An Epistemic Probabilistic Logic with Conditional Probabilities

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Abstract. We present a proof-theoretical and model-theoretical approach to reasoning about knowledge and conditional probability. We extend both the language of epistemic logic and the language of linear weight formulas, allowing statements like "Agent Ag knows that the probability of A given B is at least a half". We axiomatize this logic, provide corresponding semantics and prove that the axiomatization is sound and strongly complete. We also show that the logic is decidable.

Keywords: Probabilistic logic · Epistemic logic · Completeness

1 Introduction

Epistemic logics are formal models designed in order to reason about the knowledge of agents and their knowledge of each other's knowledge. During the last couple of decades, they have found applications in various fields such as game theory, the analysis of multi-agent systems in computer science and artificial intelligence, and for analyzing the behavior and interaction of agents in a distributed system [7,8,24]. In parallel, uncertain reasoning has emerged as one of the main fields in artificial intelligence, with many different tools developed for representing and reasoning with uncertain knowledge. A particular line of research concerns the formalization in terms of logic, and the questions of providing an axiomatization and decision procedure for *probabilistic logic* attracted the attention of researchers and triggered investigation about formal systems for probabilistic reasoning [1,6,9–11,19,20].

Fagin and Halpern [5] emphasised the need for combining those two fields for many application areas, and in particular in distributed systems applications, when one wants to analyze randomized or probabilistic programs. They developed a joint framework for reasoning about knowledge and probability, proposed a complete axiomatization and investigated decidability of the framework. Based on the seminal paper by Fagin, Halpern and Meggido [6], they extended the propositional epistemic language with formulas which express linear combinations of probabilities, called *linear weight formulas*, i.e., the formulas of the

W. Faber et al. (Eds.): JELIA 2021, LNAI 12678, pp. 279–293, 2021.
https://doi.org/10.1007/978-3-030-75775-5_19

form $a_1w(\alpha_1) + ... + a_kw(\alpha_k) \geq r$, where a_j's and r are rational numbers. They proposed a finitary axiomatization and proved weak completeness, using a small model theorem.

In this paper, we extend the logic from [5] by also allowing formulas that can represent conditional probability. Thus, our language contains both knowledge operators K_i (one for each agent i) and conditional probability formulas of the form $a_1w_i(\alpha_1, \beta_1) + ... + a_kw_i(\alpha_k, \beta_k) \geq r$. The expressions of the form $w_i(\alpha, \beta)$ represent conditional probabilities that agent i places on events according to Kolmogorov definition: $P(A|B) = \frac{P(A \cap B)}{P(B)}$ if $P(B) > 0$, while $P(A|B)$ is undefined when $P(B) = 0$. The corresponding semantics consists of enriched Kripke models, with a probability measure assigned to every agent in each world.

Our main results are a sound and complete axiomatization for the logic and decidability result. We prove strong completeness (every consistent set of formulas is satisfiable) using an adaptation of Henkin's construction, modifying some of our earlier methods [2–4,16,18,19,21]. Our axiom system contains infinitary rules of inference, whose premises and conclusions are in the form of k-nested implications (Definition 6). This form of infinitary rules is a technical solution already used in probabilistic, epistemic and temporal logics for obtaining various *strong necessitation* results [13,15,17,22,23]. An obvious alternative to an infinitary axiomatization would be to develop a finitary system which would be weakly complete (strong completeness of a finitary system is impossible due to the noncompactness phenomena for probability logics, see [11]). We do not know a finitary axiomatization for this rich language. Moreover, even for logics which need to express conditional probabilities only (i.e., without knowledge operators), the task of developing a finitary system turned out to be very hard to accomplish. Fagin, Halpern and Meggido [6] faced problems when they tried to represent conditional probabilities by adding multiplication to the syntax of linear weight formulas, and they needed to introduce a first-order extension of the language in order to obtain completeness. The only finitary axiomatization we are aware of is the fuzzy approach of Marchioni and Godo [14], who consider the probability of a conditional event of the form "α given β" as the truth-value of the fuzzy proposition $P(\alpha|\beta)$ which is read as "$P(\alpha|\beta)$ is probable."

In the last part of this paper, we prove that satisfiability problem for our logic is decidable. From the technical point of view, we combine the method of filtration [12] and a reduction to a system of inequalities.

2 Syntax and Semantics

Let $\mathcal{P} = \{p, q, r, \ldots\}$ be a set of propositional letters and let $\mathbf{A}$ be a finite set of agents. Let $\mathcal{Q}$ denote the set of all rational numbers and let $[0,1]_Q = [0,1] \cap \mathcal{Q}$.

Definition 1 (Formula). *The set For of all formulas of the logic is the smallest set such that:*

- $\mathcal{P} \subset For$;
- *If* $\alpha \in For$ *then* $K_i\alpha \in For$.

– *For any* $i \in \mathbf{A}$ *and* $k \geq 1$, *if* $\alpha_1, \alpha_1', \ldots, \alpha_k, \alpha_k' \in For$ *and* $a_1, \ldots, a_k, r \in \mathcal{Q}$, *then* $a_1 w_i(\alpha_1, \alpha_1') + \cdots + a_k w_i(\alpha_k, \alpha_k') \geq r \in For$,
– *If* α *and* β *are formulas then* $\neg\alpha, \alpha \wedge \beta \in For$.

The meaning of formula $K_i\alpha$ is "agent i knows α", while the expression $w_i(\alpha, \beta)$ denotes conditional probability of α given β, according to the agent i.

An expression of the form $a_1 w_i(\alpha_1, \alpha_1') + \cdots + a_k w_i(\alpha_k, \alpha_k')$ is called *term*. Following [5], we do not allow appearance of multiple agents inside of a term. We denote terms with f_i, g_i and h_i.

The propositional connectives, $\vee$, $\rightarrow$ and $\leftrightarrow$, are introduced as abbreviations, in the usual way. We define $\top$ to be an abbreviation for the formula $p \vee \neg p$ where p is a propositional letter, while $\bot$ is $\neg\top$. We also use abbreviations to define other types of inequalities; for example: $w_i(\alpha, \beta) \geq w_i(\alpha', \beta')$ as an abbreviation for $w_i(\alpha, \beta) - w_i(\alpha', \beta') \geq 0$, $w_i(\alpha, \beta) \leq w_i(\alpha', \beta')$ for $w_i(\alpha', \beta') \geq w_i(\alpha, \beta)$, $w_i(\alpha, \beta) = w_i(\alpha', \beta')$ for $(w_i(\alpha, \beta) \geq w_i(\alpha', \beta')) \wedge (w_i(\alpha, \beta) \leq w_i(\alpha', \beta'))$, and $w_i(\alpha, \beta) > w_i(\alpha', \beta')$ for $(w_i(\alpha, \beta) \geq w_i(\alpha', \beta')) \wedge \neg(w_i(\alpha, \beta) = w_i(\alpha', \beta'))$.

Now we introduce the semantics of our logic CKL.

Definition 2 (CKL-structure). *A* CKL-*structure is a tuple* $(W, \mathcal{K}, Prob, v)$ *where:*

1. W *is a non-empty set of objects called* worlds.
2. $v : W \times \mathcal{P} \rightarrow \{true, false\}$ *assigns to each world* $u \in W$ *a two-valued evaluation* $v(u, \cdot)$ *of propositional letters,*
3. $\mathcal{K} = \{\mathcal{K}_i \mid i \in \mathbf{A}\}$ *is a set of binary equivalence relations on* W. *We denote* $\mathcal{K}_i(u) = \{u' \mid (u', u) \in \mathcal{K}_i\}$, *and write* $u\mathcal{K}_i u'$ *if* $u' \in \mathcal{K}_i(u)$,
4. $Prob$ *assigns to every* $i \in \mathbf{A}$ *and* $u \in W$ *a probability space* $Prob(i, u) = (W_i(u), H_i(u), \mu_i(u))$, *where*
   – $W_i(u)$ *is a non-empty subset of* W,
   – $H_i(u)$ *is an algebra of subsets of* $W_i(u)$, *i.e. a set such that*
     (a) $W_i(u) \in H_i(u)$,
     (b) *if* $A \in H_i(u)$, *then* $W_i(u) \setminus A \in H_i(u)$, *and*
     (c) *if* $A, B \in H_i(u)$, *then* $A \cup B \in H_i(u)$.
   – $\mu_i(u) : H_i(u) \longrightarrow [0, 1]$ *is a finitely additive measure, i.e.,*
     (a) $\mu_i(u)(W_i(u)) = 1$,
     (b) $\mu_i(u)(A \cup B) = \mu_i(u)(A) + \mu_i(u)(B)$, *whenever* $A \cap B = \emptyset$.

The elements of $H_i(u)$ *are called measurable sets.*

Definition 3 (Satisfiability). *Let* M *be a* CKL-*structure and let* u *be some world from* M. *The* satisfiability relation $\models$ *is defined recursively as follows:*

1. *If* $\alpha \in \mathcal{P}$ *then* $M, u \models \alpha$ *iff* $v(u, \alpha) = true$,
2. $M, u \models K_i\alpha$ *iff* $M, u' \models \alpha$ *for all* $u' \in K_i(u)$,
3. $M, u \models \sum_{k=1}^{n} a_k w_i(\alpha_k, \beta_k) \geq r$ *if* $\mu_i(u)(\{u' \in W_i(u) \mid M, u' \models \beta_k\}) > 0$ *for every* $k \in \{1, \ldots, n\}$ *and* $\sum_{k=1}^{n} a_k \mu_i(u)(\{u' \in W_i(u) \mid M, u' \models \alpha_k\} | \{u' \in W_i(u) \mid M, u' \models \beta_k\}) \geq r$,
4. $M, u \models \neg\alpha$ *iff* $M, u \not\models \alpha$,
5. $M, u \models \alpha \wedge \beta$ *iff* $M, u \models \alpha$ *and* $M, u \models \beta$.

We denote by $[\alpha]_{i,M,u}$ the set of all worlds from $W_i(u)$ in which α holds, i.e.,

$$[\alpha]_{i,M,u} = \{u' \in W_i(u) \mid M, u' \models \alpha\}.$$

We write $[\alpha]$ instead of $[\alpha]_{i,M,u}$ when i, M and u are clear from the context. Note that the satisfiability relation defined in Definition 3 is a partial relation, i.e., it is not in general defined for all formulas. The reason is that a formula $\sum_{k=1}^{n} a_k w_i(\alpha_k, \beta_k) \geq r$ can be evaluated in u only if all the sets $[\alpha_k]_{i,M,u}$ and $[\beta_k]_{i,M,u}$ are measurable. In order to keep the relation $\models$ *total* (i.e., well-defined for all the formulas), in this paper we consider only the models in which all those sets are indeed measurable.

Definition 4 (CKL-measurable structure). *A* CKL*-structure M is* CKL*-measurable iff $[\alpha]_{i,u} \in H(u)$ for every world u from M, every $\alpha \in For$ and every $i \in \mathbf{A}$. We denote the set of all measurable structures with* $\mathrm{CKL}_{\mathrm{Meas}}$.

Note that, according to Definition 3, the formula $w_i(\alpha, \beta) \geq r \vee w_i(\alpha, \beta) \leq r$ is not necessary satisfied in a model; the reason is that unconditional probability is simply undefined if probability of the condition is zero.

Definition 5 (Model, entailment). *For an $M = (W, Prob, K, v) \in \mathrm{CKL}_{\mathrm{Meas}}$, $u \in W$ and a set of formulas T, we say that M, u is a* model *of T, and write $M, u \models T$, iff $M, u \models \alpha$ for every $\alpha \in T$. The set T is* satisfiable*, if there is $M \in \mathrm{CKL}_{\mathrm{Meas}}$ and a world u from M such that $M, u \models T$. Formula α is valid if $\neg\alpha$ is not satisfiable. We say that T* entails *α and write $T \models \alpha$, if for every $M = (W, Prob, K, v) \in \mathrm{CKL}_{\mathrm{Meas}}$ and every $u \in W$ if $M, u \models T$ then $M, u \models \alpha$.*

3 Axiomatization

In this section we present an axiomatization of our logic, which we denote Ax(CKL). First we need to introduce a useful notion which we use for the proof of Theorem 2.

Definition 6 (k-nested implication). *Let $\alpha \in For$ be a formula and let $k \in \mathbb{N}$. Let $\theta = (\theta_0, \ldots, \theta_k)$ be a sequence of k formulas, and $X = (X_1, \ldots, X_k)$ a sequence of knowledge operators from $\{K_i \mid i \in \mathbf{A}\}$. The k-nested implication formula $\Phi_{k,\theta,X}(\alpha)$ is defined recursively as follows:*

$$\Phi_{0,\theta,X}(\alpha) = \theta_0 \to \alpha$$

$$\Phi_{k,\theta,X}(\alpha) = \theta_k \to X_k \Phi_{k-1,\theta_{j=0}^{k-1},X_{j=0}^{k-1}}(\alpha)$$

For example, if $X = (K_a, K_b, K_c)$, $a, b, c \in \mathbf{A}$, then $\Phi_{3,\theta,X}(\alpha) = \theta_3 \to K_c(\theta_2 \to K_b(\theta_1 \to K_a(\theta_0 \to \top)))$.

$Ax(\mathrm{CKL})$ contains the following axiom schemas and inference rules. It is straightforward to check that $Ax(\mathrm{CKL})$ is sound with respect to $\mathrm{CKL}_{\mathrm{Meas}}$.

Axiom and rule for propositional reasoning

(A1) All instances of classical propositional tautologies.
(R1) From $\{\alpha, \alpha \to \beta\}$ infer β

Axioms and rules for reasoning about knowledge

(A2) $(K_i\alpha \wedge K_i(\alpha \to \beta)) \to K_i\beta$, for every $i \in G$
(A3) $K_i\alpha \to \alpha$,
(A4) $K_i\alpha \to K_iK_i\alpha$,
(A5) $\neg K_i\alpha \to K_i\neg K_i\alpha$,
(R2) From α infer $K_i\alpha$.

Axioms for reasoning about linear inequalities

(A6) $((a_1w_i(\alpha_1,\alpha'_1) + \cdots + a_kw_i(\alpha_k,\alpha'_k) \leq r) \wedge (w_i(\alpha'_{k+1},\top) > 0)) \leftrightarrow (a_1w_i(\alpha_1,\alpha'_1) + \cdots + a_kw_i(\alpha_k,\alpha'_k) + 0w_i(\alpha_{k+1},\alpha'_{k+1}) \leq r)$
(A7) $(a_1w_i(\alpha_1,\alpha'_1) + \cdots + a_kw_i(\alpha_k,\alpha'_k) \leq r) \to (a_{j_1}w_i(\alpha_{j_1},\alpha'_{j_1}) + \cdots + a_{j_k}w_i(\alpha_{j_k},\alpha'_{j_k}) \leq r)$ where $j_1,\dots j_k$ is a permutation of $1,\dots k$.
(A8) $(a_1w_i(\alpha_1,\alpha'_1)+\cdots+a_kw_i(\alpha_k,\alpha'_k) \leq r) \wedge (a'_1w_i(\alpha_1,\alpha'_1)+\cdots+a'_kw_i(\alpha_k,\alpha'_k) \leq r') \to ((a_1+a'_1)w_i(\alpha_1,\alpha'_1) + \cdots + (a_k+a'_k)w_i(\alpha_k,\alpha'_k) \leq r + r')$
(A9) $(a_1w_i(\alpha_1,\alpha'_1) + \cdots + a_kw_i(\alpha_k,\alpha'_k) \leq r) \leftrightarrow (da_1w_i(\alpha_1,\alpha'_1) + \cdots + da_kw_i(\alpha_k,\alpha'_k) \leq dr)$ where $d > 0$.
(A10) $\bigwedge_{i=0}^{n} w_i(\alpha'_i,\top) > 0 \to ((a_1w_i(\alpha_1,\alpha'_1) + \cdots + a_kw_i(\alpha_k,\alpha'_k) \leq r) \vee (a_1w_i(\alpha_1,\alpha'_1) + \cdots + a_kw_i(\alpha_k,\alpha'_k) \geq r)$
(A11) $(f_i \geq r) \to (f_i > r')$ for $r > r'$

Axioms and rule for reasoning about probabilities

(A12) $w_i(\alpha,\top) \geq 0$
(A13) $w_i(\alpha \wedge \beta,\top) + w_i(\alpha \wedge \neg\beta,\top) = w_i(\alpha,\top)$
(A14) $w_i(\alpha,\top) = w_i(\beta,\top)$ if $\alpha \leftrightarrow \beta$ is an instance of propositional tautology
(A15) $\sum_{j=1}^{n} a_jw_i(\alpha_j,\beta_j) \geq r \to w_i(\beta_j,\top) > 0$ for every $j \in \{1,\dots,n\}$
(A16) $(w_i(\beta,\top) \geq s \wedge w_i(\alpha,\beta) \geq r) \to w_i(\alpha \wedge \beta,\top) \geq sr$
(R3) From α infer $w_i(\alpha,\top) \geq 1$
(R4) From the set of premises $\{\Phi_{k,\theta,X}(f_i \geq r - \frac{1}{k}) \mid k \in \mathbb{N}\}$ infer $\Phi_{k,\theta,X}(f_i \geq r)$
(R5) From the set of premises $\{\Phi_{k,\theta,X}(w_i(\beta,\top) > 0)\} \cup \{\Phi_{k,\theta,X}((w_i(\beta,\top) \geq s \to w_i(\alpha \wedge \beta,\top) \geq rs) \mid s \in [0,1]_Q\}$ infer $\Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r)$

The given axioms and rules are divided into four groups, according to the type of reasoning. The axioms A6–A14 are adapted from axiom system from [5] to our approach to conditional probabilities. The axioms A15 and A16, together with the rule R5 properly capture the third condition of Definition 3. The rules R4 and R5 are infinitary inference rules. R4 is a variant of so called Archimedean rule, whose role is to prevent nonstandard values. Intuitively, it says that is the value of a term is infinitely close to r, then it must be equal to r.

Let us now define some basic notions of proof theory.

Definition 7 (Theorem, proof). *A formula α is a* theorem*, denoted by* $\vdash \alpha$*, if there is a sequence of formulas* $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda+1}$ *(λ is finite or countable ordinal), such that* $\alpha_{\lambda+1} = \alpha$ *and every* α_i*,* $i \leq \lambda + 1$*, is an axiom, or it is derived from the preceding formulas by an inference rule.*

A formula α is deducible from a set $T \subseteq For$ $(T \vdash_{Ax(\mathrm{CKL})} \alpha)$ *if there is a sequence of formulas* $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda+1}$ *(λ is finite or countable ordinal), such that* $\alpha_{\lambda+1} = \alpha$ *and every* α_i *is an axiom or a formula from* T*, or it is derived from the preceding formulas by an inference rule, with the exception that e R2 and R3 can be applied to the theorems only. The sequence* $\alpha_0, \alpha_1, \ldots, \alpha$ *is a* proof *of* α *from* T*. We write* $\vdash$ *instead of* $\vdash_{Ax_{\mathrm{CKL}}}$ *when it is clear from context.*

Note that the length of a proof is any countable successor ordinal.

Definition 8 (Consistency). *A set of formulas* T *is* inconsistent *if* $T \vdash \bot$*, otherwise it is* consistent*.* T *is a* maximal consistent set *(mcs) of formulas if it is consistent and every proper superset of* T *is inconsistent.*

4 Completeness

In this section we show that the axiomatization $Ax(\mathrm{CKL})$ is strongly complete for the logic CKL, i.e., we prove that every consistent set of formulas has a model. First we prove several auxiliary statements.

Theorem 1 (Deduction theorem). *Let* T *be a set of formula and* α *and* β *a formulas. Then*

$$T \cup \{\alpha\} \vdash \beta \text{ iff } T \vdash \alpha \to \beta.$$

Deduction theorem can be proven using transfinite induction on the length of the inference. For the cases when we apply infinitary inference rules, we refer the reader to [23], when a similar proof is presented, using the form of k-nested implications in the infinitary rules.

Theorem 2 (Strong necessitation). *If* T *is a set of formulas and* $T \vdash \alpha$*, then* $K_i T \vdash K_i \alpha$*, for all* $i \in \mathbf{A}$*, where* $K_i T = \{K_i \alpha \mid \alpha \in T\}$*.*

Proof. Let $T \vdash \alpha$. We will prove the theorem by using the transfinite induction on the length of the proof of $T \vdash \alpha$. Here we will only consider the application of the rule R5. Let α be the formula $\Phi_{k,\theta,X}(w_i(\gamma, \beta) \geq r)$ which was obtained by the rule R5. Then we have

$T \vdash \Phi_{k,\theta,X}(w_i(\beta, \top) > 0)$
$T \vdash \Phi_{k,\theta,X}(w_i(\beta, \top) \geq s \to w_i(\gamma \wedge \beta, \top) \geq rs)$ for all $s \in [0,1]_Q$
$K_i T \vdash K_i \Phi_{k,\theta,X}(w_i(\beta, \top) > 0)$ by IH
$K_i T \vdash K_i \Phi_{k,\theta,X}(w_i(\beta, \top) \geq s \to w_i(\gamma \wedge \beta, \top) \geq rs)$ for all $s \in [0,1]_Q$, by IH
$K_i T \vdash \top \to K_i \Phi_{k,\theta,X}(w_i(\beta, \top) > 0)$
$K_i T \vdash \top \to K_i \Phi_{k,\theta,X}(w_i(\beta, \top) \geq s \to w_i(\gamma \wedge \beta, \top) \geq rs)$ for all $s \in [0,1]_Q$
$K_i T \vdash \Phi_{k+1,\overline{\theta},\overline{X}}(w_i(\beta, \top) > 0)$, $\overline{\theta} = (\theta, \top), \overline{X} = (X, K_i)$

$K_iT \vdash \Phi_{k+1,\overline{\theta},\overline{X}}(w_i(\beta,\top) \geq s \rightarrow w_i(\gamma \wedge \beta,\top) \geq rs)$ for all $s \in [0,1]_Q$,
$K_iT \vdash \Phi_{k+1,\overline{\theta},\overline{X}}(w_i(\gamma,\beta) \geq r)$, by R5
$K_iT \vdash \top \rightarrow K_i\Phi_{k,\theta,X}(w_i(\gamma,\beta) \geq r)$
$K_iT \vdash K_i\alpha$. □

Next we prove some crucial statements which we need for the proof of the completeness theorem.

Theorem 3 (Lindenbaum's Theorem). *Every consistent set of formulas can be extended to a maximal consistent set.*

Proof. Let T be an arbitrary consistent set of formulas. Assume that $\{\gamma_i \mid i = 0,1,2,\dots\}$ is an enumeration of all formulas from For. We construct the set T^* recursively, in the following way:

1. $T_0 = T$.
2. If the formula γ_i is consistent with T_i, then $T_{i+1} = T_i \cup \{\gamma_i\}$.
3. If the formula γ_i is not consistent with T_i, then:
   (a) If $\gamma_i = \Phi_{k,\theta,X}(f_i \geq r)$ and $f_i = w_i(\alpha,\beta)$, then we define $T_{i+1} = T_i \cup \{\neg\gamma_i, \neg\Phi_{k,\theta,X}(f_i \geq r - \frac{1}{m}), \gamma"_i\}$ where
   $\gamma"_i = \neg\Phi_{k,\theta,X}(w_i(\beta,\top) > 0)$, if $T_i \cup \{\neg\Phi_{k,\theta,X}(w_i(\beta,\top) > 0\} \nvdash \bot$
   $\gamma"_i = \neg\Phi_{k,\theta,X}(w_i(\beta,\top) \geq s \rightarrow w_i(\alpha \wedge \beta,\top) \geq sr)$, otherwise,
   for some $m \in \mathbb{N}$ and $s \in [0,1]_Q$ such that T_{i+1} is consistent.
   (b) If $\gamma_i = \Phi_{k,\theta,X}(f_i \geq r)$ and $f_i \neq w_i(\alpha,\beta)$ then we define $T_{i+1} = T_i \cup \{\neg\gamma_i, \neg\Phi_{k,\theta,X}(f_i \geq r - \frac{1}{m})\}$ for some $m \in \mathbb{N}$, such that T_{i+1} is consistent.
   (c) Otherwise, $T_{i+1} = T_i \cup \{\neg\gamma_i\}$.
4. $T^* = \bigcup_{n=0}^{\infty} T_n$.

First we will show that the set T^* is correctly defined, i.e., there exist $m \in \mathbb{N}$ from (3a) and (3b) and rational number s from the step (3a) of the construction. Let us prove correctness in step (3a) exists.

Let us assume that $T_i' = T_i \cup \{\Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r)\}$ is inconsistent. From Theorem 1 we obtain $T_i \vdash \neg\Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r)$. Suppose that the set $T_i \cup \{\neg\Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r - \frac{1}{m})\}$ inconsistent for every $m \in \mathbb{N}$. By Theorem 1, we have $T_i \vdash \Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r - \frac{1}{m})$ for every $m \in \mathbb{N}$. Then by the rule R3 we have $T_i \vdash \Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r)$. Contradiction. Now suppose that the set $T_i' \cup \{\neg\Phi_{k,\theta,X}(w_i(\beta,\top) > 0)\}$ is inconsistent, and that the set $T_i' \cup \{\neg\Phi_{k,\theta,X}(w_i(\beta,\top) \geq s \rightarrow w_i(\alpha \wedge \beta,\top) \geq sr)\}$ is inconsistent for every s. By Theorem 1, we obtain that $T_i' \vdash \Phi_{k,\theta,X}(w_i(\beta,\top) > 0)$ and $T_i' \vdash \Phi_{k,\theta,X}(w_i(\beta,\top) \geq s \rightarrow w_i(\alpha \wedge \beta,\top) \geq sr)$, for every s. By the rule R4 we have $T_i' \vdash \Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r)$. Contradiction.

Next we prove that T^* is a maximal consistent set. Note that every T_i is consistent by the construction. This still doesn't imply consistency of $T^* = \bigcup_{n=0}^{\infty} T_n$, since we have infinitary rules. First we show that for every $\gamma' \in For$ either $\gamma' \in T^*$ or $\neg\gamma' \in T^*$ holds. Let i and j be the nonnegative integers such that $\gamma_i = \gamma'$ and $\gamma_j = \neg\gamma'$. Then, either γ' or $\neg\gamma'$ is consistent with $T_{max\{i,j\}}$. If $T_{max\{i,j\}}$ is not consistent with γ' and $\neg\gamma'$ then by Theorem 1, $T_{max\{i,j\}}$ will be inconsistent. Then either $\gamma' \in T_{i+1}$ or $\neg\gamma' \in T_{j+1}$, so either $\gamma' \in T^*$ or $\neg\gamma' \in T^*$.

In order to prove the consistency of T^*, we will show that T^* is deductively closed. If the formula γ is an instance of some axiom, then $\gamma \in T^*$ by the construction of T^*. Here we show that T^* is closed under the rule R5; the other cases are similar. Suppose $T^* \vdash \Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r)$ was obtained by R5, where $\Phi_{k,\theta,X}(w_i(\beta,\top) > 0) \in T^*$ and $\Phi_{k,\theta,X}(w_i(\beta,\top) \geq s \rightarrow w_i(\alpha \wedge \beta,\top) \geq sr) \in T^*$ for all $s \in [0,1]_Q$. Assume that $\Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r) \notin T^*$. Let j be the positive integer such that $\gamma_j = \Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r)$. Then, $T_j \cup \{\gamma_j\}$ is inconsistent, since otherwise $\Phi_{k,\theta,X}(w_i(\alpha,\beta) \geq r) \in T_{j+1} \subset T^*$. By the step (3a) $\neg\Phi_{k,\theta,X}(w_i(\beta,\top) > 0) \in T_{j+1}$ or there is $s' \in [0,1]_Q$ such that $\neg\Phi_{k,\theta,X}(w_i(\beta,\top) \geq s' \rightarrow w_i(\alpha \wedge \beta,\top) \geq s'r) \in T_{j+1}$. Suppose $\neg\Phi_{k,\theta,X}(w_i(\beta,\top) > 0) \in T_{j+1}$ and from $\Phi_{k,\theta,X}(w_i(\beta,\top) > 0) \in T^*$ there is nonegative integer k such that $\Phi_{k,\theta,X}(w_i(\beta,\top) > 0) \in T_k$. Then $T_{\max\{k,j+1\}} \vdash \perp$, a contradiction.

Now suppose that $\neg\Phi_{k,\theta,X}(w_i(\beta,\top) \geq s' \rightarrow w_i(\alpha\wedge\beta,\top) \geq s'r) \in T_{j+1}$, where $s' \in [0,1]_Q$. We have that $\Phi_{k,\theta,X}(w_i(\beta,\top) \geq s \rightarrow w_i(\alpha \wedge \beta,\top) \geq sr) \in T^*$ for all $s \in [0,1]_Q$, so we have $\Phi_{k,\theta,X}(w_i(\beta,\top) \geq s' \rightarrow w_i(\alpha \wedge \beta,\top) \geq s'r) \in T^*$. Then, there is nonegative integer k' such that $\Phi_{k,\theta,X}(w_i(w_i(\beta,\top) \geq s' \rightarrow w_i(\alpha\wedge\beta,\top) \geq s'r) \in T'_k$. Then $T_{\max\{k',j+1\}} \vdash \perp$, a contradiction. Consequently, the set T^* is deductively closed.

From the fact that T^* is deductively closed we can prove that T^* is consistent. Indeed, if T^* is inconsistent, there is $\gamma' \in For$ such that $T^* \vdash \gamma' \wedge \neg\gamma'$. But then there is a nonnegative integer i such that $\gamma' \wedge \neg\gamma' \in T_i$, a contradiction. □

Next we introduce some notation, that we use in definition of the canonical model. For a given $T \subseteq For$ and $i \in \mathbf{A}$, we define the set T/K_i as follows:

$$T/K_i = \{\alpha \mid K_i\alpha \in T\}.$$

Definition 9 (Canonical model). *The canonical model $M_C = (W, \mathcal{K}, Prob, v)$ is defined as follows:*

- $W = \{u \mid u$ *is maximal consistent set*$\}$,
- *for every world u and every propositional letter $p \in \mathcal{P}$, $v(u,p) = true$ iff $p \in u$,*
- $\mathcal{K} = \{\mathcal{K}_i \mid i \in \mathbf{A}\}$ *where* $\mathcal{K}_i = \{(u',u) \mid u'/K_i \subset u\}$
- $Prob(i,u) = (W_i(u), H_i(u), \mu_i(u))$ *such that:*
  - $W_i(u) = W$,
  - $H_i(u) = \{\{u' \in W \mid \alpha \in u'\} \mid \alpha \in For\}$,
  - $\mu_i(u) : H_i(u) \rightarrow [0,1]$ *such that* $\mu_i(u)(\{u' \in W \mid \alpha \in u'\}) = \sup\{r \in [0,1]_Q \mid w_i(\alpha,\top) \geq r \in u\}$.

We use the following notation to refer to the elements of $H_i(u)$ from the canonical model:

$$[\![\alpha]\!] = \{u' \in W \mid \alpha \in u'\}.$$

Lemma 1. *Let u be a world of M_C. If $f_i = a_1 w_i(\alpha_1,\alpha'_1) + \cdots + a_k w_i(\alpha_k,\alpha'_k)$ then $a_1\mu_i(u)([\![\alpha_1]\!]|[\![\alpha'_1]\!]) + \cdots + a_k\mu_i(u)([\![\alpha_k]\!]|[\![\alpha'_k]\!]) = \sup\{s \mid u \vdash f_i \geq s\}$.*

Proof. First we will show that $\mu_i(u)([\![\alpha]\!]|[\![\beta]\!]) = \sup\{r \in [0,1]_Q \mid w_i(\alpha,\beta) \geq r \in u\}$. Note that if $\mu_i(u)([\![\beta]\!]) = 0$ then both $\mu_i(u)([\![\alpha]\!]|[\![\beta]\!])$ and $\sup\{r \in [0,1]_Q \mid w_i(\alpha,\beta) \geq r \in u\}$ are undefined.

Suppose that $w_i(\alpha,\beta) \geq r \in u$ and let $\{s_n \mid n \in \mathbb{N}\}$ be strictly increasing sequence of numbers from $[0,1]_Q$, such that $\lim_{n\to\infty} s_n = \mu_i(u)([\![\beta]\!])$. Let n be any number from $\mathbb{N}$. Then $u \vdash w_i(\beta,\top) \geq s_n$. Using the assumption $w_i(\alpha,\beta) \geq r \in u$, the axioms A15 and A16 and propositional reasoning, we obtain $u \vdash w_i(\beta,\top) > 0$ and $u \vdash w_i(\alpha\wedge\beta,\top) \geq rs_n$. Finally, by Definition 9 we have $\mu_i(u)([\![\beta]\!]) > 0$ and $\mu_i(u)([\![\alpha\wedge\beta]\!]) \geq \lim_{n\to\infty} rs_n = r\mu_i(u)([\![\beta]\!])$, i.e., $\mu_i(u)([\![\beta]\!]) > 0$ and $\mu_i(u)([\![\alpha]\!]|[\![\beta]\!]) \geq r$. We can conclude that $\mu_i(u)([\![\alpha]\!]|[\![\beta]\!]) \geq \sup\{r \in [0,1]_Q \mid w_i(\alpha,\beta) \geq r \in u\}$.

Let now $\mu_i(u)([\![\alpha]\!]|[\![\beta]\!]) \geq t$ and $\mu_i(u)([\![\beta]\!]) > 0$. We want to show that $u \vdash w_i(\beta,\top) > 0$ and $u \vdash w_i(\beta,\top) \geq s \to w_i(\alpha\wedge\beta,\top) \geq ts$ for all $s \in [0,1]_Q$.

If $u \nvdash w_i(\beta,\top) > 0$ then $u \vdash w_i(\beta,\top) = 0$, i.e., $\mu_i(u)([\![\beta]\!]) = 0$, contradiction.

If $s > \mu_i(u)([\![\beta]\!])$, than $u \vdash \neg(w_i(\beta,\top) \geq s)$, so $u \vdash w_i(\beta,\top) \geq s \to w_i(\alpha\wedge\beta,\top) \geq ts$. Let now $s \leq \mu_i(u)([\![\beta]\!])$, then $st \leq \mu_i(u)([\![\alpha\wedge\beta]\!])$, so $u \vdash w_i(\alpha\wedge\beta,\top) \geq ts$. Now, we have that for every $s \in [0,1]_Q$, $u \vdash w_i(\beta,\top) \geq s \to w_i(\alpha\wedge\beta,\top) \geq ts$, by the rule R5 we get $u \vdash w_i(\alpha,\beta) \geq t$. So $\mu_i(u)([\![\alpha]\!]|[\![\beta]\!]) \leq \sup\{r \in [0,1]_Q \mid w_i(\alpha,\beta) \geq r \in u\}$.

Let $f_i = a_1 w_i(\alpha_1,\alpha_1') + \cdots + a_k w_i(\alpha_k,\alpha_k')$. By the properties of supremum and A8, $a_1\mu_i(u)([\![\alpha_1]\!]|[\![\alpha_1']\!]) + \cdots + a_k\mu_i(u)([\![\alpha_k]\!]|[\![\alpha_k']\!]) = a_1 \sup\{s_1 \mid u \vdash w_i(\alpha_1,\alpha_1') \geq s_1\} + \cdots + a_k \sup\{s_k \mid u \vdash w_i(\alpha_k,\alpha_k') \geq s_k\} = \sup\{s \mid u \vdash f_i \geq s\}$. $\square$

Lemma 2. *The canonical model M_C is a* CKL*-structure.*

Proof. The proof that every $H_i(u)$ from M_C is an algebra of sets is trivial. The fact that every $\mu_i(u)$ is a finitely additive probability measure follows from the axioms for reasoning about probabilities and Lemma 1. $\square$

On the other hand, in order to show that $M_C \in \mathrm{CKL}_{\mathrm{Meas}}$, we need to prove that $[\alpha]_{i,M_C,u} = [\![\alpha]\!]$, for every i and u. This follows form the following lemma.

Lemma 3 (Truth lemma). *Let M_C be the canonical model and $\gamma \in For$. Then for every world u from M_C*

$$\gamma \in u \textit{ iff } M_C, u \models \gamma.$$

Proof. We use induction on the complexity of the formula γ. If γ is a propositional letter, the statement follows from the construction of M_C. The cases when γ is a conjunction or a negation are straightforward.

Suppose $\gamma = K_i\beta$. Let $K_i\beta \in u$. Since $\beta \in u/K_i$, then $\beta \in u'$ for every u' such that $(u,u') \in \mathcal{K}_i$ (by the definition of $\mathcal{K}_i$). Therefore, $M_C, u' \models \beta$ by induction hypothesis (β is subformula of $K_i\beta$), and then $M_C, u \models K_i\beta$.

Let now $M_C, u \models K_i\beta$. Assume the opposite, that $K_i\beta \notin u$. Then, $u/K_i \cup \{\neg\beta\}$ must be consistent. If it would not be consistent, then $u/K_i \vdash \beta$ by the Deduction theorem and $u \supset K_i(u/K_i) \vdash K_i\beta$ by Theorem 2, i.e., $K_i\beta \in u$, which is a contradiction. Therefore, $u/K_i \cup \{\neg\beta\}$ can be extended to a maximal consistent U, so $u\mathcal{K}_iU$. Since $\neg\beta \in U$, then $M_C, U \models \neg\beta$ by induction hypothesis, so we get the contradiction $M_C, u \not\models K_i\beta$.

Let $f_i = a_1 w_i(\alpha_1, \alpha_1') + \cdots + a_k w_i(\alpha_k, \alpha_k')$. We suppose that $f_i \geq r \in u$, then $r \leq \sup\{s \mid u \vdash f_i \geq s\}$ and $w_i(\alpha_j', \top) > 0 \in u$ for every $j \in \{1, \ldots, k\}$. Then by Lemma 1, $M_C, u \models f_i \geq r$.

For the other direction, assume that $M_C, u \models f_i \geq r$. Suppose that $f_i \geq r \notin u$. Then we have $w_i(\alpha_j', \top) = 0 \in u$ for some $j \in \{1, \ldots, k\}$ or $f_i < r \in u$. If $w_i(\alpha_j', \top) = 0$ for some j then $M_C, u \not\models f_i \geq r$, a contradiction. Let $f_i < r \in u$, then, reasoning as above we conclude $M_C, u \models f_i < r$, a contradiction. □

Consequently, we have shown that for every $\alpha \in For$, every $i \in \mathbb{A}$ and every world u from M_C the equality $[\alpha]_{i,M_C,u} = [\![\alpha]\!]$ holds, so M_C is a CKL-measurable structure.

Theorem 4 (Strong completeness of CKL). *A set of formulas T is consistent iff T is $\mathrm{CKL_{Meas}}$-satisfiable.*

Proof. The direction form right to left is straightforward. For the other direction, suppose that T is a consistent set of formulas. By Theorem 3, there is a maximal consistent superset T^* of T. Since $M_C \in \mathrm{CKL_{Meas}}$, we only need to show that M_C is a model of T^*. By Lemma 3, if T is consistent set we know that T^* is a world in M_C, so we obtain $M_C, T^* \models T$. □

5 Decidability of CKL

In this section, we prove that the logic CKL is decidable. Recall the satisfiability problem: given a CKL-formula α, we want to determine if there exists a world u in a $\mathrm{CKL_{Meas}}$-model M such that $M, u \models \alpha$. First, we show that a CKL-formula is satisfiable iff it is satisfiable in a measurable structure with a finite number of worlds.

For a formula α we denote $Subf(\alpha)$ the set of all subformulas of α.

Theorem 5. *If a CKL-formula α is satisfiable in a model $M \in \mathrm{CKL_{Meas}}$, then it is satisfied in a model $M^* \in \mathrm{CKL_{Meas}}$ with at most $2^{|Subf(\alpha)|}$ number of worlds.*

Proof. Let s be a world from M such that $M, s \models \alpha$. Let $Subf(\alpha)$ be the set of all subformulas of α and $k = |Subf(\alpha)|$. By $\sim$ we denote the equivalence relation over $W \times W$, where $s \sim s'$ iff for every $\beta \in Subf(\alpha)$, $M, s \models \beta$ iff $M, s' \models \beta$. The quotient set $W_{/\sim}$ is finite and $|W_{/\sim}| \leq 2^{|Subf(\alpha)|}$. Now, for every class C_i we choose an element and denote it s_i^*. We consider the model $M^* = (W^*, \mathcal{K}^*, Prob^*, v^*)$, where:

- $W^* = \{s_i^* \mid C_i \in W_{/\sim}\}$,
- $\mathcal{K}^* = \{\mathcal{K}_a^* \mid a \in \mathbf{A}\}$ is a set of binary relations on W^* where $(s_i^*, s_j^*) \in \mathcal{K}_a^*$ iff for every $K_a\phi \in Subf(\alpha)$, $M, s_i^* \models K_a\phi$ iff $M, s_j^* \models K_a\phi$
- For every agent a and $s_i^* \in W^*$, $Prob^*(a, s_i^*) = (W_a^*(s_i^*), H_a^*(s_i^*), \mu_a^*(s_i^*))$ is defined as follows:

- $W^*_a(s^*_i) = \{s^*_j \in W^* \mid (\exists u \in C_j) u \in W_a(s_i)\}$,
- $H^*_a(s^*_i)$ is the power set of $W^*_a(s^*_i)$,
- $\mu^*_a(s^*_i)(\{s^*_j\}) = \mu_a(s^*_i)(C_j(s^*_i))$, where $C_j(s^*_i) = C_j \cap W^*_a(s^*_i)$ and for any $D \in H^*_a(s^*_i)$, $\mu^*_a(s^*_i)(D) = \sum_{s^*_j \in D} \mu^*_a(s^*_i)(\{s^*_j\})$,

– $v^*(s_i, p) = v(s_i, p)$.

It can be shown that $M^* \in \mathrm{CKL}_{\mathrm{Meas}}$.

Finally, using induction on the complexity of the formulas, one can show that for any $\beta \in Subf(\alpha)$, $M, s \models \beta$ iff $M^*, s^*_i \models \beta$ where s^*_i represents C_s in M^*. □

Note that there are infinitely many finite models from $\mathrm{CKL}_{\mathrm{Meas}}$ with at most $2^{|Subf(\alpha)|}$ worlds, because there are infinitely many possibilities for real-valued probabilities. Thus, the previous theorem does not directly imply decidability, and the further complementary steps are needed. In order to show decidability we will translate the problem of satisfiability of a formula to the problem of satisfiability of finite sets of equations and inequalities.

Theorem 6. *Satisfiability problem for* CKL *is decidable.*

Proof. Let α be a CKL-formula. We want to check whether there is a $\mathrm{CKL}_{\mathrm{Meas}}$-structure M and a world s form M such that $M, s \models \alpha$. Using the previous theorem, we will consider only the structures with l worlds, where $l \leq 2^{|Subf(\alpha)|}$.

The idea is to see is there any structure with at least l worlds whom we can join a valuation, a set of binary equivalence relations and finitely additive probabilities such that the formula α is satisfied in some world of the structure. For this we will use potential structures which we call pre-structures. In pre-structures we do not specify probability measures (in order to avoid infinitely many cases), but we want to specify enough information about measures from which we can determine satisfiability of all subformulas of α.

Let $Subf(\alpha)$ be the set of subformulas of α, let $\mathcal{P}^\alpha = \mathcal{P} \cap Subf(\alpha)$ and let $SubP(\alpha)$ be the set of all subformulas of α of the form $\sum_{k=1}^n a_k w_i(\alpha_k, \beta_k) \geq r$. For every $l \leq 2^{|Subf(\alpha)|}$ we consider pre-structures $\overline{M} = (\overline{W}, \overline{\mathcal{K}}, \overline{S}, \overline{v})$ such that:

– $\overline{W}$ is a set of worlds such that $|\overline{W}| = l$
– $\overline{v} : \overline{W} \times \mathcal{P}^\alpha \to \{true, false\}$.
– $\overline{\mathcal{K}} = \{\overline{\mathcal{K}}_a \mid a \in \mathbf{A}\}$ on $\overline{W}$.
– $\overline{S} : \overline{W} \times SubP(\alpha) \to \{true, false\}$.

Note that for every number l we have finitely many possibilities for the choice of pre-structures, i.e., we have finite number of choices of valuation, binary equivalence relations and function $\overline{S}$. This pre-structure is not a CKL-structure, but we can check if a subformula of α holds in a world of a pre-structure $\overline{M}$ using the relation $\Vdash$, defined as follows:

1. If $\gamma \in \mathcal{P}^\alpha$ then $\overline{M}, s \Vdash \gamma$ iff $\overline{v}(s, \gamma) = true$,
2. $\overline{M}, s \Vdash \overline{K}_a \gamma$ iff $\overline{M}, s' \Vdash \gamma$ for all $s' \in \overline{\mathcal{K}}_a(s)$,
3. $\overline{M}, s \Vdash \sum_{k=1}^n a_k w_a(\gamma_k, \beta_k) \geq r$ iff $\overline{S}(s, \sum_{k=1}^n a_k w_a(\gamma_k, \beta_k) \geq r) = true$
4. $\overline{M}, s \Vdash \neg\gamma$ iff $\overline{M}, s \not\Vdash \gamma$,
5. $\overline{M}, s \Vdash \gamma \wedge \beta$ iff $\overline{M}, s \Vdash \gamma$ and $\overline{M}, s \Vdash \gamma$.

We will consider only those $\overline{M} = (\overline{W}, \overline{\mathcal{K}}, \overline{S}, \overline{v})$ such that $\overline{M}, s \Vdash \alpha$ for some world $s \in \overline{W}$. For each such $\overline{M}$ we want to check whether $\overline{M}$ can be extended to a structure, i.e., whether there is a measurable structure $M = (\overline{W}, \overline{\mathcal{K}}, Prob, v)$ such that $\overline{v}$ is a restriction of v and for every agent a and every $s \in \overline{W}$ and $\sum_{k=1}^{n} a_k w_a(\gamma_k, \beta_k) \geq r \in SubP(\alpha)$ we have $M, s \models \sum_{k=1}^{n} a_k w_a(\gamma_k, \beta_k) \geq r$ iff $\overline{S}(s, \sum_{k=1}^{n} a_k w_a(\gamma_k, \beta_k) \geq r) = true$. It is straightforward to check that for such M we have $M, s \models \beta$ iff $\overline{M}, s \Vdash \beta$ holds for every $\beta \in Subf(\alpha)$. Since the way v extends $\overline{v}$ is irrelevant, it suffices to check whether $\overline{S}$ can be replaced with $Prob$ in some $\overline{M} = (\overline{W}, \overline{\mathcal{K}}, \overline{S}, \overline{v})$ such that $\overline{M}, s \Vdash \alpha$ for some world $s \in \overline{W}$. For that purpose, for each such $\overline{M}$ we consider specific equations and inequalities, that we describe below. We chose the variables of the form y_{a,s_i,s_j} which represent the values $\mu_a(s_i)(\{s_j\})$. Now we state the equations and inequalities:

$$(1) \quad y_{a,s_i,s_j} \geq 0, \text{ for every world } s_j$$

$$(2) \quad \sum_{s_j \in \overline{M}} y_{a,s_i,s_j} = 1$$

$$(3) \quad \sum_{w_j : M^l, w_j \Vdash \beta_k} y_{a,s_i,s_j} > 0 \text{ for every } k \in \{1, \ldots, n\}, \text{ and}$$

$$\sum_{k=1}^{n} \Big(a_k \sum_{s_j : \overline{M}, s_j \Vdash \beta_k \wedge \gamma_k} y_{a,s_i,s_j} \prod_{t \neq k, t=1}^{n} \sum_{s_j : \overline{M}, s_j \Vdash \beta_t} y_{a,s_i,s_j} \Big) \geq$$

$$r \prod_{k=1}^{n} \sum_{s_j : \overline{M}, s_j \Vdash \beta_k} y_{a,s_i,s_j}, \text{ for every formula} \sum_{k=1}^{n} a_k w_a(\gamma_k, \beta_k) \geq r$$

$$\text{such that } \overline{S}(s_i, \sum_{k=1}^{n} a_k w_a(\gamma_k, \beta_k) \geq r) = true$$

$$(4) \bigvee_{k=1}^{n} \Big(\sum_{s_j : \overline{M}, s_j \Vdash \beta_k} y_{a,s_i,s_j} = 0 \Big) \text{ or}$$

$$\sum_{k=1}^{n} \Big(a_k \sum_{s_j : \overline{M}, s_j \Vdash \beta_k \wedge \gamma_k} y_{a,s_i,s_j} \prod_{t \neq k, t=1}^{n} \sum_{s_j : \overline{M}, s_j \Vdash \beta_t} y_{a,s_i,s_j} \Big) <$$

$$r \prod_{k=1}^{n} \sum_{s_j : \overline{M}, s_j \Vdash \beta_k} y_{a,s_i,s_j}, \text{ for every formula} \sum_{k=1}^{n} a_k w_a(\gamma_k, \beta_k) \geq r$$

$$\text{such that } \overline{S}(s_i, \sum_{k=1}^{n} a_k w_a(\gamma_k, \beta_k) \geq r) = false$$

The inequality (1) above assures that all the probability measures are non-negative, and the equality (2) states that the probability of the set of all possible worlds has to be equal to 1. The equality (3) states that the probabilities of the sets of all evidences in a formula are greater than 0 and the linear combination of probabilities is greater than r, from the corresponding formula. It is easy to see that (3) corresponds to the third condition of the satisfiability relation from Definition 3, after we clean the denominators. Similarly, (4), corresponds to the combination of the fourth and the third condition from Definition 3.

The equations and inequalities (1)–(4) form not one, but a number of finite systems of equations and inequalities. Note that adding (4) to any system Sys of equations and inequalities results with a disjunction of at least two different extensions of Sys. For the purpose of this proof, the fact that we always have finitely many systems is sufficient, and it is enough if one of the systems is solvable. Those systems are represented in the language of real closed fields, and it is well known that the theory of real closed fields is decidable. Since we have finitely many possibilities for the choice of l, and for every l finitely many possibilities for the choice of pre-structure, our logic is decidable as well. $\square$

6 Conclusion

We have investigated a propositional logic of knowledge and conditional probability that allows explicit reasoning about probabilities. We have been able to obtain strongly complete axiomatization and decision procedure for our logic. Following [5], we proposed the most general case, where no semantic relationship is posed between the modalities for knowledge and probability. Fagin and Halpern [5] also consider some modification of the semantics, by posing relations between the sample spaces $W_i(u)$ and possible worlds $\mathcal{K}_i(u)$, which model some typical situations in the multi-agent systems. For example, they consider a natural assumption $W_i(u) \subseteq \mathcal{K}_i(u)$, which forbids an agent to place positive probabilities to the events she knows to be false. The paper [5] provides characterization of all those semantic assumptions in terms of corresponding axioms. (for example, $W_i(u) \subseteq \mathcal{K}_i(u)$ corresponds to $K_i\alpha \rightarrow w_i(\alpha) = 1$). Adding those axioms to our system would also make it complete for the considered semantics.

Acknowledgments. This work has been partially funded by the Science Fund of the Republic of Serbia through the project Advanced artificial intelligence techniques for analysis and design of system components based on trustworthy BlockChain technology - AI4TrustBC (the first and the third author).

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