Equivariant Enumerative Geometry and Donaldson-Thomas Theory

Beoordelingscommissie:
Prof. dr. Jim Bryan, University of British Columbia
Prof. dr. Gunther L. M. Cornelissen, Universiteit Utrecht
Prof. dr. Alina Marian, Northeastern University
Dr. Adrien Sauvaget, Université de Cergy-Pontoise
Prof. dr. Alessandro Tanzini, Scuola Internazionale Superiore di Studi Avanzati, Trieste

Omslag: Babak Atefi
DOI: https://doi.org/10.33540/1119.
Printing: Ridderprint — www.ridderprint.nl.

Equivariant Enumerative Geometry and Donaldson-Thomas Theory

Equivariante aftellende meetkunde en Donaldson-Thomas-theorie

(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de
Universiteit Utrecht
op gezag van de
rector magnificus, prof. dr. H. R. B. M. Kummeling,
ingevolge het besluit van het college voor promoties
in het openbaar te verdedigen op

maandag 13 juni 2022 des middags te 12.15 uur

door

Sergej Monavari

geboren op 18 april 1994 te Camposampiero, Italië.

Promotor:
Prof. dr. C. F. Faber
Copromotor:
Dr. M. Kool

Ai miei genitori,
ed ai miei nonni

Contents

Chapter 1. Introduction 9
- 1.1 A bit of history 9
- 1.2 Donaldson-Thomas theory 11
- 1.3 Pandharipande-Thomas theory of local curves 13
- 1.4 Higher rank K-theoretic Donaldson-Thomas theory of points 14
- 1.5 Donaldson-Thomas theory of Calabi-Yau 4-folds 16
- 1.6 Nested Quot schemes 20
- 1.7 Organization of the thesis 22
- 1.8 Cross-reference 23
- 1.9 Conventions 23

I Donaldson-Thomas theory in dimension three 25

Chapter 2. Localization in Equivariant Enumerative Geometry 27
- 2.1 *Une promenade equivariante* 27
- 2.2 *Une promenade virtuelle* 39

Chapter 3. Double nested Hilbert schemes and the local PT theory of curves 43
- 3.1 Introduction 43
- 3.2 Double nested Hilbert schemes 49
- 3.3 Moduli space of stable pairs 58
- 3.4 K-theory class of the perfect obstruction theory 62
- 3.5 Universality 66
- 3.6 The leading term 74
- 3.7 Toric computations 76
- 3.8 Summing the theory up 82
- 3.9 K-theoretic refinement 85
- 3.10 Appendix: The combinatorial identities 88

Chapter 4. Higher rank K-theoretic Donaldson-Thomas theory of points 93
- 4.1 Introduction 93
- 4.2 The local Quot scheme: critical and equivariant structure 97
- 4.3 Invariants attached to the local Quot scheme 106
- 4.4 Higher rank vertex on the local Quot scheme 108
- 4.5 The higher rank K-theoretic DT partition function 112

4.6 The higher rank cohomological DT partition function 118
4.7 Elliptic Donaldson–Thomas invariants 121
4.8 Higher rank DT invariants of compact toric 3-folds 127

II Donaldson-Thomas theory in dimension four 137

Chapter 5. Donaldson-Thomas theory of Calabi-Yau 4-folds 139
5.1 Taking square roots 139
5.2 Oh-Thomas virtual structures 142

Chapter 6. K-theoretic DT/PT for toric Calabi-Yau 4-folds 149
6.1 Introduction 149
6.2 K-theoretic vertex formalism 158
6.3 Limits of K-theoretic conjecture 173
6.4 Hilbert schemes of points 178
6.5 Local resolved conifold 180
6.6 Appendix: Combinatorial sign rule 183

Chapter 7. Stable pair invariants of local Calabi-Yau 4-folds 195
7.1 Introduction 195
7.2 Moduli spaces 202
7.3 Invariants 212
7.4 Appendix: Tables 221

III Nested Quot schemes 225

Chapter 8. On the motive of the nested Quot scheme of points on a curve 227
8.1 Introduction 227
8.2 Nested Quot schemes of points 228
8.3 Tangent space and smoothness in the case of curves 232
8.4 Białynicki-Birula decomposition 233
8.5 The motive of the nested Quot scheme on a curve 236
8.6 Appendix: Motivic exponentials 240

Chapter 9. A note on the smoothness of the nested Quot scheme of points 243
9.1 Introduction 243
9.2 Properties of the moduli space 244
9.3 Proof of the theorem 245

Samenvatting 251

Acknowledgements 255

Curriculum vitae 259

Bibliography 261

CHAPTER 1

Introduction

Qual è 'l geomètra che tutto s'affige
per misurar lo cerchio, e non ritrova,
pensando, quel principio ond'elli
indige,
tal era io a quella vista nova:
veder voleva come si convenne
l'imago al cerchio e come vi s'indova

Paradiso, Canto XXXIII, Dante

1.1. A bit of history

Enumerative Geometry deals with the classical problem of counting — or better, enumerating — geometric objects. Some simple problems were already addressed and solved by the Greeks, such as the number of circles tangent to three given circles in general positions (Apollonius' problem) or the number of quadrics tangent to five given lines in the plane. These classical problems were solved with ad hoc techniques, which would not generalize to a more comprehensive framework. At the end of the nineteenth's century, these problems had been formulated as finding invariants of a *moduli space* — often a projective space, where one can employ Bézout's Theorem. Despite not being fully rigorous, this was the first appearance of *intersection theory*, which allowed to systematically formulate counting problems as intersection of classes in cohomology. This program reached its apex with *Schubert calculus* and its later rigorous treatment (Hilbert's fifteenth problem), which essentially dealt with the intersection theory of Grassmannians — something that we might consider a precursor of *Gromov-Witten theory*. Nevertheless, soon it was realized that as soon as the moduli space is no longer smooth, studying its geometry and intersection theory gets incredibly hard and explicit results were difficult to produce. As an example, a question that remained unsolved for decades was the number of rational plane curves of degree d passing through $3d-1$ points in general position. For $d=1$, there is only one line passing through 2 points, for $d=2$ there is only one smooth quadric passing through 5 points and for $d=3$ there are precisely 12 (singular!) cubics passing through 8 points; explicit computations were done for $d = 4, 5$. The question remained dormant for some time, until Kontsevich proved in 1994 that

$$N_d = \sum_{d_A + d_B = d} N_{d_A} N_{d_B} d_A^2 d_B \left(d_B \binom{3d-4}{3d_A - 2} - d_A \binom{3d-4}{3d_A - 1} \right),$$

where we denote by N_d the enumerative answer we are looking for. Quite amazingly, this formula recursively reduces to the elementary fact that through 2 points passes a unique line, and more importantly reflects the *associativity* of a certain *quantum product*. That is, the classical problem is solved by changing point of view and merging Enumerative Geometry with Mathematical Physics and String Theory. This is the first interesting example of a *Gromov-Witten invariant*.

A different source of interesting invariants to consider are *topological invariants*, which are invariants that depend on the topology of a geometric object, rather than on its algebraic structure, as *intersection numbers*. The most elementary example is the following. Let X be a a smooth algebraic variety, and consider its symmetric powers $X^{(n)}$. If we package together all topological Euler characteristics, we obtain

$$\sum_{n \geq 0} e(X^{(n)}) q^n = \prod_{n \geq 1} (1-q)^{-e(X)},$$

which is known as the *Macdonald's formula* [**124**]. Symmetric powers parametrize unordered collection of points (possibly coinciding) on a variety, but do not keep track of the possible non-reduced subscheme structure of these *fat points*, and are singular if the dimension of X is at least two. The *Hilbert scheme of points* $X^{[n]}$ of a variety X parametrizes zero-dimensional subschemes of length n, and maps to the symmetric power via the Hilbert-Chow morphism $X^{[n]} \to X^{(n)}$ by *forgetting* the subscheme structure. Equivalently, the Hilbert scheme parametrizes ideal sheaves with fixed Chern character. The Hilbert scheme of points is a nicer moduli space for studying configurations of points on a variety; in dimension one the Hilbert-Chow is an isomorphism, and in dimension two is a resolution of singularities. Their generating series of topological Euler characteristics is

$$\sum_{n \geq 0} e(X^{[n]}) q^n = M_{\dim X}(q)^{e(X)},$$

where $M_d(q)$ is the generating series of *d-dimensional partitions*. The topological Euler characteristic features *motivic* behaviour, meaning that it behaves well under cut-and-paste operations. All above formulas admit natural refinements to *motivic invariants*, and we shall study some of those in Chapter 8.

In dimension three, the Hilbert scheme of points is in general singular and its *naive* motivic invariants are not the correct answer, meaning that they do not match the predictions from String Theory, in analogy with Gromov-Witten theory. The correct framework is to consider the *virtual structure* on the Hilbert scheme on a threefold and its *virtual intersection numbers*. This overcomes at once the issue of performing intersection theory on a singular space — acting as if it were smooth — and matching predictions from Physics. This is the first instance of *Donaldson-Thomas theory*, introduced in Thomas' thesis [**176**].

What is extraordinary and surprising is that the two theories we introduced — Gromov-Witten and Donaldson-Thomas — correspond and fit in a broader framework of theories, which conjecturally relate *curve-counting* invariants to *sheaf-counting* invariants

$$
\begin{tikzcd}
\mathsf{GW} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \mathsf{DT} \arrow[d, leftrightarrow] \\
\mathsf{GV} \arrow[r, leftrightarrow] & \mathsf{PT}
\end{tikzcd}
$$

We briefly sketch the main actors playing a rôle in this diagram. Gromov-Witten (GW) invariants count *stable* curves mapped to a target variety. Gopakumar-Vafa (GV) invariants are mysteriously defined in String Theory by counting BPS bound states [**82, 83**] and relate to GW with a multiple cover formula. Donaldson-Thomas (DT) invariants count *Gieseker stable* sheaves and relate to GW via a certain change of variable. Pandharipande-Thomas (PT) invariants count *stable pairs* in the derived category and can be expressed in terms of DT through wall-crossing. See Pandharipande-Thomas [**154**] for a nice survey on the relation among these counting theories.

The left-hand-side of the diagram (GW/GV) counts curves and depends on the symplectic structure of the target variety; the right-hand-side (DT/PT) counts sheaves and depends a priori on the complex structure of the target variety. This mysterious correspondence should be seen as an expression of S-duality in String Theory, intertwining A-model and B-model of Topological String Theory[1]. This is perhaps an analogue of the relation between Fukaya categories and derived categories of coherent sheaves on Calabi-Yau varieties predicted by Mirror Symmetry, though here S-duality does not swap the variety with its mirror dual.

In this thesis, we explore the right-hand-side of this story, dealing with sheaf-counting theories, with two goals in mind: provide closed expressions for refined DT and PT invariants and find new refined relations among the four counting theories playing a major rôle in modern Enumerative Geometry.

1.2. Donaldson-Thomas theory

Donaldson-Thomas theory is a sheaf-theoretic enumerative theory originally designed for *smooth projective threefolds*, which first appeared in Thomas' thesis [**176**]. Roughly speaking, the theory is defined as follows. Let $(X, \mathcal{O}(1))$ be a smooth projective variety with a fixed polarization and let $\gamma \in H^*(X, \mathbb{Q})$ be a fixed Chern character. The moduli space $M(X, \gamma)$ of *Gieseker stable* sheaves on X with Chern character γ is — under reasonable assumptions — a projective moduli scheme, whose deformations-obstructions

[1] See the following impassioned discussion on MathOverflow [**126**].

at every point $[E] \in M(X,\gamma)$ are controlled by the Ext groups

$$\mathrm{Ext}^1(E,E), \quad \mathrm{Ext}^2(E,E), \quad \mathrm{Ext}^3(E,E), \quad \dots$$

When the deformation theory is controlled by only two terms — that is, there are no higher obstructions — $M(X,\gamma)$ is endowed with a *perfect obstruction theory* in the sense of Behrend-Fantechi and Li-Tian [**12, 123**], induced by the truncated Atiyah class [**95**]. See [**94**] for a complete introduction on Gieseker stability. This is usually the case if X is a Calabi-Yau or Fano threefold or for suitable choices of the Chern character. The perfect obstruction theory induces a *virtual fundamental class* $[M(X,\gamma)]^{\mathrm{vir}} \in A_*(M(X,\gamma))$ and one defines invariants by integrating Chow cohomology classes $\alpha \in A^*(M(X,\gamma),\mathbb{Q})$ against the virtual fundamental class

$$\int_{[M(X,\gamma)]^{\mathrm{vir}}} \alpha \in \mathbb{Q}.$$

We call such intersection numbers *Donaldson-Thomas invariants* of X. In particular, DT invariants are deformation invariants, as the virtual class does not vary if we deform X in a family.

In this thesis we are interested in capturing invariants of smooth *quasi-projective* varieties. Unfortunately, without properness assumption, defining invariants through intersection theory is not directly possible. To remedy this deficiency, we assume that X is acted on by an algebraic torus $\mathbf{T}$, whose action canonically lifts to the moduli space $M(X,\gamma)$. If the fixed locus $M(X,\gamma)^{\mathbf{T}}$ is proper, we define the invariants through Graber-Pandharipande virtual localization formula

$$\int_{[M(X,\gamma)]^{\mathrm{vir}}} \alpha := \int_{[M(X,\gamma)^{\mathbf{T}}]^{\mathrm{vir}}} \frac{\alpha|_{M(X,\gamma)^{\mathbf{T}}}}{e^{\mathbf{T}}(N^{\mathrm{vir}})} \in H^*_{\mathbf{T}}(\mathsf{pt})_{\mathrm{loc}}.$$

Here α is a $\mathbf{T}$-equivariant Chow cohomology class, N^{vir} is the *virtual normal bundle*, $e^{\mathbf{T}}(-)$ is the *equivariant Euler class* and $H^*_{\mathbf{T}}(\mathsf{pt})_{\mathrm{loc}} \subset \mathbb{Q}(s_1,\dots,s_r)$, where r is the rank of $\mathbf{T}$ — see Chapter 2 for an introduction to equivariant localization. If $M(X,\gamma)$ was already proper, the left-hand-side would be well-defined and the equality would follow by the virtual localization formula.

A refinement of such invariants are *K-theoretic invariants*. In simple words, the perfect obstruction theory induces a *virtual structure sheaf* $\mathcal{O}^{\mathrm{vir}} \in K_0(M(X,\gamma))$, and instead of pushing-forward classes in (equivariant) cohomology, we push-forward class in (equivariant) *K-theory*, which means computing *holomorphic Euler characteristics* $\chi(M(X,\gamma),\mathcal{O}^{\mathrm{vir}} \otimes V) \in K_0(\mathsf{pt})$, where $V \in K_0(M(X,\gamma))$. As before, if X is only quasi-projective but endowed with an action of an algebraic torus, we define the invariants equivariantly by the virtual localization formula.

We will mostly deal with fine moduli spaces of coherent sheaves on quasi-projective varieties that do not arise from a Gieseker stability condition, but that can be explicitly characterized and whose deformation theory behaves similarly to the one of $M(X,\gamma)$. In Chapter 4 we study the *Quot scheme*, which is a generalization of the Hilbert scheme to higher rank and in Chapter 3 we consider the moduli space of *stable pairs*.

1.3. Pandharipande-Thomas theory of local curves

We study from a new perspective the *local Pandharipande-Thomas theory of curves*. A *local curve* $X = \mathrm{Tot}_C(L_1 \oplus L_2)$ is the total space of two line bundles L_1, L_2 on a smooth projective curve C, and if $L_1 \otimes L_2 \cong K_C$, we say that X is *Calabi-Yau*. Let $n \in \mathbb{Z}$ and $d \geq 1$. A *stable pair* on X is a pair (F, s) where F is a pure, 1-dimensional sheaf with proper support $[\mathrm{Supp}(F)] = d[C]$, $\chi(F) = n$ and s is a section with 0-dimensional cokernel. The moduli space of stable pairs $P_n(X, d)$ was introduced and studied by Le Potier [121] for any smooth quasi-projective variety, while later Pandharipande-Thomas [152] reinterpreted $P_n(X, d)$ as parametrizing complexes $[\mathcal{O}_X \xrightarrow{s} F] \in \mathbf{D}^{\mathrm{b}}(X)$ with trivial determinant in the derived category. Their main motivation was to give a geometric understanding of the Gromov-Witten/Donaldson-Thomas correspondence studied in [127] — we are going to review and formulate such a correspondence directly in the language of stable pairs.

As in the case of Donaldson-Thomas theory, the moduli space $P_n(X, d)$ is represented by a quasi-projective scheme, whose deformation theory at a point $I^\bullet = [\mathcal{O}_X \xrightarrow{s} F] \in P_n(X, d)$ is controlled by

$$\mathrm{Ext}^1(I^\bullet, I^\bullet)_0, \quad \mathrm{Ext}^2(I^\bullet, I^\bullet)_0,$$

where $(-)_0$ denotes the trace-free part. In particular, $P_n(X, d)$ is endowed with a perfect obstruction theory, which induces a virtual structure by which we define *Pandharipande-Thomas (or stable pairs) invariants*. The local curve X is acted on by the torus $\mathbf{T} = (\mathbb{C}^*)^2$ which *scales* the fibers of the line bundles, and the action naturally lifts on $P_n(X, d)$ making the perfect obstruction theory $\mathbf{T}$-equivariant. Define the partition function of stable pair invariants equivariantly as

$$\mathsf{PT}_d(X; q) := \sum_{n \in \mathbb{Z}} q^n \cdot \int_{[P_n(X,d)]^{\mathrm{vir}}} 1 \in \mathbb{Q}(s_1, s_2)(\!(q)\!),$$

where s_1, s_2 are the equivariant parameter. Analogously, Bryan-Pandharipande [29] studied the generating series of Gromov-Witten invariants $\mathsf{GW}_d(X; u)$ for a genus g curve C, where u keeps track of the genus, and conjectured that

$$(-i)^{d(2-2g+\deg L_1+\deg L_2)} \cdot \mathsf{GW}_d(X; u) = (-q)^{-\frac{1}{2}\cdot d(2-2g+\deg L_1+\deg L_2)} \mathsf{PT}_d(X, q),$$

after the change of variable $q = -e^{iu}$. An important part of the conjecture is that the right-hand-side is a *rational* function in q, so that the change of variable is well-defined. More precisely, Bryan-Pandharipande [29] conjectured a GW/DT correspondence, motivated by the analogous GW/DT correspondence conjectured in [127] for projective Calabi-Yau threefolds, as PT invariants had not been introduced yet. This conjecture — in its GW/DT formulation — was solved in combination with the results of Okounkov-Pandharipande [148], who study the Donaldson-Thomas theory of a local curve. Their techniques involve a TQFT approach and a degeneration argument to reduce to relative GW/DT invariants with $C = \mathbb{P}^1$.

Our main result is that the generating series $\mathsf{PT}_d(X; q)$ of such invariants is controlled by some universal series and determine them under the anti-diagonal restriction $s_1 + s_2 = 0$.

Theorem 1.3.1 (Theorem 3.1.3). *There are universal series* $A_\lambda(q), B_\lambda(q), C_\lambda(q) \in \mathbb{Q}(s_1, s_2)[\![q]\!]$ *such that*

$$\mathsf{PT}_d(X;q) = \sum_{\lambda \vdash d} \left(q^{-|\lambda|}A_\lambda(q)\right)^{g-1} \cdot \left(q^{-n(\lambda)}B_\lambda(q)\right)^{\deg L_1} \cdot \left(q^{-n(\overline{\lambda})}C_\lambda(q)\right)^{\deg L_2},$$

where λ *is a partition of* d, $\overline{\lambda}$ *is the conjugate partition of* λ, $n(\lambda) = \sum_{i=0}^{l(\lambda)} i \cdot \lambda_i$ *and* $g = g(C)$. *Moreover, under the anti-diagonal restriction* $s_1 + s_2 = 0$

$$\begin{aligned}
A_\lambda(q, s_1, -s_1) &= (-s_1^2)^{|\lambda|} \cdot \prod_{\Box \in \lambda} h(\Box)^2, \\
B_\lambda(-q, s_1, -s_1) &= (-1)^{n(\lambda)} \cdot s_1^{-|\lambda|} \cdot \prod_{\Box \in \lambda} h(\Box)^{-1} \cdot \prod_{\Box \in \lambda}(1 - q^{h(\Box)}), \\
C_\lambda(-q, s_1, -s_1) &= (-1)^{n(\overline{\lambda})} \cdot (-s_1)^{-|\lambda|} \cdot \prod_{\Box \in \lambda} h(\Box)^{-1} \cdot \prod_{\Box \in \lambda}(1 - q^{h(\Box)}).
\end{aligned}$$

In particular, combined with [**29**] *the GW/PT correspondence holds for* $s_1 + s_2 = 0$.

The novelty of our proof is the different approach which only relies on the Graber-Pandharipande localization — without degenerating the curve C and invoking relative invariants. In fact, we identify the connected components of the fixed locus $P_n(X,d)^{\mathbf{T}}$ with *double nested Hilbert schemes of points* $C^{[\mathbf{n}_\lambda]}$, which parametrize flags of zero-dimensional subscheme of C nesting in two directions dictated by a *reversed plane partition* $\mathbf{n}_\lambda$. On each $C^{[\mathbf{n}_\lambda]}$ we construct a virtual fundamental class by realizing $C^{[\mathbf{n}_\lambda]}$ as the zero locus of a section of a vector bundle on a smooth ambient space in a natural way, and prove that it coincides with the one induced by the deformation theory of stable pairs. Each localized contribution to the invariants is expressed in terms of three universal series, which reduce the computations to the case of $C = \mathbb{P}^1, L_1 = L_2 = \mathcal{O}_{\mathbb{P}^1}$ — the trivial vector bundle case — and $L_1 \otimes L_2 = K_{\mathbb{P}^1}$ — the Calabi-Yau case. In both cases, the invariants are computed by further localizing the invariants with the respect the $\mathbb{C}^*$-action on $\mathbb{P}^1$. Contrary to the degeneration approach of [**148**], our methods easily generalize to include insertions and refinement to K-theoretic invariants.

As a byproduct, we introduce and study *double nested Hilbert schemes of points*, which are a generalization of the *nested Hilbert scheme of points* widely studied in the literature and whose geometry we believe to be of independent interest.

1.4. Higher rank K-theoretic Donaldson-Thomas theory of points

Let X be a smooth quasi-projective threefold and F a locally free sheaf on X. We consider the *Quot scheme* $\mathrm{Quot}_X(F, n)$ which parametrizes classes of quotients $[F \twoheadrightarrow Q]$, where the support of Q is zero-dimensional with prescribed length $\chi(Q) = n$ and we identify two quotients if their kernels coincide. In general, Q is not the structure sheaf of a subscheme of X, unless F is a line bundle and in such case the Quot scheme is simply isomorphic to the Hilbert scheme of points $\mathrm{Hilb}^n(X)$. The Hilbert scheme of points on a threefold is in general a very singular scheme, but it admits a perfect obstruction theory — induced by the deformation of the ideal sheaves — and therefore a virtual structure.

In the higher rank case — that is, for the general Quot scheme — it was shown by Beentjes-Ricolfi [**9**] and Ricolfi [**165**] that a perfect obstruction theory exists if $X = \mathbb{C}^3$ and $F = \mathcal{O}_{\mathbb{C}^3}^{\oplus r}$, if X is a projective Calabi-Yau threefold and F simple and rigid, or if $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ and F is exceptional. Moreover, the perfect obstruction theory is *symmetric* as soon as X is Calabi-Yau. In particular, in the local case — for $\mathbb{C}^3$ — the Quot scheme is globally a critical locus

$$\mathrm{Quot}_{\mathbb{C}^3}(\mathcal{O}_{\mathbb{C}^3}^{\oplus r}, n) \cong Z(df),$$

where f is a function (called the *superpotential*) on a smooth ambient variety (called the *non-commutative Quot scheme*). A nice feature of critical loci is that, if we denote by K_{vir} the *virtual canonical bundle*, they naturally admit a *square root* $K_{\mathrm{vir}}^{1/2}$ as a genuine line bundle. Inspired by M-Theory, Nekrasov-Okounkov [**142**] define the *twisted virtual structure sheaf*

$$\widehat{\mathcal{O}}^{\mathrm{vir}} := \mathcal{O}^{\mathrm{vir}} \otimes K_{\mathrm{vir}}^{1/2}.$$

Furthermore, in the local case the Quot scheme is acted on by the torus $(\mathbb{C}^*)^3 \times (\mathbb{C}^*)^r$, which *moves* the support of the quotients and *scales* the fibers of $\mathcal{O}_{\mathbb{C}^3}^{\oplus r}$. Therefore we define the *higher rank K-theoretic Donaldson-Thomas invariants* by

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{C}^3, q, t, w) = \sum_{n \geq 0} q^n \chi(\mathrm{Quot}_{\mathbb{C}^3}(\mathcal{O}^{\oplus r}, n), \widehat{\mathcal{O}}^{\mathrm{vir}}) \in \mathbb{Z}(\!(t, (t_1t_2t_3)^{\frac{1}{2}}, w)\!)[\![q]\!],$$

where $t = (t_1, t_2, t_3), w = (w_1, \dots, w_r)$ are the equivariant parameters. These invariants should be considered as an algebro-geometric analogue of the $\widehat{A}$-*genus* of a spin manifold, with the twist $K_{\mathrm{vir}}^{1/2}$ playing the rôle of the spin structure. Our main theorem is the following.

Theorem 1.4.1 (Theorem 4.1.1)**.** *The rank r K-theoretic DT partition function of $\mathbb{C}^3$ is given by*

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{C}^3, (-1)^r q, t, w) = \mathrm{Exp}\left(\frac{[t_1t_2][t_1t_3][t_2t_3]}{[t_1][t_2][t_3]} \frac{[\mathfrak{t}^r]}{[\mathfrak{t}][\mathfrak{t}^{\frac{r}{2}}q][\mathfrak{t}^{\frac{r}{2}}q^{-1}]}\right),$$

where $\mathfrak{t} = t_1t_2t_3$ *is the* Calabi-Yau weight *and* $[x] = x^{1/2} - x^{-1/2}$.

Here the *plethystic exponential* is defined for an arbitrary power series $f(p_1, \dots, p_l)$ as

$$\mathrm{Exp}\,(f) = \exp\left(\sum_{n>0} \frac{1}{n} f(p_1^n, \dots, p_l^n)\right).$$

This expression had been initially conjectured in rank one by Nekrasov [**138**] in the context of String Theory, and later proved by Okounkov [**149**], while the higher rank case was conjectured by Awata-Kanno [**6**], again in String Theory. In string-theoretic terms, K-theoretic DT invariants describe the quantum mechanics of a system of D0-D6 branes, where r D6-branes *wrap* the target threefold. Notice that these invariants should in principle depend on the framing parameters w, but they do not. Something remarkable and mysterious in its nature is that the K-theoretic invariants feature the same factorization properties and a similar structure of *motivic* DT invariants studied by Behrend-Bryan-Szendrői [**11**] and Ricolfi [**163**].

The proof essentially goes as follows. To each fixed point of the Quot scheme corresponds a *tuple of plane partitions*. We develop a *higher rank vertex formalism*, which is a generalization of the classical *topological vertex*, in terms of which we express the localized contribution of the invariants. We crucially prove that the generating series does not depend on the framing parameters, which allows us to send the framing parameters w to infinity, which recovers our final expression.

The K-theoretic partition function contains *refined* information and recovers many other interesting result. For instance, define the *cohomological* DT invariants as

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{C}^3, q, s, v) := \sum_{n\geq 0} q^n \int_{[\mathrm{Quot}_{\mathbb{C}^3}(\mathcal{O}^{\oplus r}, n)]^{\mathrm{vir}}} 1 \in \mathbb{Q}(\!(s, v)\!)[\![q]\!].$$

By taking a *limit* of the K-theoretic DT partition function, we obtain the following result on the *cohomological DT partition function.*

Theorem 1.4.2 (Theorem 4.1.2)**.** *The rank r cohomological DT partition function of $\mathbb{C}^3$ is given by*

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{C}^3, q, s) = \mathsf{M}((-1)^r q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}},$$

where $\mathsf{M}(q) = \prod_{m\geq 1}(1-q^m)^{-m}$ is the MacMahon function.

This solves a conjecture of Szabo [**173**], once more in String Theory. By glueing the contribution of toric charts, we extend this formula to any projective toric variety and choice of equivariant exceptional vector bundle, partially proving a conjecture of Ricolfi [**165**].

Finally, we introduce the *virtual chiral elliptic genus*, which is a variation of elliptic genus which includes the twist $K_{\mathrm{vir}}^{1/2}$, and recovers the *elliptic* DT invariants studied by Benini-Bonelli-Poggi-Tanzini [**16**] in String Theory.

1.5. Donaldson-Thomas theory of Calabi-Yau 4-folds

As we already mentioned, Donaldson-Thomas theory is classically defined for smooth projective *threefolds*, essentially because the deformation theory of sheaves is controlled by only two terms — at least in the Calabi-Yau or Fano case. If X is a *Calabi-Yau 4-fold*, the deformations-obstructions at every point $[E] \in M(X,\gamma)$ are controlled by the Ext groups

$$\mathrm{Ext}^1(E,E), \quad \mathrm{Ext}^2(E,E), \quad \mathrm{Ext}^3(E,E)$$

and a perfect obstruction theory does not exist in the sense of Behrend-Fantechi. Already Donaldson-Thomas [**66**] wondered about extending Donaldson-Thomas theory to four *complex* dimensions, motivated by gauge theory. Their gauge-theoretic approach was later considered by Cao-Leung [**38**], who constructed DT theory of Calabi-Yau 4-folds in special examples. The key point in their construction is a notion of *orientability* of the moduli space of coherent sheaves, without which the theory could not be considered.

The first complete theory appeared in Borisov-Joyce seminal work [**23**], based on Derived Differential Geometry and Pantev-Töen-Vaquié-Vezzosi [**160**] theory of shifted symplectic structures. The key idea of Borisov-Joyce is to choose at every point $E \in M(X,\gamma)$ a half-dimensional real subspace

$$\mathrm{Ext}^2_+(E,E) \subseteq \mathrm{Ext}^2(E,E)$$

of the usual obstruction space $\mathrm{Ext}^2(E,E)$, on which the quadratic form Q defined by Serre duality is real and positive definite. Then one glues local Kuranishi-type models of the form

$$\kappa_+ = \pi_+ \circ \kappa : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2_+(E,E),$$

where κ is the Kuranishi map for $M(X,\gamma)$ at E and π_+ denotes projection on the first factor of the decomposition $\mathrm{Ext}^2(E,E) = \mathrm{Ext}^2_+(E,E) \oplus \sqrt{-1}\cdot \mathrm{Ext}^2_+(E,E)$. This glueing procedure is where the technicality is hidden, making it hard to apply their machinery.

More recently — after most of this thesis was written — Oh-Thomas [**147**] constructed an *algebraic* virtual fundamental class on the moduli space of coherent sheaves on a Calabi-Yau 4-fold, which lifts the Borisov-Joyce class, modulo the technical assumption to invert 2. Morally, the local model of their construction is the following

$$\begin{tikzcd} & \mathcal{E} \arrow[d] \\ M(X,\gamma) = Z(s) \arrow[r, hook] & \mathcal{A}, \arrow[u, bend right, "s"'] \end{tikzcd}$$

where s is an *isotropic* section of a $SO(r,\mathbb{C})$-bundle $\mathcal{E}$ over a smooth ambient space $\mathcal{A}$. This would be true globally from a gauge-theoretic perspective, where $\mathcal{A}$ is allowed to be infinite-dimensional, but unfortunately such a model is forbidden in the algebraic setting.

By the same construction of Behrend-Fantechi [**12**], the obstruction theory determines an *isotropic* cone $\mathfrak{C} \subset \mathcal{E}$. The virtual fundamental class and *twisted* virtual structure sheaf are defined via the *square root* Gysin-type maps

$$\begin{aligned} [M(X,\gamma)]^{\mathrm{vir}} &:= \sqrt{0^!_{\mathcal{E}}}[\mathfrak{C}] \in A_*\left(M(X,\gamma), \mathbb{Z}\left[\tfrac{1}{2}\right]\right), \\ \widehat{\mathcal{O}}^{\mathrm{vir}}_{M(X,\gamma)} &:= \sqrt{0^*_{\mathcal{E}}}[\mathcal{O}_{\mathfrak{C}}] \otimes K_{\mathcal{A}}^{1/2} \in K_0\left(M(X,\gamma), \mathbb{Z}\left[\tfrac{1}{2}\right]\right), \end{aligned}$$

where $0_{\mathcal{E}} \colon X \hookrightarrow \mathcal{E}$ is the zero section of the vector bundle E. As expected, both the virtual fundamental class and the twisted virtual structure sheaf are deformation invariants. The square-root operator $\sqrt{\cdot}$ is defined using Edidin-Graham square-root characteristic classes [**68**] and represents the trickiest aspect of DT theory for Calabi-Yau 4-folds. Notice that, in particular, only the *twisted* virtual structure sheaf is well-defined, and that we need to invert 2.

1.5.1. DT/PT correspondence For a Calabi-Yau 4-fold X, a curve class $\beta \in H_2(X,\mathbb{Z})$ and $n \in \mathbb{Z}$, the machinery of Oh-Thomas induces a virtual structure on the Hilbert scheme of *points* and *curves* $\mathrm{Hilb}^n(X,\beta)$ and the moduli space of stable pairs $P_n(X,\beta)$. In both cases, the virtual dimension is n, so we morally need to include insertions. To any line bundle L on X there is an associated *tautological* rank n complex $L^{[n]}$ on $\mathrm{Hilb}^n(X,\beta)$ and $P_n(X,\beta)$. We define K*-theoretic invariants* by

$$\mathrm{I}_\beta(X,L,y) := \sum_{n\geq 0} q^n \cdot \chi\Big(\mathrm{Hilb}^n(X,\beta), \widehat{\mathcal{O}}^{\mathrm{vir}} \otimes \widehat{\Lambda}^\bullet(L^{[n]}\otimes y^{-1})\Big) \in \frac{\mathbb{Q}(t_1^{\pm 1/2},t_2^{\pm 1/2},t_3^{\pm 1/2},t_4^{\pm 1/2})}{(t_1t_2t_3t_4-1)}[\![q]\!],$$

and similarly $P_\beta(X,L,y)$ for stable pairs, where $\widehat{\Lambda}^\bullet(\cdot) := \Lambda^\bullet(\cdot)\otimes \det(\cdot)^{-1/2}$. If the moduli spaces involved are only *quasi*-projective, the invariants are defined equivariantly by means of virtual localization.

This definition is motivated by the work of Nekrasov and Nekrasov-Piazzalunga [**140, 143**] in String Theory, who considered the case of $\mathrm{Hilb}^n(\mathbb{C}^4)$. In fact, from a string-theoretic perspective these invariants appear as the result of supersymmetric localization of $U(1)$ super-Yang–Mills theory with matter on a Calabi–Yau 4-fold and describe the quantum mechanics of a system of D0-D2-D8 branes, with the tautological insertion of $L^{[n]}$ corresponding to a matter bundle. Nekrasov conjectures the following closed formula.

Conjecture 1.5.1 (Conjecture 6.1.5)**.** *The K-theoretic partition function for $\mathbb{C}^4$ is*

$$I_0(\mathbb{C}^4,\mathcal{O}_{\mathbb{C}^4},y) = \mathrm{Exp}\left(\frac{[t_1t_2][t_1t_3][t_2t_3][y]}{[t_1][t_2][t_3][t_4][y^{\frac{1}{2}}q][y^{\frac{1}{2}}q^{-1}]}\right),$$

where $[x] = x^{1/2} - x^{-1/2}$.

The proof of this conjecture will appear in the upcoming work of Kool-Rennemo [**115**].

Inspired by the DT/PT correspondence for Calabi-Yau threefolds, we conjecture the following DT/PT correspondence for K-theoretic invariants with tautological insertions.

Conjecture 1.5.2 (Conjecture 6.1.7)**.** *Let X be a Calabi-Yau 4-fold and L a line bundle. Then there exist orientations such that*

$$I_\beta(X,L,y) = I_0(X,L,y)\cdot P_\beta(X,L,y).$$

See Chapter 6 for the proper notation and assumptions of this conjecture. To be precise, in Chapter 6 we formulate our conjectures always in the *toric* setting, but it is natural to believe such a correspondence should hold for the projective case as well. We develop a vertex/edge formalism which computes the invariants through the virtual localization and check the conjecture in some cases up to some orders. We remark that it is important that L is a *line* bundle, as other analogous tautological insertions would feature a DT/PT correspondence — cf. Remark 6.2.17. Our DT/PT correspondence contains, as a specialization, many other conjectural DT/PT correspondences. In fact, the *cohomological* limit of this conjecture is the DT/PT correspondence for tautological insertions studied by Cao-Kool [**33**] and a further limit is the DT/PT correspondence with no insertions. Instead, if we specialize $y = t_4$, we *dimensionally reduce* our theory and recover Nekrasov-Okounkov DT/PT correspondence for Calabi-Yau threefolds [**142**].

The nastiest aspect of this theory is the localization formula. While for DT theory of threefolds the localization formula reduces the invariants to some localized contributions expressed by the vertex formalism, in DT theory of Calabi-Yau 4-folds this happens only *up to a sign*, for each component of the fixed locus. Existence of a unique suitable choice of signs is guaranteed by Oh-Thomas work, but any not-so-low-order computation is not feasible and leads to cumbersome computations, as the number of fixed points rapidly grows. To remedy this deficiency, we propose several sign rules that conjecturally compute DT invariants and prove they are *canonical*. A proof of the sign rule for $\mathrm{Hilb}^n(\mathbb{C}^4)$ will appear in Kool-Rennemo [**115**].

During the final writing of this thesis, Bae-Kool-Park [**7**] exploited Oh-Thomas machinery to develop *surface* sheaf-counting DT/PT theories. Among their many results, they extend our vertex formalism to include surfaces and conjecture a series of DT/PT-type correspondences, both in the toric and projective setting. For another recent development in surface sheaf-counting theories see Gholampour-Jiang-Lo [**80**].

1.5.2. PT invariants of local surfaces Gromov-Witten invariants are rational numbers, which are virtual counts of stable maps from curves to a fixed algebraic variety. Due to multiple cover contributions, they are in general not integers. For a Calabi-Yau 4-fold X, Klemm-Pandharipande [**108**] defined Gopakumar-Vafa type invariants using Gromov-Witten theory and conjectured their integrality — cf. [**102**] for the analogues conjectures for Calabi-Yau threefolds. For $\gamma \in H^4(X, \mathbb{Z})$, the *genus zero Gopakumar-Vafa type* invariants

$$n_{0,\beta}(\gamma) \in \mathbb{Q}$$

are defined by the identity

$$\sum_{\beta>0} \mathrm{GW}_{0,\beta}(\gamma)\, q^\beta = \sum_{\beta>0} n_{0,\beta}(\gamma) \sum_{d=1}^{\infty} d^{-2} q^{d\beta},$$

where $\mathrm{GW}_{0,\beta}(\gamma)$ are the usual Gromov-Witten invariants. With a similar multiple-cover type formula Klemm-Pandharipande define *genus one* Gopakumar-Vafa type invariants $n_{1,\beta}$, where no insertions are needed. For dimensional reasons, genus $g \geq 2$ invariants automatically vanish.

Inspired by its threefold analogue, Cao-Maulik-Toda [**39, 40**] proposed a *sheaf-theoretic* interpretation of GV invariants of Calabi-Yau 4-folds as Donaldson-Thomas and Pandharipande-Thomas (stable pair) invariants, using Borisov-Joyce virtual fundamental class. The *stable pair* invariants of X with *primary insertions* are defined by

$$P_{n,\beta}(\gamma) := \int_{[P_n(X,\beta)]^{\mathrm{vir}}} \tau(\gamma)^n,$$

see Chapter 7 for the definition of $\tau(\cdot)$. Cao-Maulik-Toda [**40**] conjecture the following GV/PT correspondence.

Conjecture 1.5.3 (Conjectures 7.1.1, 7.1.2). *Let X be a smooth projective Calabi-Yau 4-fold, $\beta \in H_2(X,\mathbb{Z})$, $\gamma \in H^4(X,\mathbb{Z})$, and $n \geqslant 1$. Then there exist orientations such that*

$$P_{n,\beta}(\gamma) = \sum_{\substack{\beta_0+\beta_1+\cdots+\beta_n=\beta \\ \beta_0,\beta_1,\ldots,\beta_n \geqslant 0}} P_{0,\beta_0} \cdot \prod_{i=1}^{n} n_{0,\beta_i}(\gamma),$$

where the sum is over all effective decompositions of β. If $n = 0$

$$\sum_{\beta \geqslant 0} P_{0,\beta}\, q^{\beta} = \prod_{\beta > 0} M\left(q^{\beta}\right)^{n_{1,\beta}},$$

where $M(q) = \prod_{k \geqslant 1}(1-q^k)^{-k}$ denotes the MacMahon function.

The first part of Conjecture 1.5.3 can be interpreted as a wall-crossing formula in the category of D0-D2-D8 bound states in Calabi-Yau 4-folds [**42**], while the second part seems to be more mysterious. In [**40**], these conjectures were verified in the some cases, mainly for irreducible curve classes and $n = 0, 1$, where the geometry turns accessible.

In this thesis we produced examples and instances of this conjecture for non-trivial cases, and develop techniques that allow to compute PT invariants for arbitrary curve class β — in particular non-irreducible — and arbitrarily high n (modulo computational complexity). We focus on *local surfaces*, that are quasi-projective Calabi-Yau 4-folds of the form $\mathrm{Tot}_S(L_1 \oplus L_2)$, where S is a toric surface and L_1, L_2 are line bundles such that $L_1 \otimes L_2 = K_S$. We classify when the moduli space of stable pairs of such local surfaces happens to be *proper* and when all the stable pairs are *scheme-theoretically* supported on the zero-section. Within this classification, we prove the following theorem, by means of which we compute all the invariants and check the Cao-Maulik-Toda conjectures.

Theorem 1.5.4 (Theorem 7.1.5). *Assume $P_n(X,\beta) \cong P_n(S,\beta)$ and some extra technical assumption on (S,β). Denote by $[\mathrm{pt}] \in H^4(X,\mathbb{Z})$ the pull-back of the Poincaré dual of the point class on S. Then there exist an orientation such that*

$$P_{n,\beta}([\mathrm{pt}]) = (-1)^{\beta \cdot L_2 + n} \int_{S^{[m]} \times \mathbb{P}^{\chi(\beta)-1}} c_m(\mathcal{O}_S(\beta)^{[m]}(1)) \, \frac{h^n (1+h)^{\chi(L_1(\beta))} (1-h)^{\chi(L_2(\beta))} \, c(T_{S^{[m]}}(L_1))}{c(L_1(\beta)^{[m]}(1)) \cdot c((L_2(\beta)^{[m]}(1))^*)},$$

where $m := n + g(\beta) - 1$ and $h := c_1(\mathcal{O}(1))$. Moreover, $P_{n,\beta}([\mathrm{pt}]) = 0$ when $\beta^2 < 0$.

Whenever the stable pairs are not scheme-theoretically supported in the zero section, we exploit the vertex formalism developed in Chapter 6 to compute the invariants by means of virtual localization. We stress once more that our techinques allow to compute PT invariants of local (toric) surfaces for possibly non-reduced curves classes.

1.6. Nested Quot schemes

Let $K_0(\mathrm{Var}_{\mathbf{k}})$ be the Grothendieck ring of varieties over an algebraically closed field $\mathbf{k}$, not necessarily of characteristic zero. If Y is a $\mathbf{k}$-variety, its *motivic zeta function*

$$\zeta_Y(q) = 1 + \sum_{n > 0} \left[Y^{(n)}\right] q^n \; \in \; K_0(\mathrm{Var}_{\mathbf{k}})[\![q]\!]$$

is a generating series introduced by Kapranov in [**101**], where he proved that for smooth curves it is a rational function in q. The motivic zeta function *universally* comprises all possible motivic invariants of $Y^{(n)}$, in particular topological Euler characteristic.

The *motive* of a moduli space — its class in $K_0(\mathrm{Var}_{\mathbf{k}})$ — provides interesting information on its geometry, altought it is usually extremely difficult to compute it. We variate on the theme of Quot schemes, by computing the motive of the *nested Quot scheme of points* $\mathrm{Quot}_C(E, \boldsymbol{n})$ on a smooth curve C, entirely in terms of $\zeta_C(q)$. Here, E is a locally free sheaf on C, and $\boldsymbol{n} = (0 \leq n_1 \leq \cdots \leq n_d)$ is a non-decreasing tuple of integers, for some fixed $d > 0$. The scheme $\mathrm{Quot}_C(E, \boldsymbol{n})$ generalises the classical Quot scheme of Grothendieck (recovered when $d = 1$): it parametrises flags of quotients $E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1$ where T_i is a 0-dimensional sheaf of length n_i. Our main result is the following.

Theorem 1.6.1 (Theorem 8.1.1)**.** *Let C be a smooth curve over $\mathbf{k}$, let E be a locally free sheaf of rank r on C. Then*

$$\sum_{0\leq n_1 \leq \cdots \leq n_d} \left[\mathrm{Quot}_C(E, \boldsymbol{n})\right] q_1^{n_1} \cdots q_d^{n_d} = \prod_{\alpha=1}^{r} \prod_{i=1}^{d} \zeta_C\left(\mathbb{L}^{\alpha-1} q_i^{d-i+1}\right) \in K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!],$$

where $\mathbb{L} = [\mathbb{A}^1_{\mathbf{k}}]$ is the Lefschetz motive. In particular, this generating function $\mathsf{Z}_{C,r,d}(\boldsymbol{q})$ is rational in $q_1, \ldots, q_d$. Moreover it can be expressed through the plethystic exponential as

$$\mathsf{Z}_{C,r,d}(\boldsymbol{q}) = \mathrm{Exp}\left(\left[C \times \mathbb{P}^{r-1}_{\mathbf{k}}\right] \sum_{i=1}^{d} q_i^{d+1-i}\right).$$

To prove this theorem we exploit the *Białynicki-Birula decomposition*, which relies on a suitable torus action on the nested Quot scheme of points and its *smoothness* over a smooth curve. Motivated by the latter, we extended Cheah's classification of smoothness of the *nested Hilbert scheme of points* to the nested Quot scheme of points in any dimension.

The cohomology ring of the nested Quot scheme on a curve has been studied by Mochizuki [**131**]. Moreover, our main result on the motive of the nested Quot scheme relates to the computation of the Voevodsky motive with rational coefficients of the scheme of iterated Hecke correspondences perfomed by Hoskins–Lehalleur [**92**, Section 3], which features a similar factorisation property as the one we proved — indeed, our nested Quot scheme can be defined via the scheme of Hecke correspondences. Their main motivation is the computation of the motive of the stack $\mathrm{Bun}_{n,d}(C)$ of vector bundles of rank n and degree d on a smooth projective curve C, but their work fits in the general framework of the geometric Langlands program and the study of Higgs bundles on a curve — see e.g. [**79, 93**], and in particular [**79**, Corollary 4.10] for a formula related to our main result.

1.7. Organization of the thesis

1.7.1. Chapter 2 We provide a gentle introduction to virtual localization formulas, which are the main technical tool of the thesis. We define equivariant analogues of cohomology, Chow groups and K-theory, and explain how to operatively deal with the localization theorems after Atiyah-Bott, Edidin-Graham and Thomason in the smooth setting. In the second part, we introduce *virtual structures* à la Behrend-Fantechi and the virtual localization formulas after Graber-Pandharipande, Fantechi-Göttsche and Qu.

1.7.2. Chapter 3 We solve the stable pair theory of local curves. We show that the fixed locus of the moduli space of stable pairs on a local curve is a disjoint union of double nested Hilbert schemes — a new moduli space we introduce and study — and compute the localized contributions to the virtual invariants via an universality argument, as predicted by the GW/DT correspondence.

1.7.3. Chapter 4 We solve the higher rank K-theoretic Donaldson-Thomas theory of points, defined as virtual intersection numbers on the Quot scheme of zero-dimensional quotients of a locally free sheaf of rank r. We develop the higher rank topological vertex, which reduces the problem to the combinatorics of plane partitions. This solves conjectures of Awata-Kanno, Szabo and Benini-Bonelli-Poggi-Tanzini (in String Theory) and Ricolfi (in the toric case).

1.7.4. Chapter 5 We summarize the novel construction of Oh-Thomas of virtual structures on moduli spaces of sheaves on Calabi-Yau 4-folds. We describe the Edidin-Graham square-root classes and how they are used by Oh-Thomas to construct a virtual fundamental class, a virtual structure sheaf and the relevant localization formulas.

1.7.5. Chapter 6 We study K-theoretic Donaldson-Thomas and Pandharipande-Thomas theory of a toric Calabi-Yau 4-fold, inspired by the work of Nekrasov in String Theory. We develop a vertex/edge formalism computing invariants with tautological insertions and conjecture a DT/PT correspondence. We apply this machinery to compute invariants of Hilbert scheme of toric varieties and propose a conjecture on the K-theoretic PT invariants of the local resolved conifold.

1.7.6. Chapter 7 We study PT invariants of local toric surfaces. We develop two techniques to compute such invariants with primary and descendent insertions, respectively with intersection theory and equivariant cohomology. This provides a larger class of examples of the Cao-Maulik-Toda GV/PT conjectural correspondence for Calabi-Yau 4-folds.

1.7.7. Chapter 8 We introduce and study the nested Quot scheme of points. We describe its tangent space and prove that the nested Quot scheme is smooth over a smooth curve. Finally we compute its motive in the Grothendieck group of varieties and show it factors in shifted motivic zeta functions.

1.7.8. Chapter 9 We classify the smoothness of the nested Quot scheme of points, extending Cheah's classification for nested Hilbert scheme of points.

1.8. Cross-reference

The original contents of this thesis are based on the following author's articles, written in collaboration with Y. Cao, N. Fasola, M. Kool and A.T. Ricolfi.

[**36**] Y. Cao, M. Kool, and S. Monavari, *K-theoretic DT/PT correspondence for toric Calabi-Yau 4-folds*, ArXiv:1906.07856.

[**37**] Y. Cao, M. Kool, and S. Monavari, *Stable pair invariants of local Calabi-Yau 4-folds*, **International Mathematics Research Notices**, (2022), no. 2022, 4753–4798.

[**76**] N. Fasola, S. Monavari, A.T. Ricolfi, *Higher rank K-theoretic Donaldson-Thomas theory of points*, **Forum of Mathematics Sigma**, Vol. 9, e15 (2021), 1–51.

[**132**] S. Monavari, *Double nested Hilbert schemes and the local stable pairs theory of curves*, to appear in **Compositio Mathematica**.

[**133**] S. Monavari, *Canonical vertex formalism in DT theory of Calabi-Yau 4-folds*, **Journal of Geometry and Physics**, 174 (2022), 104466.

[**134**] S. Monavari and A. T. Ricolfi, *On the motive of the nested Quot scheme of points on a curve*, ArXiv: 2106.11817.

[**135**] S. Monavari and A. T. Ricolfi, *Sur la lissité du schéma Quot ponctuel emboîté*, **Canadian Mathematical Bulletin**, (2022).

1.9. Conventions

We work over $\mathbb{C}$. A *scheme* is a separated scheme of finite type over $\mathbb{C}$. We denote by $\mathbb{C}^*$ the multiplicative group $\mathbb{G}_m$. If X is a scheme, we let $\mathbf{D}^{[a,b]}(X)$ (resp. $\mathbf{D}^{\mathrm{b}}(X)$) denote the derived category of coherent sheaves on X, whose objects are complexes with vanishing cohomology sheaves outside the interval $[a,b]$ (resp. some interval). Chow groups $A^*(X)$ and cohomology groups $H^*(X)$ are taken with rational coefficients. We denote by $(\cdot)^\vee$ the derived dual and by $(\cdot)^*$ the dual of a coherent sheaf. Whenever it is clear from the context, we may omit pullbacks from the notation, in particular for the case of restrictions and projections.

PART I

Donaldson-Thomas theory in dimension three

CHAPTER 2

Localization in Equivariant Enumerative Geometry

¿Es o no es
el sueño que olvidé
antes del alba?

Haiku, J.L. Borges

2.1. *Une promenade equivariante*

2.1.1. Equivariant cohomology Equivariant cohomology is a generalization of singular cohomology, that takes into consideration topological spaces and group actions on them. Let G be a topological group and M be a topological space with a G-action. Naively, one may hope to recover relevant information looking at the space of G-orbits, that is the topological quotient M/G, and its cohomology. This approach has several drawbacks:

- Any geometric structure we may consider on M (smooth manifold, algebraic variety, etc.) could be lost when passing to the quotient M/G, whose cohomology would not be as well behaved as expected.
- If the G-action on M is trivial then the cohomology of the quotient completely forgets about the group acting:

$$H^*(M/G) = H^*(M).$$

The problem is solved by substituting the topological quotient M/G by the *homotopical quotient*. Recall that a principal G-bundle $P \to Q$ is called *universal* if any principal G-bundle $E \to M$ fits into a cartesian diagram

$$\begin{array}{ccc} E & \longrightarrow & P \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & Q, \end{array}$$

for a suitable map $f : M \to Q$, unique up to homotopy. In other words, any principal G-bundle E can be realized as the pullback $E \cong f^*P$ of a universal G-bundle (again, up to homotopy). Remarkably, as soon as P is contractible, the principle G-bundle $P \to Q$ is universal, and unique up to homotopy — see [**169**] for the construction of universal principal bundles.

Definition 2.1.1. Let $EG \to BG$ be the universal principal G-bundle. The *homotopical quotient* of M is

$$M_G = EG \times_G M = (EG \times M)/\sim,$$

where G acts on the right of EG and on the left on M, and the equivalent relation is defined identifying $(pg, q) \sim (p, gq)$ for $p \in EG, q \in M, g \in G$.

This definition induces a natural fibration

$$\pi : M_G \to BG$$

with fiber M, fitting in the commutative diagram

$$\begin{array}{ccccc} EG & \longleftarrow & EG \times M & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ BG & \xleftarrow[\pi]{} & M_G & \xrightarrow{\sigma} & M/G. \end{array}$$

Here σ will not be in general a fibration: the fiber of an equivalence class of an element $m \in M$ is

$$\sigma^{-1}([m]) = EG/G_m \cong BG_m,$$

where G_m is the stabilizer of m. This shows how the homotopical quotient M_G carries refined information and controls how far the G-action is from being free.

Definition 2.1.2. Let M be a topological space with the action of a topological group G. The *equivariant cohomology* of M is

$$H_G^*(M) := H^*(M_G).$$

The definition is independent of the choice of the universal bundle $EG \to BG$, as any two such universal bundles are homotopically equivalent.

Equivariant cohomology satisfies all expected functorial features one would expect from a cohomology theory. The most basic object is the equivariant cohomology of the point, that is

$$H_G^* := H_G^*(\mathsf{pt}) = H^*(BG).$$

The natural projection $M \to \mathsf{pt}$ induces a pullback $H_G^*(\mathsf{pt}) \to H_G^*(M)$ and therefore a structure of H_G^*-module on $H_G^*(M)$. If $f : M \to N$ is a map of compact oriented manifolds with the action of a compact Lie group, there is a natural *equivariant pushfoward*

$$f_* : H_G^*(M) \to H_G^{*-q}(N),$$

where $q = \dim M - \dim N$. If $N = \mathsf{pt}$, this map is the *equivariant integration* and denoted by

$$\int_X : H_G^*(M) \to H_G^*.$$

The inclusion of any fiber $M \hookrightarrow M_G$ induces a map

$$H_G^*(M) \to H^*(M),$$

which we should think of as the restriction of some *equivariant data* over M_G to some *ordinary data* over its fiber M. Be careful that in general it is not true that $H_G^*(M) \cong H^*(M) \otimes H_G$. When this happens, the G-action on M is said to be *formal*.

Example 2.1.3. Let M be a topological space with the trivial G-action. Then

$$H_G^*(M) \cong H^*(M) \otimes H_G.$$

Finally there are equivariant version of *characteristic classes*, such as Chern classes and Euler classes, satisfying all the usual axioms and properties.

2.1.2. Equivariant cohomology of tori As an intermezzo, we compute the equivariant cohomology of real tori $\mathbb{T} = (S^1)^r$ and algebraic tori $\mathbf{T} = (\mathbb{C}^*)^r$.

The universal principal S^1-bundle is obtained by the direct limit of the bundles $S^{2n+1} \to \mathbb{P}^n_{\mathbb{C}}$

$$\begin{array}{ccccc} ES^1 & = & S^\infty & = & \varinjlim S^{2n+1} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ BS^1 & = & \mathbb{P}^\infty_{\mathbb{C}} & = & \varinjlim \mathbb{P}^n_{\mathbb{C}}, \end{array}$$

while the universal principal $\mathbb{C}^*$-bundle is obtained by the direct limit of the bundles $\mathbb{C}^{n+1} \setminus \{ 0 \} \to \mathbb{P}^n_{\mathbb{C}}$

$$\begin{array}{ccccc} E\mathbb{C}^* & = & \mathbb{C}^\infty \setminus \{ 0 \} & = & \varinjlim \mathbb{C}^{n+1} \setminus \{ 0 \} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ B\mathbb{C}^* & = & \mathbb{P}^\infty_{\mathbb{C}} & = & \varinjlim \mathbb{P}^n_{\mathbb{C}}. \end{array}$$

In both cases, the equivariant cohomology of the point is

$$H^*_{S^1} = H^*_{\mathbb{C}^*} = H^*(\mathbb{P}^\infty_{\mathbb{C}}) = \varprojlim \frac{\mathbb{Q}[\lambda]}{(\lambda^{n+1})} = \mathbb{Q}[\lambda].$$

In the higher rank case it follows by the Kunneth decomposition

$$H^*_{\mathbb{T}} = H^*_{\mathbf{T}} = \bigotimes_{i=1}^r \mathbb{Q}[\lambda_i] = \mathbb{Q}[\lambda_1, ..., \lambda_r].$$

Remark 2.1.4. The fact that the S^1 and $\mathbb{C}^*$-equivariant cohomologies coincide should not come as a surprise, as the algebraic torus $\mathbb{C}^*$ is simply the complexification of the compact Lie group S^1.

2.1.3. Atiyah-Bott localization formula We consider now only smooth complex projective varieties X acted by an algebraic torus $\mathbf{T} = (\mathbb{C}^*)^r$. For simplicity, we consider $\mathbf{T}$-equivariant cohomology with coefficients in $\mathbb{C}$, but analogues statements can be deduced for suitable algebraic extensions of $\mathbb{Q}$.

The inclusion of the $\mathbf{T}$-fixed locus $\iota : X^{\mathbf{T}} \hookrightarrow X$ — by a result of Iversen [99], the $\mathbf{T}$-fixed locus is still a smooth projective variety — induces equivariant pushforward and pullback in cohomology

$$\begin{aligned}\iota_* &: H^*_{\mathbf{T}}(X^{\mathbf{T}}) \to H^{*-q}_{\mathbf{T}}(X)\\ \iota^* &: H^*_{\mathbf{T}}(X) \to H^*_{\mathbf{T}}(X^{\mathbf{T}}),\end{aligned}$$

where $q = \dim X - \dim X^{\mathbf{T}}$, and the equivariant version of the self-intersection formula reads

$$\iota^*\iota_* 1 = e^{\mathbf{T}}(N_{X^{\mathbf{T}}/X}),$$

where $N_{X^{\mathbf{T}}/X}$ is the normal bundle and $e^{\mathbf{T}}(\cdot)$ is the equivariant Euler class. We start with the classical statement of Atiyah-Bott abstract localization formula [5].

Theorem 2.1.5 (Atiyah-Bott abstract localization)**.** *Let X be a smooth projective variety with a $\mathbf{T}$-action, and denote by $H^*_{\mathbf{T},\mathrm{loc}}$ the fraction field of $H^*_{\mathbf{T}}$. Then there is an isomorphism*

$$\iota_* : H^*_{\mathbf{T}}(X^{\mathbf{T}}) \otimes H^*_{\mathbf{T},\mathrm{loc}} \xrightarrow{\sim} H^*_{\mathbf{T}}(X) \otimes H^*_{\mathbf{T},\mathrm{loc}}.$$

Clearly, the inverse of the isomorphism of Theorem 2.1.5 is given by

$$\frac{1}{e^{\mathbf{T}}(N_{X^{\mathbf{T}}/X})} \cdot \iota^* : H^*_{\mathbf{T}}(X) \otimes H^*_{\mathbf{T},\mathrm{loc}} \xrightarrow{\sim} H^*_{\mathbf{T}}(X^{\mathbf{T}}) \otimes H^*_{\mathbf{T},\mathrm{loc}}.$$

Composing the isomorphism induced by ι_* with its inverse, Atiyah-Bott localization expresses every cohomological class on X as a sum of *localized* contribution on its $\mathbf{T}$-fixed locus $X^{\mathbf{T}}$.

Corollary 2.1.6 (Atiyah-Bott localization formula)**.** *Let X be a smooth projective variety with a $\mathbf{T}$-action and denote by X_i the connected components of the $\mathbf{T}$-fixed locus $X^{\mathbf{T}}$. Then the fundamental class can be expressed as*

$$[X] = \sum_i \iota_* \frac{[X_i]}{e^{\mathbf{T}}(N_{X_i/X})},$$

and any equivariant cohomology class $\alpha \in H^*_{\mathbf{T}}(X)$ can be expressed as

$$\alpha = \sum_i \iota_* \frac{\iota^*\alpha}{e^{\mathbf{T}}(N_{X_i/X})}.$$

In particular, the following integration formula holds

$$\int_X \alpha = \sum_i \int_{X_i} \frac{\alpha|_{X_i}}{e^{\mathbf{T}}(N_{X_i/X})}.$$

The integration formula in Corollary 2.1.6 can be applied to a wide range of situations.

Example 2.1.7. Atiyah-Bott localization formula can be applied to compute ordinary integrals. In fact, consider the commutative diagram

$$\begin{tikzcd} H^*_{\mathbf{T}}(X) \arrow[r, "\pi_*^{\mathbf{T}}"] \arrow[d] & H^*_{\mathbf{T}}(\mathsf{pt}) \cong \mathbb{C}[\lambda_1, \dots, \lambda_r] \arrow[d, "\lambda_i=0"] \\ H^*(X) \arrow[r, "\pi_*"] & H^*(\mathsf{pt}) \cong \mathbb{C}, \end{tikzcd}$$

where $\pi_*, \pi_*^{\mathbf{T}}$ denote the (equivariant) integration maps and the vertical maps are *forgetting* the equivariant structure. Say that there is a lift of an ordinary cohomology class $\alpha \in H^*(X)$ to an equivariant class $\tilde{\alpha} \in H^*_{\mathbf{T}}(X)$ — unfortunately, the map from equivariant to ordinary cohomology is not always surjective. Then we can compute the integral as

$$\begin{aligned} \int_X \alpha &= \left(\int_X \tilde{\alpha} \right)\Big|_{\lambda_i=0} \\ &= \left(\sum_i \int_{X_i} \frac{\tilde{\alpha}|_{X_i}}{e^{\mathbf{T}}(N_{X_i/X})} \right)\Bigg|_{\lambda_i=0}. \end{aligned}$$

Remark 2.1.8. Often the restriction of the equivariant parameters $\lambda_i = 0$ is not needed — a phenomenon we refer to as *rigidity principle*. For instance, suppose that $X^{\mathbf{T}}$ consists in finitely many fixed points $\{ p_i \}_i$. Then we have

$$\int_X \alpha = \left(\sum_i \frac{p_i(\lambda)}{q_i(\lambda)} \right)\Bigg|_{\lambda_i=0},$$

where $p_i(\lambda), q_i(\lambda) \in \mathbb{C}[\lambda_1, \dots, \lambda_r]$ are some homogeneous polynomials in the equivariant parameters and $q_i(\lambda)$ have degree at most $\dim X$. Moreover, the sum $\sum_i \frac{p_i(\lambda)}{q_i(\lambda)}$ has *no poles* in $\lambda_i = 0$. Therefore, if $\deg p_i(\lambda) \leq \dim X$, the sum has to be already a constant *before* setting $\lambda_i = 0$.

Example 2.1.9. If X is not proper, the integration over X does not make sense. Suppose instead that X is acted by an algebraic torus $\mathbf{T}$ with proper $\mathbf{T}$-fixed locus $X^{\mathbf{T}}$. We may formally define

$$\int_X \alpha := \int_{X^{\mathbf{T}}} \frac{\alpha|_{X^{\mathbf{T}}}}{e^{\mathbf{T}}(N_{X^{\mathbf{T}}/X})} \in H^*_{\mathbf{T},\mathrm{loc}},$$

where the right-hand-side is defined equivariantly. If X is proper, this definition is consistent with the usual integration over X by the Atiyah-Bott localization. Clearly, if X is not proper, we cannot specialize the equivariant parameters $\lambda_i = 0$.

Example 2.1.10 (Topological Euler characteristic)**.** It is classically know that the Euler characteristic of a variety X with a $\mathbf{T}$-action is the same of its $\mathbf{T}$-fixed locus. Atiyah-Bott localization formula provides a simple proof in the case the variety is smooth and projective.

The inclusion $X^{\mathbf{T}} \hookrightarrow X$ induces an exact sequence of vector bundles on $X^{\mathbf{T}}$

$$0 \to T_{X^{\mathbf{T}}} \to T_X|_{X^{\mathbf{T}}} \to N_{X^{\mathbf{T}}/X} \to 0,$$

and we can compute the topological Euler characteristic of X by the Poincaré-Hopf Theorem

$$\begin{aligned} e_{\mathrm{top}}(X) &= \int_X e(T_X) \\ &= \int_{X^{\mathbf{T}}} \frac{e^{\mathbf{T}}(T_X|_{X^{\mathbf{T}}})}{e^{\mathbf{T}}(N_{X^{\mathbf{T}}/X})} \\ &= \int_{X^{\mathbf{T}}} e(T_{X^{\mathbf{T}}}) \\ &= e_{\mathrm{top}}(X^{\mathbf{T}}). \end{aligned}$$

2.1.3.1. A vademecum on Atiyah-Bott localization formula We try to give a rough account on how to operatively use the Atiyah-Bott localization, at least when integrating equivariant Chern classes of vector bundles on a smooth projective variety.

Say that X carries a trivial $\mathbf{T}$-action, for $\mathbf{T} = (\mathbb{C}^*)^r$ — the main scenario is when X is a $\mathbf{T}$-fixed locus. Remember that the character group of a torus is $\widehat{\mathbf{T}} = \mathbb{Z}^r$, and denote by $\mathfrak{t}_1, \dots, \mathfrak{t}_r$ the irreducible characters corresponding to the standard basis of $\mathbb{Z}^r$. Any $\mathbf{T}$-equivariant vector bundle $\mathcal{E}$ on X admits an *eigenbundle decomposition*

$$\mathcal{E} = \bigoplus_{\mu \in \mathbb{Z}^r} \mathcal{E}_\mu \otimes \mathfrak{t}^\mu,$$

where $\mathcal{E}_\mu$ is the vector bundle corresponding to the weight $\mu \in \widehat{\mathbf{T}}$ and we introduced the short-cut

$$\mathfrak{t}^\mu = \mathfrak{t}_1^{\mu_1} \dots \mathfrak{t}_r^{\mu_r}.$$

The equivariant Chern classes of $\mathcal{E}$

$$c_i^{\mathbf{T}}(\mathcal{E}) \in H^*_{\mathbf{T}}(X) \cong H^*(X) \otimes H^*_{\mathbf{T}}(\mathsf{pt})$$

satisfy the same multiplicative properties of ordinary Chern classes, by seeing the irreducible characters $\mathfrak{t}^\mu$ as equivariant line bundles $\mathcal{O}_X \otimes \mathfrak{t}^\mu$. For example, the equivariant Euler class of $\mathcal{E}$ is computed as

$$e^{\mathbf{T}}(\mathcal{E}) = \prod_{\mu \in \mathbb{Z}^r} \sum_{i=0}^{\operatorname{rk} \mathcal{E}_\mu} c_i(\mathcal{E}_\mu) \cdot c_1^{\mathbf{T}}(\mathfrak{t}^\mu)^{\operatorname{rk} \mathcal{E}_\mu - i} \in H^*(X)[\lambda_1, \dots, \lambda_r].$$

The first Chern class of the equivariant line bundles are the generators of the equivariant cohomology

$$c_1^{\mathbf{T}}(\mathfrak{t}_i) = \lambda_i \in \mathbb{Q}[\lambda_1, \dots, \lambda_r],$$

and the previous expression can be simplified to

$$e^{\mathbf{T}}(\mathcal{E}) = \prod_{\mu \in \mathbb{Z}^r} \sum_{i=0}^{\operatorname{rk} \mathcal{E}_\mu} c_i(\mathcal{E}_\mu) \cdot (\mu \cdot \lambda)^{\operatorname{rk} \mathcal{E}_\mu - i},$$

where $\mu \cdot \lambda$ denotes the standard scalar product.

Example 2.1.11. How do we perform *explictly* equivariant integration? Let's consider the projective space $\mathbb{P}^n$ for a concrete example. Consider the action of the torus $\mathbf{T} = (\mathbb{C}^*)^{n+1}$, acting on the homogeneous coordinates of $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ as

$$(t_0, \dots, t_n) \cdot [x_0 : \dots : x_n] = [t_0 x_0 : \dots : t_n x_n].$$

The $n+1$ $\mathbf{T}$-fixed points are $p_i = [0 : \dots : 0 : 1 : 0 \dots : 0]$ (where the 1 is in position i), for $i = 0, \dots, n$. The normal bundle at every fixed point splits as

$$N_{p_i} = T_{\mathbb{P}^n, p_i} = \bigoplus_{\substack{j=0 \\ j \neq i}}^{n} \mathbb{C} \cdot \frac{\partial}{\partial x_{ij}},$$

where $\left(x_{ij} = \frac{x_j}{x_i}\right)_{j \neq i}$ are local affine coordinates around p_i and $\frac{\partial}{\partial x_{ij}}$ is the dual of the Kähler differential dx_{ij}. Equivariantly it decomposes as

$$N_{p_i} = \bigoplus_{\substack{j=0 \\ j \neq i}}^{n} \mathbb{C} \cdot \mathsf{t}_i \, \mathsf{t}_j^{-1},$$

therefore its equivariant Euler class is

$$e^{\mathbf{T}}(N_{p_i}) = \prod_{\substack{j=0 \\ j \neq i}}^{n} (\lambda_i - \lambda_j) \in \mathbb{Q}[\lambda_0, \dots, \lambda_n].$$

The line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ fits into the Euler exact sequence

$$0 \to \Omega^1_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n} \otimes \mathbb{C}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0,$$

which presents $\mathcal{O}_{\mathbb{P}^n}(1)$ as a quotient of $\mathcal{O}_{\mathbb{P}^n} \otimes \mathbb{C}^{n+1}$, by which we deduce that the fibers over the fixed points have weight

$$\mathcal{O}_{\mathbb{P}^n}(1)|_{p_i} = \mathbb{C} \otimes \mathsf{t}_i \, .$$

Performing the simplest intersection number

$$\int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1)^k) = \begin{cases} 0, & k < n, \\ 1, & k = n, \end{cases}$$

yields the nice combinatorial identity

$$\sum_{i=0}^{n} \frac{\lambda_i^k}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (\lambda_i - \lambda_j)} = \begin{cases} 0, & k < n, \\ 1, & k = n. \end{cases}$$

The left-hand-side is a homogeneous rational expression of total degree less than 0, which cannot have poles in $\lambda_i - \lambda_j = 0$, by which we conclude that the expression is constant on λ_i *before* the specialization $\lambda_i = 0$. This is an instance of the *rigidity principle* described in Remark 2.1.8.

2.1.4. Equivariant Chow groups All the contents of the previous section could be formulated in a purely algebraic fashion. We briefly summarize the construction of *equivariant Chow groups* and *localization formula in Chow groups*, following Edidin-Graham [**69, 70**].

Let X be a scheme acted by a g-dimensional algebraic group G and V be a G-representation of dimension l, containing an open $U \subset V$ such that a principal bundle quotient $U \to U/G$ exists, and that $V \setminus U$ has codimension more than $n - i$. Then G acts freely on $X \times U$ and — under some mild hypotheses satisfied in our case of interest, see [**69**, Prop. 23] — the quotient $X_G = (X \times U)/G$ exists as a scheme.

Definition 2.1.12. The i-th equivariant Chow group is

$$A_i^G(X) := A_{i+l-g}(X_G).$$

This definition is independent of V and U as long as $V \setminus U$ has sufficiently high codimension.

Equivariant Chow groups enjoy — once more — all the functorial features of ordinary Chow groups, which comprises the equivariant upgrade of proper pushforward, flat pullback, and Chern classes. Using equivariant Chern classes, one may naturally define *Chow cohomology* (or *operational Chow group*) $A_G^*(X)$, which comes equipped with an intersection product turning $A_G^*(X)$ into a ring. If X is smooth of dimension n, this is simply $A_G^*(X) \cong A_{n-*}^G(X)$. If X is a projective variety, equivariant Chow cohomology comes equipped with a *cycle map*

$$A_G^*(X) \to H_G^*(X),$$

in complete analogy with the ordinary case, which is an isomorphism in many cases of interest [**78**, Example 19.1.11], among which there are toric varieties. In the case of an algebraic torus $\mathbf{T} = (\mathbb{C}^*)^r$ we have

$$A_{\mathbf{T}}^*(\mathsf{pt}) \cong \mathbb{Q}[\lambda_1, \dots, \lambda_r].$$

2.1.4.1. Edidin-Graham localization An algebraic version of Theorem 2.1.5 holds. Let X be a scheme acted by an algebraic torus $\mathbf{T} = (\mathbb{C}^*)^r$ and $\iota : X^{\mathbf{T}} \hookrightarrow X$ be the inclusion of the $\mathbf{T}$-fixed locus.

Theorem 2.1.13 (Edidin-Graham localization)**.** *Denote by $A_{\mathbf{T},\mathrm{loc}}^*$ the fraction field of $A_{\mathbf{T}}^*(\mathsf{pt})$. Then there is an isomorphism*

$$\iota_* : A_{\mathbf{T}}^*(X^{\mathbf{T}}) \otimes A_{\mathbf{T},\mathrm{loc}}^* \xrightarrow{\sim} A_{\mathbf{T}}^*(X) \otimes A_{\mathbf{T},\mathrm{loc}}^*.$$

Moreover if X is a smooth variety, the fundamental class can be expressed as

$$[X] = \sum_i \iota_* \frac{[X_i]}{e^{\mathbf{T}}(N_{X_i/X})},$$

and any Chow class $\alpha \in A_{\mathbf{T}}^*(X)$ can be explicitly written as

$$\alpha = \sum_i \iota_* \frac{\iota^* \alpha}{e^{\mathbf{T}}(N_{X_i/X})},$$

where X_i are the connected components of the fixed locus $X^{\mathbf{T}}$. In particular, if X is a smooth projective variety, the integration formula holds

$$\int_X \alpha = \sum_i \int_{X_i} \frac{\alpha|_{X_i}}{e^{\mathbf{T}}(N_{X_i/X})}.$$

2.1.5. Equivariant K-theory For a complete discussion of equivariant algebraic K-theory we refer the reader to Chriss-Ginzburg [**57**, Sec. 5] and Okounkov's wonderful lectures [**149**, Sec. 2]. Let X be an scheme acted by a (linearly) reductive group G. We denote by $\mathrm{Coh}_G(X)$ the category of G-equivariant sheaves on X and by $\mathrm{D}^{\mathrm{b}}_G(X)$ the bounded derived category of G-equivariant sheaves on X. By *equivariant K-theory* of X we refer to the Grothendieck group of equivariant coherent (resp. locally free) sheaves $K_0^G(X)$ (resp. $K^0_G(X)$). There is always a map relating the two Grothendieck groups

$$K^0_G(X) \to K_0^G(X),$$

which is an isomorphism if X is a smooth. A G-equivariant coherent sheaf over a point is the same as a finite-dimensional G-representation, thus the K-theory of the point is simply the representation ring

$$K^0_G(\mathsf{pt}) \cong \mathbb{Z}[\widehat{G}],$$

which consists of *virtual representations*, that is the formal difference of two G-representations. We already observed that the character group of the algebraic torus $\mathbf{T} = (\mathbb{C}^*)^r$ is $\widehat{\mathbf{T}} \cong \mathbb{Z}^r$, yielding

$$K^0_{\mathbf{T}}(\mathsf{pt}) \cong \mathbb{Z}[\mathfrak{t}_1^{\pm 1}, \ldots, \mathfrak{t}_r^{\pm 1}],$$

where $\mathfrak{t}_1, \ldots, \mathfrak{t}_r$ — the coordinates of $\mathbf{T}$ — are seen as irreducible 1-dimensional complex representations of $\mathbf{T}$.

Equivariant K-theory enjoys all the usual functorial properties, which descent from the functoriality in $\mathrm{Coh}_G(X)$ and $\mathbf{D}^{\mathrm{b}}_G(G)$. For example, let $f : X \to Y$ be a flat morphism and $\mathcal{F}$ a sheaf on Y. The *equivariant flat pullback* of $[\mathcal{F}]$ is the class $[f^*\mathcal{F}] \in K_0^G(X)$. If f is not flat, but only finitely many $\mathbf{L}^i f^*\mathcal{F}$ are non-zero, the equivariant pullback in K-theory is defined as

$$\mathbf{L}f^*[\mathcal{F}] = \sum_{i=0}^{\infty} (-1)^i [\mathbf{L}^i f^*\mathcal{F}] \in K_0^G(X).$$

Similarly, let $f : X \to Y$ be a proper map of quasi-projective schemes. The *equivariant proper pushforward* of a sheaf $\mathcal{F}$ on X is

$$\mathbf{R}f_*[\mathcal{F}] = \sum_{i=0}^{\infty} (-1)^i [\mathbf{R}^i f_*\mathcal{F}] \in K_0^G(Y),$$

which is a well-defined class as only finitely many $\mathbf{R}^i f_*\mathcal{F}$ can be non-zero. For simplicity, we will often denote $\mathbf{R}f_*$ by f_* keeping in mind that it comes from a derived functor[1].

[1] Often in the literature $\mathbf{R}f_*$ is denoted by $f_!$, see e.g. [**78**].

If $Y = \mathsf{pt}$, f_* plays the rôle of the integration map in equivariant K-theory and we denote the pushforward by the *equivariant homolorphic Euler characteristic*

$$\chi(X, \mathcal{F}) := f_*[\mathcal{F}] = \sum_{i=0}^{\infty} (-1)^i H^i(X, \mathcal{F}) \in \mathbb{Z}[\widehat{G}],$$

in complete analogy with the ordinary case. Here the cohomology groups $H^i(X, \mathcal{F})$ are seen as finite-dimensional complex G-representations, leading to much more refined invariants with respect to the bare numbers computed by the ordinary holomorphic Euler characteristic.

Remark 2.1.14. If the map is not proper, we may not directly define the pushforward in (ordinary) K-theory. Equivariantly, there is still hope. In fact, suppose that $\mathbf{R}^i f_* \mathcal{F}$ decomposes in (possibly infinitely-many) finite-dimensional weight spaces. Then we may define the pushforward $f_*[\mathcal{F}]$ in a completion of $K_0^G(Y)$ as the (possibly infinite) direct sum of all the finite-dimensional weight spaces. For instance, this situation happens whenever X is a quasi-projective variety with a proper $\mathbf{T}$-fixed locus, generalizing the situation of Example 2.1.9.

2.1.5.1. K-theoretic localization Let X be a quasi-projective scheme and $\mathcal{E}$ a locally free sheaf on X. Define the *total wedge power*

$$\Lambda_p \mathcal{E} = \sum_{i=0}^{\operatorname{rk} \mathcal{E}} p^i \Lambda^i \mathcal{E} \in K_G^0(X),$$

which satisfies $\Lambda_p(\mathcal{E} \oplus \mathcal{E}') = \Lambda_p(\mathcal{E}) \otimes \Lambda_p(\mathcal{E}')$ for any two locally free sheaves $\mathcal{E}, \mathcal{E}'$ and extend it linearly to any class $\mathcal{E} \in K_G^0(X)$. If $p = -1$, we denote it by $\Lambda_{-1} = \Lambda^\bullet$. If $X = \mathsf{pt}$, $\mathbf{T} = (\mathbb{C}^*)^r$ and $V = \sum_\mu \mathsf{t}^\mu$ is a $\mathbf{T}$-representation, it satisfies

$$\Lambda_p V = \prod_\mu (1 - p\, \mathsf{t}^\mu) \in \mathbb{Z}[\mathsf{t}_1^{\pm 1}, \ldots, \mathsf{t}_r^{\pm 1}].$$

A version of the localization theorem in K-theory has been proven by Thomason[2] [**179**, Thm. 2.1], taking place in the localization of $K_{\mathbf{T}}^0(\mathsf{pt})$

$$K_{\mathbf{T}}^0(\mathsf{pt})_{\mathrm{loc}} := \mathbb{Z}[\mathsf{t}_1^{\pm 1}, \ldots, \mathsf{t}_r^{\pm 1}] \left[\frac{1}{1 - \mathsf{t}^\mu} : \mu \in \widehat{\mathbf{T}} \right].$$

Theorem 2.1.15 (K-theoretic localization)**.** *Let X be a scheme acted by an algebraic torus $\mathbf{T}$, with $\mathbf{T}$-fixed locus $\iota : X^{\mathbf{T}} \hookrightarrow X$. Then there is an isomorphism*

$$\iota_* : K_0^{\mathbf{T}}(X^{\mathbf{T}}) \otimes K_{\mathbf{T},\mathrm{loc}}^0 \xrightarrow{\sim} K_0^{\mathbf{T}}(X) \otimes K_{\mathbf{T},\mathrm{loc}}^0.$$

Moreover if X is a smooth, any K-theory class $V \in K_0^{\mathbf{T}}(X)$ can be explicitly written as

$$V = \sum_i \iota_* \frac{\iota^* V}{\Lambda^\bullet(N_{X_i/X}^*)},$$

[2] See also Edidin-Graham [**71**] for a nonabelian version of the K-theoretic localization formula.

where X_i are the connected components of the fixed locus $X^{\mathbf{T}}$. In particular, if X is a smooth projective variety, the integration formula holds

$$\chi(X,V)=\sum_i \chi\left(X_i, \frac{V|_{X_i}}{\Lambda^\bullet(N^*_{X_i/X})}\right).$$

As for the case of equivariant cohomology, Theorem 2.1.15 can be applied to compute ordinary Euler holomorphic characteristics. Say that $V \in K_0(X)$ admits a lift $\tilde{V} \in K_0^{\mathbf{T}}(X)$, then thanks to the commutative diagram

$$\begin{tikzcd} K_0^{\mathbf{T}}(X) \arrow[r, "\pi_*^{\mathbf{T}}"] \arrow[d] & K^0_{\mathbf{T}}(\mathsf{pt}) \cong \mathbb{Z}[\mathfrak{t}_1^{\pm 1},\dots,\mathfrak{t}_r^{\pm 1}] \arrow[d, "\mathfrak{t}_i=1"] \\ K_0(X) \arrow[r, "\pi_*"] & K^0(\mathsf{pt}) \cong \mathbb{Z}, \end{tikzcd}$$

and the ordinary holomorphic Euler characteristic is computed as

$$\begin{aligned}\chi(X,V) &= \chi(X,\tilde{V})\Big|_{\mathfrak{t}_i=1}\\ &= \sum_i \chi\left(X_i, \frac{\tilde{V}|_{X_i}}{\Lambda^\bullet(N^*_{X_i/X})}\right)\Bigg|_{\mathfrak{t}_i=1}.\end{aligned}$$

Now differently than the case of equivariant cohomology, the rigidity of the equivariant parameters in K-theory features much rarely and essentially depends on which elements of the form $1-\mathfrak{t}^\mu$ we actually need to invert. Arbesfeld [**3**, Sec. 3.1] proved a nice criterion to determine the denominators in the localization.

Let X be a smooth quasi-projective scheme acted by a torus $\mathbf{T}$ with proper $\mathbf{T}$-fixed locus $X^{\mathbf{T}}$ — equivariant holomorphic Euler characteristic is well-defined by Remark 2.1.14. We say that a weight $\mathfrak{t}^\mu \in \widehat{\mathbf{T}}$ is *compact* if the $\mathbf{T}$-fixed locus $X^{\mathbf{T}_{\mathfrak{t}^\mu}}$ is proper, and *non-compact* otherwise. Here, $\mathbf{T}_{\mathfrak{t}^\mu} \subset \mathbf{T}$ denote the maximal torus contained in $\ker(\mathfrak{t}^\mu) \subset \mathbf{T}$. In particular, for every non-zero $n \in \mathbb{Z}$, we have $\mathbf{T}_{\mathfrak{t}^\mu} = \mathbf{T}_{\mathfrak{t}^{n\mu}}$.

Proposition 2.1.16 ([**3**, Prop. 3.2])**.** *Let $\mathcal{F} \in K_0^{\mathbf{T}}(X)$ be a class in K-theory. We have*[3]

$$\chi(X,\mathcal{F}) \in \mathbb{Z}[\mathfrak{t}_1^{\pm 1},\dots,\mathfrak{t}_r^{\pm 1}]\left[\frac{1}{1-\mathfrak{t}^\mu} : \mathfrak{t}^\mu \text{ non-compact weight}\right].$$

In other words, we can express

$$\chi(X,\mathcal{F})=\frac{p(\mathfrak{t})}{q(\mathfrak{t})},$$

where $p(\mathfrak{t}), q(\mathfrak{t}) \in \mathbb{Z}[\mathfrak{t}_1^{\pm 1},\dots,\mathfrak{t}_r^{\pm 1}]$ are Laurent polynomials and $q(\mathfrak{t})$ is a product of elements of the form $(1-\mathfrak{t}^\mu)$, with $\mathfrak{t}^\mu$ a non-compact weight.

We end the section with some examples.

[3]This result holds in more generality in the virtual setting described in Section 2.2.

Example 2.1.17 (A local example)**.** Consider $\mathbb{A}^n$ with the standard action of $\mathbf{T} = (\mathbb{C}^*)^n$. As an infinite-dimensional vector space, the global sections of $\mathcal{O}_{\mathbb{A}^n}$ are

$$\begin{aligned} H^0(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}) &= \bigoplus_{k_1,\dots,k_n \geq 0} \mathbb{C} \cdot x_1^{n_1} \cdots x_n^{k_n} \\ &= \frac{\mathbb{C}}{(1-x_1)\cdots(1-x_n)}, \end{aligned}$$

where in the last line we just formally expand the geometric series. As there is no higher cohomology, the equivariant holomorphic Euler characteristic is simply the global sections seen as a $\mathbf{T}$-representation

$$\chi(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}) = \frac{1}{(1-\mathsf{t}_1)\cdots(1-\mathsf{t}_n)}.$$

The same quantity can be also computed with the K-theoretic localization, as the only $\mathbf{T}$-fixed point of $\mathbb{A}^n$ is the origin, which also confirms that the non-compact weights of $\mathbb{A}^n$ are of the form t_i^k, for $k \in \mathbb{Z}$.

Example 2.1.18 (A global example)**.** With the same notation as in Example 2.1.11, consider $\mathbb{P}^n$ with the action of $\mathbf{T} = (\mathbb{C}^*)^{n+1}$ and let $d \geq 0$. As a $\mathbb{C}$-vector space, the global section of $\mathcal{O}_{\mathbb{P}^n}(d)$ are

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \mathbb{C}[x_0, \dots, x_n]_{(d)} = \bigoplus_{d_0+\dots+d_n=d} \mathbb{C} \cdot x_0^{d_0} \cdots x_n^{d_n}.$$

As there is no higher cohomology, the equivariant holomorphic Euler characteristic is simply the global sections seen as a $\mathbf{T}$-representation

$$\chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \bigoplus_{d_0+\dots+d_n=d} \mathbb{C} \cdot \mathsf{t}_0^{d_0} \cdots \mathsf{t}_n^{d_n}.$$

The right-hand-side can be computed via the K-theoretic localization theorem

$$\chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \sum_{i=0}^n \frac{\mathsf{t}_i^d}{\prod_{\substack{j=0 \\ j\neq i}}^n (1-\mathsf{t}_i^{-1}\mathsf{t}_j)},$$

yielding the identity

$$\sum_{i=0}^n \frac{\mathsf{t}_i^d}{\prod_{\substack{j=0 \\ j\neq i}}^n (1-\mathsf{t}_i^{-1}\mathsf{t}_j)} = \sum_{d_0+\dots+d_n=d} \mathsf{t}_0^{d_0} \cdots \mathsf{t}_n^{d_n}.$$

In fact, by Proposition 2.1.16 the left-hand-side cannot have poles of the form $(1-\mathsf{t}_i^{-1}\mathsf{t}_j)$ - as confirmed by the right-hand-side - as all weights are compact. If we specialize $\mathsf{t}_i = 1$, we recover the ordinary holomorphic Euler characteristic

$$\chi(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{n}.$$

2.2. *Une promenade virtuelle*

2.2.1. Perfect obstruction theories Performing intersection theory on a singular scheme is a hard job. A way to bypass this problem (algebraically!) is to introduce *virtual structures* à la Behrend-Fantechi and Li-Tian [**12, 123**].

Definition 2.2.1. A *perfect obstruction theory* on a scheme X is the datum of a morphism

$$\phi\colon \mathbb{E} \to \mathbb{L}_X$$

in $\mathbf{D}^{[-1,0]}(X)$, where $\mathbb{E}$ is a perfect complex of perfect amplitude contained in $[-1,0]$, such that

- $h^0(\phi)$ is an isomorphism,
- $h^{-1}(\phi)$ is surjective.

Here, $\mathbb{L}_X = \tau_{\geq -1}L_X^\bullet$ is the cut-off at -1 of the full cotangent complex $L_X^\bullet \in \mathbf{D}^{[-\infty,0]}(X)$ introduced by Illusie [**96**]. A perfect obstruction theory is called *symmetric* (see [**13**]) if there exists an isomorphism $\theta\colon \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee[1]$ such that $\theta = \theta^\vee[1]$. The *virtual dimension* of X with respect to $(\mathbb{E},\phi)$ is the integer $\mathrm{vd} = \operatorname{rk}\mathbb{E}$. This is just $\operatorname{rk} E^0 - \operatorname{rk} E^{-1}$ if one can write $\mathbb{E} = [E^{-1} \to E^0]$, where E^0, E^{-1} are locally free sheaves on X.
A perfect obstruction theory determines a cone

$$\mathfrak{C} \hookrightarrow E_1 = (E^{-1})^*.$$

Letting $0_{E_1}\colon X \hookrightarrow E_1$ be the zero section of the vector bundle E_1, the induced *virtual fundamental class* on X is the refined intersection

$$[X]^{\mathrm{vir}} = 0_{E_1}^![\mathfrak{C}] \in A_{\mathrm{vd}}(X).$$

By a result of Siebert [**171**, Thm. 4.6], the virtual fundamental class depends only on the K-theory class of $\mathbb{E}$.

On the K-theoretic side, it was observed in [**12**, Sec. 5.4] that a perfect obstruction theory induces also a *virtual structure sheaf*

$$\mathcal{O}_X^{\mathrm{vir}} = [\mathbf{L}0_{E_1}^*\,\mathcal{O}_{\mathfrak{C}}] \in K_0(X).$$

Its construction first appeared in [**110, 58**] in the context of dg-manifolds and in [**75**] in the (algebraic) language of perfect obstruction theories. More recently, Thomas gave a description of $\mathcal{O}_X^{\mathrm{vir}}$ in terms of the *K-theoretic Fulton class*, showing that it only depends on the K-theory class of $\mathbb{E}$ [**177**, Cor. 4.5]. Both the virtual class and the virtual structure sheaf are deformation invariants.

Locally, all perfect obstruction theories are of the following form.

Example 2.2.2 (Kuranishi global model[4])**.** Let a scheme Z

$$
\begin{tikzcd}
 & \mathcal{E} \arrow[d, shift right] \\
Z := Z(s) \arrow[r, hook, "\iota"] & A, \arrow[u, bend right, "s"']
\end{tikzcd}
$$

be the zero locus of a section $s \in \Gamma(A, \mathcal{E})$, where $\mathcal{E}$ is a vector bundle over a smooth quasi-projective variety A. Then there exists a induced perfect obstruction theory on Z

$$
\begin{array}{ccc}
\mathbb{E} & = & \left[\mathcal{E}^*|_Z \xrightarrow{\mathrm{d}s^*} \Omega_A|_Z\right] \\
{\scriptstyle\phi}\big\downarrow & & {\scriptstyle s^*}\big\downarrow \qquad\qquad \big\downarrow{\scriptstyle\mathrm{id}} \\
\mathbb{L}_Z & = & \left[\mathcal{I}/\mathcal{I}^2 \xrightarrow{\mathrm{d}} \Omega_A|_Z\right]
\end{array}
$$

in $\mathbf{D}^{[-1,0]}(Z)$, where we represented the truncated cotangent complex by means of the exterior derivative d constructed out of the ideal sheaf $\mathcal{I} \subset \mathcal{O}_Z$ of the inclusion $Z \hookrightarrow A$. Moreover

$$
\begin{aligned}
\iota_*[Z]^{\mathrm{vir}} &= e(\mathcal{E}) \cap [A] \in A_*(A), \\
\iota_* \mathcal{O}_Z^{\mathrm{vir}} &= \Lambda^\bullet \mathcal{E}^* \in K^0(A).
\end{aligned}
$$

In other words, the virtual fundamental class $[Z]^{\mathrm{vir}}$ and the virtual structure sheaf $\mathcal{O}_Z^{\mathrm{vir}}$ push to the smooth ambient space A as if the section is regular, that is as if the intersection is transverse.

Example 2.2.3 (Perturbing the equations)**.** The simplest instance of the Kuranishi global model is the following affine situation

$$
\begin{tikzcd}
 & \mathcal{O}_{\mathbb{A}^2} \oplus \mathcal{O}_{\mathbb{A}^2} \arrow[d, shift right] \\
\mathbb{A}^1 = Z(s) \arrow[r, hook] & \mathbb{A}^2, \arrow[u, bend right, "s"']
\end{tikzcd}
$$

where $s = (x, x) \in \mathbb{C}[x, y]^{\oplus 2}$. The section s is clearly not regular, as the two equations are not linearly independent; however, we shall *perturbe* it to a regular section s'

$$(x, x) \rightsquigarrow (x, x + \varepsilon \cdot y),$$

where $0 \neq \varepsilon \in \mathbb{C}$. Now, the zero locus of s' is simply the origin $(0, 0) \in \mathbb{A}^2$ with its Kozsul resolution

$$\mathcal{O}_{(0,0)} = \Lambda^\bullet \mathbb{C}[x, y]^{\oplus 2} = [\mathbb{C}[x, y] \to \mathbb{C}[x, y]^{\oplus 2} \to \mathbb{C}[x, y]].$$

We remarked before that the virtual structure sheaf just depends on the K-theory class of the perfect obstruction theory: in other words, we may forget the maps of the complex defining it. Therefore we showed that the induced virtual structure sheaf on $\mathbb{A}^1$ satisfies

$$\iota_* \mathcal{O}_{\mathbb{A}^1}^{\mathrm{vir}} = \mathcal{O}_{(0,0)} \in K^0(\mathbb{A}^2).$$

[4]I borrowed this terminology from Richard Thomas, but it may be not standard in Algebraic Geometry.

The slogan of this example is

"The virtual structure sheaf of X is the class of the Kozsul resolution of a perturbation of the section defining X."

Remark 2.2.4. The *Kuranishi global model* of Example 2.2.2 enjoys a more conceptual understanding in the framework of *Derived Algebraic Geometry* — see [**182**] for an introduction to the subject. The zero section $Z(s) \hookrightarrow A$ is realized as the fiber product

$$\begin{tikzcd} Z(s) \arrow[r, hook] \arrow[d, hook] & A \arrow[d, "s"] \\ A \arrow[r, "0"] & \mathcal{E}. \end{tikzcd}$$

If we consider the fiber product in the category of schemes Sch, then $Z(s)$ is an (ordinary) scheme with structure sheaf $\mathcal{O}_{Z(s)}$. However, we can enlarge our category of schemes to the *∞-category of derived schemes* dSch and take the fiber product as a derived scheme. This results in a derived scheme Z, whose truncation recovers the ordinary zero section $Z(s)$, but comes naturally equipped with a sheaf (in the derived sense) $\mathcal{O}_Z^{\mathrm{der}}$ satisfying (cf. [**182**, pag. 192])

$$\mathcal{O}_Z^{\mathrm{der}} = \mathcal{O}_{Z(s)}^{\mathrm{vir}} \in K_0(Z(s)),$$

and whose truncation recovers the structure sheaf $\mathcal{O}_{Z(s)}$. In other words, we should more naturally see the virtual structure sheaf of an ordinary scheme as the ordinary structure sheaf of a derived scheme (as long as such a derived enhancing is possible).

2.2.2. Virtual localization formulas Let X be a scheme acted by an algebraic torus $\mathbf{T}$ with a $\mathbf{T}$-equivariant perfect obstruction theory $\mathbb{E}$ and denote by $T_X^{\mathrm{vir}} := \mathbb{E}^\vee \in K^0(X)$ the *virtual tangent bundle*[5]. Graber-Pandharipande [**85**, Prop. 1] showed that there exists an induced perfect obstruction theory on the $\mathbf{T}$-fixed locus $X^{\mathbf{T}}$, with virtual tangent bundle $T_{X^{\mathbf{T}}}^{\mathrm{vir}} = T_X^{\mathrm{vir}}|_{X^{\mathbf{T}}}^{\mathsf{fix}}$ the $\mathbf{T}$-fixed part of the virtual tangent bundle. Denote by $N_{X^{\mathbf{T}}/X}^{\mathrm{vir}} := T_X^{\mathrm{vir}}|_{X^{\mathbf{T}}}^{\mathsf{mov}}$ — the movable part of the virtual tangent bundle — the *virtual normal bundle*. Graber-Pandharipande [**85**] generalized the Atiyah-Bott localization formula in the virtual setting[6].

Theorem 2.2.5 (Graber-Pandharipande virtual localization)**.** *Denote by $\iota : X^{\mathbf{T}} \hookrightarrow X$ the inclusion of the fixed locus. The virtual fundamental class can be expressed as*

$$[X]^{\mathrm{vir}} = \sum_i \iota_* \frac{[X_i]^{\mathrm{vir}}}{e^{\mathbf{T}}(N_{X_i/X}^{\mathrm{vir}})},$$

[5] Strictly speaking, for this to be a virtual bundle one should at least ask for a resolution $[N_0 \to N_1]$ of $N_{X/X^{\mathbf{T}}}^{\mathrm{vir}}$, which could a priori not exist [**52**, Rem. 3.6], [**107**, Ass. 5.4]. For instance, without this assumption, $e^{\mathbf{T}}(N_{X/X^{\mathbf{T}}}^{\mathrm{vir}})$ would not be well-defined.

[6] See also Chang-Kiem-Li [**52**] for an independent proof using Kiem-Li cosection localization [**103**], where some global assumptions of [**85**] are weakened, and Kresch [**120**, Thm. 5.3.5].

where X_i are the connected components of the fixed locus $X^{\mathbf{T}}$. In particular, if X is a proper scheme, for any Chow class $\alpha \in A^*_{\mathbf{T}}(X)$ the integration formula holds

$$\int_{[X]^{\mathrm{vir}}} \alpha = \sum_i \int_{[X_i]^{\mathrm{vir}}} \frac{\alpha|_{X_i}}{e^{\mathbf{T}}(N^{\mathrm{vir}}_{X_i/X})}.$$

In the K-theoretic setting, the virtual localization formula has been proved[7] by Fantechi-Göttsche [**75**, Prop. 7.1] and Qu [**162**, Thm. 3.3].

Theorem 2.2.6 (K-theoretic virtual localization)**.** *Denote by $\iota : X^{\mathbf{T}} \hookrightarrow X$ the inclusion of the fixed locus. The virtual structure sheaf can be explicitly written as*

$$\mathcal{O}_X^{\mathrm{vir}} = \sum_i \iota_* \frac{\mathcal{O}^{\mathrm{vir}}_{X_i}}{\Lambda^\bullet(N^{\mathrm{vir},*}_{X_i/X})},$$

where X_i are the connected components of the fixed locus $X^{\mathbf{T}}$. In particular, if X is a proper scheme, for any K-theory class $V \in K_0^{\mathbf{T}}(X)$ the integration formula holds

$$\chi(X, V \otimes \mathcal{O}_X^{\mathrm{vir}}) = \sum_i \chi\left(X_i, \frac{V|_{X_i} \otimes \mathcal{O}^{\mathrm{vir}}_{X_i}}{\Lambda^\bullet(N^{\mathrm{vir},*}_{X_i/X})}\right).$$

[7]See also Kiem-Savvas [**107**] for an independent proof using K-theoretic cosection localization for almost perfect obstruction theories, where some global assumptions of [**75**] are weakened, and Kiem-Park [**106**] where the torus localization is proved for a general *virtual intersection theory.*

CHAPTER 3

Double nested Hilbert schemes and the local PT theory of curves

Ombre sui colori volano
sussulta in silenzio
la nostalgia di un ciliegio

Haiku, Andrea Pavlov

3.1. Introduction

3.1.1. Double nested Hilbert scheme of points Let X be a quasi-projective scheme over $\mathbb{C}$. We denote by $X^{[n]}$ the *Hilbert scheme of n points* on X, which parametrizes 0-dimensional closed subschemes $Z \subset X$ of length n. Given a tuple of non-decreasing integers $\mathbf{n} = (n_0 \leq \cdots \leq n_d)$, the *nested Hilbert scheme of points* $X^{[\mathbf{n}]}$ parametrizes flags of zero-dimensional subschemes $(Z_0 \subset \cdots \subset Z_d)$ of X, where each Z_i has length n_i. The scheme structure of these moduli spaces has been intensively studied in the literature, see for example [**54, 135**].
We propose a variation of this moduli space, by parametrizing flags of subschemes nesting in two directions. Let λ be a Young diagram and $\mathbf{n}_\lambda = (n_\square)_{\square \in \lambda}$ a reversed plane partition, that is a labelling of λ by non-negative integers non-decreasing in rows and columns. We denote by $X^{[\mathbf{n}_\lambda]}$ the *double nested Hilbert scheme of points*, the moduli space parametrizing flags of 0-dimensional closed subschemes $(Z_\square)_{\square \in \lambda} \subset X$

$$\begin{array}{ccccccccc}
Z_{00} & \subset & Z_{01} & \subset & Z_{02} & \subset & Z_{03} & \subset & \dots \\
\cap & & \cap & & \cap & & \cap & & \\
Z_{10} & \subset & Z_{11} & \subset & Z_{12} & \subset & Z_{13} & \subset & \dots \\
\cap & & \cap & & \cap & & \cap & & \\
Z_{20} & \subset & Z_{21} & \subset & Z_{22} & \subset & \dots & & \\
\cap & & \cap & & \cap & & & & \\
\dots & & \dots & & \dots & & & &
\end{array}$$

where each $Z_\square$ has length $n_\square$. If λ is a horizontal or vertical Young diagram, the nesting is linear and we recover the usual nested Hilbert scheme of points.
The scheme structure of these moduli spaces is interesting already in dimension one, for a smooth curve C. Cheah proved [**54**] that the nested Hilbert scheme $C^{[\mathbf{n}]}$ is smooth, being isomorphic to a product of symmetric powers of C via a Hilbert-Chow type morphism. However, as soon as we allow double nestings, $C^{[\mathbf{n}_\lambda]}$ can have several irreducible components (see Example 3.2.6), therefore failing to be smooth.
Our first result is a closed formula for the generating series of topological Euler characteristic of $C^{[\mathbf{n}_\lambda]}$ in terms of the hook-lengths $h(\square)$ of λ.

Theorem 3.1.1 (Theorem 3.2.10)**.** *Let C be a smooth quasi-projective curve and λ a Young diagram. Then*

$$\sum_{\mathbf{n}_\lambda} e(C^{[\mathbf{n}_\lambda]})q^{|\mathbf{n}_\lambda|} = \prod_{\square\in\lambda}(1-q^{h(\square)})^{-e(C)}.$$

This is achieved by exploiting the power structure on the Grothendieck ring of varieties $K_0(\mathrm{Var}_\mathbb{C})$, by which we reduce to the combinatorial problem of counting the number of reversed plane partitions of a given Young diagram, which was solved by Stanley and Hillman-Grassl [**172, 91**]. Motivic analogues of this formula are studied in [**134**].

3.1.2. Virtual fundamental class The double nested Hilbert scheme $C^{[\mathbf{n}_\lambda]}$ is in general singular, making it hard to perform intersection theory. To remedy this, we show that $C^{[\mathbf{n}_\lambda]}$ admits a *perfect obstruction theory* in the sense of Behrend-Fantechi and Li-Tian [**12, 123**]. In fact, we can (globally!) realize $C^{[\mathbf{n}_\lambda]}$ as the zero locus of a section of a vector bundle over a smooth ambient space.

Theorem 3.1.2 (Theorem 3.2.7)**.** *Let C be an irreducible smooth quasi-projective curve. There exists a section s of a vector bundle $\mathcal{E}$ over a smooth scheme $A_{C,\mathbf{n}_\lambda}$ such that*

$$\begin{tikzcd} & \mathcal{E} \arrow[d] \\ C^{[\mathbf{n}_\lambda]} \cong Z(s) \arrow[r, hook] & A_{C,\mathbf{n}_\lambda}. \arrow[u, bend right, "s"'] \end{tikzcd}$$

By this construction $C^{[\mathbf{n}_\lambda]}$ naturally admits a perfect obstruction theory (see Example 2.2.2) and in particular carries a virtual fundamental class $[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}$, which recovers the usual fundamental class in the case where the nesting is linear. We pause a moment to explain this construction in the easiest interesting example, that is for the reversed plane partition

n_{00}	n_{01}
n_{10}	n_{11}

The embedding in the smooth ambient space is

$$C^{[\mathbf{n}_\lambda]} \hookrightarrow A_{C,\mathbf{n}_\lambda} := C^{[n_{00}]}\times C^{[n_{10}-n_{00}]}\times C^{[n_{11}-n_{01}]}\times C^{[n_{01}-n_{00}]}\times C^{[n_{11}-n_{10}]},$$

$$(Z_{00},Z_{01},Z_{10},Z_{11}) \mapsto (Z_{00},Z_{10}-Z_{00},Z_{11}-Z_{01},Z_{01}-Z_{00},Z_{11}-Z_{10}).$$

In other words, $A_{C,\mathbf{n}_\lambda}$ records the subscheme in position $(0,0)$ and all possible vertical and horizontal differences of subschemes, where sum and difference are well-defined by seeing the closed subschemes Z_{ij} as divisors on C. At the level of closed points, the image of the embedding is given by all $(Z_{00}, X_1, X_2, Y_1, Y_2) \in A_{C,\mathbf{n}_\lambda}$ such that $X_1 + Y_2 = Y_1 + X_2$ — again, as divisors. Notice that $X_1 + Y_2$ and $Y_1 + X_2$ are effective divisors of the same degree, therefore they are equal if and only if one is contained into the other, say $X_1 + Y_2 \subset Y_1 + X_2$.

This relation is encoded into a section of a vector bundle $\mathcal{E}$, as we now explain. Denote by $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ the universal divisors on $A_{C,\mathbf{n}_\lambda} \times C$ and set

$$\Gamma^1 = \mathcal{Y}_1 + \mathcal{X}_2,$$
$$\Gamma^2 = \mathcal{X}_1 + \mathcal{Y}_2.$$

The vector bundle $\mathcal{E}$ is defined as

$$\mathcal{E} = \pi_* \mathcal{O}_{\Gamma^2}(\Gamma^1),$$

where $\pi : A_{C,\mathbf{n}_\lambda} \times C \to A_{C,\mathbf{n}_\lambda}$ is the projection. The section s of $\mathcal{E}$ is the one induced — via π_* — by the section of $\mathcal{O}_{A_{C,\mathbf{n}_\lambda} \times C}(\Gamma^1)$ which vanishes on Γ^1 and then restricted to Γ^2.

3.1.3. Stable pair invariants of local curves Let C be a smooth projective curve and L_1, L_2 two line bundles over C. We denote by *local curve* the total space $X = \mathrm{Tot}_C(L_1 \oplus L_2)$ with its natural $\mathbf{T} = (\mathbb{C}^*)^2$-action on the fibers.

For $d > 0$ and $n \in \mathbb{Z}$, we denote by $P_X = P_n(X, d[C])$ the moduli space of stable pairs $[\mathcal{O}_X \xrightarrow{s} F] \in \mathrm{D^b}(X)$ with curve class $d[C]$ and $\chi(F) = n$. The moduli space P_X has a perfect obstruction theory [**152**], but is in general non-proper. Still, the $\mathbf{T}$-action on X induces one on P_X with proper $\mathbf{T}$-fixed locus $P_X^{\mathbf{T}}$, therefore we can define invariants via Graber-Pandharipande virtual localization [**85**]

$$\mathsf{PT}_{d,n}(X) := \int_{[P_X^{\mathbf{T}}]^{\mathrm{vir}}} \frac{1}{e^{\mathbf{T}}(N^{\mathrm{vir}})} \in \mathbb{Q}(s_1, s_2),$$

where s_1, s_2 are the generators of the $\mathbf{T}$-equivariant cohomology and N^{vir} is the virtual normal bundle. We denote its generating series by

$$\mathsf{PT}_d(X; q) := \sum_{n \in \mathbb{Z}} q^n \cdot \mathsf{PT}_{d,n}(X) \in \mathbb{Q}(s_1, s_2)(\!(q)\!).$$

Pandharipande-Pixton extensively studied stable pair theory on local curves [**157, 156**] using degeneration techniques and relative invariants, focusing on the rationality of the generating series, including the case of descendent insertions. The novelty here is the different approach which only relies on the Graber-Pandharipande localization — without degenerating the curve C — and the virtual structure constructed on the double nested Hilbert schemes $C^{[\mathbf{n}_\lambda]}$. This is in particular useful to address the K-theoretic generalizations of stable pair invariants (cf. Section 3.1.8).

Our main result is that the generating series $\mathsf{PT}_d(X; q)$ of such invariants is controlled by some universal series and determine them under the anti-diagonal restriction $s_1 + s_2 = 0$.

Theorem 3.1.3 (Theorems 3.8.2, 3.8.1). *There are universal series $A_\lambda(q), B_\lambda(q), C_\lambda(q) \in \mathbb{Q}(s_1, s_2)[\![q]\!]$ such that*

$$\mathsf{PT}_d(X;q) = \sum_{\lambda \vdash d} \left(q^{-|\lambda|} A_\lambda(q)\right)^{g-1} \cdot \left(q^{-n(\lambda)} B_\lambda(q)\right)^{\deg L_1} \cdot \left(q^{-n(\overline{\lambda})} C_\lambda(q)\right)^{\deg L_2},$$

where $\overline{\lambda}$ is the conjugate partition of λ, $n(\lambda) = \sum_{i=0}^{l(\lambda)} i \cdot \lambda_i$ and $g = g(C)$. Moreover, under the anti-diagonal restriction $s_1 + s_2 = 0$

$$\begin{aligned} A_\lambda(q, s_1, -s_1) &= (-s_1^2)^{|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^2, \\ B_\lambda(-q, s_1, -s_1) &= (-1)^{n(\lambda)} \cdot s_1^{-|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^{-1} \cdot \prod_{\square \in \lambda} (1 - q^{h(\square)}), \\ C_\lambda(-q, s_1, -s_1) &= (-1)^{n(\overline{\lambda})} \cdot (-s_1)^{-|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^{-1} \cdot \prod_{\square \in \lambda} (1 - q^{h(\square)}). \end{aligned}$$

We sketch now the main steps required in proving Theorem 3.1.3.

3.1.4. Proof of the main theorem The connected components of the $\mathbf{T}$-fixed locus $P_n(X, d[C])^{\mathbf{T}}$ are double nested Hilbert schemes of points $C^{[\mathbf{n}_\lambda]}$, for suitable reversed plane partitions $\mathbf{n}_\lambda$ and Young diagram λ. In fact, pushing forward via $X \to C$ a $\mathbf{T}$-fixed stable pair $[\mathcal{O}_X \xrightarrow{s} F]$, corresponds a decomposition $\bigoplus_{(i,j)\in\mathbb{Z}^2} [\mathcal{O}_C \xrightarrow{s_{ij}} F_{ij}]$ on C, where every F_{ij} is a line bundle with section s_{ij}. These data produce divisors $Z_{ij} \subset C$ satisfying the nesting conditions dictated by λ, in other words an element of $C^{[\mathbf{n}_\lambda]}$.

On each connected component, there is an induced virtual fundamental class $[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}_{\mathsf{PT}}$, coming from the deformation of stable pairs. This virtual cycle coincides with the one constructed by the zero-locus construction of Theorem 3.1.2. By determining the class in K-theory of the virtual normal bundle, stable pair invariants on X are reduced to ($\mathbf{T}$-equivariant) virtual intersection numbers on $C^{[\mathbf{n}_\lambda]}$, namely

$$\int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) \in \mathbb{Q}(s_1, s_2). \tag{3.1.1}$$

The generating series of these invariants, for every fixed Young diagram λ, is controlled by three universal series (Theorem 3.5.1)

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) = A_\lambda^{g-1} \cdot B_\lambda^{\deg L_1} \cdot C_\lambda^{\deg L_2} \in \mathbb{Q}(s_1, s_2)[\![q]\!].$$

This universal structure is proven by following the strategy of [**72**]. In fact, these invariants are *multiplicative* on triples of the form $(C, L_1, L_2) = (C' \sqcup C'', L_1' \oplus L_1'', L_2' \oplus L_2'')$ and are polynomial in the Chern numbers of (C, L_1, L_2). The latter is obtained by pushing the virtual intersection number to $C^{[\mathbf{n}_\lambda]}$ on the smooth ambient space $A_{C,\mathbf{n}_\lambda}$ — a product of symmetric powers of C — and later to a product of Jacobians $\mathrm{Pic}^{n_i}(C)$, where the integrand is a polynomial on well-behaved cohomology classes.

By the universal structure any computation is reduced to a basis of the three-dimensional $\mathbb{Q}$-vector space of Chern numbers of triples (C, L_1, L_2). A simple basis consists of the Chern numbers of $(\mathbb{P}^1, \mathcal{O}, \mathcal{O})$ and any two $(\mathbb{P}^1, L_1, L_2)$ with $L_1 \otimes L_2 = K_{\mathbb{P}^1}$.

In both cases, the invariants are explictly determined under the anti-diagonal restriction $s_1 + s_2 = 0$ by further applying the virtual localization formula.

3.1.5. Toric computations The $\mathbb{C}^*$-action on $\mathbb{P}^1$ canonically lifts to the double nested Hilbert scheme $\mathbb{P}^{1[\mathbf{n}_\lambda]}$, with only finitely many $\mathbb{C}^*$-fixed points, therefore we can further $\mathbb{C}^*$-localize the invariants (3.1.1) to obtain

$$\int_{\left[\mathbb{P}^{1[\mathbf{n}_\lambda]}\right]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{\mathbb{P}^1,L_1,L_2}) = \left(\sum_{\underline{Z}\in \mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}} e^{\mathbf{T}\times\mathbb{C}^*}(-T^{\mathrm{vir}}_{\underline{Z}} - N^{\mathrm{vir}}_{\mathbb{P}^1,L_1,L_2,\underline{Z}})\right)\Bigg|_{s_3=0},$$

where s_3 is the generator of the $\mathbb{C}^*$-equivariant cohomology and $T^{\mathrm{vir}}_{\underline{Z}}$ is the virtual tangent bundle of $\mathbb{P}^{1[\mathbf{n}_\lambda]}$ at the fixed point $\underline{Z}$.

Under the anti-diagonal restriction $s_1 + s_2 = 0$, this translates the computation of the invariants into a purely combinatorial problem, which we explictly solve in the trivial vector bundle case $L_1 = L_2 = \mathcal{O}_{\mathbb{P}^1}$ and in the Calabi-Yau case $L_1 \otimes L_2 = K_{\mathbb{P}^1}$. A few remarks are in order. In the trivial vector bundle case, the solution is equivalent to the vanishing

$$\int_{[\mathbb{P}^{1[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{\mathbb{P}^1,\mathcal{O},\mathcal{O}})\Big|_{s_1+s_2=0} = 0,$$

for every reversed plane partition of positive size $|\mathbf{n}_\lambda| > 0$. This relies on the vanishing $e^{\mathbf{T}\times\mathbb{C}^*}(-T^{\mathrm{vir}}_{\underline{Z}} - N^{\mathrm{vir}}_{\mathbb{P}^1,L_1,L_2,\underline{Z}}) = 0$, which comes from a simple vanishing property of the topological vertex in stable pair theory proved in [**130**].

In the Calabi-Yau case, the invariants turn out to be *topological*, under the anti-diagonal restriction.

Theorem 3.1.4 (Theorem 3.7.3)**.** *Let X be Calabi-Yau. Then the generating series of the invariants* (3.1.1) *coincides, up to a sign, with the generating series of the topological Euler characteristic*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|}\cdot\left(\int_{\left[\mathbb{P}^{1[\mathbf{n}_\lambda]}\right]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{\mathbb{P}^1,L_1,L_2})\right)\Bigg|_{s_1+s_2=0} = (-1)^{\deg L_1(c_\lambda+|\lambda|)+|\lambda|}\cdot\sum_{\mathbf{n}_\lambda}(-q)^{|\mathbf{n}_\lambda|}e\left(\mathbb{P}^{1[\mathbf{n}_\lambda]}\right),$$

where $c_\lambda = \sum_{(i,j)\in\lambda}(j-i)$.

This happens as, under the anti-diagonal restriction, each $\mathbb{C}^*$-fixed point $\underline{Z}$ contributes with a sign

$$e^{\mathbf{T}\times\mathbb{C}^*}(-T^{\mathrm{vir}}_{\underline{Z}} - N^{\mathrm{vir}}_{\mathbb{P}^1,L_1,L_2,\underline{Z}})\Big|_{s_1+s_2=0} = (-1)^{\deg L_1(c_\lambda+|\lambda|)+|\lambda|+|\mathbf{n}_\lambda|}, \tag{3.1.2}$$

which is independent of $\underline{Z}$ and the invariants amount to a (signed) count of the $\mathbb{C}^*$-fixed points. It is not a priori clear how to obtain the the same sign through the vertex formalism for stable pairs developed by Pandharipande-Thomas [**155**].

Nevertheless, the topological nature of the invariants in the Calabi-Yau case is not surprising also for a non-toric curve C. If X is Calabi-Yau and $P_n(X, d[C])$ is proper — which happens only in rare cases — the anti-diagonal restriction would compute its virtual Euler characteristic and Behrend's weighted Euler characteristic, which is a purely topological invariant of a scheme with a symmetric perfect obstruction theory [**10**].

3.1.6. Gromov-Witten/stable pairs correspondence In the seminal work [**127**], a conjectural correspondence - known as the MNOP conjecture - between Gromov-Witten invariants and Donaldson-Thomas invariants of projective threefolds is formulated, proven for toric varieties in [**127, 128, 129**] for primary insertions. By defining the GW/DT invariants via equivariant residues, the conjecture has been extended to local curves in [**29**] and proven by combining the results of [**29, 148**].
Stable pair invariants were later introduced by Pandharipande-Thomas [**152**] to give a more natural geometric interpretation of the MNOP conjecture through the DT/PT correspondence proved by Toda and Bridgeland in [**180, 27**] using wall-crossing and Hall algebra techniques. The Gromov-Witten/stable pairs correspondence has been subsequently extended to include descendent insertions and to quasi-projective varieties whenever invariants can be defined through virtual localization. The correspondence had been confirmed by Pandharipande-Pixton for Calabi-Yau and Fano complete intersections in product of projective spaces and toric varieties [**159, 158**] and had been recently addressed in [**145**]. See [**151**] for a complete survey on the subject.

3.1.7. The local GW theory of curves For $X = \mathrm{Tot}_C(L_1 \oplus L_2)$ a local curve, let $\overline{M}^\bullet_h(X, d[C])$ denote the moduli space of stable maps (with possibly disconnected domain) of genus h and degree $d[C]$. Define the partition function of Gromov-Witten invariants of X (with a shifted exponent)

$$\mathsf{GW}_d(g|\deg L_1, \deg L_2; u) = u^{2-2g+\deg L_1+\deg L_2} \sum_{h\in\mathbb{Z}} u^{2h-2} \int_{[\overline{M}^\bullet_h(X,d[C])^{\mathbf{T}}]^{\mathrm{vir}}} \frac{1}{e^{\mathbf{T}}(N^{\mathrm{vir}})} \in \mathbb{Q}(s_1, s_2)(\!(u)\!),$$

where the dependence is only on the genus $g = g(C)$, the degrees of the line bundles and the degree d. The Gromov-Witten theory of local curves had been solved by Bryan-Pandharipande [**29**, Theorem 7.1] using a TQFT approach. Moreover they deduced an explicit closed formula for the partition function under the anti-diagonal restriction $s_1 + s_2 = 0$.

Theorem 3.1.5 (Bryan-Pandharipande)**.** *The partition function of Gromov-Witten invariants satisfies*

$$\begin{aligned}\mathsf{GW}_d(g|k_1, k_2; u)|_{s_1+s_2=0} = (-1)^{d(g-1-k_2)} s_1^{d(2g-2-k_1-k_2)} \cdot \\ \sum_{\lambda\vdash d} Q^{\frac{1}{2}c_\lambda(k_1-k_2)} \prod_{\square\in\lambda} h(\square)^{2g-2-k_1-k_2} \cdot i^{-k_1-k_2} \left(Q^{\frac{h(\square)}{2}} - Q^{-\frac{h(\square)}{2}}\right)^{k_1+k_2},\end{aligned}$$

where we set $Q = e^{iu}$ and $i = \sqrt{-1}$.

With this explicit expression it is immediate to check the Gromov-Witten/stable pairs correspondence under the anti-diagonal restriction.

Corollary 3.1.6 (Corollary 3.8.3)**.** *Let X be a local curve. Under the anti-diagonal restriction $s_1 + s_2 = 0$ the GW/stable pair correspondence holds*

$$(-i)^{d(2-2g+k_1+k_2)} \cdot \mathsf{GW}_d(g|k_1, k_2; u) = (-q)^{-\frac{1}{2}\cdot d(2-2g+k_1+k_2)} \mathsf{PT}_d(X, q),$$

after the change of variable $q = -e^{iu}$ and $k_i = \deg L_i$.

3.1.8. K-theoretic refinement K-theoretic refinement of Donaldson-Thomas theory and stable pair theory attracted much attention recently, both in Mathematics and String Theory: see for example [**178, 1, 3, 76**] for Calabi-Yau threefolds, [**140, 143, 36**] for Calabi-Yau fourfolds and [**142, 149, 109, 150**] for local curves.
A scheme X with a perfect obstruction theory is endowed not only with a virtual fundamental class, but also with a *virtual structure sheaf* $\mathcal{O}_X^{\mathrm{vir}} \in K_0(X)$. If X is proper, K-theoretic invariants are simply of the form

$$\chi(X, \mathcal{O}_X^{\mathrm{vir}} \otimes V) \in \mathbb{Z},$$

where $V \in K_0(X)$. If X is a local curve the moduli space of stable pairs P_X is in general not proper and K-theoretic stable pair invariants are defined by virtual localization [**75**] on the proper $\mathbf{T}$-fixed locus $P_X^{\mathbf{T}}$, that is one set

$$\chi(P_X, \mathcal{O}_{P_X}^{\mathrm{vir}} \otimes V) := \chi\left(P_X^{\mathbf{T}}, \frac{\mathcal{O}_{P_X^{\mathbf{T}}}^{\mathrm{vir}} \otimes V|_{P_X^{\mathbf{T}}}}{\Lambda^\bullet N^{\mathrm{vir},*}}\right) \in \mathbb{Q}(\mathfrak{t}_1, \mathfrak{t}_2).$$

In Section 3.9 we show that, also in the K-theoretic setting, the invariants are controlled by universal series.
The naive generalization of cohomological invariants is for $V = \mathcal{O}_X$, that is no insertions. However, we learn from Nekrasov-Okounkov [**142**] that it is more natural to consider the *twisted virtual structure sheaf*

$$\widehat{\mathcal{O}}^{\mathrm{vir}} = \mathcal{O}^{\mathrm{vir}} \otimes K_{\mathrm{vir}}^{1/2},$$

where $K_{\mathrm{vir}}^{1/2}$ is a square root[1] of the *virtual canonical bundle*. Denote by $\mathsf{PT}_d^{\widehat{K}}(X; q)$ the generating series of K-theoretic invariants with $V = K_{\mathrm{vir}}^{1/2}$.

Theorem 3.1.7 (Corollary 3.9.5)**.** *There exist universal series* $A_{\widehat{K},\lambda}(q), B_{\widehat{K},\lambda}(q), C_{\widehat{K},\lambda}(q) \in \mathbb{Q}(\mathfrak{t}_1^{1/2}, \mathfrak{t}_2^{1/2})[\![q]\!]$ *such that*

$$\mathsf{PT}_d^{\widehat{K}}(X; q) = \sum_{\lambda \vdash d} \left(q^{-|\lambda|} A_{\widehat{K},\lambda}(q)\right)^{g-1} \cdot \left(q^{-n(\lambda)} B_{\widehat{K},\lambda}(q)\right)^{\deg L_1} \cdot \left(q^{-n(\overline{\lambda})} C_{\widehat{K},\lambda}(q)\right)^{\deg L_2}.$$

Moreover, the universal series are explicitly computed under $\mathfrak{t}_1 \mathfrak{t}_2 = 1$.

We are not aware of a K-theoretic Gromov-Witten refinement for which a *refined* GW/stable pairs correspondence holds.

3.2. Double nested Hilbert schemes

3.2.1. Young diagrams By definition, a *partition* λ of $d \in \mathbb{Z}_{\geq 0}$ is a finite sequence of positive integers

$$\lambda = (\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \dots)$$

where

$$|\lambda| = \sum_i \lambda_i = d.$$

[1] This square root may not exist as a line bundle, but it does exists as a class in K-theory after inverting 2.

The number of parts of λ is called the *length* of λ and is denoted by $l(\lambda)$. A partition λ can be equivalently described by its associated *Young diagram*, which is the collection of d boxes in $\mathbb{Z}^2$ located at (i, j) where $0 \leq j < \lambda_i$.[2]
Given a partition λ, a *reversed plane partition* $\mathbf{n}_\lambda = (n_\square)_{\square\in\lambda} \in \mathbb{Z}_{\geq 0}$ is a collection of non-negative integers such that $n_\square \leq n_{\square'}$ for any $\square, \square' \in \lambda$ such that $\square \leq \square'$. In other words, a reversed plane partition is a Young diagram labelled with non-negative integers which are non-decreasing in rows and columns. The *size* of a reversed plane partition is

$$|\mathbf{n}_\lambda| = \sum_{\square\in\lambda} n_\square.$$

The *conjugate partition* $\overline{\lambda}$ is obtained by reflecting the Young diagram of λ about the

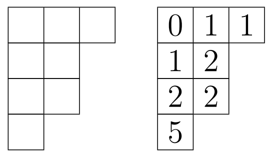

FIGURE 1. On the left, a Young diagram of size 8. On the right, a reversed plane partition of size 14.

$i = j$ line.

In the chapter we will require the following standard quantities. Given a box in the Young diagram λ, define the *content* $c(\square) = j - i$ and the *hooklength* $h(\square) = \lambda_i + \overline{\lambda}_j - i - j - 1$. The total content

$$c_\lambda = \sum_{\square\in\lambda} c(\square)$$

satisfies the following identities (cf [**125**, pag. 11]):

$$\sum_{\square\in\lambda} h(\square) = n(\lambda) + n(\overline{\lambda}) + |\lambda|, \quad c_\lambda = n(\overline{\lambda}) - n(\lambda), \tag{3.2.1}$$

where

$$n(\lambda) = \sum_{i=0}^{l(\lambda)} i \cdot \lambda_i.$$

For any Young diagram λ there is an associated graph, where any box of λ corresponds to a vertex and any face common to two boxes correspond to an edge connecting the corresponding vertices. A *square* of this graph is a circuit made of four different edges.

Lemma 3.2.1. *Let λ be a Young diagram and denote by V, E, Q respectively the number of vertices, edges and squares of the associated graph. Then*

$$V - E + Q - 1 = 0.$$

PROOF. We prove the claim by induction on the size of λ. If $|\lambda| = 1$, this is clear. Suppose it holds for all λ with $|\lambda| \leq n - 1$. Then we construct λ of size n by adding a box with lattice coordinates (i, j) to a Young diagram $\tilde{\lambda}$ of size $n - 1$. There are two

[2]This notation was borrowed by [**29**, Sec. 3.1]; however, in our conventions, (i, j) labels the box's corner closest to the origin.

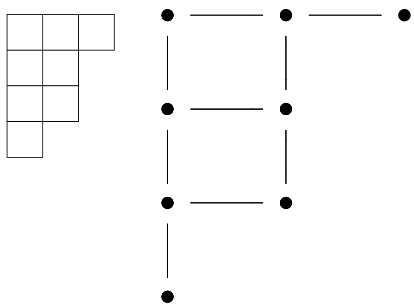

FIGURE 2. A Young diagram and its associated graph, with 8 vertices, 9 edges and 2 squares.

possibilities: either one of i, j is zero, so we added one vertex and one edge, or both i, j are non-zero, so we added one vertex, one square and two edges. In both cases the claim is proved. $\square$

3.2.2. Double nested Hilbert schemes Let X be a projective scheme and $\mathcal{O}(1)$ a fixed ample line bundle. The *Hilbert polynomial* of a closed subscheme $Y \subset X$ is defined by

$$m \mapsto \chi(\mathcal{O}_Y \otimes \mathcal{O}(m)).$$

Given a polynomial $p(m)$, the *Hilbert scheme* is the moduli space parametrizing closed subschemes $Y \subset X$ with Hilbert polynomial $p(m)$, which is representable by a projective scheme (e.g. by [87]). We consider here a more general situation, where we replace closed subschemes by flags of closed subschemes, satisfying certain nesting conditions dictated by Young diagrams.

Let λ be a Young diagram and $\mathbf{p}_\lambda = (p_\square)_{\square \in \lambda} \in \mathbb{Z}[x]$ be a collection of polynomials indexed by λ. If all $p_\square$ are non-negative integers which are non-decreasing in rows and columns, $\mathbf{p}_\lambda = \mathbf{n}_\lambda$ is a reversed plane partition.

Definition 3.2.2. Let X be a projective scheme and $\mathbf{p}_\lambda$ as above. The *double nested Hilbert functor* of X of type $\mathbf{p}_\lambda$ is the moduli functor

$$\underline{\mathrm{Hilb}}^{\mathbf{p}_\lambda}(X) : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Sets},$$

$$\mathrm{T} \mapsto \left\{ (\mathcal{Z}_\square)_{\square \in \lambda} \subset X \times T \;\middle|\; \begin{array}{l} \mathcal{Z}_\square \text{ a } T\text{-flat closed subscheme with Hilbert polynomial } p_\square, \\ \text{such that } \mathcal{Z}_\square \subset \mathcal{Z}_{\square'} \text{ for } \square \leq \square'. \end{array} \right\}.$$

Remark 3.2.3. If $|\lambda| = 1$ we recover the classical Hilbert scheme, while if λ is a horizontal (or vertical) Young diagram we recover the *nested Hilbert scheme*, already widely studied in the literature.

Proposition 3.2.4. *Let X be a projective scheme and $\mathbf{p}_\lambda$ as above. Then $\underline{\mathrm{Hilb}}^{\mathbf{p}_\lambda}(X)$ is representable by a projective scheme* $\mathrm{Hilb}^{\mathbf{p}_\lambda}(X)$*, which we call the* double nested Hilbert scheme.

PROOF. We prove our claim in the case $\mathbf{p}_\lambda$ is

$$\begin{array}{|c|c|}\hline p_{00} & p_{01} \\ \hline p_{10} & p_{11} \\ \hline \end{array}$$

as the general case will follow by an analogous reasoning. There are forgetful maps between nested Hilbert functors

$$\underline{\mathrm{Hilb}}^{[p_{00},p_{01},p_{11}]}(X) \to \underline{\mathrm{Hilb}}^{[p_{00},p_{11}]}(X)$$
$$\underline{\mathrm{Hilb}}^{[p_{00},p_{10},p_{11}]}(X) \to \underline{\mathrm{Hilb}}^{[p_{00},p_{11}]}(X),$$

which forget the second subscheme of the corresponding flag. Consider their fiber product

$$\begin{tikzcd}
\underline{\mathrm{Hilb}}^{[p_{00},p_{01},p_{11}]}(X) \times_{\underline{\mathrm{Hilb}}^{[p_{00},p_{11}]}(X)} \underline{\mathrm{Hilb}}^{[p_{00},p_{10},p_{11}]}(X) \arrow[r] \arrow[d] & \underline{\mathrm{Hilb}}^{[p_{00},p_{01},p_{11}]}(X) \arrow[d] \\
\underline{\mathrm{Hilb}}^{[p_{00},p_{01},p_{11}]}(X) \arrow[r] & \underline{\mathrm{Hilb}}^{[p_{00},p_{11}]}(X).
\end{tikzcd}$$

There is an obvious morphism of functors

$$\underline{\mathrm{Hilb}}^{\mathbf{p}_\lambda}(X) \to \underline{\mathrm{Hilb}}^{[p_{00},p_{01},p_{11}]}(X) \times_{\underline{\mathrm{Hilb}}^{[p_{00},p_{11}]}(X)} \underline{\mathrm{Hilb}}^{[p_{00},p_{10},p_{11}]}(X),$$

which is easily checked to be an isomorphism by comparing each flat family of flags over every scheme T. We conclude by the fact that the nested Hilbert functors (and their fiber products) are representable by a projective scheme by [**170**, Thm. 4.5.1]. $\square$

Thanks to representability, double nested Hilbert schemes are equipped with universal subschemes, for any $\square \in \lambda$,

$$\mathcal{Z}_\square \subset X \times \mathrm{Hilb}^{\mathbf{p}_\lambda}(X),$$

such that the fiber over a point $\underline{Z} = (Z_\square)_{\square\in\lambda} \in \mathrm{Hilb}^{\mathbf{p}_\lambda}(X)$ is

$$\mathcal{Z}_\square|_{\underline{Z}} = Z_\square \subset X.$$

Remark 3.2.5. If $\mathbf{p}_\lambda = \mathbf{n}_\lambda$, Definition 3.2.2 generalizes to X quasi-projective. In fact, let $X \subset \overline{X}$ be any compactification of X. We define the *double nested Hilbert scheme points* as the open subscheme

$$X^{[\mathbf{n}_\lambda]} := \mathrm{Hilb}^{\mathbf{n}_\lambda}(X) \subset \mathrm{Hilb}^{\mathbf{n}_\lambda}(\overline{X})$$

consisting of the 0-dimensional subschemes supported on $X \subset \overline{X}$.

Double nested Hilbert schemes of points are rarely smooth varieties. Some smooth examples consist of

- $|\lambda| = 1$, X a smooth quasi-projective curve or surface (see e.g. [**137**]),
- λ a vertical/horizontal Young diagram, X a smooth quasi-projective curve (see e.g. [**54**]).

In general, $X^{[\mathbf{n}_\lambda]}$ is singular even for X a smooth quasi-projective curve.

0	1
1	2

Example 3.2.6. Let C be a smooth curve and consider the reversed plane partition $\mathbf{n}_\lambda$

There are two types of flags of divisors, of the form

$$\begin{array}{ccc} \varnothing & \subset & P \\ \cap & & \cap \\ Q & \subset & P+Q, \end{array} \qquad \begin{array}{ccc} \varnothing & \subset & P \\ \cap & & \cap \\ P & \subset & P+Q, \end{array}$$

where $P, Q \in C$. Therefore its reduced scheme structure consists of two irreducible components $C \times C \cup C \times C$, intersecting at the diagonals of $C \times C$.

Singularities make it hard to perform intersection theory on $X^{[\mathbf{n}_\lambda]}$. To remedy this we construct, in special cases, *virtual fundamental classes* in $A_*(X^{[\mathbf{n}_\lambda]})$, using the machinery described in Section 2.2.

3.2.3. Points on Curves Let C be an irreducible smooth quasi-projective curve and $\mathbf{n}_\lambda$ a reversed plane partition. In this section we show that $C^{[\mathbf{n}_\lambda]}$ is the zero locus of a section of a vector bundle over a smooth ambient space, and therefore admits a perfect obstruction theory as in Example 2.2.2.
We define

$$A_{C,\mathbf{n}_\lambda} = C^{[n_{00}]} \times \prod_{\substack{(i,j)\in\lambda \\ i\geq 1}} C^{[n_{ij}-n_{i-1,j}]} \times \prod_{\substack{(l,k)\in\lambda \\ k\geq 1}} C^{[n_{lk}-n_{l,k-1}]}.$$

As $C^{[n]} \cong C^{(n)}$ is a symmetric product via the Hilbert-Chow morphism, $A_{C,\mathbf{n}_\lambda}$ is a smooth quasi-projective variety of dimension

$$\dim(A_{C,\mathbf{n}_\lambda}) = n_{00} + \sum_{\substack{(i,j)\in\lambda \\ i\geq 1}} (n_{ij} - n_{i-1,j}) + \sum_{\substack{(l,k)\in\lambda \\ k\geq 1}} (n_{lk} - n_{l,k-1}).$$

To ease the notation, we denote its elements by $\underline{Z} = ((Z_{00}, X_{ij}, Y_{lk}))_{ij,lk} \in A_{C,\mathbf{n}_\lambda}$, where $Z_{00} \subset C$ is a divisor of length n_{00} and $X_{ij} \subset C$ (resp. $Y_{lk} \subset C$) is a divisor of length $n_{ij} - n_{i-1,j}$ (resp. $n_{lk} - n_{l,k-1}$).
$A_{C,\mathbf{n}_\lambda}$ comes equipped with universal divisors, which we denote by

$$\mathcal{Z}_{00}, \mathcal{X}_{ij}, \mathcal{Y}_{lk} \subset C \times A_{C,\mathbf{n}_\lambda},$$

with fibers are

$$\begin{aligned} \mathcal{Z}_{00}|_{\underline{Z}} &= Z_{00}, \\ \mathcal{X}_{ij}|_{\underline{Z}} &= X_{ij}, \quad i \geq 1, \\ \mathcal{Y}_{lk}|_{\underline{Z}} &= Y_{lk}, \quad k \geq 1. \end{aligned}$$

For every $(i,j) \in \lambda$ with $i, j \geq 1$ define the universal effective divisors

$$\begin{aligned} \Gamma^1_{ij} &= \mathcal{X}_{i,j} + \mathcal{Y}_{i-1,j}, \\ \Gamma^2_{ij} &= \mathcal{Y}_{i,j} + \mathcal{X}_{i,j-1}. \end{aligned}$$

Theorem 3.2.7. *Let C be an irreducible smooth quasi-projective curve, $\pi : C \times A_{C,\mathbf{n}_\lambda} \to A_{C,\mathbf{n}_\lambda}$ be the natural projection and define the vector bundle*

$$\mathcal{E} = \bigoplus_{\substack{(i,j)\in\lambda \\ i,j\geq 1}} \pi_* \mathcal{O}_{\Gamma^2_{ij}}(\Gamma^1_{ij}).$$

Then there exists a section s of $\mathcal{E}$ whose zero set is isomorphic to $C^{[\mathbf{n}_\lambda]}$

$$\begin{tikzcd} & \mathcal{E} \arrow[d] \\ C^{[\mathbf{n}_\lambda]} \cong Z(s) \arrow[r, hook] & A_{C,\mathbf{n}_\lambda}. \arrow[u, bend right, "s"'] \end{tikzcd}$$

Proof. Notice that $\mathcal{E}$ is a vector bundle, as by cohomology and base change all higher direct images vanish

$$\mathbf{R}^k\pi_* \mathcal{O}_{\Gamma^2_{ij}}(\Gamma^1_{ij}) = 0, \quad k > 0.$$

There is a closed immersion

$$C^{[\mathbf{n}_\lambda]} \hookrightarrow A_{C,\mathbf{n}_\lambda},$$

given on closed points $(Z_{ij})_{(i,j)\in\lambda} \in C^{[\mathbf{n}_\lambda]}$ as

$$(Z_{ij})_{(i,j)\in\lambda} \mapsto (Z_{00}, (Z_{ij} - Z_{i-1,j}), (Z_{ij} - Z_{i,j-1}))_{(i,j\in\lambda)} \in A_{C,\mathbf{n}_\lambda}.$$

Define sections $\tilde{s}_{ij} \in H^0(C \times A_{C,\mathbf{n}_\lambda}, \mathcal{O}_{\Gamma^2_{ij}}(\Gamma^1_{ij}))$ as the composition

$$\tilde{s}_{ij} : \mathcal{O}_{C\times A_{C,\mathbf{n}_\lambda}} \xrightarrow{s'_{ij}} \mathcal{O}_{C\times A_{C,\mathbf{n}_\lambda}}(\Gamma^1_{ij}) \to \mathcal{O}_{\Gamma^2_{ij}}(\Gamma^1_{ij}),$$

where s'_{ij} is the section vanishing on Γ^1_{ij} while the second morphism is the restriction along $j : \Gamma^2_{ij} \to A_{C,\mathbf{n}_\lambda} \times C$; in other words, $\tilde{s}_{ij} = j_*j^*s'_{ij}$. The sections $\tilde{s}_{ij}$ induce sections $s_{ij} = \pi_*\tilde{s}_{ij}$ of $\pi_* \mathcal{O}_{\Gamma^2_{ij}}(\Gamma^1_{ij})$ and we set $s = (s_{ij})_{ij} \in H^0(A_{C,\mathbf{n}_\lambda}, \mathcal{E})$. We claim that

$$C^{[\mathbf{n}_\lambda]} \cong Z(s).$$

To prove it, we follow the strategy of [**33**, Prop. 2.4]. For $(i,j) \in \lambda$ with $i,j \geq 1$, consider the universal divisors

$$\begin{tikzcd} & \Gamma^2_{ij} \arrow[dl] \arrow[r, hook, "j"] \arrow[dr, "\tilde{\pi}"] & A_{C,\mathbf{n}_\lambda} \times C \arrow[d, "\pi"] \\ C & & A_{C,\mathbf{n}_\lambda}. \end{tikzcd}$$

Let $\underline{Z}_T = (Z^T_{00}, X^T_{ij}, Y^T_{lk})_{ij,lk}$ be any T-flat family with corresponding classifying morphism $f : T \to A_{C,\mathbf{n}_\lambda}$, where $Z^T_{00}, X^T_{ij}, Y^T_{lk} \subset T \times C$ have zero-dimensional fibers of appropriate length. Consider the commutative diagram

$$\begin{tikzcd} & \Gamma^{2,T}_{ij} \arrow[dl] \arrow[r, hook, "j_T"] \arrow[dr, "\tilde{\pi}_T"] & T \times C \arrow[d, "\pi_T"] \\ C & & T, \end{tikzcd}$$

where $\Gamma^{1,T}_{ij}, \Gamma^{2,T}_{ij}$ are the pullbacks of the universal divisors $\Gamma^1_{ij}, \Gamma^2_{ij}$ along $f \times \mathrm{id}_C$. To prove the claim it suffices to show that $\underline{Z}_T$ is a T-point of $C^{[\mathbf{n}_\lambda]}$ if and only if f factors

through $Z(s)$.
Now $\underline{Z}_T$ is a T-point of $C^{[\mathbf{n}_\lambda]}$ if and only if $\Gamma^{2,T}_{ij} = \Gamma^{1,T}_{ij}$ for all $(i,j) \in \lambda$ such that $(i,j) \geq 1$. Notice that the inclusion $\Gamma^{2,T}_{ij} \subset \Gamma^{1,T}_{ij}$ is enough to have the equality, as all fibers are divisors in C of the same degree.
On the other hand, we have that f factors through $Z(s)$ if and only if f^*s is the zero section of $f^*\mathcal{E}$, i.e. if $f^*s_{ij} = 0$ for all $(i,j) \in \lambda$ such that $(i,j) \geq 1$. Repeatedly applying flat base change, we obtain

$$f^*s_{ij} = \tilde{\pi}_* j_T^* (f \times \mathrm{id}_C)^* s'_{ij}.$$

Therefore f^*s_{ij} is the zero section if and only if $\Gamma^{2,T}_{ij} \subset \Gamma^{1,T}_{ij}$ as required. □

Thanks to Theorem 3.2.7, $C^{[\mathbf{n}_\lambda]}$ falls in the situation of Example 2.2.2 and we obtain a virtual fundamental class.

Corollary 3.2.8. *Let C be a smooth quasi-projective curve and $\mathbf{n}_\lambda$ a reversed plane partition. Then $C^{[\mathbf{n}_\lambda]}$ has a perfect obstruction theory*

$$[\mathcal{E}^*|_{C^{[\mathbf{n}_\lambda]}} \to \Omega^1_{A_{C,\mathbf{n}_\lambda}}|_{C^{[\mathbf{n}_\lambda]}}] \to \mathbb{L}_{C^{[\mathbf{n}_\lambda]}}. \tag{3.2.2}$$

In particular there exists a virtual fundamental class

$$[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}} \in A_*(C^{[\mathbf{n}_\lambda]}).$$

3.2.4. Topological Euler characteristic Recall that we can view Euler characteristic weighted by a constructible function as a Lebesgue integral, where the measurable sets are constructible sets, measurable functions are constructible functions and the measure of a set is given by its Euler characteristic (cf. [**28**, Sec. 2]). In this language we have

$$e(X) = \int_X 1 \cdot de,$$

for any constructible set X. The following lemma is reminiscent of the existence of a power structure on the Grothendieck ring of varieties.

Lemma 3.2.9 ([**28**, Lemma 32])**.** *Let B be a scheme of finite type over $\mathbb{C}$ and $e(B)$ its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ be any function with $g(0) = 1$. Let $G : \mathrm{Sym}^n B \to \mathbb{Z}(\!(p)\!)$ be the constructible function defined by*

$$G(\mathbf{ax}) = \prod_i g(a_i),$$

for all $\mathbf{ax} = \sum_i a_i x_i \in \mathrm{Sym}^n B$, where $x_i \in B$ are distinct closed points. Then

$$\sum_{n=0}^{\infty} q^n \int_{\mathrm{Sym}^n B} G \cdot de = \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(B)}.$$

Using Lemma 3.2.9 we compute the topological Euler characteristic of double nested Hilbert schemes of points of any quasi-projective smooth curve.

Theorem 3.2.10. *Let C be a smooth quasi-projective curve and λ a Young diagram. Then*

$$\sum_{\mathbf{n}_\lambda} e(C^{[\mathbf{n}_\lambda]})q^{|\mathbf{n}_\lambda|} = \prod_{\square\in\lambda}(1-q^{h(\square)})^{-e(C)}.$$

Proof. Consider the constructible map

$$\rho_n : \coprod_{|\mathbf{n}_\lambda|=n} C^{[\mathbf{n}_\lambda]} \to \operatorname{Sym}^n C,$$

defined, for $\underline{Z} = (Z_\square)_{\square\in\lambda} \in C^{[\mathbf{n}_\lambda]}$, by

$$\rho_n(\underline{Z}) = \sum_{\square\in\lambda} Z_\square \in \operatorname{Sym}^n C.$$

In other words, ρ just forgets the distribution and the nesting of the divisor $\sum_{\square\in\lambda} Z_\square$ among all $\square \in \lambda$.
Let $\mathbf{ax} = \sum_i a_i x_i \in \operatorname{Sym}^n C$, with x_i different to each other. The fiber $\rho_n^{-1}(\mathbf{ax})$ is clearly 0-dimensional and satisfies

$$\rho_n^{-1}(\mathbf{ax}) \cong \prod_i \rho_{a_i}^{-1}(a_i x_i). \tag{3.2.3}$$

In particular the Euler characteristic of the fiber $\rho_n(nx)$ does not depend on the point $x \in C$ and counts the number of reversed plane partition of size n and underlying Young diagram λ

$$e(\rho_n^{-1}(nx)) = \sum_{|\mathbf{n}_\lambda|=n} 1. \tag{3.2.4}$$

Consider now

$$\int_{\coprod_{|\mathbf{n}_\lambda|=n} C^{[\mathbf{n}_\lambda]}} 1\cdot de = \int_{\operatorname{Sym}^n C} \rho_{n*}1\cdot de,$$

where for any $\mathbf{ax} \in \operatorname{Sym}^n C$ with x_i different to each other, using (3.2.3) and (3.2.4)

$$\begin{aligned}\rho_{n*}1(\mathbf{ax}) &= e(\rho_n^{-1}(\mathbf{ax}))\\ &= \prod_i \sum_{|\mathbf{n}_\lambda|=a_i} 1.\end{aligned}$$

Now, $g(a) = \sum_{|\mathbf{n}_\lambda|=a} 1$ and $G(\mathbf{ax}) = \rho_{n*}1(\mathbf{ax})$ satisfy the hypotheses of Lemma 3.2.9 and therefore

$$\begin{aligned}\sum_{n=0}^\infty \sum_{|\mathbf{n}_\lambda|=n} e(C^{[\mathbf{n}_\lambda]})q^n &= \sum_{n=0}^\infty q^n \int_{\coprod_{|\mathbf{n}_\lambda|=n} C^{[\mathbf{n}_\lambda]}} 1\cdot de\\ &= \sum_{n=0}^\infty q^n \int_{\operatorname{Sym}^n C} \rho_{n*}1\cdot de\\ &= \left(\sum_{n=0}^\infty \sum_{|\mathbf{n}_\lambda|=n} q^{|\mathbf{n}_\lambda|}\right)^{e(C)}.\end{aligned}$$

A closed formula for the generating series of reversed plane partitions was given by Stanley [172, Prop. 18.3] and by Hillman-Grassl [91, Thm. 1] using hook-lengths

$$\sum_{n=0}^{\infty}\sum_{|\mathbf{n}_\lambda|=n} q^{|\mathbf{n}_\lambda|} = \prod_{\square\in\lambda}(1-q^{h(\square)})^{-1},$$

by which we conclude the proof. $\square$

3.2.5. Double nesting of divisors We conclude this section with a generalization of the zero-locus construction of Theorem 3.2.7.

Let X be a smooth projective variety of dimension d and $\boldsymbol{\beta}_\lambda = (\beta_\square)_{\square\in\lambda}$ be a collection of homology classes $\beta_\square \in H_{2d-2}(X,\mathbb{Z})$. Denote by $H_{\boldsymbol{\beta}_\lambda}$ the double nested Hilbert scheme of effective divisors on X, which parametrizes flags of divisors $(Z_\square)_{\square\in\lambda} \subset X$ satisfying the nesting condition dictated by $\boldsymbol{\beta}_\lambda$. Denote by

$$A_{X,\boldsymbol{\beta}_\lambda} = H_{\beta_{00}} \times \prod_{\substack{(i,j)\in\lambda\\ i\geq 1}} H_{\beta_{ij}-\beta_{i-1,j}} \times \prod_{\substack{(l,k)\in\lambda\\ k\geq 1}} H_{\beta_{lk}-\beta_{l,k-1}},$$

where H_β is the usual Hilbert scheme of divisors on X of class β. Analogously to Section 3.2.3, $A_{X,\boldsymbol{\beta}_\lambda}$ comes equipped with universal (Cartier) divisors $\mathcal{Z}_{00}, \mathcal{X}_{ij}, \mathcal{Y}_{lk} \subset X \times A_{X,\boldsymbol{\beta}_\lambda}$ and for every $(i,j)\in\lambda$ with $i,j\geq 1$ we define the universal effective divisors

$$\begin{aligned}\Gamma^1_{ij} &= \mathcal{X}_{i,j} + \mathcal{Y}_{i-1,j},\\ \Gamma^2_{ij} &= \mathcal{Y}_{i,j} + \mathcal{X}_{i,j-1}.\end{aligned}$$

Define the coherent sheaf

$$\mathcal{E} = \bigoplus_{\substack{(i,j)\in\lambda\\ i,j\geq 1}} \pi_*\mathcal{O}_{\Gamma^2_{ij}}(\Gamma^1_{ij}),$$

where $\pi : X\times A_{X,\boldsymbol{\beta}_\lambda} \to A_{X,\boldsymbol{\beta}_\lambda}$ is the natural projection. Under some extra assumptions on X and $\boldsymbol{\beta}_\lambda$ Theorem 3.2.7 generalizes.

Proposition 3.2.11. *Assume that $A_{X,\boldsymbol{\beta}_\lambda}$ is smooth and $\mathcal{E}$ is a vector bundle. Then there exists a section s of $\mathcal{E}$ such that*

$$\begin{tikzcd} & \mathcal{E} \arrow[d] \\ H_{\boldsymbol{\beta}_\lambda} \cong Z(s) \arrow[r, hook] & A_{X,\boldsymbol{\beta}_\lambda}. \arrow[u, bend right, "s"'] \end{tikzcd}$$

In particular, $H_{\boldsymbol{\beta}_\lambda}$ has a perfect obstruction theory.

Corollary 3.2.12. *Let $X = \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_s}$. Then there exists a virtual fundamental class $[H_{\boldsymbol{\beta}_\lambda}]^{\mathrm{vir}} \in A_*(H_{\boldsymbol{\beta}_\lambda})$.*

PROOF. The smoothness of $A_{X,\boldsymbol{\beta}_\lambda}$ follows by the smoothness of the space $H_\beta \cong \mathbb{P}(H^0(X,\mathcal{O}_X(\beta)))$ for every $\beta\in H_{n-2}(X,\mathbb{Z})$. Let D_1, D_2 be two effective divisors on

X such that $[D_1] = [D_2] \in H_{n-2}(X, \mathbb{Z})$; in particular, D_1, D_2 are linearly equivalent. Combining the long exact sequence in cohomology of the short exact sequence

$$0 \to \mathcal{O}(D_1 - D_2) \to \mathcal{O}(D_1) \to O_{D_2}(D_1) \to 0$$

and the vanishings

$$H^k(X, \mathcal{O}(D_1)) = 0, \quad k = 1, \dots, \dim X - 1,$$
$$H^k(X, \mathcal{O}(D_1 - D_2)) = 0, \quad k = 2, \dots \dim X,$$

yields that $H^k(X, \mathcal{O}_{D_2}(D_1)) = 0$ for $k \geq 1$; cohomology and base implies that $\mathbf{R}^k\pi_* \mathcal{O}_{\Gamma^2_{ij}}(\Gamma^1_{ij}) = 0$ for $k \geq 1$. Finally, We have that, if $\dim X \geq 2$,

$$\dim H^0(X, \mathcal{O}_{D_2}(D_1)) = \dim H^0(X, \mathcal{O}_X(D_1)) - 1,$$

which depends only on the degree $[D_1] = [D_2] \in H_2(X, \mathbb{Z})$ and implies that the dimension of the fibers of $\mathcal{E}$ is constant, thus $\mathcal{E}$ is a vector bundle. □

3.3. Moduli space of stable pairs

3.3.1. Moduli space of stable pairs Moduli spaces of stable pairs were introduced by Pandharipande-Thomas [**152**] in order to give a geometric interpretation of the MNOP conjectures [**127**], through the DT/PT correspondence proved by Toda (for Euler characteristic) and Bridgeland in [**180, 27**] using wall-crossing and Hall algebra techniques.

For a smooth quasi-projective threefold X, a curve class $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we define $P_n(X, \beta)$ to be the moduli space of pairs

$$I^\bullet = [\mathcal{O}_X \xrightarrow{s} F] \in \mathbf{D}^{\mathrm{b}}(X)$$

in the derived category of X where F is a pure 1-dimensional sheaf with proper support $[\mathrm{supp}(F)] = \beta$ with $\chi(F) = n$ and s is a section with 0-dimensional cokernel.

By the work of Huybrechts-Thomas [**95**], the Atiyah class gives a *perfect obstruction theory* on $P_n(X, \beta)$

$$\mathbb{E} = \mathbf{R}\mathscr{H}om_\pi(\mathbb{I}, \mathbb{I})_0^\vee[-1] \to \mathbb{L}_{P_n(X,\beta)}, \tag{3.3.1}$$

where $(\cdot)_0$ denotes the trace-free part, $\pi : X \times P_n(X, \beta) \to P_n(X, \beta)$ is the canonical projection and $\mathbb{I}^\bullet = [\mathcal{O} \to \mathbb{F}]$ is the universal stable pair on $X \times P_n(X, \beta)$.

If X is projective, the perfect obstruction theory induces a virtual fundamental class $[P_n(X, \beta)]^{\mathrm{vir}} \in A_*(P_n(X, \beta))$ and one defines *stable pair* (or *PT*) *invariants* by integrating cohomology classes $\gamma \in H^*(P_n(X, \beta), \mathbb{Z})$ against the virtual fundamental class

$$\mathsf{PT}_{\beta,n}(X, \gamma) = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} \gamma \in \mathbb{Z}. \tag{3.3.2}$$

We focus here in the case of X a *local curve*, i.e. $X = \mathrm{Tot}_C(L_1 \oplus L_2)$ the total space of the direct sum of two line bundles L_1, L_2 on a smooth projective curve C and $\beta = d[C] \in H_2(X, \mathbb{Z}) \cong H_2(C, \mathbb{Z})$ a multiple of the zero section of $X \to C$.

X is a smooth *quasi-projective* threefold, therefore the moduli space of stable pairs $P_n(X, \beta)$ is hardly ever a proper scheme and one cannot define invariants as in (3.3.2).

Nevertheless, the algebraic torus $\mathbf{T} = (\mathbb{C}^*)^2$ acts on X by scaling the fibers and the action naturally lifts to $P_n(X, d[C])$, making the perfect obstruction theory naturally $\mathbf{T}$-equivariant by [**166**, Example 4.6]. Moreover, the $\mathbf{T}$-fixed locus $P_n(X, d[C])^{\mathbf{T}}$ is proper (cf. Prop. 3.3.1), therefore by Graber-Pandharipande [**85**] there is naturally an induced perfect obstruction theory on $P_n(X, d[C])^{\mathbf{T}}$ and a virtual fundamental class $[P_n(X, d[C])^{\mathbf{T}}]^{\mathrm{vir}} \in A_*(P_n(X, d[C])^{\mathbf{T}})$. We define $\mathbf{T}$-*equivariant stable pair invariants* by Graber-Pandharipande virtual localization formula

$$\mathsf{PT}_{d,n}(X) = \int_{[P_n(X,d[C])^{\mathbf{T}}]^{\mathrm{vir}}} \frac{1}{e^{\mathbf{T}}(N^{\mathrm{vir}})} \in \mathbb{Q}(s_1, s_2), \tag{3.3.3}$$

where s_1, s_2 are the generators of $\mathbf{T}$-equivariant cohomology and the virtual normal bundle

$$N^{\mathrm{vir}} = (\mathbb{E}|^{\vee}_{P_n(X,d[C])^{\mathbf{T}}})^{\mathsf{mov}} \in K^0_{\mathbf{T}}(P_n(X, d[C])^{\mathbf{T}}) \tag{3.3.4}$$

is the $\mathbf{T}$-moving part of the restriction of the dual of the perfect obstruction theory. Stable pair invariants with descendent insertions on local curves have been studied in [**156, 157, 144**].

3.3.2. The fixed locus In this section we prove that the $\mathbf{T}$-fixed locus $P_n(X, d[C])^{\mathbf{T}}$ is a disjoint union of double nested Hilbert schemes of points $C^{[\mathbf{n}_\lambda]}$, for suitable reversed plane partitions $\mathbf{n}_\lambda$, where λ are Young diagram of size $|\lambda| = d$. Our strategy is similar to Kool-Thomas [**119**, Sec. 4] for local surfaces.

Given a $\mathbf{T}$-equivariant coherent sheaf on X, its pushdown along $p : X \to C$ decomposes into weight spaces (e.g. by [**90**, Ex. II.5.17, II.5.18])

$$p_*F = \bigoplus_{(i,j)\in\mathbb{Z}^2} F_{ij} \otimes \mathfrak{t}_1^i\, \mathfrak{t}_2^j,$$

where F_{ij} is a coherent sheaf on C and $\mathfrak{t}_1, \mathfrak{t}_2$ are the generators of $K^0_{\mathbf{T}}(\mathsf{pt})$. For example

$$p_*\mathcal{O}_X = \bigoplus_{i,j\geq 0} L_1^{-i} \otimes L_2^{-j} \otimes \mathfrak{t}_1^{-i}\, \mathfrak{t}_2^{-j}\,.$$

Since p is affine, the pushdown does not lose any information, and we recover the $\mathcal{O}_X$-module structure of F by the $p_*\mathcal{O}_X$-action that p_*F carries. This is generated by the action of the -1 pieces $L_1^{-1} \otimes \mathfrak{t}_1^{-1}, L_2^{-1} \otimes \mathfrak{t}_2^{-1}$, so we find that the $\mathcal{O}_X$-module structure is determined by the maps

$$\begin{aligned}
\left(\bigoplus_{i,j} F_{ij} \otimes \mathfrak{t}_1^i\, \mathfrak{t}_2^j\right) \otimes L_1^{-1} \otimes \mathfrak{t}_1^{-1} &\to \bigoplus_{i,j} F_{ij} \otimes \mathfrak{t}_1^i\, \mathfrak{t}_2^j,\\
\left(\bigoplus_{i,j} F_{ij} \otimes \mathfrak{t}_1^i\, \mathfrak{t}_2^j\right) \otimes L_2^{-1} \otimes \mathfrak{t}_2^{-1} &\to \bigoplus_{i,j} F_{ij} \otimes \mathfrak{t}_1^i\, \mathfrak{t}_2^j,
\end{aligned} \tag{3.3.5}$$

which commute with both the actions of $\mathcal{O}_C$ and $\mathbf{T}$. In other words, (3.3.5) are $\mathbf{T}$-equivariant maps of $\mathcal{O}_C$-modules. By $\mathbf{T}$-equivariance, they are sums of maps

$$\begin{aligned}
F_{ij} \otimes L_1^{-1} &\to F_{i-1,j},\\
F_{ij} \otimes L_2^{-1} &\to F_{i,j-1}.
\end{aligned} \tag{3.3.6}$$

Let now $(F, s) \in P_n(X, d[C])^{\mathbf{T}}$ be a $\mathbf{T}$-fixed stable pair. Then s is a $\mathbf{T}$-equivariant section of a $\mathbf{T}$-equivariant coherent sheaf F on X. Applying p_* to $\mathcal{O}_X \xrightarrow{s} F$ gives a graded map which commutes with the maps (3.3.6). Writing

$$G_{ij} = F_{-i,-j} \otimes L_1^i \otimes L_2^j,$$

we find that the $\mathbf{T}$-fixed stable pair (F, s) on X is equivalent to the following data of sheaves and commuting maps on C

$$\begin{tikzcd}
 & & & & \mathcal{O}_C \arrow[rr, equal] \arrow[dl, equal] \arrow[ddd] & & \mathcal{O}_C \arrow[rr, equal] \arrow[dl, equal] \arrow[ddd] & & \mathcal{O}_C \arrow[r, equal] \arrow[dl, equal] \arrow[ddd] & \cdots \\
 & & & \mathcal{O}_C \arrow[rr, equal] \arrow[dl, equal] \arrow[ddd] & & \mathcal{O}_C \arrow[rr, equal] \arrow[dl, equal] \arrow[ddd] & & \mathcal{O}_C \arrow[r, equal] \arrow[dl, equal] \arrow[ddd] & \cdots & \\
 & & \cdots & & \cdots & & \cdots & & & \\
 & \cdots \arrow[r] & G_{0,-1} \arrow[rr] \arrow[dl] & & G_{00} \arrow[rr] \arrow[dl] & & G_{01} \arrow[rr] \arrow[dl] & & G_{02} \arrow[r] \arrow[dl] & \cdots \\
\cdots \arrow[r] & G_{1,-1} \arrow[rr] & & G_{10} \arrow[rr] \arrow[dl] & & G_{11} \arrow[rr] \arrow[dl] & & G_{12} \arrow[r] \arrow[dl] & \cdots & \\
 & & \cdots & & \cdots & & \cdots & & &
\end{tikzcd} \tag{3.3.7}$$

By the purity of F, each G_{ij} is either zero or a pure 1-dimensional coherent sheaf on C, and the "vertical" maps are generically isomorphisms. In particular, for every (i, j) such that either $i < 0$ or $j < 0$, it follows that G_{ij} is zero-dimensional and therefore vanishes by the purity assumption. Moreover, if G_{ij} is non-zero, it is a rank 1 torsion-free sheaf on a smooth curve, that is a line bundle on C (with a section). Finally, any $\mathbf{T}$-equivariant stable pair on X is set-theoretically supported on C, thus is properly supported on X and only finitely many G_{ij} can be non-zero. This results in a diagram of the following shape

$$\begin{tikzcd}
 & & \mathcal{O}_C \arrow[rr, equal] \arrow[dl, equal] \arrow[ddd] & & \mathcal{O}_C \arrow[rr, equal] \arrow[dl, equal] \arrow[ddd] & & \mathcal{O}_C \arrow[r, equal] \arrow[dl, equal] \arrow[ddd] & \cdots \\
 & \mathcal{O}_C \arrow[rr, equal] \arrow[dl, equal] \arrow[ddd] & & \mathcal{O}_C \arrow[rr, equal] \arrow[dl, equal] \arrow[ddd] & & \mathcal{O}_C \arrow[r, equal] \arrow[dl, equal] \arrow[ddd] & \cdots & \\
\cdots & & \cdots & & \cdots & & & \\
 & & \mathcal{O}_C(Z_{00}) \arrow[rr, hook] \arrow[dl, hook] & & \mathcal{O}_C(Z_{01}) \arrow[rr, hook] \arrow[dl, hook] & & \mathcal{O}_C(Z_{02}) \arrow[r, hook] \arrow[dl, hook] & \cdots \\
 & \mathcal{O}_C(Z_{10}) \arrow[rr, hook] \arrow[dl, hook] & & \mathcal{O}_C(Z_{11}) \arrow[rr, hook] \arrow[dl, hook] & & \mathcal{O}_C(Z_{12}) \arrow[r, hook] \arrow[dl, hook] & \cdots & \\
\cdots & & \cdots & & \cdots & & &
\end{tikzcd} \tag{3.3.8}$$

where Z_{ij} are divisors on C and all "horizontal" maps are injections of line bundles. Therefore, a $\mathbf{T}$-fixed stable pair (F, s) is equivalent to a nesting of divisors

$$\begin{array}{ccccccccc} Z_{00} & \subset & Z_{01} & \subset & Z_{02} & \subset & Z_{03} & \subset & \dots \\ \cap & & \cap & & \cap & & \cap & & \\ Z_{10} & \subset & Z_{11} & \subset & Z_{12} & \subset & Z_{13} & \subset & \dots \\ \cap & & \cap & & \cap & & \cap & & \\ Z_{20} & \subset & Z_{21} & \subset & Z_{22} & \subset & \dots & & \\ \cap & & \cap & & \cap & & & & \\ \dots & & \dots & & \dots & & & & \end{array} \tag{3.3.9}$$

where the nesting is dictated by a Young diagram λ. This results into a point of the double nested Hilbert scheme $C^{[\mathbf{n}_\lambda]}$, where by Riemann-Roch

$$\chi(F) = |\mathbf{n}_\lambda| + \mathbf{f}_{\lambda,g}(\deg L_1, \deg L_2),$$

where for a Young diagram λ and $g, k_1, k_2 \in \mathbb{Z}$ we define

$$\begin{aligned} \mathbf{f}_{\lambda,g}(k_1, k_2) &= \sum_{(i,j)\in\lambda} (1 - g - i \cdot k_1 - j \cdot k_2) \\ &= |\lambda|(1-g) - k_1 \cdot n(\lambda) - k_2 \cdot n(\overline{\lambda}). \end{aligned} \tag{3.3.10}$$

Conversely, any nesting of divisors as in (3.3.9) corresponds to a diagram of sheaves as in (3.3.8), which corresponds to a $\mathbf{T}$-fixed stable pair on X. Therefore we have a bijection of set:

$$P_n(X, d[C])^{\mathbf{T}} = \bigsqcup_{\lambda\vdash d} \bigsqcup_{\mathbf{n}_\lambda} C^{[\mathbf{n}_\lambda]}, \tag{3.3.11}$$

where the disjoint union is over all Young diagrams λ of size d and all reversed plane partitions $\mathbf{n}_\lambda$ satisfying $n = |\mathbf{n}_\lambda| + \mathbf{f}_{\lambda,g}(\deg L_1, \deg L_2)$. We mimic [**119**, Prop. 4.1] to prove that the above bijection on sets is an isomorphism of schemes.

Proposition 3.3.1. *There exists an isomorphism of schemes*

$$P_n(X, d[C])^{\mathbf{T}} = \bigsqcup_{\lambda\vdash d} \bigsqcup_{\mathbf{n}_\lambda} C^{[\mathbf{n}_\lambda]},$$

where the disjoint union is over all Young diagrams λ of size d and all reversed plane partitions $\mathbf{n}_\lambda$ satisfying

$$n = |\mathbf{n}_\lambda| + \mathbf{f}_{\lambda,g}(\deg L_1, \deg L_2).$$

In particular, $P_n(X, d[C])^{\mathbf{T}}$ is proper.

Proof. Let B be any (connected) scheme over $\mathbb{C}$. We need to adapt the construction of this section to a $\mathbf{T}$-fixed stable pair on $X \times B$, flat over B. Pushing down by the affine map $p : X \times B \to C \times B$ gives a graded sheaf $\bigoplus_{i,j} F_{ij}$ on $C \times B$, flat over B (therefore so are all its weight spaces F_{ij}). The original sheaf F on $X \times B$ can be reconstructed from the maps (3.3.6). Therefore a $\mathbf{T}$-fixed pair (F, s) on $X \times B$, flat over B, is equivalent to the data (3.3.7), with each G_{ij} on $C \times B$, flat over B.

If (F, s) is a *stable pair*, over each closed fiber $C \times \{b\}$, where $b \in B$, we showed that each (non-zero) G_{ij} is a line bundle. By [**94**, Lemma 2.1.7], this shows that each (non-zero) G_{ij} is a line bundle on $C \times B$. Together with its non-zero section, this defines divisors $Z_{ij} \subset C \times B$, flat over B, satisfying the nesting condition of (3.3.9), which yields a B-point $B \to \coprod_{\mathbf{n}_\lambda} C^{[\mathbf{n}_\lambda]}$. Conversely, any B-point $B \to \coprod_{\mathbf{n}_\lambda} C^{[\mathbf{n}_\lambda]}$ defines a diagram (3.3.8), equivalent to a $\mathbf{T}$-fixed stable pair (F, s) on $X \times B$, flat over B. $\square$

As a corollary, we compute the generating series of the topological Euler characteristic of the moduli space of stable pairs on a local curve.

Corollary 3.3.2. *Let C be a smooth projective curve of genus g, L_1, L_2 line bundle on C and set $X = \mathrm{Tot}_C(L_1 \oplus L_2)$. Then for any $d > 0$ we have*

$$\sum_{n \in \mathbb{Z}} e(P_n(X, d[C])) \cdot q^n = \sum_{|\lambda| = d} q^{|\lambda|(1-g) - \deg L_1 n(\lambda) - \deg L_2 n(\overline{\lambda})} \prod_{\square \in \lambda} (1 - q^{h(\square)})^{2g-2}.$$

Proof. The topological Euler characteristic of a $\mathbf{T}$-scheme is the same of its $\mathbf{T}$-fixed locus, therefore

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} e(P_n(X, d[C])) \cdot q^n &= \sum_{n \in \mathbb{Z}} e\left(P_n(X, d[C])^{\mathbf{T}}\right) \cdot q^n \\
&= \sum_{n \in \mathbb{Z}} \sum_{|\lambda| = d} \sum_{\substack{|\mathbf{n}_\lambda| = n - \\ \mathbf{f}_{\lambda,g}(\deg L_1, \deg L_2)}} q^n \cdot e(C^{[\mathbf{n}_\lambda]}) \\
&= \sum_{|\lambda| = d} q^{\mathbf{f}_{\lambda,g}(\deg L_1, \deg L_2)} \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \cdot e(C^{[\mathbf{n}_\lambda]}) \\
&= \sum_{|\lambda| = d} q^{\mathbf{f}_{\lambda,g}(\deg L_1, \deg L_2)} \prod_{\square \in \lambda} (1 - q^{h(\square)})^{2g-2},
\end{aligned}$$

where in the second line we applied Proposition 3.3.1 and in the last line Theorem 3.2.10. $\square$

3.4. K-theory class of the perfect obstruction theory

Let C be an irreducible smooth projective curve. On each connected component $C^{[\mathbf{n}_\lambda]} \subset P_n(X, \beta)^{\mathbf{T}}$ of the $\mathbf{T}$-fixed locus there exists and induced virtual fundamental class $[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}_{\mathrm{PT}}$ coming from the perfect obstruction theory (3.3.1). We show in this section that $[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}_{\mathrm{PT}}$ agrees with the virtual fundamental class constructed in Corollary 3.2.8 and compute the class in K-theory of the virtual normal bundle N^{vir}.

We start by describing the class in K-theory of (the restriction of) the perfect obstruction theory $\mathbb{E} \in K_0^{\mathbf{T}}(C^{[\mathbf{n}_\lambda]})$. To ease readability we will omit various pullbacks whenever they are clear from the context. Recall the following identities in K-theory

$$\begin{aligned}
\mathbb{E} &= -\mathbf{R}\pi_* \mathbf{R}\mathscr{H}om(\mathbb{I}, \mathbb{I})_0^\vee \in K_0^{\mathbf{T}}(P_n(X, \beta)^{\mathbf{T}}), \\
\mathbb{F} &= \sum_{(i,j) \in \lambda} i_* \mathcal{O}_{C \times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{ij}) \otimes L_1^{-i} \otimes L_2^{-j} \otimes \mathfrak{t}_1^{-i} \mathfrak{t}_2^{-j} \in K_0^{\mathbf{T}}(X \times C^{[\mathbf{n}_\lambda]}), \\
\mathbb{I} &= \mathcal{O}_{X \times P_n(X, \beta)^{\mathbf{T}}} - \mathbb{F} \in K_0^{\mathbf{T}}(X \times P_n(X, \beta)^{\mathbf{T}}),
\end{aligned}$$

where the various maps fit in the diagram

$$
\begin{tikzcd}
C \arrow[r, bend left, "i"] & X \arrow[l, "p"] & X \times C^{[\mathbf{n}_\lambda]} \arrow[l] \arrow[r, "\pi"] & C^{[\mathbf{n}_\lambda]} \\
& & C \times C^{[\mathbf{n}_\lambda]} \arrow[ull] \arrow[u, bend left, "i"] \arrow[ur, "\pi"] &
\end{tikzcd}
$$

Again, to ease notation, we keep denoting $i \times \mathrm{id}_{C^{[\mathbf{n}_\lambda]}}, p \times \mathrm{id}_{C^{[\mathbf{n}_\lambda]}}$ by i, p and by π the composition $\pi \circ i$. We compute

$$
\begin{aligned}
\mathbb{E}^\vee &= \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathcal{O}, \mathbb{F}) + \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathbb{F}, \mathcal{O}) - \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathbb{F}, \mathbb{F}) \\
&= \mathbf{R}\pi_*\mathbb{F} - (\mathbf{R}\pi_*(\mathbb{F} \otimes K_X))^\vee \otimes \mathfrak{t}_1\, \mathfrak{t}_2 - \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathbb{F}, \mathbb{F}),
\end{aligned}
$$

where in the second equality we applied Grothendieck duality[3] and the projection formula and $K_X = K_C \otimes L_1^{-1} \otimes L_2^{-1}$. Define $\Lambda^\bullet(V) = \sum_{i=0}^{\mathrm{rk}\, V} (-1)^i \Lambda^i V$ for a locally free sheaf V and extend it by linearity to any class in $K^0_{\mathbf{T}}(C)$. By [**57**, Lemma 5.4.9] , for every $\mathbf{T}$-equivariant coherent sheaf $\mathcal{F} \in K_0^{\mathbf{T}}(C)$, we have

$$
\mathbf{L}i^* i_* \mathcal{F} = \Lambda^\bullet N^*_{C/X} \otimes \mathcal{F} \in K_0^{\mathbf{T}}(C),
$$

where $N_{C/X} = L_1 \otimes \mathfrak{t}_1 \oplus L_2 \otimes \mathfrak{t}_2$ is the $\mathbf{T}$-equivariant normal bundle of $i : C \to X$; an analogous statement holds for $i : C \times C^{[\mathbf{n}_\lambda]} \to X \times C^{[\mathbf{n}_\lambda]}$. Therefore

$$
\begin{aligned}
\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathbb{F}, \mathbb{F}) &= \sum_{(i,j),(l,k)\in\lambda} \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om_X(i_*\, \mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{ij}), i_*\, \mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{lk}) \otimes L_1^{i-l} L_2^{j-k})\, \mathfrak{t}_1^{i-l}\, \mathfrak{t}_2^{j-k} \\
&= \sum_{(i,j),(l,k)\in\lambda} \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om_C(\mathbf{L}i^* i_*\, \mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{ij}), \mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{lk}) \otimes L_1^{i-l} L_2^{j-k})\, \mathfrak{t}_1^{i-l}\, \mathfrak{t}_2^{j-k} \\
&= \sum_{(i,j),(l,k)\in\lambda} \mathbf{R}\pi_* \left((\mathcal{O} - L_1\, \mathfrak{t}_1 - L_2\, \mathfrak{t}_2 + L_1 L_2\, \mathfrak{t}_1\, \mathfrak{t}_2)(\mathcal{Z}_{lk} - \mathcal{Z}_{ij}) \otimes L_1^{i-l} L_2^{j-k})\, \mathfrak{t}_1^{i-l}\, \mathfrak{t}_2^{j-k} \right),
\end{aligned}
$$

where in the second line we used adjunction in the derived category. To simplify the notation, for any $(i,j),(l,k) \in \lambda$ set $\Delta_{ij;lk} = \mathcal{Z}_{lk} - \mathcal{Z}_{ij}$, which is an effective divisor if $(i,j) \le (l,k)$. Putting all together we have the following identity in $K_0^{\mathbf{T}}(C^{[\mathbf{n}_\lambda]})$

$$
\begin{aligned}
\mathbb{E}^\vee = &\sum_{(i,j)\in\lambda} \mathbf{R}\pi_*(\mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{ij}) \otimes L_1^{-i} L_2^{-j})\, \mathfrak{t}_1^{-i}\, \mathfrak{t}_2^{-j} \\
&- \sum_{(i,j)\in\lambda} \left(\mathbf{R}\pi_*(\mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{ij}) \otimes K_X \otimes L_1^{-i} L_2^{-j}\, \mathfrak{t}_1^{-i}\, \mathfrak{t}_2^{-j})\right)^\vee \otimes \mathfrak{t}_1\, \mathfrak{t}_2 \\
&- \sum_{(i,j),(l,k)\in\lambda} \mathbf{R}\pi_* \left((\mathcal{O} - L_1\, \mathfrak{t}_1 - L_2\, \mathfrak{t}_2 + L_1 L_2\, \mathfrak{t}_1\, \mathfrak{t}_2)(\Delta_{ij;lk}) \otimes L_1^{i-l} L_2^{j-k})\, \mathfrak{t}_1^{i-l}\, \mathfrak{t}_2^{j-k} \right).
\end{aligned}
\tag{3.4.1}
$$

Theorem 3.4.1. *There is an identity of virtual fundamental classes*

$$
[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}_{\mathrm{PT}} = [C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}} \in A_*(C^{[\mathbf{n}_\lambda]}),
$$

where the class on the left-hand-side is induced by (3.3.1) *by Graber-Pandharipande localization and the one on the right-hand-side is constructed in Corollary 3.2.8.*

[3]Even though X is not proper, we can pass to a compactification $X \hookrightarrow \overline{X}$ and use Grothendieck duality exploiting the fact that the sheaves involved have proper support, see [**33**, footnote 4].

Proof. By a result of Siebert [171, Thm. 4.6] any two virtual fundamental classes coincide if the classes in K-theory of their perfect obstruction theory agree. The class in K-theory of the dual of the induced PT perfect obstruction theory is the $\mathbf{T}$-fixed part of $\mathbb{E}^\vee$ by [85, Prop. 1]

$$(\mathbb{E}^\vee)^{\mathsf{fix}} = \mathbf{R}\pi_* \mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{00}) - \sum_{(i,j)\in\lambda} \mathbf{R}\pi_* \mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}} + \sum_{\substack{(i,j)\in\lambda\\ i\geq 1}} \mathbf{R}\pi_* \mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\Delta_{i-1,j;ij})$$
$$+ \sum_{\substack{(i,j)\in\lambda\\ j\geq 1}} \mathbf{R}\pi_* \mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\Delta_{i,j-1;ij}) - \sum_{\substack{(i,j)\in\lambda\\ i,j\geq 1}} \mathbf{R}\pi_* \mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\Delta_{i-1,j-1;ij}).$$

We explained in Section 3.2.1 how to associate a graph to any Young diagram λ. Notice that boxes $(i,j)\in\lambda$ are in bijection with the vertices V, boxes $(i,j)\in\lambda$, such that $i\geq 1$ (resp. $j\geq 1$) are in bijection with vertical (resp. horizontal) edges E and boxes $(i,j)\in\lambda$, such that $i,j\geq 1$ are in bijection with squares Q of the associated graph. By Lemma 3.2.1

$$V - E + Q - 1 = 0. \tag{3.4.2}$$

Combining this identity with the universal exact sequences

$$\begin{aligned} &0\to\mathcal{O}\to\mathcal{O}(\mathcal{Z}_{00})\to\mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00})\to 0,\\ &0\to\mathcal{O}\to\mathcal{O}(\Delta_{ij;lk})\to\mathcal{O}_{\Delta_{ij;lk}}(\Delta_{ij;lk})\to 0,\end{aligned} \tag{3.4.3}$$

whenever $\Delta_{ij,lk}$ is an effective divisor, one gets

$$(\mathbb{E}^\vee)^{\mathsf{fix}} = \mathbf{R}\pi_* \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00}) + \sum_{\substack{(i,j)\in\lambda\\ i\geq 1}} \mathbf{R}\pi_* \mathcal{O}_{\Delta_{i-1,j;ij}}(\Delta_{i-1,j;ij})$$
$$+ \sum_{\substack{(i,j)\in\lambda\\ j\geq 1}} \mathbf{R}\pi_* \mathcal{O}_{\Delta_{i,j-1;ij}}(\Delta_{i,j-1;ij}) - \sum_{\substack{(i,j)\in\lambda\\ i,j\geq 1}} \mathbf{R}\pi_* \mathcal{O}_{\Delta_{i-1,j-1;ij}}(\Delta_{i-1,j-1;ij}).$$

Moreover, in the expression above, all higher direct images $\mathbf{R}^k\pi_*$ vanish for $k>0$ by cohomology and base change, therefore

$$(\mathbb{E}^\vee)^{\mathsf{fix}} = \pi_* \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00}) + \sum_{\substack{(i,j)\in\lambda\\ i\geq 1}} \pi_* \mathcal{O}_{\Delta_{i-1,j;ij}}(\Delta_{i-1,j;ij})$$
$$+ \sum_{\substack{(i,j)\in\lambda\\ j\geq 1}} \pi_* \mathcal{O}_{\Delta_{i,j-1;ij}}(\Delta_{i,j-1;ij}) - \sum_{\substack{(i,j)\in\lambda\\ i,j\geq 1}} \pi_* \mathcal{O}_{\Delta_{i-1,j-1;ij}}(\Delta_{i-1,j-1;ij}).$$

We finally show that this is precisely the same class in K-theory as

$$T_{A_{C,\mathbf{n}_\lambda}}|_{C^{[\mathbf{n}_\lambda]}} - \mathcal{E}|_{C^{[\mathbf{n}_\lambda]}} \in K_0(C^{[\mathbf{n}_\lambda]}),$$

where $\mathcal{E}\to A_{C,\mathbf{n}_\lambda}$ is the vector bundle constructed in Theorem 3.2.7. In fact, in the notation of Section 3.2.3, we have

$$\begin{aligned} \mathcal{X}_{ij}|_{C^{[\mathbf{n}_\lambda]}\times C} &= \Delta_{i-1,j;ij}, \quad i\geq 1,\\ \mathcal{Y}_{ij}|_{C^{[\mathbf{n}_\lambda]}\times C} &= \Delta_{i,j-1;ij}, \quad j\geq 1,\\ \Gamma^1_{ij}|_{C^{[\mathbf{n}_\lambda]}\times C} = \Gamma^2_{ij}|_{C^{[\mathbf{n}_\lambda]}\times C} &= \Delta_{i-1,j-1;ij}, \quad i,j\geq 1.\end{aligned}$$

Moreover, the explicit description of the tangent bundle of the Hilbert scheme of points on a smooth curve in terms of its universal subscheme [**2**, Lemma IV.2.3] yields

$$T_{A_{C,\mathbf{n}_\lambda}} = \pi_* \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00}) \oplus \bigoplus_{\substack{(i,j)\in\lambda \\ i\geq 1}} \pi_* \mathcal{O}_{\mathcal{X}_{ij}}(\mathcal{X}_{ij}) \oplus \bigoplus_{\substack{(i,j)\in\lambda \\ j\geq 1}} \pi_* \mathcal{O}_{\mathcal{Y}_{ij}}(\mathcal{Y}_{ij}),$$

by which we conclude that

$$(\mathbb{E}^\vee)^{\mathsf{fix}} = T_{A_{C,\mathbf{n}_\lambda}}|_{C^{[\mathbf{n}_\lambda]}} - \mathcal{E}|_{C^{[\mathbf{n}_\lambda]}} \in K_0(C^{[\mathbf{n}_\lambda]}).$$

□

In virtue of Proposition 3.4.1, we denote now on by $T^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}}$ the class in K-theory of the dual of the perfect obstruction theory (3.2.2) and the one induced by the fixed part of (3.3.1), which we showed to agree.
In order to compute stable pair invariants (3.3.3) one needs to express the virtual normal bundle (3.3.4) in terms of K-theoretic classes which are easier to handle. For instance, we could express N^{vir} in terms of pullbacks of the line bundles L_1, L_2, K_C and the universal divisors $\Delta_{ij;lk}$, but that would lead to cumbersome expressions difficult to manipulate.

Example 3.4.2. As a concrete example, we compute the weight space of $\mathbb{E}^\vee$ relative to the character $\mathfrak{t}_1\,\mathfrak{t}_2$, which we denote by $\mathbb{E}^\vee_{\mathfrak{t}_1\,\mathfrak{t}_2}$.

An application of Grothendieck duality and projection formula on $\pi : C \times C^{[\mathbf{n}_\lambda]} \to C^{[\mathbf{n}_\lambda]}$ yields

$$\mathbf{R}\pi_* L(\Delta) = -(\mathbf{R}\pi_*(K_C \otimes L^{-1}(-\Delta)))^\vee \in K_0(C^{[\mathbf{n}_\lambda]}), \tag{3.4.4}$$

for any divisor $\Delta \subset C \times C^{[\mathbf{n}_\lambda]}$ and any line bundle L on C. As in the proof of Proposition 3.4.1, combining (3.4.2), (3.4.3), (3.4.4), the identity $K_X = K_C \otimes L_1^{-1} \otimes L_2^{-1}$ and some vanishing of higher direct images yields

$$\begin{aligned}\mathbb{E}^\vee_{\mathfrak{t}_1\,\mathfrak{t}_2} &= -(\pi_*(K_X \otimes \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00})))^\vee - \sum_{\substack{(i,j)\in\lambda \\ i\geq 1}} (\pi_*(K_X \otimes \mathcal{O}_{\Delta_{i-1,j;ij}}(\Delta_{i-1,j;ij})))^\vee \\ &- \sum_{\substack{(i,j)\in\lambda \\ j\geq 1}} (\pi_*(K_X \otimes \mathcal{O}_{\Delta_{i,j-1;ij}}(\Delta_{i,j-1;ij})))^\vee + \sum_{\substack{(i,j)\in\lambda \\ i,j\geq 1}} (\pi_*(K_X \otimes \mathcal{O}_{\Delta_{i-1,j-1;ij}}(\Delta_{i-1,j-1;ij})))^\vee.\end{aligned}$$

The situation notably simplifies if we impose X to be Calabi-Yau.

Proposition 3.4.3. *If X is Calabi-Yau, we have an identity*

$$N^{\mathrm{vir}} = -T^{\mathrm{vir},\vee}_{C^{[\mathbf{n}_\lambda]}} \otimes \mathfrak{t}_1\,\mathfrak{t}_2 + \Omega - \Omega^\vee \otimes \mathfrak{t}_1\,\mathfrak{t}_2 \in K^0_{\mathbf{T}}(C^{[\mathbf{n}_\lambda]}),$$

where $\Omega, \Omega^\vee \in K^0_{\mathbf{T}}(C^{[\mathbf{n}_\lambda]})$ have no weight spaces corresponding to the characters $(\mathfrak{t}_1\,\mathfrak{t}_2)^0, \mathfrak{t}_1\,\mathfrak{t}_2$.

Proof. If X is Calabi-Yau, the perfect obstruction theory (3.3.1) satisfies

$$\mathbb{E}^\vee = -\mathbb{E} \otimes \mathfrak{t}_1\,\mathfrak{t}_2 \in K^0_{\mathbf{T}}(C^{[\mathbf{n}_\lambda]}) \tag{3.4.5}$$

by $\mathbf{T}$-equivariant Serre duality. Setting $\mathbb{E}^\vee = W_+ - W_-$, where $W_+, W_- \in K^0_{\mathbf{T}}(C^{[\mathbf{n}_\lambda]})$ are classes of $\mathbf{T}$-equivariant vector bundles, (3.4.5) implies that

$$W_- = W_+^\vee \otimes \mathfrak{t}_1\, \mathfrak{t}_2\,.$$

We have that $(\mathbb{E}^\vee)^{\mathsf{fix}} = T^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}}$, therefore

$$\mathbb{E}^\vee = T^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}} - T^{\mathrm{vir},\vee}_{C^{[\mathbf{n}_\lambda]}} \otimes \mathfrak{t}_1\, \mathfrak{t}_2 + \Omega - \Omega^\vee \otimes \mathfrak{t}_1\, \mathfrak{t}_2,$$

which concludes the argument. □

Remark 3.4.4. A simple computation shows that we could take Ω to be of the form

$$\begin{aligned}\Omega = \sum_{\substack{(i,j)\in\lambda\\ (i,j)\neq(0,0)}} &\mathbf{R}\pi_*\left(\mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{ij})\otimes L_1^{-i}L_2^{-j}\right)\mathfrak{t}_1^{-i}\,\mathfrak{t}_2^{-j}\\ &- \sum_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)\neq(l,k)\\ (i,j)\neq(l+1,k+1)}} \mathbf{R}\pi_*\left(\mathcal{O}(\Delta_{ij;lk})\otimes L_1^{i-l}L_2^{j-k}\right)\mathfrak{t}_1^{i-l}\,\mathfrak{t}_2^{j-k}\\ &\qquad + \sum_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)\neq(l-1,k)\\ (i,j)\neq(l,k+1)}} \mathbf{R}\pi_*\left(\mathcal{O}(\Delta_{ij;lk})\otimes L_1^{i-l+1}L_2^{j-k}\right)\mathfrak{t}_1^{i-l+1}\,\mathfrak{t}_2^{j-k}\,.\end{aligned}$$

All other choices $\tilde{\Omega}$ must be of the form

$$\tilde{\Omega} = \Omega + A + A^\vee \otimes \mathfrak{t}_1\, \mathfrak{t}_2,$$

for any $A \in K^0_{\mathbf{T}}(C^{[\mathbf{n}_\lambda]})$ having no weight spaces corresponding to the characters $(\mathfrak{t}_1\,\mathfrak{t}_2)^0, \mathfrak{t}_1\,\mathfrak{t}_2$. In particular, this implies that the parity of $\operatorname{rk}\Omega$ is independent by the choice of Ω.

3.5. Universality

3.5.1. Universal expression In this section we fix a Young diagram λ. In the previous sections, given a triple (C, L_1, L_2) with C an irreducible smooth projective curve and L_1, L_2 line bundles on C, we reduced stable pair invariants (with no insertions) of $\mathrm{Tot}_C(L_1 \oplus L_2)$ to the computation of

$$\int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) \in \mathbb{Q}(s_1, s_2), \tag{3.5.1}$$

where $\mathbf{n}_\lambda$ is a reversed plane partition and the virtual normal bundle $N^{\mathrm{vir}}_{C,L_1,L_2}$ is the $\mathbf{T}$-moving part of the class in K-theory (3.4.1).

We state our main results, describing the generating series of (3.5.1) in terms of three universal functions exploiting the universality techniques used in [**72**, Thm. 4.2] for surfaces. Furthermore, we find explicit expressions for these universal series under the anti-diagonal restriction $s_1 + s_2 = 0$.

Theorem 3.5.1. *Let C be a genus g smooth irreducible projective curve and L_1, L_2 line bundles over C. We have an identity*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) = A_\lambda^{g-1} \cdot B_\lambda^{\deg L_1} \cdot C_\lambda^{\deg L_2} \in \mathbb{Q}(s_1,s_2)[\![q]\!],$$

where $A_\lambda, B_\lambda, C_\lambda \in \mathbb{Q}(s_1,s_2)[\![q]\!]$ are fixed universal series for $i = 1,2,3$ which only depend on λ. Moreover

$$\begin{aligned} A_\lambda(s_1,s_2) &= A_{\overline{\lambda}}(s_2,s_1), \\ B_\lambda(s_1,s_2) &= C_{\overline{\lambda}}(s_2,s_1). \end{aligned}$$

Proof. The proof is similar to [**72**, Thm. 4.2]. Consider the map

$$Z : \mathcal{K} := \{\, (C, L_1, L_2) : C \text{ curve}, L_1, L_2 \text{ line bundles} \,\} \to \mathbb{Q}(s_1,s_2)[\![q]\!]$$

given by

$$Z(C,L_1,L_2) = C^{-1}_{g,\deg L_1 \deg L_2} \cdot \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}),$$

where $C_{g,\deg L_1 \deg L_2}$ is the leading term of the generating series of the integrals (3.5.1).

By Proposition 3.5.2 the integral (3.5.1) is *multiplicative* and by Corollary 3.5.4 it is a polynomial on $g, \deg L_1, \deg L_2$. This implies that Z factors through

$$\mathcal{K} \xrightarrow{\gamma} \mathbb{Q}^3 \xrightarrow{Z'} \mathbb{Q}(s_1,s_2)[\![q]\!],$$

where $\gamma(C,L_1,L_2) = (g-1, \deg L_1, \deg L_2)$ and Z' is a linear map.

A basis of $\mathbb{Q}^3$ is given by the images

$$e_1 = \gamma(\mathbb{P}^1, \mathcal{O}, \mathcal{O}), \quad e_2 = \gamma(\mathbb{P}^1, \mathcal{O}(1), \mathcal{O}), \quad e_3 = \gamma(\mathbb{P}^1, \mathcal{O}, \mathcal{O}(1)),$$

and the image of a generic triple (C, L_1, L_2) can be written as

$$\gamma(C,L_1,L_2) = (1 - g - \deg L_1 - \deg L_2) \cdot e_1 + \deg L_1 \cdot e_2 + \deg L_2 \cdot e_3.$$

We conclude that

$$Z'(C,L_1,L_2) = Z'(e_1)^{1-g} \cdot (Z'(e_1)^{-1} Z'(e_2))^{\deg L_1} \cdot (Z'(e_1)^{-1} Z'(e_3))^{\deg L_2},$$

which gives the universal series we were looking for. The second claim just follows by interchanging the role of L_1 and L_2. $\square$

We devote the remainder of Section 3.5 to prove the multiplicativity and polynomiality of (3.5.1). In Section 3.6 we compute the leading term of the generating series of (3.5.1), while in Section 3.7 we explicitly compute the integral (3.5.1) in the toric case under the anti-diagonal restriction. These computations will lead to the proof of the second part of the main Theorem 3.1.3 (see Theorem 3.8.1).

3.5.2. Multiplicativity We show now that the integral (3.5.1) is multiplicative. First of all, notice that if $C = C' \sqcup C''$ is a smooth projective curve with two connected components, the construction of Theorem 3.2.7 does not directly work and we need to adjust it to define a virtual fundamental class.
For any reversed plane partition $\mathbf{n}_\lambda$ there is an induced stratification

$$C^{[\mathbf{n}_\lambda]} = \bigsqcup_{\mathbf{n}'_\lambda + \mathbf{n}''_\lambda = \mathbf{n}_\lambda} C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}.$$

We set

$$A_{C,\mathbf{n}_\lambda} := \bigsqcup_{\mathbf{n}'_\lambda + \mathbf{n}''_\lambda = \mathbf{n}_\lambda} A_{C',\mathbf{n}'_\lambda} \times A_{C'',\mathbf{n}''_\lambda},$$

which is a smooth projective variety. Let $\mathcal{E}_{C',\mathbf{n}'_\lambda}, \mathcal{E}_{C'',\mathbf{n}''_\lambda}$ denote the vector bundles over $A_{C',\mathbf{n}'_\lambda}, A_{C'',\mathbf{n}''_\lambda}$ of Theorem 3.2.7. We define a vector bundle $\mathcal{E}_{C,\mathbf{n}_\lambda}$ over $A_{C,\mathbf{n}_\lambda}$ by declaring its restriction to any connected component $A_{C',\mathbf{n}'_\lambda} \times A_{C'',\mathbf{n}''_\lambda}$ to be

$$\mathcal{E}_{C,\mathbf{n}_\lambda}|_{A_{C',\mathbf{n}'_\lambda} \times A_{C'',\mathbf{n}''_\lambda}} = \mathcal{E}_{C',\mathbf{n}'_\lambda} \boxplus \mathcal{E}_{C'',\mathbf{n}''_\lambda}.$$

By Theorem 3.2.7, there exists a section s of $\mathcal{E}_{C,\mathbf{n}_\lambda}$ such that

$$C^{[\mathbf{n}_\lambda]} \cong Z(s) \hookrightarrow A_{C,\mathbf{n}_\lambda},$$

and therefore an induced virtual fundamental class $[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}$ satisfying

$$[C^{[\mathbf{n}_\lambda]}]|^{\mathrm{vir}}_{C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}} = [C'^{[\mathbf{n}'_\lambda]}]^{\mathrm{vir}} \boxtimes [C''^{[\mathbf{n}''_\lambda]}]^{\mathrm{vir}}. \tag{3.5.2}$$

By iterating this construction, there exists a natural virtual fundamental class on $C^{[\mathbf{n}_\lambda]}$ for any smooth projective curve C (with any number of connected components).

Proposition 3.5.2. *Let (C, L_1, L_2) be a triple where $C = C' \sqcup C''$ and $L_i = L'_i \oplus L''_i$ for $i = 1, 2$, where L'_i are line bundles on C' and L''_i are line bundles on C''. Then*

$$\begin{aligned}\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} &\int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) \\ &= \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C'^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C',L'_1,L'_2}) \cdot \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C''^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C'',L''_1,L''_2}).\end{aligned}$$

PROOF. Let $\mathbf{n}_\lambda$ be a fixed reversed plane partition. We claim that the restriction of the virtual normal bundle to the connected component $C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]} \subset C^{[\mathbf{n}_\lambda]}$ decomposes as

$$N^{\mathrm{vir}}_{C,L_1,L_2}|_{C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}} = N^{\mathrm{vir}}_{C',L'_1,L'_2} \boxplus N^{\mathrm{vir}}_{C'',L''_1,L''_2}. \tag{3.5.3}$$

In fact, $N^{\mathrm{vir}}_{C,L_1,L_2}$ is a linear combination of K-theoretic classes of the form

$$\mathbf{R}\pi_*(\mathcal{O}_{C \times C^{[\mathbf{n}_\lambda]}}(\Delta) \otimes L_1^a L_2^b) \otimes \mathfrak{t}^\mu \in K^0_{\mathbf{T}}(C^{[\mathbf{n}_\lambda]}),$$

where Δ is a $\mathbb{Z}$-linear combination of the universal divisors $\mathcal{Z}_{ij}$ on $C \times C^{[\mathbf{n}_\lambda]}$, $a, b \in \mathbb{Z}$ and $\mathfrak{t}^\mu$ is a $\mathbf{T}$-character and notice that

$$L_1^a L_2^b = {L'_1}^a {L'_2}^b \oplus {L''_1}^a {L''_2}^b \in \mathrm{Pic}(C' \sqcup C'').$$

Consider the induced stratification

$$C \times C^{[\mathbf{n}_\lambda]} = \bigsqcup_{\mathbf{n}'_\lambda + \mathbf{n}''_\lambda = \mathbf{n}_\lambda} (C' \sqcup C'') \times C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}.$$

Denote by Δ', Δ'' the corresponding universal divisor on $C' \times C'^{[\mathbf{n}'_\lambda]}, C'' \times C''^{[\mathbf{n}''_\lambda]}$ and consider the projection maps

$$q_1 : C' \times C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]} \to C' \times C'^{[\mathbf{n}'_\lambda]},$$
$$q_2 : C'' \times C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]} \to C'' \times C''^{[\mathbf{n}''_\lambda]}.$$

On every component $(C' \sqcup C'') \times C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}$ we have

$$\mathcal{O}_{C \times C^{[\mathbf{n}_\lambda]}}(\Delta)|_{(C' \sqcup C'') \times C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}} = q_1^* \mathcal{O}_{C' \times C'^{[\mathbf{n}'_\lambda]}}(\Delta') \oplus q_2^* \mathcal{O}_{C'' \times C''^{[\mathbf{n}''_\lambda]}}(\Delta''),$$

and similarly

$$\mathcal{O}_{C \times C^{[\mathbf{n}_\lambda]}}(\Delta) \otimes L_1^a L_2^b|_{(C' \sqcup C'') \times C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}} = \\ q_1^* \left(\mathcal{O}_{C' \times C'^{[\mathbf{n}'_\lambda]}}(\Delta') \otimes {L'_1}^a {L'_2}^b \right) \oplus q_2^* \left(\mathcal{O}_{C'' \times C''^{[\mathbf{n}''_\lambda]}}(\Delta'') \otimes {L''_1}^a {L''_2}^b \right).$$

Consider the cartesian diagram given by the natural projections

$$\begin{tikzcd} C' \times C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]} \arrow[r, "q_1"] \arrow[d, "\pi"] & C' \times C'^{[\mathbf{n}'_\lambda]} \arrow[d, "\pi_1"] \\ C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]} \arrow[r, "\tilde{q}_1"] & C'^{[\mathbf{n}'_\lambda]}. \end{tikzcd}$$

Flat base change yields

$$\mathbf{R}\pi_* q_1^* \left(\mathcal{O}_{C' \times C'^{[\mathbf{n}'_\lambda]}}(\Delta') \otimes {L'_1}^a {L'_2}^b \right) = \tilde{q}_1^* \mathbf{R}\pi_{1*} \left(\mathcal{O}_{C' \times C'^{[\mathbf{n}'_\lambda]}}(\Delta') \otimes {L'_1}^a {L'_2}^b \right),$$

and analogously for C'', which implies

$$\mathbf{R}\pi_*(\mathcal{O}_{C \times C^{[\mathbf{n}_\lambda]}}(\Delta) \otimes L_1^a L_2^b) \otimes \mathfrak{t}^\mu |_{C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}} \\ = \mathbf{R}\pi_{1*}(\mathcal{O}_{C' \times C'^{[\mathbf{n}'_\lambda]}}(\Delta') \otimes {L'_1}^a {L'_2}^b) \otimes \mathfrak{t}^\mu \boxplus \mathbf{R}\pi_{2*}(\mathcal{O}_{C'' \times C''^{[\mathbf{n}''_\lambda]}}(\Delta'') \otimes {L''_1}^a {L''_2}^b) \otimes \mathfrak{t}^\mu,$$

and proves the claim (3.5.3). Combining (3.5.2) and (3.5.3) yields

$$\int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) = \sum_{\mathbf{n}'_\lambda + \mathbf{n}''_\lambda = \mathbf{n}_\lambda} \int_{[C'^{[\mathbf{n}'_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C',L'_1,L'_2}) \cdot \int_{[C''^{[\mathbf{n}''_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C'',L''_1,L''_2}),$$

which concludes the proof. □

3.5.3. Chern numbers dependence We now show that the integral 3.5.1 is a polynomial in the Chern numbers of the triple (C, L_1, L_2). Our strategy is to express the integral on a product of Picard varieties $\mathrm{Pic}^n(C)$ — via the Abel-Jacobi map — where the integrand is a polynomial expression on tautological classes. Through this section, we follow the notation as in [**2**, Sec. VIII.2] and [**119**, Sec. 9, 10.1].

3.5.3.1. Tautological integrals on $\mathrm{Pic}^n(C)$ Let C be a smooth curve of genus g. If $n > 2g-2$, the Abel-Jacobi map

$$\begin{aligned} \mathrm{AJ} : C^{(n)} &\to \mathrm{Pic}^n(C) \\ Z &\to [\mathcal{O}_C(Z)] \end{aligned}$$

is a projective bundle. In fact, consider the diagram

$$\begin{tikzcd} C^{(n)} \times C \arrow[r, "\mathrm{AJ}\times\mathrm{id}"] \arrow[d, "\pi"] & \mathrm{Pic}^n(C) \times C \arrow[d, "\overline{\pi}"] \\ C^{(n)} \arrow[r, "\mathrm{AJ}"] & \mathrm{Pic}^n(C) \end{tikzcd}$$

and the Poincaré line bundle $\mathcal{P}$ on $\mathrm{Pic}^n(C) \times C$, normalized by fixing

$$\mathcal{P}|_{\mathrm{Pic}^n(C)\times\{c\}} \cong \mathcal{O}_{\mathrm{Pic}^n(C)},$$

for a certain $c \in C$. Then

$$C^{(n)} \cong \mathbb{P}(\overline{\pi}_*\mathcal{P}).$$

The universal divisor $\mathcal{Z} \subset C^{(n)} \times C$ satisfies [**119**, eqn. (67)]

$$\mathcal{O}(\mathcal{Z}) \cong (\mathrm{AJ}\times\mathrm{id})^*\mathcal{P} \otimes \pi^*\,\mathcal{O}(1) \text{ and } \mathcal{O}(1) \cong \mathcal{O}(\mathcal{Z})|_{C^{(n)}\times\{c\}},$$

and we denote the first Chern class of the latter by

$$\omega := c_1(\mathcal{O}(1)) \in H^2(C^{(n)}, \mathbb{Z}).$$

Consider now the product of Abel-Jacobi maps

$$\mathrm{AJ} : C^{(n_1)} \times \cdots \times C^{(n_s)} \to \mathrm{Pic}^{n_1}(C) \times \cdots \times \mathrm{Pic}^{n_s}(C),$$

where each $n_i > 2g-2$. We denote by $\mathcal{P}_i$ (the pullback of) the Poincaré line bundle on $\mathrm{Pic}^{n_i}(C) \times C$, each normalized at a point $c_i \in C$, and by ω_i the first Chern classes of the tautological bundles on $C^{(n_i)}$. Finally we denote by $\mathcal{Z}_i \subset C^{(n_i)} \times C$ the universal divisors and by $\mathcal{I}_i$ (the pullback of) their ideal sheaves, which in this case are line bundles.
We are interested in studying integrals of the form

$$\int_{C^{(n_1)}\times\cdots\times C^{(n_s)}} f, \tag{3.5.4}$$

where f is a polynomial in the Chern classes of the K-theoretic classes

$$\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om\left(\bigotimes_{i\in I}\mathcal{I}_i, \bigotimes_{j\in J}\mathcal{I}_j \otimes L_k\right),$$

L_k are line bundles on C and I, J are families of indices (possibly with repetitions).
We assume now that $n_i \gg 0$ for all $i = 1, \ldots, s$. Applying the projection formula and flat base change yields

$$\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om\left(\bigotimes_{i\in I}\mathcal{I}_i, \bigotimes_{j\in J}\mathcal{I}_j \otimes L_k\right)$$

$$= \mathbf{R}\pi_* \mathbf{R}\mathscr{H}om\left(\bigotimes_{i\in I}(\mathrm{AJ}\times\mathrm{id})^*\mathcal{P}_i^* \otimes \pi^* \mathcal{O}_{C^{(n_i)}}(-1), \bigotimes_{j\in J}(\mathrm{AJ}\times\mathrm{id})^*\mathcal{P}_j^* \otimes \pi^* \mathcal{O}_{C^{(n_j)}}(-1) \otimes L_k\right)$$

$$= \mathcal{F} \otimes \mathbf{R}\pi_* \left(\bigotimes_{i\in I}(\mathrm{AJ}\times\mathrm{id})^*\mathcal{P}_i \otimes \bigotimes_{j\in J}(\mathrm{AJ}\times\mathrm{id})^*\mathcal{P}_j^* \otimes L_k\right)$$

$$= \mathcal{F} \otimes \mathrm{AJ}^* \,\mathbf{R}\overline{\pi}_* \left(\bigotimes_{i\in I}\mathcal{P}_i \otimes \bigotimes_{j\in J}\mathcal{P}_j^* \otimes L_k\right),$$

where $\mathcal{F} = \bigotimes_{i\in I} \mathcal{O}_{C^{(n_i)}}(1) \otimes \bigotimes_{j\in J} \mathcal{O}_{C^{(n_j)}}(-1)$. The Chern classes of the last expression are a linear combination of

$$\prod_{i=1}^{s} \omega_i^{m_i} \cdot \mathrm{AJ}^* \mathrm{ch}_l \left(\mathbf{R}\overline{\pi}_* \left(\bigotimes_{i\in I}\mathcal{P}_i \otimes \bigotimes_{j\in J}\mathcal{P}_j^* \otimes L_k\right)\right),$$

where ch denotes the Chern character for certain $m_i \in \mathbb{Z}_{\geq 0}$. Integrating this class yields

$$\int_{C^{(n_1)}\times\cdots\times C^{(n_s)}} \prod_{i=1}^{s} \omega_i^{m_i} \cdot \mathrm{AJ}^* \mathrm{ch}_l \left(\mathbf{R}\overline{\pi}_* \left(\bigotimes_{i\in I}\mathcal{P}_i \otimes \bigotimes_{j\in J}\mathcal{P}_j^* \otimes L_k\right)\right)$$

$$= \int_{\mathrm{Pic}^{n_1}(C)\times\cdots\times\mathrm{Pic}^{n_s}(C)} \mathrm{AJ}_* \prod_{i=1}^{s} \omega_i^{m_i} \cdot \mathrm{ch}_l \left(\mathbf{R}\overline{\pi}_* \left(\bigotimes_{i\in I}\mathcal{P}_i \otimes \bigotimes_{j\in J}\mathcal{P}_j^* \otimes L_k\right)\right).$$

Using a standard identity [**78**, Sec. 3.1] we can express the pushforward $\mathrm{AJ}_* \omega_i^{m_i}$ in terms of Segre classes (and therefore Chern characters) of $\overline{\pi}_* \mathcal{P}_i$. These Chern characters appearing in the integral are computed by Grothendieck-Riemann-Roch

$$\mathrm{ch}\left(\mathbf{R}\overline{\pi}_* \left(\bigotimes_{i\in I}\mathcal{P}_i \otimes \bigotimes_{j\in J}\mathcal{P}_j^* \otimes L_k\right)\right) = \overline{\pi}_* \left(\prod_{i\in I}\mathrm{ch}(\mathcal{P}_i) \prod_{j\in J}\mathrm{ch}(\mathcal{P}_j^*) \cdot e^{c_1(L_k)} \cdot \mathrm{td}(T_C)\right).$$

The Chern character of the Poincaré line bundle is (cf. [**2**, pag. 335])

$$\mathrm{ch}(\mathcal{P}_i) = 1 + n_i[c_i] + \gamma_i - \theta_i[c_i] \in H^*(\mathrm{Pic}^{n_i}(C)\times C, \mathbb{Z}).$$

Here, in the decomposition

$$H^2(\mathrm{Pic}^{n_i}(C)\times C) \cong H^2(\mathrm{Pic}^{n_i}(C)) \oplus \big(H^1(\mathrm{Pic}^{n_i}(C)) \otimes H^1(C)\big) \oplus H^2(C),$$

we have

$$[c_i] \in H^2(C),$$
$$\theta_i \in H^2(\mathrm{Pic}^{n_i}(C)),$$
$$\gamma_i = [\Delta]^{1,1} \in H^1(C)\otimes H^1(C) \cong H^1(\mathrm{Pic}^{n_i}(C)) \otimes H^1(C),$$

where $\Delta \subset C \times C$ is the diagonal and θ_i is the theta divisor. All of this results in

$$\int_{C^{(n_1)}\times\cdots\times C^{(n_s)}} f = \int_{\mathrm{Pic}^{n_1}(C)\times\cdots\times\mathrm{Pic}^{n_s}(C)\times C} \tilde{f},$$

where $\tilde{f}$ is a polynomial in the classes

$$\theta_i,\ [c_i],\ \gamma_i,\ c_1(T_C),\ c_1(L_k).$$

3.5.3.2. Extension to all n In the previous section we assumed that all $n_i > 2g-2$; we explain now how to remove this assumption, following closely [**119**, Sec. 10.1].

Let $n \in \mathbb{Z}_{\geq 0}$ and $N > 2g-2$. Then $C^{(N)} \cong \mathbb{P}(\pi_*\mathcal{P})$, where $\mathcal{P}$ is the Poincaré line bundle on $\operatorname{Pic}^N(C) \times C$ normalized at $c \in C$. We can embed $C^{(n)}$ in $C^{(N)}$ as the zero section of a vector bundle

$$\begin{tikzcd} & \pi_* \mathcal{O}(\mathcal{W})|_{C^{(N)}\times(N-n)c} \arrow[d] \\ C^{(n)} \cong Z(s) \arrow[r, hook, "\iota"] & C^{(N)}, \arrow[u, bend right, "s"'] \end{tikzcd}$$

by sending $Z \mapsto Z + (N-n)c$, where $\mathcal{W} \subset C^{(N)} \times C$ is the universal divisor and $(N-n)c \subset C$ is an Artinian thickened point. Moreover, the section is regular, therefore

$$\iota_*[C^{(n)}] = e(\mathcal{G}) \cap [C^{(N)}] \in H_*(C^{(N)}),$$

where $\mathcal{G} = \pi_* \mathcal{O}(\mathcal{W})|_{C^{(N)}\times(N-n)c}$. Finally, if we denote by $\mathcal{Z} \subset C^{(n)} \times C$ the universal divisor, we have that $(\iota \times \mathrm{id})^*\mathcal{W} = \mathcal{Z}((N-n)c)$.

Recall that we are interested in the integrals (3.5.4). Choose $N_i > 2g-2$, denote by $\mathcal{I}_i'$ the universal ideal sheaves of the universal divisors $\mathcal{W}_i$ on $C^{(N_1)} \times \cdots \times C^{(N_s)} \times C$ and by π' the projection map. By base change we can write

$$\begin{aligned} &\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om\left(\bigotimes_{i\in I}\mathcal{I}_i, \bigotimes_{j\in J}\mathcal{I}_j \otimes L_k\right) = \\ &\qquad \mathbf{L}\iota^*\mathbf{R}\pi'_*\mathbf{R}\mathscr{H}om\left(\bigotimes_{i\in I}\mathcal{I}_i'(-(N_i-n_i)c_i), \bigotimes_{j\in J}\mathcal{I}_j'(-(N_j-n_j)c_j) \otimes L_k\right), \end{aligned}$$

therefore

$$\int_{C^{(n_1)}\times\cdots\times C^{(n_s)}} f = \int_{C^{(N_1)}\times\cdots\times C^{(N_s)}} f' \cdot \prod_{i=1}^{s} e(\mathcal{G}_i),$$

where each $\mathcal{G}_i = \pi_* \mathcal{O}(\mathcal{W}_i)|_{C^{(N_i)}\times(N_i-n_i)c_i}$ and f' is a polynomial in the Chern classes of

$$\mathbf{R}\pi'_*\mathbf{R}\mathscr{H}om\left(\bigotimes_{i\in I}\mathcal{I}_i', \bigotimes_{j\in J}\mathcal{I}_j' \otimes \bigotimes_{i\in I}\mathcal{O}((N_i-n_i)c_i) \otimes \bigotimes_{j\in J}\mathcal{O}(-(N_j-n_j)c_j) \otimes L_k\right).$$

The exact sequence

$$0 \to \mathcal{O}(\mathcal{W}_i - (N_i-n_i)c_i) \to \mathcal{O}(\mathcal{W}_i) \to \mathcal{O}(\mathcal{W}_i)|_{C^{(N_i)}\times(N_i-n_i)c_i} \to 0$$

yields the identity in K-theory

$$\mathcal{G}_i = \mathbf{R}\pi'_*\mathcal{I}_i'^* - \mathbf{R}\pi'_*\left(\mathcal{I}_i'^* \otimes \mathcal{O}((N_i-n_i)c_i)\right),$$

by which we conclude that we can apply the construction in Section 3.5.3.1 to express

$$\int_{C^{(n_1)}\times\cdots\times C^{(n_s)}} f = \int_{\operatorname{Pic}^{N_1}(C)\times\cdots\times\operatorname{Pic}^{N_s}(C)\times C} \tilde{f},$$

where $\tilde{f}$ is a polynomial in the classes

$$\theta_i,\ [c_i],\ \gamma_i,\ c_1(T_C),\ c_1(L_k),\ c_1(\mathcal{O}((N_i - n_i)c_i)). \tag{3.5.5}$$

Smooth projective curves of the same genus are diffeomorphic to each other, therefore $\mathrm{Pic}^n(C)$ is diffeomorphic to a g-dimensional complex torus. By the intersection theory on $\mathrm{Pic}^n(C)$ developed in [**2**, Sec. VIII.2] we immediately obtain the following result.

Proposition 3.5.3. *Let f be a polynomial in the Chern classes of the K-theory classes*

$$\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om\left(\bigotimes_{i\in I}\mathcal{I}_i, \bigotimes_{j\in J}\mathcal{I}_j\otimes L_k\right),$$

where L_k are line bundles on C and I, J are families of indices (possibly with repetitions). Then

$$\int_{C^{(n_1)}\times\cdots\times C^{(n_s)}} f = \int_{\mathrm{Pic}^{n_1}(C)\times\cdots\times\mathrm{Pic}^{n_s}(C)\times C}\tilde{f},$$

where $\tilde{f}$ is a polynomial in the classes (3.5.5). In particular, the integral is a polynomial in the genus $g = g(C)$ and the degrees of the line bundles $\{L_k\}_k$.

As a corollary, we obtain that the localized contributions on $C^{[\mathbf{n}_\lambda]}$ only depend on the Chern numbers of (C, L_1, L_2).

Corollary 3.5.4. *Let C be a genus g irreducible smooth projective curve and L_1, L_2 line bundles on C. Then the intersection numbers (3.5.1) are polynomials in $g, \deg L_1, \deg L_2$.*

Proof. Let $i : C^{[\mathbf{n}_\lambda]} \hookrightarrow A_{C,\mathbf{n}_\lambda}$ be the embedding of Theorem 3.2.7 and to ease the notation set $A_{C,\mathbf{n}_\lambda} = C^{(n_1)}\times\cdots\times C^{(n_s)}$. We claim that

$$N^{\mathrm{vir}}_{C,L_1,L_2} = i^*\tilde{N}^{\mathrm{vir}}_{C,L_1,L_2},$$

for a certain class $\tilde{N}^{\mathrm{vir}}_{C,L_1,L_2} \in K^0_{\mathbf{T}}(A_{C,\mathbf{n}_\lambda})$. In fact, $N^{\mathrm{vir}}_{C,L_1,L_2}$ is a linear combination of classes in K-theory of the form

$$\mathbf{R}\pi_*(\mathcal{O}_{C\times C^{[\mathbf{n}_\lambda]}}(\Delta)\otimes L_1^a L_2^b)\otimes \mathfrak{t}^\mu \in K^0_{\mathbf{T}}(C^{[\mathbf{n}_\lambda]}),$$

where Δ is a $\mathbb{Z}$-linear combination of the universal divisors $\mathcal{Z}_{ij}$ on $C\times C^{[\mathbf{n}_\lambda]}$, $a, b\in\mathbb{Z}$ and $\mathfrak{t}^\mu$ is a $\mathbf{T}$-character. Each of such universal divisors Δ can be expressed, in the Picard group of $C^{[\mathbf{n}_\lambda]}$, as a linear combination of the divisors $i^*\mathcal{Z}_{00}, i^*\mathcal{X}_{ij}, i^*\mathcal{Y}_{ij}$, with notation as in Section 3.2.3; applying base change proves the claim. Therefore the integral (3.5.1) can be expressed as an intersection number on the product of symmetric powers of curves $A_{C,\mathbf{n}_\lambda}$

$$\int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) = \int_{A_{C,\mathbf{n}_\lambda}} e^{\mathbf{T}}(\mathcal{E} - \tilde{N}^{\mathrm{vir}}_{C,L_1,L_2}),$$

where $\mathcal{E}$ is the vector bundle of Theorem 3.2.7. In particular, the K-theory class of $\mathcal{E} - \tilde{N}^{\mathrm{vir}}_{C,L_1,L_2}$ is a linear combination of classes of the form

$$\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om\left(\bigotimes_{i\in I}\mathcal{I}_i, \bigotimes_{j\in J}\mathcal{I}_j\otimes L_k\right)\otimes\mathfrak{t}^\mu,$$

where L_k are line bundles on C and I, J are families of indices (possibly with repetitions). By Proposition 3.5.3 this integral is a polynomial in g and the degrees of L_k. We conclude the proof by noticing that all line bundles L_k possibly occuring are a linear combination of L_1, L_2, K_C. $\square$

3.6. The leading term

We compute the leading term of the generating series of the integrals (3.5.1), which is essential for the computation of the full generating series in Theorem 3.8.1. Recall that the fixed a Young diagram λ.

Proposition 3.6.1. *Let C be a smooth projective curve of genus g and L_1, L_2 line bundles on C. Then under the anti-diagonal restriction $s_1 + s_2 = 0$ we have*

$$\int_{C^{[\mathbf{0}_\lambda]}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) = (-1)^{|\lambda|(g-1+\deg L_2)+n(\lambda)\deg L_1+n(\bar{\lambda})\deg L_2}\left(s_1^{|\lambda|}\cdot\prod_{\square\in\lambda}h(\square)\right)^{2g-2-\deg L_1-\deg L_2},$$

where $\mathbf{0}_\lambda$ is the unique reversed plane partition of size 0 and underlying Young diagram λ.

Proof. We have that $C^{[\mathbf{0}_\lambda]} \cong \mathsf{pt}$ and $[C^{[\mathbf{0}_\lambda]}]^{\mathrm{vir}} = [\mathsf{pt}] \in A_*(\mathsf{pt})$, therefore $\mathbb{E}^\vee$ is completely $\mathbf{T}$-movable and

$$\int_{C^{[\mathbf{0}_\lambda]}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) = e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) \in \mathbb{Q}(s_1, s_2),$$

where by (3.4.1), (3.4.4) we express the class in K-theory of the virtual normal bundle as

$$\begin{aligned}N^{\mathrm{vir}}_{C,L_1,L_2} = &\sum_{(i,j)\in\lambda}\left(\mathbf{R}\Gamma(L_1^{-i}L_2^{-j})\,\mathfrak{t}_1^{-i}\,\mathfrak{t}_2^{-j} - \left(\mathbf{R}\Gamma(K_C L_1^{-i-1}L_2^{-j-1})\,\mathfrak{t}_1^{-i}\,\mathfrak{t}_2^{-j}\right)^\vee\otimes\mathfrak{t}_1\,\mathfrak{t}_2\right)\\ &-\sum_{(i,j),(l,k)\in\lambda}\Bigg(\mathbf{R}\Gamma((\mathcal{O}_C - L_1\,\mathfrak{t}_1)L_1^{i-l}L_2^{j-k})\,\mathfrak{t}_1^{i-l}\,\mathfrak{t}_2^{j-k}\\ &\qquad - \left(\mathbf{R}\Gamma((\mathcal{O}_C - L_1\,\mathfrak{t}_1)K_C L_1^{i-l-1}L_2^{j-k-1})\,\mathfrak{t}_1^{i-l}\,\mathfrak{t}_2^{j-k}\right)^\vee\otimes\mathfrak{t}_1\,\mathfrak{t}_2\Bigg).\end{aligned}$$

Applying Riemann-Roch, every line bundle L on C satisfies

$$\mathbf{R}\Gamma(L) = \mathbb{C}^{\deg L+1-g} \in K^0(\mathsf{pt}) \cong \mathbb{Z},$$

therefore we can write the virtual normal bundle as

$$\begin{aligned}N^{\mathrm{vir}}_{C,L_1,L_2} = &\sum_{\mu}(\mathbb{C}^{m_\mu+1-g}\,\mathfrak{t}^\mu - \mathbb{C}^{m_\mu+g-1-\deg L_1-\deg L_2}\,\mathfrak{t}^{-\mu}\,\mathfrak{t}_1\,\mathfrak{t}_2)\\ &-\sum_{\nu}(\mathbb{C}^{m_\nu+1-g}\,\mathfrak{t}^\nu - \mathbb{C}^{m_\nu+g-1-\deg L_1-\deg L_2}\,\mathfrak{t}^{-\nu}\,\mathfrak{t}_1\,\mathfrak{t}_2),\end{aligned}\tag{3.6.1}$$

where the weights μ range among

(3.6.2)

$(-i,-j) \in \mathbb{Z}^2$ such that $(i,j) \in \lambda, (i,j) \neq (0,0)$,

$(i-l+1, j-k) \in \mathbb{Z}^2$ such that $((i-j),(l-k)) \in \lambda, (i,j) \neq ((l-1,k)(l,k+1))$, the weights ν range among

(3.6.3)

$$(i-l, j-k) \in \mathbb{Z}^2 \text{ such that } ((i-j),(l-k)) \in \lambda, (i,j) \neq ((l,k),(l+1,k+1)),$$

and, for a weight $\mu = (\mu_1, \mu_2)$, we set $m_\mu = \mu_1 \deg L_1 + \mu_2 \deg L_2$. In particular, the weights $1, \mathfrak{t}_1\,\mathfrak{t}_2$ do not appear in (3.6.1), as the virtual tangent bundle of the $\mathbf{T}$-fixed locus has rank 0 and by the explicit description of the weight space of $\mathfrak{t}_1\,\mathfrak{t}_2$ in Example 3.4.2. For every weight μ, we compute

$$e^{\mathbf{T}}(\mathbb{C}^{m_\mu+1-g}\,\mathfrak{t}^\mu - \mathbb{C}^{m_\mu+g-1-\deg L_1-\deg L_2}\,\mathfrak{t}^{-\mu}\,\mathfrak{t}_1\,\mathfrak{t}_2) = \frac{(\mu\cdot s)^{m_\mu+1-g}}{(-\mu\cdot s+s_1+s_2)^{m_\mu+g-1-\deg L_1-\deg L_2}},$$

where $s=(s_1,s_2)$ and $\mu\cdot s$ is the standard inner product. With a simple manipulation we end up with

$$\begin{aligned} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) &= (-1)^{\sum_{\nu\neq(a,a)} m_\nu + \sum_{\mu\neq(b,b)} m_\mu + (1-g+\deg L_1+\deg L_2)(\#\{\,\nu\neq(a,a)\,\}-\#\{\,\mu\neq(b,b)\,\})} \\ &\quad\cdot \frac{\prod_{\nu\neq(a,a)}(\nu\cdot s)^{m_\nu+1-g}(\nu\cdot s-s_1-s_2)^{-m_\nu+1-g+\deg L_1+\deg L_2}}{\prod_{\mu\neq(b,b)}(\mu\cdot s)^{m_\mu+1-g}(\mu\cdot s-s_1-s_2)^{-m_\mu+1-g+\deg L_1+\deg L_2}} \\ &\cdot \frac{\prod_{\nu=(a,a)} a^{a(\deg L_1+\deg L_2)+1-g}(1-a)^{(1-a)(\deg L_1+\deg L_2)+1-g}(s_1+s_2)^{2-2g+\deg L_1+\deg L_2}}{\prod_{\mu=(b,b)} b^{b(\deg L_1+\deg L_2)+1-g}(1-b)^{(1-b)(\deg L_1+\deg L_2)+1-g}(s_1+s_2)^{2-2g+\deg L_1+\deg L_2}}. \end{aligned}$$

Following the proof of Lemma 3.10.1, we see that for any weight $\nu=(a,a)$ (with $a\in\mathbb{Z}$) there is either a weight $\mu=(a,a)$ or $\mu=(1-a,1-a)$ (and viceversa). This implies that

$$\#\{\,\nu=(a,a)\,\} = \#\{\,\mu=(b,b)\,\}$$

and

$$\frac{\prod_{\nu=(a,a)} a^{a(\deg L_1+\deg L_2)+1-g}(1-a)^{(1-a)(\deg L_1+\deg L_2)+1-g}(s_1+s_2)^{2-2g+\deg L_1+\deg L_2}}{\prod_{\mu=(b,b)} b^{b(\deg L_1+\deg L_2)+1-g}(1-b)^{(1-b)(\deg L_1+\deg L_2)+1-g}(s_1+s_2)^{2-2g+\deg L_1+\deg L_2}} = 1.$$

In particular, the anti-diagonal restriction $s_1+s_2=0$ is well-defined; we get

$$\begin{aligned} &\left.\frac{\prod_{\nu\neq(a,a)}(\nu\cdot s)^{m_\nu+1-g}(\nu\cdot s-s_1-s_2)^{-m_\nu+1-g+\deg L_1+\deg L_2}}{\prod_{\mu\neq(b,b)}(\mu\cdot s)^{m_\mu+1-g}(\mu\cdot s-s_1-s_2)^{-m_\mu+1-g+\deg L_1+\deg L_2}}\right|_{s_1+s_2=0} \\ &\qquad\qquad = \left(s_1^{\#\nu-\#\mu}\cdot\frac{\prod_{\nu=(\nu_1,\nu_2)}(\nu_1-\nu_2)}{\prod_{\mu=(\mu_1,\mu_2)}(\mu_1-\mu_2)}\right)^{2-2g+\deg L_1+\deg L_2} \end{aligned}$$

where the product is over all $\mu,\nu\neq(a,a)$. Moreover, it is immediate to check that $\#\mu-\#\nu=|\lambda|$. By Lemma 3.10.2, we conclude that the last expression equals to

$$\sigma(\lambda)^{\deg L_1+\deg L_2}\cdot\left(s_1^{-|\lambda|}\cdot\prod_{\Box\in\lambda}h(\Box)^{-1}\right)^{2-2g+\deg L_1+\deg L_2},$$

where $\sigma(\lambda)$ is defined in Lemma 3.10.2. We are only left with a sign computation; we conclude by Lemma 3.10.3

$$\begin{aligned} (-1)^{\sum_{\nu\neq(a,a)} m_\nu+\sum_{\mu\neq(b,b)} m_\mu+(1-g+k_1+k_2)|\lambda|}\sigma(\lambda)^{k_1+k_2} &= (-1)^{\rho(\lambda)+|\lambda|(k_1+k_2+g-1)}\sigma(\lambda)^{k_1+k_2} \\ &= (-1)^{|\lambda|(k_2+g-1)+n(\lambda)k_1+n(\overline{\lambda})k_2}, \end{aligned}$$

where $\rho(\lambda)$ was defined is Lemma 3.10.3 and we set $k_i = \deg L_i$. □

3.7. Toric computations

3.7.1. Torus action Let U_0, U_∞ be the standard open cover of $\mathbb{P}^1$. The torus $\mathbb{C}^*$ acts on the coordinate functions of $\mathbb{P}^1$ as $t \cdot x = tx$ (resp. $t \cdot x = t^{-1}x$) in the chart U_0 (resp. U_∞). The $\mathbb{C}^*$-representation of the tangent space at the $\mathbb{C}^*$-fixed points of $\mathbb{P}^1$ is

$$\begin{aligned} T_{\mathbb{P}^1,0} &= \mathbb{C} \otimes \mathfrak{t}^{-1}, \\ T_{\mathbb{P}^1,\infty} &= \mathbb{C} \otimes \mathfrak{t}\,. \end{aligned}$$

We prove some identities of $\mathbb{C}^*$-representations that will be useful later in this section.

Lemma 3.7.1. *Let $Z = Z_0 \sqcup Z_\infty \subset \mathbb{P}^1$ be a closed subscheme, where Z_0 (resp. Z_∞) is a closed subscheme of length n_0 (resp. n_∞) supported on 0 (resp. ∞). For any $a \in \mathbb{Z}$, we have the following identities in $K^0_{\mathbb{C}^*}(\mathsf{pt})$*

$$\begin{aligned} \mathbf{R}\Gamma(K^a_{\mathbb{P}^1}) &= \begin{cases} \sum_{i=a}^{-a} \mathfrak{t}^i & a \leq 0, \\ -\sum_{i=-a+1}^{a-1} \mathfrak{t}^i & a \geq 1, \end{cases} \\ \mathbf{R}\Gamma(\mathcal{O}_Z(Z)) &= \sum_{i=1}^{n_0} \mathfrak{t}^{-i} + \sum_{i=1}^{n_\infty} \mathfrak{t}^i, \\ \mathbf{R}\Gamma(\mathcal{O}_Z \otimes K^a_{\mathbb{P}^1}) &= \sum_{i=0}^{n_0-1} \mathfrak{t}^{a+i} + \sum_{i=0}^{n_\infty-1} \mathfrak{t}^{-a-i}, \\ \mathbf{R}\operatorname{Hom}(\mathcal{O}_Z, K^a_{\mathbb{P}^1}) &= -\sum_{i=0}^{n_0-1} \mathfrak{t}^{a-i-1} - \sum_{i=0}^{n_\infty-1} \mathfrak{t}^{-a+i+1}\,. \end{aligned}$$

Proof. We have that

$$\begin{aligned} \mathbf{R}\Gamma(K^a_{\mathbb{P}^1}) &= \chi(\mathbb{P}^1, K^a_{\mathbb{P}^1}) \\ &= \frac{\mathfrak{t}^a}{1-\mathfrak{t}} - \frac{\mathfrak{t}^{-a}}{1-\mathfrak{t}^{-1}} \\ &= \begin{cases} \sum_{i=a}^{-a} \mathfrak{t}^i & a \leq 0, \\ -\sum_{i=-a+1}^{a-1} \mathfrak{t}^i & a \geq 1, \end{cases} \end{aligned}$$

where in the second line we applied the classical K-theoretic localization [**179**] on $\mathbb{P}^1$. Secondly, we have

$$\begin{aligned} \mathbf{R}\Gamma(\mathcal{O}_Z(Z)) &= H^0(\mathbb{P}^1, \mathcal{O}_Z(Z)) \\ &= H^0(U_0, \mathcal{O}_{Z_0}(Z_0)|_{U_0}) + H^0(U_\infty, \mathcal{O}_{Z_\infty}(Z_\infty)|_{U_\infty}) \\ &= H^0(U_0, \mathcal{O}_{Z_0} \otimes K_{\mathbb{P}^1}|_{U_0})^* + H^0(U_\infty, \mathcal{O}_{Z_\infty} \otimes K_{\mathbb{P}^1}|_{U_\infty})^* \\ &= \sum_{i=1}^{n_0} \mathfrak{t}^{-i} + \sum_{i=1}^{n_\infty} \mathfrak{t}^i, \end{aligned}$$

where in the second line we used Čech cohomology and in the third line we used [**149**, Ex. 3.4.5].

Thirdly, by applying Čech cohomology as before we have

$$\begin{aligned}\mathbf{R}\Gamma(\mathcal{O}_Z \otimes K_{\mathbb{P}^1}^a) &= H^0(U_0, \mathcal{O}_{Z_0} \otimes K_{\mathbb{P}^1}^a|_{U_0}) + H^0(U_\infty, \mathcal{O}_{Z_\infty} \otimes K_{\mathbb{P}^1}^a|_{U_\infty}) \\ &= \sum_{i=0}^{n_0-1} \mathfrak{t}^{a+i} + \sum_{i=0}^{n_\infty-1} \mathfrak{t}^{-a-i}.\end{aligned}$$

Finally, combining Serre duality and the previous result yields

$$\begin{aligned}\mathbf{R}\operatorname{Hom}(\mathcal{O}_Z, K_{\mathbb{P}^1}^a) &= -\mathbf{R}\Gamma(\mathcal{O}_Z \otimes K_{\mathbb{P}^1}^{1-a})^* \\ &= -\sum_{i=0}^{n_0-1} \mathfrak{t}^{a-i-1} - \sum_{i=0}^{n_\infty-1} \mathfrak{t}^{-a+i+1}.\end{aligned}$$

□

The $\mathbb{C}^*$-action on $\mathbb{P}^1$ naturally lifts to a $\mathbb{C}^*$-action on the Hilbert scheme of point $\mathbb{P}^{1[n]}$, whose $\mathbb{C}^*$-fixed locus consists of length n closed subschemes $Z \subset \mathbb{P}^1$ supported on $0, \infty$. Therefore, there is an induced $\mathbb{C}^*$-action on $A_{\mathbb{P}^1, \mathbf{n}_\lambda}$, whose $\mathbb{C}^*$-fixed locus is 0-dimensional and reduced. This $\mathbb{C}^*$-action restricts to a $\mathbb{C}^*$-action on $\mathbb{P}^{1[\mathbf{n}_\lambda]}$, whose $\mathbb{C}^*$-fixed locus is necessarily 0-dimensional and reduced. Moreover, the perfect obstruction theory (3.2.2) is naturally $\mathbb{C}^*$-equivariant, as all the ingredients of Theorem 3.2.7 are.

Proposition 3.7.2. *Let $\underline{Z} \in \mathbb{P}^{1[\mathbf{n}_\lambda], \mathbb{C}^*}$ be a $\mathbb{C}^*$-fixed point. Then $T^{\mathrm{vir}}_{\mathbb{P}^{1[\mathbf{n}_\lambda]}, \underline{Z}}$ is completely $\mathbb{C}^*$-movable. In particular, the induced perfect obstruction theory on $\mathbb{P}^{1[\mathbf{n}_\lambda], \mathbb{C}^*}$ is trivial.*

Proof. We need to show that the class in K-theory

$$T_{A_{C,\mathbf{n}_\lambda}}|_{\underline{Z}} - \mathcal{E}|_{\underline{Z}} \in K^0_{\mathbb{C}^*}(\mathsf{pt})$$

does not have $\mathbb{C}^*$-fixed part. Recall that we have an identity in $K^0_{\mathbb{C}^*}(\mathsf{pt})$

$$T_{A_{C,\mathbf{n}_\lambda}}|_{\underline{Z}} = \mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{Z_{00}}(Z_{00})) + \sum_{\substack{(i,j)\in\lambda \\ i\geq 1}} \mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{X_{ij}}(X_{ij})) + \sum_{\substack{(i,j)\in\lambda \\ j\geq 1}} \mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{Y_{ij}}(Y_{ij})),$$

where for simplicity we denoted by $X_{ij} = Z_{ij} - Z_{i-1,j}$ and by $Y_{ij} = Z_{ij} - Z_{i,j-1}$. Moreover we have

$$\mathcal{E}|_{\underline{Z}} = \sum_{\substack{(i,j)\in\lambda \\ i,j\geq 1}} \mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{W_{ij}}(W_{ij})) \in K^0_{\mathbb{C}^*}(\mathsf{pt}),$$

where for simplicity we denoted by $W_{ij} = Z_{ij} - Z_{i-1,j-1}$. Therefore the virtual tangent space is a sum of classes of the form $\mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_Z(Z))$, with $Z \subset \mathbb{P}^1$ a closed subscheme, which is entirely $\mathbb{C}^*$-movable by the description in Lemma 3.7.1. □

3.7.2. Case I: Calabi-Yau We compute the integral (3.5.1) for $C = \mathbb{P}^1$ in the case of $L_1 \otimes L_2 = K_{\mathbb{P}^1}$, showing that it coincides (up to a sign) with the topological Euler characteristic $e\left(\mathbb{P}^{1[\mathbf{n}_\lambda]}\right)$.

Theorem 3.7.3. *Let L_1, L_2 be line bundles on $\mathbb{P}^1$ such that $L_1 \otimes L_2 = K_{\mathbb{P}^1}$. For any reversed plane partition $\mathbf{n}_\lambda$, we have*

$$\left.\left(\int_{\left[\mathbb{P}^{1[\mathbf{n}_\lambda]}\right]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{\mathbb{P}^1,L_1,L_2})\right)\right|_{s_1+s_2=0} = (-1)^{\deg L_1(c_\lambda+|\lambda|)+|\lambda|+|\mathbf{n}_\lambda|}\cdot e\left(\mathbb{P}^{1[\mathbf{n}_\lambda]}\right).$$

Proof. By Graber-Pandharipande [85], there is an induced perfect obstruction theory and virtual fundamental class on the $\mathbb{C}^*$-fixed locus $\mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}$, both trivial by Proposition 3.7.2. By Proposition 3.4.3

$$N^{\mathrm{vir}}_{\mathbb{P}^1,L_1,L_2} = -T^{\mathrm{vir},\vee}_{\mathbb{P}^{1[\mathbf{n}_\lambda]}}\otimes \mathfrak{t}_1\,\mathfrak{t}_2 + \Omega - \Omega^\vee\otimes\mathfrak{t}_1\,\mathfrak{t}_2 \in K^0_{\mathbf{T}}\left(\mathbb{P}^{1[\mathbf{n}_\lambda]}\right),$$

and applying the virtual localization formula with respect to the $\mathbb{C}^*$-action yields

$$\begin{aligned}\int_{\left[\mathbb{P}^{1[\mathbf{n}_\lambda]}\right]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{\mathbb{P}^1,L_1,L_2}) &= \left.\left(\sum_{\underline{Z}\in\mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}} \frac{e^{\mathbf{T}\times\mathbb{C}^*}(T^{\mathrm{vir},\vee}_{\mathbb{P}^{1[\mathbf{n}_\lambda]},\underline{Z}}\otimes\mathfrak{t}_1\,\mathfrak{t}_2)}{e^{\mathbf{T}\times\mathbb{C}^*}(T^{\mathrm{vir}}_{\mathbb{P}^{1[\mathbf{n}_\lambda]},\underline{Z}})}\cdot\frac{e^{\mathbf{T}\times\mathbb{C}^*}(\Omega|_{\underline{Z}}^\vee\otimes\mathfrak{t}_1\,\mathfrak{t}_2)}{e^{\mathbf{T}\times\mathbb{C}^*}(\Omega|_{\underline{Z}})}\right)\right|_{s_3=0}\\ &= \left.\left(\sum_{\underline{Z}\in\mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}} \frac{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}_{\underline{Z}}^\vee\otimes\mathfrak{t}_1\,\mathfrak{t}_2)}{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}_{\underline{Z}})}\right)\right|_{s_3=0},\end{aligned}$$

where s_3 is the generator of the equivariant cohomology $H^*_{\mathbb{C}^*}(\mathsf{pt})$ and we denoted by

$$\mathsf{V}_{\underline{Z}} = T^{\mathrm{vir}}_{\mathbb{P}^{1[\mathbf{n}_\lambda]},\underline{Z}} + \Omega|_{\underline{Z}} \in K^0_{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt})$$

the $\mathbb{C}^*$-equivariant lift of the K-theoretic class $T^{\mathrm{vir}}_{\mathbb{P}^{1[\mathbf{n}_\lambda]}+\underline{Z}}, \Omega|_{\underline{Z}} \in K^0_{\mathbf{T}}(\mathsf{pt})$. Under the Calabi-Yau restriction $s_1+s_2=0$, we have by Lemma 3.7.6

$$\left.\left(\frac{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}_{\underline{Z}}^\vee\otimes\mathfrak{t}_1\,\mathfrak{t}_2)}{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}_{\underline{Z}})}\right)\right|_{s_1+s_2=0} = (-1)^{\mathrm{rk}\,\mathsf{V}_{\underline{Z}}}.$$

Moreover, by Lemma 3.7.5

$$\mathrm{rk}\,\mathsf{V}_{\underline{Z}} = \deg L_1(c_\lambda+|\lambda|)+|\lambda|+|\mathbf{n}_\lambda| \mod 2.$$

Therefore, we conclude that

$$\begin{aligned}\left.\left(\sum_{\underline{Z}\in\mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}} \frac{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}_{\underline{Z}}^\vee\otimes\mathfrak{t}_1\,\mathfrak{t}_2)}{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}_{\underline{Z}})}\right)\right|_{s_+s_2=0} &= \sum_{\underline{Z}\in\mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}}(-1)^{\mathrm{rk}\,\mathsf{V}_{\underline{Z}}}\\ &= (-1)^{\deg L_1(c_\lambda+|\lambda|)+|\lambda|+|\mathbf{n}_\lambda|}\cdot e\left(\mathbb{P}^{1[\mathbf{n}_\lambda]}\right),\end{aligned}$$

as the Euler characteristic of a $\mathbb{C}^*$-scheme coincides with the Euler characteristic of its $\mathbb{C}^*$-fixed locus (in our case the number of fixed points). □

Exploiting the close formula for the generating series of the topological Euler characteristic proved in Theorem 3.2.10, we derive the following close expression in the Calabi-Yau case.

Corollary 3.7.4. *Let L_1, L_2 be line bundles on $\mathbb{P}^1$ such that $L_1\otimes L_2 = K_{\mathbb{P}^1}$ and λ be a Young diagram. We have*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|}\cdot\left.\left(\int_{\left[\mathbb{P}^{1[\mathbf{n}_\lambda]}\right]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2})\right)\right|_{s_1+s_2=0} = (-1)^{\deg L_1(c_\lambda+|\lambda|)+|\lambda|}\cdot\prod_{\Box\in\lambda}(1-(-q)^{h(\Box)})^{-2},$$

where the sum is over all reversed plane partition $\mathbf{n}_\lambda$.

We devote the remainder of this section to prove the technical lemmas we used in Theorem 3.7.3.

Lemma 3.7.5. *Let C be a smooth projective curve of genus g, L_1, L_2 be line bundles on C such that $L_1 \otimes L_2 = K_C$ and let $\Omega \in K^0_{\mathbf{T}}(C^{[\mathbf{n}_\lambda]})$ be the K-theory class of Remark 3.4.4. Then*

$$\operatorname{rk}\left(T^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}} + \Omega\right) = \deg L_1(c_\lambda + |\lambda|) + (1-g)|\lambda| + |\mathbf{n}_\lambda| \mod 2.$$

Proof. Let $\underline{Z} \in C^{[\mathbf{n}_\lambda]}$. Then

$$\begin{aligned}\operatorname{rk} T^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}} &= \operatorname{rk} T^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}}|_{\underline{Z}} \\ &= n_{00} + \sum_{\substack{(i,j)\in\lambda \\ i\geq 1}} (n_{ij} - n_{i-1,j}) + \sum_{\substack{(i,j)\in\lambda \\ j\geq 1}} (n_{ij} - n_{i,j-1}) - \sum_{\substack{(i,j)\in\lambda \\ i,j\geq 1}} (n_{ij} - n_{i-1,j-1})\end{aligned}$$

For any line bundle L on C and Δ a $\mathbb{Z}$-linear combination of the universal divisors $\mathcal{Z}_{ij}$ on $C \times C^{[\mathbf{n}_\lambda]}$, Riemann-Roch yields

$$\begin{aligned}\operatorname{rk} \mathbf{R}\pi_*(\mathcal{O}(\Delta)\otimes L) &= \chi(C, L \otimes \mathcal{O}(\Delta|_{\underline{Z}})) \\ &= \deg L + \deg \Delta|_{\underline{Z}} + 1 - g.\end{aligned}$$

Since Ω is a sum of K-theoretic classes of the form $\mathbf{R}\pi_*(\mathcal{O}(\Delta)\otimes L)$, for suitable L, Δ, we have

$$\begin{aligned}\operatorname{rk}\Omega = &\sum_{\substack{(i,j)\in\lambda \\ (i,j)\neq(0,0)}} (n_{ij} - i\cdot \deg L_1 - j\cdot(2g-2-\deg L_1) + 1 - g) \\ &- \sum_{\substack{(i,j),(l,k)\in\lambda \\ (i,j)\neq(l,k) \\ (i,j)\neq(l+1,k+1)}} (n_{lk} - n_{ij} - (i-l)\deg L_1 - (j-k)(2g-2-\deg L_1) + 1 - g) \\ &+ \sum_{\substack{(i,j),(l,k)\in\lambda \\ (i,j)\neq(l-1,k) \\ (i,j)\neq(l,k+1)}} (n_{lk} - n_{ij} - (i-l+1)\deg L_1 - (j-k)(2g-2-\deg L_1) + 1 - g).\end{aligned}$$

Denote by $\equiv$ the congruence modulo 2. We have

$$n_{00} - \sum_{\substack{(i,j)\in\lambda \\ (i,j)\neq(0,0)}} n_{ij} \equiv |\mathbf{n}_\lambda|.$$

Denote by V, E, Q respectively the number of vertices, edges and squares of the graph associated to λ as in Lemma 3.2.1. We have

$$\begin{aligned}\sum_{\substack{(i,j)\in\lambda \\ (i,j)\neq(0,0)}} 1 - \sum_{\substack{(i,j),(l,k)\in\lambda \\ (i,j)\neq(l,k) \\ (i,j)\neq(l+1,k+1)}} 1 + \sum_{\substack{(i,j),(l,k)\in\lambda \\ (i,j)\neq(l-1,k) \\ (i,j)\neq(l,k+1)}} 1 &= V - 1 - (|\lambda|^2 - V - Q) + (|\lambda|^2 - E) \\ &= |\lambda|,\end{aligned}$$

where in the last line we used Lemma 3.2.1. The coefficient of $\deg L_1$ in $\operatorname{rk}\Omega$ is

$$\begin{aligned}
&\sum_{\substack{(i,j)\in\lambda\\(i,j)\neq(0,0)}}(i-j)-\sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)\neq(l,k)\\(i,j)\neq(l+1,k+1)}}(i-j-l+k)+\sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)\neq(l-1,k)\\(i,j)\neq(l,k+1)}}(i-j+l-k+1)\\
&\equiv\sum_{(i,j)\in\lambda}(i+j)+\sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)=(l,k)\\(i,j)=(l+1,k+1)}}(i+j+l+k)+\sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)=(l-1,k)\\(i,j)=(l,k+1)}}(i+j+l+k)+\sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)\neq(l-1,k)\\(i,j)\neq(l,k+1)}}1\\
&\equiv c_\lambda+2E+|\lambda|\\
&\equiv c_\lambda+|\lambda|.
\end{aligned}$$

Finally

$$\begin{aligned}
&\sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)\neq(l,k)\\(i,j)\neq(l+1,k+1)}}(n_{lk}-n_{ij})+\sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)\neq(l-1,k)\\(i,j)\neq(l,k+1)}}(n_{lk}-n_{ij})\\
&\qquad+\sum_{\substack{(i,j)\in\lambda\\i\geq1}}(n_{ij}-n_{i-1,j})+\sum_{\substack{(i,j)\in\lambda\\j\geq1}}(n_{ij}-n_{i,j-1})-\sum_{\substack{(i,j)\in\lambda\\i,j\geq1}}(n_{ij}-n_{i-1,j-1})\equiv0.
\end{aligned}$$

Combining all these identities together, we conclude that

$$\operatorname{rk}\left(T^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}}+\Omega\right)=\deg L_1(c_\lambda+|\lambda|)+|\lambda|(1-g)+|\mathbf{n}_\lambda|\mod 2.$$

□

Lemma 3.7.6. *Let $\underline{Z}\in\mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}$ be a $\mathbb{C}^*$-fixed point and set*

$$\mathsf{V}_{\underline{Z}}=T^{\mathrm{vir}}_{\mathbb{P}^{1[\mathbf{n}_\lambda]}}|_{\underline{Z}}+\Omega|_{\underline{Z}}\in K^0_{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt}),$$

where $\Omega\in K^0_{\mathbf{T}}(\mathbb{P}^{1[\mathbf{n}_\lambda]})$ is as in Remark 3.4.4. Then we have

$$\left.\left(\frac{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}^\vee_{\underline{Z}}\otimes\mathfrak{t}_1\,\mathfrak{t}_2)}{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}_{\underline{Z}})}\right)\right|_{s_1+s_2=0}=(-1)^{\operatorname{rk}\mathsf{V}_{\underline{Z}}}.$$

Proof. Denote by $\mathsf{V}^{CY}_{\underline{Z}}$ the sub-representation of $\mathsf{V}_{\underline{Z}}$ consisting of weight spaces corresponding to the characters $(\mathfrak{t}_1\,\mathfrak{t}_2)^a$, for all $a\in\mathbb{Z}$, with respect to the $\mathbf{T}\times\mathbb{C}^*$-action. We claim that $\mathsf{V}^{CY}_{\underline{Z}}$ is of the form

$$\mathsf{V}^{CY}_{\underline{Z}}=A_{\underline{Z}}+A^\vee_{\underline{Z}}\otimes\mathfrak{t}_1\,\mathfrak{t}_2,$$

for a suitable $A_{\underline{Z}}\in K^0_{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt})$.
Step I: Assuming the claim, set

$$\tilde{\mathsf{V}}_{\underline{Z}}=\mathsf{V}_{\underline{Z}}-\mathsf{V}^{CY}_{\underline{Z}}$$

and $\tilde{\mathsf{V}}_{\underline{Z}}=\sum_\mu\mathfrak{t}^\mu-\sum_\nu\mathfrak{t}^\nu\in K^0_{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt})$, where none of the characters $\mathfrak{t}^\mu,\mathfrak{t}^\nu$ is a power of $\mathfrak{t}_1\,\mathfrak{t}_2$. Write $e^{\mathbf{T}\times\mathbb{C}^*}(\mathfrak{t}^\mu)=\mu\cdot s$, where $s=(s_1,s_2,s_3)$. Then we conclude that

$$\left.\left(\frac{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}^\vee_{\underline{Z}}\otimes\mathfrak{t}_1\,\mathfrak{t}_2)}{e^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{V}_{\underline{Z}})}\right)\right|_{s_1+s_2=0}=\left.\left(\frac{e^{\mathbf{T}\times\mathbb{C}^*}(\tilde{\mathsf{V}}^\vee_{\underline{Z}}\otimes\mathfrak{t}_1\,\mathfrak{t}_2)}{e^{\mathbf{T}\times\mathbb{C}^*}(\tilde{\mathsf{V}}_{\underline{Z}})}\right)\right|_{s_1+s_2=0}$$

$$= \left(\prod_{\mu} \frac{-\mu \cdot s + s_1 + s_2}{\mu \cdot s} \cdot \prod_{\nu} \frac{\nu \cdot s}{-\nu \cdot s + s_1 + s_2} \right) \Bigg|_{s_1 + s_2 = 0}$$
$$= (-1)^{\operatorname{rk} \mathsf{V}_{\underline{Z}}},$$

where we used that no $\mu \cdot s, \nu \cdot s$ is a multiple of $s_1 + s_2$ and that $\operatorname{rk} \mathsf{V}_{\underline{Z}} = \operatorname{rk} \tilde{\mathsf{V}}_{\underline{Z}} \mod 2$.
Step II: We prove now our claim on $\mathsf{V}^{CY}_{\underline{Z}}$. Firstly, by Proposition 3.7.2 $T^{\mathrm{vir}}_{\mathbb{P}^{1[\mathbf{n}_\lambda]}, \underline{Z}}$ is $\mathbb{C}^*$-movable, which implies that there are no weight spaces corresponding to a power of $\mathfrak{t}_1 \mathfrak{t}_2$.
It is clear that the $\mathbf{T} \times \mathbb{C}^*$-weight spaces of $\Omega|_{\underline{Z}}$ relative to the characters $(\mathfrak{t}_1 \mathfrak{t}_2)^a$ are given by $\Omega'|_{\underline{Z}}$, where

$$\Omega' = \sum_{\substack{(a,a) \in \lambda \\ a \neq 0}} \mathbf{R}\pi_* \left(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^{1[\mathbf{n}_\lambda]}}(\mathcal{Z}_{aa}) \otimes K_{\mathbb{P}^1}^{-a} \right) (\mathfrak{t}_1 \mathfrak{t}_2)^{-a}$$
$$- \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a, k+a) \\ a \neq 0,1}} \mathbf{R}\pi_* \left(\mathcal{O}(\Delta_{ij;lk}) \otimes K_{\mathbb{P}^1}^{a} \right) (\mathfrak{t}_1 \mathfrak{t}_2)^a + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a-1, k+a) \\ a \neq 0,1}} \mathbf{R}\pi_* \left(\mathcal{O}(\Delta_{ij;lk}) \otimes K_{\mathbb{P}^1}^{a} \right) (\mathfrak{t}_1 \mathfrak{t}_2)^a,$$

as we just considered the weight spaces of $(\mathfrak{t}_1 \mathfrak{t}_2)^a$ in Ω of Remark 3.4.4. We notice that $\Omega'|_{\underline{Z}}$ is a sum of $\mathbf{T} \times \mathbb{C}^*$-representations of the form

$$\mathbf{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(Z_a) \otimes K_{\mathbb{P}^1}^{-a}) \otimes (\mathfrak{t}_1 \mathfrak{t}_2)^{-a}, \text{ if } a \geq 1, \text{ or}$$
$$\mathbf{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(-Z_a) \otimes K_{\mathbb{P}^1}^{a}) \otimes (\mathfrak{t}_1 \mathfrak{t}_2)^{a}, \text{ if } a \geq 2,$$

where $Z_a \subset \mathbb{P}^1$ are effective divisors. We have the following identities of $\mathbb{C}^*$-representations

$$\mathbf{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(Z_a) \otimes K_{\mathbb{P}^1}^{-a}) = \mathbf{R}\Gamma(K_{\mathbb{P}^1}^{-a}) - \mathbf{R}\operatorname{Hom}(\mathcal{O}_{Z_a}, K_{\mathbb{P}^1}^{-a}), \quad a \geq 1,$$
$$\mathbf{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(-Z_a) \otimes K_{\mathbb{P}^1}^{a}) = \mathbf{R}\Gamma(K_{\mathbb{P}^1}^{a}) - \mathbf{R}\Gamma(\mathcal{O}_{Z_a} \otimes K_{\mathbb{P}^1}^{a}), \quad a \geq 2.$$

By Lemma 3.7.1, their $\mathbb{C}^*$-fixed part is

$$(\mathbf{R}\Gamma(K_{\mathbb{P}^1}^{a}))^{\mathsf{fix}} = \begin{cases} 1 & a \leq -1, \\ -1 & a \geq 2, \end{cases}$$
$$(\mathbf{R}\operatorname{Hom}(\mathcal{O}_{Z_a}, K_{\mathbb{P}^1}^{-a}))^{\mathsf{fix}} = 0, \quad a \geq 1,$$
$$(\mathbf{R}\Gamma(\mathcal{O}_{Z_a} \otimes K_{\mathbb{P}^1}^{a}))^{\mathsf{fix}} = 0, \quad a \geq 2.$$

Combining everything together, we conclude that

$$\mathsf{V}^{CY}_{\underline{Z}} = \sum_{\substack{(a,a) \in \lambda \\ a \neq 0}} (\mathfrak{t}_1 \mathfrak{t}_2)^{-a} + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a, k+a) \\ a \neq 0,1}} \operatorname{sgn}(a) (\mathfrak{t}_1 \mathfrak{t}_2)^a - \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a-1, k+a) \\ a \neq 0,1}} \operatorname{sgn}(a) (\mathfrak{t}_1 \mathfrak{t}_2)^a,$$

where sgn is the usual sign function. Our claim follows by Lemma 3.10.1. □

3.7.3. Case II: trivial vector bundle If $L_1 = L_2 = \mathcal{O}_{\mathbb{P}^1}$, the integral (3.5.1) amounts to the leading term computation of Section 3.6 and a vanishing result, which relies on a combinatorial fact about the topological vertex formalism for stable pairs developed in [**155**].

Proposition 3.7.7. *Under the anti-diagonal restriction $s_1+s_2=0$ we have an identity*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|}\int_{[\mathbb{P}^{1[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{\mathbb{P}^1,\mathcal{O},\mathcal{O}})\Big|_{s_1+s_2=0} = (-s_1^2)^{-|\lambda|}\cdot\prod_{\square\in\lambda} h(\square)^{-2}.$$

Proof. The leading term is computed in Proposition 3.6.1, therefore we just need to prove that the integral vanishes for $|\mathbf{n}_\lambda|>0$. We apply Graber-Pandharipande virtual localization [85] with respect to the $\mathbb{C}^*$-action on $\mathbb{P}^{1[\mathbf{n}_\lambda]}$

$$\int_{[\mathbb{P}^{1[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{\mathbb{P}^1,\mathcal{O},\mathcal{O}}) = \left(\sum_{\underline{Z}\in\mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}} e^{\mathbf{T}\times\mathbb{C}^*}(-T^{\mathrm{vir}}_{\mathbb{P}^{1[\mathbf{n}_\lambda]},\underline{Z}} - N^{\mathrm{vir}}_{\mathbb{P}^1,\mathcal{O},\mathcal{O},\underline{Z}})\right)\Bigg|_{s_3=0},$$

where s_3 is the generator of $H^*_{\mathbb{C}^*}(\mathsf{pt})$. The $\mathbb{C}^*$-action on $\mathbb{P}^{1[\mathbf{n}_\lambda]}$ is just the restriction of the natural $\mathbf{T}\times\mathbb{C}^*$-action on $P_n(\mathbb{P}^1\times\mathbb{C}^2, d[\mathbb{P}^1])$, where $d=|\lambda|$ and $n=|\lambda|+|\mathbf{n}_\lambda|$. This means that, as $\mathbf{T}\times\mathbb{C}^*$-representation,

$$\mathbb{E}^\vee|_{\underline{Z}} = T^{\mathrm{vir}}_{\mathbb{P}^{1[\mathbf{n}_\lambda]},\underline{Z}} + N^{\mathrm{vir}}_{\mathbb{P}^1,\mathcal{O},\mathcal{O},\underline{Z}} \in K^0_{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt})$$

can be described via the topological vertex formalism of Pandharipande-Thomas [155, Thm. 3], which states

$$\mathbb{E}^\vee|_{\underline{Z}} = \mathsf{V}_{0,\underline{Z}} + \mathsf{V}_{\infty,\underline{Z}} + \mathsf{E}_{\underline{Z}} \in K^0_{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt}),$$

where $\mathsf{V}_{0,\underline{Z}}, \mathsf{V}_{\infty,\underline{Z}}$ are the vertex terms corresponding to the two toric charts of $\mathbb{P}^1\times\mathbb{C}^2$ and $\mathsf{E}_{\underline{Z}}$ is the edge term. By [130, Lemma 22] we have that $e^{\mathbf{T}\times\mathbb{C}^*}(-\mathsf{V}_{0,\underline{Z}}-\mathsf{V}_{\infty,\underline{Z}})$ is divisible by (s_1+s_2) if $|\mathbf{n}_\lambda|>0$, while $e^{\mathbf{T}\times\mathbb{C}^*}(-\mathsf{E}_{\underline{Z}})$ is easily seen to be coprime with (s_1+s_2).[4] This implies that the anti-diagonal restriction $s_1+s_2=0$ is well-defined on every localized term and satisfies

$$e^{\mathbf{T}\times\mathbb{C}^*}(\mathbb{E}^\vee|_{\underline{Z}})\big|_{s_1+s_2=0} = 0,$$

by which we conclude the required vanishing. $\square$

Remark 3.7.8. We could prove Proposition 3.7.7 without relying on the vertex formalism for stable pairs, by simply carrying out a detailed (but probably longer) analysis of the weight space of $\mathfrak{t}_1\mathfrak{t}_2$ as in Lemma 3.7.6.

3.8. Summing the theory up

3.8.1. Anti-diagonal restriction We combine the computations in Section 3.6, 3.7 to prove the second part of Theorem 3.1.3 from the Introduction.

Theorem 3.8.1. *Under the anti-diagonal restriction $s_1+s_2=0$ the three universal series are*

$$A_\lambda(q,s_1,-s_1) = (-s_1^2)^{|\lambda|}\cdot\prod_{\square\in\lambda} h(\square)^2,$$

[4] In fact, $\mathsf{E}_{\underline{Z}}$ is the $\mathbf{T}$-representation of the tangent space at a $\mathbf{T}$-fixed point of $\mathrm{Hilb}^{|\lambda|}(\mathbb{C}^2)$. The claim follows by noticing that the $\mathbf{T}$-fixed locus coincides with the fixed locus of the subtorus $\{t_1t_2=1\}\subset\mathbf{T}$ preserving the Calabi-Yau form of $\mathbb{C}^2$, which is 0-dimensional and reduced.

$$B_\lambda(-q, s_1, -s_1) = (-1)^{n(\lambda)} \cdot s_1^{-|\lambda|} \cdot \prod_{\Box \in \lambda} h(\Box)^{-1} \cdot \prod_{\Box \in \lambda} (1 - q^{h(\Box)}),$$
$$C_\lambda(-q, s_1, -s_1) = (-1)^{n(\overline{\lambda})} \cdot (-s_1)^{-|\lambda|} \cdot \prod_{\Box \in \lambda} h(\Box)^{-1} \cdot \prod_{\Box \in \lambda} (1 - q^{h(\Box)}).$$

Proof. Let $C_{g,\deg L_1 \deg L_2} \in \mathbb{Q}(s_1, s_2)$ be the leading term of the generating series of Theorem 3.5.1. We can write

$$\begin{aligned} C^{-1}_{g,\deg L_1 \deg L_2} \cdot \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) &= \tilde{A}^{g-1}_{\lambda,1} \cdot \tilde{A}^{\deg L_1}_{\lambda,2} \cdot \tilde{A}^{\deg L_2}_{\lambda,3} \\ &= \exp\left((g-1) \cdot \log \tilde{A}_{\lambda,1} + \deg L_1 \cdot \log \tilde{A}_{\lambda,2} + \deg L_2 \cdot \log \tilde{A}_{\lambda,3}\right), \end{aligned}$$

for suitable $\tilde{A}_{\lambda,i} \in 1 + \mathbb{Q}(s_1, s_2)[\![q]\!]$, where $\log \tilde{A}_i$ are well-defined as $\tilde{A}_i$ are power series starting with 1. The claim is therefore reduced to the computation of the leading term (under the anti-diagonal restriction $s_1 + s_2 = 0$) and to the solution of a linear system. The leading term is computed in Proposition 3.6.1, which also shows that it is invertible in $\mathbb{Q}(s_1, s_2)$. The linear system is solved by computing the generating series of the integrals (3.5.1), under the anti-diagonal restriction, on a basis of $\mathbb{Q}^3$. The classes

$$\gamma(\mathbb{P}^1, \mathcal{O}, \mathcal{O}), \quad \gamma(\mathbb{P}^1, \mathcal{O}(-1), \mathcal{O}(-1)), \quad \gamma(\mathbb{P}^1, \mathcal{O}, \mathcal{O}(-2))$$

are linearly independent and we computed their generating series in Proposition 3.7.7 and Corollary 3.7.4. $\Box$

3.8.2. Degree 1 In degree 1, we consider the Young diagram consisting of a single box and the corresponding double nested Hilbert scheme is $C^{[n]} \cong C^{(n)}$, the symmetric power of a smooth projective curve C, with universal subscheme $\mathcal{Z} \subset C \times C^{(n)}$. Given line bundles L_1, L_2 on C, the class (3.4.1) in K-theory of the virtual normal bundle is

$$\begin{aligned} N^{\mathrm{vir}}_{C,L_1,L_2} &= -\mathbf{R}\pi_*(L_1L_2 \otimes \mathcal{O}_{\mathcal{Z}}) \otimes \mathfrak{t}_1\,\mathfrak{t}_2 + \mathbf{R}\pi_* L_1 \otimes \mathfrak{t}_1 + \mathbf{R}\pi_* L_2 \otimes \mathfrak{t}_2 \\ &= -(L_1L_2)^{[n]} \otimes \mathfrak{t}_1\,\mathfrak{t}_2 + \mathcal{O}^{\deg L_1 + 1 - g}_{C^{(n)}} \otimes \mathfrak{t}_1 + \mathcal{O}^{\deg L_2 + 1 - g}_{C^{(n)}} \otimes \mathfrak{t}_2, \end{aligned}$$

where $(L_1L_2)^{[n]}$ is the tautological bundle with fibers $(L_1L_2)^{[n]}|_Z = H^0(C, L_1L_2 \otimes \mathcal{O}_Z)$. This yields

$$\int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} e^{\mathbf{T}}(-N^{\mathrm{vir}}_{C,L_1,L_2}) = s_1^{g-1-\deg L_1} s_2^{g-1-\deg L_2} \int_{C^{(n)}} e((L_1L_2)^{[n]}).$$

By the universal structure of Theorem 3.5.1, we just need to compute the (generating series of the) last integral for $L_1 = L_2 = \mathcal{O}_C$ and $L_1 \otimes L_2 \cong K_C$, which yields the explicit universal series

$$\begin{aligned} A_\Box(q, s_1, s_2) &= s_1 s_2, \\ B_\Box(q, s_1, s_2) &= s_1^{-1}(1+q), \\ C_\Box(q, s_1, s_2) &= s_2^{-1}(1+q). \end{aligned}$$

3.8.3. GW/PT correspondence Let $X = \mathrm{Tot}_C(L_1 \oplus L_2)$ be a local curve. Combining Theorem 3.5.1, Theorem 3.8.1 and the description of the $\mathbf{T}$-fixed locus of $P_n(X, d[C])$ in Proposition 3.3.1, we obtain our main result.

Theorem 3.8.2. *The generating series of stable pair invariants satisfies*

$$\mathsf{PT}_d(X;q) = \sum_{\lambda \vdash d} \left(q^{-|\lambda|} A_\lambda(q)\right)^{g-1} \cdot \left(q^{-n(\lambda)} B_\lambda(q)\right)^{\deg L_1} \cdot \left(q^{-n(\overline{\lambda})} C_\lambda(q)\right)^{\deg L_2},$$

where $A_{\lambda,i}$ are the universal series of Theorem 3.5.1. Moreover, under the anti-diagonal restriction $s_1 + s_2 = 0$

$$\mathsf{PT}_d(X;-q)|_{s_1+s_2=0} =$$
$$(-1)^{d \deg L_2} \sum_{\lambda \vdash d} q^{d(1-g) - \deg L_1 n(\lambda) - \deg L_2 n(\overline{\lambda})} \prod_{\square \in \lambda} (s_1 h(\square))^{2g-2-\deg L_1 - \deg L_2} (1 - q^{h(\square)})^{\deg L_1 + \deg L_2}.$$

Comparing with Bryan-Pandharipande's results — cf. Theorem 3.1.5 and [**29**, Sec. 8] for the fully equivariant result in degree 1 — we obtain a proof of the Gromov-Witten/stable pairs correspondence for local curves.

Corollary 3.8.3. *Let X be a local curve. Under the anti-diagonal restriction $s_1 + s_2 = 0$ the GW/stable pair correspondence holds*

$$(-i)^{d(2-2g+\deg L_1 + \deg L_2)} \cdot \mathsf{GW}_d(g | \deg L_1, \deg L_2; u) = (-q)^{-\frac{1}{2} \cdot d(2-2g+\deg L_1+\deg L_2)} \mathsf{PT}_d(X, q),$$

after the change of variable $q = -e^{iu}$. Moreover it holds fully equivariantly in degree 1.

3.8.4. Resolved conifold In some cases, the moduli space $P_n(X, d[C])$ happens to be proper, for example whenever $H^0(C, L_i) = 0$ for $i = 1, 2$; see [**37**, Prop. 3.1] for a similar setting for local surfaces. Under this assumption, the invariants are computed as

$$\mathsf{PT}_d(X;q) = \mathsf{PT}_d(X;q)|_{s_1=s_2=0}$$

and can be deduced from the anti-diagonal restriction. The resulting invariants are interesting - that is, non-zero - only in the Calabi-Yau case; in this case, they coincide with the virtual Euler characteristic and Behrend's weighted Euler characteristic of $P_n(X, d[C])$ [**10**]. Applying Riemann-Roch, this situation may appear only when $H^l(C, L_i) = 0$ for $l = 0, 1$ and $\deg L_i = g - 1$ for $i = 1, 2$.

An interesting example is the resolved conifold $X = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$; in this case, the invariants can be further packaged into a generating series

$$\begin{aligned}
1 + \sum_{d \geq 1} Q^d \cdot \mathsf{PT}_d(X;-q) &= \sum_{d \geq 0} (-Qq)^d \sum_{\lambda \vdash d} q^{n(\lambda)+n(\overline{\lambda})} \prod_{\square \in \lambda} (1 - q^{h(\square)})^{-2} \\
&= \sum_{\lambda} (-Qq)^{|\lambda|} \frac{q^{n(\lambda)}}{\prod_{\square \in \lambda}(1 - q^{h(\square)})} \frac{q^{n(\overline{\lambda})}}{\prod_{\square \in \overline{\lambda}}(1 - q^{h(\square)})} \\
&= \sum_{\lambda} (-Qq)^{|\lambda|} s_\lambda(q) s_{\overline{\lambda}}(q) \\
&= \prod_{n \geq 1} (1 - Qq^n)^n,
\end{aligned}$$

where we used some identities involving the Schur function s_λ (see e.g. [125]). This last generating series agrees with the expression of the unrefined limit of the topological vertex of Iqbal-Kozçaz-Vafa [98] and can be seen as a specialization both of the motivic invariants of Morrison-Mozgovoy-Nagao-Szendrői [136] and of the K-theoretic invariants of Kononov-Okounkov-Osinenko [109].

3.9. K-theoretic refinement

3.9.1. K-theoretic invariants Let $X = \mathrm{Tot}_C(L_1 \oplus L_2)$ be a local curve. The perfect obstruction theory on $P_X := P_n(X, d[C])$ induces a ($\mathbf{T}$-equivariant) virtual structure sheaf $\mathcal{O}_{P_X}^{\mathrm{vir}} \in K_0^{\mathbf{T}}(P_X)$ [75] which depends only on the K-theory class of the perfect obstruction theory [177, Cor. 4.5]. K-theoretic PT invariants are defined by virtual K-theoretic localization [75] for any $V \in K_0^{\mathbf{T}}(P_X)$

$$\begin{aligned}\mathsf{PT}^K_{d,n}(X,V) &:= \chi(P_n(X,d[C]), \mathcal{O}_{P_X}^{\mathrm{vir}} \otimes V)\\ &:= \chi\left(P_n(X,d[C])^{\mathbf{T}}, \frac{\mathcal{O}^{\mathrm{vir}}_{P_X^{\mathbf{T}}} \otimes V|_{P_X^{\mathbf{T}}}}{\Lambda^\bullet N^{\mathrm{vir},*}}\right) \in \mathbb{Q}(\mathfrak{t}_1,\mathfrak{t}_2),\end{aligned}$$

where $\Lambda^\bullet(V) := \sum_{i=0}^{\mathrm{rk}\, V}(-1)^i \Lambda^i V$ is defined for every locally free sheaf V and then extended by linearity to any class in K-theory.

Remark 3.9.1. Differently than the case of equivariant cohomology, $\chi(M,\mathcal{F}) \in K_0^{\mathbf{T}}(\mathsf{pt})_{\mathrm{loc}}$ is well-defined for any $\mathbf{T}$-equivariant coherent sheaf $\mathcal{F}$ on a non-proper scheme M, as long as the weight spaces of $\mathcal{F}$ are finite-dimensional; in this case, the virtual localization formula is an actual theorem, rather than an ad-hoc definition of the invariants.

Using the description of the $\mathbf{T}$-fixed locus $(P_n(X,d[C))^{\mathbf{T}}$ of Proposition 3.3.1, K-theoretic stable pair invariants on X (with no insertions) are reduced to intersection numbers on $C^{[\mathbf{n}_\lambda]}$. The same techniques of Section 3.5 can be applied in this setting, yielding the following result.

Proposition 3.9.2. *Let C be a genus g smooth irreducible projective curve and L_1, L_2 line bundles over C. We have an identity*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|}\chi\left(C^{[\mathbf{n}_\lambda]}, \frac{\mathcal{O}^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}}}{\Lambda^\bullet N^{\mathrm{vir},*}_{C,L_1,L_2}}\right) = A_{K,\lambda}^{g-1}\cdot B_{K,\lambda}^{\deg L_1}\cdot C_{K,\lambda}^{\deg L_2} \in \mathbb{Q}(\mathfrak{t}_1,\mathfrak{t}_2)[\![q]\!],$$

where $A_{K,\lambda}, B_{K,\lambda}, C_{K,\lambda} \in \mathbb{Q}(\mathfrak{t}_1,\mathfrak{t}_2)[\![q]\!]$ are fixed universal series for $i=1,2,3$, only depending on λ. Moreover

$$\begin{aligned}A_{K,\lambda}(\mathfrak{t}_1,\mathfrak{t}_2) &= A_{K,\overline{\lambda}}(\mathfrak{t}_2,\mathfrak{t}_1),\\ B_{K,\lambda}(\mathfrak{t}_1,\mathfrak{t}_2) &= C_{K,\overline{\lambda}}(\mathfrak{t}_2,\mathfrak{t}_1).\end{aligned}$$

Proof. The proof follows the same strategy as Theorem 3.5.1. We just need to notice that $\Lambda^\bullet(\cdot)$ is multiplicative and that, via virtual Hirzebruch-Riemann-Roch [75], we can express

$$\chi\left(C^{[\mathbf{n}_\lambda]}, \frac{\mathcal{O}^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}}}{\Lambda^\bullet N^{\mathrm{vir},*}_{C,L_1,L_2}}\right) = \int_{[C^{[\mathbf{n}_\lambda]}]^{\mathrm{vir}}} \mathrm{ch}(-\Lambda^\bullet N^{\mathrm{vir},*}_{C,L_1,L_2})\cdot \mathrm{td}(T^{\mathrm{vir}}_{C^{[\mathbf{n}_\lambda]}})$$
$$= \int_{A_{C,\mathbf{n}_\lambda}} f,$$

where f is a polynomial expression of classes of the same form as in Proposition 3.5.3. □

Denote by $\mathsf{PT}^K_d(X;q) = \sum_{n\in\mathbb{Z}} q^n \mathsf{PT}^K_{d,n}(X)$ the generating series of K-theoretic stable pair invariants.

Corollary 3.9.3. *Let $X = \mathrm{Tot}_C(L_1\oplus L_2)$ be a local curve. We have*

$$\mathsf{PT}^K_d(X;q) = \sum_{\lambda\vdash d}\left(q^{-|\lambda|}A_{K,\lambda}(q)\right)^{g-1}\cdot\left(q^{-n(\lambda)}B_{K,\lambda}(q)\right)^{\deg L_1}\cdot\left(q^{-n(\overline{\lambda})}C_{K,\lambda}(q)\right)^{\deg L_2}.$$

3.9.2. Nekrasov-Okounkov Let M be a scheme with a perfect obstruction theory $\mathbb{E}$. Define the virtual canonical bundle to be $K_{\mathrm{vir}} = \det\mathbb{E}\in \mathrm{Pic}(M)$. Assume that K_{vir} admits a *square root* $K^{1/2}_{\mathrm{vir}}$, that is a line bundle such that $(K^{1/2}_{\mathrm{vir}})^{\otimes 2}\cong K_{\mathrm{vir}}$. Nekrasov-Okounkov [**142**] teach us that it is much more natural to consider the *twisted virtual structure sheaf*

$$\widehat{\mathcal{O}}^{\mathrm{vir}}_M = \mathcal{O}^{\mathrm{vir}}_M\otimes K^{1/2}_{\mathrm{vir}}.$$

In [**142**], Nekrasov-Okounkov show existence of square roots for P_X (and uniqueness, up to 2-torsion). Nevertheless, even if square roots could not exists on P_X as *line bundles*, they exist as a class $K^{1/2}_{\mathrm{vir},P_X}\in K^0(P_X,\mathbb{Z}[\frac{1}{2}])$ and are unique [**147**, Lemma 5.1]. In our setting, we define *Nekrasov-Okounkov K-theoretic stable pair invariants* as[5]

$$\mathsf{PT}^{\widehat{K}}_{d,n}(X) := \mathsf{PT}^K_{d,n}(X, K^{1/2}_{\mathrm{vir},P_X})$$
$$= \chi\left(P_n(X,d[C])^{\mathbf{T}}, \frac{\mathcal{O}^{\mathrm{vir}}_{P_X^{\mathbf{T}}}\otimes K^{1/2}_{\mathrm{vir},P_X}|_{P_X^{\mathbf{T}}}}{\Lambda^\bullet N^{\mathrm{vir},*}}\right)\in\mathbb{Q}(\mathfrak{t}_1^{1/2},\mathfrak{t}_2^{1/2}),$$

which are an algebro-geometric analogue of the $\widehat{A}$-genus of a spin manifold. On the $\mathbf{T}$-fixed locus, we have an identity in $K^0_{\mathbf{T}}(P_X^{\mathbf{T}},\mathbb{Z}[\frac{1}{2}])_{\mathrm{loc}}$

$$\frac{\mathcal{O}^{\mathrm{vir}}_{P_X^{\mathbf{T}}}\otimes K^{1/2}_{\mathrm{vir},P_X}|_{P_X^{\mathbf{T}}}}{\Lambda^\bullet N^{\mathrm{vir},*}} = \frac{\mathcal{O}^{\mathrm{vir}}_{P_X^{\mathbf{T}}}\otimes K^{1/2}_{\mathrm{vir},P_X^{\mathbf{T}}}\otimes\left(\det N^{\mathrm{vir},*}\right)^{1/2}}{\Lambda^\bullet N^{\mathrm{vir},*}}$$
$$= \frac{\widehat{\mathcal{O}}^{\mathrm{vir}}_{P_X^{\mathbf{T}}}}{\widehat{\Lambda}^\bullet N^{\mathrm{vir},*}},$$

where we define $\widehat{\Lambda}^\bullet(\cdot) = \Lambda^\bullet(\cdot)\otimes\det(\cdot)^{-1/2}$. Again, the same techniques of Section 3.5 and Proposition 3.9.2 and can be applied in this setting, yielding the following result.

[5]To take square roots *equivariantly*, we need to replace the torus $\mathbf{T}$ with the minimal cover where the characters $\mathfrak{t}_1^{1/2},\mathfrak{t}_2^{1/2}$ are well-defined, see [**142**, Sec. 7.2.1].

Theorem 3.9.4. *Let C be a genus g smooth irreducible projective curve and L_1, L_2 line bundles over C. We have an identity*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|}\chi\left(C^{[\mathbf{n}_\lambda]}, \frac{\widehat{\mathcal{O}}^{\mathrm{vir}}_{P_X^{\mathbf{T}}}}{\widehat{\Lambda}^\bullet N^{\mathrm{vir},*}_{C,L_1,L_2}}\right) = A^{g-1}_{\widehat{K},\lambda}\cdot B^{\deg L_1}_{\widehat{K},\lambda}\cdot C^{\deg L_2}_{\widehat{K},\lambda} \in \mathbb{Q}(\mathsf{t}_1^{1/2},\mathsf{t}_2^{1/2})[\![q]\!],$$

where $A_{\widehat{K},\lambda}, B_{\widehat{K},\lambda}, C_{\widehat{K},\lambda} \in \mathbb{Q}(\mathsf{t}_1^{1/2},\mathsf{t}_2^{1/2})[\![q]\!]$ are fixed universal series for $i = 1, 2, 3$. Moreover

$$\begin{aligned} A_{\widehat{K},\lambda}(\mathsf{t}_1,\mathsf{t}_2) &= A_{\widehat{K},\overline{\lambda}}(\mathsf{t}_2,\mathsf{t}_1),\\ B_{\widehat{K},\lambda}(\mathsf{t}_1,\mathsf{t}_2) &= C_{\widehat{K},\overline{\lambda}}(\mathsf{t}_2,\mathsf{t}_1). \end{aligned}$$

Denote by $\mathsf{PT}^{\widehat{K}}_d(X;q) = \sum_{n\in\mathbb{Z}} q^n \mathsf{PT}^{\widehat{K}}_{d,n}(X)$ the generating series of Nekrasov-Okounkov K-theoretic stable pair invariants.

Corollary 3.9.5. *Let $X = \mathrm{Tot}_C(L_1 \oplus L_2)$ be a local curve. We have*

$$\mathsf{PT}^{\widehat{K}}_d(X;q) = \sum_{\lambda\vdash d}\left(q^{-|\lambda|}A_{\widehat{K},\lambda}(q)\right)^{g-1}\cdot\left(q^{-n(\lambda)}B_{\widehat{K},\lambda}(q)\right)^{\deg L_1}\cdot\left(q^{-n(\overline{\lambda})}C_{\widehat{K},\lambda}(q)\right)^{\deg L_2}.$$

The techniques of Section 3.6, 3.7 can be adapted to compute the generating series of Nekrasov-Okounkov K-theoretic invariants under the anti-diagonal restriction $\mathsf{t}_1\mathsf{t}_2 = 1$. In fact, as in the proof of Theorem 3.8.1, we just need to compute the leading term of the generating series and the cases $g = 0$, $L_1 \otimes L_2 = K_{\mathbb{P}^1}$ and $L_1 = L_2 = \mathcal{O}_{\mathbb{P}^1}$. Similarly to the proof of Theorem 3.7.3, as we work with $C \cong \mathbb{P}^1$, applying the K-theoretic virtual localization formula [**75**] on $(\mathbb{P}^1)^{[\mathbf{n}_\lambda]}$ yields

$$\chi\left(\mathbb{P}^{1[\mathbf{n}_\lambda]}, \frac{\widehat{\mathcal{O}}^{\mathrm{vir}}_{P_X^{\mathbf{T}}}}{\widehat{\Lambda}^\bullet N^{\mathrm{vir},*}_{C,L_1,L_2}}\right) = \left(\sum_{\underline{Z}\in\mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}}\chi\left(\underline{Z}, \frac{1}{\widehat{\Lambda}^\bullet(T^{\mathrm{vir},*}_{\mathbb{P}^{1[\mathbf{n}_\lambda]},\underline{Z}} + N^{\mathrm{vir},*}_{\mathbb{P}^1,L_1,L_2,\underline{Z}})}\right)\right)\Bigg|_{\mathsf{t}_3=1},$$

where t_3 is the equivariant parameter of the $\mathbb{C}^*$-action. For a character $\mathsf{t}^\mu \in K^0_{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt})$, define the operator $[\mathsf{t}^\mu] = \mathsf{t}^{\frac{\mu}{2}} - \mathsf{t}^{-\frac{\mu}{2}}$ and extend it by linearity to any $V \in K^0_{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt})$. It is proven in [**76**, Sec. 6.1] that

$$\chi(\mathsf{pt}, \widehat{\Lambda}^\bullet(V^*)) = [V],$$

which satisfies $[V^*] = (-1)^{\mathrm{rk}\, V}[V]$, therefore

$$\chi\left(\mathbb{P}^{1[\mathbf{n}_\lambda]}, \frac{\widehat{\mathcal{O}}^{\mathrm{vir}}_{P_X^{\mathbf{T}}}}{\widehat{\Lambda}^\bullet N^{\mathrm{vir},*}_{C,L_1,L_2}}\right) = \left(\sum_{\underline{Z}\in\mathbb{P}^{1[\mathbf{n}_\lambda],\mathbb{C}^*}}[-T^{\mathrm{vir}}_{\mathbb{P}^{1[\mathbf{n}_\lambda]},\underline{Z}} - N^{\mathrm{vir}}_{\mathbb{P}^1,L_1,L_2,\underline{Z}}]\right)\Bigg|_{\mathsf{t}_3=1}.$$

Explicit computations yields

$$\begin{aligned} A_{\widehat{K},\lambda}(q,\mathsf{t}_1,\mathsf{t}_1^{-1}) &= (-1)^{|\lambda|}\cdot F_\lambda^{-2},\\ B_{\widehat{K},\lambda}(-q,\mathsf{t}_1,\mathsf{t}_1^{-1}) &= (-1)^{n(\lambda)}\cdot F_\lambda\cdot\prod_{\Box\in\lambda}(1-q^{h(\Box)}),\\ C_{\widehat{K},\lambda}(-q,\mathsf{t}_1,\mathsf{t}_1^{-1}) &= (-1)^{|\lambda|+n(\overline{\lambda})}F_\lambda\cdot\prod_{\Box\in\lambda}(1-q^{h(\Box)}), \end{aligned}$$

where

$$F_\lambda = \prod_{\substack{(i,j)\in\lambda\\ (i,j)\neq(a,a)}} \frac{1}{\mathfrak{t}_1^{\frac{|j-i|}{2}} - \mathfrak{t}_1^{-\frac{|j-i|}{2}}} \cdot \frac{\prod_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)\neq(l+a,k+a)}} \left(\mathfrak{t}_1^{\frac{|i-j+k-l|}{2}} - \mathfrak{t}_1^{-\frac{|i-j+k-l|}{2}}\right)}{\prod_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)\neq(l+a-1,k+a)}} \left(\mathfrak{t}_1^{\frac{|1+i-j+k-l|}{2}} - \mathfrak{t}_1^{-\frac{|1+i-j+k-l|}{2}}\right)}.$$

3.10. Appendix: The combinatorial identities

In this appendix we collect the proofs of some technical results on the combinatorics of Young diagrams we have used.

Lemma 3.10.1. *Let λ be a Young diagram and consider the Laurent polynomials in $\mathbb{Z}[\mathfrak{t}^{\pm1}]$*

$$g_\lambda(\mathfrak{t}) = \sum_{\substack{(a,a)\in\lambda\\ a\neq 0, a\in\mathbb{Z}}} \mathfrak{t}^{-a} + \sum_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)=(l+a,k+a)\\ a\neq 0,1, a\in\mathbb{Z}}} \operatorname{sgn}(a)\,\mathfrak{t}^{a} - \sum_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)=(l+a-1,k+a)\\ a\neq 0,1, a\in\mathbb{Z}}} \operatorname{sgn}(a)\,\mathfrak{t}^{a},$$

$$h_\lambda(\mathfrak{t}) = \sum_{\substack{(a,a)\in\lambda\\ a\neq 0, a\in\mathbb{Z}}} \mathfrak{t}^{-a} - \sum_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)=(l+a,k+a)\\ a\neq 0,1, a\in\mathbb{Z}}} \mathfrak{t}^{a} + \sum_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)=(l+a-1,k+a)\\ a\neq 0,1, a\in\mathbb{Z}}} \mathfrak{t}^{a}.$$

Then we have

$$\begin{aligned} g_\lambda(\mathfrak{t}) &= A_\lambda(\mathfrak{t}) + A_\lambda(\mathfrak{t}^{-1})\,\mathfrak{t},\\ h_\lambda(\mathfrak{t}) &= B_\lambda(\mathfrak{t}) - B_\lambda(\mathfrak{t}^{-1})\,\mathfrak{t}, \end{aligned}$$

where $A_\lambda(\mathfrak{t}), B_\lambda(\mathfrak{t}) \in \mathbb{Z}[\mathfrak{t}^{\pm1}]$.

Proof. We prove the first claim by induction on the size of λ. If $|\lambda| = 1$ this is clear. Suppose now the claim holds for all Young diagrams of size $|\lambda| = n$ and consider a Young diagram of size $|\tilde{\lambda}| = n+1$ obtained by adding to a Young diagram λ a box whose lattice coordinates are $(i,j)\in\mathbb{Z}^2$.

- $(i,j) = (i,0)$ or $(i,j) = (0,j)$, with $i,j\neq 0$. We have
$$g_{\tilde\lambda} = g_\lambda.$$
- $(i,j) = (i,1), i\geq 1$. We have
$$\begin{aligned} g_{\tilde\lambda} &= g_\lambda + \mathfrak{t}^{-1} - \mathfrak{t}^{-1}\\ &= g_\lambda. \end{aligned}$$
- $(i,j) = (1,j), j\geq 2$. We have
$$\begin{aligned} g_{\tilde\lambda} &= g_\lambda - \mathfrak{t}^{-1} - \mathfrak{t}^{2}\\ &= g_\lambda - \mathfrak{t}^{-1} - (\mathfrak{t}^{-1})^{-1}\,\mathfrak{t}. \end{aligned}$$
- $(i,j) = (i,i), i\geq 2$. We have
$$\begin{aligned} g_{\tilde\lambda} &= g_\lambda + \mathfrak{t}^{-i} - \sum_{l=1}^{i}\mathfrak{t}^{-l} + \sum_{l=2}^{i}\mathfrak{t}^{l} - \left(\sum_{l=2}^{i}\mathfrak{t}^{l} - \sum_{l=1}^{i-1}\mathfrak{t}^{-l}\right)\\ &= g_\lambda. \end{aligned}$$

- $(i,j), i > j \geq 2$. We have

$$g_{\tilde{\lambda}} = g_\lambda - \sum_{l=1}^{j} \mathsf{t}^{-l} + \sum_{l=2}^{j} \mathsf{t}^{l} - \left(\sum_{l=2}^{j} \mathsf{t}^{l} - \sum_{l=1}^{j} \mathsf{t}^{-l} \right)$$
$$= g_\lambda .$$

- $(i,j), j > i \geq 2$. We have

$$g_{\tilde{\lambda}} = g_\lambda - \sum_{l=1}^{i} \mathsf{t}^{-l} + \sum_{l=2}^{i} \mathsf{t}^{l} - \left(\sum_{l=2}^{i+1} \mathsf{t}^{l} - \sum_{l=1}^{i-1} \mathsf{t}^{-l} \right)$$
$$= g_\lambda - \mathsf{t}^{-i} - \mathsf{t}^{i+1} .$$

Therefore the induction step is proven in all possible cases and we conclude the proof. With an analogous analysis one proves the second claim as well. $\square$

Lemma 3.10.2. *Let λ be a Young diagram. Then the following identity holds*

$$\prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(a,a)}} (j-i) \cdot \frac{\prod_{\substack{(i,j),(l,k)\in\lambda \\ (i,j)\neq(l+a-1,k+a)}} (1+i-j+k-l)}{\prod_{\substack{(i,j),(l,k)\in\lambda \\ (i,j)\neq(l+a,k+a)}} (i-j+k-l)} = \sigma(\lambda) \cdot \prod_{\square\in\lambda} h(\square),$$

where

$$\sigma(\lambda) = \prod_{\substack{(i,j)\in\lambda \\ (i,j)\neq(a,a)}} \operatorname{sgn}(j-i) \cdot \prod_{\substack{(i,j),(l,k)\in\lambda \\ (i,j)\neq(l+a-1,k+a)}} \operatorname{sgn}(1+i-j+k-l) \cdot \prod_{\substack{(i,j),(l,k)\in\lambda \\ (i,j)\neq(l+a,k+a)}} \operatorname{sgn}(i-j+k-l).$$

Proof. The sign $\sigma(\lambda)$ is easily determined, so we just need to compute the absolute value. To ease the notation, we adopt the following convention for the remainder of the proof: we set $|0| = 1$, which is merely a formal shortcut to include in the productory trivial factors we would have otherwise excluded. The claim therefore becomes

$$\prod_{(i,j)\in\lambda} |i-j| \cdot \prod_{(i,j),(l,k)\in\lambda} \frac{|1+i-j+k-l|}{|i-j+k-l|} = \prod_{\square\in\lambda} h(\square). \tag{3.10.1}$$

Denote the left-hand-side of (3.10.1) by H_λ. We prove this claim on the induction on the size of λ. If $|\lambda| = 1$, the claim is trivially satisfied. Assume it holds for all Young diagrams of size n and consider a Young diagram λ' of size $n+1$ obtained by a Young diagram of size λ by adding a box with lattice coordinates $(i,j) \in \mathbb{Z}^2$. We have

$$H_{\lambda'} = H_\lambda \cdot |i-j| \cdot \prod_{(l,k)\in\lambda} \frac{|1+i-j+k-l|}{|i-j+k-l|} \cdot \frac{|-1+i-j+k-l|}{|i-j+k-l|}.$$

To avoid confusion, we denote by $h(\square)$ (resp. $h'(\square)$) the hooklength of $\square \in \lambda$ (resp. $\square \in \lambda'$). The strategy now is to divide the boxes of λ' in sub-collections and compute separately each contribution of the product on the right-hand-side.

Step I: Fix a box $(\tilde{i}, j) \in \lambda$, with $\tilde{i} < i$. The contribution of all boxes on the right (on the same row) of $(\tilde{i}, j)$ is

$$\prod_{k=j}^{\lambda_{\tilde{i}}-1} \frac{|i-j+k-\tilde{i}+1| \cdot |i-j+k-\tilde{i}-1|}{|i-j+k-\tilde{i}| \cdot |i-j+k-\tilde{i}|}$$
$$= \frac{|i-\tilde{i}+1| \cdot |i-\tilde{i}-1|}{|i-\tilde{i}| \cdot |i-\tilde{i}|} \cdot \frac{|i-\tilde{i}+2| \cdot |i-\tilde{i}|}{|i-\tilde{i}+1| \cdot |i-\tilde{i}+1|} \cdots \frac{|i-\tilde{i}+\lambda_{\tilde{i}}-j| \cdot |i-\tilde{i}+\lambda_{\tilde{i}}-j-2|}{|i-\tilde{i}+\lambda_{\tilde{i}}-j-1| \cdot |i-\tilde{i}+\lambda_{\tilde{i}}-j-1|}$$

$$= \frac{|i-\tilde{i}-1|\cdot|i-\tilde{i}+\lambda_{\tilde{i}}-j|}{|i-\tilde{i}|\cdot|i-\tilde{i}+\lambda_{\tilde{i}}-j-1|}$$
$$= \frac{|i-\tilde{i}-1|}{|i-\tilde{i}|}\cdot\frac{h'(\tilde{i},j)}{h(\tilde{i},j)}.$$

We multiply now the last expression for all boxes $(\tilde{i},j)$ with $\tilde{i}=0,\dots,i-1$

$$\begin{aligned}\prod_{\tilde{i}=0}^{i-1}\frac{|i-\tilde{i}-1|}{|i-\tilde{i}|}\cdot\frac{h'(\tilde{i},j)}{h(\tilde{i},j)} &= \frac{|i-1|}{|i|}\cdot\frac{|i-2|}{|i-1|}\cdots\frac{|0|}{|1|}\cdot\prod_{\tilde{i}=0}^{i-1}\frac{h'(\tilde{i},j)}{h(\tilde{i},j)}\\ &= \frac{1}{|i|}\cdot\prod_{\tilde{i}=0}^{i-1}\frac{h'(\tilde{i},j)}{h(\tilde{i},j)}.\end{aligned}$$

This is the contribution of all boxes $(l,k)\in\lambda$ such that $k\geq j$. By symmetry, we get that the contribution of all boxes $(l,k)\in\lambda$ such that $l\geq i$ is given by

$$\frac{1}{|j|}\cdot\prod_{\tilde{j}=0}^{j-1}\frac{h'(i,\tilde{j})}{h(i,\tilde{j})}.$$

Step II: The contribution of the remaining boxes is given by

$$\begin{aligned}&|i-j|\cdot\prod_{\substack{(l,k)\in\lambda\\ l<i\\ k<j}}\frac{|1+i-j+k-l|}{|i-j+k-l|}\cdot\frac{|-1+i-j+k-l|}{|i-j+k-l|}\\ &\qquad= |i-j|\cdot\prod_{l=0}^{i-1}\prod_{k=0}^{j-1}\frac{|1+i-j+k-l|}{|i-j+k-l|}\cdot\frac{|-1+i-j+k-l|}{|i-j+k-l|}\\ &\qquad\qquad= |i-j|\cdot\prod_{l=0}^{i-1}\frac{|i-l-j-1|\cdot|i-l|}{|i-l-j|\cdot|i-l-1|} = |i-j|\cdot\frac{|i|\cdot|j|}{|i-j|} = |i|\cdot|j|.\end{aligned}$$

Step III: Using Step I,II and the induction step we have

$$\begin{aligned}H_{\lambda'} &= \prod_{\Box\in\lambda}h(\Box)\cdot\prod_{\tilde{i}=0}^{i-1}\frac{h'(\tilde{i},j)}{h(\tilde{i},j)}\cdot\prod_{\tilde{j}=0}^{j-1}\frac{h'(i,\tilde{j})}{h(i,\tilde{j})}\\ &= \prod_{\Box\in\lambda'}h'(\Box),\end{aligned}$$

which concludes the proof. $\Box$

Lemma 3.10.3. *Let λ be a Young diagram and $k_1,k_2\in\mathbb{Z}$. Set*

$$\begin{aligned}\rho(\lambda) = \sum_{\substack{(i,j)\in\lambda\\ (i,j)\neq(a,a)}}(ik_1+jk_2) + \sum_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)\neq(l+a-1,k+a)}}((i-l+1)k_1+(j-k)k_2)\\ + \sum_{\substack{(i,j),(l,k)\in\lambda\\ (i,j)\neq(l+a,k+a)}}((i-l)k_1+(j-k)k_2).\end{aligned}$$

We have

$$(-1)^{\rho(\lambda)+|\lambda|(k_1+k_2)}\sigma(\lambda)^{k_1+k_2} = (-1)^{|\lambda|k_2+n(\lambda)k_1+n(\overline{\lambda})k_2},$$

where $\sigma(\lambda)$ was defined in Lemma 3.10.2.

PROOF. Let $\equiv$ denote congruence modulo 2. We have

$$\rho(\lambda)+|\lambda|(k_1+k_2) \equiv \sum_{(i,j)\in\lambda}(ik_1+jk_2)+(k_1+k_2)\left(\sum_{\substack{(i,j)\in\lambda\\(i,j)=(a,a)}} a + \sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)=(l+a-1,k+a)}} a + \sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)=(l+a,k+a)}} a\right)$$

$$+ k_1 \sum_{(i,j),(l.k)\in\lambda} 1 + |\lambda|(k_1+k_2)$$

$$\equiv n(\lambda)k_1 + n(\overline{\lambda})k_2 + |\lambda|k_2 + (k_1+k_2)\left(\sum_{\substack{(i,j)\in\lambda\\(i,j)=(a,a)}} a + \sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)=(l+a-1,k+a)}} a + \sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)=(l+a,k+a)}} a\right),$$

therefore the statement of the lemma reduces to the following claim

$$(-1)^{\sum_{\substack{(i,j)\in\lambda\\(i,j)=(a,a)}} a+\sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)=(l+a-1,k+a)}} a+\sum_{\substack{(i,j),(l,k)\in\lambda\\(i,j)=(l+a,k+a)}} a} \cdot \sigma(\lambda) = 1. \tag{3.10.2}$$

Given a lattice point $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2$ define

$$\tau(\mu) = \begin{cases} (-1)^{\mu_1} & \mu_1 = \mu_2, \\ \operatorname{sgn}(\mu_1 - \mu_2) & \mu_1 \neq \mu_2. \end{cases}$$

Notice that the left-hand-side of (3.10.2) can be rewritten as

$$\prod_{(i,j)\in\lambda} \tau(-i,-j) \prod_{(i,j),(l,k)\in\lambda} \tau(i-l, j-k)\tau(1+i-l, j-k),$$

and denote this last expression by F_λ. We prove the claim (3.10.2) on induction on the size of λ. If $|\lambda| = 1$, the result is clear. Assume it holds for all Young diagrams of size n and consider a Young diagram λ' of size $n+1$ obtained from a Young diagram of size λ by adding a box with lattice coordinates $(i,j) \in \mathbb{Z}^2$; we have

$$F_{\lambda'} = F_\lambda \cdot \tau(-i,-j) \prod_{(l,k)\in\lambda} \tau(i-l, j-k)\tau(l-i, k-j)\tau(1+i-l, j-k)\tau(1+l-i, k-j).$$

We analyze the contribution of every box $(l,k) \in \lambda$ in the product above. We say that a box (l,k) is in the same diagonal as (i,j) if it is of the form

$$(l,k) = (i+a, j+a), \quad a \in \mathbb{Z}.$$

The contribution of the boxes on the same diagonal of (i,j) is

$$\tau(-a,-a)\tau(a,a)\tau(1-a,-a)\tau(1+a,a) = 1.$$

We say that a box (l,k) is in the *lower diagonal* of (i,j) if it is of the form

$$(l,k) = (i+a, j+a-1), \quad a \in \mathbb{Z}.$$

The contribution of the boxes in the lower diagonal of (i,j) is

$$\tau(-a, 1-a)\tau(a, a-1)\tau(1-a, 1-a)\tau(1+a, a-1) = (-1)^a.$$

We say that a box (l,k) is in the *upper diagonal* of (i,j) if it is of the form

$$(l,k) = (i+a-1, j+a), \quad a \in \mathbb{Z}.$$

The contribution of the boxes in the upper diagonal of (i, j) is

$$\tau(1-a,-a)\tau(a-1,a)\tau(2-a,-a)\tau(a,a) = (-1)^{a+1}.$$

A completely analogous analysis shows that all other boxes $(i, j) \in \lambda$ do not contribute to the product. Therefore the contribution to the sign in the product is just given by the boxes in the upper or lower diagonal of (i, j), as displayed in the picture

1	1	1	1	1
1	-1	1	-1	
1	1	1	ij	
1	1			
1				

FIGURE 3. The label ± 1 in every box represents the contribution to the final sign, with $(i, j) = (2, 3)$.

If we denote by δ_+, δ_- respectively the number of boxes in the upper diagonal and lower diagonal, we conclude that

$$\prod_{(l,k)\in\lambda} \tau(i-l, j-k)\tau(l-i, k-j)\tau(1+i-l, j-k)\tau(1+l-i, k-j) = (-1)^{\lceil \frac{\delta_+}{2} \rceil + \lfloor \frac{\delta_-}{2} \rfloor},$$

where $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$ denote the ususal *ceiling* and *floor* functions. One readily proves that

$$(\delta_+, \delta_-) = \begin{cases} (j+1, j) & i > j, \\ (i, i) & i = j, \\ (i, i+1) & i < j. \end{cases}$$

With a direct analysis we can show that

$$(-1)^{\lceil \frac{\delta_+}{2} \rceil + \lfloor \frac{\delta_-}{2} \rfloor} \cdot \tau(-i, -j) = 1$$

in all the three cases, by which we conclude the inductive step. $\square$

CHAPTER 4

Higher rank K-theoretic Donaldson-Thomas theory of points

Black hole sun
won't you come
and wash away the rain?
Black hole sun
won't you come
won't you come

Black Hole Sun, Soundgarden

4.1. Introduction

4.1.1. Overview Classical Donaldson–Thomas (DT in short) invariants of a smooth complex projective Calabi–Yau 3-fold Y, introduced in [**176**], are integers that virtually count stable torsion free sheaves on Y, with fixed Chern character γ. However, the theory is much richer than what the bare DT numbers

$$\mathsf{DT}(Y,\gamma) \in \mathbb{Z} \tag{4.1.1}$$

can capture: there are extra symmetries subtly hidden in the local structure of the moduli spaces of sheaves giving rise to the classical DT invariants (4.1.1). This idea has been present in the physics literature for some time [**98, 65**].

These hidden symmetries suggest that there should exist more *refined* invariants, of which the DT numbers (4.1.1) are just a shadow. These branch out in two main directions:

- *motivic* Donaldson–Thomas invariants, and
- *K-theoretic* Donaldson–Thomas invariants.

For the former, which at date includes a number of interesting subbranches, the papers by Kontsevich and Soibelman [**111, 112**] are a good starting point, and Szendrői's survey [**174**] contains an extensive bibliography on the subject. For the latter, see some recent developments after Nekrasov–Okounkov [**142**], such as [**149, 3, 178**] and

[**140, 143, 36**] for a generalisation to Calabi–Yau 4-folds. In this chapter we deal with K-theoretic DT theory. The relationship between motivic and K-theoretic, which we briefly sketch in Section 4.5.3, will be investigated in future work.

The subtle structure of the DT moduli spaces is most evident in the *local case*, i.e. when the theory is applied to the simplest Calabi–Yau 3-fold of all, namely the affine space $\mathbb{A}^3$. See [**11, 61**] for the rank 1 motivic DT theory of $\mathbb{A}^3$, and [**165**] for a higher rank version. We solve the K-theoretic Donaldson–Thomas theory of points of $\mathbb{A}^3$. In [**9**] it is shown that the main player in the theory, the Quot scheme

$$\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n) \tag{4.1.2}$$

of length n quotients of the free sheaf $\mathscr{O}^{\oplus r}$, is a global *critical locus*, i.e. it can be realised as $\{ \mathrm{d}f = 0 \}$, for f a regular function on a smooth scheme. This structural result, revealing in bright light the symmetries we were talking about, is used to define the *higher rank K-theoretic DT theory of points* that is the central character in this chapter. The rank 1 theory, corresponding to $\operatorname{Hilb}^n(\mathbb{A}^3)$, was already defined, and it was solved by Okounkov [**149**, Sec. 3], proving a conjecture by Nekrasov [**138**]. Our first main result (Theorem 4.1.1) can be seen as an upgrade of his calculation, completing the study of the degree 0 K-theoretic DT theory of $\mathbb{A}^3$.

In physics, remarkably, the definition of the K-theoretic DT invariants studied here already existed, and gave rise to a conjecture that we — again, Theorem 4.1.1 — prove mathematically. More precisely, our formula for the K-theoretic DT partition function $\mathsf{DT}_r^{\mathrm{K}}$ of $\mathbb{A}^3$ was first conjectured by Nekrasov [**138**] for $r = 1$ and by Awata and Kanno [**6**] for arbitrary r as the partition function of a quiver matrix model describing instantons of a topological $U(r)$ gauge theory on D6 branes.

We also study higher rank *cohomological DT invariants* of $\mathbb{A}^3$. As we show in Corollary 4.6.1, these can be obtained as a suitable limit of the K-theoretic invariants. Motivated by [**6, 139**], a closed formula for their generating function $\mathsf{DT}_r^{\mathrm{coh}}$ was conjectured by Szabo [**173**, Conj. 4.10] as a generalisation of the $r = 1$ case established by Maulik–Nekrasov–Okounkov–Pandharipande [**128**, Thm. 1]. We prove this conjecture as our Theorem 4.1.2. To get there, in Section 4.4 we develop a *higher rank topological vertex* formalism based on the combinatorics of *r-colored plane partitions*,[1] generalising the classical vertex formalism of [**127, 128**].

We pause for a second to explain a key step in this chapter. The Quot scheme (4.1.2), which gives rise to most of the invariants we study here, is acted on by an algebraic torus

$$\mathbf{T} = (\mathbb{C}^*)^3 \times (\mathbb{C}^*)^r,$$

and by their very definition, both the K-theoretic and the cohomological DT invariants depend, a priori, on the sets $t = (t_1, t_2, t_3)$ and $w = (w_1, \ldots, w_r)$ of equivariant parameters of $\mathbf{T}$. A technical result, which is proved as Theorem 4.5.5, states that

(4.1.3) The K-theoretic DT invariants do not depend on the framing parameters w.

[1] In this chapter, an r-colored plane partition is an r-tuple of classical plane partitions, see Definition 4.2.10.

This will allow us to take arbitrary limits to evaluate our formulae. We emphasise that this independence, automatic if $r = 1$ (see Remark 4.5.2), is quite surprising and highly nontrivial if $r > 1$.

4.1.2. Main results We briefly outline here the main results obtained in this chapter.

4.1.2.1. K-theoretic DT invariants As we mentioned above, the Quot scheme (4.1.2) is a critical locus, thus it carries a natural symmetric ($\mathbf{T}$-equivariant, as we prove) perfect obstruction theory in the sense of Behrend–Fantechi [**12, 13**]. As we recall in Section 4.3.1.1, there is also an induced *twisted virtual structure sheaf* $\widehat{\mathcal{O}}^{\mathrm{vir}} \in K_0^{\mathbf{T}}(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n))$, which is a twist, by a square root of the virtual canonical bundle, of the ordinary virtual structure sheaf $\mathcal{O}^{\mathrm{vir}}$. The rank r *K-theoretic DT partition function* of the Quot scheme of points of $\mathbb{A}^3$, encoding the rank r K-theoretic DT invariants of $\mathbb{A}^3$, is defined as

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w) = \sum_{n \geq 0} q^n \chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n), \widehat{\mathcal{O}}^{\mathrm{vir}}) \in \mathbb{Z}(\!(t, (t_1 t_2 t_3)^{\frac{1}{2}}, w)\!)[\![q]\!],$$

where the half power is caused by the twist by the chosen square root of the virtual canonical bundle (this choice does not affect the invariants, cf. Remark 4.3.3).

Granting Theorem 4.5.5, stated informally in (4.1.3), we shall write $\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t) = \mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w)$, ignoring the framing parameters w. In Section 4.5.2 we determine a closed formula for this series, proving the conjecture by Awata–Kanno [**6**]. This conjecture was checked for low number of instantons in [**16**, Sec. 4].

To state our first main result, we need to recall the definition of the *plethystic exponential*. Given an arbitrary power series

$$f = f(p_1, \ldots, p_e; u_1, \ldots, u_\ell) \in \mathbb{Q}(p_1, \ldots, p_e)[\![u_1, \ldots, u_\ell]\!],$$

one sets

$$\mathrm{Exp}\,(f) = \exp\left(\sum_{n>0} \frac{1}{n} f(p_1^n, \ldots, p_e^n; u_1^n, \ldots, u_\ell^n)\right), \tag{4.1.4}$$

viewed as an element of $\mathbb{Q}(p_1, \ldots, p_e)[\![u_1, \ldots, u_\ell]\!]$. Consider, for a formal variable x, the operator $[x] = x^{1/2} - x^{-1/2}$. In Section 4.5.1 we consider this operator on $K_0^{\mathbf{T}}(\mathsf{pt})$. We are now able to state our first main result.

Theorem 4.1.1 (Theorem 4.5.3)**.** *The rank r K-theoretic DT partition function of $\mathbb{A}^3$ is given by*

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, (-1)^r q, t) = \mathrm{Exp}\,(\mathsf{F}_r(q, t_1, t_2, t_3)), \tag{4.1.5}$$

where, setting $\mathfrak{t} = t_1 t_2 t_3$, one defines

$$\mathsf{F}_r(q, t_1, t_2, t_3) = \frac{[\mathfrak{t}^r]}{[\mathfrak{t}][\mathfrak{t}^{\frac{r}{2}} q][\mathfrak{t}^{\frac{r}{2}} q^{-1}]} \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]}.$$

The case $r = 1$ of Theorem 4.1.1 was proved by Okounkov in [**149**]. The general case was proposed conjecturally in [**6, 16**].

It is interesting to notice that Formula (4.1.5) is equivalent to the product decomposition

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3,(-1)^r q,t)=\prod_{i=1}^{r}\mathsf{DT}_1^{\mathrm{K}}(\mathbb{A}^3,-q\mathfrak{t}^{\frac{-r-1}{2}+i},t), \tag{4.1.6}$$

that we obtain in Theorem 4.5.9. This is precisely the product formula [**143**, Formula (3.14)] appearing as a limit of the (conjectural) 4-fold theory developed by Nekrasov and Piazzalunga.

Formula (4.1.6) is also related to its motivic cousin: as we observe in Section 4.5.3, the motivic partition function $\mathsf{DT}_r^{\mathrm{mot}}$ of the Quot scheme of points of $\mathbb{A}^3$ (see [**9**, Prop. 2.7] and the references therein) satisfies the same product formula (4.1.6), after the transformation $\mathfrak{t}^{1/2}\to-\mathbb{L}^{1/2}$.

4.1.2.2. Cohomological DT invariants The generating function of *cohomological DT invariants* is defined as

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s,v)=\sum_{n\geq 0}q^n\int_{[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)]^{\mathrm{vir}}}1\,\in\,\mathbb{Q}(\!(s,v)\!)[\![q]\!]$$

where $s=(s_1,s_2,s_3)$ and $v=(v_1,\ldots,v_r)$, with $s_i=c_1^{\mathbf{T}}(t_i)$ and $v_j=c_1^{\mathbf{T}}(w_j)$ respectively, and the integral is defined in Equation (4.3.2) via $\mathbf{T}$-equivariant residues. It is a consequence of (4.1.3) that $\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s,v)$ does not depend on v, so we will shorten it as $\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s)$. In Section 4.6.1 we explain how to recover the cohomological invariants out of the K-theoretic ones. This is the limit formula (Corollary 4.6.1)

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s)=\lim_{b\to 0}\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3,q,e^{bs}),$$

essentially a formal consequence of our explicit expression for the K-theoretic *higher rank vertex* (cf. Section 4.4) attached to the Quot scheme $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)$.

Theorem 4.1.2 (Theorem 4.6.2)**.** *The rank r cohomological DT partition function of $\mathbb{A}^3$ is given by*

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s)=\mathsf{M}((-1)^r q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3}},$$

where $\mathsf{M}(t)=\prod_{m\geq 1}(1-t^m)^{-m}$ is the MacMahon function.

The case $r=1$ of Theorem 4.1.2 was proved by Maulik–Nekrasov–Okounkov–Pandharipande [**128**, Thm. 1]. Theorem 4.1.2 was conjectured by Szabo in [**173**] and confirmed for $r\leq 8$ and $n\leq 8$ in [**16**]. The specialisation $\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s)\big|_{s_1+s_2+s_3=0}=\mathsf{M}((-1)^r q)^r$ was already computed in physics [**59**].

4.1.2.3. Elliptic DT invariants In Section 4.7 we define the *virtual chiral elliptic genus* for any scheme with a perfect obstruction theory, which recovers as a special case the virtual elliptic genus defined in [**75**]. By means of this new invariant we introduce a refinement $\mathsf{DT}_r^{\mathrm{ell}}$ of the generating series $\mathsf{DT}_r^{\mathrm{K}}$, providing a mathematical definition of the *elliptic DT invariants* studied in [**16**]. We propose a conjecture (Conjecture 4.7.8) about the behaviour of $\mathsf{DT}_r^{\mathrm{ell}}$ and, granting this conjecture, we obtain a proof of a conjecture formulated by Benini–Bonelli–Poggi–Tanzini (Theorem 4.7.10).

4.1.2.4. Global geometry So far we have only discussed results concerning *local* geometry. When X is a *projective* toric 3-fold and F is an equivariant exceptional locally free sheaf, by [**165**, Thm. A] there is a 0-dimensional torus equivariant (cf. Proposition 4.8.2) perfect obstruction theory on $\mathrm{Quot}_X(F,n)$. Therefore the higher rank Donaldson–Thomas invariants

$$\mathsf{DT}_{F,n} = \int_{[\mathrm{Quot}_X(F,n)]^{\mathrm{vir}}} 1 \in \mathbb{Z}$$

can be computed via the Graber–Pandharipande virtual localisation formula [**85**]. The next result confirms (in the toric case) a prediction [**165**, Conj. 3.5] for their generating function. Before stating it, recall that an *exceptional sheaf* on a variety X is a coherent sheaf $F \in \mathrm{Coh}\, X$ such that $\mathrm{Hom}(F,F) = \mathbb{C}$ (i.e. F is simple), and $\mathrm{Ext}^i(F,F) = 0$ for $i > 0$.

Theorem 4.1.3 (Theorem 4.8.7)**.** *Let (X,F) be a pair consisting of a smooth projective toric 3-fold X along with an exceptional equivariant locally free sheaf F of rank r. Then*

$$\sum_{n\geq 0} \mathsf{DT}_{F,n} q^n = \mathsf{M}((-1)^r q)^{r\int_X c_3(T_X\otimes K_X)}.$$

The corresponding formula in the Calabi–Yau case was proved in [**165**, Sec. 3.2], whereas the general rank 1 case was proved in [**128**, Thm. 2] and [**122**, Thm. 0.2].

4.1.3. Plan of the chapter Sections 4.2 – 4.7 are devoted to the "local Quot scheme" $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)$. In Section 4.2 we recall its critical structure and we define a $\mathbf{T}$-action on it, whose fixed locus is parametrised by the finitely many r-colored plane partitions (Proposition 4.2.12); we study the equivariant critical obstruction theory on the Quot scheme and prove that the induced virtual class on the $\mathbf{T}$-fixed locus is trivial (Corollary 4.2.15). In Section 4.3.3 we introduce cohomological and K-theoretic DT invariants of $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)$. In Section 4.4 we develop a higher rank vertex formalism which we exploit to write down a formula (Proposition 4.4.2) for the virtual tangent space of a $\mathbf{T}$-fixed point in the Quot scheme. In Section 4.5 we prove Theorem 4.1.1 as well as Formula (4.1.6). In Section 4.6 we prove Theorem 4.1.2 and we show that $\mathsf{DT}_r^{\mathrm{coh}}$ does not depend on any choice of possibly nontrivial $(\mathbb{C}^*)^3$-weights on $\mathscr{O}^{\oplus r}$. In Section 4.7, we give a mathematically rigorous definition of a "chiral" version of the virtual elliptic genus of [**75**] and use it in Section 4.7.2 to define elliptic DT invariants. In Section 4.7.3 we also give closed formulae for elliptic DT invariants in some limiting cases, based on the conjectural independence on the elliptic parameter — see Conjecture 4.7.8 and Remark 4.7.11. In particular Theorem 4.7.10 proves a conjecture recently appeared in the physics literature [**16**, Formula (3.20)]. In Section 4.8 we prove Theorem 4.1.3 by gluing vertex contributions from the toric charts of a projective toric 3-fold.

4.2. The local Quot scheme: critical and equivariant structure

4.2.1. Overview In this section we start working on the local Calabi–Yau 3-fold $\mathbb{A}^3$. Fix integers $r \geq 1$ and $n \geq 0$. Our focus will be on the *local Quot scheme*

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n),$$

whose points correspond to short exact sequences

$$0 \to S \to \mathcal{O}^{\oplus r} \to T \to 0$$

where T is a 0-dimensional $\mathcal{O}_{\mathbb{A}^3}$-module with $\chi(T) = n$.

We shall use the following notation throughout.

Notation 4.2.1. If F is a locally free sheaf on a variety X, and $F \twoheadrightarrow T$ is a surjection onto a 0-dimensional sheaf of length n, with kernel $S \subset F$, we denote by

$$[S] \in \mathrm{Quot}_X(F, n)$$

the corresponding point in the Quot scheme.

In this section, we will:

- recall from [9] the description of the Quot scheme as a critical locus (Sec. 4.2.2),
- describe a $\mathbf{T}$-action (for $\mathbf{T} = (\mathbb{C}^*)^3 \times (\mathbb{C}^*)^r$ a torus of dimension $3 + r$) on $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$, with isolated fixed locus consisting of direct sums of monomial ideals (Sec. 4.2.3),
- reinterpret the fixed locus $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}}$ in terms of colored partitions (Sec. 4.2.4),
- prove that the critical perfect obstruction theory on $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$ is $\mathbf{T}$-equivariant (Lemma 4.2.13), and that the induced $\mathbf{T}$-fixed obstruction theory on the fixed locus is trivial (Corollary 4.2.15).

The content of this section is the starting point for the definition (see Sec. 4.3.3) of virtual invariants on $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$, as well as our construction (see Sec. 4.4) of the *higher rank vertex formalism.*

4.2.2. The critical structure on the Quot scheme Let V be an n-dimensional complex vector space. Consider the space $R_{r,n} = \mathrm{Rep}_{(n,1)}(\widetilde{\mathsf{L}}_3)$ of r-framed $(n, 1)$-dimensional representations of the 3-loop quiver L_3, depicted in Figure 1. The notation "$(n, 1)$" means that the main vertex (the one belonging to the 3-loop quiver, labelled "0" in the figure) carries a copy of V, whereas the framing vertex (labelled "∞") carries a copy of $\mathbb{C}$.

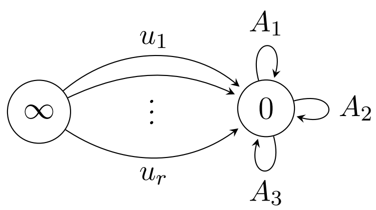

FIGURE 1. The r-framed 3-loop quiver $\widetilde{\mathsf{L}}_3$.

We have that $R_{r,n}$ is an affine space of dimension $3n^2 + rn$, with an explicit description as

$$R_{r,n} = \{ (A_1, A_2, A_3, u_1, \dots, u_r) \mid A_j \in \mathrm{End}(V),\ u_i \in V \}$$

$$= \mathrm{End}(V)^{\oplus 3} \oplus V^{\oplus r}.$$

By [9, Prop. 2.4], there exists a stability parameter θ on the 3-loop quiver such that θ-stable framed representations $(A_1, A_2, A_3, u_1, \ldots, u_r) \in R_{r,n}$ are precisely those satisfying the condition:

$$\text{the vectors } u_1, \ldots, u_r \in V \text{ jointly generate } (A_1, A_2, A_3) \in \mathrm{Rep}_n(\mathsf{L}_3).$$

Imposing this stability condition on $R_{r,n}$ we obtain an open subscheme

$$U_{r,n} \subset R_{r,n}$$

on which $\mathrm{GL}(V)$ acts freely by the rule

$$g \cdot (A_1, A_2, A_3, u_1, \ldots, u_r) = (gA_1g^{-1}, gA_2g^{-1}, gA_3g^{-1}, gu_1, \ldots, gu_r).$$

The quotient

$$\mathrm{Quot}_r^n = U_{r,n} / \mathrm{GL}(V) \tag{4.2.1}$$

is a smooth quasiprojective variety of dimension $2n^2 + rn$. In [9] the scheme Quot_r^n is referred to as the *non-commutative Quot scheme*, by analogy with the *non-commutative Hilbert scheme*, i.e. the moduli space of left ideals of codimension n in $\mathbb{C}\langle x_1, x_2, x_3\rangle$ (which of course exists for an arbitrary number of free variables).

On $R_{r,n}$ one can define the function

$$h_n \colon R_{r,n} \to \mathbb{A}^1, \quad (A_1, A_2, A_3, u_1, \ldots, u_r) \mapsto \mathrm{Tr}\, A_1[A_2, A_3],$$

induced by the superpotential $\mathsf{W} = A_1[A_2, A_3]$ on the 3-loop quiver. Note that this function

- is symmetric under cyclic permutations of A_1, A_2 and A_3, and
- does not touch the vectors u_i, which are only used to define its domain.

Moreover, $h_n|_{U_{r,n}}$ is $\mathrm{GL}(V)$-invariant, and thus descends to a regular function

$$f_n \colon \mathrm{Quot}_r^n \to \mathbb{A}^1. \tag{4.2.2}$$

Proposition 4.2.2 ([9, Thm. 2.6])**.** *There is an identity of closed subschemes*

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) = \mathrm{crit}(f_n) \subset \mathrm{Quot}_r^n.$$

In particular, $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$ *carries a symmetric perfect obstruction theory.*

We use the notation $\mathrm{crit}(f)$ for the zero scheme $\{\, \mathrm{d}f = 0 \,\}$, for f a function on a smooth scheme. The embedding of the Quot scheme inside a non-commutative quiver model had appeared (conjecturally, and in a slightly different language) in the physics literature [59].

Every critical locus $\mathrm{crit}(f)$ has a canonical symmetric obstruction theory, determined by the Hessian complex attached to the function f. It will be referred to as the *critical* *obstruction theory* throughout. In the case of $\mathrm{Q} = \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$, this symmetric obstruction theory is the morphism

$$
\begin{tikzcd}
\mathbb{E}_{\mathrm{crit}} = \arrow[d, "\phi"'] & \big[T_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}} \arrow[r, "\mathsf{Hess}(f_n)"] \arrow[d, "(\mathrm{d}f_n)^\vee|_{\mathrm{Q}}"'] & \Omega_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}}\big] \arrow[d, "\mathrm{id}"] \\
\mathbb{L}_{\mathrm{Q}} = & \big[\mathcal{I}/\mathcal{I}^2 \arrow[r, "\mathrm{d}"] & \Omega_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}}\big]
\end{tikzcd}
\tag{4.2.3}
$$

in $\mathbf{D}^{[-1,0]}(\mathrm{Q})$, where we represented the truncated cotangent complex by means of the exterior derivative d constructed out of the ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathrm{Quot}_r^n}$ of the inclusion $\mathrm{Q} \hookrightarrow \mathrm{Quot}_r^n$.

Remark 4.2.3. As proved by Cazzaniga-Ricolfi in [50], for any integer $m \geq 3$, the Quot scheme $\mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, n)$ is canonically isomorphic to the moduli space of *framed sheaves* on $\mathbb{P}^m$, i.e. the moduli space of pairs (E, φ) where E is a torsion free sheaf on $\mathbb{P}^m$ with Chern character $(r, 0, \ldots, 0, -n)$ and $\varphi\colon E|_D \xrightarrow{\sim} \mathcal{O}_D^{\oplus r}$ is an isomorphism, for $D \subset \mathbb{P}^m$ a fixed hyperplane.

4.2.3. Torus actions on the local Quot scheme In this section we define a torus action on the Quot scheme. Set

$$
\mathbb{T}_1 = (\mathbb{C}^*)^3, \quad \mathbb{T}_2 = (\mathbb{C}^*)^r, \quad \mathbf{T} = \mathbb{T}_1 \times \mathbb{T}_2. \tag{4.2.4}
$$

The torus $\mathbb{T}_1$ acts on $\mathbb{A}^3$ by the standard action

$$
t \cdot x_i = t_i x_i, \tag{4.2.5}
$$

and this action lifts to an action on $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$. At the same time, the torus $\mathbb{T}_2 = (\mathbb{C}^*)^r$ acts on the Quot scheme by scaling the fibres of $\mathcal{O}^{\oplus r}$. Thus we obtain a $\mathbf{T}$-action on $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$.

Remark 4.2.4. The fixed locus $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbb{T}_1}$ is proper. Indeed, a $\mathbb{T}_1$-invariant surjection $\mathcal{O}^{\oplus r} \twoheadrightarrow T$ necessarily has the quotient T entirely supported at the origin $0 \in \mathbb{A}^3$. Hence

$$
\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbb{T}_1} \hookrightarrow \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)_0
$$

sits inside the *punctual Quot scheme* as a closed subscheme. But $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)_0$ is proper, since it is a fibre of the Quot-to-Chow morphism $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) \to \mathrm{Sym}^n \mathbb{A}^3$, which is a proper morphism.

We recall, verbatim from [9, Lemma 2.10], the description of the full $\mathbf{T}$-fixed locus induced by the product action on the local Quot scheme.

Lemma 4.2.5. *There is an isomorphism of schemes*

$$
\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}} = \coprod_{n_1 + \cdots + n_r = n} \prod_{i=1}^{r} \mathrm{Hilb}^{n_i}(\mathbb{A}^3)^{\mathbb{T}_1}. \tag{4.2.6}
$$

In particular, the $\mathbf{T}$-fixed locus is isolated and compact. Moreover, letting $\mathbf{T}_0 \subset \mathbf{T}$ be the subtorus defined by $t_1 t_2 t_3 = 1$, one has a scheme-theoretic identity

$$
\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}_0} = \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}}. \tag{4.2.7}
$$

PROOF. The main result proved by Bifet in [**18**] (in greater generality) implies that

$$\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbb{T}_2} = \coprod_{n_1+\cdots+n_r=n} \prod_{i=1}^{r} \operatorname{Hilb}^{n_i}(\mathbb{A}^3). \tag{4.2.8}$$

The isomorphism (4.2.6) follows by taking $\mathbb{T}_1$-invariants. Since $\operatorname{Hilb}^k(\mathbb{A}^3)^{\mathbb{T}_1}$ is isolated (a disjoint union of reduced points, each corresponding to a monomial ideal of colength k), the first claim follows. Let now $\mathbb{T}_0 \subset \mathbb{T}_1$ be the subtorus defined by $t_1t_2t_3 = 1$, so that $\mathbf{T}_0 = \mathbb{T}_0 \times \mathbb{T}_2$. Equation (4.2.7) follows combining Equation (4.2.8) and the isomorphism $\operatorname{Hilb}^k(\mathbb{A}^3)^{\mathbb{T}_1} \cong \operatorname{Hilb}^k(\mathbb{A}^3)^{\mathbb{T}_0}$ proved in [**13**, Lemma 4.1]. $\square$

Remark 4.2.6. The $\mathbf{T}$-action on $\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)$ just described can be seen as the restriction of a $\mathbf{T}$-action on the larger space Quot_r^n. Indeed, a torus element $\mathbf{t} = (t_1, t_2, t_3, w_1, \ldots, w_r) \in \mathbf{T}$ acts on $(A_1, A_2, A_3, u_1, \ldots, u_r) \in \operatorname{Quot}_r^n$ by

$$\mathbf{t} \cdot P = (t_1A_1, t_2A_2, t_3A_3, w_1u_1, \ldots, w_ru_r). \tag{4.2.9}$$

The critical locus $\operatorname{crit}(f_n)$ is $\mathbf{T}$-invariant, and the induced action is precisely the one we described earlier in this section.

4.2.4. Combinatorial description of the T-fixed locus The $\mathbf{T}$-fixed locus $\operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbf{T}}$ is described purely in terms of *r-colored plane partitions* of size n, as we now explain.

We first recall the definition of a partition of arbitrary dimension.

Definition 4.2.7. Let $d \geq 1$ and $n \geq 0$ be integers. A $(d-1)$-dimensional partition of n is a collection of n points $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbb{Z}^d_{\geq 0}$ with the following property: if $\mathbf{a}_i = (a_{i1}, \ldots, a_{id}) \in \mathcal{A}$, then whenever a point $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{Z}^d_{\geq 0}$ satisfies $0 \leq y_j \leq a_{ij}$ for all $j = 1, \ldots, d$, one has $\mathbf{y} \in \mathcal{A}$. The integer $n = |\mathcal{A}|$ is called the *size* of the partition.

There is a bijective correspondence between the sets of

- $(d-1)$-dimensional partitions of size n,
- $(\mathbb{C}^*)^d$-fixed points of $\operatorname{Hilb}^n(\mathbb{A}^d)$, and
- monomial ideals $I \subset \mathbb{C}[x_1, \ldots, x_d]$ of colength n.

We will be interested in the case $d = 3$, corresponding by definition to *plane partitions*. These can be visualised (cf. Figure 2) as configurations of n boxes stacked in the corner of a room (with gravity pointing in the $(-1,-1,-1)$ direction).

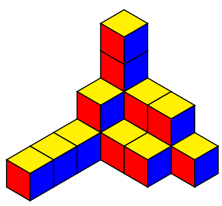

FIGURE 2. A plane partition of size $n = 16$.

Example 4.2.8. If $\mathcal{A}$ is a $(d-1)$-dimensional partition of size n as in Definition 4.2.7, the associated monomial ideal is

$$I_{\mathcal{A}} = \left\langle x_1^{i_1}\cdots x_d^{i_d} \,\middle|\, (i_1,\ldots,i_d)\in \mathbb{Z}_{\geq 0}^d \setminus \mathcal{A}\right\rangle \subset \mathbb{C}[x_1,\ldots,x_d].$$

For instance, if $d=3$, in the case of the plane partition pictured in Figure 2, the associated monomial ideal of colength 16 is generated by the monomials shaping the staircase of the partition, and is thus equal to

$$\left\langle x_3^4, x_1x_3^2, x_1^2x_3, x_1^5, x_1^2x_2, x_1x_2x_3, x_2x_3^2, x_1x_2^3, x_2^3x_3, x_2^4\right\rangle \subset \mathbb{C}[x_1,x_2,x_3].$$

Here is an alternative definition of plane partitions.

Definition 4.2.9. A (finite) *plane partition* is a sequence $\pi = \{\,\pi_{ij} \mid i,j\geq 0\,\} \subset \mathbb{Z}_{\geq 0}$ such that $\pi_{ij}=0$ for $i,j\gg 0$ and

$$\pi_{ij}\geq \pi_{i+1,j}, \quad \pi_{ij}\geq \pi_{i,j+1} \quad \text{for all } i,j\geq 0.$$

Definition 4.2.10. An *r-colored plane partition* is a tuple $\overline{\pi} = (\pi_1,\ldots,\pi_r)$, where each π_i is a plane partition.

Denote by

$$|\pi| = \sum_{i,j\geq 0}\pi_{ij}$$

the *size* of a plane partition (i.e. the number n in Definition 4.2.7) and by $|\overline{\pi}| = \sum_{i=1}^r |\pi_i|$ the size of an r-colored plane partition.

In the light of Definition 4.2.9, the monomial ideal associated to a plane partition π is

$$I_\pi = \left\langle x_1^i x_2^j x_3^{\pi_{ij}} \,\middle|\, i,j\geq 0\right\rangle \subset \mathbb{C}[x_1,x_2,x_3].$$

It is clear that the colength of the ideal I_π is $|\pi|$.

Remark 4.2.11. A general plane partition may have infinite legs, each shaped by (i.e. asymptotic to) a standard (1-dimensional) partition, or Young diagram. We are not concerned with infinite plane partitions here, since we only deal with quotients $\mathscr{O}^{\oplus r} \twoheadrightarrow T$ with finite support.

Proposition 4.2.12. *There is a bijection between* $\mathbf{T}$*-fixed points* $[S]\in \operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)^{\mathbf{T}}$ *and r-colored plane partitions $\overline{\pi}$ of size n.*

Proof. For $r=1$ this is well known: as we recalled above, monomial ideals $I\subset \mathbb{C}[x_1,x_2,x_3]$ are in bijective correspondence with plane partitions. Similarly, to each r-colored plane partition $\overline{\pi}=(\pi_1,\ldots,\pi_r)$ there corresponds a subsheaf $S_{\overline{\pi}} = \bigoplus_{i=1}^r I_{\pi_i} \subset \mathscr{O}^{\oplus r}$. But these are the $\mathbf{T}$-fixed points of the Quot scheme by Lemma 4.2.5. ☐

4.2.4.1. Computing the trace of a monomial ideal Recall the map tr sending a torus representation to its weight space decomposition. Consider the 3-dimensional torus $\mathbb{T}_1$ acting on the coordinate ring $R=\mathbb{C}[x_1,x_2,x_3]$ of $\mathbb{A}^3$. Then we have

$$\operatorname{tr}_R = \sum_{\square\,\in\mathbb{Z}_{\geq 0}^3} t^{\square} = \sum_{(i,j,k)\,\in\mathbb{Z}_{\geq 0}^3} t_1^i t_2^j t_3^k = \frac{1}{(1-t_1)(1-t_2)(1-t_3)}.$$

For a cyclic monomial ideal $\mathfrak{m}_{abc} = x_1^a x_2^b x_3^c \cdot R \subset R$, one has

$$\mathrm{tr}_{\mathfrak{m}_{abc}} = \sum_{i\geq a}\sum_{j\geq b}\sum_{k\geq c} t_1^i t_2^j t_3^k = \frac{t_1^a t_2^b t_3^c}{(1-t_1)(1-t_2)(1-t_3)}.$$

More generally, for a monomial ideal $I_\pi \subset \mathbb{C}[x_1, x_2, x_3]$, one has

$$\mathrm{tr}_{I_\pi} = \sum_{(i,j,k)\notin\pi} t_1^i t_2^j t_3^k. \tag{4.2.10}$$

These are the building blocks needed to compute tr_S for an arbitrary sheaf $S = \bigoplus_{i=1}^r I_{\pi_i}$ corresponding to a $\mathbf{T}$-fixed point $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}}$.

4.2.5. The T-fixed obstruction theory Recall from [85, Prop. 1] that a torus equivariant obstruction theory on a scheme Y induces a canonical perfect obstruction theory, and hence a virtual fundamental class, on each component of the torus fixed locus. In this subsection we show that the reduced isolated locus

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}} \hookrightarrow \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$$

carries the trivial $\mathbf{T}$-fixed perfect obstruction theory, so the induced virtual fundamental class agrees with the actual (0-dimensional) fundamental class.

We first need to check the equivariance of the critical obstruction theory $\mathbb{E}_{\mathrm{crit}}$ obtained in Proposition 4.2.2. In fact, this follows from the general fact that the critical obstruction theory on $\mathrm{crit}(f) \subset Y$, for f a function on a smooth scheme Y, acted on by an algebraic torus $\mathbf{T}$, is naturally $\mathbf{T}$-equivariant as soon as f is $\mathbf{T}$-homogeneous. However, for the sake of completeness, we include a direct proof below for the case at hand.

Lemma 4.2.13. *The critical obstruction theory on* $\mathrm{Q} = \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$ *is* $\mathbf{T}$*-equivariant.*

Proof. We start with two observations:

(1) The potential $f_n = \mathrm{Tr}\, A_1[A_2, A_3]$ recalled in (4.2.2) is homogeneous (of degree 3) in the matrix coordinates of the non-commutative Quot scheme.
(2) The potential f_n satisfies the relation

$$f_n(t\cdot P) = t_1 t_2 t_3 \cdot f_n(P) \tag{4.2.11}$$

for every $t = (t_1, t_2, t_3) \in \mathbb{T}_1$ and $P \in \mathrm{Quot}_r^n$.

Fix a point $p \in \mathrm{Q} = \mathrm{crit}(f_n) \subset \mathrm{Quot}_r^n$. Then, setting $N = 2n^2 + rn = \dim \mathrm{Quot}_r^n$, let $x_1, \ldots, x_N$ be local holomorphic coordinates of Quot_r^n around p. Let the torus $\mathbf{T}$ act on these coordinates as prescribed by Equation (4.2.9), i.e. t_1 (resp. t_2 and t_3) rescales each x_k corresponding to the entries of the first (resp. second and third) matrix, and w_l rescales the coordinates of the vector u_l, for $l = 1, \ldots, r$. Formally, for a matrix coordinate x_k, we set

$$(t_1, t_2, t_3, w_1, \ldots, w_r) \cdot x_k = t_{\ell(k)} x_k$$

where $\ell(k) \in \{1, 2, 3\}$ depends on whether x_k comes from an entry of A_1, A_2 or A_3. We also have to prescribe an action on tangent vectors and 1-forms. For a matrix coordinate x_k, we set

$$(t_1,t_2,t_3,w_1,\ldots,w_r)\cdot\frac{\partial}{\partial x_k}=\frac{t_1t_2t_3}{t_{\ell(k)}}\frac{\partial}{\partial x_k} \tag{4.2.12}$$
$$(t_1,t_2,t_3,w_1,\ldots,w_r)\cdot \mathrm{d}x_k=t_{\ell(k)}\mathrm{d}x_k.$$

If x_k comes from a vector component of the l-th vector, we set

$$(t_1,t_2,t_3,w_1,\ldots,w_r)\cdot\frac{\partial}{\partial x_k}=w_l^{-1}\frac{\partial}{\partial x_k} \tag{4.2.13}$$
$$(t_1,t_2,t_3,w_1,\ldots,w_r)\cdot \mathrm{d}x_k=w_l\mathrm{d}x_k.$$

However, the $\mathbb{T}_2$-action (4.2.13) will be invisible in the Hessian since the function f_n does not touch the vectors.

The Hessian can be seen as a section

$$\mathsf{Hess}(f_n)\in\Gamma\left(\mathrm{Q}, T^*_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}}\otimes T^*_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}}\right).$$

In checking the equivariance relation

$$\mathbf{t}\cdot\mathsf{Hess}(f_n)(\xi)=\mathsf{Hess}(f_n)(\mathbf{t}\cdot\xi),\quad \mathbf{t}\in\mathbf{T},$$

we may ignore local coordinates x_k corresponding to vector entries, because the Hessian is automatically equivariant in these coordinates (equivariance translates into the identity $0=0$).

So, let us fix an x_k coming from one of the matrices. The (i,j)-component of the Hessian applied to $\partial/\partial x_k$ is given by

$$\mathsf{Hess}_{ij}(f_n)\left(\frac{\partial}{\partial x_k}\right)=\frac{\partial^2 f_n}{\partial x_i\partial x_j}(x_1,\ldots,x_N)\,\mathrm{d}x_j.$$

This will vanish unless $k\in\{\,i,j\,\}$. Without loss of generality we may assume $k=i$. In this case we obtain, up to a sign convention,

$$\mathsf{Hess}_{ij}(f_n)\left(\frac{t_1t_2t_3}{t_{\ell(k)}}\frac{\partial}{\partial x_k}\right)=\frac{t_1t_2t_3}{t_{\ell(k)}}\frac{\partial^2 f_n}{\partial x_k\partial x_j}(x_1,\ldots,x_N)\,\mathrm{d}x_j. \tag{4.2.14}$$

On the other hand, combining the observations (1) and (2) with (4.2.12), we obtain

$$\begin{aligned}\mathbf{t}\cdot\mathsf{Hess}_{ij}(f_n)\left(\frac{\partial}{\partial x_k}\right)&=\frac{\partial^2 f_n}{(\partial t_{\ell(k)}x_k)(\partial t_{\ell(j)}x_j)}(t_{\ell(1)}x_1,\ldots,t_{\ell(N)}x_N)t_{\ell(j)}\mathrm{d}x_j\\&=\frac{t_1t_2t_3}{t_{\ell(k)}t_{\ell(j)}}t_{\ell(j)}\frac{\partial^2 f_n}{\partial x_k\partial x_j}(x_1,\ldots,x_N)\,\mathrm{d}x_j,\end{aligned}$$

which agrees with the right hand side of Equation (4.2.14). Thus we conclude that the Hessian complex is $\mathbf{T}$-equivariant, as well as the morphism (4.2.3) to the cotangent complex. This finishes the proof. □

The property (4.2.11) of f_n exhibits the differential $\mathrm{d}f_n$ as a GL_3-equivariant section

$$\mathrm{d}f_n\otimes\mathfrak{t}^{-1}\colon\ \mathscr{O}_{\mathrm{Quot}_r^n}\to\Omega_{\mathrm{Quot}_r^n}\otimes\mathfrak{t}^{-1},$$

where $\mathfrak{t}^{-1} = (t_1t_2t_3)^{-1}$ is the determinant representation of $\mathbb{C}^3 = \bigoplus_{1\leq i\leq 3} t_i^{-1}\cdot\mathbb{C}$. Therefore, explicitly, the morphism in $\mathbf{D}^{[-1,0]}(\mathrm{Coh}_{\mathrm{Q}}^{\mathbf{T}})$ lifting the critical obstruction theory (4.2.3) is

$$\begin{tikzcd}
\left[\mathfrak{t}\otimes T_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}} \arrow[r, "\mathsf{Hess}(f_n)"] \arrow[d, "(\mathrm{d}f_n)^\vee|_{\mathrm{Q}}"'] & \Omega_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}}\right] \arrow[d, "\mathrm{id}"] \\
\left[\mathcal{I}/\mathcal{I}^2 \arrow[r, "\mathrm{d}"] & \Omega_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}}\right]
\end{tikzcd} \tag{4.2.15}$$

where $\mathcal{I}$ is the ideal sheaf, so that, in particular, the equivariant K-theory class of the virtual tangent bundle attached to the (equivariant) perfect obstruction theory (4.2.15) is

$$T_{\mathrm{Q}}^{\mathrm{vir}} = T_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}} - \Omega_{\mathrm{Quot}_r^n}\big|_{\mathrm{Q}}\otimes\mathfrak{t}^{-1} \in K_{\mathbf{T}}^0(\mathrm{Q}). \tag{4.2.16}$$

This fact will be recalled and used in Propositions 4.4.2 and 4.8.3.

Lemma 4.2.13 implies the existence of a "$\mathbf{T}$-fixed" obstruction theory

$$\mathbb{E}_{\mathrm{crit}}\big|^{\mathsf{fix}}_{\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}} \to \mathbb{L}_{\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}} \tag{4.2.17}$$

on the fixed locus of the Quot scheme. We proved in Lemma 4.2.5 that this fixed locus is 0-dimensional, isolated and reduced. The next result will imply that the virtual fundamental class induced by (4.2.17) on the fixed locus agrees with the actual fundamental class.

Proposition 4.2.14. *Let $[S]\in\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}$ be a torus fixed point. The deformations and obstructions of $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)$ at $[S]$ are entirely $\mathbf{T}$-movable. In particular, the virtual tangent space at $[S]$ can be written as*

$$T_S^{\mathrm{vir}} = \mathbb{E}_{\mathrm{crit}}^\vee\big|^{\mathsf{mov}}_{[S]} = T_{\mathrm{Q}}\big|_{[S]} - \Omega_{\mathrm{Q}}\big|_{[S]}\otimes\mathfrak{t}^{-1} \in K_{\mathbf{T}}^0(\mathsf{pt}). \tag{4.2.18}$$

Proof. The perfect obstruction theory $\mathbb{E}_{\mathrm{crit}}$ on $\mathrm{Q}=\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)$, made explicit in Diagram (4.2.15), satisfies $\mathbb{E}_{\mathrm{crit}}\cong\mathbb{E}_{\mathrm{crit}}^\vee[1]\otimes\mathfrak{t}$. Its equivariant K-theory class is therefore

$$\mathbb{E}_{\mathrm{crit}} = \Omega_{\mathrm{Q}} - T_{\mathrm{Q}}\otimes\mathfrak{t} \in K_0^{\mathbf{T}}(\mathrm{Q}).$$

We know by Equation (4.2.7) in Lemma 4.2.5 that no power of $\mathfrak{t}$ is a weight of $T_{\mathrm{Q}}|_p$ for any fixed point $p\in\mathrm{Q}^{\mathbf{T}}$, which implies that

$$\left(T_{\mathrm{Q}}\big|_p\otimes\mathfrak{t}\right)^{\mathsf{fix}} = 0, \quad \Omega_{\mathrm{Q}}\big|_p^{\mathsf{fix}} = 0. \tag{4.2.19}$$

The claim follows. □

Corollary 4.2.15. *There is an identity*

$$\left[\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}\right]^{\mathrm{vir}} = \left[\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}\right] \in A_0^{\mathbf{T}}(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}).$$

4.3. Invariants attached to the local Quot scheme

In this section we introduce cohomological and K-theoretic DT invariants of $\mathbb{A}^3$, the main object of study of this chapter, starting from the Quot scheme $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$ studied in the previous section. We first need to introduce some notation and terminology.

4.3.1. Some notation Recall the tori $\mathbb{T}_1 = (\mathbb{C}^*)^3$ and $\mathbb{T}_2 = (\mathbb{C}^*)^r$ from (4.2.4). We let t_1, t_2, t_3 and $w_1, \ldots, w_r$ be the generators of the representation rings $K^0_{\mathbb{T}_1}(\mathsf{pt})$ and $K^0_{\mathbb{T}_2}(\mathsf{pt})$, respectively. Then one can write the equivariant cohomology rings of $\mathbb{T}_1$ and $\mathbb{T}_2$ as

$$H^*_{\mathbb{T}_1} = \mathbb{Q}[s_1, s_2, s_3], \quad H^*_{\mathbb{T}_2} = \mathbb{Q}[v_1, \ldots, v_r],$$

where $s_i = c_1^{\mathbb{T}_1}(t_i)$ and $v_j = c_1^{\mathbb{T}_2}(w_j)$. For a virtual $\mathbf{T}$-module $V \in K^0_{\mathbf{T}}(\mathsf{pt})$, we let

$$\mathrm{tr}_V \in \mathbb{Z}(\!(t_1, t_2, t_3, w_1, \ldots, w_r)\!)$$

denote its character, i.e. its decomposition into weight spaces. We denote by $\overline{(\cdot)}$ the involution defined on $\mathbb{Z}(\!(t_1, t_2, t_3, w_1, \ldots, w_r)\!)$ by

$$\overline{P}(t_1, t_2, t_3, w_1, \ldots, w_r) = P(t_1^{-1}, t_2^{-1}, t_3^{-1}, w_1^{-1}, \ldots, w_r^{-1}).$$

4.3.1.1. Twisted virtual structure sheaf For any scheme X endowed with a perfect obstruction theory $\mathbb{E} \to \mathbb{L}_X$, define as in [**75**, Def. 3.12], the *virtual canonical bundle*

$$\mathcal{K}_{X,\mathrm{vir}} = \det \mathbb{E} = \det(T_X^{\mathrm{vir}})^\vee.$$

This is just $\det E^0 \otimes (\det E^{-1})^\vee$ if $\mathbb{E} = E^0 - E^{-1} \in K^0(X)$. We will simply write $\mathcal{K}_{\mathrm{vir}}$ when X is clear from the context.

Lemma 4.3.1. *Let A be a smooth variety equipped with a regular function $f \colon A \to \mathbb{A}^1$, and let $X = \mathrm{crit}(f) \subset A$ be the critical locus of f, with its critical (symmetric) perfect obstruction theory $\mathbb{E}_{\mathrm{crit}} \to \mathbb{L}_X$. Then $\mathcal{K}_{X,\mathrm{vir}} \in \mathrm{Pic}(X)$ admits a square root, i.e. there exists a line bundle*

$$\mathcal{K}_{X,\mathrm{vir}}^{\frac{1}{2}} \in \mathrm{Pic}(X)$$

whose second tensor power equals $\det \mathbb{E}_{\mathrm{crit}}$.

Proof. The K-theory class of the critical perfect obstruction theory is

$$\mathbb{E}_{\mathrm{crit}} = \Omega_A\big|_X - T_A\big|_X,$$

and by definition one has

$$\mathcal{K}_{X,\mathrm{vir}} = \frac{\det \Omega_A|_X}{\det T_A|_X} = \frac{\det \Omega_A|_X}{(\det \Omega_A|_X)^{-1}} = K_A\big|_X \otimes K_A\big|_X. \qquad \square$$

Let X be a scheme endowed with a perfect obstruction theory, and let $\mathcal{O}_X^{\mathrm{vir}} \in K_0(X)$ be the induced virtual structure sheaf. Assume the virtual canonical bundle admits a square root. Following [**142**], we define the *twisted* (or *symmetrised*) *virtual structure sheaf* as

$$\widehat{\mathcal{O}}_X^{\mathrm{vir}} = \mathcal{O}_X^{\mathrm{vir}} \otimes \mathcal{K}_{X,\mathrm{vir}}^{\frac{1}{2}}.$$

In case X carries a torus action, we will see in Remark 4.3.2 that $\widehat{\mathcal{O}}_X^{\mathrm{vir}}$ acquires a canonical weight.

4.3.2. Classical enumerative invariants This chapter is concerned with the general theory of *Quot schemes*, hence in the (virtual) enumeration of 0-dimensional quotients of locally free sheaves on 3-folds.

The naive (topological) Euler characteristic of the Quot scheme $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$ is computed via the Gholampour–Kool formula [**81**, Prop. 2.3]

$$\sum_{n\geq 0} e(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n))q^n = \mathsf{M}(q)^r,$$

where $\mathsf{M}(q) = \prod_{m\geq 1}(1-q^m)^{-m}$ is the MacMahon function, the generating function for the number of plane partitions of non-negative integers. On the other hand, the Behrend weighted Euler characteristic of the Quot scheme can be computed via the formula

$$\sum_{n\geq 0} e_{\mathrm{vir}}(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n))q^n = \mathsf{M}((-1)^r q)^r, \tag{4.3.1}$$

proved in [**9**, Cor. 2.8]. For a complex scheme Z of finite type over $\mathbb{C}$, we have set $e_{\mathrm{vir}}(Z) = e(Z, \nu_Z)$, where ν_Z is Behrend's constructible function [**10**]. See [**9**, Thm. A] for a proof of the analogue of (4.3.1) for an arbitrary pair (Y, F) consisting of a smooth 3-fold Y along with a locally free sheaf F on it. It was shown by Toda [**181**] that, on a projective Calabi–Yau 3-fold Y, the wall-crossing factor in the higher rank DT/PT correspondence is precisely $\mathsf{M}((-1)^r q)^{re(Y)}$. The relationship between Toda's wall-crossing formula [**181**] and the Gholampour–Kool's formula for Euler characteristics of Quot schemes on 3-folds [**81**] was clarified in [**9**] via a Hall algebra argument.

4.3.3. Virtual invariants of the Quot scheme The scheme $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$ is not proper, but carries a torus action with proper fixed locus. Thus we may define virtual invariants via equivariant residues, by setting

$$\int_{[\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)]^{\mathrm{vir}}} 1 := \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}} \frac{1}{e^{\mathbf{T}}(T_S^{\mathrm{vir}})} \in \mathbb{Q}(\!(s,v)\!), \tag{4.3.2}$$

where $s = (s_1, s_2, s_3)$ and $v = (v_1, \ldots, v_r)$ are the equivariant parameters of the torus $\mathbf{T}$ and T_S^{vir} is the virtual tangent space (4.2.18). The sum runs over all $\mathbf{T}$-fixed points $[S]$, which are isolated, reduced and with the trivial perfect obstruction theory induced from the critical obstruction theory on the Quot scheme (cf. Corollary 4.2.15). We refer to these invariants as (degree 0) *cohomological rank r DT invariants*, as they take value in (an extension of) the fraction field $\mathbb{Q}(s, v)$ of the $\mathbf{T}$-equivariant cohomology ring $H^*_{\mathbf{T}}$. We will study their generating function

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v) = \sum_{n\geq 0} q^n \int_{[\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)]^{\mathrm{vir}}} 1 \in \mathbb{Q}(\!(s,v)\!)[\![q]\!] \tag{4.3.3}$$

in Section 4.6. On the other hand, K-theoretic invariants arise as natural refinements of their cohomological counterpart. Naively, one would like to study the virtual holomorphic Euler characteristic

$$\chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n),\mathcal{O}^{\mathrm{vir}}) = \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}} \mathrm{tr}\left(\frac{1}{\Lambda^\bullet T_S^{\mathrm{vir},*}}\right) \in \mathbb{Z}(\!(t,w)\!),$$

where $t=(t_1,t_2,t_3)$, $w=(w_1,\ldots,w_r)$, and via the trace map tr we identify a (possibly infinite-dimensional) virtual $\mathbf{T}$-module with its decomposition into weight spaces. It turns out that guessing a closed formula for these invariants is incredibly difficult and, after all, not what one should look at. Instead, Nekrasov–Okounkov [**142**] teach us that we should focus our attention on

(4.3.4)
$$\chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n),\widehat{\mathcal{O}}^{\mathrm{vir}}) = \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}} \mathrm{tr}\left(\frac{\mathcal{K}_{\mathrm{vir}}^{\frac{1}{2}}}{\Lambda^\bullet T_S^{\mathrm{vir},*}}\right) \in \mathbb{Z}(\!(t,(t_1t_2t_3)^{\frac{1}{2}},w)\!),$$ [2]

where the *twisted virtual structure sheaf* $\widehat{\mathcal{O}}^{\mathrm{vir}}$ is defined in Section 4.3.1.1 — a square root of the virtual canonical bundle exists by Lemma 4.3.1 and Proposition 4.2.2. The generating function of rank r K-theoretic DT invariants

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3,q,t,w) = \sum_{n\geq 0} q^n \chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n),\widehat{\mathcal{O}}^{\mathrm{vir}}) \in \mathbb{Z}(\!(t,(t_1t_2t_3)^{\frac{1}{2}},w)\!)[\![q]\!] \tag{4.3.5}$$

will be studied in Section 4.5.

Remark 4.3.2. To be precise, we should replace the torus $\mathbf{T}$ with its double cover $\mathbf{T}_{\mathfrak{t}}$, the minimal cover of $\mathbf{T}$ where the character $\mathfrak{t}^{-1/2}$ is defined, as in [**142**, Sec. 7.1.2]. Then $\mathcal{K}_{\mathrm{vir}}^{1/2}$ is a $\mathbf{T}_{\mathfrak{t}}$-equivariant sheaf with character $\mathfrak{t}^{-(\dim \mathrm{Quot}_r^n)/2}$. To ease the notation, we keep denoting the torus acting as $\mathbf{T}$.

Remark 4.3.3. As remarked in [**142, 3**], choices of square roots of $\mathcal{K}_{\mathrm{vir}}$ differ by a 2-torsion element in the Picard group, which implies that $\chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n),\widehat{\mathcal{O}}^{\mathrm{vir}})$ does not depend on such choices of square roots. Thus there is no ambiguity in (4.3.5).

4.4. Higher rank vertex on the local Quot scheme

4.4.1. The virtual tangent space of the local Quot scheme By Lemma 4.2.5, we can represent the sheaf corresponding to a $\mathbf{T}$-fixed point

$$[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}$$

as a direct sum of ideal sheaves

$$S=\bigoplus_{\alpha=1}^r \mathcal{I}_{Z_\alpha} \subset \mathcal{O}^{\oplus r},$$

with $Z_\alpha\subset\mathbb{A}^3$ a finite subscheme of length n_α supported at the origin, and satisfying $n=\sum_{1\leq\alpha\leq r} n_\alpha$. In this subsection we derive a formula for the character of

$$\chi(\mathcal{O}^{\oplus r},\mathcal{O}^{\oplus r})-\chi(S,S)\in K_{\mathbf{T}}^0(\mathsf{pt}),$$

[2]In theory all equivariant weights could appear with half powers. However, by Remark 4.3.2, $(t_1t_2t_3)^{\frac{1}{2}}$ is the only one that truly appears.

where for F and G in $K^0_{\mathbf{T}}(\mathsf{pt})$, we set

$$\chi(F,G) = \mathbf{R}\operatorname{Hom}(F,G) = \sum_{i\geq 0}(-1)^i \operatorname{Ext}^i(F,G).$$

Our method follows the approach of [**127**, Sec. 4.7]. Moreover, we show in Proposition 4.4.2 that such character agrees with the virtual tangent space T^{vir}_S induced by the critical obstruction theory.

Let Q_α be the $\mathbb{T}_1$-character of the α-summand of $\mathcal{O}^{\oplus r}/S$, i.e. (cf. Equation (4.2.10))

$$\mathsf{Q}_\alpha = \operatorname{tr}_{\mathcal{O}_{Z_\alpha}} = \sum_{(i,j,k)\,\in\,\pi_\alpha} t_1^i t_2^j t_3^k,$$

where $\pi_\alpha \subset \mathbb{Z}^3_{\geq 0}$ is the plane partition corresponding to the monomial ideal $\mathcal{I}_{Z_\alpha} \subset R = \mathbb{C}[x,y,z]$. Let $\mathsf{P}_\alpha(t_1,t_2,t_3)$ be the Poincaré polynomial of $\mathcal{I}_{Z_\alpha}$. This can be computed via a $\mathbb{T}_1$-equivariant free resolution

$$0 \to E_{\alpha,s} \to \cdots \to E_{\alpha,1} \to E_{\alpha,0} \to \mathcal{I}_{Z_\alpha} \to 0.$$

Writing

$$E_{\alpha,i} = \bigoplus_j R(d_{\alpha,ij}), \quad d_{\alpha,ij} \in \mathbb{Z}^3,$$

one has, independently of the chosen resolution, the formula

$$\mathsf{P}_\alpha(t_1,t_2,t_3) = \sum_{i,j}(-1)^i t^{d_{\alpha,ij}}.$$

By [**127**, Sec. 4.7] we know that there is an identity

$$\mathsf{Q}_\alpha = \frac{1+\mathsf{P}_\alpha}{(1-t_1)(1-t_2)(1-t_3)}. \tag{4.4.1}$$

For each $1 \leq \alpha, \beta \leq r$, we can compute

$$\begin{aligned}\chi(\mathcal{I}_{Z_\alpha}, \mathcal{I}_{Z_\beta}) &= \sum_{i,j,k,l}(-1)^{i+k}\operatorname{Hom}_R(R(d_{\alpha,ij}), R(d_{\beta,kl})) \\ &= \sum_{i,j,k,l}(-1)^{i+k} R(d_{\beta,kl} - d_{\alpha,ij}),\end{aligned}$$

which immediately yields the identity

$$\operatorname{tr}_{\chi(\mathcal{I}_{Z_\alpha},\mathcal{I}_{Z_\beta})} = \frac{\mathsf{P}_\beta \overline{\mathsf{P}}_\alpha}{(1-t_1)(1-t_2)(1-t_3)} \in \mathbb{Z}(\!(t_1,t_2,t_3)\!).$$

It follows that, as a $\mathbf{T}$-representation, one has

$$\begin{aligned}\chi(S,S) &= \chi\left(\sum_\alpha w_\alpha \otimes \mathcal{I}_{Z_\alpha}, \sum_\beta w_\beta \otimes \mathcal{I}_{Z_\beta}\right) \\ &= \sum_{1\leq \alpha,\beta\leq r} \chi(w_\alpha \otimes \mathcal{I}_{Z_\alpha}, w_\beta \otimes \mathcal{I}_{Z_\beta}),\end{aligned}$$

which yields

$$\operatorname{tr}_{\chi(S,S)} = \sum_{1\leq\alpha,\beta\leq r} \frac{w_\alpha^{-1} w_\beta \cdot \mathsf{P}_\beta \overline{\mathsf{P}}_\alpha}{(1-t_1)(1-t_2)(1-t_3)}. \tag{4.4.2}$$

On the other hand,

$$\mathrm{tr}_{\chi(\mathcal{O}^{\oplus r},\mathcal{O}^{\oplus r})} = \sum_{1\le\alpha,\beta\le r} \frac{w_\alpha^{-1}w_\beta}{(1-t_1)(1-t_2)(1-t_3)}. \tag{4.4.3}$$

Combining Formulae (4.4.2) and (4.4.3) with Formula (4.4.1) yields the following result.

Proposition 4.4.1. *Let $[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}$ be a torus fixed point. There is an identity*

$$\mathrm{tr}_{\chi(\mathcal{O}^{\oplus r},\mathcal{O}^{\oplus r})-\chi(S,S)} = \sum_{1\le\alpha,\beta\le r} w_\alpha^{-1}w_\beta\left(\mathsf{Q}_\beta - \frac{\overline{\mathsf{Q}}_\alpha}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}\mathsf{Q}_\beta\overline{\mathsf{Q}}_\alpha\right) \tag{4.4.4}$$

in $\mathbb{Z}(\!(t_1,t_2,t_3,w_1,\ldots,w_r)\!)$.

For every $\mathbf{T}$-fixed point $[S]$ we define associated *vertex* terms

$$\mathsf{V}_{ij} = w_i^{-1}w_j\left(\mathsf{Q}_j - \frac{\overline{\mathsf{Q}}_i}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}\mathsf{Q}_j\overline{\mathsf{Q}}_i\right) \tag{4.4.5}$$

for every $i,j=1,\ldots,r$. It is immediate to see that for $r=1$ (forcing $i=j$) we recover the vertex formalism developed in [**127**].

Proposition 4.4.1 can then be restated as

$$\mathrm{tr}_{\chi(\mathcal{O}^{\oplus r},\mathcal{O}^{\oplus r})} - \mathrm{tr}_{\chi(S,S)} = \sum_{1\le i,j\le r}\mathsf{V}_{ij}.$$

We now relate this to the virtual tangent space T_S^{vir} of a point $[S]\in\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}$.

Proposition 4.4.2. *Let $[S]\in\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}$ be a $\mathbf{T}$-fixed point. Let $T_S^{\mathrm{vir}} = \mathbb{E}_{\mathrm{crit}}^\vee\big|_{[S]}$ be the virtual tangent space induced by the $\mathbf{T}$-equivariant critical obstruction theory. Then there are identities*

$$T_S^{\mathrm{vir}} = \chi(\mathcal{O}^{\oplus r},\mathcal{O}^{\oplus r}) - \chi(S,S) = \sum_{1\le i,j\le r}\mathsf{V}_{ij}\in K_0^{\mathbf{T}}(\mathsf{pt}).$$

Proof. Let Quot_r^n be the non-commutative Quot scheme defined in (4.2.1). The superpotential $f_n\colon \mathrm{Quot}_r^n\to\mathbb{A}^1$ defined in (4.2.2) is equivariant with respect to the character $(t_1,t_2,t_3)\mapsto t_1t_2t_3$, so it gives rise to a GL_3-equivariant section

$$\mathcal{O}_{\mathrm{Quot}_r^n}\xrightarrow{\mathrm{d}f_n\otimes\mathfrak{t}^{-1}}\Omega_{\mathrm{Quot}_r^n}\otimes\mathfrak{t}^{-1} \tag{4.4.6}$$

where, starting from the representation $\mathbb{C}^3=\bigoplus_{1\le i\le 3}t_i^{-1}\cdot\mathbb{C}\in K_{\mathbb{T}_1}^0(\mathsf{pt})$, we have set

$$\mathfrak{t}^{-1} = \det\mathbb{C}^3 = (t_1t_2t_3)^{-1}.$$

Here, and throughout this proof, we are identifying a representation with its own character. The zero locus of the section (4.4.6) is our Quot scheme

$$\mathrm{Q} = \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n),$$

endowed with the $\mathbf{T}$-action described in Section 4.2.3. According to Equation (4.2.16), the virtual tangent space computed with respect to the critical $\mathbf{T}$-equivariant obstruction theory on Q is

$$T_S^{\mathrm{vir}} = \left(T_{\mathrm{Quot}_r^n} - \Omega_{\mathrm{Quot}_r^n} \otimes \mathfrak{t}^{-1}\right)\big|_{[S]}. \tag{4.4.7}$$

The tangent space to the smooth scheme Quot_r^n can be written, around a point $S \hookrightarrow \mathcal{O}^{\oplus r} \twoheadrightarrow V$, as

$$T_{\mathrm{Quot}_r^n}\big|_{[S]} = (\mathbb{C}^3 - 1) \otimes (\overline{V} \otimes V) + \bigoplus_{\alpha=1}^{r} \mathrm{Hom}(w_\alpha \mathbb{C}, V), \tag{4.4.8}$$

where

$$\bigoplus_{\alpha=1}^{r} \mathrm{Hom}(w_\alpha \mathbb{C}, V) = \bigoplus_{\alpha=1}^{r} w_\alpha^{-1} \otimes V$$

represents the r framings on the 3-loop quiver. Let V be written as a direct sum of structure sheaves

$$V = \bigoplus_{\alpha=1}^{r} \mathcal{O}_{Z_\alpha},$$

where the α-summand has $\mathbf{T}$-character $w_\alpha \mathsf{Q}_\alpha$. Substituting

$$\mathbb{C}^3 - 1 = t_1^{-1} + t_2^{-1} + t_3^{-1} - 1 = \frac{t_1t_2 + t_1t_3 + t_2t_3 - t_1t_2t_3}{t_1t_2t_3}$$

$$V = \sum_{\alpha=1}^{r} w_\alpha \mathsf{Q}_\alpha$$

into Formula (4.4.8) yields

$$T_{\mathrm{Quot}_r^n}\big|_{[S]} = \frac{t_1t_2 + t_1t_3 + t_2t_3 - t_1t_2t_3}{t_1t_2t_3} \sum_{1\le \alpha,\beta\le r} w_\alpha^{-1} w_\beta \overline{\mathsf{Q}}_\alpha \mathsf{Q}_\beta + \sum_{1\le \alpha,\beta\le r} w_\alpha^{-1} w_\beta \mathsf{Q}_\beta,$$

and hence

$$\left(\Omega_{\mathrm{Quot}_r^n} \otimes \mathfrak{t}^{-1}\right)\big|_{[S]} = \frac{t_1 + t_2 + t_3 - 1}{t_1t_2t_3} \sum_{1\le \alpha,\beta\le r} w_\alpha^{-1} w_\beta \overline{\mathsf{Q}}_\alpha \mathsf{Q}_\beta + \sum_{1\le \alpha,\beta\le r} w_\alpha^{-1} w_\beta \frac{\overline{\mathsf{Q}}_\alpha}{t_1t_2t_3},$$

which by Formula (4.4.7) yields

$$T_S^{\mathrm{vir}} = \sum_{1\le \alpha,\beta\le r} w_\alpha^{-1} w_\beta \left(\mathsf{Q}_\beta - \frac{\overline{\mathsf{Q}}_\alpha}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}\overline{\mathsf{Q}}_\alpha \mathsf{Q}_\beta\right).$$

The right hand side is shown to be equal to $\chi(\mathcal{O}^{\oplus r}, \mathcal{O}^{\oplus r}) - \chi(S,S)$ in Proposition 4.4.1. $\square$

4.4.2. A small variation of the vertex formalism All locally free sheaves on $\mathbb{A}^3$ are trivial, but this is not true equivariantly. For example, we have $K_{\mathbb{A}^3} = \mathcal{O}_{\mathbb{A}^3} \otimes t_1t_2t_3 \in K^0_{\mathbb{T}_1}(\mathbb{A}^3)$, even though the ordinary canonical bundle is trivial. Consider

$$F = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{A}^3} \otimes \lambda_i \in K^0_{\mathbb{T}_1}(\mathbb{A}^3) \tag{4.4.9}$$

where $\lambda = (\lambda_i)_i$ are weights of the $\mathbb{T}_1$-action, i.e. monomials in the representation ring of $\mathbb{T}_1$. Let $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(F,n)^{\mathbf{T}}$ be a $\mathbf{T}$-fixed point. It decomposes as $S = \bigoplus_{i=1}^{r} \mathcal{I}_{Z_i} \otimes \lambda_i \in K^0_{\mathbb{T}_1}(\mathbb{A}^3)$, where the weights λ_i are naturally inherited from F. This generalises the discussion in Section 4.4.1, which can be recovered by setting all weights λ_i to be trivial. Just as in Proposition 4.4.2, we can compute

$$T^{\mathrm{vir}}_{S,\lambda} = \chi(F,F) - \chi(S,S) \in K^0_{\mathbf{T}}(\mathsf{pt}).$$

We find

$$T^{\mathrm{vir}}_{S,\lambda} = \chi\left(\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{A}^3} \otimes \lambda_i w_i, \bigoplus_{j=1}^r \mathcal{O}_{\mathbb{A}^3} \otimes \lambda_j w_j\right) - \chi\left(\bigoplus_{j=1}^r \mathcal{I}_{Z_i} \otimes \lambda_i w_i, \bigoplus_{i=1}^r \mathcal{I}_{Z_j} \otimes \lambda_j w_j\right).$$

Therefore we derive the same expression for the vertex formalism as before, just substituting w_i with $\lambda_i w_i$. This variation will be crucial in Section 4.8.3 where we indentify the Quot scheme of the local model with the restriction of $\mathrm{Quot}_X(F,n)$ to an open toric chart, with X a toric projective 3-fold and F a $\mathbb{T}_1$-equivariant exceptional locally free sheaf.

Define, for $\lambda = (\lambda_1, \ldots, \lambda_r)$ as above and F as in (4.4.9), the equivariant integral

$$\int_{[\mathrm{Quot}_{\mathbb{A}^3}(F,n)]^{\mathrm{vir}}} 1 := \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)^{\mathbf{T}}} \frac{1}{e^{\mathbf{T}}(T^{\mathrm{vir}}_{S,\lambda})} \in \mathbb{Q}(\!(s,v)\!), \tag{4.4.10}$$

and let

$$\mathsf{DT}^{\mathrm{coh}}_r(\mathbb{A}^3, q, s, v)_\lambda = \sum_{n\geq 0} q^n \int_{[\mathrm{Quot}_{\mathbb{A}^3}(F,n)]^{\mathrm{vir}}} 1 \tag{4.4.11}$$

be the generating function of the invariants (4.4.10). We shall see (cf. Corollary 4.6.4) that this expression does not depend on the equivariant weights λ_i.

4.5. The higher rank K-theoretic DT partition function

4.5.1. Symmetrised exterior algebras and brackets We recall some constructions is equivariant K-theory which will be used to prove Theorem 4.1.1. For a recent and more complete reference, the reader may consult [**149**, Sec. 2].

Let $\mathbf{T}$ be a torus, $V = \sum_\mu t^\mu$ a $\mathbf{T}$-module. Assume that $\det(V)$ is a square in $K^0_{\mathbf{T}}(\mathsf{pt})$. Define the *symmetrised exterior algebra* of V as

$$\widehat{\Lambda}^\bullet V := \frac{\Lambda^\bullet V}{\det(V)^{\frac{1}{2}}}.$$

It satisfies the relation

$$\widehat{\Lambda}^\bullet V^\vee = (-1)^{\mathrm{rk}\, V} \widehat{\Lambda}^\bullet V.$$

Define the operator $[\,\cdot\,]$ by

$$[t^\mu] = t^{\frac{\mu}{2}} - t^{-\frac{\mu}{2}}.$$

One can compute

$$\mathrm{tr}(\widehat{\Lambda}^\bullet V^\vee) = \prod_\mu \frac{1-t^{-\mu}}{t^{-\frac{\mu}{2}}} = \prod_\mu (t^{\frac{\mu}{2}} - t^{-\frac{\mu}{2}}) = \prod_\mu [t^\mu],$$

For a virtual $\mathbf{T}$-representation $V = \sum_\mu t^\mu - \sum_\nu t^\nu$, where the weight $\nu = 0$ never appears, we extend $\widehat{\Lambda}^\bullet$ and $[\,\cdot\,]$ by linearity and find

$$\mathrm{tr}(\widehat{\Lambda}^\bullet V^\vee) = \frac{\prod_\mu [t^\mu]}{\prod_\nu [t^\nu]}.$$

4.5.2. Proof of Theorem 4.1.1 By the description of the $\mathbf{T}$-fixed locus given in § 4.2.4, every colored plane partition $\overline{\pi} = (\pi_1, \ldots, \pi_r)$ corresponds to a unique $\mathbf{T}$-fixed point $S = \bigoplus_{i=1}^r \mathcal{I}_{Z_i} \in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}}$, for which we defined in Equation (4.4.5) the vertex terms V_{ij} by

$$\mathsf{V}_{ij} = w_i^{-1} w_j \left(\mathsf{Q}_j - \frac{\overline{\mathsf{Q}}_i}{t_1 t_2 t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} \mathsf{Q}_j \overline{\mathsf{Q}}_i \right)$$

with notation as in Section 4.4.1. The generating function (4.3.3) of higher rank cohomological DT invariants can be rewritten in a purely combinatorial fashion as

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v) = \sum_{\overline{\pi}} q^{|\overline{\pi}|} \prod_{i,j=1}^r e^{\mathbf{T}}(-\mathsf{V}_{ij}).$$

Similarly, the generating function (4.3.5) of the K-theoretic invariants can be rewritten as

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w) = \sum_{\overline{\pi}} q^{|\overline{\pi}|} \prod_{i,j=1}^r [-\mathsf{V}_{ij}].$$

A closed formula for $\mathsf{DT}_1^{\mathrm{K}}(\mathbb{A}^3, q, t, w)$ was conjectured in [**138**] and has recently been proven by Okounkov.

Theorem 4.5.1 ([**149**, Thm. 3.3.6])**.** *The rank* 1 *K-theoretic DT partition function of* $\mathbb{A}^3$ *is given by*

$$\mathsf{DT}_1^{\mathrm{K}}(\mathbb{A}^3, -q, t, w) = \mathrm{Exp}\left(\mathsf{F}_1(q, t_1, t_2, t_3)\right)$$

where, setting $\mathfrak{t} = t_1 t_2 t_3$*, one defines*

$$\mathsf{F}_1(q, t_1, t_2, t_3) = \frac{1}{[\mathfrak{t}^{\frac{1}{2}} q][\mathfrak{t}^{\frac{1}{2}} q^{-1}]} \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]}.$$

Remark 4.5.2. It is clear from the expression of the vertex in rank 1 that there is no dependence on w_1. This can in fact be seen as a shadow of the fact that $\mathbb{T}_2 = \mathbb{C}^*$ acts trivially on Quot_1^n and on $\mathrm{d}f_n$. Surprisingly, the same phenomenon occurs in the higher rank case (cf. Theorem 4.5.5).

We devote the rest of this section to proving a generalisation of Theorem 4.5.1 to higher rank.

Theorem 4.5.3. *The rank* r *K-theoretic DT partition function of* $\mathbb{A}^3$ *is given by*

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, (-1)^r q, t, w) = \mathrm{Exp}\left(\mathsf{F}_r(q, t_1, t_2, t_3)\right),$$

where, setting $\mathfrak{t} = t_1 t_2 t_3$*, one defines*

$$\mathsf{F}_r(q, t_1, t_2, t_3) = \frac{[\mathfrak{t}^r]}{[\mathfrak{t}][\mathfrak{t}^{\frac{r}{2}} q][\mathfrak{t}^{\frac{r}{2}} q^{-1}]} \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]}.$$

Remark 4.5.4. This result was conjectured in [**6**] by Awata and Kanno, who also proved it modq^4, i.e. up to 3 instantons. The conjecture was confirmed numerically up to some order by Benini–Bonelli–Poggi–Tanzini [**16**].

The proof of Theorem 4.5.3 will follow essentially by taking suitable limits of the weights w_i. To perform such limits, we prove the slogan (4.1.3), already anticipated in the Introduction.

Theorem 4.5.5. *The generating function* $\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w)$ *does not depend on the weights* $w_1, \ldots, w_r$.

PROOF. The n-th coefficient of $\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w)$ is a sum of contributions

$$[-T_{\overline{\pi}}^{\mathrm{vir}}], \qquad |\overline{\pi}| = n.$$

A simple manipulation shows that

$$[-T_{\overline{\pi}}^{\mathrm{vir}}] = A(t) \prod_{1 \le i < j \le r} \frac{\prod_{\mu_{ij}} w_i - w_j t^{\mu_{ij}}}{\prod_{\nu_{ij}} w_i - w_j t^{\nu_{ij}}} = A(t) \prod_{1 \le i < j \le r} \frac{\prod_{\mu_{ij}} (1 - w_i^{-1} w_j t^{\mu_{ij}})}{\prod_{\nu_{ij}} (1 - w_i^{-1} w_j t^{\nu_{ij}})}, \tag{4.5.1}$$

where $A(t) \in \mathbb{Q}(\!(t_1, t_2, t_3, (t_1t_2t_3)^{\frac{1}{2}})\!)$ and the number of weights μ_{ij} and ν_{ij} is the same. Thus, $\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w)$ is a homogeneous rational expression of total degree 0 with respect to the variables $w_1, \ldots, w_r$. We aim to show that $\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w)$ has no poles of the form $1 - w_i^{-1} w_j t^{\nu_{ij}}$, implying that it is a degree 0 polynomial in the w_i, hence constant in the w_i. This generalises the strategy of [**6**, Sec. 4].

Set $\mathsf{w} = w_i^{-1} w_j t^{\nu}$ for fixed $i < j$ and $\nu \in \widehat{\mathbb{T}}_1$. To see that $1 - \mathsf{w}$ is not a pole, we use [**3**, Prop. 3.2], which asserts the following: if M is a quasiprojective $\mathbf{T}$-scheme with a $\mathbf{T}$-equivariant perfect obstruction theory and proper (nonempty) fixed locus, then for any $V \in K_0^{\mathbf{T}}(M)$, the only poles of the form $(1 - \mathsf{w})$ that *may* appear in $\chi(M, V \otimes \mathcal{O}^{\mathrm{vir}})$ arise from *noncompact weights* $\mathsf{w} \in \widehat{\mathbf{T}}$. A weight w is called compact if the fixed locus $M^{\mathbf{T}_{\mathsf{w}}} \subset M$ is proper, where $\mathbf{T}_{\mathsf{w}}$ is the maximal torus in $\ker(\mathsf{w}) \subset \mathbf{T}$, and is called noncompact otherwise [**3**, Def. 3.1].

We of course want to apply [**3**, Prop. 3.2] to $M = \mathrm{Q} = \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$, $\mathbf{T} = \mathbb{T}_1 \times \mathbb{T}_2$ and $V = \mathcal{K}_{\mathrm{vir}}^{1/2}$. By Equation (4.5.1), our goal is to prove that $\mathsf{w} = w_i^{-1} w_j t^{\nu}$ is a compact weight for all $i < j$ and $\nu \in \widehat{\mathbb{T}}_1$.

First of all, we observe that $\mathbf{T}_{\mathsf{w}} = \ker(\mathsf{w})$. Indeed w is not a product of powers of weights of $\mathbf{T}$, hence

$$\mathcal{O}(\ker \mathsf{w}) = \mathcal{O}(\mathbf{T})/(w_i^{-1} w_j t^{\nu} - 1) \cong \mathbb{C}\left[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, w_1^{\pm 1}, \ldots, w_{i-1}^{\pm 1}, w_{i+1}^{\pm 1}, \ldots, w_r^{\pm 1}\right],$$

which shows that $\ker(\mathsf{w})$ is itself a torus (of dimension $3 + r - 1$). Next, consider the automorphism $\tau_\nu \colon \mathbf{T} \xrightarrow{\sim} \mathbf{T}$ defined by

$$(t_1, t_2, t_3, w_1, \ldots, w_r) \mapsto (t_1, t_2, t_3, w_1, \ldots, w_i t^{-\nu}, \ldots, w_j, \ldots, w_r).$$

It maps $\mathbf{T}_{\mathsf{w}} \subset \mathbf{T}$ isomorphically onto the subtorus $\mathbb{T}_1 \times \{\, w_i = w_j \,\} \subset \mathbf{T}$. This yields an inclusion of tori

$$\mathbb{T}_1 \xrightarrow{\sim} \mathbb{T}_1 \times \{\, (1, \ldots, 1) \,\} \hookrightarrow \tau(\mathbf{T}_{\mathsf{w}}). \tag{4.5.2}$$

We consider the action $\sigma_\nu \colon \mathbf{T} \times \mathrm{Q} \to \mathrm{Q}$ where $\mathbb{T}_1$ translates the support of the quotient sheaf in the usual way, the i-th summand of $\mathcal{O}^{\oplus r}$ gets scaled by $w_i t^{\nu}$ and all other summands by w_k for $k \neq i$. In other words, in terms of the matrix-and-vectors description of Q, we set

$$\sigma_\nu(\mathbf{t}, (A_1, A_2, A_3, u_1, \dots, u_r)) = (t_1A_1, t_2A_2, t_3A_3, w_1u_1, \dots, w_it^\nu u_i, \dots, w_ru_r),$$

just a variation of Equation (4.2.9) in the i-th vector component. Then, upon restricting this action to $\mathbf{T}_{\mathsf{w}}$, we have a commutative diagram

$$\begin{tikzcd} \mathbf{T}_{\mathsf{w}} \times \mathrm{Q} \arrow[r, "\sigma"] \arrow[d, "\tau_\nu \times \mathrm{id}"'] & \mathrm{Q} \arrow[d, equal] \\ \tau_\nu(\mathbf{T}_{\mathsf{w}}) \times \mathrm{Q} \arrow[r, "\sigma_\nu"] & \mathrm{Q} \end{tikzcd}$$

where σ is the restriction of the usual action (4.2.9). This diagram induces a natural isomorphism $\mathrm{Q}^{\mathbf{T}_{\mathsf{w}}} \xrightarrow{\sim} \mathrm{Q}^{\tau_\nu(\mathbf{T}_{\mathsf{w}})}$, which combined with (4.5.2) yields an inclusion

$$\mathrm{Q}^{\mathbf{T}_{\mathsf{w}}} \xrightarrow{\sim} \mathrm{Q}^{\tau_\nu(\mathbf{T}_{\mathsf{w}})} \hookrightarrow \mathrm{Q}^{\mathbb{T}_1},$$

where $\mathrm{Q}^{\mathbb{T}_1}$ is the fixed locus with respect to the action σ_ν. But by the same reasoning as in Remark 4.2.4, this fixed locus is proper (because, again, a $\mathbb{T}_1$-fixed surjection $\mathscr{O}^{\oplus r} \twoheadrightarrow T$ necessarily has the quotient T entirely supported at the origin $0 \in \mathbb{A}^3$). Thus w is a compact weight, and the result follows. $\square$

Remark 4.5.6. After a first draft of this thesis was already finished, we were informed of an alternative way to prove Theorem 4.5.5, which, in a nutshell, goes as follows: one exploits the (proper) Quot-to-Chow morphism $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n) \to \mathrm{Sym}^n \mathbb{A}^3$ to express the K-theoretic DT invariants as equivariant holomorphic Euler characteristics on $\mathrm{Sym}^n \mathbb{A}^3$, where the framing torus $\mathbb{T}_2$ is acting trivially on the symmetric product. One concludes by an application of Okounkov's *rigidity principle* [**149**, Sec. 2.4.1]. This strategy will be carried out in [**4**].

Thanks to Theorem 4.5.5, we may now specialise $w_1, \dots, w_r$ to arbitrary values and take arbitrary limits. We set $w_i = L^i$ for $i = 1, \dots, r$ and compute the limit for $L \to \infty$.

Lemma 4.5.7. *Let $i < j$. Then we have*

$$\lim_{L\to\infty} [-\mathsf{V}_{ij}][-\mathsf{V}_{ji}]\big|_{w_i=L^i} = (-\mathfrak{t}^{\frac{1}{2}})^{|\pi_j|-|\pi_i|}.$$

Proof. Notice that all monomials in V_{ij} are of the form $w_i^{-1}w_j\lambda$ for λ a monomial in t_1, t_2, t_3. Then

$$[w_i^{-1}w_j\lambda]\big|_{w_i=L^i} = (L^{j-i}\lambda)^{\frac{1}{2}}(1 - L^{i-j}\lambda^{-1}).$$

Write $\mathsf{Q}_i = \sum_\mu t^\mu$ and $\mathsf{Q}_j = \sum_\nu t^\nu$. Taking limits, we obtain

$$\begin{aligned} \lim_{L\to\infty} [-\mathsf{V}_{ij}]\big|_{w_i=L^i} &= \lim_{L\to\infty} [-w_i^{-1}w_j(\mathsf{Q}_j - \overline{\mathsf{Q}}_i\mathfrak{t}^{-1} + \overline{\mathsf{Q}}_i\mathsf{Q}_j\mathfrak{t}^{-1}(1-t_1)(1-t_2)(1-t_3))]\big|_{w_i=L^i} \\ &= \lim_{L\to\infty} L^{\frac{j-i}{2}(|\pi_i|-|\pi_j|)} \frac{\prod_\mu (t^{-\frac{\mu}{2}}\mathfrak{t}^{-\frac{1}{2}})}{\prod_\nu t^{\frac{\nu}{2}}}. \end{aligned}$$

Similarly, we obtain

$$\lim_{L\to\infty} [-\mathsf{V}_{ji}]\big|_{w_i=L^i} = (-1)^{\mathrm{rk}(-\mathsf{V}_{ji})} \lim_{L\to\infty} [-\overline{\mathsf{V}}_{ji}]\big|_{w_i=L^i}$$

$$= (-1)^{|\pi_i|-|\pi_j|} \lim_{L\to\infty} [-w_i^{-1} w_j(\overline{\mathsf{Q}}_i - \mathsf{Q}_j \mathsf{t} - \overline{\mathsf{Q}}_i \mathsf{Q}_j (1-t_1)(1-t_2)(1-t_3))]\big|_{w_i=L^i}$$

$$= (-1)^{|\pi_i|-|\pi_j|} \lim_{L\to\infty} L^{\frac{j-i}{2}(|\pi_j|-|\pi_i|)} \frac{\prod_\nu (t^{\frac{\nu}{2}} \mathsf{t}^{\frac{1}{2}})}{\prod_\mu t^{-\frac{\mu}{2}}}.$$

We conclude, as required, that

$$\lim_{L\to\infty} [-\mathsf{V}_{ij}][-\mathsf{V}_{ji}]\big|_{w_i=L^i} = (-\mathsf{t}^{\frac{1}{2}})^{|\pi_j|-|\pi_i|}. \qquad \square$$

Lemma 4.5.8. *Let x be a variable and $c_i \in \mathbb{Z}$, for $i = 1, \dots, r$. Then we have*

$$\prod_{1\le i<j\le r} x^{c_j - c_i} = \prod_{i=1}^r x^{(-r-1+2i)c_i}.$$

PROOF. The assertion holds for $r = 1$ as the productory on the left hand side is empty. Assume it holds for $r-1$. Then we have:

$$\begin{aligned}
\prod_{1\le i<j\le r} x^{c_j-c_i} &= \prod_{1\le i<j\le r-1} x^{c_j-c_i} \prod_{i=1}^{r-1} x^{c_r-c_i} \\
&= x^{(r-1)c_r} \prod_{i=1}^{r-1} x^{(-r-1+2i)c_i} \\
&= \prod_{i=1}^{r} x^{(-r-1+2i)c_i}. \qquad \square
\end{aligned}$$

Combining Lemma 4.5.7 with Lemma 4.5.8 we can express the rank r K-theoretic DT theory of $\mathbb{A}^3$ as a product of r copies of the rank 1 K-theoretic DT theory. This product formula already appeared as a limit of the (conjectural) 4-fold theory developed by Nekrasov and Piazzalunga [**143**, Formula (3.14)].[3]

Theorem 4.5.9. *There is an identity*

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, (-1)^r q, t, w) = \prod_{i=1}^r \mathsf{DT}_1^{\mathrm{K}}(\mathbb{A}^3, -q\mathsf{t}^{\frac{-r-1}{2}+i}, t).$$

PROOF. Set $w_i = L^i$. The generating series $\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w)$ can be computed in the limit $L \to \infty$:

$$\begin{aligned}
\lim_{L\to\infty} \mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w) &= \lim_{L\to\infty} \sum_{\overline{\pi}} q^{|\overline{\pi}|} \prod_{i,j=1}^r [-\mathsf{V}_{ij}] \\
&= \lim_{L\to\infty} \sum_{\overline{\pi}} \prod_{i=1}^r q^{|\pi_i|}[-\mathsf{V}_{ii}] \prod_{1\le i<j\le r} [-\mathsf{V}_{ij}][-\mathsf{V}_{ji}] \\
&= \sum_{\overline{\pi}} \prod_{i=1}^r q^{|\pi_i|}[-\mathsf{V}_{ii}] \prod_{1\le i<j\le r} (-\mathsf{t}^{\frac{1}{2}})^{|\pi_j|-|\pi_i|} \\
&= \sum_{\overline{\pi}} \prod_{i=1}^r q^{|\pi_i|}[-\mathsf{V}_{ii}] \prod_{i=1}^r (-\mathsf{t}^{\frac{1}{2}})^{(-r-1+2i)|\pi_i|}
\end{aligned}$$

[3]Typo warning: N. Piazzalunga kindly pointed out to us that in [**143**, Formula (3.14)] one should read '$\frac{N+1}{2} - l$' instead of '$N + 1 - 2l$'.

$$
\begin{aligned}
&= \sum_{\overline{\pi}} \prod_{i=1}^{r} [-\mathsf{V}_{ii}] q^{|\pi_i|} (-1)^{(r+1)|\pi_i|} \mathsf{t}^{(\frac{-r-1}{2}+i)|\pi_i|} \\
&= \sum_{\overline{\pi}} \prod_{i=1}^{r} [-\mathsf{V}_{ii}] \left((-1)^{(r+1)} q \mathsf{t}^{\frac{-r-1}{2}+i} \right)^{|\pi_i|} \\
&= \prod_{i=1}^{r} \mathsf{DT}_1^{\mathrm{K}}(\mathbb{A}^3, (-1)^{(r+1)} q \mathsf{t}^{\frac{-r-1}{2}+i}, t). \qquad \square
\end{aligned}
$$

We can now prove Theorem 4.5.3 (i.e. Theorem 4.1.1 from the Introduction).

Proof of Theorem 4.5.3. Define

$$
G_{r,i}(q, t_1, t_2, t_3) = \mathsf{F}_1(q \mathsf{t}^{\frac{-r-1}{2}+i}, t_1, t_2, t_3).
$$

We have

$$
\begin{aligned}
\mathsf{DT}_1^{\mathrm{K}}(\mathbb{A}^3, -q\mathsf{t}^{\frac{-r-1}{2}+i}, t) &= \exp\left(\sum_{n\geq 1} \frac{1}{n} \frac{1}{[\mathsf{t}^{\frac{n}{2}} q^n \mathsf{t}^{n(\frac{-r-1}{2}+i)}][\mathsf{t}^{\frac{n}{2}} q^{-n} \mathsf{t}^{n(\frac{r+1}{2}-i)}]} \frac{[t_1^n t_2^n][t_1^n t_3^n][t_2^n t_3^n]}{[t_1^n][t_2^n][t_3^n]} \right) \\
&= \operatorname{Exp}\left(G_{r,i}(q, t_1, t_2, t_3) \right).
\end{aligned}
$$

By Theorem 4.5.9 and Theorem 4.5.1 it is enough to show that $\mathsf{F}_r = \sum_{i=1}^r G_{r,i}$, or equivalently

$$
\sum_{i=1}^{r} \frac{1}{[\mathsf{t}^{\frac{1}{2}} q \mathsf{t}^{\frac{-r-1}{2}+i}][\mathsf{t}^{\frac{1}{2}} q^{-1} \mathsf{t}^{\frac{r+1}{2}-i}]} = \frac{[\mathsf{t}^r]}{[\mathsf{t}][\mathsf{t}^{\frac{r}{2}} q][\mathsf{t}^{\frac{r}{2}} q^{-1}]}.
$$

It is easy to check this is true for $r = 1, 2$. Let now $r \geq 3$: we perform induction separately on even and odd cases. Assume the claimed identity holds for $r-2$. In both cases we have

$$
\begin{aligned}
&\sum_{i=1}^{r} \frac{1}{[\mathsf{t}^{\frac{1}{2}} q \mathsf{t}^{\frac{-r-1}{2}+i}][\mathsf{t}^{\frac{1}{2}} q^{-1} \mathsf{t}^{\frac{r+1}{2}-i}]} \\
&= \sum_{i=1}^{r-2} \frac{1}{[\mathsf{t}^{\frac{1}{2}} q \mathsf{t}^{\frac{-(r-2)-1}{2}+i}][\mathsf{t}^{\frac{1}{2}} q^{-1} \mathsf{t}^{\frac{(r-2)+1}{2}-i}]} + \frac{1}{[q\mathsf{t}^{-\frac{r}{2}+1}][q^{-1}\mathsf{t}^{\frac{r}{2}}]} + \frac{1}{[q\mathsf{t}^{\frac{r}{2}}][q^{-1}\mathsf{t}^{-\frac{r}{2}+1}]} \\
&= \frac{[\mathsf{t}^{r-2}]}{[\mathsf{t}][\mathsf{t}^{\frac{r-2}{2}} q][\mathsf{t}^{\frac{r-2}{2}} q^{-1}]} - \frac{1}{[\mathsf{t}^{\frac{r-2}{2}} q^{-1}][\mathsf{t}^{\frac{r}{2}} q^{-1}]} - \frac{1}{[\mathsf{t}^{\frac{r}{2}} q][\mathsf{t}^{\frac{r-2}{2}} q]} \\
&= \frac{1}{[\mathsf{t}][\mathsf{t}^{\frac{r}{2}} q][\mathsf{t}^{\frac{r}{2}} q^{-1}]} \cdot \frac{[\mathsf{t}^{r-2}][\mathsf{t}^{\frac{r}{2}} q][\mathsf{t}^{\frac{r}{2}} q^{-1}] - [\mathsf{t}][\mathsf{t}^{\frac{r}{2}} q][\mathsf{t}^{\frac{r-2}{2}} q] - [\mathsf{t}][\mathsf{t}^{\frac{r}{2}} q^{-1}][\mathsf{t}^{\frac{r-2}{2}} q^{-1}]}{[\mathsf{t}^{\frac{r-2}{2}} q][\mathsf{t}^{\frac{r-2}{2}} q^{-1}]} \\
&= \frac{[\mathsf{t}^r]}{[\mathsf{t}][\mathsf{t}^{\frac{r}{2}} q][\mathsf{t}^{\frac{r}{2}} q^{-1}]}
\end{aligned}
$$

by which we conclude the proof. $\square$

4.5.3. Comparison with motivic DT invariants Let $f \colon U \to \mathbb{A}^1$ be a regular function on a smooth scheme U, and let $\hat{\mu}$ be the group of all roots of unity. The critical locus $Z = \operatorname{crit}(f) \subset U$ inherits a canonical *virtual motive* [11], i.e. a $\hat{\mu}$-equivariant motivic class

$$
[Z]_{\mathrm{vir}} = -\mathbb{L}^{-\frac{\dim U}{2}} [\phi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} = K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})\big[\mathbb{L}^{-\frac{1}{2}}\big]
$$

such that $e[Z]_{\mathrm{vir}} = e_{\mathrm{vir}}(Z)$, where $e_{\mathrm{vir}}(-)$ is Behrend weighted Euler characteristic and the Euler number specialisation prescribes $e(\mathbb{L}^{-1/2}) = -1$. The motivic class $[\phi_f]$ is the (absolute) motivic vanishing cycle class introduced by Denef-Loeser [**63**].

The virtual motive of $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) = \mathrm{crit}(f_n)$, with respect to the critical structure of Proposition 4.2.2, was computed in [**163**, Prop. 2.3.6]. The result is as follows. Let $\mathsf{DT}_r^{\mathrm{mot}}(\mathbb{A}^3, q) \in \mathcal{M}_{\mathbb{C}}[\![q]\!]$ be the generating function of the virtual motives $[\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)]_{\mathrm{vir}}$. Then one has

$$\mathsf{DT}_r^{\mathrm{mot}}(\mathbb{A}^3, q) = \prod_{m\geq 1} \prod_{k=0}^{rm-1} \left(1 - \mathbb{L}^{2+k-\frac{rm}{2}} q^m\right)^{-1}. \tag{4.5.3}$$

The case $r = 1$ was computed in [**11**]. The general proof of Formula (4.5.3) is obtained in a similar fashion in [**163, 47**], and via a wall-crossing technique in [**49**]. Moreover, it is immediate to verify that $\mathsf{DT}_r^{\mathrm{mot}}$ satisfies a product formula analogous to the one proved in Theorem 4.5.9 for the K-theoretic invariants: we have

$$\mathsf{DT}_r^{\mathrm{mot}}(\mathbb{A}^3, q) = \prod_{i=1}^{r} \mathsf{DT}_1^{\mathrm{mot}}\left(\mathbb{A}^3, q\mathbb{L}^{\frac{-r-1}{2}+i}\right). \tag{4.5.4}$$

In particular, up to the substitution $\mathfrak{t}^{\frac{1}{2}} \to -\mathbb{L}^{\frac{1}{2}}$, the factorisation (4.5.4) is equivalent to the K-theoretic one (Theorem 4.5.9). As observed in [**165**, Sec. 4], the (signed) motivic partition function admits an expression in terms of the motivic exponential, namely

$$\mathsf{DT}_r^{\mathrm{mot}}(\mathbb{A}^3, (-1)^r q) = \mathrm{Exp}\left(\frac{(-1)^r q \mathbb{L}^{\frac{3}{2}}}{\left(1-(-1)^r q\mathbb{L}^{\frac{r}{2}}\right)\left(1-(-1)^r q\mathbb{L}^{-\frac{r}{2}}\right)} \frac{\mathbb{L}^{\frac{r}{2}} - \mathbb{L}^{-\frac{r}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\right). \tag{4.5.5}$$

Given their structural similarities, we believe it is an interesting problem to compare the K-theoretic partition function with the motivic one.

It is worth noticing that Formula (4.5.5) can be recovered from the factorisation (4.5.4), just as we discovered in the K-theoretic case during the proof of Theorem 4.5.3. This fact follows immediately from the properties of the plethystic exponential.

Remark 4.5.10. A virtual motive for $\mathrm{Quot}_Y(F, n)$ was defined in [**165**, Sec. 4] for every locally free sheaf F on a 3-fold Y. Just as in the case of the naive motives of the Quot scheme [**164**], the resulting partition function $\mathsf{DT}_r^{\mathrm{mot}}(Y, q)$ only depends on the motivic class $[Y] \in K_0(\mathrm{Var}_{\mathbb{C}})$ and on $r = \mathrm{rk}\, F$. See also [**49**] for calculations of motivic higher rank DT and PT invariants in the presence of nonzero curve classes: the generating function $\mathsf{DT}_r^{\mathrm{mot}}(Y, q)$, computed easily starting with Formula (4.5.3), is precisely the DT/PT wall-crossing factor.

4.6. The higher rank cohomological DT partition function

4.6.1. Cohomological reduction One should think of K-theoretic invariants as refinements of the cohomological ones, as by taking suitable limits one fully recovers $\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s)$ from $\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t)$. We make this precise in the remainder of this section.

Let $\mathbf{T} \cong (\mathbb{C}^*)^g$ be an algebraic torus and let $t_1, \ldots, t_g$ be its coordinates. Recall that the Chern character gives a natural transformation from (equivariant) K-theory to the (equivariant) Chow group with rational coefficients by sending $t_i \mapsto e^{s_i}$, where $s_i = c_1^{\mathbf{T}}(t_i)$. We can formally extend it to

$$\begin{array}{ccc} \mathbb{Z}[t_1^{\pm 1}, \ldots, t_g^{\pm 1}] & \xrightarrow{\mathrm{ch}} & \mathbb{Q}[\![s_1, \ldots, s_g]\!] \\ \downarrow & & \downarrow \\ \mathbb{Z}[t_1^{\pm b}, \ldots, t_g^{\pm b} | b \in \mathbb{C}] & \xrightarrow{\mathrm{ch}} & \mathbb{C}[\![s_1, \ldots, s_g]\!] \end{array}$$

by sending $t_i^b \mapsto e^{bs_i}$, where $b \in \mathbb{C}$.

In Section 4.5.1 we defined the symmetrised transformation $[t^\mu] = t^{\frac{\mu}{2}} - t^{-\frac{\mu}{2}}$. We set $[\mathrm{ch}(t^{b\mu})] = e^{\frac{b\mu \cdot s}{2}} - e^{-\frac{b\mu \cdot s}{2}}$ as an expression in rational cohomology, which enjoys the following *linearisation* property:

$$[\mathrm{ch}(t^{b\mu})] = e^{\frac{b\mu \cdot s}{2}}(1 - e^{-b\mu \cdot s}) = be^{\mathbf{T}}(t^\mu) + o(b^2).$$

In other words, $e^{\mathbf{T}}(\,\cdot\,)$ is the first-order approximation of $[\,\cdot\,]$ in $\mathbf{T}$-equivariant Chow groups. For a virtual representation $V = \sum_\mu t^\mu - \sum_\nu t^\nu \in K_0^{\mathbf{T}}(\mathsf{pt})$, denote by $V^b = \sum_\mu t^{b\mu} - \sum_\nu t^{b\nu}$ the virtual representation where we formally substitute each weight t^μ with $t^{b\mu}$. We have the identity

$$[\mathrm{ch}(V^b)] = \frac{\prod_\mu [\mathrm{ch}(t^{b\mu})]}{\prod_\nu [\mathrm{ch}(t^{b\nu})]} = b^{\mathrm{rk}\, V} \frac{\prod_\mu (e^{\mathbf{T}}(t^\mu) + o(b))}{\prod_\nu (e^{\mathbf{T}}(t^\nu) + o(b))}.$$

If $\mathrm{rk}\, V = 0$, by taking the limit for $b \to 0$ we conclude

$$\lim_{b \to 0} [\mathrm{ch}(V^b)] = e^{\mathbf{T}}(V). \tag{4.6.1}$$

It is clear from the definition of $\mathrm{ch}(\,\cdot\,)$ and $[\,\cdot\,]$ that these two transformations commute with each other. This proves the following relation between K-theoretic invariants and cohomological invariants of the local model.

Corollary 4.6.1. *There is an identity*

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v) = \lim_{b \to 0} \mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, e^{bs}, e^{bv}).$$

Proof. Follows from the description of the generating series of K-theoretic invariants as

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, q, t, w) = \sum_{n \geq 0} q^n \sum_{[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r}, n)^{\mathbf{T}}} [-T_S^{\mathrm{vir}}]$$

and by noticing that $\mathrm{rk}\, T_S^{\mathrm{vir}} = 0$. □

Thanks to the v-independence, we can now rename

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s) = \mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s, v).$$

We are ready to prove Theorem 4.1.2 from the Introduction.

Theorem 4.6.2. *The rank r cohomological DT partition function of $\mathbb{A}^3$ is given by*

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3, q, s) = \mathsf{M}((-1)^r q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}}.$$

Proof. By Corollary 4.6.1 and Theorem 4.5.3, we just need to compute the limit

$$\lim_{b\to 0}\mathsf{DT}_r^{\mathrm{K}}\left(\mathbb{A}^3,(-1)^r q, e^{bs}\right)=\lim_{b\to 0}\operatorname{Exp}\left(\mathsf{F}_r(q,t_1^b,t_2^b,t_3^b)\right).$$

Denote for ease of notation $\mathfrak{s}=c_1^{\mathbf{T}}(\mathfrak{t})=s_1+s_2+s_3$. By the definition of plethystic exponential, recalled in (4.1.4), we have

$$\begin{aligned}
&\lim_{b\to 0}\operatorname{Exp}\left(\mathsf{F}_r(q,t_1^b,t_2^b,t_3^b)\right)\\
&\qquad=\exp\sum_{k\geq 1}\frac{1}{k}\left(\lim_{b\to 0}\frac{[e^{bkr\mathfrak{s}}]}{[e^{bk\mathfrak{s}}][e^{\frac{bkr}{2}\mathfrak{s}}q^k][e^{\frac{bkr}{2}\mathfrak{s}}q^{-k}]}\frac{[e^{bk(s_1+s_2)}][e^{bk(s_1+s_3)}][e^{bk(s_2+s_3)}]}{[e^{bks_1}][e^{bks_2}][e^{bks_3}]}\right).
\end{aligned}$$

We have

$$\lim_{b\to 0}\frac{[e^{bk(s_1+s_2)}][e^{bk(s_1+s_3)}][e^{bk(s_2+s_3)}]}{[e^{bks_1}][e^{bks_2}][e^{bks_3}]}=\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3},$$

and

$$\lim_{b\to 0}\frac{[e^{bkr\mathfrak{s}}]}{[e^{bk\mathfrak{s}}][e^{\frac{bkr}{2}\mathfrak{s}}q^k][e^{\frac{bkr}{2}\mathfrak{s}}q^{-k}]}=\frac{r}{[q^k][q^{-k}]}=-r\cdot\frac{q^k}{(1-q^k)^2}.$$

Recall the plethystic exponential form of the MacMahon function

$$\mathsf{M}(q)=\prod_{n\geq 1}(1-q^n)^{-n}=\operatorname{Exp}\left(\frac{q}{(1-q)^2}\right).$$

We conclude

$$\begin{aligned}
\lim_{b\to 0}\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3,(-1)^r q,e^{bs})&=\exp\left(-r\cdot\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3}\sum_{k\geq 1}\frac{1}{k}\frac{q^k}{(1-q^k)^2}\right)\\
&=\mathsf{M}(q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3}}.
\end{aligned}$$

□

Thus we proved Szabo's conjecture [**173**, Conj. 4.10].

Remark 4.6.3. The specialisation

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathsf{A}^3,q,s)\big|_{s_1+s_2+s_3=0}=\mathsf{M}((-1)^r q)^r,$$

recovering Formula (4.3.1), was already known in physics, see e.g. [**59**].

We end this section with a small variation of Theorem 4.1.2.

Corollary 4.6.4. *Fix an r-tuple $\lambda=(\lambda_1,\ldots,\lambda_r)$ of $\mathbb{T}_1$-equivariant line bundles on $\mathbb{A}^3$. Then there is an identity*

$$\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s)=\mathsf{DT}_r^{\mathrm{coh}}(\mathbb{A}^3,q,s,v)_\lambda,$$

where the right hand side was defined in (4.4.11).

Proof. We have

$$T_{S,\lambda}^{\mathrm{vir}}=\sum_{i,j}\lambda_i^{-1}\lambda_j\mathsf{V}_{ij}.$$

Let $\mathsf{V}_{ij} = \sum_\mu w_i^{-1} w_j t^\mu$ be the decomposition into weight spaces. A monomial in $T^{\mathrm{vir}}_{S,\lambda}$ is of the form $\lambda_i^{-1}\lambda_j w_i^{-1} w_j t^\mu$ and its Euler class is

$$\begin{aligned} e^{\mathbf{T}}(\lambda_i^{-1}\lambda_j w_i^{-1} w_j t^\mu) &= \mu \cdot s + v_j + c_1^{\mathbf{T}}(\lambda_j) - v_i - c_1^{\mathbf{T}}(\lambda_i) \\ &= \mu \cdot s + \overline{v}_j - \overline{v}_i \end{aligned}$$

where we define $\overline{v}_i = v_i + c_1^{\mathbf{T}}(\lambda_i)$. We conclude that

$$\mathsf{DT}^{\mathrm{coh}}_r(\mathbb{A}^3, q, s, v)_\lambda = \mathsf{DT}^{\mathrm{coh}}_r(\mathbb{A}^3, q, s, \overline{v}),$$

which does not depend on $\overline{v}$ by Theorem 4.5.5. □

Example 4.6.5. Set $r = 2$, $n = 1$, so that the only $\mathbf{T}$-fixed points in $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus 2}, 1)$ are the direct sums of ideal sheaves

$$S_1 = \mathscr{I}_{\mathsf{pt}} \oplus \mathcal{O} \subset \mathcal{O}^{\oplus 2}, \quad S_2 = \mathcal{O} \oplus \mathscr{I}_{\mathsf{pt}} \subset \mathcal{O}^{\oplus 2},$$

where $\mathsf{pt} = (0,0,0) \in \mathbb{A}^3$ is the origin. One computes

$$\begin{aligned} T^{\mathrm{vir}}_{S_1} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_1^{-1}w_2\frac{1}{t_1t_2t_3} + w_2^{-1}w_1 \\ T^{\mathrm{vir}}_{S_2} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_2^{-1}w_1\frac{1}{t_1t_2t_3} + w_1^{-1}w_2. \end{aligned}$$

Therefore, the cohomological DT invariant is

$$\int_{[\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus 2},1)]^{\mathrm{vir}}} 1 = e^{\mathbf{T}}(-T^{\mathrm{vir}}_{S_1}) + e^{\mathbf{T}}(-T^{\mathrm{vir}}_{S_2}).$$

The part that could possibly depend on the framing parameters v_1 and v_2 is, in fact, constant:

$$\mathrm{e}^{\mathbf{T}}\left(w_1^{-1}w_2\tfrac{1}{t_1t_2t_3} - w_2^{-1}w_1\right) + e^{\mathbf{T}}\left(w_2^{-1}w_1\tfrac{1}{t_1t_2t_3} - w_1^{-1}w_2\right) = \tfrac{-v_1+v_2-\mathfrak{s}}{v_1-v_2} + \tfrac{v_1-v_2-\mathfrak{s}}{v_2-v_1} = -2.$$

Let λ_1 and λ_2 be two $\mathbb{T}_1$-equivariant line bundles. After the substitutions $w_i \to w_i\lambda_i$, and setting $\overline{v}_i = v_i + c_1(\lambda_i)$, the final sum of Euler classes depending on $\overline{v}$ becomes

$$2\frac{\overline{v}_2 - \overline{v}_1}{\overline{v}_1 - \overline{v}_2} = -2.$$

4.7. Elliptic Donaldson–Thomas invariants

4.7.1. Chiral elliptic genus In [16] an elliptic (string-theoretic) generalisation of the K-theoretic DT invariants is given, which we formalize mathematically.

Let X be a scheme carrying a perfect obstruction theory $\mathbb{E} \to \mathbb{L}_X$ of virtual dimension $\mathrm{vd} = \mathrm{rk}\,\mathbb{E}$.

Definition 4.7.1. If F is a rank r vector bundle on X we define

$$\mathcal{E}_{1/2}(F) = \bigotimes_{n \geq 1} \mathrm{Sym}^\bullet_{p^n}(F \oplus F^\vee) \in 1 + p \cdot K^0(X)[\![p]\!] \tag{4.7.1}$$

where the total symmetric algebra $\mathrm{Sym}^\bullet_p(F) = \sum_{i\geq 0} p^i[S^iF] \in K^0(X)[\![p]\!]$ satisfies $\mathrm{Sym}^\bullet_p(F) = 1/\Lambda^\bullet_{-p}(F)$. Note that $\mathcal{E}_{1/2}$ defines a homomorphism from the additive group $K^0(X)$ to the multiplicative group $1 + p \cdot K^0(X)[\![p]\!]$. Set

$$\mathcal{E}\ell\ell_{1/2}(F;p) = (-p^{-\frac{1}{12}})^{\mathrm{rk}\,F}\mathrm{ch}\left(\mathcal{E}_{1/2}(F)\right) \cdot \mathrm{td}(F) \in A^*(X)[\![p]\!][p^{\pm\frac{1}{12}}], \tag{4.7.2}$$

where $\mathrm{td}(-)$ is the Todd class, so that $\mathcal{E}\ell\ell_{1/2}(-;p)$ extends to a group homomorphism from $K^0(X)$ to the multiplicative group of units in $A^*(X)[\![p]\!][p^{\pm\frac{1}{12}}]$.

We can then define the virtual chiral elliptic genus as follows.

Definition 4.7.2. Let X be a proper scheme with a perfect obstruction theory and $V \in K^0(X)$. The *virtual chiral elliptic genus* is defined as

$$\mathrm{Ell}^{\mathrm{vir}}_{1/2}(X, V; p) = (-p^{-\frac{1}{12}})^{\mathrm{vd}} \chi^{\mathrm{vir}}\left(X, \mathcal{E}_{1/2}(T^{\mathrm{vir}}_X) \otimes V\right) \in \mathbb{Z}[\![p]\!][p^{\pm\frac{1}{12}}].$$

By the virtual Riemann–Roch theorem of [75] we can also compute the virtual chiral elliptic genus as

$$\mathrm{Ell}^{\mathrm{vir}}_{1/2}(X, V; p) = \int_{[X]^{\mathrm{vir}}} \mathcal{E}\ell\ell_{1/2}(T^{\mathrm{vir}}_X; p) \cdot \mathrm{ch}(V).$$

Remark 4.7.3. One may give a more general definition by adding a "mass deformation" and defining $\mathcal{E}^{(y)}_{1/2}(F)$ for $F \in K^0(X)$ as

$$\mathcal{E}^{(y)}_{1/2}(F; p) = \bigotimes_{n\geq 1} \mathrm{Sym}^{\bullet}_{y^{-1}p^n}(F) \otimes \mathrm{Sym}^{\bullet}_{yp^n}(F^{\vee}) \in 1 + p \cdot K^0(X)[y, y^{-1}][\![p]\!],$$

so we recover the standard definition of virtual elliptic genus by taking $\mathcal{E}(F) = \mathcal{E}^{(1)}_{1/2}(F;p) \otimes \mathcal{E}^{(y)}_{1/2}(-F;p)$, cf. [75, Sec. 6].

Proposition 4.7.4. *Let X be a proper scheme with a perfect obstruction theory and let $V \in K^0(X)$. Then the virtual chiral elliptic genus $\mathrm{Ell}^{\mathrm{vir}}_{1/2}(X, V; p)$ is deformation invariant.*

Proof. The statement follows directly from Definition 4.7.2 and [75, Theorem 3.15]. □

Let now $V = \sum_\mu t^\mu$ be a $\mathbf{T}$-module as in Section 4.5.1. The trace of its symmetric algebra is given by

$$\mathrm{tr}\left(\mathrm{Sym}^{\bullet}_p(V)\right) = \mathrm{tr}\left(\frac{1}{\Lambda^{\bullet}_{-p}(V)}\right) = \prod_\mu \frac{1}{1 - pt^\mu}.$$

Let us now assume as in Section 4.5.1 that $\det V$ is a square in $K^0_{\mathbf{T}}(\mathsf{pt})$ and $\mu = 0$ is not a weight of V. We can then compute the trace of the symmetric product in (4.7.1) as

$$\mathrm{tr}\left(\bigotimes_{n\geq 1} \mathrm{Sym}^{\bullet}_{p^n}(V \oplus V^{\vee})\right) = \prod_\mu \prod_{n\geq 1} \frac{1}{(1 - p^n t^\mu)(1 - p^n t^{-\mu})},$$

$$\begin{aligned}\mathrm{tr}\left(\frac{\bigotimes_{n\geq 1} \mathrm{Sym}^{\bullet}_{p^n}(V \oplus V^{\vee})}{\Lambda^{\bullet} V^{\vee}}\right) &= \prod_\mu \frac{1}{1 - t^{-\mu}} \prod_{n\geq 1} \frac{1}{(1 - p^n t^\mu)(1 - p^n t^{-\mu})} \\ &= \left(-ip^{1/8}\phi(p)\right)^{\mathrm{rk}\, V} \prod_\mu \frac{t^{\mu/2}}{\theta(p; t^\mu)},\end{aligned}$$

where $\phi(p)$ is the Euler function, i.e. $\phi(p) = \prod_n (1-p^n)$, and $\theta(p;y)$ denotes the Jacobi theta function

$$\theta(p;y) = -ip^{1/8}(y^{1/2}-y^{-1/2})\prod_{n=1}^{\infty}(1-p^n)(1-yp^n)(1-y^{-1}p^n).$$

Combining everything together we get the identity

$$\begin{aligned}(-p^{-\frac{1}{12}})^{\operatorname{rk} V}\operatorname{tr}\left(\frac{\bigotimes_{n\geq 1}\operatorname{Sym}^{\bullet}_{p^n}(V\oplus V^\vee)\otimes\det(V^\vee)^{1/2}}{\Lambda^\bullet V^\vee}\right) &= (-p^{-\frac{1}{12}})^{\operatorname{rk} V}\left(-ip^{1/8}\phi(p)\right)^{\operatorname{rk} V}\prod_{\mu}\frac{1}{\theta(p;t^\mu)}\\ &= \prod_\mu i\frac{\eta(p)}{\theta(p;t^\mu)},\end{aligned}$$

where $\eta(p)$ is the Dedekind eta function

$$\eta(p) = p^{\frac{1}{24}}\prod_{n\geq 1}(1-p^n).$$

For a virtual $\mathbf{T}$-representation $V = \sum_\mu t^\mu - \sum_\nu t^\nu \in K^0_{\mathbf{T}}(\mathsf{pt})$ where $\mu = 0$ is not a weight of V, we compute

$$(-p^{-\frac{1}{12}})^{\operatorname{rk} V}\operatorname{tr}\left(\frac{\bigotimes_{n\geq 1}\operatorname{Sym}^{\bullet}_{p^n}(V\oplus V^\vee)\otimes\det(V^\vee)^{1/2}}{\Lambda^\bullet V^\vee}\right) = (i\cdot\eta(p))^{\operatorname{rk} V}\frac{\prod_\nu \theta(p;t^\nu)}{\prod_\mu\theta(p;t^\mu)}.$$

For the remainder of the section we set $p = e^{2\pi i\tau}$, with $\tau\in\mathbb{H} = \{\,\tau\in\mathbb{C}\mid\Im(\tau)>0\,\}$. Denoting $\theta(\tau|z) = \theta(e^{2\pi i\tau};e^{2\pi iz})$, θ enjoys the modular behaviour

$$\theta(\tau|z+a+b\tau) = (-1)^{a+b}e^{-2\pi ibz}e^{-i\pi b^2\tau}\theta(\tau|z),\quad a,b\in\mathbb{Z}.$$

Analogously to the measure $[\,\cdot\,]$ for K-theoretic invariants, we define the *elliptic measure*

$$\theta[V] = (i\cdot\eta(p))^{-\operatorname{rk} V}\frac{\prod_\mu\theta(p;t^\mu)}{\prod_\nu\theta(p;t^\nu)},$$

which satisfies $\theta[\overline{V}] = (-1)^{\operatorname{rk} V}\theta[V]$. Notice that, if $\operatorname{rk} V = 0$, the elliptic measure refines both $[\,\cdot\,]$ and $e^{\mathbf{T}}(\,\cdot\,)$

$$\theta[V]\xrightarrow{p\to 0}[V]\xrightarrow{b\to 0}e^{\mathbf{T}}(V)$$

where the second limit was dicussed in Section 4.6.1.

4.7.2. Elliptic DT invariants

Definition 4.7.5. The generating series of elliptic DT invariants $\mathsf{DT}^{\mathrm{ell}}_r(\mathbb{A}^3,q,t,w;p)$ is defined as

$$\mathsf{DT}^{\mathrm{ell}}_r(\mathbb{A}^3,q,t,w;p) = \sum_{n\geq 0}q^n\operatorname{Ell}^{\mathrm{vir}}_{1/2}(\operatorname{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n),\mathcal{K}^{\frac{1}{2}}_{\mathrm{vir}};p)\in\mathbb{Z}(\!(t,\mathfrak{t}^{\frac{1}{2}},w)\!)[\![p,q]\!].$$

Being that $\operatorname{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r},n)$ is not projective, but nevertheless carries the action of an algebraic torus $\mathbf{T}$ with proper $\mathbf{T}$-fixed locus, we define the invariants by means of virtual localisation.

At each $\mathbf{T}$-fixed point $[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}}$, the localised contribution is

$$\operatorname{tr}\left(\frac{\bigotimes_{n\geq 1}\mathrm{Sym}^{\bullet}_{p^n}\left(T_S^{\mathrm{vir}}\oplus T_S^{\mathrm{vir},\vee}\right)}{\widehat{\Lambda}^{\bullet}T_S^{\mathrm{vir},\vee}}\right)$$

from which we deduce that we can recover the invariants $\chi(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n), \widehat{\mathcal{O}}^{\mathrm{vir}})$ in K-theory in the limit $p \to 0$. As for K-theoretic invariants, we have

$$\begin{aligned}\mathsf{DT}_r^{\mathrm{ell}}(\mathbb{A}^3, q, t, w; p) &= \sum_{n\geq 0} q^n \sum_{[S]\in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}}} \theta[-T_S^{\mathrm{vir}}]\\ &= \sum_{\overline{\pi}} q^{|\overline{\pi}|}\prod_{i,j=1}^{r}\theta[-\mathsf{V}_{ij}],\end{aligned}$$

where $\overline{\pi}$ runs over all r-colored plane partitions.

Contrary to the case of K-theoretic and cohomological invariants, there exists no conjectural closed formula for elliptic DT invariants yet, even for the rank 1 case. Moreover, the generating series depends on the equivariant parameters of the framing torus, as shown in the following example.

Example 4.7.6. Consider $Q_1^3 = \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus 3}, 1)$, whose only $\mathbf{T}$-fixed points are

$$S_1 = \mathcal{I}_{\mathsf{pt}} \oplus \mathcal{O}\oplus\mathcal{O} \subset \mathcal{O}^{\oplus 3}, \quad S_2 = \mathcal{O}\oplus\mathcal{I}_{\mathsf{pt}}\oplus\mathcal{O}\subset\mathcal{O}^{\oplus 3}, \quad S_3 = \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{I}_{\mathsf{pt}}\subset\mathcal{O}^{\oplus 3},$$

with $\mathsf{pt} = (0,0,0) \in \mathbb{A}^3$ as in Example 4.6.5. We have

$$\begin{aligned}T_{S_1}^{\mathrm{vir}} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_1^{-1}w_2\frac{1}{t_1t_2t_3} + w_2^{-1}w_1 - w_1^{-1}w_3\frac{1}{t_1t_2t_3} + w_3^{-1}w_1\\ T_{S_2}^{\mathrm{vir}} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_2^{-1}w_1\frac{1}{t_1t_2t_3} + w_1^{-1}w_2 - w_2^{-1}w_3\frac{1}{t_1t_2t_3} + w_3^{-1}w_2\\ T_{S_3}^{\mathrm{vir}} &= 1 - \frac{1}{t_1t_2t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3} - w_3^{-1}w_1\frac{1}{t_1t_2t_3} + w_1^{-1}w_3 - w_3^{-1}w_2\frac{1}{t_1t_2t_3} + w_2^{-1}w_3\end{aligned}$$

by which we may compute the corresponding elliptic invariant. Set $w_j = e^{2\pi i v_j}$ and $t_\ell = e^{2\pi i s_\ell}$, so that

$$\begin{aligned}\mathrm{Ell}_{1/2}^{\mathrm{vir}}(Q_1^3, \mathcal{K}_{\mathrm{vir}}^{\frac{1}{2}}, t, w; p) = \diamondsuit\cdot\Bigg(&\frac{\theta(\tau|v_2-v_1-s)\theta(\tau|v_3-v_1-s)}{\theta(\tau|v_1-v_2)\theta(\tau|v_1-v_3)}\\ &+\frac{\theta(\tau|v_1-v_2-s)\theta(\tau|v_3-v_2-s)}{\theta(\tau|v_2-v_1)\theta(\tau|v_2-v_3)} + \frac{\theta(\tau|v_1-v_3-s)\theta(\tau|v_2-v_3-s)}{\theta(\tau|v_3-v_1)\theta(\tau|v_3-v_2)}\Bigg),\end{aligned}$$

where $s = s_1+s_2+s_3$, with the overall factor

$$\diamondsuit = \frac{\theta(\tau|s_1+s_2)\theta(\tau|s_1+s_3)\theta(\tau|s_2+s_3)}{\theta(\tau|s_1)\theta(\tau|s_2)\theta(\tau|s_3)}.$$

Moreover, by evaluating residues in $v_i - v_j = 0$ one can realise that $\mathrm{Ell}_{1/2}^{\mathrm{vir}}(Q_1^3, \mathcal{K}_{\mathrm{vir}}^{1/2}, t, w; p)$ has no poles in v_i. Indeed

$$\mathrm{Res}_{v_1-v_2=0}\,\mathrm{Ell}_{1/2}^{\mathrm{vir}}(Q_1^3, \mathcal{K}_{\mathrm{vir}}^{\frac{1}{2}}; p) = \diamondsuit\cdot\left(\frac{\theta(\tau|-s)\theta(\tau|v_3-v_2-s)}{\theta(\tau|v_2-v_3)} - \frac{\theta(\tau|-s)\theta(\tau|v_3-v_2-s)}{\theta(\tau|v_2-v_3)}\right) = 0,$$

and the same occurs for any other pole involving the v_i's. However, this does not imply the independence of the elliptic invariants from v, as we now suggest.

Set $\overline{v}_i = v_i + a_i + b_i\tau$, with $a_i, b_i \in \mathbb{Z}$, for $i = 1, 2, 3$. Applying the quasi-periodicity of theta functions, we get

$$\begin{aligned}
&\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q_1^3, \mathcal{K}_{\mathrm{vir}}^{\frac{1}{2}}, t, \overline{w}; p) = \\
&\qquad \frac{\diamondsuit}{\theta(\tau|\overline{v}_1 - \overline{v}_2)\theta(\tau|\overline{v}_1 - \overline{v}_3)\theta(\tau|\overline{v}_2 - \overline{v}_3)} \cdot (\theta(\tau|\overline{v}_2 - \overline{v}_1 - s)\theta(\tau|\overline{v}_3 - \overline{v}_1 - s)\theta(\tau|\overline{v}_2 - \overline{v}_3) \\
&\quad -\theta(\tau|\overline{v}_1 - \overline{v}_2 - s)\theta(\tau|\overline{v}_3 - \overline{v}_2 - s)\theta(\tau|\overline{v}_1 - \overline{v}_3) + \theta(\tau|\overline{v}_1 - \overline{v}_3 - s)\theta(\tau|\overline{v}_2 - \overline{v}_3 - s)\theta(\tau|\overline{v}_1 - \overline{v}_2)) \\
&= \frac{\diamondsuit}{\theta(\tau|v_1 - v_2)\theta(\tau|v_1 - v_3)\theta(\tau|v_2 - v_3)} \cdot \Big(e^{2\pi i s(b_2+b_3-2b_1)}\theta(\tau|v_2 - v_1 - s)\theta(\tau|v_3 - v_1 - s)\theta(\tau|v_2 - v_3) \\
&\qquad\qquad -e^{2\pi i s(b_1+b_3-2b_2)}\theta(\tau|v_1 - v_2 - s)\theta(\tau|v_3 - v_2 - s)\theta(\tau|v_1 - v_3) \\
&\qquad\qquad\qquad +e^{2\pi i s(b_1+b_2-2b_3)}\theta(\tau|v_1 - v_3 - s)\theta(\tau|v_2 - v_3 - s)\theta(\tau|v_1 - v_2)\Big).
\end{aligned}$$

Notice that for general values of s the above expression is different from the value of $\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q_1^3, \mathcal{K}_{\mathrm{vir}}^{1/2}, t, w; p)$. However, if we specialise $s \in \frac{1}{3}\mathbb{Z}$, we see that in the previous example $\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q_1^3, \mathcal{K}_{\mathrm{vir}}^{1/2}, t, w; p)$ becomes constant and periodic with respect to v on the lattice $\mathbb{Z} + 3\tau\mathbb{Z}$ and is holomorphic in v, from which we conclude that it is constant on v under this specialisation. Therefore, by choosing $w_j = e^{2\pi i \frac{j}{3}}$ to be third roots of unity, one can show

$$\mathrm{Ell}^{\mathrm{vir}}_{1/2}(Q_1^3, \mathcal{K}_{\mathrm{vir}}^{\frac{1}{2}}, t, w; p)\Big|_{\mathfrak{t}=e^{2\pi i \frac{k}{3}}} = \begin{cases} (-1)^{m+1}3, & \text{if } k = 3m, \quad m \in \mathbb{Z} \\ 0, & \text{if } k \notin 3\mathbb{Z}. \end{cases}$$

4.7.3. Limits of elliptic DT invariants Even if a closed formula for the higher rank generating series of elliptic DT invariants is not available, we can still study its behaviour by looking at some particular limits of the variables p, t_i, w_j.

It is easy to see that, under the Calabi–Yau restriction $\mathfrak{t} = 1$, the generating series of elliptic DT invariants does not carry any more refined information than the cohomological one; in particular, we have no more dependence on the framing parameters w_j and the elliptic parameter p. We generalise this phenomenon in the following setting. Denote by $\mathbf{T}_k \subset \mathbb{T}_1$ the subtorus where $\mathfrak{t}^{\frac{1}{2}} = e^{\pi i k/r}$, $k \in \mathbb{Z}$. Define by

$$\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p) = \mathsf{DT}^{\mathrm{ell}}_{r}(\mathbb{A}^3, q, t, w; p)\big|_{\mathbf{T}_k}$$

the restriction of the generating series to the subtorus $\mathbf{T}_k \subset \mathbb{T}_1$, which is well-defined as no powers of the Calabi–Yau weight appear in the vertex terms (4.4.5) by Lemma 4.2.5.

Proposition 4.7.7. *If $k = rm \in r\mathbb{Z}$, then*

$$\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p) = \mathsf{M}((-1)^{r(m+1)}q)^r.$$

In particular, the dependence on t_i, w_j and p drops.

Proof. Let $S \in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbf{T}}$. Denote $T_S^{\mathrm{vir}} = T_S^{nc} - \overline{T_S^{nc}}\mathfrak{t}^{-1}$ as in Equation (4.2.16), where T_S^{nc} is the tangent space of Quot_r^n at S. Denote by $T_{S,l}^{nc}$ the sub-representation of T_S^{nc} corresponding to $\mathfrak{t}^l$, with $l \in \mathbb{Z}$. As there are no powers of the Calabi–Yau weight in T_S^{vir}, we have an identity $T_{S,l}^{nc} = \overline{T_{S,-l-1}^{nc}}\mathfrak{t}^{-1}$. Set

$$W = T_S^{nc} - \sum_{n\in\mathbb{Z}} T_{S,n}^{nc}.$$

We have that

$$T_S^{nc} - \overline{T_S^{nc}}\mathfrak{t}^{-1} = W - \overline{W}\mathfrak{t}^{-1}$$

and, in particular, neither W nor $\overline{W}\mathfrak{t}^{-1}$ contain constant terms and powers of the Calabi-Yau weight. Using the quasi-periodicity of the theta function $\theta(\tau|z)$, we have

$$\theta[-T_S^{\mathrm{vir}}] = \frac{\theta[\overline{W}\mathfrak{t}^{-1}]}{\theta[W]} = (-1)^{m \operatorname{rk} W}\frac{\theta[\overline{W}]}{\theta[W]} = (-1)^{\operatorname{rk} W(m+1)}.$$

We conclude by noticing that

$$\operatorname{rk} W = \operatorname{rk} T_S^{nc} = rn \mod 2. \qquad \square$$

Motivated by Example 4.7.6 and Proposition 4.7.7, we propose the following conjecture.

Conjecture 4.7.8. *The series* $\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p)$ *does not depend on the elliptic parameter* p.

Remark 4.7.9. Notice that the independence from the elliptic parameter p implies that we can reduce our invariants to the K-theoretic ones by setting $p = 0$, i.e.

$$\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p) = \mathsf{DT}^{\mathrm{K}}_{r}(\mathbb{A}^3, q, t)\big|_{\mathbf{T}_k},$$

which in particular do not depend on the framing parameters.

Assuming Conjecture 4.7.8, we derive a closed expression for $\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p)$, which was conjectured in [**16**, Equation (3.20)], motivated by string-theoretic phenomena.

Theorem 4.7.10. *Assume Conjecture 4.7.8 holds and let* $k \in \mathbb{Z}$. *Then there is an identity*

$$\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3, q, t, w; p) = \mathsf{M}\left((-1)^{kr}((-1)^r q)^{\frac{r}{\gcd(k,r)}}\right)^{\gcd(k,r)}.$$

Proof. Assuming Conjecture 4.7.8, by Remark 4.7.9 we just have to prove the result for K-theoretic invariants. By Theorem 4.5.3,

$$\mathsf{DT}^{\mathrm{K}}_{r}(\mathbb{A}^3, (-1)^r q, t) =$$
$$\exp \sum_{n\geq 1} \frac{1}{n}\frac{(1-t_1^{-n}t_2^{-n})(1-t_1^{-n}t_3^{-n})(1-t_2^{-n}t_3^{-n})}{(1-t_1^{-n})(1-t_2^{-n})(1-t_3^{-n})}\frac{1-\mathfrak{t}^{-rn}}{1-\mathfrak{t}^{-n}}\frac{1}{(1-\mathfrak{t}^{-\frac{rn}{2}}q^{-n})(1-\mathfrak{t}^{-\frac{rn}{2}}q^{n})}.$$

Assume now that $\mathfrak{t}^{\frac{1}{2}} = e^{\pi i \frac{k}{r}}$, with $k \in \mathbb{Z}$; we have clearly that $\mathfrak{t}^{-\frac{rn}{2}} = (-1)^{kn}$. Moreover, we have

$$\frac{1-\mathfrak{t}^{-rn}}{1-\mathfrak{t}^{-n}} = \begin{cases} r, & \text{if } n \in \frac{r}{\gcd(r,k)}\mathbb{Z} \\ 0, & \text{if } n \notin \frac{r}{\gcd(r,k)}\mathbb{Z}\end{cases}$$

In particular, if $n \in \frac{r}{\gcd(r,k)}\mathbb{Z}$, we have

$$\frac{(1-t_1^{-n}t_2^{-n})(1-t_1^{-n}t_3^{-n})(1-t_2^{-n}t_3^{-n})}{(1-t_1^{-n})(1-t_2^{-n})(1-t_3^{-n})} = -1$$

Setting $n = \frac{r}{\gcd(r,k)} m$, with $m \in \mathbb{Z}$, we have

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, (-1)^r q, t) = \exp \sum_{m \geq 1} \frac{1}{m} \gcd(r,k) \cdot \frac{-1}{(1-\overline{q}^{-m})(1-\overline{q}^{m})}$$

where to ease notation we have set $\overline{q} = ((-1)^k q)^{\frac{r}{\gcd(r,k)}}$. We conclude by using the description of the MacMahon function as a plethystic exponential

$$\mathsf{DT}_r^{\mathrm{K}}(\mathbb{A}^3, (-1)^r q, t) = \mathsf{M}\left((-1)^{kr} q^{\frac{r}{\gcd(r,k)}}\right)^{\gcd(r,k)}. \qquad \square$$

Remark 4.7.11. A key technical point in the proof of the conjecture proposed in [**16**, Equation (3.20)] is the assumption of the independence of $\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3)$ on p, as in Conjecture 4.7.8. We strongly believe it should be possible to prove this assumption by exploiting modular properties of the generating series of elliptic DT invariants. One could proceed by considering the integral representation of the DT invariants given in [**16**]. The analysis of the K-theoretic case, which we carried out in the proof of Theorem 4.7.10, shows that no dependence whatsoever is present in the limit $\mathfrak{t}^{1/2} = e^{\pi i k/r}$. As elliptic DT invariants take the form of meromorphic Jacobi forms, given by quotients of theta functions, poles in the equivariant parameters are only given by shifts along the lattice $\mathbb{Z} + \tau\mathbb{Z}$ of the poles found in K-theoretic DT invariants. Then $\mathsf{DT}^{\mathrm{ell}}_{r,k}(\mathbb{A}^3)$, as a function of each of the equivariant parameters v_i, $i = 1, \ldots, r$, and s_j, $j = 1,2,3$, is holomorphic on the torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, so it also carries no dependence on them.

4.8. Higher rank DT invariants of compact toric 3-folds

Let X be a smooth *projective* toric 3-fold, along with an exceptional locally free sheaf F of rank r. By [**165**, Thm. A], the Quot scheme $\mathrm{Quot}_X(F, n)$ has a 0-dimensional perfect obstruction theory, so that the rank r Donaldson–Thomas invariant

$$\mathsf{DT}_{F,n} = \int_{[\mathrm{Quot}_X(F,n)]^{\mathrm{vir}}} 1 \in \mathbb{Z}$$

is well defined. In this section we confirm the formula

$$\sum_{n \geq 0} \mathsf{DT}_{F,n} q^n = \mathsf{M}((-1)^r q)^{r \int_X c_3(T_X \otimes K_X)},$$

suggested in [**165**, Conj. 3.5], in the case where F is *equivariant*. This will prove Theorem 4.1.3 from the Introduction. The next subsection is an interlude on how to induce a torus action on the Quot scheme and on the associated universal short exact sequence starting from an equivariant structure on F. More details are given in [**166**], including a proof that the obstruction theory obtained in [**165**, Thm. A] is equivariant.

4.8.1. The (equivariant) obstruction theory Let X be a quasiprojective toric variety with torus $\mathbb{T} \subset X$. Let $\sigma_X \colon \mathbb{T} \times X \to X$ denote the action. If F is a $\mathbb{T}$-equivariant coherent sheaf on X, and $\mathrm{Q} = \mathrm{Quot}_X(F, n)$, then σ_X has a canonical lift

$$\sigma_{\mathrm{Q}} \colon \mathbb{T} \times \mathrm{Q} \to \mathrm{Q},$$

as proved in [**113**, Prop. 4.1].

Throughout this subsection, F denotes an exceptional locally free sheaf of rank r on a smooth projective toric 3-fold X. In other words, F is simple, i.e. $\mathrm{Hom}(F,F)=\mathbb{C}$, and $\mathrm{Ext}^i(F,F)=0$ for $i>0$.

By [**165**, Thm. A], there is a 0-dimensional perfect obstruction theory

$$\mathbb{E}\to\mathbb{L}_{\mathrm{Quot}_X(F,n)}, \tag{4.8.1}$$

governed by

$$\mathrm{Def}\,\big|_{[S]}=\mathrm{Ext}^1(S,S),\quad \mathrm{Obs}\,\big|_{[S]}=\mathrm{Ext}^2(S,S)$$

around a point $[S]\in\mathrm{Quot}_X(F,n)$. We set $\mathrm{Q}=\mathrm{Quot}_X(F,n)$ for brevity, and we denote by π_{Q} and π_X the projections from $X\times\mathrm{Q}$. Note that $\omega_{\pi_{\mathrm{Q}}}=\pi_X^*\omega_X$.

We have

$$\mathbb{E}=\mathbf{R}\pi_{\mathrm{Q}*}(\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S})_0\otimes\omega_{\pi_{\mathrm{Q}}})[2]$$

where $\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S})_0$ is the shifted cone of the trace map $\mathrm{tr}\colon\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S})\to\mathcal{O}_{X\times\mathrm{Q}}$.

Proposition 4.8.1. *There is an identity*

$$\mathbb{E}^\vee=\mathbf{R}\pi_{\mathrm{Q}*}\mathbf{R}\mathscr{H}om(F_{\mathrm{Q}},F_{\mathrm{Q}})-\mathbf{R}\pi_{\mathrm{Q}*}\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S})\in K_0(\mathrm{Quot}_X(F,n)).$$

Proof. As in the proof of [**165**, Theorem 2.5] we have

$$\begin{aligned}\mathbb{E}&=(\mathbf{R}\pi_{\mathrm{Q}*}\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S})_0)^\vee[-1]\\&=(\mathbf{R}\pi_{\mathrm{Q}*}\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S}))^\vee[-1]-(\mathbf{R}\pi_{\mathrm{Q}*}\mathbf{R}\mathscr{H}om(\mathcal{O},\mathcal{O}))^\vee[-1]\\&=(\mathbf{R}\pi_{\mathrm{Q}*}\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S}))^\vee[-1]-(\mathbf{R}\pi_{\mathrm{Q}*}\mathbf{R}\mathscr{H}om(F_{\mathrm{Q}},F_{\mathrm{Q}}))^\vee[-1],\end{aligned}$$

where the last identity uses that F is an exceptional sheaf. $\square$

The following result is proved in [**166**, Thm. B] in greater generality. Denote by $\mathbb{T}=(\mathbb{C}^*)^3$ the torus of X.

Proposition 4.8.2. *Let (X,F) be a pair consisting of a smooth projective toric 3-fold X along with an exceptional locally free $\mathbb{T}$-equivariant sheaf F. Then the perfect obstruction theory* (4.8.1) *on $\mathrm{Quot}_X(F,n)$ is $\mathbb{T}$-equivariant.*

We let $\Delta(X)$ denote the set of vertices in the Newton polytope of the toric 3-fold X. Then

$$X^{\mathbb{T}}=\{\,p_\alpha\mid\alpha\in\Delta(X)\,\}\subset X$$

will denote the fixed locus of X. For a given vertex α, let $U_\alpha\cong\mathbb{A}^3$ be the canonical chart containing the fixed point p_α. The $\mathbb{T}$-action on this chart can be taken to be the standard action (4.2.5). For every α, there is a $\mathbb{T}$-equivariant open immersion

$$\iota_{n,\alpha}\colon\ \mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n)\hookrightarrow\mathrm{Q}=\mathrm{Quot}_X(F,n)$$

parametrising quotients whose support is contained in U_α. We think of $F|_{U_\alpha}$ as an equivariant sheaf on $\mathbb{A}^3$, hence of the form described in (4.4.9). We denote by $\mathbb{E}^{\mathrm{crit}}_{n,\alpha}$ the critical obstruction theory on $\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n)$ from Proposition 4.2.2.

It is natural to ask whether the restriction of the global perfect obstruction theory (4.8.1) along $\iota_{n,\alpha}$ agrees with the critical symmetric perfect obstruction theory described in Section 4.2.2 (see Conjecture 4.8.8). However, what we really need is the following weaker result.

Proposition 4.8.3. *Let $\mathbb{E} \in K_0(\mathrm{Q})$ be the class of the global perfect obstruction theory (4.8.1). Then*

$$\mathbb{E}^{\mathrm{crit}}_{n,\alpha} \;=\; \iota^*_{n,\alpha}\mathbb{E} \;\in\; K_0(\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n)).$$

Considering the two obstruction theories as $\mathbb{T}$-equivariant, the same identity holds in equivariant K-theory:

$$\mathbb{E}^{\mathrm{crit}}_{n,\alpha} \;=\; \iota^*_{n,\alpha}\mathbb{E} \;\in\; K_0^{\mathbb{T}}(\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n)).$$

Proof. The chart U_α is Calabi–Yau, so by [**165**, Prop. 2.9] the induced perfect obstruction theory $\iota^*_{n,\alpha}\mathbb{E}$ is symmetric. Since all symmetric perfect obstruction theories share the same class in K-theory, the first statement follows.

To prove the $\mathbb{T}$-equivariant equality, we need a slightly more refined analysis. Just for this proof, let us shorten

$$\mathbb{E}_{\mathrm{cr}} = \mathbb{E}^{\mathrm{crit}}_{n,\alpha} \quad \text{and} \quad \mathbb{E} = \iota^*_{n,\alpha}\mathbb{E},$$

to ease notation. We know by Diagram (4.2.15) that we can write

$$\mathbb{E}_{\mathrm{cr}} = \left[\mathfrak{t}\otimes T_{\mathrm{Quot}^n_r}\big|_{\mathrm{Q}} \to \Omega_{\mathrm{Quot}^n_r}\big|_{\mathrm{Q}}\right] = \Omega - \mathfrak{t}\otimes T \;\in\; K_0^{\mathbb{T}}(\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n)), \tag{4.8.2}$$

where Ω (resp. T) denotes the cotangent sheaf (resp. the tangent sheaf) of $\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n)$, equipped with its natural equivariant structure.

Let $\pi\colon U_\alpha \times \mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n) \to \mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n)$ be the projection, let S be the universal kernel living on the product and set $\mathfrak{t}_\pi = \pi^*\mathfrak{t}^{-1}$. By definition,

$$\mathbb{E} = \mathbf{R}\pi_*(\mathbf{R}\mathscr{H}om(S,S)_0 \otimes \omega_\pi)[2].$$

The equivariant isomorphism $\omega_\pi \xrightarrow{\sim} \mathcal{O}\otimes\mathfrak{t}_\pi^{-1}$ along with the projection formula yield

$$\mathfrak{t}^{-1}\otimes\mathbb{E} \xrightarrow{\sim} \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(S,S)_0[2]. \tag{4.8.3}$$

We next show the right hand side is canonically isomorphic to $\mathbb{E}^\vee[1]$. We have

$$\begin{aligned}
\mathbb{E}^\vee[1] &= \mathbf{R}\mathscr{H}om(\mathbf{R}\pi_*(\mathbf{R}\mathscr{H}om(S,S)_0\otimes\omega_\pi),\mathcal{O})[-1] && \text{definition of } (-)^\vee\\
&= \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathbf{R}\mathscr{H}om(S,S)_0\otimes\omega_\pi,\omega_\pi[3])[-1] && \text{Grothendieck duality}\\
&= \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathbf{R}\mathscr{H}om(S,S)_0,\mathcal{O})[2] && \text{shift}\\
&= \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(S,S)_0^\vee[2] && \text{definition of } (-)^\vee\\
&= \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(S,S)_0[2] && \mathbf{R}\mathscr{H}om(S,S)_0 \text{ is self-dual}
\end{aligned}$$

in the derived category of $\mathbb{T}$-equivariant coherent sheaves on $\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n)$, which by (4.8.3) proves that

$$\mathfrak{t}^{-1}\otimes\mathbb{E} \cong \mathbb{E}^\vee[1].$$

We thus have

$$\Omega \cong h^0(\mathbb{E}) \cong \mathfrak{t}\otimes h^0(\mathbb{E}^\vee[1]) \cong \mathfrak{t}\otimes \mathscr{E}xt^2_\pi(S,S),$$

where we use the standard notation $\mathscr{E}xt^i_\pi(-,-)$ for the ith derived functor of $\pi_* \circ \mathscr{H}om(-,-)$. We conclude

$$\begin{aligned}\mathbb{E} &= h^0(\mathbb{E}) - h^{-1}(\mathbb{E})\\ &= \Omega - h^1(\mathbb{E}^\vee)^\vee\\ &= \Omega - h^0(\mathbb{E}^\vee[1])^\vee\\ &= \Omega - \mathscr{E}xt^2_\pi(S,S)^\vee\\ &= \Omega - (\mathfrak{t}^{-1}\otimes\Omega)^\vee\\ &= \Omega - \mathfrak{t}\otimes T\\ &= \mathbb{E}_{\mathrm{cr}}.\end{aligned}$$

□

4.8.2. The fixed locus of the Quot scheme and its virtual class In this subsection we describe $\mathrm{Quot}_X(F,n)^{\mathbb{T}}$ and we compute its virtual fundamental class, obtained via Proposition 4.8.2.

If $\mathbf{n}$ denotes a generic tuple $\{\, n_\alpha \mid \alpha\in\Delta(X)\,\}$ of non-negative integers, we set $|\mathbf{n}| = \sum_{\alpha\in\Delta(X)} n_\alpha$.

Lemma 4.8.4. *There is a scheme-theoretic identity*

$$\mathrm{Quot}_X(F,n)^{\mathbb{T}} = \coprod_{|\mathbf{n}|=n}\prod_{\alpha\in\Delta(X)} \mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}.$$

Proof. Let B be a (connected) scheme over $\mathbb{C}$. Let F_B be the pullback of F along the first projection $X\times B\to X$, and fix a B-flat family of quotients

$$\rho\colon F_B \twoheadrightarrow \mathcal{T}$$

defining a B-valued point $B\to\mathrm{Quot}_X(F,n)^{\mathbb{T}}$. Then, by restriction, we obtain, for each $\alpha\in\Delta(X)$, a B-flat family of quotients

$$\rho_\alpha\colon F_B\big|_{U_\alpha\times B} \twoheadrightarrow \mathcal{T}_\alpha = \mathcal{T}\big|_{U_\alpha\times B}, \tag{4.8.4}$$

and we let n_α be the length of the fibres of $\mathcal{T}_\alpha$. Each ρ_α corresponds to a B-valued point $g_\alpha\colon B\to\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}$, thus we obtain a B-valued point

$$(g_\alpha)_\alpha\colon B\to\prod_{\alpha\in\Delta(X)}\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}.$$

Note that the original family $\mathcal{T}$ is recovered as the direct sum $\bigoplus_\alpha\mathcal{T}_\alpha$, in particular $n=\sum_\alpha n_\alpha$. Conversely, suppose given a tuple of B-families of $\mathbb{T}$-fixed quotients

$$\left((F|_{U_\alpha})_B \twoheadrightarrow \mathcal{T}_\alpha\right)_\alpha.$$

We obtain B-valued points

$$B\to\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}\subset\mathrm{Quot}_X(F,n_\alpha)^{\mathbb{T}}.$$

Since the support of these families is disjoint, we can form the direct sum

$$\mathcal{T}=\bigoplus_\alpha\mathcal{T}_\alpha$$

to obtain a new B-flat family, representing a B-valued point of $\mathrm{Quot}_X(F,n)^{\mathbb{T}}$, as required. $\square$

Our next goal is to show that, under the identification of Lemma 4.8.4, the induced virtual fundamental class of the $\mathbf{n}$-th connected component of $\mathrm{Quot}_X(F,n)^{\mathbb{T}}$ is the box product of the virtual fundamental classes of $\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}$, whose perfect obstruction theory is the $\mathbb{T}$-fixed part of the critical one, studied in Section 4.2.2. For the rest of the section we restrict our attention to each connected component

$$\mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}} = \prod_{\alpha\in\Delta(X)} \mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}} \subset \mathrm{Quot}_X(F,n)^{\mathbb{T}}, \tag{4.8.5}$$

and we denote by

$$\begin{array}{ccc} \mathcal{S}\hookrightarrow\mathcal{F}\twoheadrightarrow\mathcal{T} & & \mathcal{S}_\alpha\hookrightarrow\mathcal{F}_\alpha\twoheadrightarrow\mathcal{T}_\alpha \\ \Big| & & \Big| \\ X\times\mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}} & & U_\alpha\times\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}} \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi_\alpha} \\ \mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}} & \xrightarrow{p_\alpha} & \mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}} \end{array} \tag{4.8.6}$$

the various universal structures and projection maps between these moduli spaces. For instance, $\mathcal{F}_\alpha$ is the pullback of $F|_{U_\alpha}$ along the projection $U_\alpha\times\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}\to U_\alpha$. Let $\mathbb{E}_{\mathbf{n}}$ be the restriction of $\mathbb{E}\in\mathbf{D}(\mathrm{Quot}_X(F,n))$ to the closed subscheme $\mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}}\subset\mathrm{Quot}_X(F,n)$.

Proposition 4.8.5. *There are identities in* $K_0^{\mathbb{T}}(\mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}})$

$$\begin{aligned} \mathbb{E}_{\mathbf{n}}^\vee &= \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathcal{F},\mathcal{F})-\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S}) \\ &= \sum_{\alpha\in\Delta(X)} p_\alpha^*\left(\mathbf{R}\pi_{\alpha*}\mathbf{R}\mathscr{H}om(\mathcal{F}_\alpha,\mathcal{F}_\alpha)-\mathbf{R}\pi_{\alpha*}\mathbf{R}\mathscr{H}om(\mathcal{S}_\alpha,\mathcal{S}_\alpha)\right). \end{aligned}$$

Proof. Exploiting the universal short exact sequence

$$0\to\mathcal{S}\to\mathcal{F}\to\mathcal{T}\to 0$$

on $X\times\mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}}\subset X\times\mathrm{Quot}_X(F,n)$, and Proposition 4.8.1, we may write

$$\begin{aligned} \mathbb{E}_{\mathbf{n}}^\vee = \mathbb{E}^\vee\big|_{\mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}}} &= \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathcal{F},\mathcal{F})-\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S}) \\ &= \mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{T})+\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathcal{T},\mathcal{S})+\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(\mathcal{T},\mathcal{T}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\mathbf{R}\pi_{\alpha*}\mathbf{R}\mathscr{H}om(\mathcal{F}_\alpha,\mathcal{F}_\alpha)-\mathbf{R}\pi_{\alpha*}\mathbf{R}\mathscr{H}om(\mathcal{S}_\alpha,\mathcal{S}_\alpha) \\ &\quad= \mathbf{R}\pi_{\alpha*}\mathbf{R}\mathscr{H}om(\mathcal{S}_\alpha,\mathcal{T}_\alpha)+\mathbf{R}\pi_{\alpha*}\mathbf{R}\mathscr{H}om(\mathcal{T}_\alpha,\mathcal{S}_\alpha)+\mathbf{R}\pi_{\alpha*}\mathbf{R}\mathscr{H}om(\mathcal{T}_\alpha,\mathcal{T}_\alpha). \end{aligned} \tag{4.8.7}$$

In the following, we write (G_1,G_2) as a shortcut for any of the three pairs of sheaves $(\mathcal{S},\mathcal{T})$, $(\mathcal{T},\mathcal{S})$ or $(\mathcal{T},\mathcal{T})$. Applying the Grothendieck spectral sequence yields

$$\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(G_1,G_2)=\sum_{i,j}(-1)^{i+j}\mathbf{R}^i\pi_*\mathscr{E}xt^j(G_1,G_2)$$

$$= \sum_j (-1)^j \pi_* \mathscr{E}xt^j(G_1, G_2),$$

where we used cohomology and base change along with the fact that $\mathbf{R}^i\pi_*$ of a 0-dimensional sheaf vanishes for $i > 0$. The standard Čech cover $\{U_\alpha\}_{\alpha\in\Delta(X)}$ of X pulls back to a Čech cover $\{V_\alpha\}_{\alpha\in\Delta(X)}$ of $X \times \mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}}$, where $V_\alpha = U_\alpha \times \mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}}$. For a finite family of indices $I \subset \mathbb{N}$, set $V_I = \bigcap_{\alpha\in I} V_\alpha$ and let $j_I \colon V_I \to X \times \mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}}$ be the natural open immersion. We have a Čech resolution $\mathscr{E}xt^j(G_1, G_2) \to \mathfrak{C}^\bullet$, where $\mathfrak{C}^\bullet$ is defined degree-wise (see e.g. [90, Lemma III.4.2]) by

$$\mathfrak{C}^k = \bigoplus_{|I|=k+1} j_{I*}j_I^*\mathscr{E}xt^j(G_1, G_2).$$

Notice that $\mathcal{T}$ vanishes on the restriction to any double intersection $U_{\alpha\beta} \times \mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}}$, where $U_{\alpha\beta} = U_\alpha \cap U_\beta$. This implies that the only contribution of the Čech cover is given by $\mathfrak{C}^0$, thus

$$\begin{aligned}\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(G_1, G_2) &= \sum_j (-1)^j \pi_* \sum_{\alpha\in\Delta(X)} j_{\alpha*}j_\alpha^*\mathscr{E}xt^j(G_1, G_2)\\ &= \sum_{\alpha\in\Delta(X)} \sum_j (-1)^j (\pi \circ j_\alpha)_* j_\alpha^*\mathscr{E}xt^j(G_1, G_2).\end{aligned}$$

Consider the following cartesian diagram

$$\begin{tikzcd}
U_\alpha \times \mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}} \arrow[r, "\widetilde{p}_\alpha"] \arrow[d, hook, "j_\alpha"] & U_\alpha \times \mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n_\alpha)^{\mathbb{T}} \arrow[d, hook] \arrow[dd, bend left, "\pi_\alpha"] \\
X \times \mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}} \arrow[r] \arrow[d, "\pi"] & X \times \mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n_\alpha)^{\mathbb{T}} \arrow[d] \\
\mathrm{Quot}_X(F,n)^{\mathbb{T}}_{\mathbf{n}} \arrow[r, "p_\alpha"] & \mathrm{Quot}_{U_\alpha}(F|_{U_\alpha}, n_\alpha)^{\mathbb{T}}
\end{tikzcd}$$

As it was already clear from the proof of Lemma 4.8.4, the universal short exact sequences in Diagram (4.8.6) satisfy $j_\alpha^*(\mathcal{S} \hookrightarrow \mathcal{F} \twoheadrightarrow \mathcal{T}) = \widetilde{p}_\alpha^*(\mathcal{S}_\alpha \hookrightarrow \mathcal{F}_\alpha \twoheadrightarrow \mathcal{T}_\alpha)$. If $(G_{1\alpha}, G_{2\alpha})$ denotes any of the pairs belonging to the set $\{ (\mathcal{S}_\alpha, \mathcal{T}_\alpha), (\mathcal{T}_\alpha, \mathcal{S}_\alpha), (\mathcal{T}_\alpha, \mathcal{T}_\alpha) \}$, we can write

$$\begin{aligned}j_\alpha^*\mathscr{E}xt^j(G_1, G_2) &= \mathbf{L}j_\alpha^*\mathscr{E}xt^j(G_1, G_2)\\ &= \mathscr{E}xt^j(\mathbf{L}j_\alpha^*G_1, \mathbf{L}j_\alpha^*G_2)\\ &= \mathscr{E}xt^j(\widetilde{p}_\alpha^*G_{1,\alpha}, \widetilde{p}_\alpha^*G_{2,\alpha})\\ &= \widetilde{p}_\alpha^*\mathscr{E}xt^j(G_{1\alpha}, G_{2\alpha}).\end{aligned}$$

We deduce, by flat base change,

$$(\pi \circ j_\alpha)_* j_\alpha^*\mathscr{E}xt^j(G_1, G_2) = (\pi \circ j_\alpha)_*\widetilde{p}_\alpha^*\mathscr{E}xt^j(G_{1\alpha}, G_{2\alpha}) = p_\alpha^*\pi_{\alpha*}\mathscr{E}xt^j(G_{1\alpha}, G_{2\alpha}).$$

Combining again the Grothendieck spectral sequence, cohomology and base change and the vanishing of higher derived pushforwards on 0-dimensional sheaves, we conclude that

$$\begin{aligned}\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om(G_1,G_2) &= \sum_{\alpha\in\Delta(X)} p_\alpha^* \sum_j (-1)^j \pi_{\alpha *}\mathscr{E}xt^j(G_{1\alpha},G_{2\alpha}) \\ &= \sum_{\alpha\in\Delta(X)} p_\alpha^* \mathbf{R}\pi_{\alpha *}\mathbf{R}\mathscr{H}om(G_{1\alpha},G_{2\alpha}).\end{aligned}$$

Now the result follows from Equation (4.8.7). □

Corollary 4.8.6. *The virtual fundamental class of* $\mathrm{Quot}_X(F,n)_{\mathbf{n}}^{\mathbb{T}}$ *is expressed as the product of the virtual fundamental classes*

$$\left[\mathrm{Quot}_X(F,n)_{\mathbf{n}}^{\mathbb{T}}\right]^{\mathrm{vir}} = \prod_{\alpha\in\Delta(X)} p_\alpha^* \left[\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}\right]^{\mathrm{vir}}.$$

Before we prove the corollary, let us explain what virtual classes are involved. The left hand side is the virtual class induced by the $\mathbb{T}$-fixed obstruction theory

$$\mathbb{E}_{\mathbf{n}}^{\mathbb{T}\text{-fix}} \to \mathbb{L}_{\mathrm{Quot}_X(F,n)_{\mathbf{n}}^{\mathbb{T}}},$$

whereas $\left[\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}\right]^{\mathrm{vir}}$ is the virtual class induced by the obstruction theory

$$\iota_{n_\alpha,\alpha}^*\mathbb{E} \to \mathbb{L}_{\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)}$$

by restricting to the $\mathbb{T}$-fixed locus and taking the $\mathbb{T}$-fixed part. Note that by Proposition 4.8.3, the perfect obstruction theory

$$\mathbb{E}_{n_\alpha,\alpha}^{\mathrm{crit}}\Big|_{\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}}^{\mathbb{T}\text{-fix}} \to \mathbb{L}_{\mathrm{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}}$$

induces the same virtual class. This follows from the general fact that the (equivariant) virtual fundamental class depends only on the class in (equivariant) K-theory of the perfect obstruction theory — cf. [**171**, Theorem 4.6], where all the ingredients are naturally equivariant.

Proof. The statement follows by taking $\mathbb{T}$-fixed parts in Proposition 4.8.5 and by Siebert's result [**171**, Theorem 4.6] mentioned above. □

4.8.3. Higher rank Donaldson–Thomas invariants of compact 3-folds For a pair (X,F) consisting of a smooth projective toric 3-fold X and an exceptional locally free sheaf F, the perfect obstruction theory (4.8.1) gives rise to a 0-dimensional virtual fundamental class

$$\left[\mathrm{Quot}_X(F,n)\right]^{\mathrm{vir}} \in A_0(\mathrm{Quot}_X(F,n)),$$

allowing one to define higher rank Donaldson–Thomas invariants

$$\mathsf{DT}_{F,n} = \int_{[\mathrm{Quot}_X(F,n)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Define the generating function

$$\mathsf{DT}_F(q) = \sum_{n\geq 0} \mathsf{DT}_{F,n} q^n.$$

We next compute this series in the case of a $\mathbb{T}$-equivariant exceptional locally free sheaf, thus proving Theorem 4.1.3 from the Introduction.

Theorem 4.8.7. *Let (X,F) be a pair consisting of a smooth projective toric 3-fold X along with an exceptional $\mathbb{T}$-equivariant locally free sheaf F. Then*

$$\mathsf{DT}_F(q)=\mathsf{M}((-1)^r q)^{r\int_X c_3(T_X\otimes K_X)}.$$

Proof. Set $Q=\operatorname{Quot}_X(F,n)$ and $Q_\alpha=\operatorname{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)$. Since by Proposition 4.8.2 the perfect obstruction theory on Q is $\mathbb{T}$-equivariant, we can apply the virtual localisation formula

$$\mathsf{DT}_{F,n}=\int_{[Q^{\mathbb{T}}]^{\mathrm{vir}}}e^{\mathbb{T}}(-N^{\mathrm{vir}}_{Q^{\mathbb{T}}/Q}),$$

where $N^{\mathrm{vir}}_{Q^{\mathbb{T}}/Q}$ is the virtual normal bundle on the $\mathbb{T}$-fixed locus computed in Lemma 4.8.4. By taking $\mathbb{T}$-moving parts in Proposition 4.8.5, we obtain the K-theoretic identity

$$N^{\mathrm{vir}}_{Q^{\mathbb{T}}/Q}=\sum_{\alpha\in\Delta(X)}p_\alpha^*N^{\mathrm{vir}}_{Q_\alpha^{\mathbb{T}}/Q_\alpha}$$

of virtual normal bundles. Thus by Corollary 4.8.6 we have

$$\int_{[Q^{\mathbb{T}}]^{\mathrm{vir}}}e^{\mathbb{T}}(-N^{\mathrm{vir}}_{Q^{\mathbb{T}}/Q})=\sum_{|\mathbf{n}|=n}\prod_{\alpha\in\Delta(X)}\int_{[\operatorname{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}_1}]^{\mathrm{vir}}}e^{\mathbb{T}}(-N^{\mathrm{vir}}_{Q_\alpha^{\mathbb{T}}/Q_\alpha}).$$

In particular, the virtual fundamental class $[\operatorname{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}]^{\mathrm{vir}}$ agrees with the one coming from the critical structure. Moreover, by the virtual localisation formula applied with respect to $(\mathbb{C}^*)^r$, we have

$$\int_{[\operatorname{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)^{\mathbb{T}}]^{\mathrm{vir}}}e^{\mathbb{T}}(-N^{\mathrm{vir}}_{Q_\alpha^{\mathbb{T}}/Q_\alpha})=\int_{[\operatorname{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)]^{\mathrm{vir}}}1,$$

where the right hand side is defined equivariantly in § 4.4.2. Finally, by Corollary 4.6.4, we have an identity

$$\int_{[\operatorname{Quot}_{U_\alpha}(F|_{U_\alpha},n_\alpha)]^{\mathrm{vir}}}1=\int_{[\operatorname{Quot}_{U_\alpha}(\mathcal{O}_{U_\alpha}^{\oplus r},n_\alpha)]^{\mathrm{vir}}}1$$

of equivariant integrals, where in the right hand side we take $\mathcal{O}_{U_\alpha}^{\oplus r}$ with the trivial $\mathbb{T}$-equivariant weights. Therefore we conclude

$$\begin{aligned}\mathsf{DT}_F(q)&=\sum_{n\geq 0}q^n\sum_{|\mathbf{n}|=n}\prod_{\alpha\in\Delta(X)}\int_{[\operatorname{Quot}_{U_\alpha}(\mathcal{O}_{U_\alpha}^{\oplus r},n_\alpha)]^{\mathrm{vir}}}1\\&=\prod_{\alpha\in\Delta(X)}\sum_{n_\alpha\geq 0}q^{n_\alpha}\int_{[\operatorname{Quot}_{U_\alpha}(\mathcal{O}_{U_\alpha}^{\oplus r},n_\alpha)]^{\mathrm{vir}}}1\\&=\prod_{\alpha\in\Delta(X)}\mathsf{M}((-1)^r q)^{-r\frac{(s_1^\alpha+s_2^\alpha)(s_1^\alpha+s_3^\alpha)(s_2^\alpha+s_3^\alpha)}{s_1^\alpha s_2^\alpha s_3^\alpha}}.\end{aligned}$$

We have used Theorem 4.6.2 to obtain the last identity, in which we have denoted $s_1^\alpha,s_2^\alpha,s_3^\alpha$ the tangent weights at p_α. We conclude taking logarithms:

$$\begin{aligned}\log\mathsf{DT}_F(q)&=\sum_{\alpha\in\Delta(X)}-r\frac{(s_1^\alpha+s_2^\alpha)(s_1^\alpha+s_3^\alpha)(s_2^\alpha+s_3^\alpha)}{s_1^\alpha s_2^\alpha s_3^\alpha}\log\mathsf{M}((-1)^r q)\\&=r\int_X c_3(T_X\otimes K_X)\cdot\log\mathsf{M}((-1)^r q)\end{aligned}$$

where the prefactor is computed through ordinary Atiyah–Bott localisation. □

We have thus proved Conjecture 3.5 in [**165**] in the toric case. The general case is still open and will be investigated in future work.

4.8.4. Conjecture: two obstruction theories are the same We close this subsection with a couple of conjectures relating the different obstruction theories appeared in the previous section.

Conjecture 4.8.8. *Let $\mathbb{E}$ be the perfect obstruction theory* (4.8.1). *Then its restriction along the open subscheme* $\iota_{n,\alpha}\colon \operatorname{Quot}_{U_\alpha}(F|_{U_\alpha}, n) \hookrightarrow \operatorname{Quot}_X(F,n)$ *agrees, as a symmetric perfect obstruction theory, with the critical obstruction theory* $\mathbb{E}_{\mathrm{crit}}$ *of Proposition 4.2.2.*

One can also ask whether $\iota_{n,\alpha}^*\mathbb{E}$ and $\mathbb{E}_{\mathrm{crit}}$ are $\mathbb{T}$-*equivariantly* isomorphic over the cotangent complex of $\operatorname{Quot}_{U_\alpha}(F|_{U_\alpha}, n)$. This is of course stronger than the statement of Proposition 4.8.3.

A similar conjecture (essentially the rank 1 specialisation of Conjecture 4.8.8) can be stated for the moduli space $\operatorname{Hilb}^n(\mathbb{A}^3) = \operatorname{Quot}_{\mathbb{A}^3}(\mathscr{O}, n)$, without reference to a compactification $\mathbb{A}^3 \subset X$. The Hilbert scheme of points has two symmetric perfect obstruction theories: the critical obstruction theory $\mathbb{E}_{\mathrm{crit}}$ (Proposition 4.2.2) and the one coming from moduli of ideal sheaves: if $\mathsf{p}\colon \mathbb{A}^3 \times \operatorname{Hilb}^n(\mathbb{A}^3) \to \operatorname{Hilb}^n(\mathbb{A}^3)$ is the projection and $\mathfrak{I}$ is the universal ideal sheaf, one has the obstruction theory

$$\mathbf{R}\mathsf{p}_*\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0[2] \to \mathbb{L}_{\operatorname{Hilb}^n(\mathbb{A}^3)}$$

obtained from the Atiyah class $\mathrm{At}_{\mathfrak{I}}$.

Conjecture 4.8.9. *There is an isomorphism of perfect obstruction theories*

$$\begin{tikzcd} \mathbb{E}_{\mathrm{crit}} \arrow[rr, "\sim"] \arrow[dr] & & \mathbf{R}\mathsf{p}_*\mathbf{R}\mathscr{H}om(\mathfrak{I},\mathfrak{I})_0[2] \arrow[dl] \\ & \mathbb{L}_{\operatorname{Hilb}^n(\mathbb{A}^3)} & \end{tikzcd}$$

on the Hilbert scheme of points $\operatorname{Hilb}^n(\mathbb{A}^3)$.

Remark 4.8.10. After a first draft of this thesis was written — but after the paper [**76**] appeared — Ricolfi-Savvas [**167**] proved Conjecture 4.8.9.

PART II

Donaldson-Thomas theory in dimension four

CHAPTER 5

Donaldson-Thomas theory of Calabi-Yau 4-folds

Yo he sido Homero; en breve, seré
Nadie, come Ulises; en breve, seré
todos: estaré muerto.

El immortal, J.L. Borges

In Section 2.2 we explained how to construct virtual fundamental classes on a scheme when the obstruction theory is controlled only by its *deformations* and *obstructions*, in other words when the obstruction theory is *perfect*. In this chapter, we summarize the results of Oh-Thomas [**147**], where virtual fundamental classes are constructed (algebraically!) in case the obstruction theory is controlled by three terms — therefore failing to be perfect of amplitude $[-1,0]$ — but is self-dual, with a suitable notation of *orientation*. The main application will be moduli spaces of sheaves on smooth quasi-projective *Calabi-Yau* 4-folds, by which one defines *Donaldson-Thomas theory of Calabi-Yau 4-folds*.

5.1. Taking square roots

5.1.1. Special orthogonal bundles Let X be a scheme and (E,q) an orthogonal bundle over X, that is a rank r vector bundle with a non-degenerate quadratic paring

$$q : E \otimes E \to \mathcal{O}_X\,.$$

The non-degenerate quadratic pairing induces an isomorphism $q : E \xrightarrow{\sim} E^*$ and therefore an isomorphism

$$\det q : (\det E)^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X\,. \tag{5.1.1}$$

An *orientation* of (E,q) (cf. [**147**, Def. 2.1]) is a trivialization

$$o : \mathcal{O}_X \xrightarrow{\sim} \det E,$$

whose square $o^{\otimes 2}$ maps to $(-1)^{\frac{r(r-1)}{2}}$ under (5.1.1).

Remark 5.1.1. From the perspective of Differential Geometry, oriented orthogonal bundle (E, q, o) are in bijection with isomorphism classes of $SO(r, \mathbb{C})$-principal bundles, as transition functions from one orthonormal basis to another preserving the orientation lie in $SO(r, \mathbb{C})$. In another words, an orientation is equivalent to a reduction of the structure group of the étale principle frame bundle from $O(r, \mathbb{C})$ to $SO(r, \mathbb{C})$ (cf. [**147**, pag. 9]).

Now on, we assume that $r = 2n$, i.e. the rank of the orthogonal bundle E is even[1]. An algebraic subbundle $\Lambda \subset (E, q)$ is called *isotropic* if $q|_\Lambda \equiv 0$ and *maximal isotropic* if it has maximal rank n. The inclusion of a maximal isotropic subbundle $\Lambda \subset E$ induces a short exact sequence

$$0 \to \Lambda \to E \to \Lambda^* \to 0. \tag{5.1.2}$$

To any maximal isotropic subbundle $\Lambda \subset (E, q, o)$ of an oriented orthogonal bundle, Oh-Thomas [**147**, Def. 2.2] associate a uniquely determined *sign* $(-1)^{|\Lambda|}$, whose precise formulation we do not need here.

5.1.2. Edidin-Gram square root Euler class Let X be a scheme and (E, q, o) be an oriented orthogonal bundle of rank $r = 2n$, and assume it admits a maximal isotropic subbundle $\Lambda \subset E$. Edidin-Graham [**68**] define[2] a characteristic class — the *square root Euler class* — by

$$\sqrt{e}(E) := (-1)^{|\Lambda|} e(\Lambda) \in A^*(X, \mathbb{Z}),$$

which is independent of the choice of the maximal isotropic subbundle Λ and satisfies by (5.1.2)

$$e(E) = (-1)^n (\sqrt{e}(E))^2.$$

In general, if E does not admit a maximal isotropic subbundle Λ, such a characteristic class could not exists in Chow groups with $\mathbb{Z}$-coefficients. However, Edidin-Graham [**68**, Sec. 6] show that such a characteristic class always exists if we replace X by a suitable cover $\tilde{X} \to X$ — where a maximal isotropic subbundle does exist — and that it descends to a class on X if we allow ourselves to invert 2. More precisely, there exists a class

$$\sqrt{e}(E) \in A^* \left(X, \mathbb{Z} \left[\tfrac{1}{2} \right] \right)$$

which satisfies

$$e(E) = (-1)^n (\sqrt{e}(E))^2,$$

which we denote by *square root Euler class*.

[1] Oh-Thomas [**147**] deal with the general case as well, but it goes beyond the scope of this thesis.

[2] The explicit choice of the sign is fixed by Oh-Thomas [**147**, eqn. (22)].

5.1.3. K-theoretic Edidin-Graham square root class Oh-Thomas [**147**, Sec. 5.1] construct a K-theoretic analogue of the square root Euler class defined by Edidin-Graham. Recall that the *K-theoretic Euler class* of a rank r vector bundle E is defined as

$$\mathfrak{e}(E) := \Lambda^\bullet E^* = \sum_{i=0}^{r} (-1)^r \Lambda^i E^* \in K^0(X, \mathbb{Z}).$$

We also introduce the symmetrized notation

$$\widehat{\Lambda}^\bullet E := \Lambda^\bullet E \otimes (\det E)^{-1/2} \in K^0\left(X, \mathbb{Z}\left[\tfrac{1}{2}\right]\right).$$

Here we are forced to invert 2 as $\det E$ could not admit a square root as a line bundle in $\operatorname{Pic}(X)$. However, it does admit a canonical square root — as a class in K-theory — if we invert 2, see [**147**, Lemma 5.1, Rem. 5.2]. In particular, if a genuine square root exists as a line bundle, any two such choices would differ in $\operatorname{Pic}(X)$ by a 2-torsion element, and therefore such a choice woud be invisible once we invert 2.

Assume now that (E, q, o) is an oriented orthogonal bundle which admits a maximal isotropic subbundle $\Lambda \subset E$. First of all notice that, as $\det E \cong \mathcal{O}_X$, we have

$$\mathfrak{e}(E) = \Lambda^\bullet E^* = \widehat{\Lambda}^\bullet E^* \in K^0\left(X, \mathbb{Z}\left[\tfrac{1}{2}\right]\right),$$

and we define the *K-theoretic square root Euler class* as

$$\sqrt{\mathfrak{e}}(E) := (-1)^{|\Lambda|} \widehat{\Lambda}^\bullet \Lambda^* \in K^0\left(X, \mathbb{Z}\left[\tfrac{1}{2}\right]\right),$$

which satisfies by (5.1.2)

$$\mathfrak{e}(E) = (-1)^n (\sqrt{\mathfrak{e}}(E))^2.$$

As before, in general such a characteristic class cannot be defined directly on X. Replacing X by its cover $\tilde{X} \to X$ and descending the class as in Section 5.1.2, Oh-Thomas construct a *K-theoretic square root Euler class*

$$\sqrt{\mathfrak{e}}(E) \in K^0\left(X, \mathbb{Z}\left[\tfrac{1}{2}\right]\right),$$

which satisfies $\sqrt{\mathfrak{e}}(E)^2 = (-1)^n \mathfrak{e}(E)$.

5.1.4. Square root Gysin map Before describing the construction of the virtual classes we need to adapt Fulton's definition of the Gysin map to the isotropic realm.

Let (E, q, o) be an orientable orthogonal bundle over a scheme X and let $Z \subset X$ be a subscheme. A *cone* $C \subset E|_Z$ is said to be *isotropic* if q, thought of as a function on the total space of E (quadratic on the fibers), vanishes on the subscheme C. If $C \subset E|_Z$ is an isotropic cone, Oh-Thomas define[3] [**147**, Def. 3.3] the *square root Gysin map*

$$\sqrt{0^!_E} : A_*\left(C, \mathbb{Z}\left[\tfrac{1}{2}\right]\right) \to A_{*-n}\left(Z, \mathbb{Z}\left[\tfrac{1}{2}\right]\right).$$

In the case (E, q, o) admits a maximal isotropic subbundle $\Lambda \subset E$ containing the cone C, the square root Gysin map is simply

$$\sqrt{0^!_E} = (-1)^{|\Lambda|} 0^!_\Lambda,$$

[3]By means of the cosection localizations of Kiem-Li [**103, 104**].

where $0^!_\Lambda$ is the usual Gysin map (and we would not have to invert 2).

The square root Gysin map admits a K-theoretic analogue as well. Given an isotropic cone $C \subset E$ as before, Oh-Thomas define [**147**, Def. 5.8]

$$\sqrt{0^*_E} : K_0\left(C, \mathbb{Z}\left[\tfrac{1}{2}\right]\right) \to K_0\left(Z, \mathbb{Z}\left[\tfrac{1}{2}\right]\right).$$

As before, if (E, q, o) admits a maximal isotropic subbundle $\Lambda \subset E$ containing the cone C, this reduces to

$$\sqrt{0^*_E} = (-1)^{|\Lambda|}\sqrt{\det \Lambda} \cdot 0^*_\Lambda,$$

where 0^*_Λ is the usual (derived) pullback.

5.1.5. Complexes All the definitions and constructions of the previous sections extend in a natural way to *complexes* $\mathbb{E} \in \mathbf{D}^{\mathrm{b}}(X)$ which are self-dual with respect to a quadratic pairing. For instance, let $\mathbb{E} \in \mathbf{D}^{\mathrm{b}}(X)$ be a virtual rank[4] r complex satisying

$$(\det \mathbb{E})^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X . \tag{5.1.3}$$

An *orientation* of $\mathbb{E}$ is a trivialization

$$o : \mathcal{O}_X \xrightarrow{\sim} \det \mathbb{E},$$

whose square $o^{\otimes 2}$ maps to $(-1)^{\frac{r(r-1)}{2}}$ under (5.1.3).

5.2. Oh-Thomas virtual structures

5.2.1. Virtual structures To define virtual structures in the *isotropic* setting, we need the following general set-up[5]. Let X be a quasi-projective scheme with a -2-shifted symplectic structure in the sense of [**160**] and an *orientation*. In more simple words, we are asking for X to be a quasi-projective scheme, with an orientable self-dual *obstruction theory*[6] $\mathbb{E} \to \mathbb{L}_X$ of amplitude contained in $[-2, 0]$. Here, the obstruction theory $\mathbb{E}$ is naturally induced by the -2-shifted symplectic structure on the derived enhancement of X, while the self-duality comes from the contraction of the -2-shifted symplectic form. More explicitly, we can display $\mathbb{E}$ through a locally free resolution

$$\mathbb{E} = [T \xrightarrow{a} E \xrightarrow{a^*} T^*] \in \mathbf{D}^{\mathrm{b}}(X),$$

where E is an *oriented orthogonal bundle*, inducing an isomorphism $E \cong E^*$ compatible with the isomorphism $\mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee[2]$.

By the same recipe of Behrend-Fantechi [**12**], the obstruction theory determines a cone

$$\mathfrak{C} \hookrightarrow E^* \cong E.$$

[4]To define the virtual rank we assume that $\mathbb{E}$ is quasi-isomorphic to a complex of locally free sheaves, which is always the case if X is quasi-projective.

[5]Kiem-Park [**105**] extended most of the construction of this section to Deligne-Mumford stacks, but we do not need this level of generality in this thesis.

[6]An obstruction theory is a complex $\mathbb{E} \in \mathbf{D}^{\mathrm{b}}(X)$ satisfying all the conditions of Definition 2.2.1 apart from being of amplitude in $[-1, 0]$.

The key point is that the cone $\mathfrak{C}$ is *isotropic* — see [**147**, Prop. 4.3] — and one therefore defines the virtual fundamental class and *twisted* virtual structure sheaf via the square root Gysin map

$$\begin{aligned} [X]^{\mathrm{vir}} &= \sqrt{0^!_E}[\mathfrak{C}] \in A_{\frac{\mathrm{vd}}{2}}\left(X, \mathbb{Z}\left[\tfrac{1}{2}\right]\right), \\ \widehat{\mathcal{O}}^{\mathrm{vir}}_X &= \sqrt{0^*_E}[\mathcal{O}_{\mathfrak{C}}] \otimes \sqrt{\det T^*} \in K_0\left(X, \mathbb{Z}\left[\tfrac{1}{2}\right]\right), \end{aligned}$$

where $0_E \colon X \hookrightarrow E$ is the zero section of the vector bundle E and $\mathrm{vd} = \mathrm{rk}\,\mathbb{E}$ (which we assumed to be even). As expected, both the virtual fundamental class and the twisted virtual structure sheaf are deformation invariants — as long as X is projective.

Remark 5.2.1. The *twisted* virtual structure sheaf had been introduced by Nekrasov-Okounkov [**142**] to properly symmetrize some invariants on moduli spaces of sheaves with a perfect obstruction theory. However, in our setting, twisting the virtual structure sheaf is essential to even define it.

Remark 5.2.2. One may wonder why we do not simply ask the datum of an *orientable self-dual 3-terms obstruction theory* on X and instead require the more complicated notion of a -2-shifted symplectic structure. Unfortunately, to prove that the cone $\mathfrak{C} \subset E$ is *isotropic*, Oh-Thomas do not know if the former condition is sufficient. In fact what they need is the extra assumption that, in an étale neighborhood U around any point $p \in X$, the obstruction theory is of the form

$$\begin{tikzcd} \left[\Omega^*_{V|U} \arrow[r, "ds"] & Q \arrow[r, "(ds)^*"] \arrow[d, "s"] & \Omega_{V|U}\right] \arrow[d, equal] & \cong & \mathbb{E}|_U \arrow[d] \\ & \left[\mathcal{I}/\mathcal{I}^2 \arrow[r, "d"] & \Omega_{V|U}\right] & \cong & \mathbb{L}_U. \end{tikzcd}$$

Here, V is an open neighborhood of $0 \in h^0(\mathbb{E}|_p^\vee)$, the section is

$$s : V \to h^1(\mathbb{E}|_p^\vee), \text{ with } ds|_0 = 0,\ q(s,s) = 0 \text{ and } s^{-1}(0) \cong U,$$

$\mathcal{I}$ is the ideal of $Z(s) \subset V$ generated by s and Q is an orientable orthogonal bundle with fiber $h^1(\mathbb{E}|_p^\vee)$ (plus all the natural compatibilities of the self-dualities involved).

This "nice local form" is automatically realized if we assume X to have a -2-shifted symplectic structure by the results of [**15, 25**].

By Remark 5.2.2, all schemes with an oriented self-dual obstruction theories (coming from a -2-shifted symplectic structure) are locally of the following form.

Example 5.2.3 (Isotropic Kuranishi global model)**.** Let a scheme Z

$$\begin{tikzcd} & \mathcal{E} \arrow[d] \\ Z := Z(s) \arrow[r, hook, "\iota"] & A, \arrow[u, bend right, "s"'] \end{tikzcd}$$

be the zero locus of an isotropic section $s \in \Gamma(A, \mathcal{E})$, where $\mathcal{E}$ is an oriented orthogonal bundle over a smooth quasi-projective variety A. Then there exists a induced obstruction theory on Z

$$\begin{tikzcd}
\big[\Omega^*_{A|Z} \arrow[r, "ds"] & \mathcal{E}^*|_Z \arrow[r, "(ds)^*"] \arrow[d, "s"] & \Omega_{A|Z}\big] \arrow[d, equal] & \cong & \mathbb{E} \arrow[d] \\
& \big[\mathcal{I}/\mathcal{I}^2 \arrow[r, "d"] & \Omega_{A|Z}\big] & \cong & \mathbb{L}_Z.
\end{tikzcd}$$

in $\mathbf{D}^{[-2,0]}(Z)$, where we represented the truncated cotangent complex by means of the exterior derivative d constructed out of the ideal sheaf $\mathcal{I} \subset \mathcal{O}_Z$ of the inclusion $Z \hookrightarrow A$. By the results of Section 5.2.1, there exist a virtual fundamental class and a virtual structure sheaf satisfying

$$\iota_*[Z]^{\mathrm{vir}} = \sqrt{e}(\mathcal{E}) \cap [A] \in A_*\left(A, \mathbb{Z}\left[\tfrac{1}{2}\right]\right),$$
$$\iota_*\widehat{\mathcal{O}}_Z^{\mathrm{vir}} = \sqrt{\mathfrak{e}}(\mathcal{E}) \otimes K_A^{1/2} \in K_0\left(A, \mathbb{Z}\left[\tfrac{1}{2}\right]\right),$$

where K_A is the canonical bundle of A.

5.2.2. Virtual localization formulas Let X be a scheme satisfying the assumption of Section 5.2.1 and acted by an algebraic torus $\mathbf{T}$, such that the obstruction theory $\mathbb{E}$ is $\mathbf{T}$-equivariantly self-dual, and denote by $T_X^{\mathrm{vir}} := \mathbb{E}^\vee \in K^0(X)$ the *virtual tangent bundle*[7]. Oh-Thomas [**147**, Eqn. (111)] showed that there exists an induced virtual structure on the $\mathbf{T}$-fixed locus $X^{\mathbf{T}}$, with virtual tangent bundle $T_{X^{\mathbf{T}}}^{\mathrm{vir}} = T_X^{\mathrm{vir}}|_{X^{\mathbf{T}}}^{\mathsf{fix}}$ the $\mathbf{T}$-fixed part of the virtual tangent bundle. Denote by $N_{X^{\mathbf{T}}/X}^{\mathrm{vir}} := T_X^{\mathrm{vir}}|_{X^{\mathbf{T}}}^{\mathsf{mov}}$ — the movable part of the virtual tangent bundle — the *virtual normal bundle*. Oh-Thomas [**147**, Thm. 7.1, 7.3] proved analogues of Graber-Pandharipande and Fantechi-Göttsche virtual localizations in the isotropic setting[8].

Theorem 5.2.4 (Oh-Thomas virtual localization). *Denote by $\iota : X^{\mathbf{T}} \hookrightarrow X$ the inclusion of the fixed locus. The virtual fundamental class can be expressed*[9] *as*

$$[X]^{\mathrm{vir}} = \sum_i \iota_* \frac{[X_i]^{\mathrm{vir}}}{\sqrt{e^{\mathbf{T}}}(N_{X_i/X}^{\mathrm{vir}})},$$

where X_i are the connected components of the fixed locus $X^{\mathbf{T}}$. In particular, if X is a proper scheme, for any Chow class $\alpha \in A_{\mathbf{T}}^*(X)$ the integration formula holds

$$\int_{[X]^{\mathrm{vir}}} \alpha = \sum_i \int_{[X_i]^{\mathrm{vir}}} \frac{\alpha|_{X_i}}{\sqrt{e^{\mathbf{T}}}(N_{X_i/X}^{\mathrm{vir}})}.$$

Theorem 5.2.5 (Oh-Thomas K-theoretic virtual localization). *Denote by $\iota : X^{\mathbf{T}} \hookrightarrow X$ the inclusion of the fixed locus. The virtual structure sheaf can be explicitly written as*

$$\widehat{\mathcal{O}}_X^{\mathrm{vir}} = \sum_i \iota_* \frac{\widehat{\mathcal{O}}_{X_i}^{\mathrm{vir}}}{\sqrt{\mathfrak{e}^{\mathbf{T}}}(N_{X_i/X}^{\mathrm{vir}})},$$

[7] As in Footnote 5, one should at least ask for a global presentation of the virtual normal bundle $N_{X/X^{\mathbf{T}}}^{\mathrm{vir}}$ for the localization formulas to be well-defined.

[8] See also Park [**161**] for an independent proof where some global assumptions of [**147**] are weakened, by using a virtual pullback formula.

[9] See [**147**, Eqn. (112)] for the precise definition of the square root Euler class of the virtual normal bundle.

where X_i are the connected components of the fixed locus $X^{\mathbf{T}}$. In particular, if X is a proper scheme, for any K-theory class $V \in K_0^{\mathbf{T}}(X)$ the integration formula holds

$$\chi(X, V \otimes \widehat{\mathcal{O}}_X^{\mathrm{vir}}) = \sum_i \chi\left(X_i, \frac{V|_{X_i} \otimes \widehat{\mathcal{O}}_{X_i}^{\mathrm{vir}}}{\sqrt{\mathfrak{e}^{\mathbf{T}}}(N_{X_i/X}^{\mathrm{vir}})}\right).$$

5.2.3. Moduli spaces of sheaves on Calabi-Yau 4-folds The main application of the virtual construction of Oh-Thomas is moduli spaces of sheaves on Calabi-Yau 4-folds.

Let X be a projective Calabi-Yau 4-fold, that is a smooth complex projective variety with trivial canonical bundle $K_X \cong \mathcal{O}_X$. Denote by M_ω a moduli space of Gieseker stable sheaves on X of fixed topological type $\omega \in H^*(X)$ with respect to a polarization $\mathcal{O}(1)$ — see [**94**] for the definition of Gieseker stability. For simplicity, let us assume that there are no strictly semistable sheaves, so that M_ω is represented by a projective scheme. By the work of Huybrechts-Thomas [**95**], the Atiyah class gives an *obstruction theory* on M_ω

$$\mathbb{E} = \tau^{[2,0]}\mathbf{R}\pi_*\mathbf{R}\mathscr{H}om_\pi(\mathcal{E}, \mathcal{E})[3] \to \mathbb{L}_{M_\omega},$$

where $\tau^{[2,0]}$ is the usual truncation, $\pi : X \times M_\omega \to M_\omega$ and $\mathcal{E}$ is any universal twisted[10] sheaf on $X \times M$. In other words, the deformation theory of any element $E \in M_\omega$ is controlled by

$$\mathrm{Ext}^1_X(E, E), \quad \mathrm{Ext}^2_X(E, E), \quad \mathrm{Ext}^3_X(E, E),$$

where the first term determines the deformations, the second term the obstructions and the third term the higher obstructions. This means that the obstruction theory is *perfect*[11] of amplitude $[-2, 0]$ rather $[-1, 0]$, making it impossible to run Behrend-Fantechi and Li-Tian machineries of perfect obstruction theories. However, by Serre duality we have the isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_X(E, E) &\cong \mathrm{Ext}^3_X(E, E)^*,\\ \mathrm{Ext}^2_X(E, E) &\cong \mathrm{Ext}^2_X(E, E)^*.\end{aligned}$$

In particular, the obstruction space $\mathrm{Ext}^2_X(E, E)$ is endowed with a non-degenerate quadratic pairing, precisely as in Section 5.2.1. This duality is reflected globally by the symmetry

$$\theta : \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee[2], \text{ such that } \theta = \theta^\vee[2], \tag{5.2.1}$$

induced by Grothendieck duality.

By the work of Pantev-Toën-Vaquié-Vezzosi [**160**], M_ω admits a -2-shifted symplectic structure that lifts the symmetry (5.2.1), and by Cao-Gross-Joyce [**32**] M_ω is *orientable*, meaning that the obstruction theory $\mathbb{E}$ is orientable. This means that we

[10] Its existence is proved by Căldăraru in [**30**]. Note that the twisting cancels in $\mathbf{R}\mathscr{H}om(\mathcal{E}, \mathcal{E})$, giving an actual complex of sheaves in $\mathbf{D}^{\mathrm{b}}(X \times M)$.

[11] Instead of truncating this complex we could simply remove the trace, which shows that the complex is perfect, see [**147**, Footnote 15].

can run the machinery of Section 5.2.1 to produce a virtual fundamental class and a twisted virtual structure sheaf

$$\begin{aligned}[M_\omega]^{\mathrm{vir}} &\in A_{\frac{\mathrm{vd}}{2}}\left(M_\omega, \mathbb{Z}\left[\tfrac{1}{2}\right]\right), \\ \widehat{\mathcal{O}}^{\mathrm{vir}}_{M_\omega} &\in K_0\left(M_\omega, \mathbb{Z}\left[\tfrac{1}{2}\right]\right).\end{aligned}$$

This is the starting point of the (algebraic!) *Donaldson-Thomas theory of Calabi-Yau 4-folds*, which studies invariants obtained integrating classes against $[M_\omega]^{\mathrm{vir}}$ and $\widehat{\mathcal{O}}^{\mathrm{vir}}_{M_\omega}$.

5.2.3.1. Generalizations This construction extends naturally to moduli spaces of *compactly supported* stable sheaves on *quasi-projective* Calabi-Yau 4-folds, where the -2-shifted symplectic structure was constructed by Brav-Dyckerhoff in [**26**] and the orientation by Bojko in [**20**]. Quasi-projective Calabi-Yau 4-folds are the object of study of the next chapters, where we focus on *toric* Calabi-Yau 4-folds, which are never projective.

Similarly, the constructions above — including the results on the -2-shifted symplectic structure and the orientations — applies to some other moduli spaces of complexes of sheaves in the derived category of X, for instance moduli spaces of Pandharipande-Thomas stable pairs and of Joyce-Song pairs. We will study the former in detail in the following chapters. See also Diaconescu-Sheshmani-Yau [**64**] for some related constructions.

5.2.4. Borisov-Joyce virtual fundamental class Let X be a Calabi-Yau 4-fold and M_ω a moduli space of sheaves on X, for some fixed topological data $\omega \in H^*(X, \mathbb{Q})$ and fixed polarization. The virtual fundamental class $[M_\omega]^{\mathrm{vir}}$ — that we constructed *algebraically* in Section 5.2.1 — had already been constructed as a (real!) class in *homology* by Borisov-Joyce [**23**] (using Derived Differential Geometry) and in special cases by Cao-Leung [**38**] (using Gauge Theory). In other words, the two virtual classes coincide via the maps [**146**]

$$\begin{tikzcd}
{[M_\omega]^{\mathrm{vir}} \in A_*\left(M_\omega, \mathbb{Z}\left[\tfrac{1}{2}\right]\right)} \arrow[r] & H_*\left(M_\omega, \mathbb{Z}\left[\tfrac{1}{2}\right]\right) \\
 & {[M_\omega]^{\mathrm{vir}} \in H_*\left(M_\omega, \mathbb{Z}\right).} \arrow[u]
\end{tikzcd}$$

This show how Donaldson-Thomas invariants of Calabi-Yau 4-folds, when defined by integrating classes $\gamma \in H^*\left(M_\omega, \mathbb{Z}\right)$ against the Oh-Thomas virtual fundamental class $[M_\omega]^{\mathrm{vir}}$, are still integer-valued, as there is no need to invert 2 while working in (co)homology.

The advantages in using Oh-Thomas machinery are that we may work in K-theory as well, by integrating K-theory classes against the twisted virtual structure sheaf $\widehat{\mathcal{O}}^{\mathrm{vir}}_{M_\omega}$, and that we may use torus localization formulas to compute invariants.

We roughly explain how the (real!) virtual fundamental class is constructed by Borisov-Joyce [**23**].

At every point $E \in M_\omega$, choose a half-dimensional real subspace

$$\mathrm{Ext}^2_+(E,E) \subseteq \mathrm{Ext}^2(E,E)$$

of the usual obstruction space $\mathrm{Ext}^2(E,E)$, on which the quadratic form Q defined by Serre duality is real and positive definite. Then one glues local Kuranishi-type models of the form

$$\kappa_+ = \pi_+ \circ \kappa : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2_+(E,E),$$

where κ is the Kuranishi map for M_ω at E and π_+ denotes projection on the first factor of the decomposition $\mathrm{Ext}^2(E,E) = \mathrm{Ext}^2_+(E,E) \oplus \sqrt{-1} \cdot \mathrm{Ext}^2_+(E,E)$.

This glueing procedure is where the technicality is hidden, relying on the theory of shifted symplectic geometry [**160**] and Joyce's theory of derived C^∞-geometry.

Example 5.2.6. When M_ω is smooth, the obstruction sheaf $\mathrm{Ob} \to M_\omega$ is a vector bundle endowed with a quadratic form Q via Serre duality. Then the virtual class is given by

$$[M_\omega]^{\mathrm{vir}} = \mathrm{PD}(e(\mathrm{Ob}, Q)),$$

where $\mathrm{PD}(\cdot)$ denotes Poincaré dual and $e(\mathrm{Ob}, Q)$ is the (real!) square root Euler class of (Ob, Q), i.e. the Euler class of its real form Ob_+. In this case, a choice of orientation of the obstruction theory is equivalent to a choice of orientation of Ob_+. The (real) square root Euler class satisfies

$$\begin{aligned} e(\mathrm{Ob},Q)^2 &= (-1)^{\frac{\mathrm{rk}(\mathrm{Ob})}{2}} e(\mathrm{Ob}), && \text{if } \mathrm{rk}(\mathrm{Ob}) \text{ is even}, \\ e(\mathrm{Ob},Q) &= 0, && \text{if } \mathrm{rk}(\mathrm{Ob}) \text{ is odd}. \end{aligned}$$

CHAPTER 6

K-theoretic DT/PT for toric Calabi-Yau 4-folds

This is not the end,
this is not the beginning
just a voice like a riot,
rocking every revision

Waiting for the end, Linkin Park

6.1. Introduction

Two recent developments are Donaldson-Thomas type invariants of Calabi-Yau 4-folds and K-theoretic virtual invariants introduced by Nekrasov-Okounkov [**142**]. Let X be a complex smooth quasi-projective variety, $\beta \in H_2(X, \mathbb{Z})$, and $n \in \mathbb{Z}$. We consider the following moduli spaces:

- $I := \mathrm{Hilb}^n(X, \beta)$ denotes the Hilbert scheme of proper closed subschemes $Z \subseteq X$ of dimension $\leqslant 1$ satisfying $[Z] = \beta$ and $\chi(\mathcal{O}_Z) = n$,
- $P := P_n(X, \beta)$ is the moduli space of stable pairs $(F, s) := [\mathcal{O}_X \xrightarrow{s} F]$ in $\mathbf{D}^{\mathrm{b}}(X)$, where F is a pure 1-dimensional sheaf on X with proper scheme theoretic support in class β, $\chi(F) = n$, and $s \in H^0(F)$ has 0-dimensional cokernel.

For projective Calabi-Yau 3-folds, both spaces have a symmetric perfect obstruction theory. The degrees of the virtual classes are known as (rank one) Donaldson-Thomas and Pandharipande-Thomas invariants. Their generating series are related by the famous DT/PT correspondence conjectured by Pandharipande-Thomas [**152**] and proved by Bridgeland [**27**] and Toda [**180**].

For projective Calabi-Yau 4-folds, I and P still have an obstruction theory

$$\mathbf{R}\mathscr{H}om_{\pi_I}(I_{\mathcal{Z}}, I_{\mathcal{Z}})_0^{\vee}[-1] \to \mathbb{L}_I,$$
$$\mathbf{R}\mathscr{H}om_{\pi_P}(\mathbb{I}^{\bullet}, \mathbb{I}^{\bullet})_0^{\vee}[-1] \to \mathbb{L}_P,$$

where $(\cdot)_0$ denotes the trace-free part, $\mathbf{R}\mathscr{H}om_\pi = \mathbf{R}\pi_* \circ \mathbf{R}\mathscr{H}om$, π_I, π_P are the natural projections and we denoted the universal objects by $\mathcal{Z} \subseteq I \times X$ and $\mathbb{I}^\bullet = [\mathcal{O}_{P\times X} \to \mathbb{F}] \in \mathbf{D}^{\mathrm{b}}(X \times P)$.

These obstruction theories are *not perfect*, so the machineries of Behrend-Fantechi [**12**] and Li-Tian [**123**] do not produce virtual classes on the moduli spaces. Nonetheless, by the construction in Section 5.2, there exist (real) virtual classes

$$[I]^{\mathrm{vir}} \in H_{2n}(I,\mathbb{Z}), \quad [P]^{\mathrm{vir}} \in H_{2n}(P,\mathbb{Z})$$

in the sense of Borisov-Joyce [**23**] and (algebraic) virtual classes and twisted virtual structure sheaves

$$\begin{aligned} [I]^{\mathrm{vir}} &\in A_n\left(I,\mathbb{Z}\left[\tfrac{1}{2}\right]\right), \quad [P]^{\mathrm{vir}} \in A_n\left(P,\mathbb{Z}\left[\tfrac{1}{2}\right]\right), \\ \widehat{\mathcal{O}}_I^{\mathrm{vir}} &\in K_0\left(I,\mathbb{Z}\left[\tfrac{1}{2}\right]\right), \quad \widehat{\mathcal{O}}_P^{\mathrm{vir}} \in K_0\left(P,\mathbb{Z}\left[\tfrac{1}{2}\right]\right) \end{aligned}$$

in the sense of Oh-Thomas [**147**]. In both cases, the virtual fundamental class depends on a choice of *orientation* of the obstruction theory, which was proven to exist by Cao-Gross-Joyce [**32**] and Bojko [**20**].

6.1.1. Nekrasov genus Throughout this chapter, X is a toric Calabi-Yau 4-fold[1]. Since X is non-proper, the moduli spaces I, P are *in general* non-proper and we define invariants by Oh-Thomas localization formula (Theorem 5.2.4, 5.2.5). There are interesting cases for which P is proper, e.g. when $X = \mathrm{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)), \mathrm{Tot}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1))$. Denote by $(\mathbb{C}^*)^4$ the dense open torus of X and let $\mathbf{T} \subseteq (\mathbb{C}^*)^4$ be the 3-dimensional subtorus preserving the Calabi-Yau volume form. Then the $\mathbf{T}$-fixed locus is

$$I^{\mathbf{T}} = I^{(\mathbb{C}^*)^4},$$

which consists of finitely many isolated reduced points [**35**, Lem. 2.2]. Roughly speaking, these are described by solid partitions (4D piles of boxes) corresponding to monomial ideals in each toric chart $U_\alpha \cong \mathbb{C}^4$ with infinite *legs* along the coordinate axes, which agree on overlaps $U_\alpha \cap U_\beta$. In general, the fixed locus $P^{(\mathbb{C}^*)^4}$ may not be isolated [**35**]. Throughout this chapter, whenever we consider a moduli space P of stable pairs, we assume:

Assumption 6.1.1. X is a toric Calabi-Yau 4-fold and $\beta \in H_2(X,\mathbb{Z})$ such that $\bigsqcup_n P_n(X,\beta)^{(\mathbb{C}^*)^4}$ is at most 0-dimensional.

When Assumption 6.1.1 is satisfied, $P_n(X,\beta)^{\mathbf{T}} = P_n(X,\beta)^{(\mathbb{C}^*)^4}$ for all n and it consists of finitely many reduced points, which are combinatorially described in [**35**, Sect. 2.2].

This assumption is equivalent to saying that in each toric chart U_α at most two infinite legs come together (Lemma 6.2.2). This is the case when X is a local curve or local surface. If in each toric chart U_α at most three legs come together, $P^{(\mathbb{C}^*)^4}$

[1] I.e. a smooth quasi-projective toric 4-fold X satisfying $K_X \cong \mathcal{O}_X$, $H^{>0}(\mathcal{O}_X) = 0$, and such that every cone of its fan is contained in a 4-dimensional cone.

is isomorphic to a disjoint union of products of $\mathbb{P}^1$'s (essentially by [**155**]). This is the case when X is a local threefold. In full generality, four infinite legs can come together in each toric chart U_α. Then $P^{(\mathbb{C}^*)^4}$ is considerably more complicated; its connected components are cut out by incidence conditions from ambient spaces of the form $\mathrm{Gr}(1,2)^\ell \times \mathrm{Gr}(1,3)^m \times \mathrm{Gr}(2,3)^n$. In order to avoid moduli, we focus on the isolated case, though we expect our results to be generalized to the general setting.

Denote the *virtual tangent spaces* of I, P by

$$T_I^{\mathrm{vir}} = \mathbf{R}\mathscr{H}om_{\pi_I}(I_{\mathcal{Z}}, I_{\mathcal{Z}})_0[1] \in K^0_{\mathbf{T}}(I), \quad T_P^{\mathrm{vir}} = \mathbf{R}\mathscr{H}om_{\pi_P}(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0[1] \in K^0_{\mathbf{T}}(P).$$

At any fixed point $x = Z \in I^{\mathbf{T}}$ or $x = (F, s) \in P^{\mathbf{T}}$, $\mathbf{T}$-equivariant Serre duality implies that the $\mathbf{T}$-equivariant K-theory classes

$$T_I^{\mathrm{vir}}|_x, T_P^{\mathrm{vir}}|_x \in K_0^{\mathbf{T}}(\mathsf{pt}) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(t_1 t_2 t_3 t_4 - 1)$$

have square roots.

Definition 6.1.2. Let $V \in K^0_{\mathbf{T}}(\mathsf{pt})$ be a virtual $\mathbf{T}$-representation. We say that $T \in K^0_{\mathbf{T}}(\mathsf{pt})$ is a *square root* of V if

$$V = T + \overline{T} \in K^0_{\mathbf{T}}(\mathsf{pt}),$$

where $\overline{(\cdot)}$ denotes the dual $\mathbf{T}$-representation.

Namely there exist $T_I^{\mathrm{vir}}|_{x,\mathrm{half}}$, $T_P^{\mathrm{vir}}|_{x,\mathrm{half}} \in K_0^{\mathbf{T}}(\mathsf{pt})$ such that

$$T_I^{\mathrm{vir}}|_x = T_I^{\mathrm{vir}}|_{x,\mathrm{half}} + \overline{T_I^{\mathrm{vir}}|_{x,\mathrm{half}}},$$

and similarly for P. These square roots are *non-unique*. We denote by *virtual canonical bundle* $K_I^{\mathrm{vir}} := \det T_I^{\mathrm{vir},\vee}$ and the $\mathbf{T}$-moving and $\mathbf{T}$-fixed parts by

$$N^{\mathrm{vir}}|_x := (T_I^{\mathrm{vir}}|_x)^{\mathsf{mov}}, \quad (T_I^{\mathrm{vir}}|_x)^{\mathsf{fix}}.$$

We use similar notations for the stable pairs case. A choice of a square root of $T_I^{\mathrm{vir}}|_x$, $T_P^{\mathrm{vir}}|_x$ induces a square root E_{half} for each of the above $\mathbf{T}$-representations E.

For any $\mathbf{T}$-equivariant line bundle L on X, we define

$$L^{[n]} := \mathbf{R}\pi_{I*}(\pi_X^* L \otimes \mathcal{O}_{\mathcal{Z}}), \quad \mathbf{R}\pi_{P*}(\pi_X^* L \otimes \mathbb{F}) \tag{6.1.1}$$

on the moduli spaces I and P.

Following [**140**][2], which deals with the case $\mathrm{Hilb}^n(\mathbb{C}^4, 0)$, we propose the following definition.

Definition 6.1.3. We define the following *Nekrasov genus* of the moduli space $I := \mathrm{Hilb}^n(X, \beta)$. Consider an extra trivial $\mathbb{C}^*$-action on X and let $\mathcal{O} \otimes y$ be the trivial line bundle with non-trivial $\mathbb{C}^*$-equivariant structure corresponding to the irreducible character y. For any $\mathbf{T}$-equivariant line bundle L on X, we define

$$I_{n,\beta}(L, y) := \chi\Big(I, \widehat{\mathcal{O}}_I^{\mathrm{vir}} \otimes \frac{\Lambda^\bullet(L^{[n]} \otimes y^{-1})}{(\det(L^{[n]} \otimes y^{-1}))^{\frac{1}{2}}}\Big)$$

---

[2] And, apparently, here [**141**].

$$= \chi\Big(I^{\mathbf{T}}, \frac{\widehat{\mathcal{O}}^{\mathrm{vir}}_{I^T}}{\sqrt{\mathfrak{e}^{\mathbf{T}}}(N^{\mathrm{vir}})} \otimes \frac{\Lambda^\bullet(L^{[n]}\otimes y^{-1})}{(\det(L^{[n]}\otimes y^{-1}))^{\frac{1}{2}}}\Big) \in \frac{\mathbb{Q}(t_1^{\pm 1/2}, t_2^{\pm 1/2}, t_3^{\pm 1/2}, t_4^{\pm 1/2})}{(t_1t_2t_3t_4 - 1)}.$$

We define $P_{n,\beta}(L,y)$ analogously replacing I by P and imposing Assumption 6.1.1.

Here the first line is a *global definition*, while the second one expresses the invariants by means of the virtual localization formula (Theorem 5.2.5) exploiting the properness of the $\mathbf{T}$-fixed locus $I^{\mathbf{T}}$. Here, $\sqrt{\mathfrak{e}}$ is the K-theoretic square root Euler class described in Section 5.1.3. For any $Z \in I^{\mathbf{T}}$ and given any $\mathbf{T}$-square root $T|^{\mathrm{vir}}_{Z,\mathrm{half}}$ of $T|^{\mathrm{vir}}_Z$, we have that

$$\sqrt{\mathfrak{e}}(T^{\mathrm{vir}}) = (-1)^{o_Z} \frac{\Lambda^\bullet(T|^{\mathrm{vir},\vee}_{Z,\mathrm{half}})}{\det(T|^{\mathrm{vir},\vee}_{Z,\mathrm{half}})^{\frac{1}{2}}}.$$

This means we can formulate[3] the Nekrasov genus without invoking the square root Euler class

$$I_{n,\beta}(L,y) = \sum_{Z\in I^{\mathbf{T}}} (-1)^{o_Z} \cdot \frac{\Lambda^\bullet(T|^{\mathrm{vir},\vee}_{Z,\mathrm{half}})}{\det(T|^{\mathrm{vir},\vee}_{Z,\mathrm{half}})^{\frac{1}{2}}} \cdot \frac{\Lambda^\bullet(L^{[n]}|_Z\otimes y^{-1})}{(\det(L^{[n]}|_Z\otimes y^{-1}))^{\frac{1}{2}}}, \tag{6.1.2}$$

at the cost of making a choice of square root[4] $T|^{\mathrm{vir}}_{Z,\mathrm{half}}$ at every fixed point $Z \in I^{\mathbf{T}}$ and paying the price of introducing a sign $(-1)^{o_Z}$.

Remark 6.1.4. During the writing of this thesis — and of [**36**] — the twisted virtual structure sheaf and K-theoretic localization formula were not established yet in the setting of Calabi-Yau 4-folds. As a consequence, we *defined* our invariants $I_{n,\beta}(L,y)$ (and $P_{n,\beta}(L,y)$) by the (expected) virtual localization formula as in (6.1.2). Luckily, Oh-Thomas [**147**] provided the correct foundational aspects of the theory as we initially conjectured them. Nevertheless, the signs appearing at each $\mathbf{T}$-fixed point are in principle implicit and little is known about them — see Appendix 6.6 for a (conjectural) sign rule.

6.1.2. *K*-theoretic DT/PT correspondence We show in Section 6.2 that the invariants $I_{n,\beta}(L,y)$ and $P_{n,\beta}(L,y)$ can be calculated by a K-theoretic vertex formalism. The case $I_{n,0}(L,y)$ was originally established by Nekrasov [**140**] — via supersymmetric localization in String Theory — and Nekrasov-Piazzalunga [**143**], who also deal with the higher rank case. Our focus is on the K-theoretic DT/PT correspondence for toric Calabi-Yau 4-folds. In Section 6.2, we define the K-theoretic DT/PT 4-fold vertex[5]

$$\mathsf{V}^{\mathsf{DT}}_{\lambda\mu\nu\rho}(t,y,q), \quad \mathsf{V}^{\mathsf{PT}}_{\lambda\mu\nu\rho}(t,y,q) \in \frac{\mathbb{Q}(t_1,t_2,t_3,t_4,y^{\frac{1}{2}})}{(t_1t_2t_3t_4-1)}(\!(q)\!),$$

[3] We use here T^{vir}_Z instead of N^{vir}_Z as both $I^{\mathbf{T}}, P^{\mathbf{T}}$ are reduced and zero-dimensional, therefore there are no positive fixed terms in T^{vir}_Z. There could be in principle negative fixed terms: in that case, $[-T^{\mathrm{vir}}_Z] = 0$, but the negative fixed terms correspond to fixed obstructions that make the localized contribution at Z vanish.

[4] When developing the vertex formalism in Section 6.2.4, we make an explicit choice of square root for each $T^{\mathrm{vir}}_I|_Z$ and $T^{\mathrm{vir}}_P|_Z$.

[5] A priori the powers of t_1,t_2,t_3,t_4 in $\mathsf{V}^{\mathsf{DT}}_{\lambda\mu\nu\rho}(t,y,q)$, $\mathsf{V}^{\mathsf{PT}}_{\lambda\mu\nu\rho}(t,y,q)$ are half-integers. We prove in Proposition 6.2.13 that they are always integers.

for any finite plane partitions (3D partitions) λ, μ, ν, ρ. In the stable pairs case, we require that at most two of λ, μ, ν, ρ are non-empty (which follows from Assumption 6.1.1 by Lemma 6.2.2). Roughly speaking, these are the generating series of $I_{n,\beta}(L, y)$, $P_{n,\beta}(L, y)$ in the case $X = \mathbb{C}^4$, $L = \mathcal{O}_{\mathbb{C}^4}$, and the underlying Cohen-Macaulay support curve is fixed and described by finite asymptotic plane partitions λ, μ, ν, ρ (see Definition 6.2.14). The series $\mathsf{V}^{\mathsf{DT}}_{\lambda\mu\nu\rho}$, $\mathsf{V}^{\mathsf{PT}}_{\lambda\mu\nu\rho}$ depend on the choice of a sign at each $\mathbf{T}$-fixed point.

Before we phrase our DT/PT vertex correspondence, we discuss a beautiful conjecture by Nekrasov for $\mathsf{V}^{\mathsf{DT}}_{\varnothing\varnothing\varnothing\varnothing}$ [**140, 143**]. We recall the definition of the plethystic exponential. For any formal power series $f(p_1, \ldots, p_r; q_1, \ldots, q_s)$ in $\mathbb{Q}(p_1, \ldots, p_r)[\![q_1, \ldots, q_s]\!]$, its plethystic exponential is defined by

$$\mathrm{Exp}(f(p_1, \ldots, p_r; q_1, \ldots, q_s)) := \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} f(p_1^n, \ldots, p_r^n; q_1^n, \ldots, q_s^n)\Big)$$

viewed as an element of $\mathbb{Q}(p_1, \ldots, p_r)[\![q_1, \ldots, q_s]\!]$. Following Nekrasov [**140**], for any formal variable x, we define the operator

$$[x] := x^{\frac{1}{2}} - x^{-\frac{1}{2}}.$$

Conjecture 6.1.5 (Nekrasov)**.** *There exist unique choices of signs such that*

$$\mathsf{V}^{\mathsf{DT}}_{\varnothing\varnothing\varnothing\varnothing}(t, y, q) = \mathrm{Exp}(\mathcal{F}(t, y; q)), \quad \mathcal{F}(t, y; q) := \frac{[t_1 t_2][t_1 t_3][t_2 t_3][y]}{[t_1][t_2][t_3][t_4][y^{\frac{1}{2}} q][y^{\frac{1}{2}} q^{-1}]},$$

where $\mathcal{F}(t, y; q) \in \frac{\mathbb{Q}(t_1, t_2, t_3, t_4, y^{\frac{1}{2}}, q)}{(t_1 t_2 t_3 t_4 - 1)}$ is expanded as a formal power series in q.

See [**140**] for the existence part. Here we conjecture the uniqueness part.

Remark 6.1.6. Recently, Kool-Rennemo [**115**] announced a proof of Nekrasov's conjecture.

We propose the following K-theoretic DT/PT 4-fold vertex correspondence:

Conjecture 6.1.7. *For any finite plane partitions λ, μ, ν, ρ, at most two of which are non-empty, there are choices of signs such that*

$$\mathsf{V}^{\mathsf{DT}}_{\lambda\mu\nu\rho}(t, y, q) = \mathsf{V}^{\mathsf{PT}}_{\lambda\mu\nu\rho}(t, y, q)\, \mathsf{V}^{\mathsf{DT}}_{\varnothing\varnothing\varnothing\varnothing}(t, y, q).$$

Suppose we choose the signs for $\mathsf{V}^{\mathsf{DT}}_{\varnothing\varnothing\varnothing\varnothing}(t, y, q)$ equal to the unique signs in Nekrasov's conjecture 6.1.5. Then, at each order in q, the choice of signs for which left-hand-side and right-hand-side agree is unique up to an overall sign.

We verify this conjecture in various cases for which $|\lambda| + |\mu| + |\nu| + |\rho| \leqslant 4$ and the number of embedded boxes is $\leqslant 3$ — for the precise statement, see Proposition 6.2.15. This conjecture and the vertex formalism imply the following:

Theorem 6.1.8. *Assume Conjecture 6.1.7 holds. Let X be a toric Calabi-Yau 4-fold and $\beta \in H_2(X, \mathbb{Z})$ such that $\bigsqcup_n P_n(X, \beta)^{(\mathbb{C}^*)^4}$ is at most 0-dimensional. Let L be a $\mathbf{T}$-equivariant line bundle on X. Then there exist choices of signs such that*

$$\frac{\sum_n I_{n,\beta}(L, y)\, q^n}{\sum_n I_{n,0}(L, y)\, q^n} = \sum_n P_{n,\beta}(L, y)\, q^n.$$

One may wonder whether there are other K-theoretic insertions for which a DT/PT correspondence similar to Conjecture 6.1.7 holds. The most natural candidates are virtual holomorphic Euler characteristics $\chi(I, \widehat{\mathcal{O}}_I^{\mathrm{vir}})$, $\chi(P, \widehat{\mathcal{O}}_P^{\mathrm{vir}})$, or replacing L in Definition 6.1.3 by a higher rank vector bundle (or even K-theory classes of negative rank). However, we have *not* found any other K-theoretic insertions that work and we believe that the insertion of Definition 6.1.3 is special (see Remark 6.2.17 for the precise statement).

In Appendix 6.6, we present expected closed formulae for the unique signs (up to overall sign) of Conjecture 6.1.7, which work for all the verifications done in this chapter. This generalizes the sign formula obtained by Nekrasov-Piazzalunga, from physics methods, for Hilbert schemes of points on $\mathbb{C}^4$ [**143**, (2.60)].

We now discuss three specializations of the K-theoretic DT/PT theory we studied so far.

6.1.3. Dimensional reduction to 3-folds Let D be a smooth toric 3-fold[6] and let $\beta \in H_2(D, \mathbb{Z})$. Consider the following generating functions

$$\sum_n \chi(\mathrm{Hilb}^n(D,\beta), \widehat{\mathcal{O}}_I^{\mathrm{vir}})\, q^n, \quad \sum_n \chi(P_n(D,\beta), \widehat{\mathcal{O}}_P^{\mathrm{vir}})\, q^n, \tag{6.1.3}$$

where $\widehat{\mathcal{O}}_I^{\mathrm{vir}} = \mathcal{O}_I^{\mathrm{vir}} \otimes (K_I^{\mathrm{vir}})^{\frac{1}{2}}$, $\widehat{\mathcal{O}}_P^{\mathrm{vir}} = \mathcal{O}_P^{\mathrm{vir}} \otimes (K_P^{\mathrm{vir}})^{\frac{1}{2}}$ are the *twisted virtual structure sheaves* of $I = \mathrm{Hilb}^n(D,\beta)$, $P = P_n(D,\beta)$ introduced in [**142**][7].

The calculation of the K-theoretic DT/PT invariants of toric 3-folds is governed by the K-theoretic 3-fold DT/PT vertex [**142, 149, 3**]

$$\mathsf{V}_{\lambda\mu\nu}^{\mathsf{3D,DT}}(t,q), \quad \mathsf{V}_{\lambda\mu\nu}^{\mathsf{3D,PT}}(t,q) \in \mathbb{Q}(t_1,t_2,t_3,(t_1t_2t_3)^{\frac{1}{2}})(\!(q)\!),$$

where λ, μ, ν are line partitions (2D partitions) determining the underlying $(\mathbb{C}^*)^3$-fixed Cohen-Macaulay curve and t_1, t_2, t_3 are the characters of the standard torus action on $\mathbb{C}^3$.

In the next theorem, λ, μ, ν are line partitions in the (x_2,x_3), (x_1,x_3), (x_1,x_2)-planes respectively. Then λ, μ, ν can be seen as plane partitions in (x_2,x_3,x_4), (x_1,x_3,x_4), (x_1,x_2,x_4)-space, respectively, by inclusion $\{x_4 = 0\} \subseteq \mathbb{C}^3$.

Any plane partition λ, μ, ν, ρ determine a $(\mathbb{C}^*)^4$-fixed Cohen-Macaulay curve on $\mathbb{C}^4$ with asymptotic profiles λ, μ, ν, ρ. The ideal sheaf of such a curve corresponds to a monomial ideal, which is described by a solid partition denoted here by $\pi(\lambda,\mu,\nu,\rho)$ (this is explained in detail in Section 6.2.1). The renormalized volume of this solid partition is denoted by $|\pi(\lambda,\mu,\nu,\rho)|$ as in (6.2.4).

Theorem 6.1.9. *Let λ, μ, ν be any line partitions in the (x_2,x_3), (x_1,x_3), (x_1,x_2)-planes respectively. For any $\mathbf{T}$-fixed subscheme $Z \subseteq \mathbb{C}^4$ with underlying maximal Cohen-Macaulay curve C determined by $\lambda, \mu, \nu, \varnothing$, we choose its sign in Definition 6.1.3 equal to*

---

[6] More precisely, a smooth quasi-projective toric 3-fold such that every cone of its fan is contained in a 3-dimensional cone.

[7] In the 3-fold case, these invariants do not depend on the choice of square roots $(K_I^{\mathrm{vir}})^{\frac{1}{2}}$, $(K_P^{\mathrm{vir}})^{\frac{1}{2}}$. This is because different choices of square roots have the same first Chern class (modulo torsion). See also [**3**, Section 2.5].

$(-1)^{|\pi(\lambda,\mu,\nu,\varnothing)|+\chi(I_C/I_Z)}$, *where* $\chi(I_C/I_Z)$ *equals the number of embedded points of* Z*. For any* $\mathbf{T}$*-fixed stable pair* (F,s) *on* $\mathbb{C}^4$ *with underlying Cohen-Macaulay curve determined by* $\lambda,\mu,\varnothing,\varnothing$*, we choose its sign in Definition 6.1.3 equal to* $(-1)^{|\pi(\lambda,\mu,\varnothing,\varnothing)|+\chi(Q)}$*, where* $\chi(Q)$ *denotes the length of the cokernel of* s*. Then*

$$(6.1.4)\qquad \mathsf{V}^{\mathsf{DT}}_{\lambda\mu\nu\varnothing}(t,y,q)|_{y=t_4} = \mathsf{V}^{\mathsf{3D,DT}}_{\lambda\mu\nu}(t,-q), \quad \mathsf{V}^{\mathsf{PT}}_{\lambda\mu\varnothing\varnothing}(t,y,q)|_{y=t_4} = \mathsf{V}^{\mathsf{3D,PT}}_{\lambda\mu\varnothing}(t,-q).$$

In particular, Conjecture 6.1.7 and compatibility of signs imply[8]

$$\mathsf{V}^{\mathsf{3D,DT}}_{\lambda\mu\varnothing}(t,q) = \mathsf{V}^{\mathsf{3D,PT}}_{\lambda\mu\varnothing}(t,q)\,\mathsf{V}^{\mathsf{3D,DT}}_{\varnothing\varnothing\varnothing}(t,q).$$

Remark 6.1.10. In all the cases for which we checked Conjecture 6.1.7 (see Proposition 6.2.15), we verified that the compatible choice of signs mentioned in Theorem 6.1.9 exists. This explains our sign choice for the $\mathbf{T}$-fixed points which are scheme theoretically supported on $\{x_4 = 0\}$.

Theorem 6.1.11. *Assume Conjecture 6.1.7 and compatibility of signs. Let* D *be a smooth toric 3-fold and* $\beta \in H_2(D,\mathbb{Z})$ *such that all* $(\mathbb{C}^*)^3$*-fixed points of* $\bigsqcup_n \mathrm{Hilb}^n(D,\beta)$*,* $\bigsqcup_n P_n(D,\beta)$ *have at most two legs in each maximal* $(\mathbb{C}^*)^3$*-invariant affine open subset of* D*, e.g.* D *is a local toric curve or local toric surface. Then the* K*-theoretic DT/PT correspondence* [**142**, Eqn. (16)] *holds:*

$$\frac{\sum_n \chi(\mathrm{Hilb}^n(D,\beta), \widehat{\mathcal{O}}^{\mathrm{vir}}_I)\, q^n}{\sum_n \chi(\mathrm{Hilb}^n(D,0), \widehat{\mathcal{O}}^{\mathrm{vir}}_I)\, q^n} = \sum_n \chi(P_n(D,\beta), \widehat{\mathcal{O}}^{\mathrm{vir}}_P)\, q^n.$$

Remark 6.1.12. The usual DT/PT correspondence on toric 3-folds [**155**] is a special case of the K-theoretic version of Nekrasov-Okounkov [**142**, Eqn. (16)]. To the author's knowledge, the later is still an open conjecture.

6.1.4. Cohomological limit I Let $t_i = e^{b\lambda_i}$, for all $i = 1,2,3,4$, and $y = e^{bm}$. We impose the Calabi-Yau relation $t_1t_2t_3t_4 = 1$, which translates into $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0$. In Section 6.3.2, we study the limit $b \to 0$. Let X be a Calabi-Yau 4-fold, $\beta \in H_2(X,\mathbb{Z})$, and L a $\mathbf{T}$-equivariant line bundle on X. Define the following invariants

$$(6.1.5)\qquad I^{\mathrm{coho}}_{n,\beta}(L,m) := \sum_{Z \in \mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}} (-1)^{o_Z} \frac{\sqrt{(-1)^{\frac{1}{2}\mathrm{ext}^2(I_Z,I_Z)} e\big(\mathrm{Ext}^2(I_Z,I_Z)\big)}}{e\big(\mathrm{Ext}^1(I_Z,I_Z)\big)} \cdot e(\mathbf{R}\Gamma(X, L\otimes \mathcal{O}_Z)^\vee \otimes e^m),$$

where $\mathrm{ext}^2(I_Z,I_Z) = \dim \mathrm{Ext}^2(I_Z,I_Z)$. The expression under the square root is a square by $\mathbf{T}$-equivariant Serre duality. Two choices of square root differ by a sign and this indeterminacy is absorbed by the choice of orientation $(-1)^{o_Z}$. These invariants take values in

$$\frac{\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,m)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)},$$

[8] Compatibility of signs means that there exist choices of signs in Conjecture 6.1.7 compatible with the choices of signs stated in this theorem. For all cases where we checked Conjecture 6.1.7 (listed in Proposition 6.2.15), the sign formulae in Appendix 6.6 satisfy this compatibility.

where $\lambda_i := c_1(t_i)$, $m := c_1(e^m)$ denote the $\mathbf{T} \times \mathbb{C}^*$-equivariant parameters. Here $\mathbb{C}^*$ corresponds to a trivial torus action with irreducible character e^m. We similarly define invariants $P^{\mathrm{coho}}_{n,\beta}(L,m)$ replacing $\mathrm{Hilb}^n(X,\beta)$ by $P_n(X,\beta)$ and $\mathbf{R}\Gamma(X, L \otimes \mathcal{O}_Z)$ by $\mathbf{R}\Gamma(X, L \otimes F)$, in which case we also require Assumption 6.1.1 holds.

Remark 6.1.13. The cohomological invariants (6.1.5) can be seen as the virtual localization of the *global* invariants

$$\int_{[\mathrm{Hilb}^n(X,\beta)]^{\mathrm{vir}}} e^{\mathbf{T}\times\mathbb{C}}((L^{[n]})^\vee \otimes e^m).$$

As the moduli space $\mathrm{Hilb}^n(X,\beta)$ is non-proper, the integral is *defined* by means of virtual localization as (6.1.5) (cf. Section 6.4). This choice of signs is related to the choice of signs needed in defining the square root Euler class $\sqrt{e}(\cdot)$ as in Section 5.1.2. A similar argument applies to the PT theory.

Theorem 6.1.14. *Let X be a toric Calabi-Yau 4-fold, $\beta \in H_2(X,\mathbb{Z})$, and let L be a $\mathbf{T}$-equivariant line bundle on X. Then*

$$\lim_{b\to 0}\Big(\sum_n I_{n,\beta}(L,y)\, q^n\Big)\Big|_{t_i=e^{b\lambda_i},\, y=e^{bm}} = \sum_n I^{\mathrm{coho}}_{n,\beta}(L,m)\, q^n,$$

$$\lim_{b\to 0}\Big(\sum_n P_{n,\beta}(L,y)\, q^n\Big)\Big|_{t_i=e^{b\lambda_i},\, y=e^{bm}} = \sum_n P^{\mathrm{coho}}_{n,\beta}(L,m)\, q^n,$$

where the choice of signs on the right-hand-side is determined by the choice of signs on the left-hand-side. For the second equality, we assume $\bigsqcup_n P_n(X,\beta)^{(\mathbb{C}^*)^4}$ is at most 0-dimensional. Hence, Conjecture 6.1.7 implies that there exist choices of signs such that

$$\frac{\sum_n I^{\mathrm{coho}}_{n,\beta}(L,m)\, q^n}{\sum_n I^{\mathrm{coho}}_{n,0}(L,m)\, q^n} = \sum_n P^{\mathrm{coho}}_{n,\beta}(L,m)\, q^n.$$

This theorem provides motivation for conjecturing the following new cohomological DT/PT correspondence for smooth projective Calabi-Yau 4-folds:

Conjecture 6.1.15. *Let X be a smooth projective Calabi-Yau 4-fold and $\beta \in H_2(X,\mathbb{Z})$. For any line bundle L on X, there exist choices of orientations such that*

$$\frac{\sum_n \int_{[\mathrm{Hilb}^n(X,\beta)]^{\mathrm{vir}}} e(L^{[n]})\, q^n}{\sum_n \int_{[\mathrm{Hilb}^n(X,0)]^{\mathrm{vir}}} e(L^{[n]})\, q^n} = \sum_n \int_{[P_n(X,\beta)]^{\mathrm{vir}}} e(L^{[n]})\, q^n.$$

Remark 6.1.16. This conjecture has been recently proven by Park [**161**] in some cases by imposing some conditions on L and β.

6.1.5. Cohomological limit II Let $t_i = e^{b\lambda_i}$, $y = e^{bm}$, $Q = mq$, where we again impose the Calabi-Yau relation $t_1t_2t_3t_4 = 1$. In Section 6.3.3, we consider the limit $b \to 0, m \to \infty$. In [**35**], the two first-named authors studied the following cohomological invariants

$$I^{\mathrm{coho}}_{n,\beta} := \sum_{Z\in \mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}} (-1)^{o_Z} \frac{\sqrt{(-1)^{\frac{1}{2}\mathrm{ext}^2(I_Z,I_Z)} e\big(\mathrm{Ext}^2(I_Z,I_Z)\big)}}{e\big(\mathrm{Ext}^1(I_Z,I_Z)\big)}, \tag{6.1.6}$$

and similar invariants $P^{\mathrm{coho}}_{n,\beta}$, where we replace $\mathrm{Hilb}^n(X,\beta)$ by $P_n(X,\beta)$ and impose Assumption 6.1.1.

Remark 6.1.17. The cohomological invariants (6.1.6) can be seen as the virtual localization of the *global* invariant

$$\int_{[\mathrm{Hilb}^n(X,\beta)]^{\mathrm{vir}}} 1,$$

similarly to Remark 6.1.13.

In [35], a vertex formalism for these invariants was established giving rise to the cohomological DT/PT vertex

$$\mathsf{V}^{\mathrm{coho,DT}}_{\lambda\mu\nu\rho}(Q), \quad \mathsf{V}^{\mathrm{coho,PT}}_{\lambda\mu\nu\rho}(Q) \in \frac{\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}(\!(Q)\!),$$

for any finite plane partitions λ,μ,ν,ρ. As above, in the stable pairs case we assume at most two of these partitions are non-empty.

The cohomological DT/PT 4-fold vertex correspondence [35] states:

Conjecture 6.1.18 (Cao-Kool). *For any finite plane partitions λ,μ,ν,ρ, at most two of which are non-empty, there are choices of signs such that*

$$\mathsf{V}^{\mathrm{coho,DT}}_{\lambda\mu\nu\rho}(Q) = \mathsf{V}^{\mathrm{coho,PT}}_{\lambda\mu\nu\rho}(Q)\, \mathsf{V}^{\mathrm{coho,DT}}_{\varnothing\varnothing\varnothing\varnothing}(Q).$$

Theorem 6.1.19. *Let X be a toric Calabi-Yau 4-fold and $\beta\in H_2(X,\mathbb{Z})$. Then*

$$\lim_{\substack{b\to 0\\ m\to\infty}} \Big(\sum_n I_{n,\beta}(\mathcal{O}_X, e^{bm})\, q^n\Big)\Big|_{t_i=e^{b\lambda_i},Q=qm} = \sum_n I^{\mathrm{coho}}_{n,\beta}\, Q^n,$$
$$\lim_{\substack{b\to 0\\ m\to\infty}} \Big(\sum_n P_{n,\beta}(\mathcal{O}_X, e^{bm})\, q^n\Big)\Big|_{t_i=e^{b\lambda_i},Q=qm} = \sum_n P^{\mathrm{coho}}_{n,\beta}\, Q^n,$$

where the choice of signs on the right-hand-side is determined by the choice of signs on the left-hand-side. For the second equality, we assume $\bigsqcup_n P_n(X,\beta)^{(\mathbb{C}^*)^4}$ is at most 0-dimensional. Moreover, Conjecture 6.1.7 implies Conjecture 6.1.18.

We summarise the above three limits in the following figure.

6.1.6. Application: local resolved conifold In order to illustrate the 4-fold vertex formalism and the three limits, we present a new conjectural formula, which can be seen as a curve analogue of Nekrasov's conjecture. Let $X = D\times\mathbb{C}$, where $D = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1)\oplus\mathcal{O}(-1))$ is the resolved conifold. Consider the generating series of K-theoretic stable pair invariants of X:

$$\mathcal{Z}_X(y,q,Q) := \sum_{n,d} P_{n,d[\mathbb{P}^1]}(\mathcal{O},y)\, q^n Q^d.$$

Conjecture 6.1.20. *Let $X = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O})$. Then there exist unique choices of signs such that*

$$\mathcal{Z}_X(y,q,Q) = \mathrm{Exp}\Big(\mathcal{F}(t,y;q,Q)\Big), \quad \mathcal{F}(t,y;q,Q) := \frac{Q\,[y]}{[t_4][y^{\frac12}q][y^{\frac12}q^{-1}]},$$

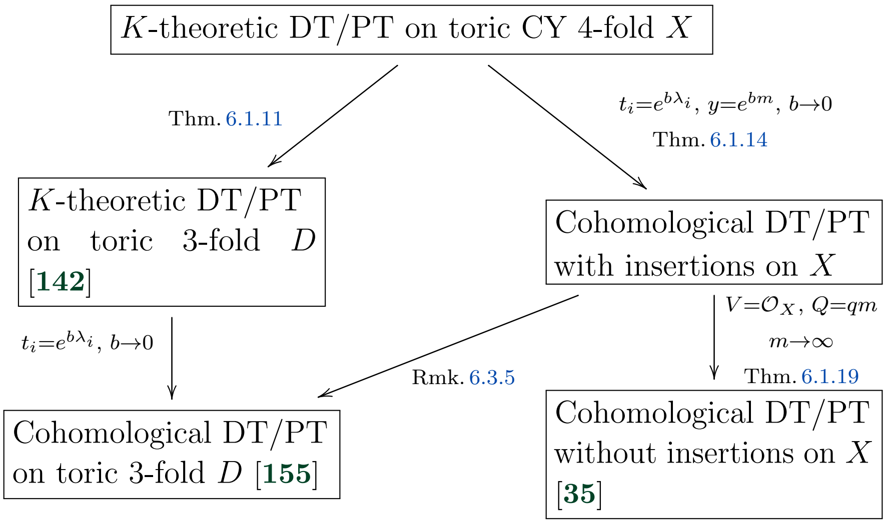

FIGURE 1. Limits of K-theoretic DT/PT on toric CY 4-folds

where t_4^{-1} denotes the torus weight of $\mathcal{O}$ over $\mathbb{P}^1$ and $\mathcal{F}(t,y;q,Q) \in \mathbb{Q}(t_4^{\frac{1}{2}}, y^{\frac{1}{2}}, q, Q)$ is expanded as a formal power series in q and Q.

This conjecture is verified modulo (more or less) Q^5q^6 using the vertex formalism. See Proposition 6.5.2 for the precise statement. Applying dimensional reduction, Conjecture 6.1.20 implies a formula for the K-theoretic stable pair invariants of the resolved conifold D recently proved by Kononov-Okounkov-Osinenko [**109**]. Applying the preferred limits discussed by Arbesfeld [**3**] to the formula of Kononov-Okounkov-Osinenko yields an expression obtained using the refined topological vertex by Iqbal-Kozçaz-Vafa [**98**]. Applying cohomological limit II yields a formula, which was recently conjectured in [**35**]. See Section 6.5 for the details.

6.1.7. Relations with other works This work is a continuation of [**35**], where Cao-Kool introduced the DT/PT correspondence (with primary insertions) for both compact and toric Calabi-Yau 4-folds. In the compact case, "DT=PT" due to insertions. In loc. cit. the authors used toric calculations to support this result and found the cohomological DT/PT 4-fold vertex correspondence (Conjecture 6.1.18), which surprisingly has the same shape as the DT/PT correspondence for Calabi-Yau 3-folds [**152**]. This motivated us to enhance Conjecture 6.1.18 to a K-theoretic version using Nekrasov's insertion (Definition 6.1.3), which specializes to (i) the cohomological DT/PT correspondence for toric Calabi-Yau 4-folds, (ii) the K-theoretic DT/PT correspondence for toric 3-folds [**142, 155**].

After the paper [**36**] was written, further studying has been done for K-theoretic and tautological invariants on Calabi-Yau 4-folds, see for example Cao-Toda [**45, 43**], Cao-Qu [**41**], Bojko [**22, 21**], Park [**161**].

6.2. K-theoretic vertex formalism

6.2.1. Fixed loci of Hilbert schemes Let X a toric Calabi-Yau 4-fold and consider the Hilbert scheme $\mathrm{Hilb}^n(X,\beta)$, parametrizing closed subschemes $Z \subset X$ with proper support in the homology class $[Z] = \beta \in H_2(X, \mathbb{Z})$ and $\chi(\mathcal{O}_Z) = n$. Denote by $\Delta(X)$ the Newton polytope of X, by $V(X)$ its set of vertices and by $E(X)$ its set of edges. Vertices $\alpha \in V(X)$ correspond to $(\mathbb{C}^*)^4$-fixed point of X, each contained in a maximal $(\mathbb{C}^*)^4$-invariant open subset $U_\alpha \subset X$. Edges $\alpha\beta \in E(X)$ correspond to $(\mathbb{C}^*)^4$-invariant lines $L_{\alpha\beta} \cong \mathbb{P}^1$, whose normal bundle is

$$N_{L_{\alpha\beta}/X} \cong \mathcal{O}(m_{\alpha\beta}) \oplus \mathcal{O}(m'_{\alpha\beta}) \oplus \mathcal{O}(m''_{\alpha\beta})$$

and satisfies $m_{\alpha\beta} + m'_{\alpha\beta} + m''_{\alpha\beta} = -2$, being X a Calabi-Yau variety. We may choose coordinates t_i on $(\mathbb{C}^*)^4$ and x_i on U_α such that the $(\mathbb{C}^*)^4$-action on U_α is determined by

$$(t_1, t_2, t_3, t_4) \cdot x_i = t_i x_i. \tag{6.2.1}$$

If the line $L_{\alpha\beta}$ is defined in these coordinates by $\{\, x_2 = x_3 = x_4 = 0 \,\}$, the transition function between the charts U_α and U_β are of the form

$$(x_1, x_2, x_3, x_4) \mapsto (x_1^{-1}, x_2 x_1^{-m_{\alpha\beta}}, x_3 x_1^{-m'_{\alpha\beta}}, x_4 x_1^{-m''_{\alpha\beta}}). \tag{6.2.2}$$

Denote by $\mathbf{T} = \{\, t_1t_2t_3t_4 = 1 \,\} \subset (\mathbb{C}^*)^4$ the subtorus preserving the Calabi-Yau form of X. The $(\mathbb{C}^*)^4$-action and the $\mathbf{T}$-action naturally lift on $\mathrm{Hilb}^n(X, \beta)$, whose fixed locus is reduced and 0-dimensional.

Lemma 6.2.1 ([**35**, Lemma 2.1, 2.2])**.** *We have an isomorphism of schemes*

$$\mathrm{Hilb}^n(X, \beta)^{\mathbf{T}} = \mathrm{Hilb}^n(X, \beta)^{(\mathbb{C}^*)^4},$$

which consists of finitely many reduced points.

We recap the description of the $(\mathbb{C}^*)^4$-fixed locus, which is completely analogous to [**127**, Sec. 4.2] in the setting of toric 3-folds. For an extensive treatment in the case of toric 4-folds, look at [**35**, Sec. 2.1].

Let $Z \in \mathrm{Hilb}^n(X, \beta)^{(\mathbb{C}^*)^4}$ and I be its ideal sheaf; $Z \subset X$ is preserved by the torus action, hence it must be supported on the $(\mathbb{C}^*)^4$-fixed points (corresponding to vertices $\alpha \in V(X)$) and $(\mathbb{C}^*)^4$-invariant lines of X (corresponding to edges $\alpha\beta \in E(X)$). Since I is $(\mathbb{C}^*)^4$-fixed on each open subset, I must be defined on U_α by a monomial ideal

$$I_\alpha = I|_{U_\alpha} \subset \mathbb{C}[x_1, x_2, x_3, x_4],$$

and may also be viewed as a solid partition π_α,

$$\pi_\alpha = \left\{ (k_1, k_2, k_3, k_4), \prod_{i=1}^{4} x_i^{k_i} \notin I_\alpha \right\} \subset \mathbb{Z}^4_{\geq 0}. \tag{6.2.3}$$

The associated subscheme of I_α is (at most) 1-dimensional. The corresponding partition π_α may be infinite in the direction of the coordinate axes. If the solid partition is viewed as a box diagram in $\mathbb{Z}^4$, the vertices in (6.2.3) are determined by the *interior* corners of the boxes, the corners closest to the origin.

The asymptotics of π_α in the coordinate directions are described by four ordinary plane partitions. In particular, in the direction of the $(\mathbb{C}^*)^4$-invariant curve $L_{\alpha\beta}$ given by $\{x_2 = x_3 = x_4 = 0\}$, we have the plane partition $\lambda_{\alpha\beta}$ with the following diagram

$$\begin{aligned}\lambda_{\alpha\beta} &= \{ (k_2,k_3,k_4) : x_1^{k_1}x_2^{k_2}x_3^{k_3}x_4^{k_4} \notin I_\alpha, \forall k_1 \geq 0 \} \\ &= \{ (k_2,k_3,k_4) : x_2^{k_2}x_3^{k_3}x_4^{k_4} \notin I_{\alpha\beta} \} \subset \mathbb{Z}^3_{\geq 0},\end{aligned}$$

where

$$I_{\alpha\beta} = I|_{U_\alpha \cap U_\beta} \subset \mathbb{C}[x_1^{\pm 1}, x_2, x_3, x_4].$$

The vertices of $\lambda_{\alpha\beta}$ defined above are the interior corners of the squares of the associated plane partition.

In summary, a $(\mathbb{C}^*)^4$-fixed ideal sheaf can be described in terms of the following data:

(i) a plane partition $\lambda_{\alpha\beta}$ assigned to each edge $\alpha\beta \in E(X)$;
(ii) a solid partition π_α assigned to each vertex $\alpha \in V(X)$, such that the asymptotics of π_α in the three coordinate directions is given by the plane partitions $\lambda_{\alpha\beta}$ assigned to the corresponding edges.

Let $Z \in \mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}$ correspond to the partition data $\{\pi_\alpha, \lambda_{\alpha\beta}\}_{\alpha,\beta}$. We see

$$\beta = \sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}|[L_{\alpha\beta}] \in H_2(X,\mathbb{Z}),$$

where $|\lambda_{\alpha\beta}|$ denotes the size of the plane partition λ, the number of the boxes in the diagram.

For a (possibly infinite) solid partitions, we define the renormalized volume $|\pi|$ as follows. Let λ_i, $i = 1,2,3,4$, be the asymptotics of π. We set

$$|\pi| = \#\{\pi \cap [0,\dots,N]^4\} - (N+1)\sum_{i=1}^4 |\lambda_i|, \quad N \gg 0. \tag{6.2.4}$$

The renormalized volume is independent of the cut-off N as long as N is sufficiently large. We will say that a solid partition is *point-like* if all the asymptotics $\lambda_i = 0$, $i = 1,\dots,4$ and *curve-like* if at least one $\lambda_i \neq 0$.

To conclude, let $\mathbf{m} = (m_2, m_3, m_4)$, λ a plane partition and set

$$f_{\mathbf{m}}(\lambda) = \sum_{(i,j,k)\in\lambda} (1 - m_2 \cdot i - m_3 \cdot j - m_4 \cdot k). \tag{6.2.5}$$

By [**35**, Lemma 2.4], if a $\mathbf{T}$-invariant closed subscheme $Z \subset X$ corresponds to a partition data $\{\pi_\alpha, \lambda_{\alpha\beta}\}_{\alpha,\beta}$, then

$$\chi(\mathcal{O}_Z) = \sum_{\alpha \in V(X)} |\pi_\alpha| + \sum_{\alpha\beta \in E(X)} f_{\mathbf{m}_{\alpha\beta}}(\lambda_{\alpha\beta}), \tag{6.2.6}$$

where $\mathbf{m}_{\alpha\beta}$ is the multidegree of the normal bundle of the $\mathbf{T}$-invariant line $L_{\alpha\beta}$.

6.2.2. Fixed loci of stable pairs The action of $(\mathbb{C}^*)^4$ on X also lifts to the moduli space $P := P_n(X, \beta)$ of stable pairs. Similar to [**155**], we give a description of the fixed locus $P^{(\mathbb{C}^*)^4}$.

For any stable pair (F, s) on X, the scheme-theoretic support $C_F := \mathrm{supp}(F)$ is a Cohen-Macaulay curve [**152**, Lem. 1.6]. Stable pairs with Cohen-Macaulay support curve C can be described as follows [**155**, Prop. 1.8]:

Let $\mathfrak{m} \subseteq \mathcal{O}_C$ be the ideal of a finite union of closed points on C. A stable pair (F, s) on X such that $C_F = C$ and $\mathrm{supp}(Q)_{\mathrm{red}} \subseteq \mathrm{supp}(\mathcal{O}_C/\mathfrak{m})$ is equivalent to a subsheaf of $\varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$.

This uses the natural inclusions

$$\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C) \hookrightarrow \mathscr{H}om(\mathfrak{m}^{r+1}, \mathcal{O}_C)$$
$$\mathcal{O}_C \hookrightarrow \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)$$

induced by $\mathfrak{m}^{r+1} \subseteq \mathfrak{m}^r \subseteq \mathcal{O}_C$.

Suppose $[(F, s)] \in P^{(\mathbb{C}^*)^4}$, then C_F is $(\mathbb{C}^*)^4$-fixed and determines $\{\pi_\alpha\}_{\alpha \in V(X)}$ with each π_α empty or a curve-like solid partition. Consider a maximal $(\mathbb{C}^*)^4$-invariant affine open subset $\mathbb{C}^4 \cong U_\alpha \subseteq X$. Denote the asymptotic plane partitions of $\pi := \pi_\alpha$ in directions $1, 2, 3, 4$ by λ, μ, ν, ρ. These correspond to $(\mathbb{C}^*)^4$-invariant ideals

$$\begin{aligned} I_{Z_\lambda} &\subseteq \mathbb{C}[x_2, x_3, x_4], \\ I_{Z_\mu} &\subseteq \mathbb{C}[x_1, x_3, x_4], \\ I_{Z_\nu} &\subseteq \mathbb{C}[x_1, x_2, x_4], \\ I_{Z_\rho} &\subseteq \mathbb{C}[x_1, x_2, x_3]. \end{aligned}$$

Define the following $\mathbb{C}[x_1, x_2, x_3, x_4]$-modules

$$\begin{aligned} M_1 &:= \mathbb{C}[x_1, x_1^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x_2, x_3, x_4]/I_{Z_\lambda}, \\ M_2 &:= \mathbb{C}[x_2, x_2^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x_1, x_3, x_4]/I_{Z_\mu}, \\ M_3 &:= \mathbb{C}[x_3, x_3^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x_1, x_2, x_4]/I_{Z_\nu}, \\ M_4 &:= \mathbb{C}[x_4, x_4^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x_1, x_2, x_3]/I_{Z_\rho}. \end{aligned}$$

Then [**155**, Sect. 2.4] gives

$$\varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{C|_{U_\alpha}}) \cong \bigoplus_{i=1}^{4} M_i =: M,$$

where $\mathfrak{m} = (x_1, x_2, x_3, x_4) \subseteq \mathbb{C}[x_1, x_2, x_3, x_4]$. Each module M_i comes from a ring, so it has a unit 1, which is homogeneous of degree $(0, 0, 0, 0)$ with respect to the character group $\widehat{(\mathbb{C}^*)^4} = \mathbb{Z}^4$. We consider the quotient

$$M/\langle(1, 1, 1, 1)\rangle. \tag{6.2.7}$$

Then $(\mathbb{C}^*)^4$-equivariant stable pairs on $U_\alpha \cong \mathbb{C}^4$ correspond to $(\mathbb{C}^*)^4$-invariant $\mathbb{C}[x_1, x_2, x_3, x_4]$-submodules of (6.2.7).

Combinatorial description of $M/\langle(1,1,1,1)\rangle$ Recall the character group of $(\mathbb{C}^*)^4$ is $\mathbb{Z}^4$. For each module M_i, the weights $w \in \mathbb{Z}^4$ of its non-zero eigenspaces determine an infinite *leg* $\mathrm{Leg}_i \subseteq \mathbb{Z}^4$ along the x_i-axis. For each weight $w \in \mathbb{Z}^4$, introduce four independent vectors $\mathbf{1}_w$, $\mathbf{2}_w$, $\mathbf{3}_w$, $\mathbf{4}_w$. Then the $\mathbb{C}[x_1,x_2,x_3,x_4]$-module structure on $M/\langle(1,1,1,1)\rangle$ is determined by

$$x_j \cdot \boldsymbol{i}_w = \boldsymbol{i}_{w+e_j},$$

where $i,j = 1,2,3,4$ and e_1,e_2,e_3,e_4 are the standard basis vectors of $\mathbb{Z}^4$. Similar to the 3-fold case [**155**, Sect. 2.5], we define regions

$$\mathrm{I}^+ \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{I}^- = \bigcup_{i=1}^{4} \mathrm{Leg}_i \subseteq \mathbb{Z}^4, \quad \text{where}$$

- I^+ consists of the weights $w \in \mathbb{Z}^4$ with all coordinates non-negative *and* which lie in precisely one leg. If $w \in \mathrm{I}^+$, then the corresponding weight space of $M/\langle(1,1,1,1)\rangle$ is 0-dimensional.
- I^- consists of all weights $w \in \mathbb{Z}^4$ with at least one negative coordinate. If $w \in \mathrm{I}^-$ is supported in Leg_i, then the corresponding weight space of $M/\langle(1,1,1,1)\rangle$ is 1-dimensional
$$\mathbb{C} \cong \mathbb{C} \cdot \boldsymbol{i}_w \subseteq M/\langle(1,1,1,1)\rangle.$$
- II consists of all weights $w \in \mathbb{Z}^4$, which lie in precisely two legs. If $w \in \mathrm{II}$ is supported in Leg_i and Leg_j, then the corresponding weight space of $M/\langle(1,1,1,1)\rangle$ is 1-dimensional
$$\mathbb{C} \cong \mathbb{C} \cdot \boldsymbol{i}_w \oplus \mathbb{C} \cdot \boldsymbol{j}_w / \mathbb{C} \cdot (\boldsymbol{i}_w + \boldsymbol{j}_w) \subseteq M/\langle(1,1,1,1)\rangle.$$
- III consists of all weights $w \in \mathbb{Z}^4$, which lie in precisely three legs. If $w \in \mathrm{III}$ is supported in Leg_i, Leg_j, and Leg_k, then the corresponding weight space of $M/\langle(1,1,1,1)\rangle$ is 2-dimensional
$$\mathbb{C}^2 \cong \mathbb{C} \cdot \boldsymbol{i}_w \oplus \mathbb{C} \cdot \boldsymbol{j}_w \oplus \mathbb{C} \cdot \boldsymbol{k}_w / \mathbb{C} \cdot (\boldsymbol{i}_w + \boldsymbol{j}_w + \boldsymbol{k}_w) \subseteq M/\langle(1,1,1,1)\rangle.$$
- IV consists of all weights $w \in \mathbb{Z}^4$, which lie in all four legs. If $w \in \mathrm{IV}$, then the corresponding weight space of $M/\langle(1,1,1,1)\rangle$ is 3-dimensional
$$\mathbb{C}^3 \cong \mathbb{C} \cdot \mathbf{1}_w \oplus \mathbb{C} \cdot \mathbf{2}_w \oplus \mathbb{C} \cdot \mathbf{3}_w \oplus \mathbb{C} \cdot \mathbf{4}_w / \mathbb{C} \cdot (\mathbf{1}_w + \mathbf{2}_w + \mathbf{3}_w + \mathbf{4}_w) \subseteq M/\langle(1,1,1,1)\rangle.$$

Box configurations. A box configuration is a finite collection of weights $B \subseteq \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{I}^-$ satisfying the following property:

if $w = (w_1,w_2,w_3,w_4) \in \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV} \cup \mathrm{I}^-$ and one of (w_1-1,w_2,w_3,w_4), (w_1,w_2-1,w_3,w_4), (w_1,w_2,w_3-1,w_4), or (w_1,w_2,w_3,w_4-1) lies in B then $w \in B$.

A box configuration determines a $(\mathbb{C}^*)^4$-invariant submodule of $M/\langle(1,1,1,1)\rangle$ and therefore a $(\mathbb{C}^*)^4$-invariant stable pair on $U_\alpha \cong \mathbb{C}^4$ with cokernel of length

$$\#(B \cap \mathrm{II}) + 2 \cdot \#(B \cap \mathrm{III}) + 3 \cdot \#(B \cap \mathrm{IV}) + \#(B \cap \mathrm{I}^-).$$

The box configurations defined in this section do *not* describe all $(\mathbb{C}^*)^4$-invariant submodules of $M/\langle(1,1,1,1)\rangle$. In this chapter, we always work with Assumption

6.1.1 from the introduction, i.e. $\bigcup_n P_n(X,\beta)^{(\mathbb{C}^*)^4}$ is at most 0-dimensional. Then the restriction of any $\mathbf{T}$-fixed stable pair (F,s) on X to any chart U_α has a Cohen-Macaulay support curve with at most two asymptotic plane partitions and is described by a box configuration as above. See [**35**, Prop. 2.5, 2.6]:

Lemma 6.2.2. *Suppose $\bigsqcup_n P_n(X,\beta)^{(\mathbb{C}^*)^4}$ is at most 0-dimensional. Then for any $[(F,s)] \in P_n(X,\beta)^{(\mathbb{C}^*)^4}$ and any $\alpha \in V(X)$, the Cohen-Macaulay curve $C_F|_{U_\alpha}$ has at most two asymptotic plane partitions.*

Lemma 6.2.3. *Suppose $\bigsqcup_n P_n(X,\beta)^{(\mathbb{C}^*)^4}$ is at most 0-dimensional. Then, for any $n \in \mathbb{Z}$, $P_n(X,\beta)^{\mathbf{T}} = P_n(X,\beta)^{(\mathbb{C}^*)^4}$ consists of finitely many reduced points.*

6.2.3. K-theory class of obstruction theory Let X be a toric Calabi-Yau 4-fold and consider the cover $\{U_\alpha\}_{\alpha\in V(X)}$ by maximal $(\mathbb{C}^*)^4$-invariant affine open subsets. We discuss the DT and PT case simultaneously. Let $E = I_Z$, with $Z \in I^{\mathbf{T}} = \mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}$, or $E = I^\bullet$, with $I^\bullet = [\mathcal{O}_X \to F] \in P^{\mathbf{T}} = P_n(X,\beta)^{\mathbf{T}}$. In the stable pairs case, we impose Assumption 6.1.1 of the introduction so $P^{\mathbf{T}} = P^{(\mathbb{C}^*)^4}$ is at most 0-dimensional by Proposition 6.2.3. We are interested in the class

$$-\mathbf{R}\,\mathrm{Hom}(E,E)_0 \in K_0^{(\mathbb{C}^*)^4}(\mathsf{pt}).$$

Note that in this section, we work with the *full* torus $(\mathbb{C}^*)^4$. In the next section, we will restrict to the Calabi-Yau torus $\mathbf{T} \subseteq (\mathbb{C}^*)^4$ when taking square roots. This class can be computed by a Čech calculation introduced for smooth toric 3-folds in [**127, 155**]. In the case of toric 4-folds, the calculation was done in [**35**, Sect. 2.4]. We briefly recall the results from loc. cit.

Consider the exact triangle

$$E \to \mathcal{O}_X \to E',$$

where $E' = \mathcal{O}_Z$ when $E = I_Z$, and $E' = F$ when $E = I^\bullet$. In both cases, E' is 1-dimensional. Define $U_{\alpha\beta} := U_\alpha \cap U_\beta$, $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, etc., and let $E_\alpha := E|_{U_\alpha}$, $E_{\alpha\beta} := E|_{U_{\alpha\beta}}$ etc. The local-to-global spectral sequence, calculation of sheaf cohomology with respect to the Čech cover $\{U_\alpha\}_{\alpha\in V(X)}$, and the fact that E' is 1-dimensional give

$$-\mathbf{R}\mathrm{Hom}_X(E,E)_0 = -\sum_{\alpha\in V(X)} \mathbf{R}\mathrm{Hom}_{U_\alpha}(E_\alpha,E_\alpha)_0 + \sum_{\alpha\beta\in E(X)} \mathbf{R}\mathrm{Hom}_{U_{\alpha\beta}}(E_{\alpha\beta},E_{\alpha\beta})_0.$$

On $U_\alpha \cong \mathbb{C}^4$, we use coordinates x_1,x_2,x_3,x_4 such that the $(\mathbb{C}^*)^4$-action is

$$t\cdot x_i = t_i x_i, \quad \text{for all } i=1,2,3,4 \text{ and } t=(t_1,t_2,t_3,t_4) \in (\mathbb{C}^*)^4.$$

Denote the $\mathbf{T}$-character of $E'|_{U_\alpha}$ by

$$Z_\alpha := \mathrm{tr}_{E'|_{U_\alpha}}.$$

In the case $E' = \mathcal{O}_Z$, the scheme $Z|_{U_\alpha}$ corresponds to a solid partition π_α as described in Section 6.2.1 and

$$Z_\alpha = \sum_{i,j,k,l\in\pi_\alpha} t_1^i t_2^j t_3^k t_4^l. \tag{6.2.8}$$

When $E' = F$, we use the short exact sequence

$$0 \to \mathcal{O}_C \to F \to Q \to 0,$$

where C is the Cohen-Macaulay support curve and Q is the cokernel. Then

$$Z_\alpha = \mathrm{tr}_{\mathcal{O}_C|_{U_\alpha}} + \mathrm{tr}_{Q|_{U_\alpha}}, \tag{6.2.9}$$

where $\mathcal{O}_C|_{U_\alpha}$ is described by a solid partition π_α and $Q|_{U_\alpha}$ is described by a box configuration $B^{(\alpha)}$ as in Section 6.2.2 (by Assumption 6.1.1). In this case, $\mathrm{tr}_{\mathcal{O}_C|_{U_\alpha}}$ is given by the right-hand-side of (6.2.8). Moreover, $\mathrm{tr}_{Q|_{U_\alpha}}$ is the sum of t^w over all $w \in B^{(\alpha)}$.

For any $\alpha\beta \in E(X)$, we consider

$$Z_{\alpha\beta} := \mathrm{tr}_{E'|_{U_{\alpha\beta}}}.$$

In both cases, $E = I_Z$ and $E = I^\bullet$, there is an underlying Cohen-Macaulay curve $C|_{U_{\alpha\beta}}$. Suppose in both charts U_α, U_β, the line $L_{\alpha\beta} \cong \mathbb{P}^1$ is given by $\{x_2 = x_3 = x_4 = 0\}$. Note that $U_{\alpha\beta} \cong \mathbb{C}^* \times \mathbb{C}^3$. Then $C|_{U_{\alpha\beta}}$ is described by a finite plane partition $\lambda_{\alpha\beta}$ (its cross-section along the x_1-axis) and

$$Z_{\alpha\beta} = \sum_{j,k,l \in \lambda_{\alpha\beta}} t_2^j t_3^k t_4^l. \tag{6.2.10}$$

Using an equivariant resolution of E_α, $E_{\alpha\beta}$, one readily obtains the following formulae for the $\mathbf{T}$-representations of $-\mathbf{R}\mathrm{Hom}(E_\alpha, E_\alpha)_0$, $\mathbf{R}\mathrm{Hom}(E_{\alpha\beta}, E_{\alpha\beta})_0$ (see [**35**, Sect. 2.4], which is based on the original calculation in [**127**])

$$\begin{aligned} \mathrm{tr}_{-\mathbf{R}\mathrm{Hom}(E_\alpha,E_\alpha)_0} &= Z_\alpha + \frac{\overline{Z}_\alpha}{t_1t_2t_3t_4} - \frac{P_{1234}}{t_1t_2t_3t_4} Z_\alpha \overline{Z}_\alpha, \\ -\mathrm{tr}_{-\mathbf{R}\mathrm{Hom}(E_{\alpha\beta},E_{\alpha\beta})_0} &= \delta(t_1)\left(-Z_{\alpha\beta} + \frac{\overline{Z}_{\alpha\beta}}{t_2t_3t_4} - \frac{P_{234}}{t_2t_3t_4} Z_{\alpha\beta}\overline{Z}_{\alpha\beta}\right), \end{aligned} \tag{6.2.11}$$

where $\overline{(\cdot)}$ is the involution on $K_0^{\mathbf{T}}(\mathsf{pt})$ mentioned in the introduction and

$$\begin{aligned} \delta(t) &:= \sum_{n \in \mathbb{Z}} t^n, \\ P_{1234} &:= (1-t_1)(1-t_2)(1-t_3)(1-t_4), \\ P_{234} &:= (1-t_2)(1-t_3)(1-t_4). \end{aligned} \tag{6.2.12}$$

As in [**127**], one has to be careful about the precise meaning of (6.2.11). For instance, in the DT case, when π_α is point-like, the formula for $\mathrm{tr}_{-\mathbf{R}\mathrm{Hom}(E_\alpha,E_\alpha)_0}$ in (6.2.11) is Laurent polynomial in the variables t_i. However, when π_α is curve-like, the infinite sum (6.2.8) for Z_α first has to be expressed as a rational function and $\mathrm{tr}_{-\mathbf{R}\mathrm{Hom}(E_\alpha,E_\alpha)_0}$ is then viewed as a rational function in the variables t_i.

The problem with (6.2.11) is that it consists of *rational functions* in t_1, t_2, t_3, t_4. From [**127**], we learn how to redistribute terms in such a ways that we obtain *Laurent* *polynomials*. Let $\beta_1, \beta_2, \beta_3, \beta_4$ be the vertices neighbouring α. Define[9]

$$\begin{aligned}
\mathsf{F}_{\alpha\beta} &:= -Z_{\alpha\beta} + \frac{\overline{Z}_{\alpha\beta}}{t_2t_3t_4} - \frac{P_{234}}{t_2t_3t_4} Z_{\alpha\beta}\overline{Z}_{\alpha\beta}, \\
\mathsf{V}_\alpha &:= \mathrm{tr}_{-\mathbf{R}\mathrm{Hom}(E_\alpha,E_\alpha)_0} + \sum_{i=1}^4 \frac{\mathsf{F}_{\alpha\beta_i}(t_{i'}, t_{i''}, t_{i'''})}{1-t_i}, \\
\mathsf{E}_{\alpha\beta} &:= t_1^{-1}\frac{\mathsf{F}_{\alpha\beta}(t_2,t_3,t_4)}{1-t_1^{-1}} - \frac{\mathsf{F}_{\alpha\beta}(t_2t_1^{-m_{\alpha\beta}}, t_3t_1^{-m'_{\alpha\beta}}, t_4t_1^{-m''_{\alpha\beta}})}{1-t_1^{-1}},
\end{aligned} \tag{6.2.13}$$

where $\{t_i, t_{i'}, t_{i''}, t_{i'''}\} = \{t_1,t_2,t_3,t_4\}$ and

$$(t_1,t_2,t_3,t_4) \mapsto (t_1^{-1}, t_2t_1^{-m_{\alpha\beta}}, t_3t_1^{-m'_{\alpha\beta}}, t_4t_1^{-m''_{\alpha\beta}})$$

corresponds to the coordinate transformation $U_\alpha \to U_\beta$ and $m_{\alpha\beta}, m'_{\alpha\beta}, m''_{\alpha\beta}$ are the weights of the normal bundle of $L_{\alpha\beta}$. Then

$$\mathrm{tr}_{-\mathbf{R}\mathrm{Hom}(E,E)_0} = \sum_{\alpha\in V(X)} \mathsf{V}_\alpha + \sum_{\alpha\beta\in E(X)} \mathsf{E}_{\alpha\beta}, \tag{6.2.14}$$

and V_α, $\mathsf{E}_{\alpha\beta}$ are Laurent polynomials for all $\alpha \in V(X)$ and $\alpha\beta \in E(X)$ by [**35**, Prop. 2.11].

Remark 6.2.4. When we want to stress the dependence on Z_α, $Z_{\alpha\beta}$ and distinguish between the DT/PT case, we write $\mathsf{V}^{\mathsf{DT}}_{Z_\alpha}$, $\mathsf{V}^{\mathsf{PT}}_{Z_\alpha}$, $\mathsf{E}^{\mathsf{DT}}_{Z_{\alpha\beta}}$, $\mathsf{E}^{\mathsf{PT}}_{Z_{\alpha\beta}}$ for the classes introduced in (6.2.13).

6.2.4. Taking square roots Let $\alpha \in V(X)$. As before, denote by $\beta_1, \ldots, \beta_4 \in V(X)$ the vertices neighbouring α and labelled such that $L_{\alpha\beta_i} = \{x_{i'} = x_{i''} = x_{i'''} = 0\}$,

[9] We only write down $\mathsf{F}_{\alpha\beta}$ and $\mathsf{E}_{\alpha\beta}$ when $L_{\alpha\beta} \cong \mathbb{P}^1$ is given by $\{x_2 = x_3 = x_4 = 0\}$, i.e. the leg along the x_1-axis. The other cases follow by symmetry.

where $\{i',i'',i'''\}=\{1,2,3,4\}\setminus\{i\}$. We define

$$
\begin{aligned}
\mathsf{v}_\alpha &:= Z_\alpha - \overline{P}_{123} Z_\alpha \overline{Z}_\alpha + \sum_{i=1}^{3} \frac{\mathsf{f}_{\alpha\beta_i}(t_{i'},t_{i''},t_{i'''})}{1-t_i} \\
&\quad + \frac{1}{(1-t_4)}\Big\{ - Z_{\alpha\beta_4} + \overline{P}_{123}\big(\overline{Z}_\alpha Z_{\alpha\beta_4} - Z_\alpha \overline{Z}_{\alpha\beta_4}\big) + \frac{\overline{P}_{123}}{1-t_4} Z_{\alpha\beta_4}\overline{Z}_{\alpha\beta_4}\Big\}, \\
\mathsf{e}_{\alpha\beta_1} &:= t_1^{-1}\frac{\mathsf{f}_{\alpha\beta_1}(t_2,t_3,t_4)}{1-t_1^{-1}} - \frac{\mathsf{f}_{\alpha\beta_1}(t_2 t_1^{-m_{\alpha\beta_1}}, t_3 t_1^{-m'_{\alpha\beta_1}}, t_4 t_1^{-m''_{\alpha\beta_1}})}{1-t_1^{-1}}, \\
\mathsf{e}_{\alpha\beta_2} &:= t_2^{-1}\frac{\mathsf{f}_{\alpha\beta_2}(t_1,t_3,t_4)}{1-t_2^{-1}} - \frac{\mathsf{f}_{\alpha\beta_2}(t_1 t_2^{-m_{\alpha\beta_2}}, t_3 t_2^{-m'_{\alpha\beta_2}}, t_4 t_2^{-m''_{\alpha\beta_2}})}{1-t_2^{-1}}, \\
\mathsf{e}_{\alpha\beta_3} &:= t_3^{-1}\frac{\mathsf{f}_{\alpha\beta_3}(t_1,t_2,t_4)}{1-t_3^{-1}} - \frac{\mathsf{f}_{\alpha\beta_3}(t_1 t_3^{-m_{\alpha\beta_3}}, t_2 t_3^{-m'_{\alpha\beta_3}}, t_4 t_3^{-m''_{\alpha\beta_3}})}{1-t_3^{-1}}, \\
\mathsf{e}_{\alpha\beta_4} &:= t_4^{-1}\frac{\mathsf{f}_{\alpha\beta_4}(t_1,t_2,t_3)}{1-t_4^{-1}} - \frac{\mathsf{f}_{\alpha\beta_4}(t_1 t_4^{-m_{\alpha\beta_4}}, t_2 t_4^{-m'_{\alpha\beta_4}}, t_3 t_4^{-m''_{\alpha\beta_4}})}{1-t_4^{-1}}, \\
\mathsf{f}_{\alpha\beta_1} &:= -Z_{\alpha\beta_1} + \frac{P_{23}}{t_2t_3} Z_{\alpha\beta_1}\overline{Z}_{\alpha\beta_1}, \\
\mathsf{f}_{\alpha\beta_2} &:= -Z_{\alpha\beta_2} + \frac{P_{13}}{t_1t_3} Z_{\alpha\beta_2}\overline{Z}_{\alpha\beta_2}, \\
\mathsf{f}_{\alpha\beta_3} &:= -Z_{\alpha\beta_3} + \frac{P_{12}}{t_1t_2} Z_{\alpha\beta_3}\overline{Z}_{\alpha\beta_3}, \\
\mathsf{f}_{\alpha\beta_4} &:= -Z_{\alpha\beta_4} + \frac{P_{12}}{t_1t_2} Z_{\alpha\beta_4}\overline{Z}_{\alpha\beta_4},
\end{aligned}
\tag{6.2.15}
$$

where $P_{23} := (1-t_2)(1-t_3)$ etc. In these formulae, there is an asymmetry with respect to the fourth leg, which we discuss below in Remark 6.2.8. Also note that these formulae become symmetric in 1,2,3 when the fourth leg is empty, i.e. $Z_{\alpha\beta_4}=0$. Recall that in the PT case, we impose Assumption 6.1.1 and there is no fourth leg. We explain how to make correct this asymmetry — in the DT case — in Appendix 6.6. We stress that, at this point, we do *not* yet impose the relation $t_1t_2t_3t_4=1$.

The first observation is that v_α and $\mathsf{e}_{\alpha\beta}$ are Laurent polynomials in the variables t_1,t_2,t_3,t_4. Indeed in the expression for $\mathsf{e}_{\alpha\beta_i}$ (6.2.15), both numerator and denominator vanish at $t_i=1$, so the pole in $t_i=1$ cancels. In order to see that v_α is a Laurent polynomial, one shows that it has no pole in $t_i=1$ for each $i=1,2,3,4$. Indeed, substituting

$$
Z_\alpha = \frac{Z_{\alpha\beta_i}}{1-t_i} + \cdots
$$

into (6.2.15), where $\cdots$ does not contain poles in $t_i=1$, one finds that all poles in $t_i=1$ cancel.

Now we restrict to the Calabi-Yau torus $t_1t_2t_3t_4=1$. Using definition (6.2.15), we find

$$
\mathsf{V}_\alpha = \mathsf{v}_\alpha + \overline{\mathsf{v}}_\alpha, \quad \mathsf{E}_{\alpha\beta} = \mathsf{e}_{\alpha\beta} + \overline{\mathsf{e}}_{\alpha\beta},
$$

for all $\alpha \in V(X)$ and $\alpha\beta \in E(X)$. This follows from a straight-forward calculation using $t_1t_2t_3t_4 = 1$ and the following two identities (and their permutations)

$$P_{123} + \overline{P}_{123} = P_{1234}, \quad -\frac{P_{23}}{t_2t_3} + t_1\frac{\overline{P}_{23}}{t_2^{-1}t_3^{-1}} = \frac{P_{234}}{t_2t_3t_4}.$$

Finally we note that, after setting $t_1t_2t_3t_4 = 1$, v_α and $\mathsf{e}_{\alpha\beta}$ do not have $\mathbf{T}$-fixed part with positive coefficient. This follows from the fact that V_α and $\mathsf{E}_{\alpha\beta}$ do not have $\mathbf{T}$-fixed terms with positive coefficient (see comments in [**35**, Def. 2.12]). We therefore established the following:

Lemma 6.2.5. *The classes v_α, $\mathsf{e}_{\alpha\beta}$ are Laurent polynomials in t_1, t_2, t_3, t_4. When $t_1t_2t_3t_4 = 1$, they satisfy the following equations*

$$\mathsf{V}_\alpha = \mathsf{v}_\alpha + \overline{\mathsf{v}}_\alpha, \quad \mathsf{E}_{\alpha\beta} = \mathsf{e}_{\alpha\beta} + \overline{\mathsf{e}}_{\alpha\beta}.$$

Moreover, for $t_1t_2t_3t_4 = 1$, v_α, $\mathsf{e}_{\alpha\beta}$ do not have $\mathbf{T}$-fixed part with positive coefficient.

Remark 6.2.6. When we want to stress the dependence on Z_α, $Z_{\alpha\beta}$ and distinguish between the DT/PT case, we write $\mathsf{v}^{\mathsf{DT}}_{Z_\alpha}$, $\mathsf{v}^{\mathsf{PT}}_{Z_\alpha}$, $\mathsf{e}^{\mathsf{DT}}_{Z_{\alpha\beta}}$, $\mathsf{e}^{\mathsf{PT}}_{Z_{\alpha\beta}}$ for the classes introduced in this subsection.

Remark 6.2.7. In the case of $\mathrm{Hilb}^n(\mathbb{C}^4, 0)$, the possibility of an explicit choice of of square root of V_α first appeared in [**143**]. The choice of square root in (6.2.15) is non-unique. For instance, in the case of $\mathrm{Hilb}^n(\mathbb{C}^4, 0)$, [**143**] instead work with $\mathsf{v}_\alpha = Z_\alpha - P_{123}Z_\alpha\overline{Z}_\alpha$. Our choice is convenient when taking limits in Section 6.3.

Remark 6.2.8. Our choice of square root (6.2.15) is asymmetric in the indices $1, 2, 3, 4$, i.e. index 4 is singled out. Later, when we add the insertion of Definition 6.1.3 (giving $\widetilde{\mathsf{v}}_\alpha$ in (6.2.16)), we want to single out the fourth direction. Putting $y = t_4$, we want $[-\widetilde{\mathsf{v}}_\alpha]$ equal to zero when Z_α is not scheme theoretically supported in the hyperplane $\{x_4 = 0\}$ and we want $[-\widetilde{\mathsf{v}}_\alpha]$ equal to the vertex of DT/PT theory of the toric 3-fold $\{x_4 = 0\} \cong \mathbb{C}^3$ when Z_α is scheme theoretically supported in $\{x_4 = 0\}$ (Proposition 6.3.1). Look at the Appendix 6.6 for a discussion of *canonicity* of this asymmetry.

6.2.5. K-theoretic insertion We turn our attention to the K-theoretic insertion in Definition 6.1.3. Using v_α, $\mathsf{e}_{\alpha\beta}$ defined in the previous section, we define

(6.2.16)

$$\begin{aligned}
\widetilde{\mathsf{v}}_\alpha &:= \mathsf{v}_\alpha - y\overline{Z}_\alpha + \sum_{i=1}^{4} \frac{y\overline{Z}_{\alpha\beta_i}(t_{i'}, t_{i''}, t_{i'''})}{1 - t_i^{-1}},\\
\widetilde{\mathsf{e}}_{\alpha\beta_1} &:= \mathsf{e}_{\alpha\beta_1} + \frac{t_1}{1-t_1} y\overline{Z}_{\alpha\beta_1}(t_2, t_3, t_4) - \frac{1}{1-t_1} y\overline{Z}_{\alpha\beta_1}(t_2t_1^{-m_{\alpha\beta_1}}, t_3t_1^{-m'_{\alpha\beta_1}}, t_4t_1^{-m''_{\alpha\beta_1}}),\\
\widetilde{\mathsf{e}}_{\alpha\beta_2} &:= \mathsf{e}_{\alpha\beta_2} + \frac{t_2}{1-t_2} y\overline{Z}_{\alpha\beta_2}(t_1, t_3, t_4) - \frac{1}{1-t_2} y\overline{Z}_{\alpha\beta_2}(t_1t_2^{-m_{\alpha\beta_2}}, t_3t_2^{-m'_{\alpha\beta_2}}, t_4t_2^{-m''_{\alpha\beta_2}}),\\
\widetilde{\mathsf{e}}_{\alpha\beta_3} &:= \mathsf{e}_{\alpha\beta_3} + \frac{t_3}{1-t_3} y\overline{Z}_{\alpha\beta_3}(t_1, t_2, t_4) - \frac{1}{1-t_3} y\overline{Z}_{\alpha\beta_3}(t_1t_3^{-m_{\alpha\beta_3}}, t_2t_3^{-m'_{\alpha\beta_3}}, t_4t_3^{-m''_{\alpha\beta_3}}),\\
\widetilde{\mathsf{e}}_{\alpha\beta_4} &:= \mathsf{e}_{\alpha\beta_4} + \frac{t_4}{1-t_4} y\overline{Z}_{\alpha\beta_4}(t_1, t_2, t_3) - \frac{1}{1-t_4} y\overline{Z}_{\alpha\beta_4}(t_1t_4^{-m_{\alpha\beta_4}}, t_2t_4^{-m'_{\alpha\beta_4}}, t_3t_4^{-m''_{\alpha\beta_4}}).
\end{aligned}$$

Then

$$\begin{aligned}
\widetilde{\mathsf{v}}_\alpha &= Z_\alpha - y\overline{Z}_\alpha - \overline{P}_{123}Z_\alpha\overline{Z}_\alpha + \sum_{i=1}^{3}\frac{\widetilde{\mathsf{f}}_{\alpha\beta_i}(t_1,t_2,t_3,t_4)}{1-t_i} \\
&\quad + \frac{1}{(1-t_4)}\Big\{ - Z_{\alpha\beta_4} - t_4 y\overline{Z}_{\alpha\beta_4} + \overline{P}_{123}\big(\overline{Z}_\alpha Z_{\alpha\beta_4} - Z_\alpha\overline{Z}_{\alpha\beta_4}\big) + \frac{\overline{P}_{123}}{1-t_4}Z_{\alpha\beta_4}\overline{Z}_{\alpha\beta_4}\Big\}, \\
\widetilde{\mathsf{e}}_{\alpha\beta_1} &= t_1^{-1}\frac{\widetilde{\mathsf{f}}_{\alpha\beta_1}(t_1,t_2,t_3,t_4)}{1-t_1^{-1}} - \frac{\widetilde{\mathsf{f}}_{\alpha\beta_1}(t_1^{-1}, t_2t_1^{-m_{\alpha\beta_1}}, t_3t_1^{-m'_{\alpha\beta_1}}, t_4t_1^{-m''_{\alpha\beta_1}})}{1-t_1^{-1}}, \\
\widetilde{\mathsf{e}}_{\alpha\beta_2} &= t_2^{-1}\frac{\widetilde{\mathsf{f}}_{\alpha\beta_2}(t_1,t_2,t_3,t_4)}{1-t_2^{-1}} - \frac{\widetilde{\mathsf{f}}_{\alpha\beta_2}(t_1t_2^{-m_{\alpha\beta_2}}, t_2^{-1}, t_3t_2^{-m'_{\alpha\beta_2}}, t_4t_2^{-m''_{\alpha\beta_2}})}{1-t_2^{-1}}, \\
\widetilde{\mathsf{e}}_{\alpha\beta_3} &= t_3^{-1}\frac{\widetilde{\mathsf{f}}_{\alpha\beta_3}(t_1,t_2,t_3,t_4)}{1-t_3^{-1}} - \frac{\widetilde{\mathsf{f}}_{\alpha\beta_3}(t_1t_3^{-m_{\alpha\beta_3}}, t_2t_3^{-m'_{\alpha\beta_3}}, t_3^{-1}, t_4t_3^{-m''_{\alpha\beta_3}})}{1-t_3^{-1}}, \\
\widetilde{\mathsf{e}}_{\alpha\beta_4} &= t_4^{-1}\frac{\widetilde{\mathsf{f}}_{\alpha\beta_4}(t_1,t_2,t_3,t_4)}{1-t_4^{-1}} - \frac{\widetilde{\mathsf{f}}_{\alpha\beta_4}(t_1t_4^{-m_{\alpha\beta_4}}, t_2t_4^{-m'_{\alpha\beta_4}}, t_3t_4^{-m''_{\alpha\beta_4}}, t_4^{-1})}{1-t_4^{-1}}, \\
\widetilde{\mathsf{f}}_{\alpha\beta_1}(t_1,t_2,t_3,t_4) &:= -Z_{\alpha\beta_1} - t_1y\overline{Z}_{\alpha\beta_1} + \frac{P_{23}}{t_2t_3}Z_{\alpha\beta_1}\overline{Z}_{\alpha\beta_1}, \\
\widetilde{\mathsf{f}}_{\alpha\beta_2}(t_1,t_2,t_3,t_4) &:= -Z_{\alpha\beta_2} - t_2y\overline{Z}_{\alpha\beta_2} + \frac{P_{13}}{t_1t_3}Z_{\alpha\beta_2}\overline{Z}_{\alpha\beta_2}, \\
\widetilde{\mathsf{f}}_{\alpha\beta_3}(t_1,t_2,t_3,t_4) &:= -Z_{\alpha\beta_3} - t_3y\overline{Z}_{\alpha\beta_3} + \frac{P_{12}}{t_1t_2}Z_{\alpha\beta_3}\overline{Z}_{\alpha\beta_3}, \\
\widetilde{\mathsf{f}}_{\alpha\beta_4}(t_1,t_2,t_3,t_4) &:= -Z_{\alpha\beta_4} - t_4y\overline{Z}_{\alpha\beta_4} + \frac{P_{12}}{t_1t_2}Z_{\alpha\beta_4}\overline{Z}_{\alpha\beta_4}.
\end{aligned}$$

Lemma 6.2.9. *The classes $\widetilde{\mathsf{v}}_\alpha$, $\widetilde{\mathsf{e}}_{\alpha\beta}$ are Laurent polynomials in t_1, t_2, t_3, t_4.*

PROOF. By Lemma 6.2.5, v_α, $\mathsf{e}_{\alpha\beta}$ are Laurent polynomials in t_1, t_2, t_3, t_4. Therefore, it suffices to only consider terms involving y, for which it is easy to see that all poles in $t_i = 1$ $(i = 1, 2, 3, 4)$ cancel. ☐

Remark 6.2.10. When we want to stress the dependence on Z_α, $Z_{\alpha\beta}$ and distinguish between the DT/PT case, we write $\widetilde{\mathsf{v}}^{\mathsf{DT}}_{Z_\alpha}$, $\widetilde{\mathsf{v}}^{\mathsf{PT}}_{Z_\alpha}$, $\widetilde{\mathsf{e}}^{\mathsf{DT}}_{Z_{\alpha\beta}}$, $\widetilde{\mathsf{e}}^{\mathsf{PT}}_{Z_{\alpha\beta}}$ for the classes introduced in (6.2.16).

We introduce some further notations. On each chart $U_\alpha \cong \mathbb{C}^4$ with coordinates $(x_1^{(\alpha)}, x_2^{(\alpha)}, x_3^{(\alpha)}, x_4^{(\alpha)})$ the $(\mathbb{C}^*)^4$-action is given by

$$t \cdot x_i^{(\alpha)} = \chi_i^{(\alpha)}(t)\, x_i^{(\alpha)}, \quad \forall\, t = (t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$$

for some characters $\chi_i^{(\alpha)} : (\mathbb{C}^*)^4 \to \mathbb{C}^*$ with $i = 1, 2, 3, 4$. We recall that both fixed loci $I^{\mathbf{T}}$, $P^{\mathbf{T}}$ consists of finitely many fixed points, giving rise to local data

$$Z = \big(\{Z_\alpha\}_{\alpha\in V(X)}, \{Z_{\alpha\beta}\}_{\alpha\beta\in E(X)}\big).$$

In DT case, Z_α are point- or curve-like solid partitions and $Z_{\alpha\beta}$ are finite plane partitions. In the PT case, Z_α are box configurations (see (6.2.9)) and $Z_{\alpha\beta}$ are finite plane partitions. Recall that in the stable pairs case we impose Assumption 6.1.1 from the introduction.

Finally, we introduce the following notation. For any $t_1^{w_1}t_2^{w_2}t_3^{w_3}t_4^{w_4}y^a \in K_0^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt})$, we set

$$[t_1^{w_1}t_2^{w_2}t_3^{w_3}t_4^{w_4}y^a] := t_1^{\frac{w_1}{2}}t_2^{\frac{w_2}{2}}t_3^{\frac{w_3}{2}}t_4^{\frac{w_4}{2}}y^{\frac{a}{2}} - t_1^{-\frac{w_1}{2}}t_2^{-\frac{w_2}{2}}t_3^{-\frac{w_3}{2}}t_4^{-\frac{w_4}{2}}y^{-\frac{a}{2}} \tag{6.2.17}$$

and we extend this definition to $K_0^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt})$ by setting

$$\Big[\sum_a \tau^a - \sum_b \tau^b\Big] := \frac{\prod_a [\tau^a]}{\prod_b [\tau^b]},$$

where we use multi-index notation $\tau := (t_1, t_2, t_3, t_4, y)$ and this expression is only defined when no index b equals 0.

Theorem 6.2.11. *Let X be a toric Calabi-Yau 4-fold, $\beta \in H_2(X,\mathbb{Z})$ and $n \in \mathbb{Z}$. Let L be a $\mathbf{T}$-equivariant line bundle on X and denote the character of $L|_{U_\alpha}$ by $\gamma^{(\alpha)}(t) \in K_0^{\mathbf{T}}(U_\alpha)$. Then*

$$I_{n,\beta}(L,y) = \sum_{Z\in \mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}} (-1)^{o_Z}\Big(\prod_{\alpha\in V(X)} [-\widetilde{\mathsf{v}}^{\mathsf{DT}}_{Z_\alpha}]\Big)\Big(\prod_{\alpha\beta\in E(X)} [-\widetilde{\mathsf{e}}^{\mathsf{DT}}_{Z_{\alpha\beta}}]\Big),$$

$$P_{n,\beta}(L,y) = \sum_{Z\in P_n(X,\beta)^{\mathbf{T}}} (-1)^{o_Z}\Big(\prod_{\alpha\in V(X)} [-\widetilde{\mathsf{v}}^{\mathsf{PT}}_{Z_\alpha}]\Big)\Big(\prod_{\alpha\beta\in E(X)} [-\widetilde{\mathsf{e}}^{\mathsf{PT}}_{Z_{\alpha\beta}}]\Big),$$

where the sums are over all $\mathbf{T}$-fixed points $Z = \big(\{Z_\alpha\}_{\alpha\in V(X)}, \{Z_{\alpha\beta}\}_{\alpha\beta\in E(X)}\big)$ and $\widetilde{\mathsf{v}}_{Z_\alpha}, \widetilde{\mathsf{e}}_{Z_{\alpha\beta}}$ are evaluated at

$$(t_1,t_2,t_3,t_4,y) = (\chi_1^{(\alpha)}(t), \chi_2^{(\alpha)}(t), \chi_3^{(\alpha)}(t), \chi_4^{(\alpha)}(t), \overline{\gamma^{(\alpha)}(t)}\cdot y).$$

Proof. We discuss the DT case; the PT case is similar. We suppose $L = \mathcal{O}_X$ and discuss the general case afterwards. We introduce the notation

$$\widehat{\Lambda}^\bullet(\cdot) := \frac{\Lambda^\bullet(\cdot)}{\det^{\frac{1}{2}}(\cdot)}.$$

For any virtual $\mathbf{T}$-representation V, we have that (eg. by [76, Sec. 6.1])

$$\widehat{\Lambda}^\bullet(V^\vee) = [V]. \tag{6.2.18}$$

At any fixed point $Z \in I^{\mathbf{T}}$, we have

$$\frac{\Lambda^\bullet(\mathcal{O}_X^{[n]}\otimes y^{-1})}{(\det(\mathcal{O}_X^{[n]}\otimes y^{-1}))^{\frac{1}{2}}}\Bigg|_Z = \frac{\Lambda^\bullet \mathbf{R}\Gamma(X,\mathcal{O}_Z\otimes y^{-1})}{(\det(\mathbf{R}\Gamma(X,\mathcal{O}_Z\otimes y^{-1}))^{\frac{1}{2}})},$$

where $\mathcal{O}_X^{[n]}$ was defined in (6.1.1). Calculation by Čech cohomology gives

$$\mathrm{tr}_{\mathbf{R}\Gamma(X,\mathcal{O}_Z\otimes y^{-1})} = \sum_{\alpha\in V(X)} \mathrm{tr}_{\Gamma(U_\alpha,\mathcal{O}_{Z_\alpha})}\otimes y^{-1} - \sum_{\alpha\beta\in E(X)} \mathrm{tr}_{\Gamma(U_{\alpha\beta},\mathcal{O}_{Z_{\alpha\beta}})}\otimes y^{-1}. \tag{6.2.19}$$

As $\mathbf{T}$-representations, $\mathrm{tr}_{\Gamma(U_\alpha,\mathcal{O}_{Z_\alpha})} = Z_\alpha$, where Z_α was defined in (6.2.8). Suppose $\mathbb{P}^1 \cong L_{\alpha\beta} = \{x_2^{(\alpha)} = x_3^{(\alpha)} = x_4^{(\alpha)} = 0\}$, i.e. leg $Z_{\alpha\beta}$ is along the $x_1^{(\alpha)}$-axis, then

$$\mathrm{tr}_{\Gamma(U_{\alpha\beta},\mathcal{O}_{Z_{\alpha\beta}})} = \delta(\chi_1^{(\alpha)}(t))\, Z_{\alpha\beta},$$

where $\chi_1^{(\alpha)}(t)$ denotes the character corresponding to the $(\mathbb{C}^*)^4$-action on the first coordinate of $U_{\alpha\beta} \cong \mathbb{C}^*\times\mathbb{C}^3$ and $\delta(t)$ was defined in (6.2.12).

By Lemma 6.2.5, we may choose square roots

$$T_I^{\mathrm{vir}}|_{Z,\mathrm{half}} = \sum_{\alpha \in V(X)} \mathsf{v}_{Z_\alpha} + \sum_{\alpha\beta \in E(X)} \mathsf{e}_{Z_{\alpha\beta}}, \tag{6.2.20}$$

which implies

$$T_I^{\mathrm{vir}}|_{Z,\mathrm{half}} - \mathbf{R}\Gamma(X, \mathcal{O}_Z \otimes y^{-1})^\vee = \sum_{\alpha \in V(X)} \widetilde{\mathsf{v}}_{Z_\alpha} + \sum_{\alpha\beta \in E(X)} \widetilde{\mathsf{e}}_{Z_{\alpha\beta}},$$

by a redistribution process analogous to the one used in defining the vertex formalism (cf. [**36**, Thm. 1.13] for more details).

The conclusion is now easily obtained by (6.2.18) which finishes the case $L = \mathcal{O}_X$. Replacing y by

$$\overline{\gamma^{(\alpha)}(t)} \cdot y, \quad \overline{\gamma^{(\alpha\beta)}(t)} \cdot y$$

establishes the general case. ☐

Remark 6.2.12. In the case $\mathrm{Hilb}^n(\mathbb{C}^4, 0)$, which is discussed in [**143**], our definition of $\widetilde{\mathsf{v}}_{Z_\alpha}$ (6.2.16) differs slightly from loc. cit., who take $(1-y^{-1})Z_\alpha - P_{123} Z_\alpha \overline{Z}_\alpha$. The difference of the second term was discussed in Remark 6.2.7. The difference of the y-term is explained as follows. For $\mathrm{Hilb}^n(\mathbb{C}^4, 0)$, Nekrasov-Piazzalunga consider the invariant defined in Definition 6.1.3 but with $L^{[n]} \otimes y^{-1}$ replaced by $(L^{[n]})^\vee \otimes y$ (and $L = \mathcal{O}_X$). Note the following two identities

$$\Lambda^\bullet E^\vee = (-1)^{\mathrm{rk}(E)} \Lambda^\bullet E \otimes \det(E)^*,$$
$$\det(E^\vee) = \det(E)^*,$$

for any $E \in K_0^{\mathbf{T}\times\mathbb{C}^*}(\mathsf{pt})$. This shows at once that our definition differs from loc. cit. by an overall factor $(-1)^n$. In the vertex formalism, it results in replacing $y\overline{Z}_\alpha$ by $y^{-1} Z_\alpha$ in (6.2.16).

We end with an observation about the powers of the equivariant parameters. The expressions $[-\widetilde{\mathsf{v}}_\alpha]$, $[-\widetilde{\mathsf{e}}_{\alpha\beta}]$ a priori involve *half-integer* powers of t_1, t_2, t_3, t_4 (formal square roots). In fact, taking a single leg of multiplicity one with weights $m_{\alpha\beta} = 0$, $m'_{\alpha\beta} = -1$, $m''_{\alpha\beta} = -1$ already shows that non-integer powers indeed occur in the edge. Nonetheless, for the vertex we have the following:

Proposition 6.2.13. *We have*

$$[-\widetilde{\mathsf{v}}_\alpha] \in \mathbb{Q}(t_1, t_2, t_3, t_4, y^{\frac{1}{2}})/(t_1 t_2 t_3 t_4 - 1).$$

Proof. We first consider the case that Z_α satisfies $Z_{\alpha\beta_1} = Z_{\alpha\beta_2} = Z_{\alpha\beta_3} = Z_{\alpha\beta_4} = 0$. As before, we will use multi-index notation for $t = (t_1, t_2, t_3, t_4)$. A monomial $\pm t^v$ in $\widetilde{\mathsf{v}}_\alpha$ contributes as follows to $[-\widetilde{\mathsf{v}}_\alpha]$:

$$[\mp t^v] = (t^{\frac{v}{2}} - t^{-\frac{v}{2}})^{\mp 1} = \Big(\frac{1 - t^{-v}}{t^{-\frac{v}{2}}}\Big)^{\mp 1}.$$

Hence, non-integer powers can only come from $t^{\mp \frac{v}{2}} = (\det(\mp t^v))^{\frac{1}{2}}$. Therefore, it suffices to calculate $(\det(\cdot))^{\frac{1}{2}}$ of

$$\widetilde{\mathsf{v}}_\alpha = Z_\alpha - y\overline{Z}_\alpha - \overline{P}_{123} Z_\alpha \overline{Z}_\alpha.$$

Writing $Z_\alpha = \sum_u t^u$, we find

$$(\det \widetilde{\mathsf{v}}_\alpha)^{\frac{1}{2}} = \prod_u \frac{t^{\frac{u}{2}}}{y^{\frac{1}{2}} t^{-\frac{u}{2}}} = \prod_u y^{-\frac{1}{2}} t^u,$$

where we used that $\det(\overline{P}_{123} Z_\alpha \overline{Z}_\alpha) = 1$.

For the general case, write $Z_\alpha = \sum_{i=1}^4 \frac{Z_{\alpha\beta_i}}{1-t_i} + W$, where W is a Laurent polynomial. Next, substitute this expression for Z_α into definition (6.2.16) of $\widetilde{\mathsf{v}}_\alpha$ and cancel all poles. Up to this point in the calculation, the relation $t_1t_2t_3t_4 = 1$ is *not* imposed. Then, similar to the calculation above, taking $(\det(\cdot))^{\frac{1}{2}}$ of the resulting Laurent polynomial gives only integer powers of t_i. $\square$

6.2.6. K-theoretic 4-fold vertex Let λ, μ, ν, ρ be plane partitions of finite size. This determines a $\mathbf{T}$-fixed Cohen-Macaulay curve $C \subseteq \mathbb{C}^4$ whose solid partition we denote this solid partition by $\pi(\lambda, \mu, \nu, \rho)$. Consider the following:

- All $\mathbf{T}$-fixed closed subschemes $Z \subseteq \mathbb{C}^4$ with underlying maximal Cohen-Macaulay subcurve C. These correspond to solid partitions π with asymptotic plane partitions λ, μ, ν, ρ in directions $1, 2, 3, 4$. We denote the collection of such solid partitions by $\Pi^{\mathsf{DT}}(\lambda, \mu, \nu, \rho)$. Any $\pi \in \Pi^{\mathsf{DT}}(\lambda, \mu, \nu, \rho)$ determines a character Z_π defined by the right-hand-side of (6.2.8) and hence, by (6.2.16), a Laurent polynomial
$$\widetilde{\mathsf{v}}_\pi^{\mathsf{DT}} \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}, y]/(t_1t_2t_3t_4 - 1).$$
- Assume at most two of λ, μ, ν, ρ are non-empty. Consider all $\mathbf{T}$-fixed stable pairs (F, s) on X with underlying Cohen-Macaulay support curve C. These correspond to box configurations as described in Section 6.2.2. We denote the collection of these box configurations by $\Pi^{\mathsf{PT}}(\lambda, \mu, \nu, \rho)$. Any $B \in \Pi^{\mathsf{PT}}(\lambda, \mu, \nu, \rho)$ determines a character Z_B defined by the right-hand-side of (6.2.9), where the Cohen-Macaulay part is given by (6.2.8) with solid partition $\pi(\lambda, \mu, \nu, \rho)$. By (6.2.16), this determines a Laurent polynomial
$$\widetilde{\mathsf{v}}_B^{\mathsf{PT}} \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}, y]/(t_1t_2t_3t_4 - 1).$$

Definition 6.2.14. Let λ, μ, ν, ρ be plane partitions of finite size. Define the DT 4-fold vertex by

$$\mathsf{V}_{\lambda\mu\nu\rho}^{\mathsf{DT}}(t, y, q) := \sum_{\pi \in \Pi^{\mathsf{DT}}(\lambda,\mu,\nu,\rho)} (-1)^{o_\pi} [-\widetilde{\mathsf{v}}_\pi^{\mathsf{DT}}]\, q^{|\pi|} \tag{6.2.21}$$

$$\in \mathbb{Q}(t_1, t_2, t_3, t_4, y^{\frac{1}{2}})/(t_1t_2t_3t_4 - 1)(\!(q)\!), \tag{6.2.22}$$

where $o_\pi = 0, 1$ denotes a choice of sign for each π, $[\cdot]$ was defined in (6.2.17), $|\pi|$ denotes renormalized volume (6.2.4), and the right-hand-side is well-defined by Lemma 6.2.5. Note that the powers of t_i are integer by Proposition 6.2.13.

Next, suppose at most two of λ, μ, ν, ρ are non-empty. Define the PT 4-fold vertex by

$$\mathsf{V}_{\lambda\mu\nu\rho}^{\mathsf{PT}}(t, y, q) := \sum_{B \in \Pi^{\mathsf{PT}}(\lambda,\mu,\nu,\rho)} (-1)^{o_B} [-\widetilde{\mathsf{v}}_B^{\mathsf{PT}}]\, q^{|B| + |\pi(\lambda,\mu,\nu,\rho)|}$$

$$\in \mathbb{Q}(t_1, t_2, t_3, t_4, y^{\frac{1}{2}})/(t_1t_2t_3t_4 - 1)((q)),$$

where $o_B = 0, 1$ denotes a choice of sign for each B, $|B|$ denotes the total number of boxes in the box configuration, and $|\pi(\lambda, \mu, \nu, \rho)|$ denotes renormalized volume.

Similarly, to any finite plane partition λ, we associate a character Z_λ defined by the right-hand-side of (6.2.10). We then define edge terms

$$\mathsf{E}^{\mathsf{DT}}_\lambda(t, y) = \mathsf{E}^{\mathsf{PT}}_\lambda(t, y) := (-1)^{o_\lambda}[-\widetilde{\mathsf{e}}_{Z_\lambda}] \in \mathbb{Q}(t_1^{\frac{1}{2}}, t_2^{\frac{1}{2}}, t_3^{\frac{1}{2}}, t_4^{\frac{1}{2}}, y^{\frac{1}{2}})/(t_1t_2t_3t_4 - 1),$$

where $\widetilde{\mathsf{e}}_{Z_\lambda}$ was defined in (6.2.16).

The vertex formalism reduces the calculation of $I_{n,\beta}(L, y), P_{n,\beta}(L, y)$ for any toric Calabi-Yau 4-fold X, $\beta \in H_2(X, \mathbb{Z})$, and $n \in \mathbb{Z}$ to a combinatorial expression involving $\mathsf{V}_{\lambda\mu\nu\rho}$ and E_λ. We illustrate this in a sufficiently general example. Let X be the total space of $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. Let $\beta = d\,[\mathbb{P}^1]$, where $\mathbb{P}^1$ lies in the zero section $\mathbb{P}^2 \subseteq X$, and let L be a $\mathbf{T}$-equivariant line bundle on X. Denote the characters of $L|_{U_\alpha}$, $L|_{U_{\alpha\beta}}$ by $\gamma^{(\alpha)}(t) \in K_0^{\mathbf{T}}(U_\alpha)$, $\gamma^{(\alpha\beta)}(t) \in K_0^{\mathbf{T}}(U_{\alpha\beta})$ for all $\alpha = 1, 2, 3$ and all $\alpha\beta$. Then (6.2.6) and Theorem 6.2.11 imply

$$\begin{aligned}
\sum_n I_{n,\beta}(L, y)\, q^n = &\sum_{\substack{\lambda,\mu,\nu \\ |\lambda|+|\mu|+|\nu|=d}} q^{f_{1,-1,-2}(\lambda)+f_{1,-1,-2}(\mu)+f_{1,-1,-2}(\nu)} \\
&\cdot \mathsf{E}^{\mathsf{DT}}_\lambda|_{(t_1,t_2,t_3,t_4,\overline{\gamma}^{(13)}(t_2,t_3,t_4)y)} \mathsf{V}^{\mathsf{DT}}_{\lambda\mu\varnothing\varnothing}|_{(t_1,t_2,t_3,t_4,\overline{\gamma}^{(1)}(t_1,t_2,t_3,t_4)y)} \mathsf{E}^{\mathsf{DT}}_\mu|_{(t_2,t_1,t_3,t_4,\overline{\gamma}^{(12)}(t_1,t_3,t_4)y)} \\
&\cdot \mathsf{V}^{\mathsf{DT}}_{\mu\nu\varnothing\varnothing}|_{(t_2^{-1},t_1t_2^{-1},t_3t_2,t_4t_2^2,\overline{\gamma}^{(2)}(t_2^{-1},t_1t_2^{-1},t_3t_2,t_4t_2^2)y)} \mathsf{E}^{\mathsf{DT}}_\nu|_{(t_1t_2^{-1},t_2^{-1},t_3t_2,t_4t_2^2,\overline{\gamma}^{(23)}(t_2^{-1},t_3t_2,t_4t_2^2)y)} \\
&\cdot \mathsf{V}^{\mathsf{DT}}_{\nu\lambda\varnothing\varnothing}|_{(t_2t_1^{-1},t_1^{-1},t_3t_1,t_4t_1^2,\overline{\gamma}^{(3)}(t_2t_1^{-1},t_1^{-1},t_3t_1,t_4t_1^2)y)},
\end{aligned}$$

where the sum is over all finite plane partitions λ, μ, ν satisfying $|\lambda| + |\mu| + |\nu| = d$. Here the choice of signs for the invariants $I_{n,\beta}(L, y)$ is determined by the choice of signs in each vertex and edge term. Replacing DT by PT, the same expression holds for the generating function of $P_{n,\beta}(L, y)$.

We conjecture that the DT/PT 4-fold vertex satisfy Conjecture 6.1.7. As above, for finite plane partitions λ, μ, ν, ρ, we denote by $\pi(\lambda, \mu, \nu, \rho)$ the curve-like solid partition corresponding to the Cohen-Macaulay curve with *asymptotics* λ, μ, ν, ρ. We normalize the DT/PT 4-fold vertex so they start with q^0 (whose coefficient is in general not equal to 1). This is achieved by multiplying by $q^{-|\pi(\lambda,\mu,\nu,\rho)|}$.

Using the vertex formalism, we verified the following cases:

Proposition 6.2.15. *There are choices of signs such that*

$$q^{-|\pi(\lambda,\mu,\nu,\rho)|}\, \mathsf{V}^{\mathsf{DT}}_{\lambda\mu\nu\rho}(t, y, q) = q^{-|\pi(\lambda,\mu,\nu,\rho)|}\, \mathsf{V}^{\mathsf{PT}}_{\lambda\mu\nu\rho}(t, y, q)\, \mathsf{V}^{\mathsf{DT}}_{\varnothing\varnothing\varnothing\varnothing}(t, y, q) \mod q^N$$

in the following cases:

- *for any* $|\lambda| + |\mu| + |\nu| + |\rho| \leqslant 1$ *and* $N = 4$,
- *for any* $|\lambda| + |\mu| + |\nu| + |\rho| \leqslant 2$ *and* $N = 4$,
- *for any* $|\lambda| + |\mu| + |\nu| + |\rho| \leqslant 3$ *and* $N = 3$,
- *for any* $|\lambda| + |\mu| + |\nu| + |\rho| \leqslant 4$ *and* $N = 3$.

In each of these cases, the uniqueness statement of Conjecture 6.1.7 holds.

Remark 6.2.16. We discuss in the Appendix 6.6 a purely combinatorial sign rule for which (conjecturally) the DT/PT correspondence holds.

We now show that Conjecture 6.1.7 implies Theorem 6.1.8 (global K-theoretic DT/PT correspondence).

PROOF OF THEOREM 6.1.8. For ease of notation, we consider the case where $X = \mathrm{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$ and $\beta = d[\mathbb{P}^1]$. The general case follows similarly. Conjecture 6.1.7 implies that there exist choices of signs such that

$$\begin{aligned}
\sum_n I_{n,\beta}(L,y)\, q^n = & \sum_{\substack{\lambda,\mu,\nu \\ |\lambda|+|\mu|+|\nu|=d}} q^{f_{1,-1,-2}(\lambda)+f_{1,-1,-2}(\mu)+f_{1,-1,-2}(\nu)} \\
& \cdot \mathsf{E}^{\mathsf{DT}}_{\lambda}|_{(t_1,t_2,t_3,t_4,\overline{\gamma}^{(13)}(t_2,t_3,t_4)y)} \mathsf{V}^{\mathsf{DT}}_{\lambda\mu\varnothing\varnothing}|_{(t_1,t_2,t_3,t_4,\overline{\gamma}^{(1)}(t_1,t_2,t_3,t_4)y)} \mathsf{E}^{\mathsf{DT}}_{\mu}|_{(t_2,t_1,t_3,t_4,\overline{\gamma}^{(12)}(t_1,t_3,t_4)y)} \\
& \cdot \mathsf{V}^{\mathsf{DT}}_{\mu\nu\varnothing\varnothing}|_{(t_2^{-1},t_1t_2^{-1},t_3t_2,t_4t_2^2,\overline{\gamma}^{(2)}(t_2^{-1},t_1t_2^{-1},t_3t_2,t_4t_2^2)y)} \mathsf{E}^{\mathsf{DT}}_{\nu}|_{(t_1t_2^{-1},t_2^{-1},t_3t_2,t_4t_2^2,\overline{\gamma}^{(23)}(t_2^{-1},t_3t_2,t_4t_2^2)y)} \\
& \cdot \mathsf{V}^{\mathsf{DT}}_{\nu\lambda\varnothing\varnothing}|_{(t_2t_1^{-1},t_1^{-1},t_3t_1,t_4t_1^2,\overline{\gamma}^{(3)}(t_2t_1^{-1},t_1^{-1},t_3t_1,t_4t_1^2)y)} \\
= & \mathsf{V}^{\mathsf{DT}}_{\varnothing\varnothing\varnothing\varnothing}\Big|_{(t_1,t_2,t_3,t_4,\overline{\gamma}^{(1)}(t_1,t_2,t_3,t_4)y)} \cdot \mathsf{V}^{\mathsf{DT}}_{\varnothing\varnothing\varnothing\varnothing}|_{(t_2^{-1},t_1t_2^{-1},t_3t_2,t_4t_2^2,\overline{\gamma}^{(2)}(t_2^{-1},t_1t_2^{-1},t_3t_2,t_4t_2^2)y)} \\
& \cdot \mathsf{V}^{\mathsf{DT}}_{\varnothing\varnothing\varnothing\varnothing}|_{(t_2t_1^{-1},t_1^{-1},t_3t_1,t_4t_1^2,\overline{\gamma}^{(3)}(t_2t_1^{-1},t_1^{-1},t_3t_1,t_4t_1^2)y)} \sum_{\substack{\lambda,\mu,\nu \\ |\lambda|+|\mu|+|\nu|=d}} q^{f_{1,-1,-2}(\lambda)+f_{1,-1,-2}(\mu)+f_{1,-1,-2}(\nu)} \\
& \cdot \mathsf{E}^{\mathsf{PT}}_{\lambda}|_{(t_1,t_2,t_3,t_4,\overline{\gamma}^{(13)}(t_2,t_3,t_4)y)} \mathsf{V}^{\mathsf{PT}}_{\lambda\mu\varnothing\varnothing}|_{(t_1,t_2,t_3,t_4,\overline{\gamma}^{(1)}(t_1,t_2,t_3,t_4)y)} \mathsf{E}^{\mathsf{PT}}_{\mu}|_{(t_2,t_1,t_3,t_4,\overline{\gamma}^{(12)}(t_1,t_3,t_4)y)} \\
& \cdot \mathsf{V}^{\mathsf{PT}}_{\mu\nu\varnothing\varnothing}|_{(t_2^{-1},t_1t_2^{-1},t_3t_2,t_4t_2^2,\overline{\gamma}^{(2)}(t_2^{-1},t_1t_2^{-1},t_3t_2,t_4t_2^2)y)} \mathsf{E}^{\mathsf{PT}}_{\nu}|_{(t_1t_2^{-1},t_2^{-1},t_3t_2,t_4t_2^2,\overline{\gamma}^{(23)}(t_2^{-1},t_3t_2,t_4t_2^2)y))} \\
& \cdot \mathsf{V}^{\mathsf{PT}}_{\nu\lambda\varnothing\varnothing}|_{(t_2t_1^{-1},t_1^{-1},t_3t_1,t_4t_1^2,\overline{\gamma}^{(3)}(t_2t_1^{-1},t_1^{-1},t_3t_1,t_4t_1^2)y)} \\
= & \Big(\sum_n I_{n,0}(L,y)\, q^n\Big) \cdot \Big(\sum_n P_{n,\beta}(L,y)\, q^n\Big).
\end{aligned}$$

□

Remark 6.2.17. Let X be a toric Calabi-Yau 4-fold and let $I := \mathrm{Hilb}^n(X,\beta)$, $P := P_n(X,\beta)$. Consider the *virtual holomorphic Euler characteristic* $\chi\left(I, \widehat{\mathcal{O}}_I^{\mathrm{vir}}\right)$ and its stable pairs analogue with I replaced by P. Then one can develop a (simpler) vertex formalism for these invariants. We checked in the case of a single leg of multiplicity one with a single embedded point that the analogue of the DT/PT correspondence (Conjecture 6.1.7) fails for all choices of signs.

Another natural thing to try is to replace L in Definition 6.1.3 by a $\mathbf{T}$-equivariant vector bundle of rank 2 or 3 or a K-theory class of rank -1 (more precisely: the class of $-L$ where L is a $\mathbf{T}$-equivariant line bundle on X). In none of these cases there exists an analogue of the DT/PT correspondence either. The special feature of the tautological insertion of Definition 6.1.3 is that, after it is absorbed in the vertex as in Section 6.2.5, the vertex $\widetilde{\mathsf{v}}_\alpha$ has rank *zero* as we will prove in Proposition 6.3.3 below.

6.3. Limits of K-theoretic conjecture

6.3.1. Dimensional reduction Let X be a toric Calabi-Yau 4-fold and $\beta \in H_2(X,\mathbb{Z})$. Let $Z = \{\{Z_\alpha\}_{\alpha\in V(X)}, \{Z_{\alpha\beta}\}_{\alpha\beta\in E(X)}\}$ be an element of either of the fixed loci

$$\bigsqcup_n \mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}, \quad \bigsqcup_n P_n(X,\beta)^{\mathbf{T}},$$

where we recall Assumption 6.1.1 from the introduction. We will work in one chart $U_\alpha \cong \mathbb{C}^4$.

Suppose the underlying Cohen-Macaulay curve corresponding to Z_α lies scheme theoretically inside the hyperplane $\{x_4 = 0\}$. In the stable pairs case, this implies Z_α is scheme theoretically supported inside $\{x_4 = 0\}$, however in the DT case Z_α may have embedded points “sticking out” of $\{x_4 = 0\}$.

Proposition 6.3.1. *If Z_α lies scheme theoretically in $\{x_4 = 0\}$, then $\widetilde{\mathsf{v}}_{Z_\alpha}|_{y=t_4} = \mathsf{V}^{\mathrm{3D}}_{Z_\alpha}$, where $\mathsf{V}^{\mathrm{3D}}_{Z_\alpha}$ is the (fully equivariant) DT/PT vertex of* [**127**, Sect. 4.7–4.9] *and* [**155**, Sect. 4.4–4.6]. *If the underlying Cohen-Macaulay curve of Z_α lies scheme theoretically in $\{x_4 = 0\}$, but Z_α does not (which can only happen in the DT case), then $[-\widetilde{\mathsf{v}}_{Z_\alpha}]|_{y=t_4} = 0$.*

Proof. When Z_α lies scheme theoretically inside $\{x_4 = 0\} \subseteq \mathbb{C}^4 =: U_\alpha$, the statement follows at once by comparing (6.2.16) to [**127**, Sect. 4.7–4.9], [**155**, Sect. 4.4–4.6].

Suppose we consider the DT case and the underlying maximal Cohen-Macaulay curve of Z_α is scheme theoretically supported in $\{x_4 = 0\}$, but Z_α is not scheme theoretically supported in $\{x_4 = 0\}$. The vertex v_{Z_α} does not have T-fixed part with positive coefficient by Lemma 6.2.5. Therefore, it suffices to consider the $\mathbf{T}$-fixed terms arising from setting $y = t_4 = (t_1t_2t_3)^{-1}$ in the y-dependent part of $\widetilde{\mathsf{v}}_{Z_\alpha}$.

As usual, we write

$$Z_\alpha = \sum_{i=1}^{3} \frac{Z_{\alpha\beta_i}}{1-t_i} + W,$$

where $\beta_1, \beta_2, \beta_3$ are the vertices in $\{x_4 = 0\}$ neighbouring α and W is a Laurent polynomial in t_1, t_2, t_3, t_4. The only terms involving y in $\widetilde{\mathsf{v}}_{Z_\alpha}$ are $-y\overline{W}$. Since Z_α is not scheme theoretically supported inside $\{x_4 = 0\}$, W contains the term $+t_4$. Setting $y = t_4$ this term contributes

$$-y\overline{t_4} = -1.$$

Furthermore, the underlying maximal Cohen-Macaulay curve of Z_α is contained in $\{x_4 = 0\}$, so all negative terms of W are of the form $-t_1^{w_1}t_2^{w_2}t_3^{w_3}$ with $w_1, w_2, w_3 \geqslant 0$. Therefore W does not contain terms of the form $-t_4$ (which equals $-t_1^{-1}t_2^{-1}t_3^{-1}$). Hence $\widetilde{\mathsf{v}}_{Z_\alpha}|_{y=t_4}$ has negative $\mathbf{T}$-fixed part and the proposition follows. □

Remark 6.3.2. Consider a chart $U_{\alpha\beta} \cong \mathbb{C}^* \times \mathbb{C}^3$ and suppose in both charts U_α, U_β, the line $L_{\alpha\beta} \cong \mathbb{P}^1$ is given by $\{x_2 = x_3 = x_4 = 0\}$. Consider a Cohen-Macaulay curve $Z_{\alpha\beta}$ which is scheme theoretically supported on $\{x_4 = 0\}$ and for which $m''_{\alpha\beta} = 0$. Then, similar to Proposition 6.3.1, $\widetilde{\mathsf{e}}_{Z_{\alpha\beta}}|_{y=t_4} = \mathsf{E}^{\mathrm{3D}}_{Z_{\alpha\beta}}$, where $\mathsf{E}^{\mathrm{3D}}_{Z_{\alpha\beta}}$ is the fully equivariant DT (and therefore PT) edge of [**127**, Sect. 4.7–4.9].

PROOF OF THEOREM 6.1.9. The first part of Theorem 6.1.9 is an immediate corollary of Proposition 6.3.1. Note that on the right-hand-side we obtain $-q$ due to our choice of signs[10].

For the second part of Theorem 6.1.9, we choose signs as in Conjecture 6.1.7 and we assume this can be done compatibly with the choice of signs of all $\mathbf{T}$-fixed points which are scheme theoretically supported on $\{x_4 = 0\}$ as stated in the theorem. Then the second part of the theorem follows. $\square$

PROOF OF THEOREM 6.1.11. We prove the claim for DT invariants, as the stable pairs case is similar.

Using the vertex formalism for K-theoretic DT invariants of toric 3-folds from [**142, 149, 3**], we have

$$\chi(\mathrm{Hilb}^n(D,\beta), \widehat{\mathcal{O}}_I^{\mathrm{vir}}) = \sum_{Z \in \mathrm{Hilb}^n(D,\beta)^{\mathbf{T}}} \Bigg(\prod_{\alpha \in V(D)} [-\mathsf{V}_{Z_\alpha}^{\mathrm{3D,DT}}] \Bigg) \Bigg(\prod_{\alpha\beta \in E(D)} [-\mathsf{E}_{Z_{\alpha\beta}}^{\mathrm{3D,DT}}] \Bigg),$$

where the sums are over all $\mathbf{T}$-fixed points $Z = \big(\{Z_\alpha\}_{\alpha \in V(D)}, \{Z_{\alpha\beta}\}_{\alpha\beta \in E(D)}\big)$ and $\mathsf{V}_{Z_\alpha}^{\mathrm{3D,DT}}, \mathsf{E}_{Z_{\alpha\beta}}^{\mathrm{3D,DT}}$ are evaluated at the characters of the $(\mathbb{C}^*)^3$-action on $U_\alpha \cap D$, $U_{\alpha\beta} \cap D$ respectively.

The generating function $\sum_n \chi(\mathrm{Hilb}^n(D,\beta), \widehat{\mathcal{O}}_I^{\mathrm{vir}})\, q^n$ is calculated using the K-theoretic 3-fold DT vertex $\mathsf{V}_{\lambda\mu\nu}^{\mathrm{3D,DT}}(t,q)$ much like in the calculation after Definition 6.2.14. Since the DT/PT edge coincide [**155**, Sect. 0.4], the result follows from Theorem 6.1.9 and a calculation similar to the proof of Theorem 6.1.8. $\square$

6.3.2. Cohomological limit I Again, let $Z = \{\{Z_\alpha\}_{\alpha \in V(X)}, \{Z_{\alpha\beta}\}_{\alpha\beta \in E(X)}\}$ be an element of either of the fixed loci

$$\bigsqcup_n \mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}, \quad \bigsqcup_n P_n(X,\beta)^{\mathbf{T}},$$

where we recall Assumption 6.1.1 from the introduction. We will work in one chart $U_\alpha \cong \mathbb{C}^4$ or $U_{\alpha\beta} \cong \mathbb{C}^* \times \mathbb{C}^3$ with standard torus action (6.2.1).

Proposition 6.3.3 ([**36**, Prop. 2.3])**.** *For any $\alpha \in V(X)$ and $\alpha\beta \in E(X)$, we have*

$$\widetilde{\mathsf{v}}_{Z_\alpha}|_{(1,1,1,1,1)} = 0, \quad \widetilde{\mathsf{e}}_{Z_{\alpha\beta}}|_{(1,1,1,1,1)} = 0,$$

i.e. $\mathrm{rk}\,\widetilde{\mathsf{v}}_{Z_\alpha} = \mathrm{rk}\,\widetilde{\mathsf{e}}_{Z_{\alpha\beta}} = 0$.

We set $t_i = e^{b\lambda_i}$ for all $i = 1,2,3,4$ and $y = e^{bm}$. The relation $t_1t_2t_3t_4 = 1$ translates into $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$. We are interested in the limit $b \to 0$.

Proposition 6.3.4. *For any $\alpha \in V(X)$ and $\alpha\beta \in E(X)$, the following limits*

$$\lim_{b\to 0} [-\widetilde{\mathsf{v}}_{Z_\alpha}]|_{t_i = e^{b\lambda_i}, y = e^{mb}}, \quad \lim_{b\to 0} [-\widetilde{\mathsf{e}}_{Z_{\alpha\beta}}]|_{t_i = e^{b\lambda_i}, y = e^{mb}}$$

exist in $\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,m)/(\lambda_1+\lambda_2+\lambda_3+\lambda_4)$.

[10]Choosing all signs of all $\mathbf{T}$-fixed points which are scheme theoretically supported on $\{x_4 = 0\}$ equal to $+1$ amounts to replacing $-q$ by q on the right-hand-side of (6.1.4).

PROOF. Using multi-index notation for $\tau := (t_1, t_2, t_3, t_4, y)$, where $t_1t_2t_3t_4 = 1$, we write

$$\widetilde{\mathsf{v}}_{Z_\alpha} = \sum_v \tau^v - \sum_w \tau^w.$$

These sums are finite by Lemma 6.2.5. This representation is unique when we require that the sequences $\{v\}$, $\{w\}$ have no elements in common. Proposition 6.3.3 implies that the number of + (i.e. "deformation") and − (i.e. "obstruction") terms in both expressions are equal, i.e.

$$\sum_v 1 - \sum_w 1 = 0.$$

Recall that $\widetilde{\mathsf{v}}_{Z_\alpha}$ has no $\mathbf{T}$-fixed part with positive coefficient (Lemma 6.2.5). When one of the w is zero, $[-\widetilde{\mathsf{v}}_{Z_\alpha}] = 0$ and the proposition is clear. Next, write the components of the weight vectors in the sum as follows

$$v = (v_1, v_2, v_3, v_4, v_m)$$

and similarly for w. Then

$$[-\widetilde{\mathsf{v}}_{Z_\alpha}]|_{t_i=e^{b\lambda_i}, y=e^{mb}} = \frac{\prod_w \big((w_1\lambda_1 + w_2\lambda_2 + w_3\lambda_3 + w_4\lambda_4 + w_m m)\, b + O(b^2)\big)}{\prod_v \big((v_1\lambda_1 + v_2\lambda_2 + v_3\lambda_3 + v_4\lambda_4 + v_m m)\, b + O(b^2)\big)}.$$

Since the number of factors in numerator and denominator is equal, say $N = \sum_v 1 = \sum_w 1$, we can divide numerator and denominator by b^N and deduce that the limit exists and equals

$$\lim_{b\to 0}[-\widetilde{\mathsf{v}}_{Z_\alpha}]|_{t_i=e^{b\lambda_i}, y=e^{mb}} = \frac{\prod_w (w_1\lambda_1 + w_2\lambda_2 + w_3\lambda_3 + w_4\lambda_4 + w_m m)}{\prod_v (v_1\lambda_1 + v_2\lambda_2 + v_3\lambda_3 + v_4\lambda_4 + v_m m)}. \tag{6.3.1}$$

The proof for $\widetilde{\mathsf{e}}_{Z_{\alpha\beta}}$ is similar. □

PROOF OF THEOREM 6.1.14. Consider the generating series

$$\sum_n I_{n,\beta}(L,y)\big|_{t_i=e^{b\lambda_i}, y=e^{bm}}\, q^n, \quad \sum_n P_{n,\beta}(L,y)\big|_{t_i=e^{b\lambda_i}, y=e^{bm}}\, q^n.$$

Both series are calculated by the vertex formalism of Theorem 6.2.11. Proposition 6.3.4 implies that the limits $b \to 0$ exist. Moreover, by the proof of Proposition 6.3.4, we have that for any virtual $\mathbf{T}\times\mathbb{C}^*$-representation $V = \sum_\mu t^\mu - \sum_\nu t^\nu$ with no negative constant term and rank $\operatorname{rk} V = 0$ that

$$\lim_{b\to 0}[V]|_{t_i=e^{b\lambda_i}, y=e^{bm}} = e^{\mathbf{T}\times\mathbb{C}^*}(V).$$

This means precisely that $\lim_{b\to 0} I_{n,\beta}(L,y)|_{t_i=e^{b\lambda_i}, y=e^{bm}}$ computes — via virtual localization — the integral

$$\int_{[\mathrm{Hilb}^n(X,\beta)]^{\mathrm{vir}}} e^{\mathbf{T}\times\mathbb{C}}((L^{[n]})^\vee \otimes e^m),$$

which by Remark 6.1.13 computes precisely our cohomological invariants.

Next assume Conjecture 6.1.7 holds. The second part of the theorem follows from Theorem 6.1.8. □

Remark 6.3.5. Taking the cohomological limit of Proposition 6.3.4 and setting $m = -\lambda_1 - \lambda_2 - \lambda_3$, one recovers the cohomological 3-fold DT/PT vertex from the cohomological 4-fold DT/PT vertex (this follows from Proposition 6.3.1). Using the vertex formalism, the 4-fold cohomological DT/PT correspondence therefore implies the 3-fold cohomological DT/PT correspondence. This gives the second diagonal arrow of Figure 1 in the introduction.

6.3.3. Cohomological limit II As before, let $Z = \{\{Z_\alpha\}_{\alpha \in V(X)}, \{Z_{\alpha\beta}\}_{\alpha\beta \in E(X)}\}$ be an element of either of the fixed loci

$$\bigsqcup_n \mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}, \quad \bigsqcup_n P_n(X,\beta)^{\mathbf{T}},$$

where we recall Assumption 6.1.1 from the introduction. We will work in one chart $U_\alpha \cong \mathbb{C}^4$ or $U_{\alpha\beta} \cong \mathbb{C}^* \times \mathbb{C}^3$ with standard torus action (6.2.1). In the DT case, Z_α is a point- or curve-like solid partition, whose renormalized volume we denote by $|Z_\alpha|$. In the stable pairs case, Z_α consists of a Cohen-Macaulay support curve $Z_{\mathcal{M},\alpha}$ together with a box configuration $B^{(\alpha)}$ (6.2.9). We denote the sum of the renormalized volume of $Z_{\mathcal{M},\alpha}$ and the length of $B^{(\alpha)}$ by $|Z_\alpha|$ as well.

In this section, we set $t_i = e^{b\lambda_i}$ for all $i = 1,2,3,4$, $y = e^{bm}$, $Q = qm$, and take the double limit $b \to 0$, $m \to \infty$. In (6.1.6), we recalled the definition of the cohomological DT/PT invariants $I^{\mathrm{coho}}_{n,\beta}, P^{\mathrm{coho}}_{n,\beta}$ studied in [**35**]. In [**35**], we defined

$$\mathsf{V}^{\mathrm{coho,DT}}_{\lambda\mu\nu\rho}, \quad \mathsf{V}^{\mathrm{coho,PT}}_{\lambda\mu\nu\rho}, \quad \mathsf{E}^{\mathrm{coho,DT}}_{\lambda}, \quad \mathsf{E}^{\mathrm{coho,PT}}_{\lambda},$$

which are defined precisely as in Definition 6.2.14 but with the Nekrasov bracket $[\cdot]$ replaced by $\mathbf{T}$-equivariant Euler class $e(\cdot)$.

Proposition 6.3.6. *For any $\alpha \in V(X)$ and $\alpha\beta \in E(X)$, we have*

$$\lim_{\substack{b\to 0\\ m\to\infty}} \left([-\widetilde{\mathsf{v}}_{Z_\alpha}]\, q^{|Z_\alpha|}\right)\Big|_{t_i = e^{b\lambda_i}, y = e^{mb}, Q = mq} = e(-\mathsf{V}^{\mathrm{coho}}_{Z_\alpha})\, Q^{|Z_\alpha|},$$

$$\lim_{\substack{b\to 0\\ m\to\infty}} \left([-\widetilde{\mathsf{e}}_{Z_{\alpha\beta}}]\, q^{f(\alpha,\beta)}\right)\Big|_{t_i = e^{b\lambda_i}, y = e^{mb}, Q = mq} = e(-\mathsf{E}^{\mathrm{coho}}_{Z_{\alpha\beta}})\, Q^{f(\alpha,\beta)},$$

where $f(\alpha,\beta)$ is defined[11] *in (6.2.2).*

Proof. We continue using the notation of the proof of Propositions 6.3.3 and 6.3.4. Let $Z_{\mathcal{M},\alpha}$ be the underlying Cohen-Macaulay curve of Z_α and denote by λ, μ, ν, ρ its asymptotic plane partitions. In the DT case, define

$$W_\alpha := \sum_{w \in Z_\alpha \setminus Z_{\mathcal{M},\alpha}} t^w + \sum_{w \in Z_{\mathcal{M},\alpha}} \big(1 - \#\{\text{legs containing } w\}\big)\, t^w.$$

In the stable pairs case, define

$$W_\alpha := \sum_{w \in B^{(\alpha)}} t^w + \sum_{w \in Z_{\mathcal{M},\alpha}} \big(1 - \#\{\text{legs containing } w\}\big)\, t^w,$$

[11]In the notation $f(\alpha,\beta)$ we suppress the choice of the multidegree $\mathbf{m}$ and of the plane partition $\lambda_{\alpha\beta}$.

where $B^{(\alpha)}$ is the box configuration corresponding to the fixed point Z_α (6.2.9). Then the terms involving y in the Laurent polynomial $\widetilde{\mathsf{v}}_{Z_\alpha}$ are $-y\overline{W}_\alpha$. We already showed

(6.3.2)
$$\lim_{b\to 0}[-\widetilde{\mathsf{v}}_{Z_\alpha}]|_{t_i=e^{b\lambda_i},y=e^{mb}}\, q^{|W_\alpha|} = \frac{\prod_w(w_1\lambda_1+w_2\lambda_2+w_3\lambda_3+w_4\lambda_4+w_m m)}{\prod_v(v_1\lambda_1+v_2\lambda_2+v_3\lambda_3+v_4\lambda_4+v_m m)}\left(\frac{Q}{m}\right)^{|W_\alpha|}.$$

In fact, y always appears in $\widetilde{\mathsf{v}}_{Z_\alpha}$, $\widetilde{\mathsf{e}}_{Z_{\alpha\beta}}$ with power $+1$, so w_m, v_m are elements of $\{0,1\}$.

As before, we set $\tau := (t_1,t_2,t_3,t_4,y)$ and we write
$$W_\alpha = \sum_a \tau^a - \sum_c \tau^c,$$
where the collections of weights $\{a\}$ and $\{c\}$ have no elements in common. Observe that the rank of W_α equals the renormalized volume $|Z_\alpha|$. By Proposition 6.3.3, the terms of (6.3.2) involving m (i.e. $w_m \neq 0$ or $v_m \neq 0$) are precisely
$$\begin{aligned}&\frac{\prod_a(-a_1\lambda_1-a_2\lambda_2-a_3\lambda_3-a_4\lambda_4+m)}{\prod_c(-c_1\lambda_1-c_2\lambda_2-c_3\lambda_3-c_4\lambda_4+m)}m^{-\sum_a 1+\sum_c 1}Q^{|Z_\alpha|}\\ &=\frac{\prod_a(-a_1\frac{\lambda_1}{m}-a_2\frac{\lambda_2}{m}-a_3\frac{\lambda_3}{m}-a_4\frac{\lambda_4}{m}+1)}{\prod_c(-c_1\frac{\lambda_1}{m}-c_2\frac{\lambda_2}{m}-c_3\frac{\lambda_3}{m}-c_4\frac{\lambda_4}{m}+1)}Q^{|Z_\alpha|}.\end{aligned}$$
Therefore, sending $m\to\infty$, this term becomes $Q^{|Z_\alpha|}$. As we saw in the proof of Theorem 6.1.14, the terms of (6.3.2) which do *not* involving m (i.e. $w_m = v_m = 0$) together are equal to $e(-\mathsf{V}^{\mathrm{coho}}_{Z_\alpha})$. So in total, we have
$$\lim_{\substack{b\to 0\\ m\to\infty}}\left([-\widetilde{\mathsf{v}}_{Z_\alpha}]\, q^{|Z_\alpha|}\right)\big|_{t_i=e^{b\lambda_i},y=e^{mb},Q=mq} = e(-\mathsf{V}^{\mathrm{coho}}_{Z_\alpha})\, Q^{|Z_\alpha|}.$$
The analysis of the edge term follows similarly (cf. [**36**, Prop. 2.6]). $\square$

Proof of Theorem 6.1.19. The first part of the theorem follows from Theorem 6.2.11 and Proposition 6.3.6. Moreover, by Proposition 6.3.6 and Definition 6.2.14 we have
$$\lim_{\substack{b\to 0\\ m\to\infty}} \mathsf{V}^{\mathsf{DT}}_{\lambda\mu\nu\rho}(t,y,q)\big|_{t_i=e^{b\lambda_i},y=e^{mb},Q=mq} = \mathsf{V}^{\mathrm{coho},\mathsf{DT}}_{\lambda\mu\nu\rho}(Q),$$
$$\lim_{\substack{b\to 0\\ m\to\infty}} \mathsf{V}^{\mathsf{PT}}_{\lambda\mu\nu\rho}(t,y,q)\big|_{t_i=e^{b\lambda_i},y=e^{mb},Q=mq} = \mathsf{V}^{\mathrm{coho},\mathsf{PT}}_{\lambda\mu\nu\rho}(Q),$$
where λ,μ,ν,ρ are finite plane partitions and in the stable pairs case, we assume at most two of them are non-empty. Moreover, the choices of signs for the right-hand-side are determined by the choices of signs for the left-hand-side. We deduce that Conjecture 6.1.7 implies Conjecture 6.1.18. $\square$

6.4. Hilbert schemes of points

In this section we consider Nekrasov's Conjecture 6.1.5 in the two cohomological limits discussed in Section 6.3.2, 6.3.3 (see also [**140, 143**]).

Let X be a toric Calabi-Yau 4-fold with $\mathbf{T}$-equivariant line bundle L. By Theorem 6.1.14, we have
$$\lim_{\substack{b\to 0\\ m\to 0}} I_{n,0}(L,y)|_{t_i=e^{b\lambda_i},y=e^{bm}} = \lim_{m\to 0}\int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} c_n((L^{[n]})^\vee\otimes e^m),$$

where the invariants on the right-hand-side are defined by localization (6.1.5). Since $L^{[n]}$ is a rank n vector bundle, we have

$$c_n((L^{[n]})^\vee \otimes e^m) = \sum_{i=0}^{n} c_i((L^{[n]})^\vee)\, m^{n-i},$$

and similarly at any $\mathbf{T}$-fixed point. Hence[12]

$$\lim_{\substack{b\to 0\\ m\to 0}} I_{n,0}(L,y)|_{t_i=e^{b\lambda_i}, y=e^{bm}} = (-1)^n \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} c_n(L^{[n]}).$$

These invariants were studied in [**33**], where it is conjectured that there exist choices of signs such that the following equation holds

$$\sum_{n=0}^{\infty} q^n \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} c_n(L^{[n]}) = M(-q)^{\int_X c_1(L)\, c_3(X)}, \tag{6.4.1}$$

where all Chern classes are $\mathbf{T}$-equivariant, $\int_X$ denotes $\mathbf{T}$-equivariant push-forward to a point, and

$$M(q) := \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$

denotes MacMahon's generating function for plane partitions.

Remark 6.4.1. This conjectural expression makes sense also in the *projective* setting and was addressed by Cao-Kool [**33**]. Recently, it has been proven in the *very ample* case (i.e. $L = \mathcal{O}(D)$ for a smooth connected divisor D on X) by Cao-Qu [**41**] and Park [**161**] and in general for any line bundle by Bojko [**22**], assuming a wall-crossing conjecture of Gross-Joyce-Tanaka [**86**].

As noted before in [**140**, Sect. 5.2], the conjectural formula (6.4.1) is a special case of Conjecture 6.1.5 as can be seen as follows[13]. For any $n \geqslant 1$, we have

$$\begin{aligned}
&\lim_{b\to 0} \frac{[t_1^n t_2^n][t_1^n t_3^n][t_2^n t_3^n][y^n]}{[t_1^n][t_2^n][t_3^n][t_4^n][y^{\frac{n}{2}} q^n][y^{\frac{n}{2}} q^{-n}]}\Big|_{t_i=e^{b\lambda_i}, y=e^{mb}} \\
&= \lim_{b\to 0} \frac{m(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(bn)^4 + O((bn)^5)}{((\lambda_1\lambda_2\lambda_3\lambda_4)(bn)^4 + O((bn)^5))(e^{\frac{bmn}{4}} q^{\frac{n}{2}} - e^{-\frac{bmn}{4}} q^{-\frac{n}{2}})(e^{\frac{bmn}{4}} q^{-\frac{n}{2}} - e^{-\frac{bmn}{4}} q^{\frac{n}{2}})} \\
&= \frac{m(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)(q^{\frac{n}{2}} - q^{-\frac{n}{2}})^2}.
\end{aligned}$$

Recall the following identity

$$\mathrm{Exp}\left(\frac{q}{(1-q)^2}\right) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}.$$

Let $L|_{\mathbb{C}^4} \cong \mathcal{O}_{\mathbb{C}^4} \otimes t_1^{d_1} t_2^{d_2} t_3^{d_3} t_4^{d_4}$. Taking $m = -(d_1\lambda_1 + d_2\lambda_2 + d_3\lambda_3 + d_4\lambda_4)$ and using Theorem 6.1.14, we see that Nekrasov's conjecture implies (6.4.1) for $X = \mathbb{C}^4$. Since

[12] Note that on a smooth projective Calabi-Yau 4-fold, $[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}$ has degree $2n$ and the limit $m \to 0$ would not be needed.

[13] Unlike [**140**], which was motivated by physics, our motivation for (6.4.1) came from our analogous conjecture on smooth projective Calabi-Yau 4-folds [**33**, Conj. 1.2].

left-hand-side and right-hand-side of (6.4.1) are *suitably multiplicative*, (6.4.1) also follows for any toric Calabi-Yau 4-fold X (see [**33**, Prop. 3.20] for details).

Finally, we consider the following limit (Theorem 6.1.19)

$$\lim_{\substack{b\to 0\\ m\to\infty}} \Big(I_{n,0}(\mathcal{O}_X, e^{bm})\, q^n\Big)\Big|_{t_i=e^{b\lambda_i}, Q=mq} = Q^n \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} 1,$$

where the right-hand-side is defined by localization, i.e. (6.1.6). For any $n \geqslant 1$, we have

$$\begin{aligned}
&\lim_{\substack{b\to 0\\ m\to\infty}} \frac{[t_1^n t_2^n][t_1^n t_3^n][t_2^n t_3^n][y^n]}{[t_1^n][t_2^n][t_3^n][t_4^n][y^{\frac{n}{2}}q^n][y^{\frac{n}{2}}q^{-n}]}\Big|_{t_i=e^{b\lambda_i}, y=e^{mb}, Q=mq} \\
&= \lim_{m\to\infty} \frac{m(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)} \frac{\left(\frac{Q}{m}\right)^n}{\left(1-\left(\frac{Q}{m}\right)^n\right)^2} \\
&= \begin{cases} \frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)}Q & \text{if } n=1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, Nekrasov's Conjecture 6.1.5 implies that there exist choices of signs such that the following identity holds

$$\sum_{n=0}^{\infty} Q^n \int_{[\mathrm{Hilb}^n(\mathbb{C}^4)]^{\mathrm{vir}}} 1 = e^{\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)}Q}.$$

This formula was also originally conjectured by Nekrasov and discussed in [**33**, App. B]. Note that the exponent appearing on right-hand-side equals $-\int_{\mathbb{C}^4} c_3(\mathbb{C}^4)$ (interpreted as a $\mathbf{T}$-equivariant integral). Therefore, Conjecture 6.1.5 and the vertex formalism together imply that there exist choices of signs such that the following equation holds

$$\sum_{n=0}^{\infty} Q^n \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} 1 = e^{-Q\int_X c_3(X)}.$$

6.5. Local resolved conifold

We start with the following lemma, which recovers [**155**, Lemma 5] after applying dimensional reduction and cohomological limit I.

Lemma 6.5.1. *There exist unique choices of signs such that*

$$\mathsf{V}^{\mathsf{PT}}_{(1),\varnothing,\varnothing,\varnothing}(t,y,q) = \mathrm{Exp}\left(\frac{[yt_1]}{[t_1]}q\right).$$

Proof. For every length n of the cokernel, there is only one $\mathbf{T}$-fixed point, and the character of the corresponding stable pair is

$$Z_n = \frac{1}{1-t_1} + \sum_{i=1}^n t_1^{-i}.$$

The corresponding vertex term is easily computed as

$$\widetilde{\mathsf{v}}_n = \sum_{i=1}^n t_1^{-i} - y\sum_{i=1}^n t_1^i$$

and we choose $(-1)^n$ for the corresponding sign. Then

$$\begin{aligned}
\mathsf{V}^{\mathsf{PT}}_{(1),\varnothing,\varnothing,\varnothing}(t,y,q) &= \sum_{n\geq 0} q^n(-1)^n[-\widetilde{\mathsf{v}}_n] \\
&= \sum_{n\geq 0}(y^{-\frac{1}{2}}q)^n \prod_{i=1}^{n} \frac{1-yt_1^i}{1-t_1^i} \\
&= \operatorname{Exp}\left(y^{-\frac{1}{2}}q\frac{1-yt_1}{1-t_1}\right) \\
&= \operatorname{Exp}\left(\frac{[yt_1]}{[t_1]}q\right),
\end{aligned}$$

where in the third line we used [**149**, Ex. 5.1.22]. □

Let $X = D\times\mathbb{C}$, where $D = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1)\oplus\mathcal{O}(-1))$ is the resolved conifold. Consider the generating series

$$\mathcal{Z}_X(y,q,Q) := \sum_{n,d} P_{n,d[\mathbb{P}^1]}(\mathcal{O},y)\, q^n Q^d.$$

Using the vertex formalism, we verified Conjecture 6.1.20 in the following cases.

Proposition 6.5.2. *Conjecture 6.1.20 holds for curve classes $\beta = d[\mathbb{P}^1]$ with $d = 1,2,3,4$ up to the following orders:*

- $d = 1$,
- $d = 2$ *modulo* q^6,
- $d = 3$ *modulo* q^6,
- $d = 4$ *modulo* q^7.

Moreover, the choices of signs in these verifications are unique and compatible with the signs in Conjecture 6.1.7 (and therefore also the signs of Theorem 6.1.9 by Remark 6.1.10).

Proof. For degree 1 the conjecture is equivalent to

$$\mathcal{Z}_{X,1}(y,q) = \frac{[y]}{[t_4][y^{\frac{1}{2}}q][y^{\frac{1}{2}}q^{-1}]}.$$

The edge term for one leg with multiplicity 1 is

$$\tilde{\mathsf{e}} = t_4 - y,$$

therefore, by Lemma 6.5.1, we conclude

$$\begin{aligned}
\mathcal{Z}_{X,1}(y,q) &= \mathsf{V}^{\mathsf{PT}}_{(1),\varnothing,\varnothing,\varnothing}(t,y,q)\cdot \mathsf{V}^{\mathsf{PT}}_{(1),\varnothing,\varnothing,\varnothing}(t,y,q)|_{t_1=t_1^{-1}}\cdot q\cdot(-1)[-\tilde{\mathsf{e}}] \\
&= -\frac{[y]}{[t_4]}q\operatorname{Exp}\left(\left(\frac{[yt_1]}{[t_1]}+\frac{[yt_1^{-1}]}{[t_1^{-1}]}\right)q\right) \\
&= -\frac{[y]}{[t_4]}q\operatorname{Exp}\left((y^{\frac{1}{2}}+y^{-\frac{1}{2}})q\right) \\
&= \frac{[y]}{[t_4][y^{\frac{1}{2}}q][y^{\frac{1}{2}}q^{-1}]}.
\end{aligned}$$

The other cases have been checked by an implementation of the vertex formalism in Mathematica. ☐

Consider the generating series of K-theoretic stable pair invariants of the resolved conifold

$$\mathcal{Z}_D(q,Q) := \sum_{n,d} \chi(P_n(Y,d[\mathbb{P}^1]), \widehat{\mathcal{O}}_P^{\mathrm{vir}})\, q^n Q^d.$$

We already saw that the 4-fold PT vertex/edge reduce to the 3-fold PT vertex/edge for all stable pairs scheme theoretically supported on $D \subseteq X$ after setting $y = t_4$ (Proposition 6.3.1 and Remark 6.3.2). By the same type of argument as in Proposition 6.3.1, one can show that all stable pairs *not* scheme theoretically supported in $D \subseteq X$ contribute zero after setting $y = t_4$. Therefore[14]

$$\mathcal{Z}_X(y,q,Q)\Big|_{y=t_4} = \mathcal{Z}_D(-q,Q).$$

Hence Conjecture 6.1.20 implies

$$\mathcal{Z}_D(-q,Q) = \mathrm{Exp}\left(\frac{-qQ}{(1-q/\kappa)(1-q\kappa)}\right), \quad \kappa := (t_1t_2t_3)^{\frac{1}{2}}.$$

This equality was recently proved by Kononov-Okounkov-Osinenko [**109**, Sect. 4], which gives good evidence for Conjecture 6.1.20 from our perspective. Applying the preferred limits discussed by Arbesfeld in [**3**, Sect. 4], this formula coincides with an expression obtained using the refined topological vertex by Iqbal-Kozçaz-Vafa [**98**, Sect. 5.1]. More precisely, setting $\widetilde{q} := q\kappa$ and $\widetilde{t} := q\kappa^{-1}$ yields [**98**, Eqn. (67)]. The formula also coincides with the generating series of motivic stable pair invariants of the resolved conifold obtained by Morrison-Mozgovoy-Nagao-Szendrői in [**136**, Prop. 4.5].

Consider the two cohomological generating series for $X = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O})$ defined by (6.1.5) and (6.1.6):

$$\mathcal{Z}_X^{\mathrm{coho}}(m,q,Q) := \sum_{n,d} P^{\mathrm{coho}}_{n,d[\mathbb{P}^1]}(\mathcal{O},m)\, q^n Q^d,$$
$$\mathcal{Z}_X^{\mathrm{coho}}(P,Q) := \sum_{n,d} P^{\mathrm{coho}}_{n,d[\mathbb{P}^1]}\, P^n Q^d.$$

Applying cohomological limits I and II (Theorem 6.1.14 and 6.1.19), Conjecture 6.1.20 and a calculation similar to the one in Appendix 6.4 imply

$$\mathcal{Z}_X^{\mathrm{coho}}(m,q,Q) = \lim_{b\to 0} \mathcal{Z}_X\Big|_{t_i=e^{b\lambda_i},y=e^{mb}} = \left(\prod_{n=1}^{\infty}(1-Qq^n)^n\right)^{\frac{m}{\lambda_4}},$$
$$\mathcal{Z}_X^{\mathrm{coho}}(P,Q) = \lim_{\substack{b\to 0\\ m\to\infty}} \mathcal{Z}_X\Big|_{t_i=e^{b\lambda_i},y=e^{mb},P=mq} = \exp\left(-\frac{PQ}{\lambda_4}\right).$$

A wall-crossing interpretation of the first formula is discussed in [**43**]. Putting $m = \lambda_4$ in the first expression yields the famous formula for the stable pair invariants (or topological string partition function) of the resolved conifold. The second formula was conjectured, and verified up to the same orders as above in [**35**, Conj. 2.22].

[14] Recall the origin of the minus sign from Theorem 6.1.9.

6.6. Appendix: Combinatorial sign rule

The goal of this appendix is to propose various square roots of the virtual tangent bundle with *sign rules*, which would make the invariants studied in this chapter effectively computable. As evidence to our proposals, we show that they

- are canonical (Theorem 6.6.6, 6.6.11, 6.6.15),
- play well with the dimensional reduction of Section 6.3.1,
- are consistent with previous computations [**33, 34, 35, 140, 143**], both in the math and physics literature.

We study here a combinatorial sign rule only for DT theory, but we expect that these techniques can be adapted to the PT side as well.

6.6.1. The vertex term: points To each fixed point $Z \in \mathrm{Hilb}^n(\mathbb{C}^4)^{\mathbf{T}}$ corresponds a solid partition π of size n. Denote by Z_π, V_π the vertex terms $Z_\alpha, \mathsf{V}_\alpha$ in (6.2.8),(6.2.13). By $\mathbf{T}$-equivariant Serre duality, we know that V_π admits a square root. We set

$$\mathsf{v}^i_\pi = Z_\pi - \overline{P_{jkl}} Z_\pi \overline{Z_\pi},$$

which enjoys

$$\mathsf{V}_\pi = \mathsf{v}^i_\pi + \overline{\mathsf{v}^i_\pi},$$

where $\{\, i,j,k,l \,\} = \{\, 1,2,3,4 \,\}$, $P_I = \prod_{a \in I}(1-t_a)$ for a set of indices I and $\overline{(\cdot)}$ is the involution in $K^0_{\mathbf{T}}(\mathsf{pt})$. For $i=4$, it recovers the square root already found in [**143**] and Section 6.2.4. The next lemma was already proven by Nekrasov-Piazzalunga [**143**], whose proof we sketch for completeness and to introduce useful notation.

Lemma 6.6.1 ([**143**, Sec. 2.4.1]). *Let π be a point-like solid partition and $i = 1, \dots 4$. Then v^i_π is $\mathbf{T}$-movable.*

Proof. Without loss of generality, suppose that $i=4$. We prove the statement by induction on the size of π. If $|\pi| = 1$, then v^4_π has no constant terms. Suppose now that the claim holds for all solid partitions π of size $|\pi| \leq n$. Consider a solid partition $\tilde{\pi}$ of size $|\tilde{\pi}| = n+1$; this can be seen as a solid partition π of size n with an extra box over it, corresponding to a $\mathbb{Z}^4$-lattice point $\mu = (A,B,C,D)$. We have

$$Z_{\tilde{\pi}} = Z_\pi + t^\mu$$

and

$$\begin{aligned}\mathsf{v}^4_{\tilde{\pi}} &= \mathsf{v}^4_\pi + t^\mu - \overline{P_{123}}(t^\mu \overline{Z_\pi} + t^{-\mu} Z_\pi + 1) \\ &= \mathsf{v}^4_\pi + t^\mu - \overline{P_{123}}(t^\mu(\overline{Z_\pi} + t^{-\mu}) + t^{-\mu}(Z_\pi + t^\mu) - 1).\end{aligned}$$

By induction, we know that v^4_π is $\mathbf{T}$-movable. Consider a subdivision $\tilde{\pi} = \pi' \sqcup \pi''$ of the boxes of $\tilde{\pi}$, where π' corresponds to the lattice points $\nu \leq \mu$ and π'' corresponds to the lattice points $\nu \not\leq \mu$. Denote by

$$\begin{aligned}Z_{\pi'} &= \sum_{\nu \in \pi'} t^\nu = \sum_{i \leq A, j \leq B, k \leq C, l \leq D} t_1^i t_2^j t_3^k t_4^l, \\ Z_{\pi''} &= \sum_{\nu \in \pi''} t^\nu.\end{aligned}$$

By construction, $Z_{\tilde{\pi}} = Z_{\pi'} + Z_{\pi''}$. We want to prove that

$$\left(t^\mu + \overline{P_{123}} - \overline{P_{123}}t^\mu \overline{Z_{\pi'}} - \overline{P_{123}}t^\mu \overline{Z_{\pi''}} - \overline{P_{123}}t^{-\mu} Z_{\pi'} - \overline{P_{123}}t^{-\mu} Z_{\pi''}\right)^{\mathsf{fix}} = 0$$

For a set of indices I, denote by δ_I the function which is 1 if only if all the indices are equal. Analyzing each piece separately, it turns out that

$$\begin{aligned}
\left(t^\mu + \overline{P_{123}}\right)^{\mathsf{fix}} &= 1 + \delta_{A,B,C,D}, \\
\left(\overline{P_{123}}t^\mu \overline{Z_{\pi''}}\right)^{\mathsf{fix}} &= 0, \\
\left(\overline{P_{123}}t^{-\mu} Z_{\pi''}\right)^{\mathsf{fix}} &= 0, \\
\left(\overline{P_{123}}t^\mu \overline{Z_{\pi'}}\right)^{\mathsf{fix}} &= \sum_{i=0}^{D} \delta_{A,B,C,i}, \\
\left(\overline{P_{123}}t^{-\mu} Z_{\pi'}\right)^{\mathsf{fix}} &= 1 - \sum_{i=0}^{D-1} \delta_{A,B,C,i},
\end{aligned}$$

by which we conclude the induction step. $\square$

We propose a sign rule for the sign in (6.2.21), relative to the square root v_π^i.

Conjecture 6.6.2. *Let π be a solid partition corresponding to a* **T***-fixed point in* $\mathrm{Hilb}^n(\mathbb{C}^4)$*. Then the sign relative to the square root v_π^i is $(-1)^{\sigma_i(\pi)}$, where*

$$\sigma_i(\pi) = |\pi| + \#\{\, (a_1, a_2, a_3, a_4) \in \pi \colon a_j = a_k = a_l < a_i \,\} \tag{6.6.1}$$

and $\{\, i, j, k, l \,\} = \{\, 1, 2, 3, 4 \,\}$.

Remark 6.6.3. For $i = 4$, the sign rule (6.6.1) was proposed by Nekrasov-Piazzalunga in the physics literature in [**143**], as the result of supersymmetric localization in string theory[15]. This sign rule is consistent with the previous computations of [**33, 34, 35, 36, 140, 143**].

Remark 6.6.4. Let π corresponds to a **T**-invariant closed subscheme $Z \subset \mathbb{C}^4$ supported in the hyperplane $\{\, x_i = 0 \,\} \subset \mathbb{C}^4$, for $i = 1, \ldots, 4$. Then $\sigma_i(\pi) = |\pi|$, which is consistent with the dimensional reduction studied in Section 6.3.1.

Remark 6.6.5. Recently, Kool-Rennemo [**115**] announced a proof of this conjecture, by explictly computing the sign appearing in the definition of the square-root Euler class of Oh-Thomas [**147**].

We prove now that the sign rule (6.6.1) is canonical, meaning that it does not really depend on choosing a preferred x_i-axis.

Theorem 6.6.6. *Let π a point-like solid partition. For every $i, j = 1, \ldots, 4$ we have*

$$(-1)^{\sigma_i(\pi)}[-\mathsf{v}_\pi^i] = (-1)^{\sigma_j(\pi)}[-\mathsf{v}_\pi^j].$$

[15] In [**143**] the sign rule is slightly different due to a slightly different choice of square root.

PROOF. v_π^i and v_π^j are both square roots of V_π and are $\mathbf{T}$-movable by Lemma 6.6.1. To prove the claim, we need to write

$$\mathsf{v}_\pi^i - \mathsf{v}_\pi^j = U_\pi - \overline{U_\pi},$$

for a $U_\pi \in K^0_{\mathbf{T}}(\mathsf{pt})$ and compute the parity of $\operatorname{rk} U_\pi^{\mathsf{mov}}$; in fact

$$\frac{[\mathsf{v}_\pi^i]}{[\mathsf{v}_\pi^j]} = \frac{[U_\pi]}{[\overline{U_\pi}]} = (-1)^{\operatorname{rk} U_\pi^{\mathsf{mov}}}.$$

Without loss of generality, suppose $i = 4$ and $j = 3$; we prove the claim by induction on the size of π. If $|\pi| = 1$, we have $Z_\pi = 1$ and

$$\begin{aligned}\mathsf{v}_\pi^4 - \mathsf{v}_\pi^3 &= \overline{P_{12}}(t_3^{-1} - t_4^{-1})\\ &= \overline{P_{12}}t_3^{-1} - \overline{\overline{P_{12}}t_3^{-1}},\end{aligned}$$

where used that $t_1t_2t_3t_4 = 1$; clearly $\operatorname{rk}(\overline{P_{12}}t_3^{-1})^{\mathsf{mov}} = 0$. Suppose now the claim holds for any partition of size $|\pi| \leq n$ and consider a solid partition $\tilde{\pi}$, obtained by a solid partition π of size n by adding a box whose lattice coordinates are $\mu = (A, B, C, D)$. We have

$$\begin{aligned}\mathsf{v}_{\tilde{\pi}}^4 - \mathsf{v}_{\tilde{\pi}}^3 &= \mathsf{v}_\pi^4 - \mathsf{v}_\pi^3 + \overline{P_{12}}(t_3^{-1} - t_4^{-1}) + \overline{P_{12}}(t_3^{-1} - t_4^{-1})(\overline{Z_\pi}t^\mu + Z_\pi t^{-\mu})\\ &= \mathsf{v}_\pi^4 - \mathsf{v}_\pi^3 - \overline{P_{12}}(t_3^{-1} - t_4^{-1}) + \overline{P_{12}}(t_3^{-1} - t_4^{-1})(\overline{Z_{\tilde{\pi}}}t^\mu + Z_{\tilde{\pi}} t^{-\mu}),\end{aligned}$$

and by the induction step

$$\begin{aligned}&e^{\mathbf{T}}(\mathsf{v}_\pi^4 - \mathsf{v}_\pi^3) = (-1)^{\sigma_4(\pi) - \sigma_3(\pi)},\\ &e^{\mathbf{T}}\left(\overline{P_{12}}(t_3^{-1} - t_4^{-1})\right) = 1.\end{aligned}$$

The final piece to compute is of the form

$$\begin{aligned}\overline{P_{12}}(t_3^{-1} - t_4^{-1})(\overline{Z_{\tilde{\pi}}}t^\mu + Z_{\tilde{\pi}} t^{-\mu}) &= (W_1 + \overline{W_1})(W_2 - \overline{W_2})\\ &= (W_1W_2 - \overline{W_1W_2}) + (\overline{W_1}W_2 - W_1\overline{W_2}),\end{aligned}$$

with

$$\begin{aligned}W_1 &= t^{-\mu}Z_{\tilde{\pi}},\\ W_2 &= t_3^{-1}\overline{P_{12}}.\end{aligned}$$

Since $\operatorname{rk} W_1W_2 = \operatorname{rk} \overline{W_1}W_2 = 0$, we just need to compute the parity of $\operatorname{rk}(W_1W_2)^{\mathsf{fix}} + \operatorname{rk}(\overline{W_1}W_2)^{\mathsf{fix}}$. As in the proof of Lemma 6.6.1, consider the subdivision $\tilde{\pi} = \pi' \sqcup \pi''$, with $Z_{\tilde{\pi}} = Z_{\pi'} + Z_{\pi''}$, where

$$\begin{aligned}Z_{\pi'} &= \sum_{i \leq A, j \leq B, k \leq C, l \leq D} t_1^i t_2^j t_3^k t_4^l,\\ Z_{\pi''} &= \sum_{\nu \in \pi''} t^\nu.\end{aligned}$$

We have

$$(W_1W_2)^{\mathsf{fix}} = \left(t_3^{-1}\overline{P_{12}}t^{-\mu}Z_{\pi'} + t_3^{-1}\overline{P_{12}}t^{-\mu}Z_{\pi''}\right)^{\mathsf{fix}}.$$

As in the proof Lemma 6.6.1 we compute

$$\left(t_3^{-1}\overline{P_{12}}t^{-\mu}Z_{\pi''}\right)^{\mathsf{fix}} = 0,$$

$$\left(t_3^{-1}\overline{P_{12}}t^{-\mu}Z_{\pi'}\right)^{\mathsf{fix}} = \sum_{k=0}^{C}\sum_{l=0}^{D-1}\delta_{A,B,k,l}.$$

Notice now that

$$\overline{W}_2 = t_4^{-1}\overline{P_{12}},$$

thus, by symmetry, we have

$$\left(\overline{W_1}W_2\right)^{\mathsf{fix}} = \sum_{k=0}^{C-1}\sum_{l=0}^{D}\delta_{A,B,k,l}.$$

We compute the parity

$$\begin{aligned}\operatorname{rk}(W_1W_2)^{\mathsf{fix}} + \operatorname{rk}(\overline{W_1}W_2)^{\mathsf{fix}} &= \sum_{k=0}^{C}\sum_{l=0}^{D-1}\delta_{A,B,k,l} + \sum_{k=0}^{C-1}\sum_{l=0}^{D}\delta_{A,B,k,l}\\ &= \sum_{l=0}^{D-1}\delta_{A,B,C,l} + \sum_{k=0}^{C-1}\delta_{A,B,k,D} \mod 2.\end{aligned}$$

Notice that

$$\sum_{l=0}^{D-1}\delta_{A,B,C,l} = \begin{cases}1 & D > A = B = C\\ 0 & \text{else}\end{cases}$$

therefore $\sigma_4(\tilde{\pi}) = \sigma_4(\pi) + \sum_{l=0}^{D-1}\delta_{A,B,C,l}$ and $\sigma_3(\tilde{\pi}) = \sigma_3(\pi) + \sum_{k=0}^{C-1}\delta_{A,B,k,D}$, by which we conclude the proof. $\square$

6.6.2. The vertex term: curves Set $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_i are finite plane partitions. Denote by $P_{\underline{\lambda}}$ the collection of (possibly infinite) solid partitions, whose asymptotic profile is given by $\underline{\lambda}$. To any solid partition $\pi \in P_{\underline{\lambda}}$ corresponds a $\mathbf{T}$-invariant closed subscheme $Z \subset \mathbb{C}^4$; denote by $Z_\pi, Z_{\lambda_i}, \mathsf{V}_\pi$ the vertex terms $Z_\alpha, Z_{\alpha\beta_i}, \mathsf{V}_\alpha$ in (6.2.8), (6.2.10), (6.2.13). $\mathbf{T}$-equivariant Serre duality implies that V_π admits a square root; set

$$\mathsf{v}_\pi^i = Z_\pi - \overline{P_{jkl}}Z_\pi\overline{Z_\pi} + \textstyle\sum_{j\neq i, j=1}^{4}\frac{\mathsf{f}_{\lambda_j}^i}{1-t_j} + \frac{1}{(1-t_i)}\left(-Z_{\lambda_i} + \overline{P_{jkl}}\left(\overline{Z_\pi}Z_{\lambda_i} - Z_\pi\overline{Z_{\lambda_i}}\right) + \frac{\overline{P_{jkl}}}{1-t_i}Z_{\lambda_i}\overline{Z_{\lambda_i}}\right),$$

$$\mathsf{f}_{\lambda_j}^i = -Z_{\lambda_j} + \overline{P_{kl}}Z_{\lambda_j}\overline{Z_{\lambda_j}},$$

which enjoy

$$\mathsf{V}_\pi = \mathsf{v}_\pi^i + \overline{\mathsf{v}_\pi^i},$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Setting $i = 4$, we recover the explicit square root choice studied in Section 5.1. We redistribute now the vertex terms in a different way. Write, for $N \gg 0$

$$Z_\pi = Z_{\pi^{\mathrm{nor}}} + \sum_{i=1}^{4}\frac{Z_{\lambda_i}}{1-t_i}t_i^{N+1},$$

$$Z_{\pi_i} = Z_{\lambda_i}\sum_{n=0}^{N}t_i^n,$$

$$\frac{Z_{\lambda_i}}{1-t_i} = Z_{\pi_i} + \frac{Z_{\lambda_i}}{1-t_i} t_i^{N+1}.$$

Here the (point-like) solid partition π^{nor} is the cut-off for $N \gg 0$ of the (possibly curve-like) solid partition π. If π is a point-like solid partition, then simply $\pi^{\mathrm{nor}} = \pi$, while π_i is simply the cut-off for $N \gg 0$ of the curve-like solid partition corresponding to the infinite leg along the x_i-axis containing π. Using the above expressions, we can express the vertex terms as

$$\mathsf{v}^i_\pi = \mathsf{v}^i_{\pi^{\mathrm{nor}}} - \sum_{a=1}^{4} \mathsf{v}^i_{\pi_a} + A^i_\pi + B^i_\pi + C^i_\pi - \overline{C^i_\pi}, \tag{6.6.2}$$

where

$$A^i_\pi = -\overline{P_{jkl}} \sum_{a\neq b, a, b\neq i} \frac{Z_{\lambda_a}}{1-t_a}\frac{\overline{Z_{\lambda_b}}}{1-t_b^{-1}}(t_a t_b^{-1})^{N+1} - \overline{P_{1234}} \sum_{a\neq i} \frac{Z_{\lambda_a}}{1-t_a}\frac{\overline{Z_{\lambda_i}}}{1-t_i^{-1}}(t_a t_i^{-1})^{N+1},$$

$$B^i_\pi = -\overline{P_{jkl}} \sum_{a\neq i} \left(\frac{Z_{\lambda_a}}{1-t_a} t_a^{N+1}(\overline{Z_{\pi^{\mathrm{nor}}}} - \overline{Z_{\pi_a}}) + \frac{\overline{Z_{\lambda_a}}}{1-t_a^{-1}} t_a^{-(N+1)}(Z_{\pi^{\mathrm{nor}}} - Z_{\pi_a}) \right)$$
$$- \overline{P_{1234}} \frac{\overline{Z_{\lambda_i}}}{1-t_i^{-1}} t_i^{-(N+1)}(Z_{\pi^{\mathrm{nor}}} - Z_{\pi_i}),$$

$$C^i_\pi = \overline{P_{123}} \left(Z_{\pi_i}(\overline{Z_{\pi^{\mathrm{nor}}}} - \overline{Z_{\pi_i}}) + \sum_{a\neq i} \frac{\overline{Z_{\lambda_a}}}{1-t_a^{-1}} t_a^{-(N+1)} Z_{\pi_i} \right),$$

where $\{ i, j, k, l \} = \{ 1, 2, 3, 4 \}$. Motivated by the above expression, we define a new square root of V_π

$$\mathsf{v}^{i\,\prime}_\pi = \mathsf{v}^i_\pi - C^i_\pi + \overline{C^i_\pi}.$$

Remark 6.6.7. It is an equivalent problem to find the sign rule for $\mathsf{v}^{i\,\prime}_\pi$ or for v^i_π. In fact, such two sign rules will differ just by $(-1)^{\mathrm{rk}(C^i_\pi)^{\mathsf{mov}}}$, which is completely determined by the solid partition π. Moreover, if the $\mathbf{T}$-invariant closed subscheme $Z \subset \mathbb{C}^4$ corresponding to a solid partition π is not supported in the $\mathbf{T}$-invariant line $\{ x_j = x_k = x_l = 0 \}$, we have that $\mathsf{v}^{i\,\prime}_\pi = \mathsf{v}^i_\pi$.

Lemma 6.6.8. *Let π be a curve-like solid partition and $i = 1, \ldots, 4$. Then $\mathsf{v}^i_\pi, \mathsf{v}^{i\,\prime}_\pi$ are $\mathbf{T}$-movable.*

Proof. By Lemma 6.6.1 $\mathsf{v}^i_{\pi^{\mathrm{red}}}$, $\mathsf{v}^i_{\pi_a}$ are $\mathbf{T}$-movable, for $a = 1, \ldots, 4$. For $N \gg 0$, we clearly have $(A^i_\pi)^{\mathsf{fix}} = (B^i_\pi)^{\mathsf{fix}} = 0$. □

We propose a sign rule for the sign in (6.2.21), relative to the square root $\mathsf{v}^{i\,\prime}_\pi$.

Conjecture 6.6.9. *Let π be a curve-like solid partition. Then the sign relative to the square root $\mathsf{v}^{i\,\prime}_\pi$ is $(-1)^{\sigma_i(\pi)}$, where*

$$\sigma_i(\pi) = |\pi| + \#\{ (a_1, a_2, a_3, a_4) \in \pi \colon a_j = a_k = a_l < a_i \}$$
$$- \sum_{\mathrm{leg}} \#\{ (a_1, a_2, a_3, a_4) \in \mathrm{leg} \colon a_j = a_k = a_l < a_i \} \tag{6.6.3}$$

and $\{i,j,k,l\} = \{1,2,3,4\}$, *where* leg *denote the curve-like solid partitions obtained by translating the plane partitions* λ_i *along the* x_i*-axis.*

Remark 6.6.10. Let π corresponds to a $\mathbf{T}$-invariant closed subscheme $Z \subset \mathbb{C}^4$ supported in the hyperplane $\{x_i = 0\} \subset \mathbb{C}^4$, for $i = 1, \dots, 4$. Then $\sigma_i(\pi) = |\pi|$, which is consistent with the dimensional reduction studied in Section 6.3.1.

We prove now that the sign rule (6.6.3) is canonical, meaning that it does not really depend on choosing a preferred x_i-axis.

Theorem 6.6.11. *Let* π *a curve-like solid partition. For every* $i, j = 1, \dots, 4$ *we have*

$$(-1)^{\sigma_i(\pi)}[-\mathsf{v}_\pi^{i\,\prime}] = (-1)^{\sigma_j(\pi)}[-\mathsf{v}_\pi^{j\,\prime}].$$

Proof. $\mathsf{v}_\pi^{i\,\prime}$ and $\mathsf{v}_\pi^{j\,\prime}$ are both square roots of V_π and are $\mathbf{T}$-movable by Lemma 6.6.8. The difference of the two sections is

$$\mathsf{v}_\pi^{i\,\prime} - \mathsf{v}_\pi^{j\,\prime} = (\mathsf{v}_{\pi^{\mathrm{nor}}}^i - \mathsf{v}_{\pi^{\mathrm{nor}}}^j) - \sum_{a=1}^4 (\mathsf{v}_{\pi_a}^i - \mathsf{v}_{\pi_a}^j) + A_\pi^i - A_\pi^j + B_\pi^i - B_\pi^j.$$

By Theorem 6.6.6, we know that

$$\begin{aligned}
[\mathsf{v}_{\pi^{\mathrm{nor}}}^i - \mathsf{v}_{\pi^{\mathrm{nor}}}^j] &= (-1)^{\sigma_i(\pi^{\mathrm{nor}}) - \sigma_j(\pi^{\mathrm{nor}})},\\
[\mathsf{v}_{\pi_a}^i - \mathsf{v}_{\pi_a}^j] &= (-1)^{\sigma_i(\pi_a) - \sigma_j(\pi_a)},
\end{aligned}$$

which satisfies

$$\sigma_i(\pi) - \sigma_j(\pi) = \sigma_i(\pi^{\mathrm{nor}}) - \sigma_j(\pi^{\mathrm{nor}}) - \sum_{a=1}^4 (\sigma_i(\pi_a) - \sigma_j(\pi_a)).$$

To conclude, with a simple computation it is possible to show that

$$\begin{aligned}
A_\pi^i - A_\pi^j &= U_A - \overline{U_A},\\
B_\pi^i - B_\pi^j &= U_B - \overline{U_B},
\end{aligned}$$

where, for $N \gg 0$, we have $\mathrm{rk}(U_A)^{\mathsf{mov}} = \mathrm{rk}(U_A)^{\mathsf{fix}} = 0$ and $\mathrm{rk}(U_B)^{\mathsf{mov}} = \mathrm{rk}(U_B)^{\mathsf{fix}} = 0$. $\square$

6.6.3. The edge term In this section we study square roots and propose a sign rule for the edge term. For simplicity, let's assume that the edge $\alpha\beta \in E(X)$ corresponds to the $\mathbb{P}^1$ given, in the local coordinates of U_α, by $\{x_2 = x_3 = x_4 = 0\}$, with normal bundle

$$N_{\mathbb{P}^1/X} \cong \mathcal{O}(m_2) \oplus \mathcal{O}(m_3) \oplus \mathcal{O}(m_4),$$

satisfying $m_2 + m_3 + m_4 = -2$; we set $\mathbf{m} = (m_2, m_3, m_4)$. We also fix a plane partition λ, corresponding to the profile of the non-reduced $\mathbf{T}$-fixed leg. The discussion for the legs along the other directions will be completely analogous. Denote here by $Z_\lambda, \mathsf{E}_\lambda$ the edge terms $Z_{\alpha\beta}, \mathsf{E}_{\alpha\beta}$ in (6.2.10), (6.2.13), where

$$Z_\lambda = \sum_{(j,k,l)\in\lambda} t_2^j t_3^k t_4^l.$$

Denote by $\widetilde{(\cdot)} : K^0_{\mathbf{T}}(\mathsf{pt}) \to K^0_{\mathbf{T}}(\mathsf{pt})$ the map sending

$$V \mapsto V(t_1^{-1}, t_2 t_1^{-m_2}, t_3 t_1^{-m_3}, t_4 t_1^{-m_4}).$$

The edge term admits a square root; set

$$\begin{aligned}\mathsf{e}^j_\lambda &= t_1^{-1}\frac{\mathsf{f}^j_\lambda}{1-t_1^{-1}} - \frac{\widetilde{\mathsf{f}^j_\lambda}}{1-t_1^{-1}},\\ \mathsf{f}^j_\lambda &= -Z_\lambda + \overline{P_{kl}} Z_\lambda \overline{Z_\lambda},\end{aligned}$$

which enjoys

$$\mathsf{E}_\lambda = \mathsf{e}^j_\lambda + \overline{\mathsf{e}^j_\lambda},$$

where $\{ j, k, l \} = \{ 2, 3, 4 \}$.

Lemma 6.6.12. *Let λ be a plane partition and $j = 2, 3, 4$. Then e^j_λ is $\mathbf{T}$-movable.*

Proof. Without loss of generality, suppose that $j = 4$. Write $\mathsf{f}^4_\lambda = \sum_\nu t^\nu$; we have

$$\frac{1}{1-t_1^{-1}}(t_1^{-1}t^\nu - t^\nu t_1^{-\mathbf{m}\nu}) = \begin{cases} -t^\nu \sum_{i=0}^{-\mathbf{m}\nu} t_1^i & \mathbf{m}\nu \leq 0,\\ 0 & \mathbf{m}\nu = 1,\\ t^\nu t_1^{-1}\sum_{i=0}^{\mathbf{m}\nu-2} t_1^{-i} & \mathbf{m}\nu \geq 2\end{cases} \tag{6.6.4}$$

where $\mathbf{m}\cdot\nu$ denotes the standard scalar product in $\mathbb{Z}^3$. Therefore the contribution to the $\mathbf{T}$-fixed part of each t^ν is

$$\operatorname{rk}\left(\frac{1}{1-t_1^{-1}}(t_1^{-1}t^\nu - t^\nu t_1^{-\mathbf{m}\nu})\right)^{\mathsf{fix}} = \begin{cases} -\sum_{i=0}^{-\mathbf{m}\nu} \delta_{i,\nu_2,\nu_3,\nu_4} & \mathbf{m}\nu \leq 0,\\ 0 & \mathbf{m}\nu = 1,\\ \sum_{i=1-\mathbf{m}\nu}^{-1} \delta_{i,\nu_2,\nu_3,\nu_4} & \mathbf{m}\nu \geq 2\end{cases} \tag{6.6.5}$$

$$= \begin{cases} -1 & \nu_2 = \nu_3 = \nu_4 \geq 0,\\ 1 & \nu_2 = \nu_3 = \nu_4 \leq -1,\\ 0 & \text{else.}\end{cases} \tag{6.6.6}$$

Denote by W_l the sub-representation of f^4_λ corresponding to the irreducible representation $(t_2t_3t_4)^l$, for $l \in \mathbb{Z}$. Equation (6.6.5) translates into

$$\operatorname{rk}\sum_\nu \left(\frac{1}{1-t_1^{-1}}(t_1^{-1}t^\nu - t^\nu t_1^{-\mathbf{m}\nu})\right)^{\mathsf{fix}} = \sum_{l\geq 0}(\operatorname{rk} W_{-l-1} - \operatorname{rk} W_l)\,.$$

Notice that $\mathsf{f}^4_\lambda - \overline{\mathsf{f}^4_\lambda}(t_2t_3t_4)^{-1}$ is the 3-fold vertex of [**127**, Eqn. (12)] in the variables t_2, t_3, t_4, which is $\mathbf{T}_0$-movable for $\mathbf{T}_0 = \{ t_2t_3t_4 = 1 \} \subset (\mathbb{C}^*)^3$ (cf. [**127**, pag. 1279]). This implies that for any $l \in \mathbb{Z}$

$$\operatorname{rk} W_l = \operatorname{rk} W_{-l-1},$$

by which we conclude the proof. □

As a corollary, we prove that the obstruction theory induced on the $\mathbf{T}$-fixed locus is trivial.

Corollary 6.6.13. *Let X be a toric Calabi-Yau 4-fold and $\beta \in H_2(X,\mathbb{Z})$. Then the induced obstruction theory on* $\mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}$ *is trivial. In particular, for* $Z \in \mathrm{Hilb}^n(X,\beta)^{\mathbf{T}}$, *the virtual tangent space* T_Z^{vir} *is* $\mathbf{T}$*-movable.*

Proof. By Lemma 6.6.8, 6.6.12 there exist $\mathbf{T}$-movable square roots $\mathsf{v}_\alpha, \mathsf{e}_{\alpha\beta}$ of $\mathsf{V}_\alpha, \mathsf{E}_{\alpha\beta}$ for any $\alpha \in V(X), \alpha\beta \in E(X)$, which implies that also $\mathsf{V}_\alpha, \mathsf{E}_{\alpha\beta}$ and T_Z^{vir} are $\mathbf{T}$-movable by (6.2.14) (see [**35**, Prop. 2.11]). We have an identity in $\mathbf{T}$-equivariant K-theory

$$T_Z^{\mathrm{vir}} = \mathrm{Ext}^1(I_Z,I_Z) - \mathrm{Ext}^2(I_Z,I_Z) + \mathrm{Ext}^3(I_Z,I_Z) \in K^0_{\mathbf{T}}(\mathsf{pt}).$$

It was proven in [**35**, Lemma 2.2] that $\mathrm{Ext}^1(I_Z,I_Z)^{\mathbf{T}} = \mathrm{Ext}^3(I_Z,I_Z)^{\mathbf{T}} = 0$, by which we conclude that $\mathrm{Ext}^2(I_Z,I_Z)^{\mathbf{T}} = 0$ as well. □

We propose a sign rule for the sign in (6.2.21), relative to the square root e^i_λ.

Conjecture 6.6.14. *Let λ be a plane partition. Then the sign relative to the square root e^i_λ is $(-1)^{\sigma_i(\lambda)}$, where*

$$\sigma_i(\lambda) = f_{\mathbf{m}}(\lambda) + |\lambda| m_i + \#\{\,(a_2,a_3,a_4) \in \lambda \colon a_j = a_k < a_i\,\}, \tag{6.6.7}$$

and $\{\,i,j,k\,\} = \{\,2,3,4\,\}$.

We prove now that the sign rule (6.6.7) is canonical, meaning that it does not really depend on choosing a preferred x_i-axis.

Theorem 6.6.15. *Let λ be a plane partition. For every $i,j = 2,3,4$ we have*

$$(-1)^{\sigma_i(\lambda)}[-\mathsf{e}^i_\lambda] = (-1)^{\sigma_j(\lambda)}[-\mathsf{e}^j_\lambda].$$

Proof. Without loss of generality assume $i=4, j=3$. Say, for $k \in \mathbb{Z}$,

$$A(k) = \begin{cases} -\sum_{i=0}^{-k} t_1^i & k \leq 0, \\ 0 & k = 1, \\ t_1^{-1}\sum_{i=0}^{k-2} t_1^{-i} & k \geq 2 \end{cases}$$

and, for a $\mathbf{T}$-representation V,

$$B(V) = \sum_{\nu \in V} t^\nu A(\mathbf{m}\nu) \in K^0_{\mathbf{T}}(\mathsf{pt}),$$

where the sum is over the weight spaces of V. We extend the definition of $B(V)$ by linearity to $K^0_{\mathbf{T}}(\mathsf{pt})$. By (6.6.4), we have

$$\mathsf{e}^4_\lambda - \mathsf{e}^3_\lambda = B(\mathsf{f}^4_\lambda - \mathsf{f}^3_\lambda),$$

Notice the decomposition

$$\begin{aligned} \mathsf{f}^4_\lambda - \mathsf{f}^3_\lambda &= W_\lambda + \overline{W_\lambda}(t_2t_3t_4)^{-1}, \\ W_\lambda &= Z_\lambda \overline{Z_\lambda}(t_4^{-1} - t_3^{-1}). \end{aligned}$$

Then

$$\begin{aligned} \mathsf{e}^4_\lambda - \mathsf{e}^3_\lambda &= B(W_\lambda) + B(\overline{W_\lambda}(t_2t_3t_4)^{-1}) \\ &= B(W_\lambda) - \overline{B(W_\lambda)}, \end{aligned}$$

by which we conclude that

$$e^{\mathbf{T}}(\mathsf{e}^4_\lambda - \mathsf{e}^3_\lambda) = (-1)^{\operatorname{rk} B(W_\lambda)^{\mathsf{mov}}}.$$

We compute the parity of $\operatorname{rk} B(W_\lambda)^{\mathsf{mov}}$ by induction on the size of λ. If $|\lambda| = 1$, we clearly have

$$\operatorname{rk} B(W_\lambda)^{\mathsf{mov}} = m_4 + m_3 \mod 2.$$

Suppose now that the claim holds for all plane partition of size $|\lambda| \leq n$ and consider a plane partition $\tilde{\lambda}$ of size $|\tilde{\lambda}| = n+1$; this can be seen as a plane partition λ of size n with an extra box over it, corresponding to a $\mathbb{Z}^3$-lattice point $\mu = (A, B, C)$. We have

$$\begin{aligned} B(W_{\tilde{\lambda}}) &= B(W_\lambda) + B(Y_4) - B(Y_3) + B(t_4^{-1} - t_3^{-1}), \\ Y_i &= t_i^{-1}(Z_\lambda t^{-\mu} + \overline{Z_\lambda} t^{\mu}) \quad i = 3, 4, \end{aligned}$$

and by the inductive step

$$\begin{aligned} \operatorname{rk} B(W_\lambda)^{\mathsf{mov}} &= \sigma_4(\lambda) - \sigma_3(\lambda) \mod 2, \\ \operatorname{rk} B(t_4^{-1} - t_3^{-1})^{\mathsf{mov}} &= m_4 - m_3 \mod 2. \end{aligned}$$

Clearly, $\operatorname{rk} B(Y_4)^{\mathsf{mov}} = \operatorname{rk} B(Y_4)^{\mathsf{fix}} \mod 2$. In fact,

$$\begin{aligned} \operatorname{rk} B(Y_4) &= \sum_{\nu \in Z_\lambda} (\mathbf{m}(\mu - \nu + (0,0,-1)) + \mathbf{m}(\nu - \mu + (0,0,-1))) \\ &= -2m_4|\lambda|. \end{aligned}$$

A simple analysis of $B(Y_4)^{\mathsf{fix}}$ as in (6.6.5) yields

$$\begin{aligned} \operatorname{rk}(B(Y_4))^{\mathsf{fix}} = \#\{ (\nu \in \lambda \colon A - \nu_2 = B - \nu_3 = C - \nu_4 + 1 \} \\ - \#\{ \nu \in \lambda \colon A - \nu_2 = B - \nu_3 = C - \nu_4 - 1 \}, \end{aligned}$$

where $\nu = (\nu_2, \nu_3, \nu_4)$; in particular, it has to satisfy $\nu \leq \mu$. Therefore we can write it as

$$\begin{aligned} \operatorname{rk}(B(Y_4))^{\mathsf{fix}} &= \sum_{i=0}^{A} \sum_{j=0}^{B} \sum_{k=0}^{C} (\delta_{A-i,B-j,C-k+1} - \delta_{A-i,B-j,C-k-1}) \\ &= \sum_{i=0}^{A} \sum_{j=0}^{B} \left(\sum_{k=-1}^{C-1} \delta_{A-i,B-j,C-k} - \sum_{k=1}^{C+1} \delta_{A-i,B-j,C-k} \right) \end{aligned}$$

By symmetry we may compute the difference

$$\begin{aligned} \operatorname{rk}(B(Y_4) - B(Y_3))^{\mathsf{fix}} = \sum_{i=0}^{A} \Bigg(\sum_{k=-1}^{C-1} \delta_{A-i,0,C-k} + \sum_{j=0}^{B-1} \delta_{A-i,B-j,C+1} - \sum_{j=-1}^{B-1} \delta_{A-i,B-j,0} \\ - \sum_{k=0}^{C-1} \delta_{A-i,B+1,C-k} - \sum_{k=1}^{C+1} \delta_{A-i,B,C-k} - \sum_{j=1}^{B} \delta_{A-i,B-j,-1} + \sum_{j=1}^{B+1} \delta_{A-i,B-j,C} + \sum_{k=1}^{C} \delta_{A-i,-1,C-k} \Bigg) \end{aligned}$$

Further analyzing which of these sums actually contribute to the rank, we finally get that

$$\mathrm{rk}(B(Y_4) - B(Y_3))^{\mathsf{fix}} = \begin{cases} 1 & A = B < C, \\ 1 & A = C < B, \\ 0 & \text{else.} \end{cases} \quad \mod 2$$

Therefore we conclude that

$$\mathrm{rk}\, B(W_{\tilde{\lambda}})^{\mathsf{mov}} = \sigma_4(\tilde{\lambda}) - \sigma_3(\tilde{\lambda}) \mod 2,$$

which finishes the inductive step. □

Remark 6.6.16. Let $X = K_Y$ be the canonical bundle of a smooth projective toric 3-fold Y and consider $\mathrm{Hilb}^n(X, \beta)$, where $\beta \in H_2(X, \mathbb{Z})$ is a class pulled-back from Y. Consider a $\mathbf{T}$-fixed point $Z \in \mathrm{Hilb}^n(X, \beta)^{\mathbf{T}}$ (corresponding to a partition data $\{\pi_\alpha, \lambda_{\alpha\beta}\}_{\alpha,\beta}$) scheme-theoretically supported on the zero section of $X \to Y$. Locally on the toric charts, label the fiber direction by x_4 and denote by $m''_{\alpha\beta}$ the degree of the normal bundle of $L_{\alpha\beta}$ in the x_4-direction. Consider the square root of T^{vir}_Z given by

$$\mathsf{v}_Z = \sum_{\alpha \in V(X)} \mathsf{v}^4_\alpha + \sum_{\alpha\beta \in E(X)} \mathsf{e}^4_{\alpha\beta}.$$

By the dimensional reduction of Section 6.3.1, the sign rules proposed for vertex and edge terms imply that the correct sign would be

$$\begin{aligned}
(-1)^{\sigma(Z, \mathsf{v}_Z)} &= \prod_{\alpha \in V(X)} (-1)^{\sigma_4(\pi_\alpha)} \cdot \prod_{\alpha\beta \in E(X)} (-1)^{\sigma_4(\lambda_{\alpha\beta})} \\
&= \prod_{\alpha \in V(X)} (-1)^{|\pi_\alpha|} \cdot \prod_{\alpha\beta \in E(X)} (-1)^{|\lambda_{\alpha\beta}| m''_{\alpha\beta} + f_{\mathbf{m}_{\alpha\beta}}(\lambda_{\alpha\beta})} \\
&= (-1)^{n + c_1(Y) \cdot \beta},
\end{aligned}$$

where the last equality follows from (6.2.6) and

$$\begin{aligned}
\sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}| m''_{\alpha\beta} &= - \sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}| \deg N_{Y/X}|_{L_{\alpha\beta}} \\
&= - \sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}| \deg K_Y|_{L_{\alpha\beta}} \\
&= \sum_{\alpha\beta \in E(X)} |\lambda_{\alpha\beta}| c_1(T_Y) \cdot [L_{\alpha\beta}] \\
&= c_1(Y) \cdot \beta.
\end{aligned}$$

The same sign was proposed in a similar setting for stable pair invariants [**37**, Prop. 4.2, Rmk. A.2], where such local geometries are studied, motivated by a choice of preferred orientation as in [**31**].

6.6.4. The vertex term: PT The analysis of the vertex term in PT theory — which makes sense only for *curves* — is trickier, but could in principle be carried in an analogues fashion as in here. This means that we could find — at least conjecturally — *canonical* choices of *square roots* with *sign rules*, and prove an analogue of Theorem 6.6.11. We just conjecture here a sign rule, inspired by Conjecture 6.6.9 and the DT/PT correspondence.

Conjecture 6.6.17. *Let B be a box configuration in PT theory with underlying solid partition π as in Section 6.2.2, with at most two non-empty profile plane partition λ_1, λ_2. Then the sign relative to the square root $\mathsf{v}_B^{\mathsf{PT}}$ (as in Section 6.2.6) is $(-1)^{\sigma(B)}$, where*

$$\sigma(B) = |\pi| + |B| + \#\{\,(a,a,a,d) \in B\colon a < d\,\} + \#\{\,(a,a,a,d) \in \pi\colon a < d\,\} - \sum_{\mathrm{leg}} \#\{\,(a,a,a,d) \in \mathrm{leg}\colon a < d\,\}, \tag{6.6.8}$$

where leg *denote the curve-like solid partitions obtained by translating the plane partitions λ_i along the x_i-axis.*

This sign rule has been verified in all the computations of this thesis.

CHAPTER 7

Stable pair invariants of local Calabi-Yau 4-folds

Desde aquel día
no he movido las piezas
en el tablero.

Haiku, J.L. Borges

7.1. Introduction

7.1.1. GW/GV invariants of Calabi-Yau 4-folds Gromov-Witten invariants are rational numbers, which are virtual counts of stable maps from curves to a fixed algebraic variety. Due to multiple cover contributions, they are in general not integers. For Calabi-Yau 4-folds, Klemm-Pandharipande [**108**] defined Gopakumar-Vafa type invariants using Gromov-Witten theory and conjectured their integrality. More specifically, let X be a smooth projective Calabi-Yau 4-fold. Gromov-Witten invariants vanish for genus $g \geqslant 2$ for dimensional reasons and one only needs to consider the genus zero and one cases.

The genus zero Gromov-Witten invariants of X for class $\beta \in H_2(X, \mathbb{Z})$ are defined using an insertion. Consider the evaluation map $\mathrm{ev}\colon \overline{M}_{0,1}(X, \beta) \to X$. For $\gamma \in H^4(X, \mathbb{Z})$, one defines

$$\mathrm{GW}_{0,\beta}(\gamma) = \int_{[\overline{M}_{0,1}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}^*(\gamma).$$

The genus zero Gopakumar-Vafa type invariants

$$n_{0,\beta}(\gamma) \in \mathbb{Q} \tag{7.1.1}$$

are defined in [**108**] by the identity

$$\sum_{\beta>0} \mathrm{GW}_{0,\beta}(\gamma)\, q^\beta = \sum_{\beta>0} n_{0,\beta}(\gamma) \sum_{d=1}^{\infty} d^{-2} q^{d\beta},$$

where the sum is over all non-zero effective classes in $H_2(X,\mathbb{Z})$. For the genus one case, the virtual dimension of $\overline{M}_{1,0}(X,\beta)$ is zero and one defines

$$\mathrm{GW}_{1,\beta} = \int_{[\overline{M}_{1,0}(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

The genus one Gopakumar-Vafa type invariants

$$n_{1,\beta} \in \mathbb{Q} \tag{7.1.2}$$

are defined in [**108**] by the identity

$$\sum_{\beta>0} \mathrm{GW}_{1,\beta}\, q^{\beta} = \sum_{\beta>0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d}\, q^{d\beta} + \frac{1}{24}\sum_{\beta>0} n_{0,\beta}(c_2(X))\log(1-q^{\beta})$$
$$-\frac{1}{24}\sum_{\beta_1,\beta_2} m_{\beta_1,\beta_2}\log(1-q^{\beta_1+\beta_2}),$$

where $\sigma(d) = \sum_{i|d} i$ and $m_{\beta_1,\beta_2} \in \mathbb{Z}$ are called meeting invariants, which can be inductively determined by the genus zero Gromov-Witten invariants of X. In [**108**], both of the invariants (7.1.1), (7.1.2) are conjectured to be integers. Using localization techniques and mirror symmetry, they calculate the Gromov-Witten invariants of X in numerous examples in support of their integrality conjecture. The genus zero integrality conjecture has been proved by Ionel-Parker using symplectic geometry [**97**, Thm. 9.2].

7.1.2. Stable pair invariants of Calabi-Yau 4-folds Stable pairs were introduced in general by Le Potier [**121**] and used by Pandharipande-Thomas to define virtual invariants of smooth projective threefolds [**152, 155, 153**]. Stable pair invariants of threefolds are related to Gromov-Witten invariants by the celebrated GW/PT correspondence [**127, 152**], which has been proved in many cases by Pandharipande-Pixton [**159, 158**].

In [**40**], Cao-Maulik-Toda studied stable pair theory of a smooth projective Calabi-Yau 4-fold X. They used stable pair invariants of X to give a sheaf theoretical interpretation of the Gopakumar-Vafa type invariants (7.1.1) and (7.1.2)[1].

Let $P_n(X,\beta)$ be the moduli space of stable pairs $[\mathcal{O}_X \xrightarrow{s} F]$ with $\mathrm{ch}(F) = (0,0,0,\beta,n)$. By the results of Section 5.2, there exists a (real[2]) virtual class

$$[P_n(X,\beta)]^{\mathrm{vir}} \in H_{2n}\big(P_n(X,\beta),\mathbb{Z}\big), \tag{7.1.3}$$

in the sense of Borisov-Joyce [**23**], which depends on the choice of an orientation of a certain (real) line bundle over $P_n(X,\beta)$ [**32**]. For $\gamma \in H^4(X,\mathbb{Z})$, we define primary insertions

$$\tau\colon H^4(X,\mathbb{Z}) \to H^2(P_n(X,\beta),\mathbb{Z}), \quad \tau(\gamma) = \pi_{P*}(\pi_X^*\gamma \cup \mathrm{ch}_3(\mathbb{F})),$$

where π_X, π_P are projections from $X \times P_n(X,\beta)$ to the corresponding factors and

$$\mathbb{I}^\bullet = [\mathcal{O} \to \mathbb{F}]$$

[1] In [**39, 44**], the authors also proposed a sheaf theoretical interpretation of (7.1.1), (7.1.2) using Donaldson-Thomas type counting invariants of one dimensional stable sheaves on X.

[2] This class can be constructed *algebraically* — again by the results of Section 5.2 — if we invert 2.

is the universal stable pair on $X \times P_n(X,\beta)$. Note that $\mathrm{ch}_3(\mathbb{F})$ is Poincaré dual to the fundamental cycle of $\mathbb{F}$. The stable pair invariants of X with primary insertions are defined by

$$P_{n,\beta}(\gamma) := \int_{[P_n(X,\beta)]^{\mathrm{vir}}} \tau(\gamma)^n. \tag{7.1.4}$$

When $n = 0$, we simply denote this invariant by $P_{0,\beta}$. We set $P_{0,0} := 1$ and $n_{0,0}(\gamma) := 0$.

Conjecture 7.1.1 (Cao-Maulik-Toda [40])**.** *Let X be a smooth projective Calabi-Yau 4-fold, $\beta \in H_2(X,\mathbb{Z})$, $\gamma \in H^4(X,\mathbb{Z})$, and $n \geqslant 1$. Then there exist choices of orientations such that*

$$P_{n,\beta}(\gamma) = \sum_{\substack{\beta_0+\beta_1+\cdots+\beta_n=\beta \\ \beta_0,\beta_1,\ldots,\beta_n \geqslant 0}} P_{0,\beta_0} \cdot \prod_{i=1}^{n} n_{0,\beta_i}(\gamma),$$

where the sum is over all effective decompositions of β.

Conjecture 7.1.2 (Cao-Maulik-Toda [40])**.** *Let X be a smooth projective Calabi-Yau 4-fold. Then there exist choices of orientations such that*

$$\sum_{\beta \geqslant 0} P_{0,\beta}\, q^\beta = \prod_{\beta > 0} M\left(q^\beta\right)^{n_{1,\beta}},$$

where $M(q) = \prod_{k \geqslant 1}(1-q^k)^{-k}$ denotes the MacMahon function.

Conjecture 7.1.1 can be interpreted as a wall-crossing formula in the category of D0-D2-D8 bound states in Calabi-Yau 4-folds [42], while Conjecture 7.1.2 seems to be more mysterious. In [40], these conjectures were verified in the following cases (modulo some minor assumptions in some of the cases). In each case, Conjecture 7.1.1 was only verified for $n = 1$.

- X is a general sextic and $\beta = [\ell], 2[\ell]$, where $\ell \subseteq X$ is a line.
- X is a Weierstrass elliptic fibration and $\beta = r[F]$, where $[F]$ is the fibre class and $r > 0$ (in the case of Conjecture 7.1.1 only for $r = 1$).
- $X = Y \times E$, where Y is a smooth projective Calabi-Yau threefold, E is an elliptic curve, and β is the push-forward of an irreducible class on $Y \times \{\mathrm{pt}\}$.
- $X = Y \times E$, where Y is a smooth projective Calabi-Yau threefold, E is an elliptic curve, and $\beta = r[E]$, where $[E]$ is the fibre class and $r > 0$ (Conjecture 7.1.2 only).

When X is either the total space of a smooth projective Fano threefold, or the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$ on $\mathbb{P}^2$, or $\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, the moduli spaces $P_n(X,\beta)$ are projective and it makes sense to consider Conjectures 7.1.1 and 7.1.2. In this setting, the conjectures were verified in some cases for irreducible curve classes in [40].

One of the main goals of this chapter is to provide more verifications for these local geometries for more general low degree curve classes.

7.1.3. Stable pair invariants of local surfaces Let S be a smooth projective surface and let L_1, L_2 be two line bundles on S satisfying $L_1 \otimes L_2 \cong K_S$. Then the total space X of $L_1 \oplus L_2$ over S is a non-proper Calabi-Yau 4-fold, which we refer to as a *local surface*. Consider the moduli space $P_n(X,\beta)$ of stable pairs (F,s) with $\chi(F) = n$ and such that F has proper scheme theoretic support in class $\beta \in H_2(X,\mathbb{Z})$. Although $P_n(X,\beta)$ is in general non-proper, it can be proper in several interesting cases (Propositions 7.2.1, 7.2.8). Then we can define virtual classes (7.1.3) and corresponding stable pair invariants (7.1.4).

Example 7.1.3. For $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$ and $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, the moduli space $P_n(X,\beta)$ is projective *for all* n, β (see Proposition 7.2.1).

Example 7.1.4. For $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,0), \mathcal{O}(-1,-2))$, $P_n(X,\beta)$ is in general non-proper. E.g. let $H_1 := \{\mathrm{pt}\} \times \mathbb{P}^1 e$, take $\beta = [H_1]$, and $n = \chi(\mathcal{O}_{H_1}) = 1$. Then $N_{H_1/X} \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-2)$ has sections in the first fibre direction, so $H_1 \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \subseteq X$ can move off the zero section $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq X$ and $P_1(X, [H_1])$ is non-proper. On the other hand, for $H_2 := \mathbb{P}^1 \times \{\mathrm{pt}\}$ and $\beta = [H_2]$, we have $\beta \cdot L_1 < 0$ and $\beta \cdot L_2 < 0$, so $P_1(X, [H_2])$ is projective by Proposition 7.2.1.

When S is toric, the local surface X is toric and the vertex formalism for calculating stable pair invariants of X has been developed in [**35, 36**] in analogy with [**155**]. Let $\mathbf{T} \subseteq (\mathbb{C}^*)^4$ denote the 3-dimensional subtorus preserving the Calabi-Yau volume form, then the fixed locus $P_n(X,\beta)^{\mathbf{T}}$ consists of finitely many isolated reduced points [**35**, Sec. 2.2], though the number of fixed points is typically very large making calculations using the vertex formalism cumbersome.

Although we perform a few new calculations using the vertex formalism as well, we mainly focus on another approach, where we use the global geometry of S. We consider the case when all stable pairs on X are scheme theoretically supported on the zero section $\iota : S \hookrightarrow X$, i.e. we have an isomorphism

$$\iota_* : P_n(S,\beta) \cong P_n(X,\beta).$$

Under this isomorphism, we have (Proposition 7.3.2)

$$[P_n(X,\beta)]^{\mathrm{vir}} = (-1)^{\beta \cdot L_2 + n} \cdot e\big(-\mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1)\big) \cdot [P_n(S,\beta)]^{\mathrm{vir}}, \tag{7.1.5}$$

where $[P_n(S,\beta)]^{\mathrm{vir}}$ is the virtual class of the pairs obstruction theory on S, $e(\cdot)$ denotes Euler class, $\pi_{P_S} \colon S \times P_n(S,\beta) \to P_n(S,\beta)$ is the projection, $\mathbf{R}\mathscr{H}om_{\pi_{P_S}} = \mathbf{R}\pi_{P_S *} \circ \mathbf{R}\mathscr{H}om$, and $\mathbb{F}$ is the universal 1-dimensional sheaf on $S \times P_n(S,\beta)$. The sign $(-1)^{\beta \cdot L_2 + n} = (-1)^{\beta \cdot c_1(Y) + n}$, where $Y = \mathrm{Tot}_S(L_1)$, comes from a preferred choice of orientation on $P_n(X,\beta)$ which was discussed in a similar situation in [**31**].

In order to use (7.1.5) for calculations, we need the fact that $P_n(S,\beta)$ is isomorphic to a relative Hilbert scheme. More precisely, assume $b_1(S) = 0$ and denote by $|\beta|$ the linear system determined by β. Denote by $\mathcal{C} \to |\beta|$ the universal curve, then [**153**, Prop. B.8] gives

$$P_n(S,\beta) \cong \mathrm{Hilb}^m(\mathcal{C}/|\beta|),$$

where $\mathrm{Hilb}^m(\mathcal{C}/|\beta|)$ denotes the relative Hilbert scheme of m points on the fibres of $\mathcal{C} \to |\beta|$ and

$$m = n + g(\beta) - 1 = n + \tfrac{1}{2}\beta(\beta + K_S).$$

This isomorphism was exploited in order to determine the surface contribution to stable pair invariants of local surfaces $\mathrm{Tot}_S(K_S)$ in [**118**]. The relative Hilbert scheme $\mathrm{Hilb}^m(\mathcal{C}/|\beta|)$ is an incidence locus in a smooth ambient space

$$\mathrm{Hilb}^m(\mathcal{C}/|\beta|) \subseteq S^{[m]} \times |\beta|,$$

where $S^{[m]}$ denotes the Hilbert scheme of m points on S. More precisely, $\mathrm{Hilb}^m(\mathcal{C}/|\beta|)$ is cut out tautologically by a section of a vector bundle on $S^{[m]} \times |\beta|$ as we recall in Section 7.3.1. This allows us to express the stable pair invariants of X in terms of intersection numbers on $S^{[m]} \times |\beta|$, or more precisely, on the "virtual" ambient space $S^{[m]} \times \mathbb{P}^{\chi(\beta)-1}$, where

$$\chi(\beta) := \chi(\mathcal{O}_S(\beta)).$$

In what follows, $\mathcal{Z} \subseteq S \times S^{[m]}$ denotes the universal subscheme and $\mathcal{I}$ is the corresponding ideal sheaf. For any line bundle $\mathcal{L}$ on S, the corresponding tautological bundle is defined by

$$\mathcal{L}^{[m]} := p_* q^* \mathcal{L},$$

where $p : \mathcal{Z} \to S^{[m]}$ and $q : \mathcal{Z} \to S$ are projections. Moreover, we consider the *twisted tangent bundle* [**46**]

$$T_{S^{[m]}}(\mathcal{L}) := \mathbf{R}\Gamma(S, \mathcal{L}) \otimes \mathcal{O} - \mathbf{R}\mathscr{H}om_\pi(\mathcal{I}, \mathcal{I} \boxtimes \mathcal{L}), \tag{7.1.6}$$

where $\pi : S \times S^{[m]} \to S^{[m]}$ denotes projection. Finally, we denote the total Chern class by c and the tautological line bundle on $\mathbb{P}^{\chi(\beta)-1}$ by $\mathcal{O}(1)$. We prove the following result (Theorem 7.3.4).

Theorem 7.1.5. *Let S be a smooth projective surface with $b_1(S) = p_g(S) = 0$ and $L_1, L_2 \in \mathrm{Pic}(S)$ such that $L_1 \otimes L_2 \cong K_S$. Suppose $\beta \in H_2(S, \mathbb{Z})$ and $n \geqslant 0$ are chosen such that $P_n(X, \beta) \cong P_n(S, \beta)$ for $X = \mathrm{Tot}_S(L_1 \oplus L_2)$. Denote by $[\mathrm{pt}] \in H^4(X, \mathbb{Z})$ the pull-back of the Poincaré dual of the point class on S. Let $P_n(X, \beta)$ be endowed with the orientation as in (7.1.5). Then*

$$P_{n,\beta}([\mathrm{pt}]) = (-1)^{\beta \cdot L_2 + n} \int_{S^{[m]} \times \mathbb{P}^{\chi(\beta)-1}} c_m(\mathcal{O}_S(\beta)^{[m]}(1)) \, \frac{h^n (1+h)^{\chi(L_1(\beta))} (1-h)^{\chi(L_2(\beta))} \, c(T_{S^{[m]}}(L_1))}{c(L_1(\beta)^{[m]}(1)) \cdot c((L_2(\beta)^{[m]}(1))^\vee)},$$

when $\beta^2 \geqslant 0$. Here $m := n + g(\beta) - 1$ and $h := c_1(\mathcal{O}(1))$. Moreover, $P_{n,\beta}([\mathrm{pt}]) = 0$ when $\beta^2 < 0$.

The main assumption in this theorem is $P_n(X, \beta) \cong P_n(S, \beta)$. For (S, L_1, L_2) with S minimal and toric, $L_1 \otimes L_2 \cong K_S$ with L_1^{-1}, L_2^{-1} non-trivial and nef, we classify all cases for which $n \geqslant 0$, $P_n(X, \beta) \cong P_n(S, \beta)$, and $P_n(S, \beta)$ is non-empty (Proposition 7.2.9, Remark 7.2.10). Note that $P_n(X, \beta) \cong P_n(S, \beta)$ more or less forces $p_g(S) = 0$, because as soon as L_1 or L_2 has non-zero sections this isomorphism does not hold. See Remark 7.3.5 for an extension to the case $b_1(S) > 0$.

7.1.4. Verifications In this chapter, we apply Theorem 7.1.5 to examples for which S is in addition toric[3]. Then the integrals on $S^{[m]}$ of Theorem 7.1.5 can be calculated using Atiyah-Bott localization for the lift of the 2-dimensional torus action from S to $S^{[m]}$ as described in Section 7.3.4. This leads to the tables for stable pair invariants in Appendix 7.4.

Denote by $[H] \in H_2(\mathbb{P}^2, \mathbb{Z})$ the class of a line and let $[H_1], [H_2] \in H_2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ be as in Example 7.1.4. In [**108**, Sect. 3], Klemm-Pandharipande determined the Gromov-Witten invariants of $X = \mathrm{Tot}_S(L_1 \oplus L_2)$ for $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$ and $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$. They tabulated the corresponding values of the Gopakumar-Vafa type invariants for $\beta = d[H]$ with $d \leqslant 10$ resp. $\beta = d_1[H_1] + d_2[H_2]$ with $d_1, d_2 \leqslant 6$. Combining their calculations and the tables in Appendix 7.4, we deduce the following:

Corollary 7.1.6. *In the following cases, Conjectures 7.1.1 and 7.1.2 are true for* $X = \mathrm{Tot}_S(L_1 \oplus L_2)$.

- $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$, $d = 1$, *and any* $n \geqslant 0$.
- $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$, $d = 2, 3, 4$, *and* $n = 0, 1$.
- $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$, $d = 2, 3$, *and* $n = 2$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, $(d_1, d_2) = (1,0), (0,1), (1,1)$, *any* $n \geqslant 0$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, $(d_1, d_2) = (0,d), (d,0)$ *with* $d \geqslant 2$, *and* $0 \leqslant n \leqslant d$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, $(d_1, d_2) = (1,d), (d,1)$ *with* $d \geqslant 2$, *and* $n = 0, 1, 2$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, $(d_1, d_2) = (2,2)$, $(2,3)$, $(3,2)$, $(2,4)$, $(4,2)$, $(3,3)$, *and* $n = 0$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, $(d_1, d_2) = (2,2), (2,3), (3,2)$, *and* $n = 1$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, $(d_1, d_2) = (2,2)$, *and* $n = 2$.

Remark 7.1.7. In all these cases $P_n(X, \beta) \cong P_n(S, \beta)$. In fact, these are *all* (S, L_1, L_2) with $L_1 \otimes L_2 \cong K_S$ for which L_1^{-1}, L_2^{-1} are ample, $n \geqslant 0$, and $P_n(X, \beta) \cong P_n(S, \beta)$ by Propositions 7.2.2 and 7.2.9. Calculations based on Theorem 7.1.5 are often more efficient than the vertex formalism [**35, 36**]. For instance, for $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, $(d_1, d_2) = (2,4)$ and $n = 0$, $P_n(X, \beta)$ has 182 $\mathbf{T}$-fixed points, whereas Theorem 7.1.5 only involves an integral over $S^{[2]} \times \mathbb{P}^{14}$.

Bousseau-Brini-van Garrel [**24**] recently determined the genus zero Gromov-Witten (and hence Gopakumar-Vafa type) invariants of several local surfaces for their verifications of the log-local principle conjectured in general in [**183**]. Combining their numbers with the tables for stable pair invariants in Appendix 7.4 allows us to provide some further verifications of Conjecture 7.1.1 as we will now describe. For any $a \geqslant 1$,

[3]To our knowledge, all local surfaces which are Calabi-Yau 4-folds and for which Gopakumar-Vafa type invariants have been calculated so far are toric.

consider the Hirzebruch surface

$$\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)).$$

We denote by $[F]$ the class of a fibre and by $[B]$ the class of the unique section satisfying $B^2 = -a$. We write $\mathcal{O}(m,n) := \mathcal{O}(mB + nF)$ and consider curve classes $\beta := d_1[B] + d_2[F]$, $d_1, d_2 \geqslant 0$.

Corollary 7.1.8. *In the following cases, Conjecture 7.1.1 is true for* $X = \mathrm{Tot}_S(L_1 \oplus L_2)$.

- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,0), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (0,1)$, *and any* $n \geqslant 1$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,0), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (0,d)$ *with* $d \geqslant 2$, *and* $n = d$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,0), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (2,2), (2,3), (1,d), (d,1)$ *with* $d \geqslant 1$, *and* $n = 1$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,0), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (1,d)$ *with* $d \geqslant 2$, *and* $n = 2$.
- $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (0,1)$, *and any* $n \geqslant 1$.
- $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (0,d)$ *with* $d \geqslant 2$, *and* $n = d$.
- $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (2,2), (2,3), (2,4), (1,d)$ *with* $d \geqslant 1$, *and* $n = 1$.
- $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (1,d)$ *with* $d \geqslant 2$, *and* $n = 2$.
- $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(0,-1), \mathcal{O}(-2,-2))$, $(d_1, d_2) = (2,2), (2,3), (1,d)$ *with* $d \geqslant 1$, $n = 1$.
- $(S, L_1, L_2) = (\mathbb{F}_2, \mathcal{O}(-1,-2), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (0,1)$, *and any* $n \geqslant 1$.
- $(S, L_1, L_2) = (\mathbb{F}_2, \mathcal{O}(-1,-2), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (0,d)$ *with* $d \geqslant 2$, *and* $n = d$.
- $(S, L_1, L_2) = (\mathbb{F}_2, \mathcal{O}(-1,-2), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (2,3), (2,4), (2,5), (1,d)$ *with* $d \geqslant 1$, *and* $n = 1$.
- $(S, L_1, L_2) = (\mathbb{F}_2, \mathcal{O}(-1,-2), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (1,d)$ *with* $d \geqslant 2$, *and* $n = 2$.

In all these cases $P_n(X,\beta) \cong P_n(S,\beta)$ and $n > 0$. Since Bousseau-Brini-van Garrel only determined the *genus zero* Gopakumar-Vafa type invariants for the above geometries, we can only verify Conjecture 7.1.1 in these cases. In fact, in Proposition 7.2.9 and Remark 7.2.10, we classify *all* cases (S, L_1, L_2) such that S is minimal toric, $L_1 \otimes L_2 \cong K_S$, L_1^{-1}, L_2^{-1} are non-trivial and nef, $n \geqslant 0$, $P_n(X,\beta) \cong P_n(S,\beta)$, and $P_n(S,\beta)$ is non-empty. Using Theorem 7.1.5, we determined the stable pair invariants in all these cases, *including* the $n = 0$ case (see Appendix 7.4).

Remark 7.1.9. For all calculations done in Appendix 7.4 for which $P_n(X,\beta) \cong P_n(S,\beta)$ and the invariant is non-zero, we have

$$P_{n,\beta}([\mathsf{pt}]) = \pm \int_{S^{[m]}} e(T_{S^{[m]}}(L_1)). \tag{7.1.7}$$

These numbers were calculated by Carlsson-Okounkov [46] and are determined by the formula

$$\sum_{m=0}^{\infty} q^m \int_{S^{[m]}} e(T_{S^{[m]}}(L_1)) = \prod_{m=1}^{\infty} (1-q^m)^{-c_2(T_S \otimes L_1)},$$

where $c_2(T_S \otimes L_1) = c_2(S) - L_1 L_2$. We do not know whether (7.1.7) is a mere coincidence.

7.1.5. Vertex calculations Although most calculations in this chapter are based on Theorem 7.1.5, we also did some computations using the vertex formalism.

Proposition 7.1.10. *For the following cases, Conjectures 7.1.1 and 7.1.2 are true.*

- $X = \mathrm{Tot}_{\mathbb{P}^3}(K_{\mathbb{P}^3})$, $d = 1$, *and any* $n \geqslant 0$.
- $X = \mathrm{Tot}_{\mathbb{P}^3}(K_{\mathbb{P}^3})$, $d = 2, 3$, *and* $n = 0, 1$.
- $X = \mathrm{Tot}_{\mathbb{P}^3}(K_{\mathbb{P}^3})$, $d = 2$, *and* $n = 2$.

For the following cases Conjecture 7.1.1 is true for $X = \mathrm{Tot}_S(L_1 \oplus L_2)$.

- $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$, $d = 2$, *and* $n = 3$.
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, $(d_1, d_2) = (0,2), (2,0), (1,2), (2,1)$, *and* $n = 3$.
- $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (0,2)$, *and* $n = 3$.

The invariants in this proposition are defined by localization on the fixed locus [35, 36]. In the cases above where $X = \mathrm{Tot}_{\mathbb{P}^3}(K_{\mathbb{P}^3})$, we have $P_n(X, \beta) \cong P_n(\mathbb{P}^3, \beta)$ and

$$[P_n(X, \beta)]^{\mathrm{vir}} = (-1)^{\beta \cdot c_1(\mathbb{P}^3) + n} \cdot [P_n(\mathbb{P}^3, \beta)]^{\mathrm{vir}}_{\mathrm{pair}},$$

where $[P_n(\mathbb{P}^3, \beta)]^{\mathrm{vir}}_{\mathrm{pair}}$ is the virtual class of the pairs perfect obstruction theory on $\mathbb{P}^3$ (discussed in (7.3.9), see also [40, Lem. 3.1] in a similar setting). The sign in this formula is a preferred choice of orientation on $P_n(X, \beta)$ similar to (7.1.5). Then the Graber-Pandharipande virtual localization formula [85] can be applied to the right hand side to show that the local invariants of Proposition 7.1.10 are equal to the global invariants (7.1.4). The same method works for the local surface case $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$, $d = 2$, $n = 3$, because then all stable pairs are scheme theoretically supported in the threefold $\mathrm{Tot}_S(L_1)$.[4] See Remark 7.4.2 for more details.

We remark that most stable pair invariants of local surfaces calculated in this chapter are small (see Section 7.4.1). For $X = \mathrm{Tot}_{\mathbb{P}^3}(K_{\mathbb{P}^3})$, the numbers are rather big:

$$P_{0,3[\ell]} = 11200, \quad P_{1,2[\ell]}([\ell]) = -820, \quad P_{1,3[\ell]}([\ell]) = -68060, \quad P_{2,2[\ell]}([\ell]) = 400,$$

where $[\ell] \in H_2(\mathbb{P}^3, \mathbb{Z}) \cong H_2(X, \mathbb{Z})$ denotes the class of a line $\ell \subseteq \mathbb{P}^3$ and we also write $[\ell] \in H^4(X, \mathbb{Z})$ for the pull-back of its Poincaré dual from $\mathbb{P}^3$ to X. This provides further good evidence for Conjectures 7.1.1 and 7.1.2.

7.2. Moduli spaces

7.2.1. Stable pair invariants of Calabi-Yau 4-folds As in [152], a stable pair (F, s) on a smooth projective Calabi-Yau 4-fold X consists of

[4] By the time a first draft of this thesis — and [37] — was written, Oh-Thomas [147] proved in full generality the localization formula, cf. Section 5.2.

- a pure dimension 1 sheaf F on X,
- a section $s \in H^0(X, F)$ with 0-dimensional or trivial cokernel.

For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, denote by $P_n(X, \beta)$ be the moduli space of stable pairs (F, s) on X such that F has scheme theoretic support with class β and $\chi(F) = n$. By [**152**], it can alternatively be seen as the moduli space parametrizing 2-term complexes

$$I^\bullet = [\mathcal{O}_X \xrightarrow{s} F] \in \mathbf{D}^b(X)$$

in the bounded derived category of coherent sheaves on X. This viewpoint produces an obstruction theory on $P_n(X, \beta)$, which is however not perfect because $\mathrm{Ext}^3(I^\bullet, I^\bullet)_0$ is in general non-vanishing. Nonetheless, by the results of Section 5.2, there is a (real) virtual fundamental class (see also [**40**, Thm. 1.4])

$$[P_n(X, \beta)]^{\mathrm{vir}} \in H_{2n}\big(P_n(X, \beta), \mathbb{Z}\big)$$

depending on a choice of orientation.

7.2.2. Compactness I In the previous section, we assumed X is a smooth *projective* Calabi-Yau 4-fold. As we will discuss in more detail in Section 7.3.2, the previous section also applies to certain cases where X is a smooth quasi-projective Calabi-Yau 4-fold and $P_n(X, \beta)$ is proper.

Suppose S is a smooth projective surface and L_1, $L_2 \in \mathrm{Pic}(S)$ satisfy

$$L_1 \otimes L_2 \cong K_S.$$

Then $X = \mathrm{Tot}_S(L_1 \oplus L_2)$ is a smooth quasi-projective Calabi-Yau 4-fold, which we refer to as a *local surface*. One way to ensure the properness of $P_n(X, \beta)$ is as follows.

Proposition 7.2.1. *Suppose S is a smooth projective surface with L_1, $L_2 \in \mathrm{Pic}(S)$ satisfying $L_1 \otimes L_2 \cong K_S$ and let $X = \mathrm{Tot}_S(L_1 \oplus L_2)$. Let $\beta \in H_2(S, \mathbb{Z})$ and suppose for any $0 \neq \beta' \leqslant \beta$,*[5] *we have $\beta' \cdot L_i < 0$ for $i = 1, 2$. Then $P_n(X, \beta)$ is projective for any $n \in \mathbb{Z}$.*

Proof. Let $[(F, s)] \in P_n(X, \beta)$. We first show that F is set theoretically supported on the zero section $S \subseteq X$. Let D be an irreducible component of the scheme theoretic support of F, then we want to show $D_{\mathrm{red}} \subseteq S$. Let $Y = \mathrm{Tot}_S(L_1)$ and consider the projection $p : X = \mathrm{Tot}_Y(L_2) \to Y$ (here and below, we suppress the pull-back of L_2 along the projection $Y \to S$). Since D_{red} is a proper irreducible reduced curve, $\mathcal{O}_{D_{\mathrm{red}}}$ is stable. By the spectral construction, it corresponds to a stable Higgs pair $(p_* \mathcal{O}_{D_{\mathrm{red}}}, \phi)$, where

$$\phi : p_* \mathcal{O}_{D_{\mathrm{red}}} \to p_* \mathcal{O}_{D_{\mathrm{red}}} \otimes L_2.$$

Denote the curve class of the scheme theoretic support of $p_* \mathcal{O}_{D_{\mathrm{red}}}$ by $\beta' \in H_2(Y, \mathbb{Z}) \cong H_2(S, \mathbb{Z})$. Then $0 \neq \beta' \leqslant \beta$, so $\beta' \cdot L_2 < 0$. Combined with stability of the Higgs pairs $(p_* \mathcal{O}_{D_{\mathrm{red}}}, \phi)$ and $(p_* \mathcal{O}_{D_{\mathrm{red}}} \otimes L_2, \phi \otimes \mathrm{id}_{L_2})$, this implies $\phi = 0$ so $D_{\mathrm{red}} \subseteq Y = \mathrm{Tot}_S(L_1)$ (see [**175**, Prop. 7.4] for a similar argument). Reversing the roles of L_1, L_2, we deduce $D_{\mathrm{red}} \subseteq Y = \mathrm{Tot}_S(L_2)$, so $D_{\mathrm{red}} \subseteq S$.

[5] The notation $\beta' \leqslant \beta$ means that there exist effective curve classes $\beta', \beta'' \in H_2(S, \mathbb{Z})$ such that $\beta = \beta' + \beta''$.

Since each element of $P_n(X,\beta)$ is set theoretically supported on S, we conclude that $P_n(X,\beta)$ is projective. Indeed, there is a $d \gg 0$ such that every element of $P_n(X,\beta)$ is scheme theoretically supported in dS, where dS denotes the d times thickening of the zero section $S \subseteq X$, i.e. the closed subscheme of X defined by $I^d \subseteq \mathcal{O}_X$, where $I \subseteq \mathcal{O}_X$ denotes the ideal of the zero section. Therefore $P_n(X,\beta) \cong P_n(dS,\beta)$, $\square$

Suppose L_1^{-1} and L_2^{-1} are ample. Then K_S^{-1} is ample, i.e. S is del Pezzo, and $P_n(X,\beta)$ is projective for all β, n by Proposition 7.2.1. As noted in [**40**, Sec. 4.2], there are only two possibilities:

Proposition 7.2.2. *Let S be a smooth projective surface and $L_1, L_2 \in \mathrm{Pic}(S)$ such that $L_1 \otimes L_2 \cong K_S$. Suppose L_1^{-1} and L_2^{-1} are ample. Then, up to permutating L_1, L_2, we only have $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$ or $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$.*

Proof. Suppose S contains a (-1)-curve C. Then the Nakai criterion and adjunction imply

$$-2 \geqslant \deg(L_1|_C) + \deg(L_2|_C) = \deg(K_S|_C) = -1,$$

so S does not contain (-1)-curves. The classification of del Pezzo surfaces yields the result. $\square$

For both geometries of this proposition, the Gromov-Witten (and hence Gopakumar-Vafa type) invariants were determined in [**108**, Sec. 3].

Let us go back to an arbitrary smooth projective surface S with L_1, $L_2 \in \mathrm{Pic}(S)$ satisfying $L_1 \otimes L_2 \cong K_S$. Consider the moduli space $P_n(S,\beta)$ of stable pairs (F,s) on S with $\chi(F) = n$ and scheme theoretic support of F in class $\beta \in H_2(S,\mathbb{Z})$. Any stable pair $I^\bullet = [\mathcal{O}_S \to F]$ gives rise to a stable pair

$$[\mathcal{O}_X \to \iota_* \mathcal{O}_S \to \iota_* F]$$

on $X = \mathrm{Tot}_S(L_1 \oplus L_2)$, where $\iota : S \hookrightarrow X$ denotes inclusion of the zero section. This gives a closed embedding

$$P_n(S,\beta) \hookrightarrow P_n(X,\beta). \tag{7.2.1}$$

We refer to elements of $P_n(X,\beta)$ in the image as "stable pairs which are scheme theoretically supported on S". Requiring $P_n(X,\beta)$ to be proper poses restrictions on n, β. The following result is very useful for finding *candidates* for proper moduli spaces $P_n(X,\beta)$ (as we will see later in this section in Proposition 7.2.8).

Proposition 7.2.3. *Let S be a smooth projective surface, L_1, $L_2 \in \mathrm{Pic}(S)$ such that $L_1 \otimes L_2 \cong K_S$ and let $X = \mathrm{Tot}_S(L_1 \oplus L_2)$. Let $\beta \in H_2(S,\mathbb{Z})$ and $n \in \mathbb{Z}$ such that $P_n(X,\beta)$ is proper and $P_n(S,\beta) \neq \varnothing$. Suppose $C_1, C_2 \subseteq S$ are effective divisors satisfying*

- *$C_1 \cong \mathbb{P}^1$ and $[C_1 + C_2] = \beta$,*
- *$L_i \cdot C_1 = 0$ for $i = 1$ or $i = 2$.*

Then

$$-\tfrac{1}{2}\beta(\beta + K_S) \leqslant n \leqslant -\tfrac{1}{2}C_2(C_2 + K_S).$$

Proof. Suppose $P_n(S,\beta) \neq \varnothing$, $P_n(X,\beta)$ is proper, and let $C_1, C_2 \subseteq S$ be as stated. Then for any element $[(F,s)] \in P_n(S,\beta)$ with underlying scheme theoretic support C, we have

$$n = \chi(F) \geqslant \chi(\mathcal{O}_C) = -\tfrac{1}{2}\beta(\beta + K_S).$$

Suppose

$$n \geqslant 1 - \tfrac{1}{2}C_2(C_2 + K_S).$$

Since $C_1 \cong \mathbb{P}^1$ and $\deg(L_i|_{C_1}) = L_i \cdot C_1 = 0$, for $i = 1$ or $i = 2$, the line bundle $L_i|_{C_1}$ is trivial. Hence we can take a nowhere vanishing section D_1 of the line bundle $L_i|_{C_1} \cong \mathbb{P}^1 \times \mathbb{C}$. In particular, D_1 and C_2 are disjoint. Therefore

$$\begin{aligned}\chi(\mathcal{O}_{D_1 \sqcup C_2}) &= \chi(\mathcal{O}_{D_1}) + \chi(\mathcal{O}_{C_2}) \\ &= 1 - \tfrac{1}{2}C_2(C_2 + K_S).\end{aligned}$$

Twisting $\mathcal{O}_{D_1 \sqcup C_2}$ by an effective divisor of appropriate length, we obtain a stable pair $[(F,s)] \in P_n(X,\beta) \setminus P_n(S,\beta)$ with underlying scheme theoretic support $D_1 \sqcup C_2$. Since D_1 does not lie in the zero-section, using the $\mathbb{C}^*$-scaling action on L_i, we get a family of stable pairs with part of the support (i.e. D_1) moving off to infinity, contradicting properness of $P_n(X,\beta)$. □

We want to apply this proposition to smooth projective surfaces S with L_1, $L_2 \in \mathrm{Pic}(S)$ such that $L_1 \otimes L_2 \cong K_S$ and L_1^{-1}, L_2^{-1} non-trivial and nef. These surfaces were recently studied in the context of the log-local principle by Bousseau-Brini-van Garrel [**24**]. In particular, they determined the genus zero Gromov-Witten (and hence Gopakumar-Vafa type) invariants of $\mathrm{Tot}_S(L_1 \oplus L_2)$ in many new cases.

Smooth projective surfaces S with K_S^{-1} nef and big are called weak del Pezzo surfaces. The weak toric del Pezzo surfaces are: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$, $\mathbb{F}_2$, or certain repeated toric blow-ups of $\mathbb{P}^2$ in at most 6 points as specified in [**168**]. In this chapter, we only consider the *minimal* cases, i.e. the first four cases. Using the notation for Hirzebruch surfaces from the introduction, the only possibilities for $L_1, L_2 \in \mathrm{Pic}(S)$ such that $L_1 \otimes L_2 \cong K_S$ with L_1^{-1}, L_2^{-1} non-trivial and nef are (up to permutations of L_1, L_2):

- $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$,
- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$ or $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,0), \mathcal{O}(-1,-2))$,
- $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-2))$ or $(\mathbb{F}_1, \mathcal{O}(0,-1), \mathcal{O}(-2,-2))$,
- $(S, L_1, L_2) = (\mathbb{F}_2, \mathcal{O}(-1,-2), \mathcal{O}(-1,-2))$.

Example 7.2.4. Suppose $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,0), \mathcal{O}(-1,-2))$. Let $H_1 = \{\mathrm{pt}\} \times \mathbb{P}^1$ and $H_2 = \mathbb{P}^1 \times \{\mathrm{pt}\}$. We define $(d_1, d_2) := d_1 H_1 + d_2 H_2$. For all $d_1, d_2 \in \mathbb{Z}$, (d_1, d_2) is effective if and only if $d_1, d_2 \geqslant 0$. For $(0, d)$ with $d \geqslant 1$, the moduli space $P_n(X, (0,d))$ is projective for all n by Proposition 7.2.1. Now suppose $d_1 > 0$, $d_2 \geqslant 0$, and $n \geqslant 0$. Let $C_1 \in |H_1|$ and $C_2 \in |(d_1 - 1)H_1 + d_2 H_2|$. Then $L_1 \cdot C_1 = 0$ and the inequalities of Proposition 7.2.3 reduce to

$$d_1 + d_2 - d_1 d_2 \leqslant n \leqslant d_1 + 2d_2 - d_1 d_2 - 1.$$

These inequalities have the following solutions:

- $(d_1, d_2) = (3,2), (2,d)$ with $d \geqslant 2$ and $n = 0$,

- $(d_1, d_2) = (d, 1), (1, d)$ with $d \geqslant 1$ and $n = 1$, or $(d_1, d_2) = (2, d)$ with $d \geqslant 1$ and $n = 1$,
- $(1, d)$ with $2 \leqslant n \leqslant d$.

Example 7.2.5. Suppose $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-2))$ and use the notation for Hirzebruch surfaces from the introduction, so $(d_1, d_2) := d_1 B + d_2 F$ for all $d_1, d_2 \in \mathbb{Z}$. Then (d_1, d_2) is effective if and only if $d_1, d_2 \geqslant 0$ (this holds for all Hirzebruch surfaces). For $(0, d)$ with $d \geqslant 1$, the moduli space $P_n(X, (0, d))$ is projective for all n by Proposition 7.2.1. Suppose $d_1 > 0$, $d_2 \geqslant 0$, and $n \geqslant 0$. Let $C_1 \in |B|$ and $C_2 \in |(d_1 - 1)B + d_2 F|$. Then $L_1 \cdot C_1 = 0$ and the inequalities of Proposition 7.2.3 reduce to

$$\tfrac{1}{2}d_1(d_1+1) - d_2(d_1-1) \leqslant n \leqslant \tfrac{1}{2}d_1(d_1-1) - d_2(d_1-2).$$

These inequalities have the following solutions:

- $(d_1, d_2) = (3, 3), (2, d)$ with $d \geqslant 3$ and $n = 0$,
- $(d_1, d_2) = (2, d)$ with $d \geqslant 2$ and $n = 1$,
- $(1, d)$ with $1 \leqslant n \leqslant d$.

Example 7.2.6. Suppose $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(0,-1), \mathcal{O}(-2,-2))$. Suppose $d_1 > 0$, $d_2 \geqslant 0$, and $n \geqslant 0$. Taking C_1, C_2 as in Example 7.2.5 leads to the same list. Additionally, we can take $d_1 \geqslant 0$, $d_2 > 0$, $n \geqslant 0$, $C_1 \in |F|$ and $C_2 \in |d_1 B + (d_2 - 1)F|$. Then $L_1 \cdot C_1 = 0$ and the inequalities of Proposition 7.2.3 reduce to

$$\tfrac{1}{2}d_1(d_1+1) - d_2(d_1-1) \leqslant n \leqslant \tfrac{1}{2}d_1(d_1+1) - (d_2-1)(d_1-1).$$

The solutions to these inequalities *and* the ones from Example 7.2.5 are:

- $(d_1, d_2) = (2, 3), (2, 4), (3, 3)$ and $n = 0$,
- $(d_1, d_2) = (2, 2), (2, 3), (1, d)$ with $d \geqslant 1$ and $n = 1$.

Example 7.2.7. Suppose $(S, L_1, L_2) = (\mathbb{F}_2, \mathcal{O}(-1,-2), \mathcal{O}(-1,-2))$. For $(0, d)$ with $d \geqslant 1$, the moduli space $P_n(X, (0, d))$ is projective for all n by Proposition 7.2.1. Suppose $d_1 > 0$, $d_2 \geqslant 0$, and $n \geqslant 0$. Let $C_1 \in |B|$ and $C_2 \in |(d_1 - 1)B + d_2 F|$. Then $L_1 \cdot C_1 = 0$ and the inequalities of Proposition 7.2.3 reduce to

$$d_1^2 - d_2(d_1 - 1) \leqslant n \leqslant (d_1 - 1)^2 - d_2(d_1 - 2).$$

These inequalities have the following solutions:

- $(d_1, d_2) = (2, d)$, $d \geqslant 4$, and $n = 0$,
- $(d_1, d_2) = (2, d)$, $d \geqslant 3$, and $n = 1$,
- $(1, d)$ with $1 \leqslant n \leqslant d$.

In these examples we listed, for given (S, L_1, L_2), all the cases for which $n \geqslant 0$ and *potentially* $P_n(X, \beta)$ is proper and $P_n(S, \beta) \neq \varnothing$ (Proposition 7.2.3). For $\beta = (0, d)$ with $d \geqslant 1$, $P_n(S, \beta) \neq \varnothing$ if and only if $n \geqslant d$, and $P_n(X, \beta)$ is proper by Proposition 7.2.1. For all other cases listed, $P_n(S, \beta)$ is also non-empty since $|\beta| \neq \varnothing$ and $n \geqslant \chi(\mathcal{O}_C)$ for any $C \in |\beta|$. Indeed adding sufficiently many points to C one obtains a stable pair (F, s) on S with $\chi(F) = n$. We now prove that in each of the cases listed, $P_n(X, \beta)$ is indeed proper.

Proposition 7.2.8. *In each of the cases listed in Examples 7.2.4–7.2.7, $P_n(X,\beta)$ is projective.*

Proof. We write out the proof for Example 7.2.4. The other cases are analogous. Recall that $H_1 := \{\mathrm{pt}\}\times\mathbb{P}^1$, $H_2 := \mathbb{P}^1\times\{\mathrm{pt}\}$, $L_1 := \mathcal{O}(-H_1)$, $L_2 := \mathcal{O}(-H_1-2H_2)$, and $\beta := d_1H_1 + d_2H_2$. Suppose n, β are as listed in Example 7.2.4. As in Proposition 7.2.1, it is enough to show that all elements of $P_n(X,\beta)$ are set theoretically supported on S. Suppose $[(F,s)] \in P_n(X,\beta)$ has scheme theoretic support C and let D be an irreducible component of C which is *not* set theoretically supported on S. Then we claim D_{red} is a proper irreducible reduced curve with class $[D_{\mathrm{red}}] \in H_2(X,\mathbb{Z}) \cong H_2(S,\mathbb{Z})$ satisfying $[D_{\mathrm{red}}]\cdot L_1 \geqslant 0$ or $[D_{\mathrm{red}}]\cdot L_2 \geqslant 0$. Indeed suppose $[D_{\mathrm{red}}]\cdot L_1 < 0$ and $[D_{\mathrm{red}}]\cdot L_2 < 0$. Using the spectral construction as in the proof of Proposition 7.2.1, stability of $\mathcal{O}_{D_{\mathrm{red}}}$, then implies $D_{\mathrm{red}} \subseteq S$ contrary to our assumption.

The only non-zero effective curve classes β' on S such that $\beta'\cdot L_1 \geqslant 0$ or $\beta'\cdot L_2 \geqslant 0$ are $\beta' = mH_1$ for some $m > 0$. Hence there exists a $\Sigma \in |H_1|$ such that

$$p|_{D_{\mathrm{red}}} : D_{\mathrm{red}} \to \Sigma \subseteq S,$$

where $p : X \to S$ denotes the projection. Note that $L_1|_\Sigma \cong \mathcal{O}$ and $L_2|_\Sigma \cong \mathcal{O}(-2)$. Since $[D_{\mathrm{red}}]\cdot L_2 < 0$, a similar argument as above shows that $D_{\mathrm{red}} \subseteq \mathrm{Tot}_\Sigma(L_1) \cong \mathbb{P}^1\times\mathbb{C}$. Therefore D_{red} is a non-zero section of $\mathrm{Tot}_\Sigma(L_1) \cong \mathbb{P}^1\times\mathbb{C}$.

Denote the irreducible components of C which are *not* set theoretically supported on S by $D_1,\ldots,D_\ell$ and let D' be the union of the remaining components. Above, we showed each $D_{i,\mathrm{red}} \cong \mathbb{P}^1$ and $D_{i,\mathrm{red}}$ is a non-zero section of $\mathrm{Tot}_{\Sigma_i}(L_1) \cong \mathbb{P}^1\times\mathbb{C}$ for some $\Sigma_i \in |H_1|$. It follows that $D_{1,\mathrm{red}},\ldots,D_{\ell,\mathrm{red}}, D'_{\mathrm{red}}$ are mutually disjoint. Denote the multiplicity of D_i at $D_{i,\mathrm{red}}$ by $\delta_i \geqslant 1$. Consider the classes $p_*[D_i], p_*[D'] \in H_2(S,\mathbb{Z})$, where $p : X \to S$ is the projection. Then

$$p_*[D_i] := \delta_i H_1, \quad p_*[D'] := \beta - \delta H_1,$$

where $\delta := \sum_{i=1}^{\ell}\delta_i$. We claim

$$\chi(\mathcal{O}_{D_i}) \geqslant 1, \quad \chi(\mathcal{O}_{D'}) \geqslant 1 - g(p_*[D']) = -\tfrac{1}{2}(\beta-\delta H_1)(\beta-\delta H_1+K_S), \tag{7.2.2}$$

for all $i = 1,\ldots,\ell$, where the last equality is by the Riemann-Roch formula. When D_i (resp. D') are reduced, these inequalities are equalities. In general, since $N_{D_{i,\mathrm{red}}/X} \cong \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(-2)$ and $N_{S/X} \cong L_1\oplus L_2$ with L_1^{-1}, L_2^{-1} nef, we have inequalities as stated [6]. From (7.2.2) and the fact that $D_1,\ldots,D_\ell, D'$ are mutually disjoint, we deduce

$$\begin{aligned} n = \chi(F) \geqslant \chi(\mathcal{O}_C) &= \sum_{i=1}^{\ell}\chi(\mathcal{O}_{D_i}) + \chi(\mathcal{O}_{D'}) \\ &\geqslant \delta - \tfrac{1}{2}(\beta-\delta H_1)(\beta-\delta H_1+K_S) \\ &= \delta - \tfrac{1}{2}(d_1-\delta)(d_2-2) - \tfrac{1}{2}d_2(d_1-\delta-2). \end{aligned}$$

[6] One way to see this is by using filtrations by thickenings of $D_{i,\mathrm{red}} \subseteq X$ and $S \subseteq X$ as in the proof of Proposition 7.2.9 below.

However, for each of the cases listed in Example 7.2.4, it is easy to see that $n \leqslant \delta - 1 - \frac{1}{2}(d_1 - \delta)(d_2 - 2) - \frac{1}{2}d_2(d_1 - \delta - 2)$ for all $1 \leqslant \delta \leqslant d_1$ by explicit calculation. We have reached a contradiction. $\square$

Conclusion. For any (S, L_1, L_2) with $L_1 \otimes L_2 \cong K_S$, L_1^{-1}, L_2^{-1} non-trivial and nef, S minimal and toric, we classified *all* $n \geqslant 0$, $\beta \in H_2(S, \mathbb{Z})$ such that $P_n(X, \beta)$ is proper and $P_n(S, \beta) \neq \varnothing$.

7.2.3. Compactness II In the previous section, we studied properness of $P_n(X, \beta)$ for local surfaces. In particular, for $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$ or for the choice $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, the moduli space $P_n(X, \beta)$ is always proper (Proposition 7.2.1). We are now interested in the cases where $P_n(X, \beta) \cong P_n(S, \beta)$, i.e. the embedding (7.2.1) is an isomorphism. For the 3-fold $\mathrm{Tot}(K_{\mathbb{P}^2})$, this question was considered by Choi-Katz-Klemm in [**56**, Prop. 2]. In the proof of the following proposition, we use some of their techniques (adapted to the 4-fold setting).

Proposition 7.2.9. *Let* $X = \mathrm{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$, $\beta = d[H]$ *with* $d \geqslant 1$, *and* $n \geqslant 0$. *Then*

$$P_n(X, \beta) \cong P_n(\mathbb{P}^2, \beta)$$

if and only if

(1) $d = 1$ *and any* $n \geqslant 0$, *or*
(2) $d = 2, 3, 4$ *and* $n = 0, 1$, *or*
(3) $d = 2, 3$ *and* $n = 2$.

Let $X = \mathrm{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1))$, $\beta = d_1[H_1] + d_2[H_2] \neq 0$ *with* $d_1, d_2 \geqslant 0$, *and* $n \geqslant 0$. *Then*

$$P_n(X, \beta) \cong P_n(\mathbb{P}^1 \times \mathbb{P}^1, \beta),$$

if and only if

(1) $(d_1, d_2) = (1,0), (0,1), (1,1)$ *and any* $n \geqslant 0$, *or*
(2) $(d_1, d_2) = (0,d), (d,0)$ *with* $d \geqslant 2$ *and* $0 \leqslant n \leqslant d$, *or*
(3) $(d_1, d_2) = (1,d), (d,1)$ *with* $d \geqslant 2$ *and* $n = 0, 1, 2$, *or*
(4) $(d_1, d_2) = (2,2), (2,3), (3,2), (2,4), (4,2), (3,3)$ *and* $n = 0$, *or*
(5) $(d_1, d_2) = (2,2), (2,3), (3,2)$ *and* $n = 1$, *or*
(6) $(d_1, d_2) = (2,2)$ *and* $n = 2$.

Proof. Let $[(F, s)] \in P_n(X, \beta)$ be a stable pair with scheme theoretic support $C := \mathrm{supp}(F)$. The stable pair (F, s) is set theoretically supported on the zero section $S \subseteq X$ by Proposition 7.2.1. Let $Y_i = \mathrm{Tot}_S(L_i)$ for $i = 1, 2$. We consider the ideals of $C \subseteq X$ and $Y_1 \subseteq X$:

$$J := I_{C \subseteq X} \subseteq \mathcal{O}_X, \quad I_2 := I_{Y_1 \subseteq X} \subseteq \mathcal{O}_X\,.$$

Note that I_2 is a line bundle on X. Since (F,s) is set theoretically supported on $S \subseteq X$ (and therefore $Y_1 \subseteq X$), there exists an $\ell \geqslant 0$ such that $J + I_2^{\ell+1} = J$ and we have

$$\chi(\mathcal{O}_C) = \sum_{j=0}^{\ell} \chi\Big(\frac{J + I_2^j}{J + I_2^{j+1}}\Big). \tag{7.2.3}$$

For each j, we have a surjective map

$$p^* L_2^{-j} \cong \frac{I_2^j}{I_2^{j+1}} \to \frac{J + I_2^j}{J + I_2^{j+1}},$$

where $p : Y_1 \to S$ denotes projection. Hence $\frac{J+I_2^j}{J+I_2^{j+1}} \cong \mathcal{O}_{C_j} \otimes p^* L_2^{-j}$ for some closed subscheme $C_j \subseteq Y_1$ of dimension $\leqslant 1$. Moreover, we have $C_j \supseteq C_{j+1}$ for all j.[7] From the fact that C is Cohen-Macaulay, it also follows that, when non-empty, C_j is not 0-dimensional.

For a fixed j, we consider the ideals of $C_j \subseteq Y_1$ and $S \subseteq Y_1$:

$$J_j := I_{C_j \subseteq Y_1} \subseteq \mathcal{O}_{Y_1}, \quad I_1 := I_{S \subseteq Y_1} \subseteq \mathcal{O}_{Y_1}.$$

Note that I_1 is a line bundle on Y_1. As above, there exists an $\ell_j \geqslant 0$ such that $J_j + I_1^{\ell_j+1} = J_j$ and we have

$$\chi(\mathcal{O}_{C_j}) = \sum_{i=0}^{\ell_j} \chi\Big(\frac{J_j + I_1^i}{J_j + I_1^{i+1}}\Big).$$

As above, for all i, we have $\frac{J_j+I_1^i}{J_j+I_1^{i+1}} \cong \mathcal{O}_{C_{ij}} \otimes L_1^{-i}$ for some closed subscheme $C_{ij} \subseteq S$ of dimension $\leqslant 1$. As above, we also have $C_{ij} \supseteq C_{i+1,j}$ for all i. This time, we leave open the possibility that C_{ij} is 0-dimensional, because C_j need not be Cohen-Macaulay. Nonetheless, denoting $\beta_{ij} := [C_{ij}]$, we have

$$\beta = \sum_{j=0}^{\ell} \sum_{i=0}^{\ell_j} \beta_{ij} \in H_2(S, \mathbb{Z}).$$

Consider the torsion filtration

$$0 \to \mathcal{T}_0 \to \mathcal{O}_{C_{ij}} \to \mathcal{O}_{C_{ij}^{\mathrm{pure}}} \to 0$$

and the exact sequence

$$0 \to \mathcal{O}_S(-C_{ij}^{\mathrm{pure}}) \to \mathcal{O}_S \to \mathcal{O}_{C_{ij}^{\mathrm{pure}}} \to 0.$$

The support of $\mathcal{T}_0$ is 0-dimensional. Applying the Hirzebruch-Riemann-Roch formula gives

$$\begin{aligned}\chi\Big(\frac{J_j + I_1^i}{J_j + I_1^{i+1}} \otimes p^* L_2^{-j}\Big) = \chi(\mathcal{O}_{C_{ij}} \otimes L_1^{-i} \otimes L_2^{-j}) \geqslant \chi(\mathcal{O}_{C_{ij}^{\mathrm{pure}}} \otimes L_1^{-i} \otimes L_2^{-j}) \\ = -\tfrac{1}{2}\beta_{ij}(\beta_{ij} + K_S) - (iL_1 + jL_2)\beta_{ij}.\end{aligned}$$

[7] This follows from the natural surjection $\mathcal{O}_{C_j} \otimes p^* L_2^{-j-1} \cong \frac{J+I_2^j}{J+I_2^{j+1}} \otimes \frac{I_2}{I_2^2} \twoheadrightarrow \frac{J+I_2^{j+1}}{J+I_2^{j+2}} \cong \mathcal{O}_{C_{j+1}} \otimes p^* L_2^{-j-1}$.

Combining with (7.2.3), we obtain

(7.2.4)
$$\begin{aligned}\chi(F) \geqslant \chi(\mathcal{O}_C) &\geqslant -\sum_{j=0}^{\ell}\sum_{i=0}^{\ell_j}(\tfrac{1}{2}\beta_{ij}(\beta_{ij}+K_S)+(iL_1+jL_2)\beta_{ij}) \\ &\geqslant -\tfrac{1}{2}\beta(\beta+K_S)-\beta(L_1+L_2) \\ &\quad+\tfrac{1}{2}\sum_{\substack{((i,j),(i',j')) \\ (i,j)\neq(i',j')}}\beta_{ij}\beta_{i'j'}+\beta_{00}(L_1+L_2)+L_1\sum_{j=1}^{\ell}\beta_{0j}+L_2\sum_{i=1}^{\ell_0}\beta_{i0},\end{aligned}$$

where we used that L_1 and L_2 are nef line bundles.

Case 1. Let $(S,L_1,L_2)=(\mathbb{P}^2,\mathcal{O}(-1),\mathcal{O}(-2))$, $\beta=d[H]$ with $d\geqslant 1$, and $n\geqslant 0$. Suppose there exists an element $[(F,s)]\in P_n(X,\beta)\backslash P_n(S,\beta)$. We use the notation above for its scheme theoretic support C and the associated schemes C_{ij}. Let $\beta_{ij}=d_{ij}[H]$, then (7.2.4) gives

$$\begin{aligned}\chi(F) &\geqslant -\tfrac{1}{2}d^2+\tfrac{9}{2}d-3d_{00}-\sum_{j=1}^{\ell}d_{0j}-2\sum_{i=1}^{\ell_0}d_{i0}+\tfrac{1}{2}\sum_{\substack{((i,j),(i',j')) \\ (i,j)\neq(i',j')}}d_{ij}d_{i'j'} \\ &\geqslant -\tfrac{1}{2}d^2+\tfrac{5}{2}d-d_{00}+\tfrac{1}{2}\sum_{\substack{((i,j),(i',j')) \\ (i,j)\neq(i',j')}}d_{ij}d_{i'j'} \\ &\geqslant -\tfrac{1}{2}d^2+\tfrac{5}{2}d,\end{aligned}$$

where the last inequality uses that there exists an $(i,j)\neq(0,0)$ with $d_{ij}\geqslant 1$, because we assumed C is not scheme theoretically supported in the zero section $S\subseteq X$. Hence $n\leqslant -\frac{1}{2}d^2+\frac{5}{2}d-1$ implies $P_n(S,\beta)\cong P_n(X,\beta)$. In particular, we find that for the cases (1)–(3) we have $P_n(S,\beta)\cong P_n(X,\beta)$. For case (1) this is obvious from Proposition 7.2.1 and the fact that β is irreducible.

For β,n other than (1)–(3), it is easy to construct a $(\mathbb{C}^*)^4$-fixed stable pair in

$$P_n(X,\beta)\setminus P_n(S,\beta)$$

using the combinatorial description of stable pairs in [**155, 35**], where $(\mathbb{C}^*)^4$ denotes the torus of the toric Calabi-Yau 4-fold X.

Case 2. Let $(S,L_1,L_2)=(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(-1,-1),\mathcal{O}(-1,-1))$, $\beta=d_1[H_1]+d_2[H_2]$ for some $d_1,d_2\geqslant 0$ not both zero, and $n\geqslant 0$. Suppose there exists an element $[(F,s)]\in P_n(X,\beta)\setminus P_n(S,\beta)$. We use the notation above for its scheme theoretic support C and the associated schemes C_{ij}. Let $\beta_{ij}=d_{1,ij}[H_1]+d_{2,ij}[H_2]$, then (7.2.4) gives

(7.2.5)

$$
\begin{aligned}
\chi(F) &\geqslant 3d_1 + 3d_2 - d_1 d_2 - 2d_{1,00} - 2d_{2,00} - \sum_{j=1}^{\ell}(d_{1,0j} + d_{2,0j}) - \sum_{i=1}^{\ell_0}(d_{1,i0} + d_{2,i0}) \\
&\quad + \tfrac{1}{2} \sum_{\substack{((i,j),(i',j')) \\ (i,j) \neq (i',j')}} (d_{1,ij} d_{2,i'j'} + d_{1,i'j'} d_{2,ij}) \\
&\geqslant 2d_1 + 2d_2 - d_1 d_2 - d_{1,00} - d_{2,00} + \sum_{\substack{((i,j),(i',j')) \\ (i,j) \neq (i',j')}} d_{1,ij} d_{2,i'j'} \\
&\geqslant d_1 + d_2 - d_1 d_2 + 1 + \sum_{\substack{((i,j),(i',j')) \\ (i,j) \neq (i',j')}} d_{1,ij} d_{2,i'j'},
\end{aligned}
$$

where the last inequality uses that C does not lie scheme theoretically in S. Suppose $d_1, d_2 \geqslant 2$, then

$$
\sum_{\substack{((i,j),(i',j')) \\ (i,j) \neq (i',j')}} d_{1,ij} d_{2,i'j'} \geqslant 2. \tag{7.2.6}
$$

Therefore $n \leqslant d_1 + d_2 - d_1 d_2 + 2$ implies $P_n(S, \beta) \cong P_n(X, \beta)$. In particular, for β, n as in (4)–(6), *except* for $(d_1, d_2) = (3, 3)$ and $n = 0$ (!), we deduce that $P_n(S, \beta) \cong P_n(X, \beta)$. Cases (1)–(3) can be found from (7.2.5) by a similar reasoning.

For $(d_1, d_2) = (3, 3)$, we still use (7.2.5), but we need to sharpen (7.2.6). Recall that the schemes C_j and C_{ij} constructed in the first part of the proof are nested. This implies $d_{1,ij} \geqslant d_{1,i+1j}$ and $d_{2,ij} \geqslant d_{2,i+1j}$ for all i, j. Using these inequalities for $(d_1, d_2) = (3, 3)$, one can show that

$$
\sum_{\substack{((i,j),(i',j')) \\ (i,j) \neq (i',j')}} d_{1,ij} d_{2,i'j'} \geqslant 3.
$$

It follows that for $[(F, s)] \in P_n(X, (3, 3)) \setminus P_n(S, (3, 3))$, we have $\chi(F) \geqslant 1$. Hence $P_0(S, (3, 3)) \cong P_0(X, (3, 3))$.

For β, n other than (1)–(6), it is easy to construct a $(\mathbb{C}^*)^4$-fixed stable pair in $P_n(X, \beta) \setminus P_n(S, \beta)$ using the combinatorial description of stable pairs in [**155, 35**]. $\square$

Remark 7.2.10. For (S, L_1, L_2) as in Examples 7.2.4–7.2.7, we found *all* cases for which $n \geqslant 0$, $P_n(X, \beta)$ is proper, and $P_n(S, \beta) \neq \varnothing$ (Propositions 7.2.3 and 7.2.8). A similar reasoning as in the proof of Proposition 7.2.9 (using (7.2.4)) can be applied to find out when $P_n(X, \beta) \cong P_n(S, \beta)$. In the following cases, we have $n \geqslant 0$, $P_n(X, \beta) \cong P_n(S, \beta)$:

- $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1, 0), \mathcal{O}(-1, -2))$, $(d_1, d_2) = (0, 1)$ and any $n \geqslant 0$, or $(d_1, d_2) = (0, d)$ for any $d \geqslant 2$ and $n = d$, or $(d_1, d_2) = (2, 2), (2, 3), (3, 2), (2, 4)$ and $n = 0$, or $(d_1, d_2) = (2, 2), (2, 3), (1, d), (d, 1)$ for any $d \geqslant 1$ and $n = 1$, or $(d_1, d_2) = (1, d)$ for any $d \geqslant 2$ and $n = 2$.
- $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(-1, -1), \mathcal{O}(-1, -2))$, $(d_1, d_2) = (0, 1)$ and any $n \geqslant 0$, or $(d_1, d_2) = (0, d)$ for any $d \geqslant 2$ and $n = d$, or $(d_1, d_2) = (2, 3), (2, 4), (3, 3), (2, 5)$ and $n = 0$, or $(d_1, d_2) = (2, 2), (2, 3), (2, 4)$ and $n = 1$, or $(d_1, d_2) = (1, d)$ for $n = 1, 2$ and any $d \geqslant n$.

- $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(0,-1), \mathcal{O}(-2,-2))$, $(d_1, d_2) = (2,3), (2,4), (3,3)$ and $n = 0$, or $(d_1, d_2) = (2,2), (2,3), (1,d)$ with $d \geqslant 1$ and $n = 1$.
- $(S, L_1, L_2) = (\mathbb{F}_2, \mathcal{O}(-1,-2), \mathcal{O}(-1,-2))$, $(d_1, d_2) = (0,1)$ and any $n \geqslant 0$, or $(d_1, d_2) = (0,d)$ for any $d \geqslant 2$ and $n = d$, or $(d_1, d_2) = (2,4), (2,5), (2,6)$ and $n = 0$, or $(d_1, d_2) = (2,3), (2,4), (2,5)$ and $n = 1$, or $(d_1, d_2) = (1,d)$ for $n = 1, 2$ and any $d \geqslant n$.

Furthermore, in all cases listed in Examples 7.2.4–7.2.7 but *not* in the above list, one can easily construct a $(\mathbb{C}^*)^4$-fixed stable pair in $P_n(X,\beta) \setminus P_n(S,\beta)$ using the combinatorial description of stable pairs on toric varieties [**155, 35**].
Conclusion. For any (S, L_1, L_2) with $L_1 \otimes L_2 \cong K_S$, L_1^{-1}, L_2^{-1} non-trivial and nef, and S minimal and toric, we have classified *all* $\beta \in H_2(S,\mathbb{Z})$ and $n \geqslant 0$ such that $P_n(X,\beta) \cong P_n(S,\beta) \neq \varnothing$.

In the next section, we develop a method to determine the stable pair invariants $P_{n,\beta}([\mathsf{pt}])$ in all of these cases (tabulated in Appendix 7.4).

7.3. Invariants

7.3.1. Virtual classes of relative Hilbert schemes For S a smooth projective surface, $\beta \in H_2(S,\mathbb{Z})$, and $n \in \mathbb{Z}$, the moduli space $P_n(S,\beta)$ has a nice description in terms of relative Hilbert schemes due to Pandharipande-Thomas [**153**]. Given a stable pair $[(F,s)] \in P_n(S,\beta)$, one has a short exact sequence

$$0 \to \mathcal{O}_C \to F \to Q \to 0,$$

where C is the scheme theoretic support of F. Dualizing on C yields a short exact sequence

$$0 \to F^* \to \mathcal{O}_C \to \mathcal{E}xt^1(Q, \mathcal{O}_C) \to 0,$$

where we used $\mathcal{E}xt^1(F, \mathcal{O}_C) = 0$ by [**153**, Lem. B.2]. Hence $\mathcal{E}xt^1(Q, \mathcal{O}_C) \cong \mathcal{O}_Z$ for some 0-dimensional subscheme $Z \subseteq C$ of length

$$m = n + g(\beta) - 1 = n + \tfrac{1}{2}\beta(\beta + K_S).$$

As shown in [**153**, Prop. B.8.], the family version of this argument gives an isomorphism

$$P_n(S,\beta) \cong \mathrm{Hilb}^m(\mathcal{C}/H_\beta), \tag{7.3.1}$$

where $\mathrm{Hilb}^m(\mathcal{C}/H_\beta)$ denotes the relative Hilbert scheme of m points on the fibres of the universal curve $\mathcal{C} \to H_\beta$ and H_β denotes the Hilbert scheme of effective divisors on S in class β. The description in terms of relative Hilbert schemes helps to establish smoothness. Although we do not need it for this chapter, we include the following observation.

Proposition 7.3.1. *In all the cases listed in Proposition 7.2.9 and Remark 7.2.10, $P_n(S,\beta)$ is smooth.*

Proof. The method in this proof was also used in [**116**]. Let $S = \mathbb{P}^2$, then for any $\beta = d[H]$ with $d \geqslant 1$, and any $n \in \mathbb{Z}$, we have a morphism

$$P_n(S,\beta) \cong \mathrm{Hilb}^m(\mathcal{C}/|\,\mathcal{O}(d)|) \to S^{[m]}, \tag{7.3.2}$$

where $m = n + \frac{1}{2}d(d-3)$. The fibre over $Z \in S^{[m]}$ is the projectivization of the kernel of the evaluation map $H^0(\mathbb{P}^2, \mathcal{O}(d)) \to H^0(Z, \mathcal{O}(d)|_Z)$. It suffices to show that for n and $d \neq 1$ as in Proposition 7.2.9, this map is surjective. Then it follows that the fibres of (7.3.2) are equi-dimensional projective spaces and $P_n(S,\beta)$ is smooth, because $S^{[m]}$ is also smooth. Surjectivity of the evaluation map for all $Z \in S^{[m]}$ is equivalent to $(m-1)$-very ampleness of $\mathcal{O}(d)$ (by definition, [**14**]). Beltrametti-Sommese showed that $\mathcal{O}(d)$ is $(m-1)$-very ample if and only if $m - 1 \leqslant d$, i.e.

$$n \leqslant d - \tfrac{1}{2}d(d-3) + 1.$$

This inequality is satisfied for all n and $d \neq 1$ in Cases (2), (3) of Proposition 7.2.9. Smoothness of $P_n(S,\beta)$ for $d = 1$ and any n is clear, because in this case the fibres of (7.3.2) are $\mathrm{Sym}^m(\mathbb{P}^1) \cong \mathbb{P}^m$.

For $S = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}(d_1H_1 + d_2H_2)$ is k-very ample if and only if $k \leqslant \min\{d_1, d_2\}$ [**14**]. For $S = \mathbb{F}_a$ (for any $a \geqslant 1$), $\mathcal{O}(d_1B + d_2F)$ is k-very ample if and only if $k \leqslant \min\{d_1, d_2 - ad_1\}$ [**14**]. The proof in the remaining cases of Proposition 7.2.9 and Remark 7.2.10 then follows similarly. $\square$

Let $\mathcal{Z} \subseteq S \times S^{[m]}$ be the universal subscheme and denote the pull-back of $\mathcal{Z}$ to $S \times S^{[m]} \times H_\beta$ by the same symbol (and similarly for $\mathcal{C} \subseteq S \times H_\beta$). Consider the rank m vector bundle

$$\mathcal{O}(\mathcal{C})^{[m]} := \pi_*(\mathcal{O}(\mathcal{C})|_{\mathcal{Z}})$$

on $S^{[m]} \times H_\beta$, where $\pi : S \times S^{[m]} \times H_\beta \to S^{[m]} \times H_\beta$ is the projection. By [**117**, App. A], there exists a tautological section s of $\mathcal{O}(\mathcal{C})^{[m]}$ cutting out $\mathrm{Hilb}^m(\mathcal{C}/H_\beta)$ from its ambient space

$$\begin{tikzcd} & \mathcal{O}(\mathcal{C})^{[m]} \arrow[d] \\ s^{-1}(0) \cong \mathrm{Hilb}^m(\mathcal{C}/H_\beta) \arrow[r, hook, "\jmath"] & S^{[m]} \times H_\beta. \arrow[u, bend right, "s"'] \end{tikzcd}$$

In general H_β is not smooth (or smooth but not of expected dimension). Therefore, this construction only provides a *relative* perfect obstruction theory on $\mathrm{Hilb}^m(\mathcal{C}/H_\beta) \to H_\beta$. The Hilbert scheme of divisors H_β has a natural perfect obstruction theory

$$(\mathbf{R}p_*\,\mathcal{O}_{\mathcal{C}}(\mathcal{C}))^\vee \to \mathbb{L}_{H_\beta},$$

where $p : S \times H_\beta \to H_\beta$ denotes projection. This is the perfect obstruction theory used to define the Poincaré/Seiberg-Witten invariants of S in [**67, 51**]. Taken together, these provide an *absolute* perfect obstruction theory on $P_n(S,\beta) \cong \mathrm{Hilb}^m(\mathcal{C}/H_\beta)$ by [**117**, App. A.3]. The virtual tangent bundle of this absolute perfect obstruction theory is

$$\mathbf{R}\mathscr{H}om_{\pi_S}(\mathbb{I}_S^\bullet, \mathbb{F})$$

where $\mathbb{I}_S^\bullet = [\mathcal{O} \to \mathbb{F}]$ denotes the universal stable pair on $S \times P_n(S,\beta)$ and $\pi_S : S \times P_n(S,\beta) \to P_n(S,\beta)$ is the projection. By [**114**, Prop. 2.1], the resulting virtual class satisfies

$$\jmath_*[\mathrm{Hilb}^m(\mathcal{C}/H_\beta)]^{\mathrm{vir}} = (S^{[m]} \times [H_\beta]^{\mathrm{vir}}) \cdot e\big(\mathcal{O}(\mathcal{C})^{[m]}\big). \tag{7.3.3}$$

The corresponding virtual class on $P_n(S,\beta)$, via the isomorphism (7.3.1), is denoted by $[P_n(S,\beta)]^{\mathrm{vir}}$.

7.3.2. Comparison of virtual classes Let S be a smooth projective surface and $L_1, L_2 \in \mathrm{Pic}(S)$ such that $L_1 \otimes L_2 \cong K_S$. We consider the local surface $X = \mathrm{Tot}_S(L_1 \oplus L_2)$, which is a Calabi-Yau 4-fold. Fix $n \in \mathbb{Z}$ and $\beta \in H_2(S,\mathbb{Z})$ such that $P_n(X,\beta)$ is proper. Then it has a virtual class

$$[P_n(X,\beta)]^{\mathrm{vir}} \in H_{2n}\big(P_n(X,\beta),\mathbb{Z}\big), \tag{7.3.4}$$

in the sense of Borisov-Joyce [**23**] (cf. Section 5.2), which depends on a choice of orientation on $P_n(X,\beta)$.

We denote by $[\mathrm{pt}] \in H^4(X,\mathbb{Z})$ the pull-back along $\pi : X \to S$ of the Poincaré dual of the point class on S. Using the same notation as in Section 7.1.2, we define stable pair invariants

$$P_{n,\beta}([\mathsf{pt}]) := \int_{[P_n(X,\beta)]^{\mathrm{vir}}} \tau([\mathsf{pt}])^n \in \mathbb{Z}. \tag{7.3.5}$$

When $n = 0$, we simply write $P_{0,\beta} := P_{0,\beta}([\mathsf{pt}])$.

Assuming $P_n(X,\beta) \cong P_n(S,\beta)$, we can compare the virtual class (7.3.4) to the virtual class on the relative Hilbert scheme (7.3.3) studied in [**117, 118**]. In Proposition 7.2.9 and Remark 7.2.10 we gave a list of examples where this assumption is satisfied.

Proposition 7.3.2. *Let S be a smooth projective surface, $L_1, L_2 \in \mathrm{Pic}(S)$ such that $L_1 \otimes L_2 \cong K_S$ and let $X = \mathrm{Tot}_S(L_1 \oplus L_2)$. Suppose $\beta \in H_2(S,\mathbb{Z})$ and $n \geqslant 0$ are chosen such that $P_n(X,\beta) \cong P_n(S,\beta)$. Then there exists a choice of orientation such that*

$$[P_n(X,\beta)]^{\mathrm{vir}} = (-1)^{\beta\cdot L_2 + n} \cdot e\big(-\mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1)\big) \cdot [P_n(S,\beta)]^{\mathrm{vir}}.$$

Here $[P_n(S,\beta)]^{\mathrm{vir}}$ is the virtual class induced from the relative Hilbert scheme (Section 7.3.1), $\mathbb{I}_S^\bullet = [\mathcal{O} \to \mathbb{F}]$ denotes the universal stable pair on $S \times P_n(S,\beta)$, and $\pi_{P_S} : S \times P_n(S,\beta) \to P_n(S,\beta)$ is the projection. The sign results from a preferred choice of orientation.

Proof. Let $Y = \mathrm{Tot}_S(L_1)$. Then $X = \mathrm{Tot}_Y(K_Y)$ is the total space of the canonical bundle of Y. By the assumption, we have isomorphisms of moduli spaces

$$P_n(S,\beta) \cong P_n(Y,\beta) \cong P_n(X,\beta). \tag{7.3.6}$$

Let $\iota : S \hookrightarrow Y$ denote the zero section. A stable pair $I_S^\bullet = [\mathcal{O}_S \xrightarrow{s} F] \in P_n(S,\beta)$ on S induces a stable pair

$$I_Y^\bullet = [\mathcal{O}_Y \to \iota_* \mathcal{O}_S \xrightarrow{\iota_* s} \iota_* F]$$

on Y. Consider the distinguished triangle

$$I_Y^\bullet \to \mathcal{O}_Y \to \iota_* F. \tag{7.3.7}$$

Applying $\mathbf{R}\mathrm{Hom}_Y(I_Y^\bullet, \cdot)$ and taking out trace gives a distinguished triangle

$$\mathbf{R}\mathrm{Hom}_Y(I_Y^\bullet, \iota_* F) \to \mathbf{R}\mathrm{Hom}_Y(I_Y^\bullet, I_Y^\bullet)_0[1] \to \mathbf{R}\mathrm{Hom}_Y(\iota_* F, \mathcal{O}_Y)[2].$$

Applying adjunction and the isomorphism

$$\mathbf{L}\iota^* I_Y^\bullet \cong I_S^\bullet \oplus F \otimes L_1^{-1} \tag{7.3.8}$$

gives a long exact sequence

$$\cdots \to \mathrm{Ext}_S^i(I_S^\bullet, F) \oplus \mathrm{Ext}_S^i(F, F \otimes L_1) \to \mathrm{Ext}_Y^{i+1}(I_Y^\bullet, I_Y^\bullet)_0 \to \mathrm{Ext}_Y^{i+2}(\iota_* F, \mathcal{O}_Y) \to \cdots.$$

Note that $\mathrm{Ext}_Y^1(\iota_* F, \mathcal{O}_Y) \cong \mathrm{Ext}_Y^2(\mathcal{O}_Y, \iota_* F \otimes K_Y)^\vee = 0$. Furthermore, the isomorphism (7.3.6) induces an isomorphism on Zariski tangent spaces

$$\mathrm{Ext}_S^0(I_S^\bullet, F) \cong \mathrm{Ext}_Y^1(I_Y^\bullet, I_Y^\bullet)_0.$$

Therefore, we deduce $\mathrm{Hom}_S(F, F \otimes L_1) = 0$ (similarly $\mathrm{Hom}_S(F, F \otimes L_2) = 0$). This vanishing allows us to conclude that the natural (Le Potier) pair obstruction theory

$$(\mathbf{R}\mathscr{H}om_{\pi_{P_Y}}(\mathbb{I}_Y^\bullet, \iota_{P_Y *}\mathbb{F}))^\vee \to \mathbb{L}_{P_n(Y,\beta)} \tag{7.3.9}$$

is *perfect*, i.e. 2-term, as we will now show[8]. Here $\mathbb{I}_Y^\bullet = [\mathcal{O} \to \iota_{P_Y *}\mathbb{F}]$ denotes the universal stable pair on $Y \times P_n(Y,\beta)$, $\iota_{P_Y} : S \times P_n(Y,\beta) \hookrightarrow Y \times P_n(Y,\beta)$ is the base change of the zero section, and $\pi_{P_Y} : Y \times P_n(Y,\beta) \to P_n(Y,\beta)$ denotes the projection.

From the distinguished triangle

$$\mathbf{R}\mathrm{Hom}_Y(\iota_* F, \iota_* F) \to \mathbf{R}\mathrm{Hom}_Y(\mathcal{O}_Y, \iota_* F) \to \mathbf{R}\mathrm{Hom}_Y(I_Y^\bullet, \iota_* F), \tag{7.3.10}$$

we obtain an exact sequence

$$0 = H^2(Y, \iota_* F) \to \mathrm{Ext}_Y^2(I_Y^\bullet, \iota_* F) \to \mathrm{Ext}_Y^3(\iota_* F, \iota_* F) \to 0 \to \mathrm{Ext}_Y^3(I_Y^\bullet, \iota_* F) \to 0.$$

Moreover, by adjunction and $\mathbf{L}\iota^* F \cong F \oplus F \otimes L_1^{-1}[1]$, we have

$$\mathrm{Ext}_Y^3(\iota_* F, \iota_* F) \cong \mathrm{Ext}_S^3(F, F) \oplus \mathrm{Ext}_S^2(F, F \otimes L_1) \cong \mathrm{Hom}_S(F, F \otimes L_2)^\vee = 0.$$

Hence $\mathrm{Ext}_Y^2(I_Y^\bullet, \iota_* F) \cong \mathrm{Ext}_Y^3(\iota_* F, \iota_* F) = 0$. Also note that $\mathrm{Hom}_Y(\iota_* F, \iota_* F) \to \mathrm{Hom}_Y(\mathcal{O}_Y, \iota_* F)$ is injective. Therefore $\mathrm{Ext}_Y^i(I_Y^\bullet, \iota_* F) = 0$ unless $i = 0, 1$ and the complex (7.3.9) is 2-term. We denote the corresponding virtual class by $[P_n(Y,\beta)]^{\mathrm{vir}}_{\mathrm{pair}}$.

We can now use the argument of [40, Prop. 4.3] to deduce that the 4-fold virtual class $[P_n(X,\beta)]^{\mathrm{vir}}$ of (7.3.4) equals the pairs virtual class $[P_n(Y,\beta)]^{\mathrm{vir}}_{\mathrm{pair}}$. For completeness, we repeat the argument. Just like pushing forward from S to Y gives (7.3.7) and (7.3.8), pushing forward further to X gives

$$\mathbf{R}\mathrm{Hom}_X(I_X^\bullet, \jmath_*\iota_* F) \to \mathbf{R}\mathrm{Hom}_X(I_X^\bullet, I_X^\bullet)_0[1] \to \mathbf{R}\mathrm{Hom}_X(\jmath_*\iota_* F, \mathcal{O}_X)[2],$$
$$\mathbf{L}\jmath^* I_X^\bullet \cong I_Y^\bullet \oplus \iota_* F \otimes K_Y^{-1},$$

where

$$I_X^\bullet = [\mathcal{O}_X \to \jmath_*\iota_* \mathcal{O}_S \xrightarrow{\jmath_*\iota_* s} \jmath_*\iota_* F]$$

and we denote the zero section by $\jmath : Y \hookrightarrow X$. Let T be the cone of the composition

$$\mathbf{R}\mathrm{Hom}_Y(I_Y^\bullet, \iota_* F) \to \mathbf{R}\mathrm{Hom}_X(I_X^\bullet, \jmath_*\iota_* F) \to \mathbf{R}\mathrm{Hom}_X(I_X^\bullet, I_X^\bullet)_0[1].$$

[8]This was proved for irreducible β in [40, Lem. 3.1].

Then T fits in the distinguished triangles

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_Y(I_Y^\bullet, \iota_* F) \to \mathbf{R}\mathrm{Hom}_X(I_X^\bullet, I_X^\bullet)_0[1] \to T,\\ &\mathbf{R}\mathrm{Hom}_Y(\iota_* F, \iota_* F \otimes K_Y) \to T \to \mathbf{R}\mathrm{Hom}_X(\jmath_*\iota_* F, \mathcal{O}_X)[2].\end{aligned} \tag{7.3.11}$$

Applying Serre duality to the first and third term of the second distinguished triangle, dualizing, and shifting gives the following distinguished triangle

$$\mathbf{R}\mathrm{Hom}_Y(\iota_* F, \iota_* F)[2] \to \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_X, \jmath_*\iota_* F)[2] \to T^\vee.$$

Comparing to (7.3.10), we obtain $T \cong \mathbf{R}\mathrm{Hom}_Y(I_Y^\bullet, \iota_* F)^\vee[-2]$. Hence from (7.3.11) we get a short exact sequence

$$0 \to \mathrm{Ext}^1_Y(I_Y^\bullet, \iota_* F) \to \mathrm{Ext}^2_X(I_X^\bullet, I_X^\bullet)_0 \to \mathrm{Ext}^1_Y(I_Y^\bullet, \iota_* F)^\vee \to 0,$$

where we crucially used $\mathrm{Ext}^2_Y(I_Y^\bullet, \iota_* F) = 0$ which was shown above. This way, we obtain a half-dimensional subspace $\mathrm{Ext}^1_Y(I_Y^\bullet, \iota_* F)$ of $\mathrm{Ext}^2_X(I_X^\bullet, I_X^\bullet)_0$. One can show that it is isotropic by the exact same argument as in the proof of [**40**, Prop. 3.3, Prop. 2.11]. From this, it is concluded in loc. cit. that

$$[P_n(X,\beta)]^{\mathrm{vir}} = (-1)^{\beta\cdot L_2 + n} \cdot [P_n(Y,\beta)]^{\mathrm{vir}}_{\mathrm{pair}}.$$

Here the sign comes from a choice of preferred orientation discussed in a similar setting in [**31**].

Finally, we express the pairs virtual class on Y in terms of the pairs virtual class on S. By adjunction, we have

$$\begin{aligned}\mathbf{R}\mathscr{H}\!om_{\pi_{P_Y}}(\mathbb{I}_Y^\bullet, \iota_{P_Y *}\mathbb{F}) &\cong \mathbf{R}\mathscr{H}\!om_{\pi_{P_S}}(\mathbf{L}\iota_{P_Y}^* \mathbb{I}_Y^\bullet, \mathbb{F})\\ &\cong \mathbf{R}\mathscr{H}\!om_{\pi_{P_S}}(\mathbb{I}_S^\bullet, \mathbb{F}) \oplus \mathbf{R}\mathscr{H}\!om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1),\end{aligned}$$

where $\pi_{P_S} : S \times P_n(S,\beta) \to P_n(S,\beta) \cong P_n(Y,\beta)$ denotes the projection. From the vanishing $\mathrm{Hom}_S(F, F \otimes L_1) = \mathrm{Hom}_S(F, F \otimes L_2) = 0$, for all $[(F,s)] \in P_n(S,\beta)$, we deduce that

$$-\mathbf{R}\mathscr{H}\!om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1) \cong \mathscr{E}xt^1_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1)$$

is locally free on $P_n(Y,\beta) \cong P_n(S,\beta)$. Hence the two virtual tangent bundles on $P_n(Y,\beta) \cong P_n(S,\beta)$ differ by a locally free sheaf (in degree 1). Therefore, by [**171**, Thm. 4.6], we have

$$[P_n(Y,\beta)]^{\mathrm{vir}}_{\mathrm{pair}} = e\big(-\mathbf{R}\mathscr{H}\!om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1)\big) \cdot [P_n(S,\beta)]^{\mathrm{vir}}. \qquad \square$$

Remark 7.3.3. Let S be a smooth projective surface satisfying $b_1(S) = p_g(S) = 0$. Let $L_1, L_2 \in \mathrm{Pic}(S)$ such that $L_1 \otimes L_2 \cong K_S$ and $X = \mathrm{Tot}_S(L_1 \oplus L_2)$. Suppose $P_n(X,\beta)$ is proper. Then by Oh-Thomas localization [**147**] there is an induced (algebraic) virtual fundamental class on each component of the fixed locus $P_n(X,\beta)^{\mathbb{C}^*}$, where $\mathbb{C}^*$ is the 1-dimensional subtorus, preserving the Calabi-Yau volume form, inside the torus $\mathbb{C}^* \times \mathbb{C}^*$ acting on the fibres of X. When $P_n(S,\beta) \subseteq P_n(X,\beta)^{\mathbb{C}^*}$ is open and closed, this gives a virtual class, which we expect to be given by (for an appropriate choice of orientation)

$$(-1)^{\beta\cdot L_2 + n} \cdot e\big(-\mathbf{R}\mathscr{H}\!om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1) \otimes \mathfrak{t}_1\big) \cdot [P_n(S,\beta)]^{\mathrm{vir}}, \tag{7.3.12}$$

where $[P_n(S,\beta)]^{\mathrm{vir}}$ is the virtual class induced from the relative Hilbert scheme (Section 7.3.1), $\mathfrak{t}_1$ is the irreducible character corresponding to the first component of the action of $\mathbb{C}^* \times \mathbb{C}^*$, and $e(\cdot)$ denotes equivariant Euler class. When $P_n(X,\beta)$ is non-proper, one could define the contribution of $P_n(S,\beta)$ to the stable pair invariants of X by (7.3.12) (capped with appropriate insertions).

7.3.3. Main theorem We are now ready to prove the theorem of the introduction. Recall from (7.1.6) that we denote by $T_{S^{[m]}}(\mathcal{L})$ the twisted (by $\mathcal{L}$) tangent bundle of $S^{[m]}$.

Theorem 7.3.4. *Let S be a smooth projective surface with $b_1(S) = p_g(S) = 0$ and $L_1, L_2 \in \mathrm{Pic}(S)$ such that $L_1 \otimes L_2 \cong K_S$. Suppose $\beta \in H_2(S,\mathbb{Z})$ and $n \geqslant 0$ are chosen such that $P_n(X,\beta) \cong P_n(S,\beta)$ for $X = \mathrm{Tot}_S(L_1 \oplus L_2)$. Denote by $[\mathrm{pt}] \in H^4(X,\mathbb{Z})$ the pull-back of the Poincaré dual of the point class on S. Let $P_n(X,\beta)$ be endowed with the orientation as in (7.1.5). Then*

$$P_{n,\beta}([\mathrm{pt}]) = (-1)^{\beta\cdot L_2+n} \int_{S^{[m]}\times\mathbb{P}^{\chi(\beta)-1}} c_m(\mathcal{O}_S(\beta)^{[m]}(1))\, \frac{h^n(1+h)^{\chi(L_1(\beta))}(1-h)^{\chi(L_2(\beta))}\, c(T_{S^{[m]}}(L_1))}{c(L_1(\beta)^{[m]}(1))\cdot c((L_2(\beta)^{[m]}(1))^\vee)},$$

when $\beta^2 \geqslant 0$. Here $m := n + g(\beta) - 1$ and $h := c_1(\mathcal{O}(1))$. Moreover, $P_{n,\beta}([\mathrm{pt}]) = 0$ when $\beta^2 < 0$.

Proof. Suppose β is an effective divisor and $m \geqslant 0$, otherwise $P_n(S,\beta) \cong \mathrm{Hilb}^m(\mathcal{C}/H_\beta) = \varnothing$ and $P_{n,\beta}([\mathrm{pt}]) = 0$. Consider the closed embedding

$$\jmath : \mathrm{Hilb}^m(\mathcal{C}/H_\beta) \hookrightarrow S^{[m]} \times H_\beta,$$

as in Section 7.3.1. Below, we will show that there exists a class $\psi \in K_0(S^{[m]} \times H_\beta)$ restricting to $e(-\mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1)) \cdot \tau([\mathsf{pt}])^n$ on $\mathrm{Hilb}^m(\mathcal{C}/H_\beta)$. By Proposition 7.3.2, it follows that

$$P_{n,\beta}([\mathsf{pt}]) = \int_{S^{[m]}\times[H_\beta]^{\mathrm{vir}}} c_m(\mathcal{O}(\mathcal{C})^{[m]}) \cdot \psi. \tag{7.3.13}$$

Since $b_1(S) = p_g(S) = 0$, we have [**67, 118**]

$$[H_\beta]^{\mathrm{vir}} = |\beta|^{\mathrm{vir}} = h^{h^1(\mathcal{O}(\beta))} \cap |\beta| \in H_{2\chi(\beta)-2}(|\beta|),$$

where h denotes the class of the hyperplane on $H_\beta = |\beta|$. Furthermore, we have

$$\mathcal{O}(\mathcal{C})^{[m]} := \pi_*(\mathcal{O}(\mathcal{C})|_{\mathcal{Z}}) \cong \mathcal{O}(\beta)^{[m]}(1),$$

which follows from the isomorphism $\mathcal{O}(\mathcal{C}) \cong \mathcal{O}_S(\beta) \boxtimes \mathcal{O}(1)$ on $S \times |\beta|$. Therefore $P_{n,\beta}([\mathsf{pt}]) = 0$, unless $\chi(\beta) \geqslant 1$, which we assume from now on.

Recall from the proof of Proposition 7.3.2 that $-\mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F}\boxtimes L_1) \cong \mathcal{E}xt^1_{\pi_{P_S}}(\mathbb{F}, \mathbb{F}\boxtimes L_1)$ is locally free on $P_n(S,\beta)$ and its rank is β^2. Therefore $P_{n,\beta}([\mathsf{pt}]) = 0$ unless $\beta^2 \geqslant 0$, which we assume from now on. Next, we extend the complex $-\mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1)$ from $P_n(S,\beta)$ to $S^{[m]} \times |\beta|$. In K-theory, we have $\mathbb{I}_S^\bullet = \mathcal{O} - \mathbb{F}$ and

$$\begin{aligned}
-\mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F}\boxtimes L_1) &= -\chi(L_1) \otimes \mathcal{O} + \mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathcal{O}, \mathbb{I}_S^\bullet \boxtimes L_1) \\
&\quad + \mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{I}_S^\bullet, L_1) - \mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{I}_S^\bullet, \mathbb{I}_S^\bullet \boxtimes L_1) \\
&= -\chi(L_1) \otimes \mathcal{O} + \mathbf{R}\mathscr{H}om_{\pi_{P_S}}((\mathbb{I}_S^\bullet)^\vee, L_1) \\
&\quad + \mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathcal{O}, (\mathbb{I}_S^\bullet)^\vee \boxtimes L_1) - \mathbf{R}\mathscr{H}om_{\pi_{P_S}}((\mathbb{I}_S^\bullet)^\vee, (\mathbb{I}_S^\bullet)^\vee \boxtimes L_1)),
\end{aligned}$$

where we suppressed some obvious pull-backs. On $S \times |\beta| \times S^{[m]}$, we have the sheaf $\mathcal{I} \boxtimes \mathcal{O}_S(\beta) \boxtimes \mathcal{O}(1)$, where we use the notation from the introduction. By [**117**, Lem. A.4], we have

$$(\mathbb{I}_S^\bullet)^\vee \cong \mathcal{I} \boxtimes \mathcal{O}_S(\beta) \boxtimes \mathcal{O}(1)|_{\mathrm{Hilb}^m(\mathcal{C}/|\beta|) \times S}.$$

Next we can replace $\mathcal{I}$ by $\mathcal{O} - \mathcal{O}_{\mathcal{Z}}$ in K-theory. Then $-\mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1)$ is the restriction of the following element in the K-group of $S^{[m]} \times |\beta|$

$$\begin{aligned}
&- \chi(L_1) \otimes \mathcal{O} + \mathbf{R}\mathscr{H}om_\pi\big((\mathcal{O} - \mathcal{O}_{\mathcal{Z}}) \boxtimes (\mathcal{O}_S(\beta) \otimes L_1^{-1}) \boxtimes \mathcal{O}(1), \mathcal{O}\big) \\
&\quad + \mathbf{R}\mathscr{H}om_\pi\big(\mathcal{O}, (\mathcal{O} - \mathcal{O}_{\mathcal{Z}}) \boxtimes (\mathcal{O}_S(\beta) \otimes L_1) \boxtimes \mathcal{O}(1)\big) - \mathbf{R}\mathscr{H}om_\pi(\mathcal{I}, \mathcal{I} \boxtimes L_1) \\
&= -\chi(L_1) \otimes \mathcal{O} + \chi(L_1(\beta)) \otimes \mathcal{O}(1) + \chi(L_2(\beta)) \otimes \mathcal{O}(-1) \\
&\quad - (L_1(\beta))^{[m]} \boxtimes \mathcal{O}(1) - \big((L_2(\beta))^{[m]}\big)^\vee \boxtimes \mathcal{O}(-1) - \mathbf{R}\mathscr{H}om_\pi(\mathcal{I}, \mathcal{I} \boxtimes L_1),
\end{aligned} \tag{7.3.14}$$

where $\pi : S \times S^{[m]} \times |\beta| \to S^{[m]} \times |\beta|$ denotes the projection and we used Serre duality, $L_1 \otimes L_2 \cong K_S$, and $L_i(\beta) := \mathcal{O}_S(\beta) \otimes L_i$.

Finally, we consider primary insertions

$$\tau \colon H^4(X, \mathbb{Z}) \to H^2(P_n(X, \beta), \mathbb{Z}), \quad \tau(\gamma) = \pi_{P*}(\pi_X^* \gamma \cup \mathrm{ch}_3(\iota_* \mathbb{F})),$$

where π_X, π_P are projections from $X \times P_n(X, \beta)$ to corresponding factors, and $\iota : S \times P_n(S, \beta) \hookrightarrow X \times P_n(S, \beta) \cong X \times P_n(X, \beta)$ is the base change of the inclusion of the zero section. Note that $\mathrm{ch}_3(\iota_* \mathbb{F})$ is Poincaré dual to the fundamental class of the scheme theoretic support of $\iota_* \mathbb{F}$, which we denote by $[\iota_* \mathbb{F}]$. The fundamental class of the scheme theoretic support of $\mathbb{F}$, which we denote by $[\mathbb{F}]$, equals $\jmath^*[\mathcal{C}]$ where $\mathcal{C} \subseteq S \times S^{[m]} \times |\beta|$ is the pullback of the universal curve over $|\beta|$. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "p"] & S \arrow[l, bend right, "\iota"'] & \\
X \times P_n(X, \beta) \arrow[u, "\pi_X"] \arrow[r, "p"] \arrow[d, "\pi_P"'] & S \times P_n(S, \beta) \arrow[l, bend right, "\iota"'] \arrow[u, "\pi_S"] \arrow[r, "\jmath"] \arrow[d, "\pi_P"'] & S \times S^{[m]} \times |\beta| \arrow[ul, "\pi_S"'] \arrow[d, "\pi"] \\
P_n(X, \beta) \arrow[r, "\cong"] & P_n(S, \beta) \arrow[r, "\jmath"] & S^{[m]} \times |\beta|
\end{tikzcd}$$

For $\gamma := p^*[\mathrm{pt}] \in H^4(X, \mathbb{Z})$, where $[\mathrm{pt}]$ denotes the Poincaré dual of the point class on S, we work our way through the diagram

$$\begin{aligned}
\tau(\gamma) &= \pi_{P*}(\pi_X^* \gamma \cup [\iota_* \mathbb{F}]) \\
&= \pi_{P*}\, p_*(p^* \pi_S^* [\mathsf{pt}] \cup \iota_* [\mathbb{F}]) \\
&= \pi_{P*}(\pi_S^* [\mathsf{pt}] \cup [\mathbb{F}]) \\
&= \jmath^*\big(\pi_*(\pi_S^* [\mathsf{pt}] \cup [\mathcal{C}])\big).
\end{aligned} \tag{7.3.15}$$

Using, once more, that on $S \times |\beta|$ we have $\mathcal{O}(\mathcal{C}) \cong \mathcal{O}_S(\beta) \boxtimes \mathcal{O}(1)$, we conclude

$$\pi_*(\pi_S^* [\mathsf{pt}] \cup [\mathcal{C}]) = h \int_S \mathsf{pt} = h.$$

Therefore $\tau([\mathsf{pt}])^n$ is simply the restriction of the class h^n on $S^{[m]} \times |\beta|$.

Since $\mathrm{rk}(-\mathbf{R}\mathscr{H}om_{\pi_{P_S}}(\mathbb{F}, \mathbb{F} \boxtimes L_1)) = \beta^2 = 2m + \chi(\beta) - 1 - m - n$, we can replace Euler class by total Chern class. The result now follows from (7.3.13), (7.3.14), and (7.3.15). □

Remark 7.3.5. For surfaces with $p_g(S) = 0$ and $b_1(S) > 0$, we can still use the formula for the virtual class from Proposition 7.3.2. Suppose $h^2(L) = 0$ for all $L \in \mathrm{Pic}_\beta(S)$. Then the virtual class $[H_\beta]^{\mathrm{vir}}$ can be calculated by fixing a sufficiently ample effective divisor A on S and considering the embedding $H_\beta \hookrightarrow H_{[A]+\beta}$ as in [**67**] (see also [**117**, Prop. A.2], [**118**]). Therefore the invariant can be expressed as an integral over $S^{[m]} \times H_{[A]+\beta}$, where $H_{[A]+\beta}$ is a projective bundle over $\mathrm{Pic}_{[A]+\beta}(S)$ via the Abel-Jacobi map. Pushing forward along the Abel-Jacobi map, the invariant can be expressed as an integral over $S^{[m]} \times \mathrm{Pic}_{A+\beta}(S)$.

7.3.4. Atiyah-Bott localization In Corollary 7.2.9, Remark 7.2.10, we gave examples of (S, L_1, L_2) for which the assumptions of Theorem 7.3.4 are satisfied. In all of these cases, S is a toric surface. As a consequence, $X = \mathrm{Tot}_S(L_1 \oplus L_2)$ is also toric, so in principle one could calculate the invariant $P_{n,\beta}([\mathrm{pt}])$ using the vertex formalism for stable pair invariants on toric Calabi-Yau 4-folds developed in [**35, 36**][9]. However, the number of $(\mathbb{C}^*)^4$-fixed points is typically very large. For instance, for $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$, $(d_1, d_2) = (2, 4)$, and $n = 0$, we have 182 fixed points, whereas Theorem 7.3.4 only involves an integral over $S^{[2]} \times \mathbb{P}^{14}$.

The calculation of intersection numbers on Hilbert schemes of points on toric surfaces is a classical subject (see e.g. [**74**]). Let S be a smooth projective toric surface with torus $\mathbf{T} = (\mathbb{C}^*)^2$. The action of $\mathbf{T}$ on S lifts to an action of $\mathbf{T}$ on $S^{[m]}$ for any m. Let P be a polynomial expression in Chern classes of

$$\mathbf{R}\Gamma(\mathcal{L}) \otimes \mathcal{O} - \mathbf{R}\mathscr{H}om_\pi(\mathcal{I}, \mathcal{I} \boxtimes \mathcal{L}), \quad \mathcal{L}^{[m]}, \tag{7.3.16}$$

for various choices of $\mathbf{T}$-equivariant line bundles $\mathcal{L}$ on S and where $\pi : S \times S^{[m]} \to S^{[m]}$ denotes the projection. Note that this includes Chern classes of the tangent bundle, which can be expressed as $-\mathbf{R}\mathscr{H}om_\pi(\mathcal{I}, \mathcal{I})_0$ (since $\mathrm{Ext}^i(I_Z, I_Z)_0 = 0$ for $Z \in S^{[m]}$ and $i \neq 1$). Suppose also that the degree of P, as a class in the Chow ring $A^*(S^{[m]})$, equals $\dim S^{[m]} = 2m$. By the Atiyah-Bott localization formula (Theorem 2.1.5), we have

$$\int_{S^{[m]}} P = \int_{(S^{[m]})^{\mathbf{T}}} \frac{P|_{(S^{[m]})^{\mathbf{T}}}}{e(N_{(S^{[m]})^{\mathbf{T}}/S^{[m]}})},$$

where $e(\cdot)$ denotes the $\mathbf{T}$-equivariant Euler class and $N_{(S^{[m]})^{\mathbf{T}}/S^{[m]}}$ is the normal bundle of the fixed point locus $(S^{[m]})^{\mathbf{T}} \subseteq S^{[m]}$. Furthermore, in this formula one has to choose a $\mathbf{T}$-equivariant lift of P. More precisely, one can choose a $\mathbf{T}$-equivariant structure on all (complexes of) sheaves appearing in P and replace all Chern classes appearing in P by $\mathbf{T}$-equivariant Chern classes.

[9]In loc. cit. it is assumed that the fixed locus $P_n(X, \beta)^{(\mathbb{C}^*)^4}$ is at most 0-dimensional. This is the case for all local Calabi-Yau 4-fold surfaces.

The fixed point locus consists of isolated reduced points, which can be described combinatorially. Consider a cover by maximal T-invariant affine open subsets:

$$\left\{U_\sigma \cong \operatorname{Spec}\mathbb{C}[x_\sigma, y_\sigma]\right\}_{\sigma=1}^{e(S)}.$$

Then the fixed locus $(S^{[m]})^{\mathbf{T}}$ precisely consists of the closed subschemes of S defined by collections of monomial ideals

$$\left\{I_\sigma \subseteq \mathbb{C}[x_\sigma, y_\sigma]\right\}_{\sigma=1}^{e(S)}$$

of total colength m. The monomial ideals of finite colength in $\mathbb{C}[x,y]$ are in bijective correspondence with partitions. Explicitly, $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_\ell)$ corresponds to the ideal

$$\left(y^{\lambda_1}, xy^{\lambda_2}, \ldots, x^{\ell-1}y^{\lambda_\ell}, x^\ell\right),$$

where $\ell(\lambda) = \ell$ is the length of λ. Hence we can index the points of the fixed locus $(S^{[m]})^T$ by collections of partitions

$$\boldsymbol{\lambda} = \left\{\lambda^{(\sigma)}\right\}_{\sigma=1}^{e(S)}$$

of total size

$$\sum_{\sigma=1}^{e(S)} |\lambda^{(\sigma)}| = \sum_{\sigma=1}^{e(S)} \sum_{i=1}^{\ell(\lambda^{(\sigma)})} \lambda_i^{(\sigma)} = m.$$

Denote the closed subscheme corresponding to $\boldsymbol{\lambda}$ by $Z_{\boldsymbol{\lambda}}$.

In order to calculate integrals such as the one in Theorem 7.3.4 by Atiyah-Bott localization, we need to consider Chern classes of

(7.3.17)

$$\mathcal{L}^{[m]}|_{Z_{\boldsymbol{\lambda}}} = H^0(\mathcal{L}|_{Z_{\boldsymbol{\lambda}}}) \in K_0^{\mathbf{T}}(\mathsf{pt}) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}],$$

$$\Big(\mathbf{R}\Gamma(\mathcal{L}) \otimes \mathcal{O} - \mathbf{R}\mathscr{H}om_\pi(\mathcal{I}, \mathcal{I} \boxtimes \mathcal{L})\Big)\Big|_{Z_{\boldsymbol{\lambda}}} = \mathbf{R}\Gamma(\mathcal{L}|_{Z_{\boldsymbol{\lambda}}}) - \mathbf{R}\mathrm{Hom}_S(I_{Z_{\boldsymbol{\lambda}}}, I_{Z_{\boldsymbol{\lambda}}} \otimes \mathcal{L}) \in K_0^{\mathbf{T}}(\mathsf{pt}),$$

where t_1, t_2 are the equivariant parameters of T.

Suppose $Z_{\boldsymbol{\lambda}}$ is a 0-dimensional T-equivariant subscheme supported entirely on a maximal T-invariant affine open subset U_σ and set $\lambda := \lambda^{(\sigma)}$. Suppose we choose coordinates such that $U_\sigma = \operatorname{Spec}\mathbb{C}[x,y]$ and the torus action (on coordinate functions) is given by $(t_1,t_2)\cdot(x,y) = (t_1x, t_2y)$. Denote the character corresponding to $\mathcal{L}|_{U_\sigma}$ by $\chi(t_1,t_2)$. Then

(7.3.18)

$$H^0(\mathcal{L}|_{Z_{\boldsymbol{\lambda}}}) = \chi(t_1,t_2)\cdot Z_\lambda,$$
$$\text{where} \quad Z_\lambda := \sum_{i=1}^{\ell(\lambda)} \sum_{j=1}^{\lambda_i} t_1^{i-1} t_2^{j-1}.$$

Now suppose $W_{\boldsymbol{\mu}}$ is a second 0-dimensional T-equivariant subscheme supported entirely on U_σ and write $\mu := \mu^{(\sigma)}$. The following formula can be deduced from a well-known calculation using Čech cohomology (e.g. see [84, Prop. 4.1]):

(7.3.19)

$$\mathbf{R}\mathrm{Hom}_S(\mathcal{O}_{W_{\boldsymbol{\mu}}}, \mathcal{O}_{Z_{\boldsymbol{\lambda}}} \otimes \mathcal{L}) = \chi(t_1,t_2)\, W_\mu^* Z_\lambda \frac{(1-t_1)(1-t_2)}{t_1t_2} \in K_0^{\mathbf{T}}(\mathsf{pt}),$$
$$\text{where} \quad W_\mu := \sum_{i=1}^{\ell(\mu)} \sum_{j=1}^{\mu_i} t_1^{i-1} t_2^{j-1}.$$

Here $(\cdot)^*$ is the involution defined by dualizing.

For arbitrary $Z_{\boldsymbol{\lambda}}$, the K-group classes of (7.3.17) can be determined from (7.3.18) and (7.3.19) by using the following equalities in K-theory

$$\mathcal{O}_{Z_{\boldsymbol{\lambda}}} = \sum_{\sigma=1}^{e(S)} \mathcal{O}_{Z_{\lambda^{(\sigma)}}},$$
$$I_{\mathcal{O}_{Z_{\boldsymbol{\lambda}}}} = \mathcal{O}_S - \mathcal{O}_{Z_{\boldsymbol{\lambda}}},$$

where $Z_{\lambda^{(\sigma)}}$ denotes the 0-dimensional closed subscheme supported on U_σ determined by $\lambda^{(\sigma)}$.

Consider Theorem 7.3.4 for the examples of (S, L_1, L_2) listed in Proposition 7.2.9 and Remark 7.2.10. In each case, we calculated the invariant $P_{n,\beta}([\mathsf{pt}])$ by first integrating out the linear system $\mathbb{P}^{\chi(\beta)-1}$. This amounts to expanding the integrand in powers of $h = c_1(\mathcal{O}(1))$ and taking the coefficient of $h^{\chi(\beta)-1}$. This gives a polynomial expression P in Chern classes of complexes of the form (7.3.16). The integral $\int_{S^{[m]}} P$ is then calculated by Atiyah-Bott localization as described. The resulting stable pair invariants are tabulated in Appendix 7.4.

With the numbers of Appendix 7.4, we are able to do various new checks of the Cao-Maulik-Toda conjectures (Conjectures 7.1.1 and 7.1.2). Combining our tables in Appendix 7.4 with the tables for $n_{0,\beta}([\mathsf{pt}]), n_{1,\beta}$ in [**108**, Sect. 3] gives Corollary 7.1.6 of the introduction.

Bousseau-Brini-van Garrel [**24**] determined all the genus zero Gromov-Witten (and therefore Gopakumar-Vafa type) invariants of $X = \mathrm{Tot}_S(L_1 \oplus L_2)$ with (S, L_1, L_2) as in Remark 7.2.10 (as well as for other cases). Note that the method of Bousseau-Brini-van Garrel does not produce genus one Gopakumar-Vafa type invariants, so we can only do verifications of Conjecture 7.1.1 in these cases. Combining the tables in Appendix 7.4 with the values for genus zero Gopakumar-Vafa type invariants provided to us by Bousseau-Brini-van Garrel gives Corollary 7.1.8 of the introduction.

Recall that for (S, L_1, L_2) with $L_1 \otimes L_2 \cong K_S$, L_1^{-1}, L_2^{-1} non-trivial and nef, and S minimal and toric, we classified all values of $n \geqslant 0$ for which $P_n(S, \beta) \cong P_n(X, \beta)$, and $P_n(S, \beta) \neq \varnothing$ (Remark 7.2.10). In these cases we therefore calculated *all* stable pair invariants $P_{n,\beta}([\mathsf{pt}])$.

7.4. Appendix: Tables

7.4.1. Local surfaces In this section, we list the stable pair invariants $P_{n,\beta}([\mathrm{pt}])$ of $X = \mathrm{Tot}_S(L_1 \oplus L_2)$ for the cases mentioned in Proposition 7.2.9 and Remark 7.2.10, and for a few additional cases. We use the following conventions and notation:

- $P_{0,0} := 1$ and $P_{n,0}([\mathsf{pt}]) = 0$ for all $n > 0$.
- Entries decorated with $\star$ were defined by a virtual localization formula on the fixed locus and have been calculated using the vertex formalism as discussed in [**35, 36**]. In these cases $P_n(X, \beta) \setminus P_n(S, \beta) \neq \varnothing$. See Remark 7.4.2 for a comparison to the globally defined invariants. All other entries have been computed using Theorem 7.3.4,

- Zeroes decorated with † have non-empty underlying moduli space $P_n(X, \beta)$. In this sense, they are "non-trivial" zeroes.

For $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$, we calculated the following values for $P_{0,d} := P_{0,d[H]}$ and $P_{n,d}([\mathsf{pt}]) := P_{n,d[H]}([\mathsf{pt}])$ (for $n > 0$).

$d \setminus n$	0	1	2	3	4
1	0	-1	$0^\dagger$	$0^\dagger$	$0^\dagger$
2	0	1	1	$0^{\star,\dagger}$	
3	-1	-1	-2		
4	2	3			

$$P_{n,1}([\mathsf{pt}]) = 0^\dagger, \ \ \forall\, n \geqslant 2.$$

For $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1), \mathcal{O}(-1,-1))$ and $P_{0,(d_1,d_2)} := P_{0,d_1[H_1]+d_2[H_2]}$ and $P_{n,(d_1,d_2)}([\mathsf{pt}]) := P_{n,d_1[H_1]+d_2[H_2]}([\mathsf{pt}])$ (for $n > 0$), where H_1, H_2 are defined in Example 7.1.4, we have

$P_{0,(d_1,d_2)}$	0	1	2	3	4
0	1	0	0	0	0
1	0	0	0	0	0
2	0	0	1	2	5
3	0	0	2	10	
4	0	0	5		

$P_{1,(d_1,d_2)}([\mathsf{pt}])$	0	1	2	3	4
0	0	1	0	0	0
1	1	1	1	1	1
2	0	1	2	5	
3	0	1	5		
4	0	1			

$P_{2,(d_1,d_2)}([\mathsf{pt}])$	0	1	2	3	4
0	0	$0^\dagger$	1	0	0
1	$0^\dagger$	2	2	2	2
2	1	2	5		
3	0	2			
4	0	2			

$P_{3,(d_1,d_2)}([\mathsf{pt}])$	0	1	2	3	4
0	0	$0^\dagger$	$0^{\star,\dagger}$	1	0
1	$0^\dagger$	$0^\dagger$	$3^\star$		
2	$0^{\star,\dagger}$	$3^\star$			
3	1				
4	0				

$$\begin{aligned}
P_{0,(1,d)} = P_{0,(d,1)} &= 0, \ \ \forall\, d \geqslant 0 \\
P_{1,(1,d)}([\mathsf{pt}]) = P_{1,(d,1)}([\mathsf{pt}]) &= 1, \ \ \forall\, d \geqslant 0 \\
P_{2,(1,d)}([\mathsf{pt}]) = P_{2,(d,1)}([\mathsf{pt}]) &= 2, \ \ \forall\, d \geqslant 1 \\
P_{n,(0,d)}([\mathsf{pt}]) = P_{n,(d,0)}([\mathsf{pt}]) &= \delta_{n,d}, \ \ \forall\, 0 \leqslant n \leqslant d \\
P_{n,(1,1)}([\mathsf{pt}]) &= 0^\dagger, \ \ \forall\, n \geqslant 3, \\
P_{n,(0,1)}([\mathsf{pt}]) = P_{n,(1,0)}([\mathsf{pt}]) &= 0^\dagger, \ \ \forall\, n \geqslant 2.
\end{aligned}$$

For $(S, L_1, L_2) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,0), \mathcal{O}(-1,-2))$, we have:

$$P_{0,(2,2)} = 1, \ \ P_{0,(2,3)} = 2, \ \ P_{0,(2,4)} = 5, \ \ P_{0,(3,2)} = 2,$$

$$P_{1,(2,2)}([\mathsf{pt}]) = 2, \;\; P_{1,(2,3)}([\mathsf{pt}]) = 5,$$
$$P_{1,(d,1)}([\mathsf{pt}]) = P_{1,(1,d)}([\mathsf{pt}]) = 1, \;\; \forall\, d \geqslant 1$$
$$P_{2,(1,d)}([\mathsf{pt}]) = 2, \;\; \forall\, d \geqslant 2,$$
$$P_{n,(0,1)}([\mathsf{pt}]) = 0^{\dagger}, \;\; \forall\, n \geqslant 2$$
$$P_{n,(0,n)}([\mathsf{pt}]) = 1, \;\; \forall\, n \geqslant 1.$$

Recall the notation for Hirzebruch surfaces from the introduction. Consider $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(-1,-1), \; \mathcal{O}(-1,-2))$. We write $P_{0,(d_1,d_2)} := P_{0,d_1[B]+d_2[F]}$ and, for $n > 0$, $P_{n,(d_1,d_2)}([\mathsf{pt}]) := P_{n,d_1[B]+d_2[F]}([\mathsf{pt}])$. In the tables below, the rows are for d_1 and the columns for d_2.

$P_{0,(d_1,d_2)}$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	1	2	5		
3	0	0	0	-1				

$P_{1,(d_1,d_2)}([\mathsf{pt}])$	0	1	2	3	4	5	6	7
0	0	1	0	0	0	0	0	0
1		-1	-1	-1	-1	-1	-1	-1
2	0	0	1	2	5			
3	0	0	0					

$$P_{1,(1,d)}([\mathsf{pt}]) = -1, \quad \forall\, d \geqslant 1$$
$$P_{2,(1,d)}([\mathsf{pt}]) = -2, \quad \forall\, d \geqslant 2$$
$$P_{3,(0,2)}([\mathsf{pt}]) = 0^{\star,\dagger},$$
$$P_{n,(0,1)}([\mathsf{pt}]) = 0^{\dagger}, \;\; \forall\, n \geqslant 2$$
$$P_{n,(0,n)}([\mathsf{pt}]) = 1, \;\; \forall\, n \geqslant 1.$$

Remark 7.4.1. Denoting the exceptional curve of $\mathbb{F}_1$ by B, we have $N_{B/X} = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(-1)$, which has sections in the direction of L_1. Therefore $P_1(X,[B])$ is non-proper, which explains the gap in the table for $P_{1,(1,0)}([\mathsf{pt}])$.

For $(S, L_1, L_2) = (\mathbb{F}_1, \mathcal{O}(0,-1), \mathcal{O}(-2,-2))$, we have

$P_{0,(d_1,d_2)}$	0	1	2	3	4
0	1	0	0	0	0
1	0	0	0	0	0
2	0	0	0	1	2
3	0	0	0	-1	

$P_{1,(d_1,d_2)}([\mathsf{pt}])$	0	1	2	3	4
0	0		0	0	0
1		-1	-1	-1	-1
2	0	0	1	2	
3	0	0	0		

$$P_{1,(1,d)}([\mathsf{pt}]) = -1, \quad \forall\, d \geqslant 1$$

For $(S, L_1, L_2) = (\mathbb{F}_2, \mathcal{O}(-1,-2), \mathcal{O}(-1,-2))$, we have

$P_{0,(d_1,d_2)}$	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	1	2	5	

$P_{1,(d_1,d_2)}([\mathsf{pt}])$	0	1	2	3	4	5	6	7
0	0	1	0	0	0	0	0	0
1		1	1	1	1	1	1	1
2	0	0	0	1	2	5		

$$P_{1,(1,d)}([\mathsf{pt}]) = 1, \quad \forall\, d \geqslant 1$$
$$P_{2,(1,d)}([\mathsf{pt}]) = 2, \quad \forall\, d \geqslant 2$$
$$P_{n,(0,1)}([\mathsf{pt}]) = 0^{\dagger}, \ \ \forall\, n \geqslant 2$$
$$P_{n,(0,n)}([\mathsf{pt}]) = 1, \ \ \forall\, n \geqslant 1.$$

7.4.2. Local $\mathbb{P}^3$ Consider $X = \mathrm{Tot}_{\mathbb{P}^3}(K_{\mathbb{P}^3})$. Let $P_{0,d} := P_{0,d[\ell]}$ and $P_{n,d}([\ell]) := P_{n,d[\ell]}([\ell])$ (for $n > 0$), where $[\ell] \in H_2(\mathbb{P}^3, \mathbb{Z}) \cong H_2(X, \mathbb{Z})$ denotes the class of a line $\ell \subseteq \mathbb{P}^3$ and we also write $[\ell] \in H^4(X, \mathbb{Z})$ for the pull-back of its Poincaré dual from $\mathbb{P}^3$ to X. Obviously, $X = \mathrm{Tot}_{\mathbb{P}^3}(K_{\mathbb{P}^3})$ is not a local surface so Theorem 7.3.4 does not apply. All stable pair invariants in this section have been calculated using the vertex formalism of [**35, 37**] (this is stressed by decorating the invariants with $\star$). We determined the following values of $P_{0,d}$ and $P_{n,d}([\ell])$.

$d \setminus n$	0	1	2	3	4
1	$0^{\star}$	$-20^{\star}$	$0^{\star,\dagger}$	$0^{\star,\dagger}$	$0^{\star,\dagger}$
2	$0^{\star}$	$-820^{\star}$	$400^{\star}$		
3	$11200^{\star}$	$-68060^{\star}$			

$$P_{n,1}([\ell]) = 0, \ \ \forall\, n \geqslant 2.$$

Remark 7.4.2. For $X = \mathrm{Tot}_{\mathbb{P}^3}(K_{\mathbb{P}^3})$ and all the cases in this table, we have $P_n(\mathbb{P}^3, \beta) \cong P_n(X, \beta)$. This can be deduced from a filtration argument similar to Proposition 7.2.9 combined with the fact that all degree 2 Cohen-Macaulay curves C on $\mathbb{P}^3$ satisfy $\chi(\mathcal{O}_C) \geqslant 1$ (see also [**44**, p. 20]). Therefore, the reasoning of Proposition 7.3.2 yields

$$[P_n(X, \beta)]^{\mathrm{vir}} = (-1)^{\beta \cdot c_1(\mathbb{P}^3) + n} \cdot [P_n(\mathbb{P}^3, \beta)]^{\mathrm{vir}}_{\mathrm{pair}}, \tag{7.4.1}$$

where $[P_n(\mathbb{P}^3, \beta)]^{\mathrm{vir}}_{\mathrm{pair}}$ is the virtual class of the pair perfect obstruction theory (7.3.9) on $\mathbb{P}^3$ (see also [**40**, Lem. 3.1] in a similar setting). The sign in this formula is a preferred choice of orientation on $P_n(X, \beta)$ as for (7.1.5). Now we are in the world of "ordinary" perfect obstruction theories and the torus action on $\mathbb{P}^3$ can be used to apply the Graber-Pandharipande virtual localization formula [**85**] to the right hand side of (7.4.1). Similar to the argument for [**35**, Thm. A.1], it then follows that the invariants in this table (defined by localization on the fixed locus [**35, 36**]) are equal to the global invariants (7.3.5). This reasoning also works for the local surface case $(S, L_1, L_2) = (\mathbb{P}^2, \mathcal{O}(-1), \mathcal{O}(-2))$, $d = 2$, $n = 3$, because then all stable pairs are scheme theoretically supported in the threefold $Y = \mathrm{Tot}_S(L_1)$.

PART III

Nested Quot schemes

CHAPTER 8

On the motive of the nested Quot scheme of points on a curve

Tuona la montagna, Etna
Soffi di fuoco
Valle di sabbia nera

Haiku, Andrea Pavlov

8.1. Introduction

Let $K_0(\mathrm{Var}_{\mathbf{k}})$ be the Grothendieck ring of varieties over an algebraically closed field $\mathbf{k}$. If Y is a $\mathbf{k}$-variety, its *motivic zeta function*

$$\zeta_Y(q) = 1 + \sum_{n>0} \left[\operatorname{Sym}^n Y\right] q^n \in K_0(\mathrm{Var}_{\mathbf{k}})[\![q]\!]$$

is a generating series introduced by Kapranov in [101], where he proved that for smooth curves it is a rational function in q.

In this chapter we compute the motive of the *nested Quot scheme of points* $\mathrm{Quot}_C(E, \boldsymbol{n})$ on a smooth curve C, entirely in terms of $\zeta_C(q)$. Here, E is a locally free sheaf on C, and $\boldsymbol{n} = (0 \leq n_1 \leq \cdots \leq n_d)$ is a non-decreasing tuple of integers, for some fixed $d > 0$. The scheme $\mathrm{Quot}_C(E, \boldsymbol{n})$ generalises the classical Quot scheme of Grothendieck (recovered when $d = 1$): it parametrises flags of quotients $E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1$ where T_i is a 0-dimensional sheaf of length n_i.

Our main result, proved in Theorem 8.5.2 in the main body, is the following.

Theorem 8.1.1. *Let C be a smooth curve over $\mathbf{k}$, let E be a locally free sheaf of rank r on C. Then*

$$\sum_{0 \leq n_1 \leq \cdots \leq n_d} \left[\mathrm{Quot}_C(E, \boldsymbol{n})\right] q_1^{n_1} \cdots q_d^{n_d} = \prod_{\alpha=1}^{r} \prod_{i=1}^{d} \zeta_C\left(\mathbb{L}^{\alpha-1} q_i^{d-i+1}\right) \in K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!],$$

where $\mathbb{L} = [\mathbb{A}^1_{\mathbf{k}}]$ is the Lefschetz motive. In particular, this generating function is rational in $q_1, \ldots, q_d$.

The statement taken with $d = 1$, thus regarding the motive $[\mathrm{Quot}_C(E, n)]$ of the usual Quot scheme of points, was proved in [**8**]. Our result is a natural generalisation, which was inspired by Mochizuki's paper on *Filt schemes* [**131**].

Our formula fits nicely in the philosophical path according to which

"rank r theories factorise in r rank 1 theories".

There are to date a number of examples of this phenomenon in Donaldson–Thomas theory, exhibiting a generating series of rank r invariants as a product of r (suitably shifted) generating series of rank 1 invariants: see for instance [**9, 165**] for enumerative DT invariants, [**76**] for K-theoretic DT invariants, [**48, 49**] for motivic DT invariants and [**143, 62**] for the parallel pictures in string theory.

The chapter is organised as follows. In Section 8.2 we introduce the *nested Quot scheme* and prove its connectedness. In Section 8.3 we describe its tangent space and prove that, for a smooth quasiprojective curve, the nested Quot scheme is smooth. Under the assumption that the locally free sheaf is split, in Section 8.4 we describe a torus framing action and its associated Białynicki-Birula decomposition. In Section 8.5 we prove that the motive of the nested Quot scheme is independent of the locally free sheaf, and exploit the Białynicki-Birula decomposition to prove Theorem 8.1.1. Our result readily implies closed formulae for the generating series of Hodge–Deligne polynomials, χ_y-genera, Poincaré polynomials, Euler characteristics, since these are all motivic measures; we provide some explicit formulae in Section 8.5.4.

After [**134**] was written, we were informed that our formula for the motive of the nested Quot scheme on a *projective* curve can be alternatively obtained, after some manipulations, from general results on the stack of iterated Hecke correspondences [**79**, Corollary 4.10] (see also [**92**, Section 3] for a related computation of the Voevodsky motive with rational coefficients). This chapter provides a direct and self-contained argument for this formula, exploiting the geometry of the nested Quot scheme.

Notation. All *schemes* are of finite type over an algebraically closed field $\mathbf{k}$. A *variety* is a reduced separated $\mathbf{k}$-scheme. If Y is a scheme and $Y_1, \ldots, Y_s$ are locally closed subschemes of Y, we say that they form a (locally closed) *stratification*, denoted '$Y = Y_1 \amalg \cdots \amalg Y_s$' with a slight abuse of notation, if the natural morphism of schemes $Y_1 \amalg \cdots \amalg Y_s \to Y$ is bijective. This is crucial in our calculations since this condition implies the identity $[Y] = [Y_1] + \cdots + [Y_s]$ in $K_0(\mathrm{Var}_{\mathbf{k}})$.

8.2. Nested Quot schemes of points

8.2.1. The moduli space Let X be a quasiprojective $\mathbf{k}$-variety and E a coherent sheaf on X. Fix an integer $d > 0$ and a non-decreasing d-tuple $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$ of non-negative integers $n_i \in \mathbb{Z}_{\geq 0}$. We define the *nested Quot functor* associated to $(X, E, \boldsymbol{n})$ to be the functor $\mathsf{Quot}_X(E, \boldsymbol{n})\colon \mathrm{Sch}_{\mathbf{k}}^{\mathrm{op}} \to \mathrm{Sets}$ sending a $\mathbf{k}$-scheme B to the set of isomorphism classes of subsequent quotients

$$E_B \twoheadrightarrow \mathcal{T}_d \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{T}_1,$$

where E_B is the pullback of E along $X \times_{\mathbf{k}} B \to X$ and $\mathcal{T}_i \in \mathrm{Coh}(X \times_{\mathbf{k}} B)$ is a B-flat family of 0-dimensional sheaves of length n_i over X for all $i = 1, \ldots, d$. Two 'nested quotients'

$$E_B \twoheadrightarrow \mathcal{T}_d \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{T}_1, \qquad E_B \twoheadrightarrow \mathcal{T}'_d \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{T}'_1$$

are considered isomorphic when $\ker(E_B \twoheadrightarrow \mathcal{T}_i) = \ker(E_B \twoheadrightarrow \mathcal{T}'_i)$ for all $i = 1, \ldots, d$.

The representability of the functor $\mathsf{Quot}_X(E, \boldsymbol{n})$ can be proved adapting the proof of [**170**, Theorem 4.5.1] or by an explicit induction on d as in [**94**, Section 2.A.1]. We define $\mathrm{Quot}_X(E, \boldsymbol{n})$ to be the moduli scheme representing the above functor. Its closed points are then in bijection with the set of isomorphism classes of nested quotients

$$E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1,$$

where each $T_i \in \mathrm{Coh}(X)$ is a 0-dimensional quotient of E of length n_i. The nested Quot scheme comes with a closed immersion

$$\mathrm{Quot}_X(E, \boldsymbol{n}) \hookrightarrow \prod_{i=1}^{d} \mathrm{Quot}_X(E, n_i) \tag{8.2.1}$$

cut out by the nesting condition $\ker(E \twoheadrightarrow T_d) \hookrightarrow \ker(E \twoheadrightarrow T_{d-1}) \hookrightarrow \cdots \hookrightarrow \ker(E \twoheadrightarrow T_1)$. In particular, it is projective as soon as X is projective. If C is a smooth proper curve over $\mathbb{C}$ and $E \in \mathrm{Coh}(C)$ is a locally free sheaf, the cohomology of $\mathrm{Quot}_C(E, \boldsymbol{n})$ was studied by Mochizuki [**131**].

Example 8.2.1. The classical Quot scheme $\mathrm{Quot}_X(E, n)$ of length n quotients of E is obtained by setting $\boldsymbol{n} = (n)$, i.e. taking $d = 1$ and $n_d = n$. If we set $\boldsymbol{n} = (1 \leq 2 \leq \cdots \leq d)$, we obtain Mochizuki's *complete Filt scheme* $\mathrm{Filt}(E, d)$, which for $d = 1$ reduce to $\mathrm{Filt}(E, 1) = \mathbb{P}(E)$ [**131**]. When $E = \mathcal{O}_X$, we use the notation $\mathrm{Hilb}^{\boldsymbol{n}}(X)$ to denote $\mathrm{Quot}_X(\mathcal{O}_X, \boldsymbol{n})$. This space is the *nested Hilbert scheme of points*, studied extensively by Cheah [**53, 54, 55**].

8.2.2. Support map and nested punctual Quot scheme Fix a variety X, a coherent sheaf E and a d-tuple of non-negative integers $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$ for some $d > 0$. Composing the embedding (8.2.1) with the usual Quot-to-Chow morphisms yields the *support map*

$$\mathsf{h}_{E,\boldsymbol{n}} \colon \mathrm{Quot}_X(E, \boldsymbol{n}) \hookrightarrow \prod_{i=1}^{d} \mathrm{Quot}_X(E, n_i) \to \prod_{i=1}^{d} \mathrm{Sym}^{n_i}(X) \tag{8.2.2}$$

recording the 0-cycles $([\mathrm{Supp}\, T_i] \in \mathrm{Sym}^{n_i}(X))_{1 \leq i \leq d}$ attached to a d-tuple $(E \twoheadrightarrow T_i)_{1 \leq i \leq d}$. Here, $\mathrm{Sym}^m X = X^m/\mathfrak{S}_m$ is the m-th symmetric power of X.

We make the following definition.

Definition 8.2.2 (Nested punctual Quot scheme)**.** Let X be a variety, $x \in X$ a point, $E \in \mathrm{Coh}(X)$ a coherent sheaf, $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$ a tuple of non-negative integers. The *nested punctual Quot scheme* attached to $(X, E, \boldsymbol{n}, x)$ is the closed subscheme

$$\mathrm{Quot}_X(E, \boldsymbol{n})_x \subset \mathrm{Quot}_X(E, \boldsymbol{n}),$$

defined as the preimage of the cycle $(n_1 x, \ldots, n_d x)$ along the support map $\mathsf{h}_{E,\boldsymbol{n}}$.

The name 'punctual' refers, as for the classical Quot schemes, to the fact that all quotients are entirely supported at a single point. We will not need the following result.

Lemma 8.2.3. *Let X be a smooth quasiprojective variety of dimension m, and let E be a locally free sheaf of rank r on X. For every d-tuple $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$, and for every $x \in X$, one has a non-canonical isomorphism*

$$\mathrm{Quot}_X(E, \boldsymbol{n})_x \cong \mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, \boldsymbol{n})_0.$$

Proof. The result follows from the isomorphism $\mathrm{Quot}_X(E,k)_x \xrightarrow{\sim} \mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, k)_0$ relating the classical punctual Quot schemes, which is proved in full detail in [**164**, Section 2.1] exploiting a choice of étale coordinates around x (which exist by the smoothness assumption, and which explain the non-canonical nature of the isomorphism). It remains to observe that the induced isomorphism

$$\prod_{i=1}^{d} \mathrm{Quot}_X(E, n_i)_x \xrightarrow{\sim} \prod_{i=1}^{d} \mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, n_i)_0$$

maps the subscheme $\mathrm{Quot}_X(E, \boldsymbol{n})_x$ isomorphically onto $\mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, \boldsymbol{n})_0$. □

8.2.3. Connectedness We prove the following connectedness result for the nested Quot scheme. A proof in the case $(r, d, \boldsymbol{n}) = (1, 1, n)$ of the classical Hilbert scheme was first given by Hartshorne [**89**], and by Fogarty in the surface case [**77**]. We shall also exploit Cheah's connectedness result for $\mathrm{Hilb}^{\boldsymbol{n}}(X)$, see [**54**, Sec. 0.4].

Theorem 8.2.4. *If X is an irreducible quasiprojective* $\mathbf{k}$*-variety and E is a locally free sheaf on X, then $\mathrm{Quot}_X(E, \boldsymbol{n})$ is connected for every $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$. In particular, the classical Quot scheme $\mathrm{Quot}_X(E, n)$ is connected for every $n \geq 0$.*

Proof. The proof consists of several steps.

Step 1: We reduce to proving the statement when $E = \mathcal{O}_X^{\oplus r}$ is trivial. Let $x = [E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1] \in \mathrm{Quot}_X(E, \boldsymbol{n})$ be a point, where E is arbitrary. Since T_d is 0-dimensional we can find an open neighbourhood $U \subset X$ of the set-theoretic support of T_d such that $E|_U = \mathcal{O}_U^{\oplus r}$ is trivial. The point x then lies in the image of the open immersion $\mathrm{Quot}_U(\mathcal{O}_U^{\oplus r}, \boldsymbol{n}) \hookrightarrow \mathrm{Quot}_X(E, \boldsymbol{n})$. By assumption, the space $\mathrm{Quot}_U(\mathcal{O}_U^{\oplus r}, \boldsymbol{n})$ is connected. Now if $x' = [E \twoheadrightarrow T'_d \twoheadrightarrow \cdots \twoheadrightarrow T'_1] \in \mathrm{Quot}_X(E, \boldsymbol{n})$ is another point, we can find another open subset $U' \subset X$ surrounding the support of T'_d and trivialising E. Since X is irreducible, $U \cap U' \neq \emptyset$, which implies $\mathrm{Quot}_U(\mathcal{O}_U^{\oplus r}, \boldsymbol{n}) \cap \mathrm{Quot}_{U'}(\mathcal{O}_{U'}^{\oplus r}, \boldsymbol{n}) \neq \emptyset$, so x and x' are connected in $\mathrm{Quot}_X(E, \boldsymbol{n})$ by any point in this intersection.

Step 2: The scheme $\mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n})$ has a framing $\mathbf{T}$-action with non-empty fixed locus, where $\mathbf{T} = \mathbb{G}_m^r$ (see Proposition 8.4.1 for an explicit description of this fixed locus: we shall exploit it in the next step). Let $x \in \mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n})$ be an arbitrary point. Then the closure of its orbit contains a $\mathbf{T}$-fixed point — this will be explained in Section 8.4. Therefore it is enough to prove that any two $\mathbf{T}$-fixed points $x, x' \in \mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n})^{\mathbf{T}}$ are connected in $\mathrm{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n})$.

Step 3: In principle, we should check connectedness for an *arbitrary* pair (x, x') of $\mathbf{T}$-fixed points

$$
\begin{aligned}
x &= [\mathcal{O}_X^{\oplus r} \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1] \ \in \ \prod_{\alpha=1}^{r} \operatorname{Hilb}^{\boldsymbol{n}_\alpha}(X) \subset \operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n})^{\mathbf{T}}, \\
x' &= [\mathcal{O}_X^{\oplus r} \twoheadrightarrow T'_d \twoheadrightarrow \cdots \twoheadrightarrow T'_1] \ \in \ \prod_{\alpha=1}^{r} \operatorname{Hilb}^{\boldsymbol{n}'_\alpha}(X) \subset \operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n})^{\mathbf{T}},
\end{aligned}
$$

where $\sum_{1\leq\alpha\leq r} \boldsymbol{n}_\alpha = \boldsymbol{n} = \sum_{1\leq\alpha\leq r} \boldsymbol{n}'_\alpha$. But since each nested Hilbert scheme $\operatorname{Hilb}^{\boldsymbol{m}}(X)$ is connected (cf. [**54**, Sec. 0.4]), we can actually choose a pair of convenient x and x'. We fix them satisfying the condition that $\operatorname{Supp}(T_d), \operatorname{Supp}(T'_d)$ consist of n_d distinct points. When viewed in the full space $\operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n})$, the points x and x' both belong to the open subset

$$
U \subset \operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n}),
$$

defined by the cartesian diagram

$$
\begin{tikzcd}
U \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \prod_{i=1}^{d} (\operatorname{Sym}^{n_i} X \setminus \Delta_{\mathrm{big}}) \arrow[d, hook, "\text{open}"] \\
\operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n}) \arrow[r, "\mathsf{h}_{E,\boldsymbol{n}}"] & \prod_{i=1}^{d} \operatorname{Sym}^{n_i} X,
\end{tikzcd}
\tag{8.2.3}
$$

where $\Delta_{\mathrm{big}} \subset \operatorname{Sym}^{n_i} X$ is the big diagonal and the bottom map is the support map (8.2.2). In other words, $U \subset \operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, \boldsymbol{n})$ is the open subscheme consisting of the flags of quotients $[\mathcal{O}_X^{\oplus r} \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1]$ where each T_i is supported on n_i different points. This yields an isomorphism

$$
U \cong \prod_{i=1}^{d} V_i,
$$

where $V_i \subset \operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, n_i - n_{i-1})$ is the open subscheme consisting of points $[\mathcal{O}_X^{\oplus r} \twoheadrightarrow T'_i]$ where the quotients T'_i are supported on $n_i - n_{i-1}$ different points (and we set $n_0 = 0$). The scheme V_i is the image of the étale map (cf. [**9**, Proposition A.3])

$$
A_i \xrightarrow{\ \oplus\ } \operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, n_i - n_{i-1})
$$

defined on the open subscheme

$$
A_i \subset \operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, 1)^{n_i - n_{i-1}}
$$

parametrising quotients $(\mathcal{O}_X^{\oplus r} \twoheadrightarrow \mathcal{O}_{x_k})_k$ with $x_k \neq x_l$ for every $k \neq l$. On the other hand,

$$
\operatorname{Quot}_X(\mathcal{O}_X^{\oplus r}, 1)^{n_i - n_{i-1}} \cong \mathbb{P}(\mathcal{O}_X^{\oplus r})^{n_i - n_{i-1}} \cong (X \times_{\mathbf{k}} \mathbb{P}^{r-1})^{n_i - n_{i-1}}
$$

is irreducible, hence A_i is connected, and in particular V_i is connected, being the image of a connected space along a continuous map. Therefore U is connected and the proof is complete. $\square$

8.3. Tangent space and smoothness in the case of curves

Fix $(X, E, \boldsymbol{n})$ as in the previous section. For any point $x \in \operatorname{Quot}_X(E, \boldsymbol{n})$ representing a d-tuple of nested quotients

$$E \longrightarrow\!\!\!\!\rightarrow T_d \xrightarrow{p_{d-1}} T_{d-1} \xrightarrow{p_{d-2}} \cdots \xrightarrow{p_2} T_2 \xrightarrow{p_1} T_1$$

we set $K_i = \ker(E \twoheadrightarrow T_i)$. We have a flag of subsheaves

$$K_d \xhookrightarrow{\iota_{d-1}} K_{d-1} \xhookrightarrow{\iota_{d-2}} \cdots \xhookrightarrow{\iota_2} K_2 \xhookrightarrow{\iota_1} K_1 \hookrightarrow E$$

and, for any $i = 1, \ldots, d-1$, maps

$$\begin{aligned} \phi_i \colon \operatorname{Hom}_X(K_i, T_i) &\to \operatorname{Hom}_X(K_{i+1}, T_i), & g &\mapsto g \circ \iota_i \\ \psi_i \colon \operatorname{Hom}_X(K_{i+1}, T_{i+1}) &\to \operatorname{Hom}_X(K_{i+1}, T_i), & h &\mapsto p_i \circ h \end{aligned}$$

which we assemble in a matrix

$$\Delta_x = \begin{pmatrix} -\phi_1 & \psi_1 & 0 & 0 & \cdots & 0 \\ 0 & -\phi_2 & \psi_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi_{d-1} & \psi_{d-1} \end{pmatrix}$$

defining a map

$$\Delta_x \colon \bigoplus_{i=1}^{d} \operatorname{Hom}_X(K_i, T_i) \longrightarrow \bigoplus_{i=1}^{d-1} \operatorname{Hom}_X(K_{i+1}, T_i).$$

The embedding (8.2.1) induces a $\mathbf{k}$-linear inclusion of tangent spaces

$$T_x \operatorname{Quot}_X(E, \boldsymbol{n}) \hookrightarrow \bigoplus_{i=1}^{d} \operatorname{Hom}_X(K_i, T_i),$$

which can be described as follows: a d-tuple of maps $(\delta_1, \ldots, \delta_d) \in \bigoplus_{i=1}^d \operatorname{Hom}_X(K_i, T_i)$ belongs to the tangent space of $\operatorname{Quot}_X(E, \boldsymbol{n})$ at x precisely when the diagram

$$\begin{tikzcd} K_d \arrow[r, hook, "\iota_{d-1}"] \arrow[d, "\delta_d"] & K_{d-1} \arrow[r, hook, "\iota_{d-2}"] \arrow[d, "\delta_{d-1}"] & \cdots \arrow[r, hook, "\iota_2"] & K_2 \arrow[r, hook, "\iota_1"] \arrow[d, "\delta_2"] & K_1 \arrow[d, "\delta_1"] \\ T_d \arrow[r, two heads, "p_{d-1}"] & T_{d-1} \arrow[r, two heads, "p_{d-2}"] & \cdots \arrow[r, two heads, "p_2"] & T_2 \arrow[r, two heads, "p_1"] & T_1 \end{tikzcd} \tag{8.3.1}$$

commutes. This is formalised in terms of the map Δ_x in the next proposition.

Proposition 8.3.1. *Set $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$. The tangent space of $\operatorname{Quot}_X(E, \boldsymbol{n})$ at a point $x = [E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1]$ is*

$$T_x \operatorname{Quot}_X(E, \boldsymbol{n}) = \ker \left(\bigoplus_{i=1}^{d} \operatorname{Hom}(K_i, T_i) \xrightarrow{\Delta_x} \bigoplus_{i=1}^{d-1} \operatorname{Hom}(K_{i+1}, T_i) \right).$$

In particular, if E is locally free of rank r on a smooth curve C, we have that $\operatorname{Quot}_C(E, \boldsymbol{n})$ is smooth of dimension $r \cdot n_d$.

Proof. Along the same lines of [**170**, Prop. 4.5.3(i)] it is easy to see that the tangent space is given by the maps making Diagram (8.3.1) commute, which is equivalent to belonging to the kernel of Δ_x.

Let Q_i be the 0-dimensional sheaf fitting in the exact sequences

$$0 \to K_i \to K_{i-1} \to Q_i \to 0$$
$$0 \to Q_i \to T_i \to T_{i-1} \to 0$$

for every $i = 1, \ldots, d$. If $X = C$ is a smooth curve, we have that each K_i is a locally free sheaf of rank r (because torsion free is equivalent to locally free on smooth curves); since Q_i is a 0-dimensional sheaf, we obtain the vanishings

$$\mathrm{Ext}^j_C(K_i, T_i) = \mathrm{Ext}^j_C(K_{i+1}, T_i) = \mathrm{Ext}^j_C(K_i, Q_i) = 0, \quad j > 0. \tag{8.3.2}$$

Therefore each ψ_i is a surjective map, which implies that Δ_x is surjective and that the dimension of the tangent space is computed as

$$\begin{aligned} \dim_{\mathbf{k}} T_x \operatorname{Quot}_C(E, \boldsymbol{n}) &= \dim_{\mathbf{k}} \left(\bigoplus_{i=1}^{d} \mathrm{Hom}_C(K_i, T_i) \right) - \dim_{\mathbf{k}} \left(\bigoplus_{i=1}^{d-1} \mathrm{Hom}_C(K_{i+1}, T_i) \right) \\ &= \sum_{i=1}^{d} r n_i - \sum_{i=1}^{d-1} r n_i \\ &= r n_d. \end{aligned}$$

Since the tangent space dimension is constant and $\operatorname{Quot}_C(E, \boldsymbol{n})$ is connected by Theorem 8.2.4, it is enough to find a smooth open subset $U \subset \operatorname{Quot}_C(E, \boldsymbol{n})$ of dimension rn_d. We shall exploit the fact that the classical Quot scheme $\operatorname{Quot}_C(E, m)$ is smooth of dimension rm, which follows from standard deformation theory and the vanishing $\mathrm{Ext}^1_C(K, T) = \mathrm{H}^1(C, K^\vee \otimes T) = 0$ for an arbitrary point $[K \hookrightarrow E \twoheadrightarrow T] \in \operatorname{Quot}_C(E, m)$.

Let $U \subset \operatorname{Quot}_C(E, \boldsymbol{n})$ be the open subscheme as in Diagram (8.2.3) (which of course exists for arbitrary E), and write $U \cong \prod_{i=1}^d V_i$ as in the proof of Theorem 8.2.4. We know that each $V_i \subset \operatorname{Quot}_C(E, n_i - n_{i-1})$ is smooth of dimension $r \cdot (n_i - n_{i-1})$, therefore U is smooth of dimension rn_d as required. □

Remark 8.3.2. The smoothness of $\operatorname{Quot}_C(E, \boldsymbol{n})$ was already proved by Mochizuki [**131**, Prop. 2.1], via a tangent-obstruction theory argument. See also [**135**] for the classification of smoothness of $\operatorname{Quot}_X(E, \boldsymbol{n})$ when X has arbitrary dimension.

8.4. Białynicki-Birula decomposition

Let E be a locally free sheaf of rank r on a variety X. Assume that $E = \bigoplus_{\alpha=1}^r L_\alpha$ splits into a sum of line bundles on X. Then $\operatorname{Quot}_X(E, \boldsymbol{n})$ admits the action of the algebraic torus $\mathbf{T} = \mathbb{G}_m^r$ as in [**18**]. Indeed, $\mathbf{T}$ acts diagonally on the product $\prod_{i=1}^d \operatorname{Quot}_X(E, n_i)$ and the closed subscheme $\operatorname{Quot}_X(E, \boldsymbol{n})$ is $\mathbf{T}$-invariant. Its fixed locus is determined by a straightforward generalisation of the main result of [**18**].

Proposition 8.4.1. *If $E = \bigoplus_{\alpha=1}^r L_\alpha$, there is a scheme-theoretic identity*

$$\operatorname{Quot}_X(E, \boldsymbol{n})^{\mathbf{T}} = \coprod_{\boldsymbol{n}_1+\cdots+\boldsymbol{n}_r=\boldsymbol{n}} \prod_{\alpha=1}^r \operatorname{Quot}_X(L_\alpha, \boldsymbol{n}_\alpha).$$

Proof. We construct a bijection on $\mathbf{k}$-valued points, which is straightforward to verify in families.

Fix tuples $\boldsymbol{n}_\alpha = (n_{\alpha,1} \leq \cdots \leq n_{\alpha,d})$ such that $n_i = \sum_{1\leq\alpha\leq r} n_{\alpha,i}$ for every $i = 1, \ldots, d$. An element of the connected component corresponding to $(\boldsymbol{n}_1, \ldots, \boldsymbol{n}_r)$ in the right hand side is a tuple of nested quotients

$$\left([L_\alpha \twoheadrightarrow T_d^{(\alpha)} \twoheadrightarrow \cdots \twoheadrightarrow T_1^{(\alpha)}]\right)_{1\leq\alpha\leq r},$$

where each $T_i^{(\alpha)}$ is the structure sheaf of a finite subscheme of X of length $n_{\alpha,i}$. By Bifet's theorem on the $\mathbf{T}$-fixed locus of ordinary Quot schemes [18], we have that

$$\bigoplus_{1\leq\alpha\leq r} \left(L_\alpha \twoheadrightarrow T_i^{(\alpha)}\right) \in \operatorname{Quot}_X(E, n_i)^{\mathbf{T}} \tag{8.4.1}$$

for each $i = 1, \ldots, d$, and since each of the original tuples of quotients was nested according to $\boldsymbol{n}$, it follows that also the tuples (8.4.1) are nested according to $\boldsymbol{n}$, and this proves that (8.4.1) define a point in $\operatorname{Quot}_X(E, \boldsymbol{n})^{\mathbf{T}}$.

The reverse inclusion follows by an analogous reasoning relying once more on Bifet's result [18]. □

Remark 8.4.2. For a locally free sheaf L of rank 1, we naturally have the isomorphism

$$\operatorname{Quot}_X(L, \boldsymbol{n}) \cong \operatorname{Hilb}^{\boldsymbol{n}}(X),$$

where $\operatorname{Hilb}^{\boldsymbol{n}}(X)$ is the nested Hilbert scheme of points, see for example [54]. Moreover, if $X = C$ is a smooth quasiprojective curve, we have (see [54, Sec. 0.2])

$$\operatorname{Hilb}^{\boldsymbol{n}}(C) \cong \operatorname{Sym}^{n_1}(C) \times \operatorname{Sym}^{n_2-n_1}(C) \times \cdots \times \operatorname{Sym}^{n_d-n_{d-1}}(C). \tag{8.4.2}$$

Assume now $X = C$ is a smooth quasiprojective curve and let $x \in \operatorname{Quot}_C(E, \boldsymbol{n})^{\mathbf{T}}$ be a $\mathbf{T}$-fixed point, corresponding to the tuple

$$\left([L_\alpha \twoheadrightarrow T_d^{(\alpha)} \twoheadrightarrow \cdots \twoheadrightarrow T_1^{(\alpha)}]\right)_\alpha \in \prod_{\alpha=1}^r \operatorname{Quot}_C(L_\alpha, \boldsymbol{n}_\alpha). \tag{8.4.3}$$

Set $K_i^{(\alpha)} = \ker(L_\alpha \twoheadrightarrow T_i^{(\alpha)})$. The tangent space at x can be written as

$$\mathrm{T}_x \operatorname{Quot}_C(E, \boldsymbol{n}) = \ker\left(\bigoplus_{1\leq\alpha,\beta\leq r} \bigoplus_{i=1}^d \operatorname{Hom}_C\left(K_i^{(\alpha)}, T_i^{(\beta)}\right) \xrightarrow{\Delta_x} \bigoplus_{1\leq\alpha,\beta\leq r} \bigoplus_{i=1}^{d-1} \operatorname{Hom}_C\left(K_{i+1}^{(\alpha)}, T_i^{(\beta)}\right)\right). \tag{8.4.4}$$

Denote by $w_1, \ldots, w_r$ the coordinates of the algebraic torus $\mathbf{T}$, which we see as irreducible $\mathbf{T}$-characters. As a $\mathbf{T}$-representation, the tangent space admits the following weight decomposition

$$\mathrm{T}_x \operatorname{Quot}_C(E, \boldsymbol{n}) = \ker\left(\bigoplus_{1\leq\alpha,\beta\leq r} \bigoplus_{i=1}^d \operatorname{Hom}_C\left(K_i^{(\alpha)} \otimes w_\alpha, T_i^{(\beta)} \otimes w_\beta\right) \xrightarrow{\Delta_x} \bigoplus_{1\leq\alpha,\beta\leq r} \bigoplus_{i=1}^{d-1} \operatorname{Hom}_C\left(K_{i+1}^{(\alpha)} \otimes w_\alpha, T_i^{(\beta)} \otimes w_\beta\right)\right).$$

We recall the classical result of Białynicki-Birula (see [17, Section 4]), by which we obtain a decomposition of $\operatorname{Quot}_X(E, \boldsymbol{n})$ in the case when E is completely decomposable.

Theorem 8.4.3 (Białynicki-Birula). *Let X be a smooth projective scheme with a $\mathbb{G}_m$-action and let $\{X_i\}_i$ be the connected components of the $\mathbb{G}_m$-fixed locus $X^{\mathbb{G}_m} \subset X$. Then there exists a locally closed stratification $X = \coprod_i X_i^+$, such that each $X_i^+ \to X_i$ is an affine fibre bundle. Moreover, for every closed point $x \in X_i$, the tangent space is given by $T_x(X_i^+) = T_x(X)^{\mathsf{fix}} \oplus T_x(X)^+$, where $T_x(X)^{\mathsf{fix}}$ (resp. $T_x(X)^+$) denotes the $\mathbb{G}_m$-fixed (resp. positive) part of $T_x(X)$. In particular, the relative dimension of $X_i^+ \to X_i$ is equal to $\dim T_x(X)^+$ for $x \in X_i$.*

The Białynicki-Birula "strata" are constructed as follows. If t denotes the coordinate of $\mathbb{G}_m$, we have

$$X_i^+ = \left\{ x \in X \;\middle|\; \lim_{t\to 0} t \cdot x \in X_i \right\}.$$

In particular, the properness assumption assures that the closure of each $\mathbb{G}_m$-orbit in X contains the $\mathbb{G}_m$-fixed point $\lim_{t\to 0} t \cdot x$. Recently Jelisiejew–Sienkiewicz [**100**] generalised Theorem 8.4.3, proving the the X_i^+ always exists even when X is not projective (or even not smooth). However, in the smooth case they cover X as long as the closure of every $\mathbb{G}_m$-orbit contains a fixed point.

We now determine a Białynicki-Birula decomposition for $\mathrm{Quot}_C(E, \boldsymbol{n})$, where C is a smooth *quasiprojective* curve. See Mochizuki's paper [**131**, Section 2.3.4] for an equivalent construction and tangent space calculation (in the projective case), using a slightly different, but technically equivalent, tangent complex.[1]

Let $\mathbb{G}_m \hookrightarrow \mathbf{T}$ be the generic 1-parameter subtorus given by $w \mapsto (w, w^2, \ldots, w^r)$; it is clear that $\mathrm{Quot}_C(E, \boldsymbol{n})^{\mathbf{T}} = \mathrm{Quot}_C(E, \boldsymbol{n})^{\mathbb{G}_m}$. Let

$$Q_{\underline{\boldsymbol{n}}} = \prod_{\alpha=1}^{r} \mathrm{Quot}_C(L_\alpha, \boldsymbol{n}_\alpha) \subset \mathrm{Quot}_C(E, \boldsymbol{n})^{\mathbb{G}_m}$$

be the connected component of the fixed locus corresponding to the r-tuple $\underline{\boldsymbol{n}} = (\boldsymbol{n}_\alpha)_{1\le\alpha\le r}$ decomposing $\boldsymbol{n}_1 + \cdots + \boldsymbol{n}_r = \boldsymbol{n}$.

Proposition 8.4.4. *Let C be a smooth quasiprojective curve and $E = \bigoplus_{\alpha=1}^r L_\alpha$. Then the nested Quot scheme admits a locally closed stratification*

$$\mathrm{Quot}_C(E, \boldsymbol{n}) = \coprod_{\underline{\boldsymbol{n}}} Q_{\underline{\boldsymbol{n}}}^+,$$

where $\underline{\boldsymbol{n}} = (\boldsymbol{n}_\alpha)_{1\le\alpha\le r}$ are such that $\boldsymbol{n}_1 + \cdots + \boldsymbol{n}_r = \boldsymbol{n}$ and $Q_{\underline{\boldsymbol{n}}}^+ \to Q_{\underline{\boldsymbol{n}}}$ is an affine fibre bundle of relative dimension $\sum_{1\le\alpha\le r}(\alpha-1)n_{\alpha,d}$.

Proof. The strata $Q_{\underline{\boldsymbol{n}}}^+$ are induced by Theorem 8.4.3 — we just need to show that the closure of every orbit contains a fixed point. Choose a compactification $C \hookrightarrow \overline{C}$, an extension $\overline{L}_\alpha$ of each line bundle L_α and consider the induced open immersion

$$\mathrm{Quot}_C\left(\bigoplus_{\alpha=1}^r L_\alpha, \boldsymbol{n}\right) \hookrightarrow \mathrm{Quot}_{\overline{C}}\left(\bigoplus_{\alpha=1}^r \overline{L}_\alpha, \boldsymbol{n}\right).$$

[1] We thank Takuro Mochizuki for kindly sharing with us a note proving that the tangent complex used in [**131**] is quasi-isomorphic to the one encoded by the map Δ_x.

The closure of every orbit must contain a fixed point in $\mathrm{Quot}_{\overline{C}}\left(\bigoplus_{\alpha=1}^r \overline{L}_\alpha, \boldsymbol{n}\right)$, but the $\mathbb{G}_m$-action does not move the support of a nested quotient, by which we conclude that such a fixed point had to be already contained in $\mathrm{Quot}_C\left(\bigoplus_{\alpha=1}^r L_\alpha, \boldsymbol{n}\right)$.

Let $x \in Q_{\underline{\boldsymbol{n}}}$ be a fixed point as in (8.4.3). The positive part of the tangent space (8.4.4) is

$$T_x^+ \mathrm{Quot}_C(E,\boldsymbol{n}) = \ker\left(\bigoplus_{\alpha<\beta}\bigoplus_{i=1}^{d}\mathrm{Hom}_C\big(K_i^{(\alpha)},T_i^{(\beta)}\big) \xrightarrow{\Delta_x^+} \bigoplus_{\alpha<\beta}\bigoplus_{i=1}^{d-1}\mathrm{Hom}_C\big(K_{i+1}^{(\alpha)},T_i^{(\beta)}\big)\right),$$

where Δ_x^+ is the restriction of the map Δ_x. Thanks to the vanishings (8.3.2), Δ_x^+ is surjective, therefore the relative dimension is computed as

$$\begin{aligned}
\dim_{\mathbf{k}} T_x^+ \mathrm{Quot}_C(E,\boldsymbol{n}) &= \dim_{\mathbf{k}}\left(\bigoplus_{\alpha<\beta}\bigoplus_{i=1}^{d}\mathrm{Hom}_C\big(K_i^{(\alpha)},T_i^{(\beta)}\big)\right) - \dim_{\mathbf{k}}\left(\bigoplus_{\alpha<\beta}\bigoplus_{i=1}^{d-1}\mathrm{Hom}_C\big(K_{i+1}^{(\alpha)},T_i^{(\beta)}\big)\right)\\
&= \sum_{\alpha<\beta}\left(\sum_{i=1}^{d} n_{\beta,i} - \sum_{i=1}^{d-1} n_{\beta,i}\right)\\
&= \sum_{\beta=1}^{r}(\beta-1)n_{\beta,d}
\end{aligned}$$

where we used $n_{\beta,i} = \dim_{\mathbf{k}}\mathrm{Hom}_C(K_i^{(\alpha)},T_i^{(\beta)})$ since $K_i^{(\alpha)} = \ker(L_\alpha \twoheadrightarrow T_i^{(\alpha)})$ has rank 1. The proof is complete. $\square$

8.5. The motive of the nested Quot scheme on a curve

8.5.1. Grothendieck ring of varieties Let B be a scheme locally of finite type over $\mathbf{k}$. The *Grothendieck group of B-varieties*, denoted $K_0(\mathrm{Var}_B)$, is defined to be the free abelian group generated by isomorphism classes $[X \to B]$ of finite type B-varieties, modulo the scissor relations, namely the identities $[h\colon X \to B] = [h|_Z\colon Z \to B] + [h|_{X\setminus Z}\colon X\setminus Z \to B]$ whenever $Z \hookrightarrow X$ is a closed B-subvariety of X. The neutral element for the addition operation is the class of the empty variety. The operation

$$[X \to B]\cdot[X' \to B] = [X \times_B X' \to B]$$

defines a ring structure on $K_0(\mathrm{Var}_B)$, with identity $1_B = [\mathrm{id}\colon B \to B]$. Therefore $K_0(\mathrm{Var}_B)$ is called the *Grothendieck ring of B-varieties*. If $B = \operatorname{Spec}\mathbf{k}$, we write $K_0(\mathrm{Var}_{\mathbf{k}})$ instead of $K_0(\mathrm{Var}_{\operatorname{Spec}\mathbf{k}})$, and we shorten $[X] = [X \to \operatorname{Spec}\mathbf{k}]$ for every $\mathbf{k}$-variety X.

The main rules for calculations in $K_0(\mathrm{Var}_{\mathbf{k}})$ are the following:

(1) If $X \to Y$ is a geometric bijection, i.e. a bijective morphism, then $[X] = [Y]$.
(2) If $X \to Y$ is Zariski locally trivial with fibre F, then $[X] = [Y]\cdot[F]$.

These are, indeed, the only properties that we will use.

The *Lefschetz motive* is the class $\mathbb{L} = [\mathbb{A}^1_{\mathbf{k}}] \in K_0(\mathrm{Var}_{\mathbf{k}})$. It can be used to express, for instance, the class of the projective space, namely $[\mathbb{P}^n_{\mathbf{k}}] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n \in K_0(\mathrm{Var}_{\mathbf{k}})$.

8.5.2. Independence of the vector bundle The following result generalises [**164**, Corollary 2.5], which in turn generalises the main theorem of [**8**] extending it from proper smooth curves to arbitrary smooth varieties.

Proposition 8.5.1. *Let E be a locally free sheaf of rank r on a $\mathbf{k}$-variety X. For every $\boldsymbol{n}$, the motivic class of $\operatorname{Quot}_X(E,\boldsymbol{n})$ is independent of E, that is*

$$\big[\operatorname{Quot}_X(E,\boldsymbol{n})\big] = \big[\operatorname{Quot}_X(\mathscr{O}_X^{\oplus r},\boldsymbol{n})\big] \in K_0(\mathrm{Var}_{\mathbf{k}}).$$

Proof. Let $(U_k)_{1\leq k\leq e}$ be a Zariski open cover trivialising E. We can refine it to a locally closed stratification $X = W_1 \amalg \cdots \amalg W_e$ such that $W_k \subset U_k$, so that in particular $E|_{W_k} = \mathscr{O}_{W_k}^{\oplus r}$ for every k. Each W_k is taken with the reduced induced scheme structure.

Let $\operatorname{Quot}_{X,W_k}(E,\boldsymbol{n}) \subset \operatorname{Quot}_X(E,\boldsymbol{n})$ be the preimage of $\operatorname{Sym}^{n_d}(W_k) \subset \operatorname{Sym}^{n_d}(X)$ along the projection

$$\mathrm{pr}_d \circ \mathsf{h}_{E,\boldsymbol{n}} \colon \operatorname{Quot}_X(E,\boldsymbol{n}) \to \prod_{i=1}^{d} \operatorname{Sym}^{n_i}(X) \to \operatorname{Sym}^{n_d}(X),$$

where $\mathsf{h}_{E,\boldsymbol{n}}$ is the support map (8.2.2). We endow $\operatorname{Quot}_{X,W_k}(E,\boldsymbol{n})$ with the reduced scheme structure. We have a geometric bijection

$$\coprod_{\boldsymbol{n}_1+\cdots+\boldsymbol{n}_e=\boldsymbol{n}} \prod_{k=1}^{e} \operatorname{Quot}_{X,W_k}(E,\boldsymbol{n}_k) \to \operatorname{Quot}_X(E,\boldsymbol{n}),$$

therefore the motive $[\operatorname{Quot}_X(E,\boldsymbol{n})]$ is computed entirely in terms of the motives $[\operatorname{Quot}_{X,W_k}(E,\boldsymbol{n}_k)]$. It is enough to prove that these are independent of E. In the cartesian diagram

$$\begin{tikzcd}
\operatorname{Quot}_{U_k,W_k}(E|_{U_k},\boldsymbol{n}_k) \arrow[r, hook, "j"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \operatorname{Quot}_{X,W_k}(E,\boldsymbol{n}_k) \arrow[d, hook] \\
\operatorname{Quot}_{U_k}(E|_{U_k},\boldsymbol{n}_k) \arrow[r, hook, "\text{open}"] & \operatorname{Quot}_X(E,\boldsymbol{n}_k)
\end{tikzcd}$$

the open immersion j is in fact surjective, hence an isomorphism. But we can repeat this process with $\mathscr{O}_X^{\oplus r}$ in the place of E. It follows that

$$\operatorname{Quot}_{X,W_k}(E,\boldsymbol{n}_k) \cong \operatorname{Quot}_{U_k,W_k}(\mathscr{O}_{U_k}^{\oplus r},\boldsymbol{n}_k) \cong \operatorname{Quot}_{X,W_k}(\mathscr{O}_X^{\oplus r},\boldsymbol{n}_k),$$

which yields the result. $\square$

8.5.3. Proof of the main theorem Let X be a smooth quasiprojective variety and E a locally free sheaf of rank r. Define

$$\mathsf{Z}_{X,r,d}(\boldsymbol{q}) = \sum_{\boldsymbol{n}} \big[\operatorname{Quot}_X(E,\boldsymbol{n})\big]\boldsymbol{q}^{\boldsymbol{n}} \in K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1,\ldots,q_d]\!],$$

where $\boldsymbol{n} = (n_1 \leq \cdots \leq n_d)$ and we use the multivariable notation $\boldsymbol{q} = (q_1,\ldots,q_d)$ and $\boldsymbol{q}^{\boldsymbol{n}} = \prod_{i=1}^{d} q^{n_i}$. The notation $\mathsf{Z}_{X,r,d}$ reflects the independence on E that we proved in Proposition 8.5.1. If $X = C$ is a smooth quasiprojective curve and $r = d = 1$, then $\mathsf{Z}_{C,1,1}(q)$ is simply the Kapranov motivic zeta function

$$\mathsf{Z}_{C,1,1}(q) = \zeta_C(q) = \sum_{n\geq 0} \big[\operatorname{Sym}^n(C)\big]q^n. \tag{8.5.1}$$

We can now prove our main theorem, first stated in Theorem 8.1.1 in the Introduction.

Theorem 8.5.2. *Let C be a smooth quasiprojective curve. The generating series $\mathsf{Z}_{C,r,d}(q)$ is a product of shifted motivic zeta functions: there is an identity*

$$\mathsf{Z}_{C,r,d}(\boldsymbol{q}) = \prod_{\alpha=1}^{r}\prod_{i=1}^{d}\zeta_C\left(\mathbb{L}^{\alpha-1}q_i^{d-i+1}\right).$$

In particular, $\mathsf{Z}_{C,r,d}(\boldsymbol{q})$ is a rational function in $q_1,\dots,q_d$.

Proof. By Proposition 8.5.1 the motive $[\mathrm{Quot}_C(E,\boldsymbol{n})]$ is independent on the vector bundle E, so we may assume $E=\mathcal{O}_C^{\oplus r}$. In this case, we may compute the motive exploiting the decomposition of $\mathrm{Quot}_C(\mathcal{O}_C^{\oplus r},\boldsymbol{n})$ given by Proposition 8.4.4. Every stratum is a Zariski locally trivial fibration over a connected component of the fixed locus, with fibre an affine space whose dimension we computed in Proposition 8.4.4.

In what follows, we denote by $\boldsymbol{n}_\alpha=(n_{\alpha,1}\le\cdots\le n_{\alpha,d})$ a nested tuple of non-negative integers and by $\boldsymbol{l}_\alpha=(l_{\alpha,1},\dots,l_{\alpha,d})$ a tuple of non-negative integers. Clearly the two collections of tuples are in bijection, by means of the correspondence

$$(n_{\alpha,1}\le\cdots\le n_{\alpha,d})\longleftrightarrow(n_{\alpha,1},n_{\alpha,2}-n_{\alpha,1},\dots,n_{\alpha,d}-n_{\alpha,d-1}). \tag{8.5.2}$$

We compute

$$\begin{aligned}\sum_{\boldsymbol{n}}\left[\mathrm{Quot}_C(\mathcal{O}_C^{\oplus r},\boldsymbol{n})\right]\boldsymbol{q}^{\boldsymbol{n}} &= \sum_{\boldsymbol{n}}\boldsymbol{q}^{\boldsymbol{n}}\sum_{\boldsymbol{n}_1+\cdots+\boldsymbol{n}_r=\boldsymbol{n}}\prod_{\alpha=1}^{r}\left[\mathrm{Quot}_C(\mathcal{O}_C,\boldsymbol{n}_\alpha)\right]\cdot\mathbb{L}^{(\alpha-1)n_{\alpha,d}}\\ &= \sum_{\boldsymbol{n}_1,\dots,\boldsymbol{n}_r}\prod_{\alpha=1}^{r}\boldsymbol{q}^{\boldsymbol{n}_\alpha}\left[\mathrm{Hilb}^{\boldsymbol{n}_\alpha}(C)\right]\cdot\mathbb{L}^{(\alpha-1)n_{\alpha,d}}\\ &= \sum_{\boldsymbol{l}_1,\dots,\boldsymbol{l}_r}\prod_{\alpha=1}^{r}\left(\prod_{i=1}^{d}q_i^{(d-i+1)l_{\alpha,i}}\right)\left[\mathrm{Hilb}^{\boldsymbol{n}_\alpha}(C)\right]\cdot\mathbb{L}^{(\alpha-1)\sum_{i=1}^{d}l_{\alpha,i}}\\ &= \sum_{\boldsymbol{l}_1,\dots,\boldsymbol{l}_r}\prod_{\alpha=1}^{r}\prod_{i=1}^{d}q_i^{(d-i+1)l_{\alpha,i}}\left[\mathrm{Sym}^{l_{\alpha,i}}(C)\right]\cdot\mathbb{L}^{(\alpha-1)l_{\alpha,i}}\\ &= \prod_{\alpha=1}^{r}\prod_{i=1}^{d}\zeta_C\left(\mathbb{L}^{\alpha-1}q_i^{d-i+1}\right).\end{aligned}$$

The rationality follows by the rationality of the Kapranov zeta function, proved in [**101**, Theorem 1.1.9]. □

Remark 8.5.3. We can reformulate our main theorem in terms of the motivic exponential, for which a minimal background is provided in Appendix 8.6. The case $r=d=1$ yields the classical expression

$$\zeta_C(q)=\mathrm{Exp}_+([C]q).$$

The general case becomes

$$\mathsf{Z}_{C,r,d}(\boldsymbol{q})=\mathrm{Exp}_+\left([C]\sum_{\alpha=1}^{r}\mathbb{L}^{\alpha-1}\sum_{i=1}^{d}q_i^{d+1-i}\right)$$

$$= \operatorname{Exp}_+\left(\left[C \times_{\mathbf{k}} \mathbb{P}^{r-1}_{\mathbf{k}}\right] \sum_{i=1}^{d} q_i^{d+1-i}\right).$$

Setting $d = 1$ we recover the calculations of [8, 164].

8.5.4. Hodge–Deligne polynomial In this subsection we work over $\mathbf{k} = \mathbb{C}$. Ring homomorphisms $K_0(\mathrm{Var}_{\mathbb{C}}) \to R$ are called *motivic measures.* A typical example of a motivic measure is the Hodge–Deligne polynomial

$$\mathsf{E}\colon K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v],$$

defined by sending the class $[Y]$ of a smooth projective variety[2] Y to

$$\mathsf{E}(Y; u, v) = \sum_{p,q\geq 0} \dim_{\mathbb{C}} \mathrm{H}^q(Y, \Omega^p_Y)(-u)^p(-v)^q.$$

Notation 8.5.4. If $f(u, v) = \sum_{i,j} p_{ij} u^i v^j \in \mathbb{Z}[u, v]$, we set

$$(1-q)^{-f(u,v)} = \prod_{i,j} \left(1 - u^i v^j q\right)^{-p_{ij}}.$$

This is actually the formula defining the *power structure* on $\mathbb{Z}[u, v]$. The motivic measure E can be proved to be a morphism of rings with power structure, see [88] for full details.

Let C be a smooth projective curve of genus g. We have

$$\begin{aligned}
\mathsf{E}(\zeta_C(q)) &= \sum_{n\geq 0} \mathsf{E}(\mathrm{Sym}^n(C); u, v) q^n = (1-q)^{-\mathsf{E}(C;u,v)} \\
&= (1-q)^{-(1-gu-gv+uv)} \qquad (8.5.3)\\
&= \frac{(1-uq)^g(1-vq)^g}{(1-q)(1-uvq)}.
\end{aligned}$$

For E a locally free sheaf of rank r over C, define

$$\mathsf{E}_{C,r,d}(\boldsymbol{q}) = \sum_{\boldsymbol{n}} \mathsf{E}(\mathrm{Quot}_C(E, \boldsymbol{n}); u, v)\boldsymbol{q}^{\boldsymbol{n}}.$$

As a direct consequence of Theorem 8.5.2, we obtain the following corollary.

Corollary 8.5.5. *There is an identity*

$$\mathsf{E}_{C,r,d}(\boldsymbol{q}) = \prod_{\alpha=1}^{r}\prod_{i=1}^{d} \frac{\left(1 - u^\alpha v^{\alpha-1} q_i^{d-i+1}\right)^g \left(1 - u^{\alpha-1} v^\alpha q_i^{d-i+1}\right)^g}{\left(1 - u^{\alpha-1} v^{\alpha-1} q_i^{d-i+1}\right)\left(1 - u^\alpha v^\alpha q_i^{d-i+1}\right)}.$$

PROOF. This follows by combining Theorem 8.5.2 and (8.5.3) with one another, after observing that E is multiplicative (being a ring homomorphism) and sends $\mathbb{L}$ to uv. ∎

[2] By a beautiful result of Bittner [19], the classes of smooth projective varieties generate the ring $K_0(\mathrm{Var}_{\mathbf{k}})$ as soon as $\operatorname{char}\mathbf{k} = 0$. But of course E can be defined on arbitrary varieties via mixed Hodge structures.

The generating function of the signed Poincaré polynomials is obtained from $\mathsf{E}_{C,r,d}(\boldsymbol{q})$ by setting $u = v$. The generating series of topological Euler characteristics is obtained from $\mathsf{E}_{C,r,d}(\boldsymbol{q})$ by setting $u = v = 1$, also in the quasiprojective case. So we obtain

$$\sum_{\boldsymbol{n}} e_{\mathrm{top}}(\mathrm{Quot}_C(E, \boldsymbol{n}))\boldsymbol{q}^{\boldsymbol{n}} = \prod_{i=1}^{d} \left(1 - q_i^{d-i+1}\right)^{-r \cdot e_{\mathrm{top}}(C)}.$$

8.6. Appendix: Motivic exponentials

If (Λ, μ, ϵ) is a commutative monoid in the category of $\mathbf{k}$-schemes, where $\mu \colon \Lambda \times \Lambda \to \Lambda$ is the multiplication map and $\epsilon \colon \operatorname{Spec} \mathbf{k} \to \Lambda$ is the identity element, then by [60, Example 3.5 (4)], one has a λ-ring structure σ_μ on the Grothendieck ring

$$K_0(\mathrm{Var}_\Lambda),$$

determined by the operations

$$\sigma_\mu^n \left[Y \xrightarrow{f} \Lambda\right] = \left[\mathrm{Sym}^n Y \xrightarrow{\mathrm{Sym}^n f} \mathrm{Sym}^n \Lambda \xrightarrow{\mu} \Lambda\right].$$

Assume $\Lambda_+ \subset \Lambda$ is a sub-monoid such that $\coprod_{n>0} \Lambda_+^{\times n} \to \Lambda$ is of finite type. Then we can define the *motivic exponential*

$$\mathrm{Exp}_\mu \colon K_0(\mathrm{Var}_{\Lambda_+}) \to K_0(\mathrm{Var}_\Lambda)^\times$$

by setting

$$\mathrm{Exp}_\mu(A) = \sum_{n \geq 0} \sigma_\mu^n(A)$$

for an effective class A, and extending via

$$\mathrm{Exp}_\mu(A - B) = \mathrm{Exp}_\mu(A) \cdot \mathrm{Exp}_\mu(B)^{-1}$$

whenever A and B are effective. The map Exp_μ is injective. See [61, Section 1] for more details.

We use this construction in the case $(\Lambda, \mu, \epsilon) = (\mathbb{N}^d, +, 0)$, and setting $\Lambda_+ = \mathbb{N}^d \setminus 0$. Of course here we are seeing $\mathbb{N}^d$ as the $\mathbf{k}$-scheme $\coprod_{\boldsymbol{n} \in \mathbb{N}^d} \operatorname{Spec} \mathbf{k}$. There is an isomorphism

$$K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!] \xrightarrow{\sim} K_0(\mathrm{Var}_{\mathbb{N}^d})$$

defined by sending

$$\sum_{\boldsymbol{n} \in \mathbb{N}^d} Y_{\boldsymbol{n}} \cdot q_1^{n_1} \cdots q_d^{n_d} \mapsto \left[\coprod_{\boldsymbol{n} \in \mathbb{N}^d} Y_{\boldsymbol{n}} \to \operatorname{Spec} \mathbf{k}(\boldsymbol{n})\right]$$

for varieties $Y_{\boldsymbol{n}}$, and extending by linearity. Under this identification, if we let $\mathfrak{m}$ be the ideal generated by $(q_1, \ldots, q_d)$ in $K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!]$, we can see Exp_+ as a group isomorphism

$$\mathrm{Exp}_+ \colon \mathfrak{m} \cdot K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!] \xrightarrow{\sim} 1 + \mathfrak{m} \cdot K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!] \subset (K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!])^\times$$

between an additive group (on the left) and a multiplicative group (on the right). In particular, one has the identity

$$\mathrm{Exp}_+\left(\sum_{\ell=1}^{s} f_\ell(q_1, \ldots, q_d)\right) = \prod_{\ell=1}^{s} \mathrm{Exp}_+(f_\ell(q_1, \ldots, q_d))$$

for $f_\ell(q_1, \ldots, q_d) \in \mathfrak{m} \cdot K_0(\mathrm{Var}_{\mathbf{k}})[\![q_1, \ldots, q_d]\!]$.

CHAPTER 9

A note on the smoothness of the nested Quot scheme of points

In viaggio il cielo
immutabile non giudica
speranze di primavera

Haiku, Andrea Pavlov

9.1. Introduction

Let X be a smooth quasiprojective variety of dimension m, defined over $\mathbb{C}$. Let E be a locally free sheaf of rank r over X. For a fixed $d > 0$ and a fixed d-tuple of non-decreasing integers $\boldsymbol{n} = (0 \leq n_1 \leq \cdots \leq n_d)$, we consider the *nested Quot scheme of points*

$$\mathrm{Quot}_X(E, \boldsymbol{n}) = \left\{ \left[E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1 \right] \,\middle|\, \dim(T_i) = 0,\ \chi(T_i) = n_i \right\}$$

where the dimension of a coherent sheaf T is, by definition, the dimension of its support.

In this note we give necessary and sufficient conditions for $\mathrm{Quot}_X(E, \boldsymbol{n})$ to be smooth. When $d = 1$, in which case we recover Grothendieck's Quot scheme, we simply write $n \in \mathbb{N}$ instead of $\boldsymbol{n} = (0 \leq n)$. To simplify the notation, we assume without loss of generality for the following theorem that $\boldsymbol{n} = (0 < n_1 < \cdots < n_d)$.

Theorem 9.1.1. *Let $(X, E, \boldsymbol{n})$ be as above. Then $\mathrm{Quot}_X(E, \boldsymbol{n})$ is smooth in the following cases:*

(1) If $m = 1$, for all choices of $(E, d, \boldsymbol{n})$,
(2) if $d = 1$ and $n = 1$,
(3) if $r = 1$, in the following cases:
  (a) $m = 2, d = 1$, for all choices of n,
  (b) $m = d = 2$ and $\boldsymbol{n} = (n, n + 1)$,
  (c) $m \geq 3, d = 1$ and $n \leq 3$,
  (d) $m \geq 3, d = 2$ and $\boldsymbol{n} = (1, 2), (2, 3)$,

In all other cases, $\operatorname{Quot}_X(E,\boldsymbol{n})$ *is singular.*

We prove Theorem 9.1.1 by first reducing to $(X,E)=(\mathbb{A}^m,\mathcal{O}^{\oplus r})$, then upgrading Cheah's complete classification of smooth nested Hilbert schemes [**54**] to arbitrary r, and finally by excluding all possible exceptions by explicitly producing singular points.

We remark that in the case $d=r=1$, corresponding to the *Hilbert scheme of* n *points* $\operatorname{Hilb}^n(X)$, it is known that smoothness occurs if and only if $m\leq 2$ or $n\leq 3$. If $r>1$, Grothendieck's Quot scheme $\operatorname{Quot}_X(\mathcal{O}^{\oplus r},n)$ is always smooth if X is a curve, and it is singular (but irreducible, and of dimension $n(r+1)$, see [**73**] and [**50**, Example 3.3]) if X is a surface.

The cohomology of the nested Quot scheme $\operatorname{Quot}_X(E,\boldsymbol{n})$ was studied in great detail by Mochizuki [**131**] when X is a smooth curve; in this case, its motive $[\operatorname{Quot}_X(E,\boldsymbol{n})]\in K_0(\mathrm{Var}_{\mathbb{C}})$ is computed explicitly in Chapter 8.

9.2. Properties of the moduli space

We fix, as in the Introduction, a triple $(X,E,\boldsymbol{n})$ consisting of a locally free sheaf E on a smooth variety X, and a d-tuple of integers $\boldsymbol{n}=(0\leq n_1\leq\cdots\leq n_d)$ for some $d>0$. Recall also that we set $m=\dim X$ and $r=\operatorname{rk}E$. Note that, if $n_d=1$, the space $\operatorname{Quot}_X(E,\boldsymbol{n})$ is isomorphic to $\mathbb{P}(E)$, which in particular is smooth of dimension $m+r-1$. This will be exploited in Section 9.2.3.

9.2.1. Tangent space As proved in Proposition 8.3.1, the tangent space of the nested Quot scheme $\operatorname{Quot}_X(E,\boldsymbol{n})$ at a point $z=[E\twoheadrightarrow T_d\twoheadrightarrow\cdots\twoheadrightarrow T_1]$ is described as the kernel of a suitable $\mathbb{C}$-linear map,

$$T_z\operatorname{Quot}_X(E,\boldsymbol{n})=\ker\left(\bigoplus_{i=1}^{d}\operatorname{Hom}(K_i,T_i)\xrightarrow{\Delta_z}\bigoplus_{i=1}^{d-1}\operatorname{Hom}(K_{i+1},T_i)\right),$$

where $K_i=\ker(E\twoheadrightarrow T_i)$. The definition of Δ_z is not relevant for the proof of our results, so we omit it (but it can be found in Section 8.3 or in an equivalent form in [**131**]).

9.2.2. Direct sum map Assume we have a decomposition $\boldsymbol{n}=\boldsymbol{n}_1+\cdots+\boldsymbol{n}_s$ for some $s>0$, where $\boldsymbol{n}_k=(n_{k1}\leq\cdots\leq n_{kd})$ are 'smaller' non-decreasing sequences of non-negative integers. The above 'sum' notation means of course that $n_i=\sum_{1\leq k\leq s}n_{ki}$ for all $i=1,\ldots,d$. Consider the open subset

$$U\hookrightarrow\prod_{1\leq k\leq s}\operatorname{Quot}_X(E,\boldsymbol{n}_k)$$

parametrising s-tuples of nested quotients

$$z_k=\left[E\twoheadrightarrow T_{kd}\twoheadrightarrow\cdots\twoheadrightarrow T_{k1}\right]\in\operatorname{Quot}_X(E,\boldsymbol{n}_k),\qquad k=1,\ldots,s$$

such that the support of T_{kd} is disjoint from the support of T_{ld} for all $1\leq k\neq l\leq s$. Then there is a well-defined morphism

$$U\xrightarrow{\ \oplus\ }\operatorname{Quot}_X(E,\boldsymbol{n})$$

sending an s-tuple $(z_1,\ldots,z_s)$ as above to the point

$$\left[E \twoheadrightarrow T_{1d}\oplus\cdots\oplus T_{sd}\twoheadrightarrow\cdots\twoheadrightarrow T_{11}\oplus\cdots\oplus T_{s1}\right]\in \operatorname{Quot}_X(E,\boldsymbol{n}).$$

An immediate application of the infinitesimal criterion shows that this map is étale.

9.2.3. Expected dimension Let $\boldsymbol{n}=(n_1\leq\cdots\leq n_d)$ and let us split it as $\boldsymbol{n}=\sum_{k=1}^{n_d}\boldsymbol{n}_k$, where each $\boldsymbol{n}_k=(n_{k1}\leq\cdots\leq n_{kd})$ satisfies $n_{kd}=1$. Inside

$$\prod_{k=1}^{n_d}\operatorname{Quot}_X(E,\boldsymbol{n}_k)\cong\mathbb{P}(E)^{n_d}$$

we consider the open subscheme $U_{\boldsymbol{n}}$ parametrising n_d-tuples of quotients with pairwise disjoint supports. Then $U_{\boldsymbol{n}}$ is smooth of dimension $n_d(m+r-1)$. Since $U_{\boldsymbol{n}}$ is étale over $\operatorname{Quot}_X(E,\boldsymbol{n})$, via the direct sum map, it makes sense to define the *expected dimension*

$$\operatorname{expdim}\operatorname{Quot}_X(E,\boldsymbol{n})=n_d(m+r-1).$$

Indeed, for sure $\operatorname{Quot}_X(E,\boldsymbol{n})$ has a smooth open subscheme (the image of $U_{\boldsymbol{n}}$) of this dimension. In the case of the classical Hilbert scheme of points $\operatorname{Hilb}^n(X)$, the image of U_n parametrises n-tuples of distinct points (up to order). Its dimension is $n\cdot\dim(X)$. This is the dimension of $\operatorname{Hilb}^n(X)$ when it is irreducible, since the Zariski closure of this open, the so-called *smoothable component*, is always an irreducible component.

9.2.4. Connectedness If X is irreducible, the scheme $\operatorname{Quot}_X(E,\boldsymbol{n})$ is connected, cf. Theorem 8.2.4. Therefore, if we find a point $z\in\operatorname{Quot}_X(E,\boldsymbol{n})$ such that

$$\dim_{\mathbb{C}}T_z\operatorname{Quot}_X(E,\boldsymbol{n})>\operatorname{expdim}\operatorname{Quot}_X(E,\boldsymbol{n})=n_d(m+r-1),$$

then z is necessarily a singular point of the nested Quot scheme.

9.3. Proof of the theorem

We start by reducing our *global* analysis of singularities, involving (X,E), to a *local* one, involving $(\mathbb{A}^m,\mathcal{O}^{\oplus r})$.

Lemma 9.3.1. *Let X be a smooth m-dimensional quasiprojective variety, E a locally free sheaf of rank r over X. Then $\operatorname{Quot}_X(E,\boldsymbol{n})$ is smooth if and only if $\operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r},\boldsymbol{n})$ is smooth.*

Proof. This follows since $\operatorname{Quot}_X(E,\boldsymbol{n})$ is locally an étale chart for $\operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r},\boldsymbol{n})$. We give full details below.

Consider the case $d=1$ first. Let $U\subset X$ be an open subvariety such that $E|_U=\mathcal{O}_U^{\oplus r}$. Suppose we have an étale map $\varphi\colon U\to\mathbb{A}^m$, and $V_{r,n}^{\varphi}$ denotes the open subscheme of $\operatorname{Quot}_U(\mathcal{O}_U^{\oplus r},n)$ parametrising quotients $[\mathcal{O}_U^{\oplus r}\twoheadrightarrow T]$ such that $\varphi|_{\operatorname{Supp}(T)}$ is injective. Then the association $[\mathcal{O}_U^{\oplus r}\twoheadrightarrow T]\mapsto[\mathcal{O}_{\mathbb{A}^m}^{\oplus r}\to\varphi_*\varphi^*\mathcal{O}_{\mathbb{A}^m}^{\oplus r}=\varphi_*\mathcal{O}_U^{\oplus r}\twoheadrightarrow\varphi_*T]$ defines an *étale* map [**9**, Prop. A.3]

$$\Phi_n\colon V_{r,n}^{\varphi}\to\operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r},n).$$

Varying $(U,\varphi\colon U\to\mathbb{A}^m)$ so to cover the whole of $\mathbb{A}^m$ confirms the result for $d=1$, as smoothness only depends on an étale neighbourhood.

Now we fix $\boldsymbol{n} = (0 < n_1 \leq \cdots \leq n_d)$ and (U, φ) as above. Taking the product of the étale morphisms Φ_{n_i} yields an étale map $\Phi_{\boldsymbol{n}}$ along with a cartesian diagram

$$\begin{tikzcd} Z_{\boldsymbol{n}}^{\varphi} \arrow[r, hook] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\square"] & \prod_{1\leq i\leq d} V_{r,n_i}^{\varphi} \arrow[d, "\Phi_{\boldsymbol{n}}"] \\ \mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r}, \boldsymbol{n}) \arrow[r, hook] & \prod_{1\leq i\leq d} \mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r}, n_i) \end{tikzcd}$$

where the horizontal arrows are closed immersions.

It is easy to verify that $Z_{\boldsymbol{n}}^{\varphi}$ also appears as the scheme-theoretic intersection

$$\begin{tikzcd} Z_{\boldsymbol{n}}^{\varphi} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \prod_{1\leq i\leq d} V_{r,n_i}^{\varphi} \arrow[d, hook, "\text{open}"] \\ \mathrm{Quot}_{U}(\mathcal{O}_{U}^{\oplus r}, \boldsymbol{n}) \arrow[r, hook, "\text{closed}"] & \prod_{1\leq i\leq d} \mathrm{Quot}_{U}(\mathcal{O}_{U}^{\oplus r}, n_i) \end{tikzcd}$$

inside a product of classical Quot schemes; since $\mathrm{Quot}_U(\mathcal{O}_U^{\oplus r}, \boldsymbol{n}) \subset \mathrm{Quot}_X(E, \boldsymbol{n})$ is open, we have exhibited an open subscheme $Z_{\boldsymbol{n}}^{\varphi} \subset \mathrm{Quot}_X(E, \boldsymbol{n})$ with an étale map down to $\mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r}, \boldsymbol{n})$. Varying $(U, \varphi\colon U \to \mathbb{A}^m)$ as in the case $d = 1$ yields the result. $\square$

We can now tackle the proof of our main result.

Proof of Theorem 9.1.1. By Lemma 9.3.1 we can assume that $(X, E) = (\mathbb{A}^m, \mathcal{O}_{\mathbb{A}^m}^{\oplus r})$. The smoothness in case $m = 1$, cf. (1), is proved in Proposition 8.3.1 and in [**131**, Prop. 2.1]. The smoothness in cases (3a)–(3d) is proved by Cheah [**54**, Thm. pag 43]. Finally, (2) follows from the isomorphism $\mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r}, 1) \cong \mathbb{A}^m \times \mathbb{P}^{r-1}$ (cf. Remark 9.3.2). It remains to prove that these are the *only* smooth nested Quot schemes.

We remark that if $\mathrm{Hilb}^{\boldsymbol{n}}(\mathbb{A}^m)$ is singular, then so is $\mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r}, \boldsymbol{n})$. Indeed, by Proposition 8.4.1, $\mathrm{Hilb}^{\boldsymbol{n}}(\mathbb{A}^m)$ appears as a connected component of the $\mathbb{G}_m^r$-fixed locus of $\mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r}, \boldsymbol{n})$, where $\mathbb{G}_m^r$ acts rescaling the fibres of $\mathcal{O}_{\mathbb{A}^m}^{\oplus r}$ one by one (see Section 8.4). Since Cheah proved that the nested Hilbert scheme $\mathrm{Hilb}^{\boldsymbol{n}}(\mathbb{A}^m)$ is singular if it does not fall in any of the cases (1),(2),(3a)–(3d), we deduce that, if $r > 1$, the scheme $\mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r}, \boldsymbol{n})$ is singular in the following cases

(1) if $d \geq 3$, for all choices of $\boldsymbol{n}$,
(2) if $m = 2, d = 2, \boldsymbol{n} = (n, n')$ with $n' - n \geq 2$,
(3) if $m \geq 3, d = 1$, $n \geq 4$,
(4) if $m \geq 3, d = 2$, $\boldsymbol{n} \neq (1, 2), (2, 3)$.

We are left to prove that $\mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r}, \boldsymbol{n})$ is singular in the following cases:

(A) if $m \geq 2$, $r \geq 2$, $d = 1$ and $n \geq 2$,
(B) if $m \geq 2$, $r \geq 2$, $d = 2$ and $\boldsymbol{n} = (n, n + 1)$.

Case (A), resp. (B), is settled in Lemma 9.3.3, resp. Lemma 9.3.4. $\square$

Remark 9.3.2. Let E be a coherent sheaf on a variety X. The natural isomorphism $\mathrm{Quot}_X(E,1)=\mathbb{P}(E)$ follows by explicitly comparing the moduli functors. However, the case $(X,E)=(\mathbb{A}^m,\mathscr{O}^{\oplus r})$, yielding $\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},1)=\mathbb{A}^m\times\mathbb{P}^{r-1}$, also follows from the explicit presentation of the Quot scheme $\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},n)$ as a closed subvariety of the *noncommutative Quot scheme*

$$\mathrm{Quot}_m^{n,r}=\left\{ (A_1,\ldots,A_m,v_1,\ldots,v_r)\in\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^m\times(\mathbb{C}^n)^r \;\middle|\; \begin{array}{c}(v_1,\ldots,v_r)\text{ is}\\ (A_1,\ldots,A_m)\text{-stable}\end{array}\right\}\Big/\,\mathrm{GL}_n,$$

where GL_n acts by conjugation on the endomorphisms and by left multiplication on the vectors, and finally the stability condition reads: the $\mathbb{C}$-linear span of all monomials in $A_1,\ldots,A_m$ applied to $v_1,\ldots,v_r$ equals the whole of $\mathbb{C}^n$. The variety $\mathrm{Quot}_m^{n,r}$ is smooth of dimension $(m-1)n^2+rn$. In the case $n=1$, the embedding (in general cut out by the relations $[A_i,A_j]=0$) is clearly trivial, and the GL_1-action is only nontrivial on the r-tuple of complex numbers $(v_1,\ldots,v_r)\in\mathbb{C}^r$, which cannot all be 0 by the stability condition. This gives a direct proof of the decomposition $\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},1)=\mathbb{A}^m\times\mathbb{P}^{r-1}$.

We now settle the two remaining cases, (A) and (B), to conclude the proof of Theorem 9.1.1.

Lemma 9.3.3. *Let $m\geq 2, r\geq 2, n\geq 2$. Then $\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}_{\mathbb{A}^m}^{\oplus r},n)$ is singular.*

Proof. We start with the case $n=2$. Let us consider a point $z\in\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},2)$ represented by a short exact sequence

$$0\to\mathfrak{m}_0^{\oplus 2}\oplus\mathscr{O}^{\oplus r-2}\to\mathscr{O}^{\oplus r}\to\mathscr{O}_0^{\oplus 2}\to 0$$

where $\mathfrak{m}_0=(x_1,\ldots,x_m)\subset\mathscr{O}=\mathbb{C}[x_1,\ldots,x_m]$ is the ideal of the origin $0\in\mathbb{A}^m$ and $\mathscr{O}_0=\mathscr{O}/\mathfrak{m}_0$ is its structure sheaf. We find

$$\begin{aligned}\dim_{\mathbb{C}}T_z\,\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},2)&=\dim_{\mathbb{C}}\mathrm{Hom}_{\mathscr{O}}(\mathfrak{m}_0^{\oplus 2}\oplus\mathscr{O}^{\oplus r-2},\mathscr{O}_0^{\oplus 2})\\&=\dim_{\mathbb{C}}\mathrm{Hom}_{\mathscr{O}}(\mathfrak{m}_0^{\oplus 2},\mathscr{O}_0^{\oplus 2})+\dim_{\mathbb{C}}\mathrm{Hom}_{\mathscr{O}}(\mathscr{O}^{\oplus r-2},\mathscr{O}_0)^{\oplus 2}\\&=4m+2(r-2),\end{aligned}$$

which is larger than $\operatorname{expdim}\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},2)=2(m+r-1)$ since $m\geq 2$. Using the connectedness of the Quot scheme (cf. Section 9.2.4), this calculation shows that z is a singular point.

Let us now assume $n\geq 3$. Consider the open subscheme

$$U\hookrightarrow\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},2)\times\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},1)^{n-2}$$

parametrising $(n-1)$-tuples of quotients with pairwise disjoint support. Inside U, let us pick a point of the form $u=(\mathfrak{m}_0^{\oplus 2}\oplus\mathscr{O}^{\oplus r-2},\mathfrak{m}_{p_1}\oplus\mathscr{O}^{\oplus r-1},\ldots,\mathfrak{m}_{p_{n-2}}\oplus\mathscr{O}^{\oplus r-1})$, where $0\neq p_i\in\mathbb{A}^m$ for all $1\leq i\leq n-2$ and $p_i\neq p_j$ for $1\leq i\neq j\leq n-2$. Via the direct sum map, the scheme U is étale over $\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},n)$, and we call v the image of the point u under the direct sum map. We find

$$\dim_{\mathbb{C}}T_v\,\mathrm{Quot}_{\mathbb{A}^m}(\mathscr{O}^{\oplus r},n)=\dim_{\mathbb{C}}T_uU$$

$$
\begin{aligned}
&= 4m + 2(r-2) + (n-2)(m+r-1) \\
&= n(m+r-1) + 2m - 2,
\end{aligned}
$$

which is larger than $\operatorname{expdim} \operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, n) = n(m+r-1)$ since $m \geq 2$. Again by Section 9.2.4, this proves the result. $\square$

Lemma 9.3.4. *Let $m \geq 2, r \geq 2$ and $\boldsymbol{n} = (n, n+1)$ for $n \geq 1$. Then $\operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}_{\mathbb{A}^m}^{\oplus r}, \boldsymbol{n})$ is singular.*

Proof. We start with the case $n = 1$. Let us consider a point $z \in \operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, (1,2))$ represented by the nested quotient

$$
\left[\mathcal{O}^{\oplus r} \twoheadrightarrow \mathcal{O}_0^{\oplus 2} \twoheadrightarrow \mathcal{O}_0\right],
$$

and denote by $\mathfrak{m}_0 = (x_1, \ldots, x_m) \subset \mathcal{O} = \mathbb{C}[x_1, \ldots, x_m]$ the ideal of the origin $0 \in \mathbb{A}^m$. By Section 9.2.1 the tangent space at z is

$$
\begin{aligned}
&T_z \operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, (1,2)) = \\
&\ker \left(\operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}_0 \oplus \mathcal{O}^{\oplus r-1}, \mathcal{O}_0) \oplus \operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}_0^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}, \mathcal{O}_0^{\oplus 2}) \xrightarrow{\Delta_z} \operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}_0^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}, \mathcal{O}_0) \right),
\end{aligned}
$$

and since the spaces involved in Δ_z satisfy

$$
\begin{aligned}
\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}_0 \oplus \mathcal{O}^{\oplus r-1}, \mathcal{O}_0) &= m + r - 1 \\
\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}_0^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}, \mathcal{O}_0^{\oplus 2}) &= 4m + 2(r-2) \\
\dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}}(\mathfrak{m}_0^{\oplus 2} \oplus \mathcal{O}^{\oplus r-2}, \mathcal{O}_0) &= 2m + r - 2,
\end{aligned}
$$

we find

$$
\begin{aligned}
\dim_{\mathbb{C}} T_z \operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, (1,2)) &\geq (m+r-1) + (4m + 2(r-2)) - (2m + r - 2) \\
&> 2(m+r-1) = \operatorname{expdim} \operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, (1,2)).
\end{aligned}
$$

Thus z is singular by our observation in Section 9.2.4.

Let us now assume $n \geq 2$. Consider the open subscheme

$$
U \hookrightarrow \operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, (1,2)) \times \operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, (1,1))^{n-1}
$$

parametrising n-tuples of nested quotients with pairwise disjoint support. Inside U, let us pick a point of the form $u = (z, z_1, \ldots, z_{n-1})$, where z is as above and z_i is represented by a nested quotient

$$
\left[\mathcal{O}^{\oplus r} \twoheadrightarrow \mathcal{O}_{p_i} \xrightarrow{\sim} \mathcal{O}_{p_i}\right], \qquad p_i \in \mathbb{A}^m.
$$

We further assume that $0 \neq p_i \in \mathbb{A}^m$ for all i and $p_i \neq p_j$ for $1 \leq i \neq j \leq n-1$. The scheme U is étale over $\operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, (n, n+1))$, and we call v the image of the point u under the direct sum map. We find

$$
\begin{aligned}
\dim_{\mathbb{C}} T_v \operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, (n, n+1)) &= \dim_{\mathbb{C}} T_u U \\
&> 2(m+r-1) + (n-1)(m+r-1) \\
&= (n+1)(m+r-1) \\
&= \operatorname{expdim} \operatorname{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, (n, n+1)).
\end{aligned}
$$

Thus v is a singular point. $\square$

Samenvatting

In het weiland staat en koe
De koe ziet en trein
En de koe kijkt naar de trein

Ongetekend, Rotterdam

Wat betekent het om een meetkundig object te *tellen*?[1] We zullen dit uitleggen aan de hand van een concreet voorbeeld, dat al bekend was bij de oude Grieken. Het volgende staat bekend als het probleem van Appolonius: hoe veel cirkels raken er aan drie gegeven cirkels die in *algemene positie* liggen? Ten eerste, in *algemene positie* betekent dat de drie cirkels niet een willekeurig drietal cirkels mogen zijn, maar voldoende algemene drietallen. Bijvoorbeeld, als we drie keer dezelfde cirkel kiezen dan is het probleem equivalent aan vragen hoeveel cirkels er raken aan *één* gegeven cirkel, en we kunnen duidelijk oneindig veel oplossingen voor dat probleem construeren – probeer maar! Als we aannemen dat onze cirkels voldoende algemeen zijn om alle pathologische voorbeelden uit te sluiten, kunnen we vragen: hangt het antwoord af van de keuze van cirkels die we gemaakt hebben? Het blijkt dat dit getal altijd constant is en *eindig*: er zijn altijd precies acht cirkels die raken aan de drie gegeven cirkels. Wat er desondanks kan gebeuren is dat sommige van deze cirkels, als we de initiële cirkels een klein beetje bewegen, samenvallen. Dat betekent dat het antwoord *minder* dan acht cirkels bevat, maar sommige van deze moeten we tellen met *multipliciteit*: in andere woorden, we willen onthouden dat sommige oplossingen verschillend kunnen worden als we ons oorspronkelijke probleem vervormen. Omdat we van ons oorspronkelijke probleem vragen dat het *algemeen genoeg* is, zouden we deze tweede soort pathologie ook kunnen vermijden (welke van andere aard is dan de eerste pathologie!)

Als je een oude Griek zou zijn, zou je tevreden kunnen zijn met de constructie van deze 8 cirkels. Vanuit het perspectief van de moderne wiskunde is de vraag niet waarom we acht cirkels krijgen, maar waarom we met drie cirkels moeten beginnen. Waarom niet twee, of vier? Spelen met de meetkunde geeft direct antwoord: als onze initiële data bestaat uit *twee* cirkels, dan kunnen we altijd oneindig veel cirkels tekenen die daar aan raken. Als we beginnen met *vier* cirkels, dan is het antwoord altijd nul. Dus, in zekere zin, het getal *drie* is een kritieke waarde dat tussen geen oplossingen en oneindig veel oplossingen in ligt. Dit heet de *verwachte dimensie* van het enumeratieve probleem.

[1] Ik ben Dirk van Bree in het bijzonder dankbaar voor zijn hulp bij het vertalen van de samenvatting.

De moderne (rond de 19e eeuw) aanpak om dit probleem systematisch aan te pakken is de volgende. We beginnen met een *ruimte* van alle mogelijke kwadratische krommen in het vlak. Deze *ruimte* heeft een *topologie*, dat een notie is van wanneer twee objecten dicht bij elkaar liggen of niet. Twee kwadratische krommen liggen dichter bij elkaar als het makkelijker is de ene te vervormen naar de andere. Om de vergelijking van een kwadratische krommen op te stellen hebben we *vijf* variabelen nodig, wat een verklaring is voor het feit dat deze *ruimte* van alle kwadratische krommen – welke we vanaf nu de *moduliruimte* zullen noemen, naar het Latijnse woord *modulus*, dat parameter betekent – dimensie vijf heeft. In de moduliruimte van kwadratische krommen zijn we geïnteresseerd in een kleinere ruimte, die alleen de cirkels parametreiseert. Om te garanderen dat een kwadratische kromme een cirkel is, moeten we twee van de vijf variabelen vast nemen: daarom heeft de moduliruimte van cirkels dimensie 3, en deze noemen we M. Het feit dat deze dimensie precies het aantal cirkels is dat we nodig hebben is geen toeval. In feite, binnen deze ruimte kunnen we de ruimte bekijken van alle cirkels die raken aan een gegeven cirkel C_1. Zo'n raakconditie afdwingen maakt de dimensie één lager. Als we de moduliruimte V_1 noemen (van cirkels die raken aan C_1), kunnen we op dezelfde manier V_2 en V_3 definiëren als de moduliruimte van cirkels die raken aan C_2, C_3. Elk van deze is een ruimte van dimensie twee, binnen de ruimte M die dimensie drie heeft. Nu bestaat de oplossing van ons oorsponkelijke probleem uit de cirkels die raken aan C_1, C_2, C_3, in andere woorden, elementen in de drievoudige doorsnede $V_1 \cap V_2 \cap V_3$. Zoals je zou verwachten, als V_1 dimensie twee heeft dan heeft $V_1 \cap V_2$ dimensie één en $V_1 \cap V_2 \cap V_3$ heeft dimensie nul: het is een verzameling van punten! Dit is de meetkundige reden waarom we drie raakeisen stelden: twee eisen zou leiden tot een ruimte met oplossingen van dimensie één (dus oneindig veel) en vier eisen stelling zou leiden tot een dimensie -1 ruimte van oplossingen – maar er zijn geen ruimtes met negatieve dimensie, dus de verzameling oplossingen is simpelweg leeg.

We hebben nu begrepen waarom we moeten beginnen met drie cirkels op een mooie meetkundige manier. Maar we moeten nog uitleggen waarom we precies acht cirkels krijgen elke keer. Het antwoord is weer meetkundig – maar er is wat algebra nodig om het precies uit te voeren. Denk er eens aan wat er zou gebeuren als je twee lijnen snijdt in het vlak: je krijgt dan één punt in de doorsnede. Als je een lijn met een kwadratische kromme snijdt, krijg je twee punten. Als je twee kwadratische krommen doorsnijdt, krijg je vier punten. Zie je een patroon? Dit heet de *stelling van Bézout* en vertelt je dat als je een krommen van *graad* d_1 doorsnijdt met een kromme van graad d_2, dat je dan $d_1 \cdot d_2$ punten in de doorsnede moet verwachten. In feite is een lijn een graad één kromme en een kwadratische kromme een graad twee kromme (tekens eens wat krommen om te zien wat er gebeurt!). Een analoog resultaat geldt in ruimtes van hogere dimensie. In ons geval is M een driedimensionale ruimte en elk van V_1, V_2, V_3 is een *graad* twee hyperoppervlak in M. Dus, als we de doorsnede nemen moeten we een totaal van $2 \cdot 2 \cdot 2 = 8$ punten in de doorsnede verwachten – precies het aantal oplossingen van het oorsponkelijke Griekse probleem.

Moderne wiskunde is niet geïnteresseerd in de vorm van het probleem, maar in de conceptuele aspecten van de ruimte van oplossingen, zijn graad of andere relevante informatie van zijn meetkunde. Helaas is het leven niet altijd eenvoudig en het is vaak niet mogelijk om een enumeratief probleem te vertalen in geschikte moduliruimten. Dit is waarom enumeratieve meetkunde niet meer gaat over het enumereren van objecten, maar het bestuderen van de diepere eigenschappen van de moduliruimten zelf – een quote van Andrei Okounkov, Fields medaillist en een van de grootste moderne enumeratieve meetkundigen.

Acknowledgements

Rido di me, di te, di tutto ció che di mortale c'é

Tanto Tanto Tanto, Jovanotti

First of all I want to thank Martijn Kool for supervising my PhD project. No line of this thesis would have been written if it was not for your constant inspiration, help and our endless discussions on the board. I realize now that it is much easier to be a PhD student than a PhD supervisor, and I am deeply indebted for all the time you spent in teaching me what being a Mathematician means (especially while not doing Maths!). I am also grateful to Carel Faber for all the mathematical (and non-mathematical) tips I received during these years, which definitely helped me finding my own path.
Scientific research is never a one-man job: I could not have written such a long thesis without the collaboration of Yalong Cao, Martijn Kool, Nadir Fasola and Andrea Ricolfi. Yalong, although the pandemic prevented me from visiting Tokyo, I really enjoyed our long email discussions throughout the years and I am looking forward to come in the future. Nadir, thanks for opening a window into the *Physics* world when I first visited Trieste. Andrea, you are a man of few words, and there is truly nothing I could add here to express my gratitude to you.
I wish to thank Jim Bryan, Gunther Cornelissen, Alina Marian, Adrien Sauvaget and Alessandro Tanzini for committing to read my PhD thesis. Adrien: thanks for the long lunch breaks, the boardgame nights and for showing me how to juggle in this crazy world, Math would have been less fun without you around.

Four years is a very long time, especially when you spend half of it without knowing whether you will be able to see your office — and colleagues! — again. There are some people who deserve a particular special mention.
Maarten, we started our PhD more or less together, and shared all the happy and not-so-happy moments of mathematical growth (which are inevitable!). Thanks for all the lunches and dinners together, the (Russian...) movie nights, the PlayStation evenings and for the relaxing walks outside the department: I learned more in these moments than while staring at my blackboard.
Luca, I regard as you as my *big PhD brother*: I was always amazed by how you could be such a great mathematician and at the very same time so humble — a rare combination I wish I featured. Thanks for all the beers, the nights at my place, the chess games but

most importantly thanks for being there when I mostly needed it.
Stefano, when you arrived in Utrecht you just finished your PhD while I just started mine. Thanks for sharing your experience with me and advice me on how to roam in this tough academic world, I would not be where I am without you.
Jan-Willem, I had the rare luck to have an amazing office mate from whom I learned (some) Math, (some) *Nederlands*, but who most importantly made my office-life lighter and chattier.
Mar, thanks for all the teas in the department, for the long evening chats (when it was clearly too late to go on working!) and for organizing all the social activities I was too lazy to do by myself.
Valentijn, thanks for all the boardgame afternoons and for the patience of teaching me again how to skate. I hope we can throw some more discs in the park!
Francesco, it is needless to say that I could not have survived the first months without all your *expertise* of the Department, I really missed you when you left Utrecht!
Davide. I guess *un puntoebbasta* is enough, you are the first person I ever met in Utrecht, and I owe you more much than you may think (really!).
Thanks to Ties, Dirk and Reinier for being my academic brothers, I enjoyed all our trips and time together.
I interacted with too many people in Utrecht and writing down something about each of them would make this thesis even longer, but if you are reading this thesis you should know that I enjoyed each single moment with you. Thanks also to all the people I met outside the department: you helped me reminding me that Math is not always the answer.

During my PhD I had the luck to travel and visit many Universities. I am grateful to Georg Oberdieck for inviting me to Bonn University and for all the insightful discussions we had. Thanks (again) to Andrea Ricolfi for inviting me in SISSA several times, and thanks to all the students that I met during these visits. I am particularly indebted to Richard Thomas for inviting me to spend few weeks in Imperial College. I spent most of my PhD reading your work, and it was a pleasure to finally discuss it together.
The month I spent in London was potentially the most disastrous of my life: I truly have to thank Riccardo for turning it into my best experience so far. Thanks also to Marta, Francesco, Federico, Lambert, Alessio, Lorenzo for this wonderful month and all the time together — I hope we are going to see each other soon again.

Earning a PhD consists in two parts: the first one, very simple, is writing a decent thesis. The most difficult aspect is going through the kafkaesque bureaucracy of the University. If you are reading these lines it means I finally reached *the Castle*, but the merit is not mine: I own everything to Vera and Cécile for their incredible help and dedition to solve all my administrative problems.

Many thanks to Andrea for sharing his *Design* expertise to help me having a nice cover, and thanks to Babak for designing it for me.

To all my friends in Padova, in Italy and in the world: you are too many to be mentioned, but there is a piece of all of you in this thesis. Nevertheless, I am obliged to mention some exceptions.
Andrea (Sangio!), thanks for having a bed ready for me in all the cities you lived in, for explaining me why four dimensions are not enough but most importantly for being always on my side, no matter what.
Rory, thanks for being the only *fixed point* in my life. When life gives you lemons, ask Rory how to do a lemonade: she will always have the answer, and I am grateful for this.

To my family, Sacha, Elisa and Majid: thanks for supporting me in every choice I made and for letting me design my own path. I could not wish for something else.
And finally thanks to you, for all the love that there has been.

Curriculum vitae

Sergej Monavari was born on the 18 April 1994 in Padova, Italy. He completed his Bachelor's in Mathematics at the Università di Padova from 2013 to 2016 with a thesis in Algebra. In 2016, he won the INDAM scholarship for the Master's program in Mathematics, ranking third at a national level. From 2016 to 2018 he completed his Master's in Mathematics (*cum laude*) jointly between the Università di Padova and Universiteit Leiden, within the framework of the ALGANT program, with a thesis in Algebraic Geometry.

In 2016, he joined Universiteit Utrecht as a PhD candidate in Mathematics under the supervision of Martijn Kool. During this period he is a research visitor at SISSA (Trieste) in December 2019 and September 2020, at HCM (Bonn) in November-December 2021 and at Imperial College (London) in March-April 2022. His PhD thesis *Equivariant Enumerative Geometry and Donaldson-Thomas Theory* is composed by seven research articles in Algebraic Geometry, and was defended on the 13 June 2022.

From September 2022, Sergej will join EPFL (Lausanne) as a *Collaborateur scientifique* at the Chair of Arithmetic Geometry.

Bibliography

1. R. Afgani, *Refinements of Kool-Thomas Invariants via Equivariant K-theoretic invariants*, ArXiv:2012.05278, 2020.
2. E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985.
3. N. Arbesfeld, *K-theoretic Donaldson-Thomas theory and the Hilbert scheme of points on a surface*, Algebr. Geom. **8** (2021), no. 5, 587–625.
4. N. Arbesfeld and Y. Kononov, *Boxcounting and Quot schemes*, in preparation.
5. M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), no. 1, 1–28.
6. H. Awata and H. Kanno, *Quiver matrix model and topological partition function in six dimensions*, J. High Energy Phys. (2009), no. 7, 076, 23.
7. Y. Bae, M. Kool, and H. Park, in preparation.
8. M. Bagnarol, B. Fantechi, and F. Perroni, *On the motive of Quot schemes of zero-dimensional quotients on a curve*, New York J. Math. **26** (2020), 138–148.
9. S. V. Beentjes and A. T. Ricolfi, *Virtual counts on Quot schemes and the higher rank local DT/PT correspondence*, Math. Res. Lett. **28** (2021), no. 4, 967–1032.
10. K. Behrend, *Donaldson-Thomas type invariants via microlocal geometry*, Ann. of Math. (2) **170** (2009), no. 3, 1307–1338.
11. K. Behrend, J. Bryan, and B. Szendrői, *Motivic degree zero Donaldson-Thomas invariants*, Invent. Math. **192** (2013), no. 1, 111–160.
12. K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88.
13. K. Behrend and B. Fantechi, *Symmetric obstruction theories and Hilbert schemes of points on threefolds*, Algebra Number Theory **2** (2008), no. 3, 313–345.
14. M. Beltrametti and A. J. Sommese, *On k-spannedness for projective surfaces*, Algebraic geometry (L'Aquila, 1988), Lecture Notes in Math., vol. 1417, Springer, Berlin, 1990, pp. 24–51.
15. O. Ben-Bassat, C. Brav, V. Bussi, and D. Joyce, *A 'Darboux theorem' for shifted symplectic structures on derived Artin stacks, with applications*, Geom. Topol. **19** (2015), no. 3, 1287–1359.
16. F. Benini, G. Bonelli, M. Poggi, and A. Tanzini, *Elliptic non-Abelian Donaldson-Thomas invariants of* $\mathbb{C}^3$, J. High Energy Phys. (2019), no. 7, 068, 41.
17. A. Biał ynicki Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.
18. E. Bifet, *Sur les points fixes du schéma* $\mathrm{Quot}_{\mathcal{O}_X^r/X/k}$ *sous l'action du tore* $\mathbf{G}^r_{m,k}$, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 9, 609–612.
19. F. Bittner, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. **140** (2004), no. 4, 1011–1032.
20. A. Bojko, *Orientations for DT invariants on quasi-projective Calabi-Yau 4-folds*, Adv. Math. **388** (2021), Paper No. 107859, 59.

21. A. Bojko, *Wall-crossing and orientations for invariants counting coherent sheaves on CY fourfolds*, PhD Thesis, University of Oxford, 2021.
22. A. Bojko, *Wall-crossing for zero-dimensional sheaves and Hilbert schemes of points on Calabi–Yau 4-folds*, ArXiv:2102.01056, 2021.
23. D. Borisov and D. Joyce, *Virtual fundamental classes for moduli spaces of sheaves on Calabi-Yau four-folds*, Geom. Topol. **21** (2017), no. 6, 3231–3311.
24. P. Bousseau, A. Brini, and M. van Garrel, *Stable maps to Looijenga pairs*, ArXiv: 2011.08830, 2021.
25. C. Brav, V. Bussi, and D. Joyce, *A Darboux theorem for derived schemes with shifted symplectic structure*, J. Amer. Math. Soc. **32** (2019), no. 2, 399–443.
26. C. Brav and T. Dyckerhoff, *Relative Calabi-Yau structures II: shifted Lagrangians in the moduli of objects*, Selecta Math. (N.S.) **27** (2021), no. 4, Paper No. 63, 45.
27. T. Bridgeland, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. **24** (2011), no. 4, 969–998.
28. J. Bryan and M. Kool, *Donaldson-Thomas invariants of local elliptic surfaces via the topological vertex*, Forum Math. Sigma **7** (2019), e7, 45.
29. J. Bryan and R. Pandharipande, *The local Gromov-Witten theory of curves*, J. Amer. Math. Soc. **21** (2008), no. 1, 101–136.
30. A. H. Căldăraru, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, ProQuest LLC, Ann Arbor, MI, 2000, Thesis (Ph.D.)–Cornell University.
31. Y. Cao, *Genus zero Gopakumar–Vafa type invariants for Calabi–Yau 4-folds II: Fano 3-folds*, Communications in Contemporary Mathematics **22** (2020), no. 07, 1950060.
32. Y. Cao, J. Gross, and D. Joyce, *Orientability of moduli spaces of* Spin(7)*-instantons and coherent sheaves on Calabi-Yau 4-folds*, Adv. Math. **368** (2020), 107134, 60.
33. Y. Cao and M. Kool, *Zero-dimensional Donaldson-Thomas invariants of Calabi-Yau 4-folds*, Adv. Math. **338** (2018), 601–648.
34. Y. Cao and M. Kool, *Counting zero-dimensional subschemes in higher dimensions*, J. Geom. Phys. **136** (2019), 119–137.
35. Y. Cao and M. Kool, *Curve counting and DT/PT correspondence for Calabi-Yau 4-folds*, Adv. Math. **375** (2020), 107371.
36. Y. Cao, M. Kool, and S. Monavari, *K-theoretic DT/PT correspondence for toric Calabi-Yau 4-folds*, ArXiv:1906.07856, 2019.
37. Y. Cao, M. Kool, and S. Monavari, *Stable pair invariants of local Calabi-Yau 4-folds*, Int. Math. Res. Not. IMRN (2022), no. 6, 4753–4798.
38. Y. Cao and N. C. Leung, *Donaldson-Thomas theory for Calabi-Yau 4-folds*, ArXiv: 1407.7659, 2014.
39. Y. Cao, D. Maulik, and Y. Toda, *Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-folds*, Adv. Math. **338** (2018), 41–92.
40. Y. Cao, D. Maulik, and Y. Toda, *Stable pairs and Gopakumar-Vafa type invariants for Calabi-Yau 4-folds*, J. Eur. Math. Soc. (JEMS) **24** (2022), no. 2, 527–581.
41. Y. Cao and F. Qu, *Tautological Hilbert scheme invariants of Calabi-Yau 4-folds via virtual pullback*, ArXiv: 2012.04415, 2020.
42. Y. Cao and Y. Toda, *Curve counting via stable objects in derived categories of Calabi-Yau 4-folds*, ArXiv: 1909.04897, 2019.
43. Y. Cao and Y. Toda, *Counting perverse coherent systems on Calabi-Yau 4-folds*, ArXiv: 2009.10909, 2020.
44. Y. Cao and Y. Toda, *Gopakumar-Vafa type invariants on Calabi-Yau 4-folds via descendent insertions*, Comm. Math. Phys. **383** (2021), no. 1, 281–310.
45. Y. Cao and Y. Toda, *Tautological stable pair invariants of Calabi-Yau 4-folds*, Adv. Math. **396** (2022), Paper No. 108176, 44.

46. E. Carlsson and A. Okounkov, *Exts and vertex operators*, Duke Math. J. **161** (2012), no. 9, 1797–1815.
47. A. Cazzaniga, *On some computations of refined Donaldson–Thomas invariants*, PhD Thesis, University of Oxford, 2015.
48. A. Cazzaniga, D. Ralaivaosaona, and A. T. Ricolfi, *Higher rank motivic Donaldson-Thomas invariants of $\mathbb{A}^3$ via wall-crossing, and asymptotics*, ArXiv: 2004.07020, 2020.
49. A. Cazzaniga and A. T. Ricolfi, *Framed motivic Donaldson-Thomas invariants of small crepant resolutions*, to appear in Math. Nachr., 2020.
50. A. Cazzaniga and A. T. Ricolfi, *Framed sheaves on projective space and Quot schemes*, Math. Z. **300** (2022), no. 1, 745–760.
51. H.-l. Chang and Y.-H. Kiem, *Poincaré invariants are Seiberg-Witten invariants*, Geom. Topol. **17** (2013), no. 2, 1149–1163.
52. H.-L. Chang, Y.-H. Kiem, and J. Li, *Torus localization and wall crossing for cosection localized virtual cycles*, Adv. Math. **308** (2017), 964–986.
53. J. Cheah, *The cohomology of smooth nested Hilbert schemes of points*, ProQuest LLC, Ann Arbor, MI, 1994, Thesis (Ph.D.)–The University of Chicago.
54. J. Cheah, *Cellular decompositions for nested Hilbert schemes of points*, Pacific J. Math. **183** (1998), no. 1, 39–90.
55. J. Cheah, *The virtual Hodge polynomials of nested Hilbert schemes and related varieties*, Math. Z. **227** (1998), no. 3, 479–504.
56. J. Choi, S. Katz, and A. Klemm, *The refined BPS index from stable pair invariants*, Comm. Math. Phys. **328** (2014), no. 3, 903–954.
57. N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Modern Birkhäuser Classics, vol. , Birkhäuser Boston, Ltd., Boston, MA, 2010, Reprint of the 1997 edition.
58. I. Ciocan-Fontanine and M. Kapranov, *Virtual fundamental classes via dg-manifolds*, Geom. Topol. **13** (2009), no. 3, 1779–1804.
59. M. Cirafici, A. Sinkovics, and R. J. Szabo, *Cohomological gauge theory, quiver matrix models and Donaldson-Thomas theory*, Nuclear Phys. B **809** (2009), no. 3, 452–518.
60. B. Davison and S. Meinhardt, *Motivic Donaldson-Thomas invariants for the one-loop quiver with potential*, Geom. Topol. **19** (2015), no. 5, 2535–2555.
61. B. Davison and A. T. Ricolfi, *The local motivic DT/PT correspondence*, J. Lond. Math. Soc. (2) (2021).
62. M. Del Zotto, N. Nekrasov, N. Piazzalunga, and M. Zabzine, *Playing with the index of M-theory*, ArXiv:2103.10271 2021.
63. J. Denef and F. Loeser, *Geometry on arc spaces of algebraic varieties*, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 327–348.
64. D.-E. Diaconescu, A. Sheshmani, and S.-T. Yau, *Atiyah class and sheaf counting on local Calabi Yau fourfolds*, Adv. Math. **368** (2020), 107132, 54.
65. T. Dimofte and S. Gukov, *Refined, motivic, and quantum*, Lett. Math. Phys. **91** (2010), no. 1, 1–27.
66. S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), Oxford Univ. Press, Oxford, 1998, pp. 31–47.
67. M. Dürr, A. Kabanov, and C. Okonek, *Poincaré invariants*, Topology **46** (2007), no. 3, 225–294.
68. D. Edidin and W. Graham, *Characteristic classes and quadric bundles*, Duke Math. J. **78** (1995), no. 2, 277–299.
69. D. Edidin and W. Graham, *Equivariant intersection theory*, Invent. Math. **131** (1998), no. 3, 595–634.
70. D. Edidin and W. Graham, *Localization in equivariant intersection theory and the Bott residue formula*, Amer. J. Math. **120** (1998), no. 3, 619–636.

71. D. Edidin and W. Graham, *Nonabelian localization in equivariant K-theory and Riemann-Roch for quotients*, Adv. Math. **198** (2005), no. 2, 547–582.
72. G. Ellingsrud, L. Göttsche, and M. Lehn, *On the cobordism class of the Hilbert scheme of a surface*, J. Algebraic Geom. **10** (2001), no. 1, 81–100.
73. G. Ellingsrud and M. Lehn, *Irreducibility of the punctual quotient scheme of a surface*, Ark. Mat. **37** (1999), no. 2, 245–254.
74. G. Ellingsrud and S. A. Strø mme, *On a cell decomposition of the Hilbert scheme of points in the plane*, Invent. Math. **91** (1988), no. 2, 365–370.
75. B. Fantechi and L. Göttsche, *Riemann-Roch theorems and elliptic genus for virtually smooth schemes*, Geom. Topol. **14** (2010), no. 1, 83–115.
76. N. Fasola, S. Monavari, and A. T. Ricolfi, *Higher rank K-theoretic Donaldson-Thomas theory of points*, Forum Math. Sigma **9** (2021), 51, e15.
77. J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), 511–521.
78. W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.
79. O. García-Prada, J. Heinloth, and A. Schmitt, *On the motives of moduli of chains and Higgs bundles*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 12, 2617–2668.
80. A. Gholampour, Y. Jiang, and J. Lo, *Stable pairs of 2-dimensional sheaves on 4-folds*, ArXiv:2110.01342, 2021.
81. A. Gholampour and M. Kool, *Higher rank sheaves on threefolds and functional equations*, Épijournal Geom. Algébrique **3** (2019), Art. 17, 29.
82. R. Gopakumar and C. Vafa, *M-Theory and Topological Strings–I*, ArXiv: hep-th/9809187, 1998.
83. R. Gopakumar and C. Vafa, *M-Theory and Topological Strings–II*, ArXiv: hep-th/9812127, 1998.
84. L. Göttsche and M. Kool, *Virtual refinements of the Vafa-Witten formula*, Comm. Math. Phys. **376** (2020), no. 1, 1–49.
85. T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518.
86. J. Gross, D. Joyce, and Y. Tanaka, *Universal structures in $\mathbb{C}$-linear enumerative invariant theories. I*, ArXiv: 2005.05637, 2020.
87. A. Grothendieck, *Techniques de construction et théoremes d'existence en géométrie algébrique. IV: Les schemas de Hilbert*, Sem. Bourbaki 13(1960/61), No. 221, 28 p. (1961)., 1961.
88. S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández, *Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points*, Michigan Math. J. **54** (2006), no. 2, 353–359.
89. R. Hartshorne, *Connectedness of the Hilbert scheme*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 5–48.
90. R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.
91. A. P. Hillman and R. M. Grassl, *Reverse plane partitions and tableau hook numbers*, J. Combinatorial Theory Ser. A **21** (1976), no. 2, 216–221.
92. V. Hoskins and S. Pepin Lehalleur, *A formula for the Voevodsky motive of the moduli stack of vector bundles on a curve*, Geom. Topol. **25** (2021), no. 7, 3555–3589.
93. V. Hoskins and S. Pepin Lehalleur, *On the Voevodsky motive of the moduli space of Higgs bundles on a curve*, Selecta Math. (N.S.) **27** (2021), no. 1, Paper No. 11, 37.
94. D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, second ed., vol. ,, Cambridge Mathematical Library, no. ,, Cambridge University Press, Cambridge, 2010.
95. D. Huybrechts and R. P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes*, Math. Ann. **346** (2010), no. 3, 545–569.

96. L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239, vol. , Springer-Verlag, Berlin-New York, 1971.
97. E.-N. Ionel and T. H. Parker, *The Gopakumar-Vafa formula for symplectic manifolds*, Ann. of Math. (2) **187** (2018), no. 1, 1–64.
98. A. Iqbal, C. Kozçaz, and C. Vafa, *The refined topological vertex*, J. High Energy Phys. (2009), no. 10, 069, 58.
99. B. Iversen, *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. **16** (1972), 229–236.
100. J. Jelisiejew and L. u. Sienkiewicz, *Białynicki-Birula decomposition for reductive groups*, J. Math. Pures Appl. (9) **131** (2019), 290–325.
101. M. Kapranov, *The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups*, ArXiv:0001005, 2000.
102. S. Katz, *Gromov-Witten, Gopakumar-Vafa, and Donaldson-Thomas invariants of Calabi-Yau threefolds*, Snowbird lectures on string geometry, Contemp. Math., vol. 401, Amer. Math. Soc., Providence, RI, 2006, pp. 43–52.
103. Y.-H. Kiem and J. Li, *Localizing virtual cycles by cosections*, J. Amer. Math. Soc. **26** (2013), no. 4, 1025–1050.
104. Y.-H. Kiem and J. Li, *Localizing virtual structure sheaves by cosections*, Int. Math. Res. Not. IMRN (2020), no. 22, 8387–8417.
105. Y.-H. Kiem and H. Park, *Localizing virtual cycles for Donaldson-Thomas invariants of Calabi-Yau 4-folds*, ArXiv: 2012.13167, 2020.
106. Y.-H. Kiem and H. Park, *Virtual intersection theories*, Adv. Math. **388** (2021), Paper No. 107858, 51.
107. Y.-H. Kiem and M. Savvas, *Localizing virtual structure sheaves for almost perfect obstruction theories*, Forum Math. Sigma **8** (2020), Paper No. e61, 36.
108. A. Klemm and R. Pandharipande, *Enumerative geometry of Calabi-Yau 4-folds*, Comm. Math. Phys. **281** (2008), no. 3, 621–653.
109. Y. Kononov, A. Okounkov, and A. Osinenko, *The 2-leg vertex in K-theoretic DT theory*, Comm. Math. Phys. **382** (2021), no. 3, 1579–1599.
110. M. Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 335–368.
111. M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, ArXiv: 0811.2435, 2008.
112. M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. **5** (2011), no. 2, 231–352.
113. M. Kool, *Fixed point loci of moduli spaces of sheaves on toric varieties*, Adv. Math. **227** (2011), no. 4, 1700–1755.
114. M. Kool, *Stable pair invariants of surfaces and Seiberg-Witten invariants*, Q. J. Math. **67** (2016), no. 3, 365–386.
115. M. Kool and J. V. Rennemo, *Proof of a magnificent conjecture*, in preparation.
116. M. Kool, V. Shende, and R. P. Thomas, *A short proof of the Göttsche conjecture*, Geom. Topol. **15** (2011), no. 1, 397–406.
117. M. Kool and R. Thomas, *Reduced classes and curve counting on surfaces I: theory*, Algebr. Geom. **1** (2014), no. 3, 334–383.
118. M. Kool and R. Thomas, *Reduced classes and curve counting on surfaces II: calculations*, Algebr. Geom. **1** (2014), no. 3, 384–399.
119. M. Kool and R. P. Thomas, *Stable pairs with descendents on local surfaces I: the vertical component*, Pure Appl. Math. Q. **13** (2017), no. 4, 581–638.
120. A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), no. 3, 495–536.
121. J. Le Potier, *Systèmes cohérents et structures de niveau*, Astérisque (1993), no. 214, 143.

122. J. Li, *Zero dimensional Donaldson-Thomas invariants of threefolds*, Geom. Topol. **10** (2006), 2117–2171.
123. J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc. **11** (1998), no. 1, 119–174.
124. I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Cambridge Philos. Soc. **58** (1962), 563–568.
125. I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, vol. , The Clarendon Press, Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.
126. MathOverflow, *Are Donaldson-Thomas invariants "A-model" or "B-model"?*, link, 2012.
127. D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory. I*, Compos. Math. **142** (2006), no. 5, 1263–1285.
128. D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory. II*, Compos. Math. **142** (2006), no. 5, 1286–1304.
129. D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande, *Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, Invent. Math. **186** (2011), no. 2, 435–479.
130. D. Maulik, R. Pandharipande, and R. P. Thomas, *Curves on $K3$ surfaces and modular forms*, J. Topol. **3** (2010), no. 4, 937–996.
131. T. Mochizuki, *The structure of the cohomology ring of the filt schemes*, ArXiv:0301184, 2003.
132. S. Monavari, *Double nested Hilbert schemes and the local stable pairs theory of curves*, ArXiv:2108.10827, 2021.
133. S. Monavari, *Canonical vertex formalism in DT theory of toric Calabi-Yau 4-folds*, J. Geom. Phys. **174** (2022), 104466.
134. S. Monavari and A. T. Ricolfi, *On the motive of the nested Quot scheme of points on a curve*, ArXiv: 2106.11817, 2021.
135. S. Monavari and A. T. Ricolfi, *Sur la lissité du schéma Quot ponctuel emboîté*, ArXiv:2108.00703, 2021.
136. A. Morrison, S. Mozgovoy, K. Nagao, and B. Szendrői, *Motivic Donaldson-Thomas invariants of the conifold and the refined topological vertex*, Adv. Math. **230** (2012), no. 4-6, 2065–2093.
137. H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999.
138. N. Nekrasov, $\mathbf{Z}$*-theory: chasing* $\mathfrak{m}/f$ *theory*, vol. 6, 2005, Strings 04. Part II, pp. 261–269.
139. N. Nekrasov, *Instanton partition functions and M-theory*, Jpn. J. Math. **4** (2009), no. 1, 63–93.
140. N. Nekrasov, *Magnificent four*, Ann. Inst. Henri Poincaré D **7** (2020), no. 4, 505–534.
141. N. Nekrasov, *Magnificent four*, Adv. Theor. Math. Phys. **24** (2020), no. 5, 1171–1202.
142. N. Nekrasov and A. Okounkov, *Membranes and sheaves*, Algebr. Geom. **3** (2016), no. 3, 320–369.
143. N. Nekrasov and N. Piazzalunga, *Magnificent four with colors*, Comm. Math. Phys. **372** (2019), no. 2, 573–597.
144. A. Oblomkov, *EGL formula for DT/PT theory of local curves*, ArXiv: 1901.03014, 2019.
145. A. Oblomkov, A. Okounkov, and R. Pandharipande, *GW/PT descendent correspondence via vertex operators*, Comm. Math. Phys. **374** (2020), no. 3, 1321–1359.
146. J. Oh and R. P. Thomas, *Counting sheaves on Calabi-Yau 4-folds, II*, in preparation.
147. J. Oh and R. P. Thomas, *Counting sheaves on Calabi-Yau 4-folds, I*, ArXiv: 2009.05542, 2020.
148. A. Okounkov and R. Pandharipande, *The local Donaldson-Thomas theory of curves*, Geom. Topol. **14** (2010), no. 3, 1503–1567.
149. A. Okounkov, *Lectures on K-theoretic computations in enumerative geometry*, Geometry of moduli spaces and representation theory, IAS/Park City Math. Ser., vol. 24, Amer. Math. Soc., Providence, RI, 2017, pp. 251–380.
150. A. Okounkov, *Takagi lectures on Donaldson-Thomas theory*, Jpn. J. Math. **14** (2019), no. 1, 67–133.

151. R. Pandharipande, *Descendents for stable pairs on 3-folds*, Modern geometry: a celebration of the work of Simon Donaldson, Proc. Sympos. Pure Math., vol. 99, Amer. Math. Soc., Providence, RI, 2018, pp. 251–287.
152. R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. **178** (2009), no. 2, 407–447.
153. R. Pandharipande and R. P. Thomas, *Stable pairs and BPS invariants*, J. Amer. Math. Soc. **23** (2010), no. 1, 267–297.
154. R. Pandharipande and R. P. Thomas, *13/2 ways of counting curves*, Moduli spaces, London Math. Soc. Lecture Note Ser., vol. 411, Cambridge Univ. Press, Cambridge, 2014, pp. 282–333.
155. R. Pandharipande and R. P. Thomas, *The 3-fold vertex via stable pairs*, Geom. Topol. **13** (2009), no. 4, 1835–1876.
156. R. Pandharipande and A. Pixton, *Descendents on local curves: stationary theory*, Geometry and arithmetic, EMS Ser. Congr. Rep., vol. , Eur. Math. Soc., Zürich, 2012, pp. 283–307.
157. R. Pandharipande and A. Pixton, *Descendents on local curves: rationality*, Compos. Math. **149** (2013), no. 1, 81–124.
158. R. Pandharipande and A. Pixton, *Gromov-Witten/pairs descendent correspondence for toric 3-folds*, Geom. Topol. **18** (2014), no. 5, 2747–2821.
159. R. Pandharipande and A. Pixton, *Gromov-Witten/Pairs correspondence for the quintic 3-fold*, J. Amer. Math. Soc. **30** (2017), no. 2, 389–449.
160. T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 271–328.
161. H. Park, *Virtual pullbacks in Donaldson-Thomas theory of Calabi-Yau 4-folds*, ArXiv: 2110.03631, 2021.
162. F. Qu, *Virtual pullbacks in K-theory*, Ann. Inst. Fourier (Grenoble) **68** (2018), no. 4, 1609–1641.
163. A. T. Ricolfi, *Local Donaldson–Thomas invariants and their refinements*, Ph.D. thesis, University of Stavanger, 2017.
164. A. T. Ricolfi, *On the motive of the Quot scheme of finite quotients of a locally free sheaf*, J. Math. Pures Appl. (9) **144** (2020), 50–68.
165. A. T. Ricolfi, *Virtual classes and virtual motives of Quot schemes on threefolds*, Adv. Math. **369** (2020), 107182, 32.
166. A. T. Ricolfi, *The equivariant Atiyah class*, C. R., Math., Acad. Sci. Paris **359** (2021), no. 3, 257–282.
167. A. T. Ricolfi and M. Savvas, *The d-critical structure on the Quot scheme of points of a Calabi-Yau 3-fold*, ArXiv:2106.16133, 2021.
168. H. Sato, *The classification of smooth toric weakened Fano 3-folds*, Manuscripta Math. **109** (2002), no. 1, 73–84.
169. G. Segal, *Classifying spaces and spectral sequences*, Publications Mathématiques de l'IHÉS **34** (1968), 105–112 (en).
170. E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006.
171. B. Siebert, *Virtual fundamental classes, global normal cones and Fulton's canonical classes*, Frobenius manifolds, Aspects Math., E36, vol. , Friedr. Vieweg, Wiesbaden, 2004, pp. 341–358.
172. R. P. Stanley, *Theory and application of plane partitions. II*, Studies in Appl. Math. **50** (1971), 259–279.
173. R. J. Szabo, $\mathcal{N} = 2$ *gauge theories, instanton moduli spaces and geometric representation theory*, J. Geom. Phys. **109** (2016), 83–121.
174. B. Szendrői, *Cohomological Donaldson-Thomas theory*, String-Math 2014, Proc. Sympos. Pure Math., vol. 93, Amer. Math. Soc., Providence, RI, 2016, pp. 363–396.
175. Y. Tanaka and R. P. Thomas, *Vafa-Witten invariants for projective surfaces I: stable case*, J. Algebraic Geom. **29** (2020), no. 4, 603–668.

176. R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations*, J. Differential Geom. **54** (2000), no. 2, 367–438.
177. R. P. Thomas, *A K-theoretic Fulton class*, ArXiv:1810.00079, 2018.
178. R. P. Thomas, *Equivariant K-theory and refined Vafa-Witten invariants*, Comm. Math. Phys. **378** (2020), no. 2, 1451–1500.
179. R. W. Thomason, *Une formule de Lefschetz en K-théorie équivariante algébrique*, Duke Math. J. **68** (1992), no. 3, 447–462.
180. Y. Toda, *Curve counting theories via stable objects I. DT/PT correspondence*, J. Amer. Math. Soc. **23** (2010), no. 4, 1119–1157.
181. Y. Toda, *Hall algebras in the derived category and higher-rank DT invariants*, Algebr. Geom. **7** (2020), no. 3, 240–262.
182. B. Toën, *Derived algebraic geometry*, EMS Surv. Math. Sci. **1** (2014), no. 2, 153–240.
183. M. van Garrel, T. Graber, and H. Ruddat, *Local Gromov-Witten invariants are log invariants*, Adv. Math. **350** (2019), 860–876.