Yet, in other tasks Tiago's reasoning developed as he continuously transformed the position of a plane inspired by the feedback. For example, in Task 8, we observed adaptive motor control behaviour to transform one plane into another. These experiences grounded the insight that only three parameters a, b and c control the position of a plane. Tiago combined his previous equations into the general equation. When asked to explain the meaning of this equation, Tiago produced a qualitative explanation clearly grounded in his actions solving previous motor tasks (6–8). This evidences that embodied, spatially articulated experience grounded the meaning of mathematical notations. His gestures referred to tilting and lifting planes, and also contained a suggestion of a correct mathematical explanation to Task 8. In Task 9, again, we observed adaptive control of motor actions and the use of the feedback to maintain a constrained motion 'rotating around an axis to maintain the steepness'. Tiago's developed notion of invariant angle associated to the steepness of a plane informed him on how to perform the task properly. We observed Tiago's meaning making in two ways: implicitly at the level of motor action, and explicitly verbally expressed, based on mathematical language and theory. The first, for example, in Task 8, the latter in Task 10, when Tiago introduced the idea of ratio $a : b = 1 : 2$ as invariant, to obtain the right direction of the height lines. Thus, we see how manipulating planes—reaching and maintaining target feedback from projected height lines—sometimes formed a ground from which mathematical reasoning emerged, but at other times was based on meaning making provided by mathematical pre-knowledge.

The student working on our tasks seems to follow a less linear path than the usual three consecutive phases: moving in new ways—signifying—reconciling (Abrahamson *et al.* 2020). Solving the sensorimotor problems seems to not exclusively rely on adaptively developed new sensorimotor perceptive structures. The student's pre-knowledge also facilitated a process of signification and reflection before and during the performance. As such the performed actions did not only give rise to new perceptual structure facilitated by feedback, but also the actions were enabled by recognized familiar structures. The meaning of mathematical notions derived both from new perceptual structures emerged in the performance, and from students pre-knowledge, now involved in new performances.

5.2. *Perception-based tasks*

The affordances of ARSB for acting in three-dimensional space appeared to be important for meaning making during perception-based tasks. Perceptional experiences such as observing the height difference (in Task 2) and marble rolling (in Task 9) played a role in reasoning about height lines and the perpendicular gradient vector. Generally, since the subject is three-dimensional geometry, Tiago benefited a lot from the three-dimensional nature of the sandbox itself. His gestures developed in interaction with the sand and the circular and rectangular planes. As he reasoned he meaningfully moved his hand around the sandbox, manipulated artefacts and pointed at things. For example, in Task 6, Tiago pointed out the triangle and angle that he saw between the ground and the plane.

Finally, an essential affordance of the ARSB was to perceive the shapes of sand as if they composed a real landscape. This naturally provoked everyday language like 'mountain' and 'steepness'. This discourse supported Tiago in making the mathematical notions like 'contour line' and 'gradient' more meaningful. The ARSB allowed Tiago to ground those notions in everyday experience represented through metaphor in the sandbox activities. The sandbox did not contain mathematical elements like a three-dimensional coordinate system, so Tiago was invited to mathematize the situation in his own pace, when needed. The feedback on action Tiago received consisted not just of a correct-incorrect signal, but of a set of mathematical potentially meaningful projected height lines, whose shape, direction and mutual distance informed Tiago how to adjust his actions. Hence, we conclude that Tiago's reasoning and meaning making are supported by the metaphorical interpretation of the sandbox as a landscape, by meaningfully interpreting feedback on his actions and by the three-dimensional nature of the sandbox (and artefacts) that allowed Tiago to point, gesture and manipulate in three dimensions.

5.3. *Incorporation-based tasks*

With respect to incorporation-based tasks (Task 1–3 and 9), we conclude that Tiago managed to incorporate ARSB and marble functionalities: drawing height lines, vertical projection, height measurements and distinguishing the steepest direction. In all these cases, Tiago first perceived—through interaction—a mathematically relevant outsourced functionality (e.g. projection of height lines), after which this specific functionality was removed from the learning environment, and he was invited to incorporate it. This process of incorporation supported Tiago in developing expressions—words and gestures—that supported his meaningful reasoning, as observed in three key episodes. First, during Task 1 the notion of angle related to steepness of a surface emerged during the eyes-closed part, when Tiago turned to touch (and maybe proprioception) to support him in following a height line. This central idea of angle—that clearly was meaningful to Tiago—guided his action and thought on many occasions during further tasks. Second, contour plots were introduced through incorporation in Task 3 (Vignette B), as Tiago incorporated the projection functionality in two ways. Later in Task 9, just after Tiago traced the marble direction in a contour plot (another incorporation task), he explained the direction perpendicular to the contour lines in terms of shortest distance travelled by the marble. Moreover, he gestured how following a height line might increase this distance. Third, in Task 2 he incorporated vertical measurement by using his thumb. This supported his investigations in Task 10 on the gradient and the proportion $a : b$. Thus, we presented some examples that support our claim that incorporation-based tasks can support mathematical exploration and meaningful reasoning. Solutions to incorporation-based tasks were possible only based on previously available feedback in interacting with the sandbox and other materials. The affordance of the ARSB as a learning environment is the possibility to perform these actions both with and without feedback.

6. Conclusion

We aimed to study the ARSB as an embodied learning environment to foster meaning making in context of bivariable calculus. Our results identify the potential of the ARSB as an embodied learning environment for concepts related to functions of two variables, like contour plots and gradients. We could distinguish two ways in which ARSB affordances were used in the learning sequence: in perception- and action-based tasks the affordances of the ARSB were immediately available and intensively involved in the interaction; in incorporation tasks on the contrary, a critical affordance was deliberately removed and the student could reproduce its functionality without technology. The student's mathematical reasoning involved perceptual structures, new grounding experiences, which he discovered acting upon the affordances of the ARSB and which therefore were meaningful to him. Further research is needed to investigate how learning trajectories in this domain can be developed for larger groups in more traditional classroom situations.

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