BRILL

JAN P. HOGENDIJK

A MATHEMATICAL CLASSIFICATION OF THE CONTENTS OF AN ANONYMOUS PERSIAN COMPENDIUM ON DECORATIVE PATTERNS

The purpose of this chapter is to provide a mathematical classification of the contents of the *Anonymous Compendium* comprising folios 180r–199r of Ms. Persan 169, located in the Bibliothèque nationale de France and referred to in this volume as the Paris Codex.[1] Ms. Persan 169 was written in a single hand, probably in the fifteenth to sixteenth centuries, and contains other mathematical texts as well.[2] Because the sixty-one patterns in the *Anonymous Compendium* occur in rather random order, it is likely that they were taken from different sources, probably dating back to different centuries.

The *Anonymous Compendium* is historically important because it differs in style and content from almost all other geometrical works that have come down to us from the Islamic tradition, including some of the other treatises in the Paris Codex. Most medieval Islamic authors who wrote on geometry were trained in the style of Euclid's *Elements* (ca. 300 B.C.), which was available in several good Arabic translations. These authors studied geometry in order to become astronomers and astrologers,[3] and in the majority of their writings no reference is made to decorative geometrical patterns—which may be surprising, as they must have often seen such patterns in palaces and mosques.

Most of the patterns appear to have been designed not by theoretically trained geometers but by craftsmen (Arabic: *ṣunna*ᶜ). The tenth-century geometer and astronomer Abu'l-Wafa' al-Buzjani wrote a short *Kitāb fī a*ᶜ*māl al-handasa* (*Book of Geometrical Constructions*, henceforth *Geometrical Constructions*), which is the subject of the chapter by Elaheh Kheirandish in the present publication.[4] In the *Geometrical Constructions*, Abu'l-Wafa' comments on the practices of craftsmen and also reports on meetings at which craftsmen and theoretically trained geometers were present.[5] Abu'l-Wafa' says that the craftsmen did not distinguish between geometrically exact and approximate constructions, and were not concerned with geometric proofs. He also complains that they only wanted their decorative patterns to look right. The rather snobbish aim of his *Geometrical Constructions* was to make sure that the craftsmen henceforth used only mathematically exact constructions.

The *Anonymous Compendium* conforms to a large extent with Abu'l-Wafa''s description of the practices of medieval Islamic craftsmen. Mathematically exact and approximate constructions occur next to one another, and are presented without proofs. The approximate character of some constructions is sometimes indicated but not always. Moreover, the *Anonymous Compendium* also uses an instrument called the *gūnyā* (set square), which is mentioned in Abu'l-Wafa''s book, but not in Euclid's *Elements*. The *Anonymous Compendium* thus preserves textual evidence of how mathematical principles were applied in practice by craftsmen who designed decorative patterns—evidence, that is, of a mathematical tradition few written traces of which have otherwise come down to us.

The author(s) of the material in the *Anonymous Compendium* were not completely unfamiliar with the way in which theoretically trained geometers were doing mathematics. In many of the diagrams, points are labeled using the letters of the Arabic alphabet in increasing numerical value (*alif*=1, *bā*'=2, *jīm*=3, *dāl*=4, etc., hence the name *abjad* for this system of numbering). This labeling system was standard in Arabic translations of Euclid's *Elements* and in works by theoretically trained medieval Islamic geometers. Indeed, the letters of the entire Arabic alphabet are presented in the *abjad* order on folio 196v. These numbers may have been written as a personal note on a blank folio, but, alternatively, their presence may indicate that the audience for which the

© KONINKLIJKE BRILL NV, LEIDEN, 2017 | DOI 10.1163/9789004315204_005

Anonymous Compendium was written was not wholly familiar with the *abjad* system. There are, however, slight differences between the *Anonymous Compendium* and the theoretically trained geometers' standard method for labeling points in cases where the number of points exceeded the number of letters in the alphabet (see fig. 11 and the commentary below).

Other differences can also be mentioned. The text in the *Anonymous Compendium* is terse, and sometimes unintelligible or defective; some of the diagrams are ambiguous; and, in one case, a single figure is used for two mutually inconsistent diagrams (fig. 11 below). By contrast, medieval Islamic geometrical works in the tradition of Euclid's *Elements* usually contain clear explanations.

Moreover, the author(s) of the geometrical constructions in the *Anonymous Compendium* often did not worry about issues that were important to theoretically trained geometers of the Islamic tradition in the tenth century and later. Many of the latter thought that all constructions should be made only by means that they considered legitimate, i.e., with an (unmarked) ruler, compass, and possibly conic sections. This restricted the range of possible constructions, as some—such as the trisection of an arbitrary arc, or the division of a circle in seven or nine equal arcs—could be made only by use of conic sections, and no legitimate solutions existed for problems such as the division of a circle into eleven equal arcs. In the *Anonymous Compendium*, by contrast, it is assumed, without any further explanation, that any arc can be trisected and that a circle can be divided into an arbitrary number of equal arcs. Works by theoretically trained medieval Islamic geometers rarely had approximate constructions, whereas the *Anonymous Compendium* contains many intricate ones. The quality of these approximations is often very good, and an approximate construction with an error of, say, one percent may produce a vastly better result than a theoretically "exact" construction that is difficult to execute with actual instruments. The *Anonymous Compendium* therefore preserves evidence of a type of practical geometrical knowledge that was perhaps not sufficiently appreciated by Abu'l-Wafa' and others trained in the theoretical geometry of the Greeks.

As for modern attempts to relate medieval Islamic patterns to mathematical concepts such as aperiodic tilings or group theory, it should be noted that the craftsmen who appear to have designed the decorative patterns were, with a only a few possible exceptions, not trained in the Euclidean deductive approach to mathematics, and so were not used to theoretical mathematical concepts and arguments. More generally, reading modern mathematical concepts into medieval texts and buildings is problematic.[6] Still, Islamic decorative patterns can be useful tools for teaching modern mathematics, even if no historical relationship between the two can be established.

The diagrams in the *Anonymous Compendium* do not represent a cross-section of the decorations on the remaining architectural monuments in Iran, and the decorative applications of some of them are uncertain at best. For example, a number of diagrams in the manuscript involve heptagons or heptagonal stars, which are found only infrequently in architectural remains in Iran, given the complexity of their design. Indeed, only one of the heptagonal diagrams in the *Anonymous Compendium* has hitherto been found on a building, namely in the North Dome of the Friday Mosque in Isfahan (see figs. 8–10 below). In one or two groups of diagrams to be discussed in Part VI, a connection between the work of the craftsmen and that of contemporaneous theoretically trained mathematicians can be demonstrated. One may well ask whether the *Anonymous Compendium*, although it contains material by craftsmen, piqued the interest of some theoretically trained geometers who were more open-minded than Abu'l-Wafa', and whether this is the reason it was copied and has come down to us. No answer can be given at this time because of the lack of written (and published) original documents about the mathematical traditions of craftsmen. Thus it is difficult to put some of the material in the *Anonymous Compendium* in its proper historical context. Libraries in Iran and other countries may well contain unexplored documents on the issue, but it may also be that parts of the tradition were transmitted orally and never put into writing.

In the following sections, the sixty-one patterns in the *Anonymous Compendium* will be classified into seven groups on mathematical grounds: (1) cut-and-paste geometrical constructions; (2) preliminary geometrical constructions and one theorem; (3) geometrical constructions using a single compass opening; (4) constructions of rectangular patterns; (5) divisions of a square; (6) difficult patterns involving kites (Persian: *turunj*, a quadrilateral with two equal opposite angles and two pairs of equal sides arranged in such a way that the sides containing each of the equal angles are unequal); and (7) *muqarnas*. Each section begins with a list of all the patterns in that group, and a brief description of each pattern with references; this is followed by a detailed discussion of one or two examples that were selected for the light they shed on some characteristics of the *Anonymous Compendium* and/or the mathematical practices of its author(s). I have refrained from giving a full mathematical commentary on all the patterns, not only because of lack of space, but also because the mathematical analysis of some of these patterns provides delightful research projects for students. My survey should be read in conjunction with Gülru Necipoğlu's survey in the present volume, which focuses on the cultural and art-historical aspects of the *Anonymous Compendium* in addition to analyzing its contents. It would also profit the reader to consult the drawings Alpay Özdural created based on the *Anonymous Compendium*, all of which have been reproduced here as well (see plates).

In this chapter (as throughout the present volume), the original geometrical constructions are cited by folio and number, as in "fol. 180v [1]" for Construction 1. Thus, in the section in this volume titled "Translation, Transcription, and Drawings," the reader can quickly locate the Persian text and English translation, as well as the commentaries summarized by Necipoğlu, which are based on Özdural's unpublished book manuscript. Using the further references provided in that section, the reader can find Özdural's extensive discussions on the constructions in his unpublished manuscript, which the editors have decided to make available to interested readers (online at http://dx.doi.org/10.1163/9789004XXXXXX). In the endnotes, I have added systematic references to the two previously published translations of the *Anonymous Compendium*, namely A.B. Vildanova's Russian translation, which is appended to Mitkhat S. Bulatov's *Geometricheskaia garmonizatsiia v arkhitekture Srednei Azii IX–XV vv*. (references are to the figures in the translation), and the modern Persian adaptation by Sayyid Ali Riza Jazbi, *Handasa-i Īrānī: Kitāb-i Tijārat: Fī mā yaḥtāju ilayhi al-ʿummāl wa-al-ṣunnāʾ min al-ashkāl al-handasiyya, yā, Kārburd-i handasa dar ʿamal*.[7]

I. CUT-AND-PASTE CONSTRUCTIONS

For each of these constructions the *Anonymous Compendium* presents a diagram that consists of two or three figures, divided into the same set of pieces. The purpose is to show how each of the two or three figures can be put together from the pieces.

The following constructions consist of figures only, without accompanying text:

Fol. 181r [3]:[8] an equilateral triangle and a hexagonal star, formed from six pieces.

Fol. 182v [10]:[9] a square and an octagonal star, formed from eight pieces.

Fol. 186bis(r) [22]:[10] a square and a regular octagon, formed from nine pieces.

Fol. 197r [58]:[11] a regular heptagon, an isosceles triangle, and a rectangle, formed from twenty pieces.

Fol. 197r [59]:[12] a regular hexagon, an equilateral triangle, and a rectangle, formed from ten pieces (this construction will be discussed below).

Fol. 197r [60]:[13] a regular octagon and an octagonal star, formed from twenty-four pieces.

The following constructions consist of a diagram together with instructions in the text:

Fol. 180r [1]:[14] two small decagons are cut into three pieces each, and these pieces plus one pentagonal star are put together to form one larger decagon.

Fol. 181r [4]:[15] the figure features a regular octagon cut into five pieces, which can be reassembled to form a square, as well as a geometrical construction. The text explains how to use this geometrical construction in order to derive the side of the square from the regular octagon.

Fol. 182r [8]:[16] the figure displays a square cut into eight pieces and a hexagonal star assembled from the same eight pieces, as well as part of a geometrical construction based on the regular hexagon. The text explains how the side of the square can be derived from the hexagonal star.

Fol. 182v [9]:[17] a geometrical construction in which an arbitrary rectangle is cut into four pieces that are reassembled to form a perfect square. The construction is based on the same idea as Euclid's *Elements* II:14, though it differs in the details.[18]

Fol. 183r [11]:[19] how a regular hexagon can be cut into twelve pieces that can be reassembled to form an equilateral triangle. Then, for an equilateral triangle, the text shows how the side of a regular hexagon with the same area can be found by an (approximate) geometrical construction.

Our example will be fol. 197r [59], a construction without any accompanying text (figs. 1–3).

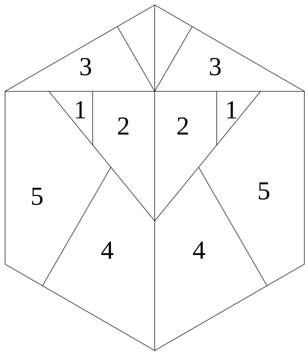

Fig. 1.

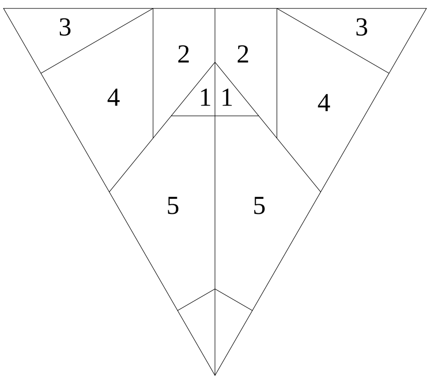

Fig. 2.

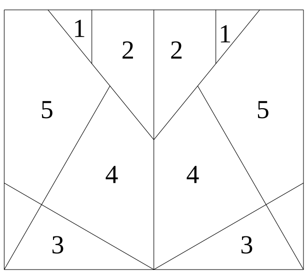

Fig. 3.

Figures 1–3 feature a regular hexagon, an isosceles triangle, and a rectangle, dissected into pieces from which all three figures can be composed. These figures have been derived from the manuscript with one difference: I have arbitrarily assumed the isosceles triangle to be equilateral, and have drawn the figures in a mathematically correct way. In the manuscript, the pieces are indicated by numbers (transcribed in the figures) so that the correspondence between the pieces is clear. There is no text in the manuscript to explain the exact way in which the pieces have to be cut; the reader is left to work out the details for himself/herself. After this exercise, she or he will probably be convinced that the *Anonymous Compendium* was meant to be studied under the guidance of a competent teacher who could supplement the visual information provided in the document.

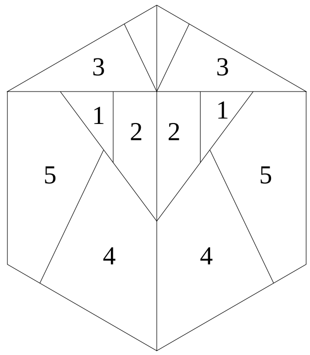

Fig. 4.

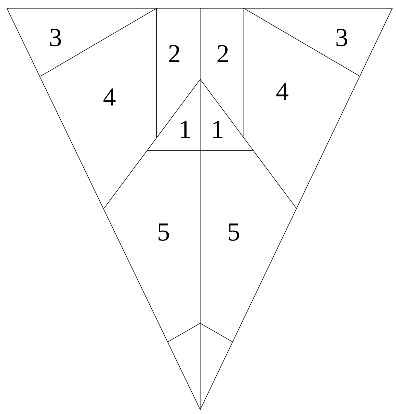

Fig. 5.

It should be noted that in the *Anonymous Compendium* the pieces I have reproduced in figures 1 and 2 are drawn in such a way that the piece numbered 1 is wider than the piece numbered 2. This does not happen in my reconstructed figure, but it may if the vertex angle of the isosceles triangle is less than 54°; figures 4 and 5 have been drawn for a vertex angle of (360/7)°. It is tempting to assume that the craftsmen had in mind the general dissection of an isosceles (rather than an equilateral) triangle, but because there is no accompanying text, one cannot be sure.

It is not necessary to assume that the fancy cut-and-paste constructions in this section were actually used in practice. Just like European arithmetic teachers in later centuries, medieval Islamic craftsmen may have challenged one another with gamelike problems that surpassed the requirements of their routine work.

II. PRELIMINARY GEOMETRICAL CONSTRUCTIONS AND ONE THEOREM

The *Anonymous Compendium* contains a few preliminary geometrical constructions and one theorem:

Fol. 181v [5]:[20] finding the center of a circle from an arc of the circle. The construction is unlike that in Euclid, *Elements* III:25.[21]

Fol. 181v [6]:[22] the completion of a full circle from the arc of a circle. The construction is not based on construction no. 5 and assumes that any arc of a circle can be trisected.

Fol. 182r [7]:[23] the division of a triangle into four pieces by means of three straight lines through the vertices, such that the small triangle is similar to the original triangle. The construction in the manuscript is only correct for an equilateral triangle.

Fol. 183v [12][24] contains the following theorem: for every triangle inscribed in a circle, the ratio of the angles is equal to the ratio of the arcs of the circumscribed circle subtended by the sides of the triangle. This theorem is stated without proof, and is illustrated by a triangle whose angles are in the proportion 1:2:4 (related to a regular heptagon). The theorem is a preliminary to the following construction, i.e., no. 13.

Fol. 184r [13]:[25] the construction of a triangle whose angles are in a given ratio (of whole numbers). The construction is illustrated by the ratios 1:2:6 (related to a regular nonagon) and 3:4:5. It is assumed that the circumference of the circle can be divided into any whole number of equal arcs. An accompanying table lists all seven triangles whose angles are integer multiples of one-ninth of a right angle (in modern terms, multiples of 20°). The text also explains how to construct a triangle with one given side whose angles are in a given ratio of whole numbers.

Fol. 186r [20]:[26] the construction of a regular pentagon with a given side (discussed below).

Fol. 192r [38]:[27] the construction of a circle through three given points (only a figure, without further explanation). The center is found as the point of intersection of the perpendicular bisectors of two of the line segments joining the three given points.

As our example we will discuss the construction of a regular pentagon with a given side in fol. 186r [20]: see figure 6, to which the words "side," "altitude," and "chord" were added in the Persian manuscript.

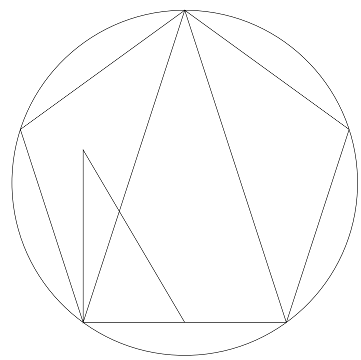

Fig. 6.

The construction is not indicated but must be inferred from figure 6 alone. In the following explanation, assume that the horizontal segment has length 1. First we construct the right-angled triangle in the figure, such that the sides containing the right angle are ½ (horizontal side) and 1 (vertical side). Then the hypotenuse will have length $\frac{1}{2}\sqrt{5}$. Now construct the isosceles triangle in the figure with (horizontal) base of length 1 and sides of length $\frac{1}{2} + \frac{1}{2}\sqrt{5}$, that is, the sum of two sides of the right-angled triangle that was constructed before. We now will have found three vertices of the regular pentagon. The two others can be found by constructing two more isosceles triangles whose sides have length 1.

III. GEOMETRICAL CONSTRUCTIONS USING A SINGLE COMPASS OPENING

This category consists of constructions using a single compass opening. Such constructions eliminate errors that are caused by successive adjustments of the instrument. The following constructions are all approximate but the errors are small and often negligible in practice. Several scholars have proposed that these constructions were meant to serve as templates for triangles corresponding to different set squares.[28]

Fol. 184v [14]:[29] The purpose of this construction is, in modern terms, to find an angle that is 1/5 times two right angles (i.e., 36°), with a fixed compass opening (discussed below).

Fol. 186r [19]:[30] construction of an equilateral and equiangular pentagon with a [fixed compass] opening [equal] to the altitude.

Fol. 186v and 186bis(r) [21]:[31] another construction of a regular pentagon, with a compass opening equal to the altitude of the pentagon.

Fol. 186bis(v) [23]:[32] construction of an equilateral and equiangular pentagon with a [fixed compass] opening equal to the side.

Fol. 186bis(v) [24]: [33] a pentagonal star is inscribed in a semicircle. No instructions are given in the text, but the figure is titled, perhaps incorrectly, "on the construction of an equilateral and equiangular pentagon, with compass opening [equal to] the side."

Fol. 187r [25]: [34] the construction of a pentagon with a [fixed compass] opening equal to the chord. Near the figure is written "by Abu Bakr al-Khalil al-Tajir."

The construction fol. 184v [14] will serve as an example (fig. 7). Here points are labeled with letters [35] that have been transcribed in the figure. The *Anonymous Compendium* also explains the construction, which is as follows:

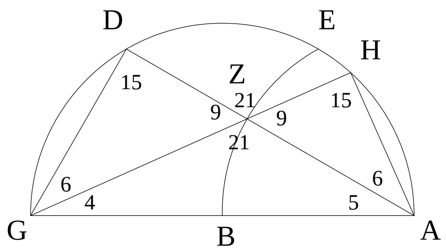

Fig. 7.

First draw a semicircle with center *B* and diameter *ABG*. The compass opening will always be equal to *r*, the radius of this circle. Draw an arc with center *A*, which will pass through *B* and meet the semicircle at point *E*. Find point *D* as the point of intersection of the semicircle and the circular arc with center *G* and radius *r*, and draw *GD*. Draw line *AD* and let *Z* be the point of intersection with arc *EB*. Draw line *GZ* to meet the circle at *H*. Draw line *AH*. Then ∠*DGZ* and ∠*HAZ* are the required angles. The text in the *Anonymous Compendium* does not inform the reader that we also obtain *DH*, the side of the regular pentagon inscribed in the semicircle, and that the construction is a good approximation but not exact.[36] Abu'l-Wafa' explains an exact construction of the regular pentagon by a single compass opening equal to the radius of the pentagon.[37] Abu'l-Wafa''s exact construction is complicated, so the easier approximation may well have given better results in practice.

The text states the purpose of this diagram as the construction of "set square (i.e., *gūnyā*) [number] 5 with a compass opening equal to the radius [of the circumscribing semicircle] from set square [number] 6." A "*gūnyā* n" is a right-angled triangle, with one angle equal to the n-th part of two right angles, that is, in modern terms, (180/n)°. Triangle *ADG* is *gūnyā-6*, and the two triangles *AZH* and *GZD* are *gūnyā-5*. The term *gūnyā* may ultimately be of Greek origin, from the word *gonia*, meaning angle, as defined in Definition 8 of Book I of Euclid's *Elements*.[38]

In the figure, the magnitude of the angles is indicated by numbers that have been transcribed in figure 7, where an angle of "1" would correspond in modern terms to 6°. The author(s) of the *Anonymous Compendium* were probably unfamiliar with the modern concept of "degrees" to measure angles (such that a right angle is 90°). Degrees with their sexagesimal subdivisions were introduced in late Babylonian astronomy to measure arcs of the ecliptic and are not found in Euclid's *Elements*. In later Greek and Islamic astronomy, degrees were used for measuring arcs on a sphere and on a circle, but not often for measuring angles.

IV. CONSTRUCTIONS OF RECTANGULAR PATTERNS TO BE USED IN DECORATIVE TILINGS

In this category, the constructions consist of a basic rectangle filled with a decorative pattern. In some cases, the whole plane can be tiled with the pattern by translating and mirroring the basic rectangle. In others, the decorative applications are less obvious. In most of the constructions, only one side of the basic rectangle is given; the other sides are determined in the construction process and usually the length and width have an irrational ratio.

Ten constructions in the manuscript belong to this category, the first seven of which have textual instructions.

Fol. 190v [35]:[39] a pattern containing a regular heptagon, and parts of a hexagonal and octagonal star.

Fol. 192r [37]:[40] another pattern containing a regular heptagon (discussed below).

Fol. 192v [42]:[41] a pattern containing two kites, as in figure 12 below, subdivided into three pieces, but with two equal angles of 75° rather than 90°.

Fol. 193r [43]:[42] a nice pattern with hexagonal symmetry; the pattern can be extended to tile the plane in a beautiful way with hexagonal figures.

Fol. 193v [44]:[43] a pattern with pentagonal stars and equilateral, but non-equiangular, pentagons in between.

Fol. 195v [54]:[44] a pattern to tile the plane with large dodecagonal and decagonal stars, with small pentagonal stars in between.

Fol. 196v [57]:[45] a division of a rectangle whose length and width are expressed as "three perpendiculars [i.e., altitudes] plus one and a half sides" and "4 and a half sides minus one perpendicular [altitude]." By "side" and "perpendicular [altitude]" the text means the hypotenuse and the intermediate side of a right-angled triangle with angles of 30°, 60°, and 90°, that is, a "set square 6" triangle. In modern terms the length and width of the basic rectangle are in the ratio $(3\sqrt{3}+3) : (9-\sqrt{3}) = 1.127\ldots$.

The last three constructions in this category lack texts:

Fol. 189r [32][46] displays a rectangle whose sides are specified as "two parts" and "square root of 5," divided into eight pieces. Six pieces form two large subdivided kites as in figure 12 below, plus two smaller kites, similar to the larger ones. The pattern resembles the "variant pattern" in Part VI below, but can be constructed by ruler and compass, by dividing a line segment in extreme and mean ratio, as in Euclid's *Elements* II:11.[47]

Fol. 196r [56]:[48] a figure only, without text, displaying a pattern for tiling the plane with dodecagonal stars and somewhat smaller decagonal stars, but without pentagonal stars.

Finally, fol. 199r [61]:[49] an intricate division of a rectangle, filled with small pentagonal stars, intermediate figures, and parts of decagonal stars; copying the rectangles and their mirror image produces a beautiful tiling of the plane. This is the last pattern in the *Anonymous Compendium*; it is separated from the rest of the compendium by several blank pages in the manuscript. The points are not labeled with letters and there is no accompanying text explaining the construction of the pattern.

My example is fol. 192r [37], the resulting pattern for which is found in the North Dome of the Friday Mosque in Isfahan and can be related to *girih* (from Persian, meaning knot) tiles, as will be discussed below.[50] Figure 8 is a transcription of the construction in the manuscript, which has points labeled with letters. It also has some broken lines that are used in the construction but do not form part of the final decorative pattern.

The procedure is as follows. The length *AB* is assumed and at points *A* and *B* two perpendiculars, *AF* and *BC*, of unspecified length, are drawn. The text prescribes that line *AG* should be drawn such that angle $\angle BAG$ is equal to "three-sevenths of the right angle." In the figure in the manuscript, *AG* is drawn as a broken line, apparently indicating that it is part of an auxiliary construction but does not belong to the final pattern. Broken lines do not often occur in works by medieval Islamic geometers trained in the style of Euclid's *Elements*. The designers do not give further details about the construction of line *AG*, and were not worried by the fact that an angle of (in modern terms) $(3 \times 90)/7°$ cannot be constructed by ruler and compass. Then point *E* is chosen on *BG* such that $BE = AG/2$, and the broken line *EZ* is drawn parallel to *AG*. According to the text, point *L* should be constructed on *EZ* in such a way that if *LK* is drawn parallel to *EB* to meet *AB* at *K*, we have $LK = EL/2$. Point *K* is found through the following auxiliary construction: through an arbitrary point *T* on *EZ* draw a line parallel to *BE*, and choose *I* on this parallel such that $TI = TE/2$ as in the figure, and draw *EI*, which will intersect *AB* at the required point *K*. Again, some of the lines appear as broken lines in the figure in the manuscript, as well as in figure 8.

Then a circular arc is drawn with center *Z* and radius *ZK* to meet *EZ* at *M*, and point *N* is chosen on the arc such that $MN = KM$. Note that point *N* is the mirror image of *K* in line *ZE*. Finally, *LN* is extended to meet the horizontal side of the rectangle through *A* at point *S*. The text says that *S* is "the center of the heptagon" and laconically concludes: "the construction should be easy, God, may He be exalted, willing."

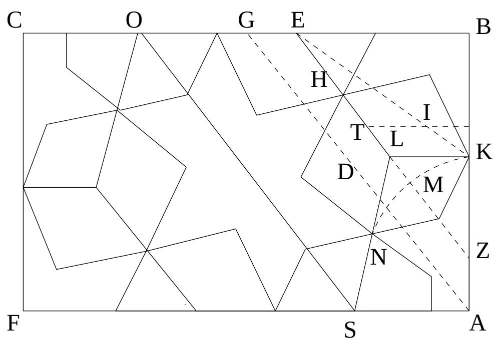

Fig. 8.

Then a few alternative methods are presented in the manuscript to find point *S*. If point *O* is on *BE*, which is extended such that $EL = EO$, then point *O* is also the center of a heptagon and *OS* is parallel to *AG*. The text does not give more information.

To analyze this pattern I use the notation R for a right angle, and $\alpha = 2R/7$; in this notation $\angle EZB = \angle GAB = 3\alpha/2$. In the pattern, rectangle *ABCF* is divided by line *SO* into two congruent trapezia, and each trapezium is divided (in the way of *ABOS*) into three quadrangles: *BKLE* (angles *R*, *R*, 5α, 2α), *AKLS* (angles *R*, *R*, 4α, 3α), and *OELS* (angles 2α, 5α, 5α, 2α). Note that $OE = EL = LS = 2LK$. To complete the pattern, we note that *N* is the midpoint of *LS*, and we use point *K*, point *N*, and the midpoints of segments *LE* and *EO*. Points *S* and *O* are centers of regular heptagons, and the other lines in the pattern can be defined on the basis of arguments of symmetry, but the manuscript does not give any details, nor does it mention that the ratio between the length and width of the rectangle is $(2 + \cos 2\alpha + \cos 3\alpha)/(\sin 2\alpha + \sin 3\alpha) = 1.62003\ldots$, although it would be useful to know this number if one wanted to draw copies of the rectangle.

Bulatov observed that copies of the rectangle and of its mirror image produce a nice tiling of the plane, as shown by the thick continuous lines in figure 9.[51] As mentioned above, this tiling appears in the North Dome in the Friday Mosque in Isfahan; figure 10 shows two rectangles together with two mirror images superimposed on the pattern in that mosque.

Using figure 9, we can relate the pattern to a procedure in which a decorative pattern is based on a set of auxiliary figures called "*girih* tiles" (mentioned above). The actual pattern was drawn through the midpoints of these *girih* tiles. The auxiliary figures themselves could then be erased. We note that the thin lines in figure 9 do not show up in the final pattern in the Friday Mosque in figure 10. Thus, the thin lines in figure 9 define a set of two hexagonal *girih* tiles (with angles 5α, 5α, 4α, 5α, 5α, 4α and 6α, 4α, 4α, 6α, 4α, 4α) on which the actual pattern in figure 10 is based. One complete *girih* tile is visible in figure 8 as the trapezium *OELS* together with its mirror image in *OS*.

In conclusion, the concise and laconic instructions in the text are insufficient to explain the design and application of the pattern. The *Anonymous Compendium* was clearly part of an oral tradition of instruction.

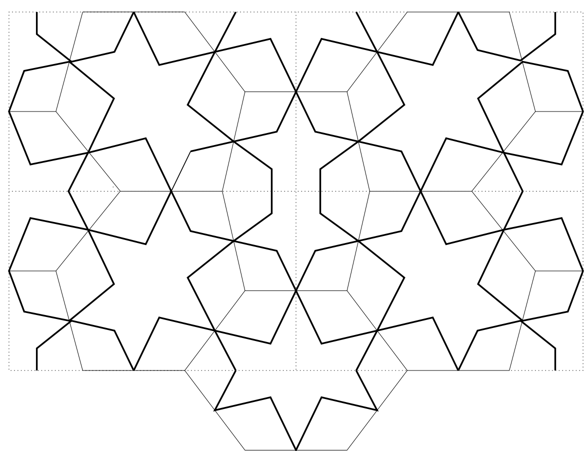

Fig. 9.

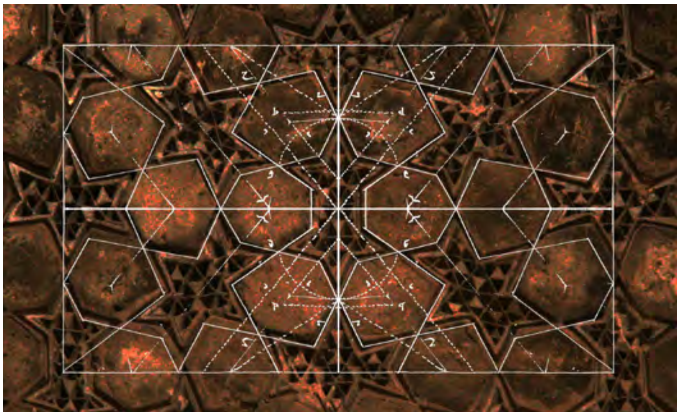

Fig. 10.

V. DIVISIONS OF SQUARES

Many constructions in the manuscript start from a square that is then divided into parts. These constructions will be listed here in subcategories according to the probable purposes for which these divisions were designed. Some of these constructions are of considerable mathematical difficulty and will be deferred to the next section, which describes kite patterns.

a. Fol. 180v [2][52] features a big square divided into 97 pieces, which can be reassembled into 2 decagons, 2 nonagons, 2 octagons, and 2 small squares. Rather than showing how to assemble the two decagons, nonagons, or octagons, the text explains how to assemble a decagonal, nonagonal, or octagonal star. The construction was probably designed in order to show how a square (for example, a colored square tile) could be cut up and reassembled to form other figures.

b. The following two patterns were drawn in a square that could then be repeated to produce a tiling for a larger area.

Fol. 194r [45]:[53] an octagonal star is constructed in the interior of a square, and parts of hexagonal stars are drawn in such a way that the centers of these stars are the vertices of the square. Using copies of this square diagram, the plane can be tiled with hexagonal and octagonal stars, as well as intermediate figures.

Fol. 194v [48]:[54] inside a square the halves of four heptagonal stars and intermediate figures are constructed. The explanation is laconic, and the diagram shows that the construction is inexact, in the sense that one of the angles of every heptagonal star protrudes beyond the square.

c. Divisions of a single square that may have been used as an independent decorative element and also to tile a larger area.

Fol. 188v [30]:[55] division of a large square into a border area and a concentric intermediate square, which is in turn subdivided into four kites, four non-convex hexagons, and a small square in the middle. The division is drawn in such a way that many line segments and distances are equal (see the explanation below).

Fol. 189r [31]:[56] another division of a larger square in a way somewhat similar to fol. 188v [30]. In fol. 189r [31] many of the angles should be, in modern terms, 60° and 120° (see the explanation below).

Fol. 192v [41]:[57] another division of a big square, somewhat similar to fol. 188v [30] and fol. 189r [31], with an intermediate square, four kites, and four quadrilaterals, but without a small square in the middle.

Fol. 196r [55]:[58] decorative division of a square by means of lines that are parallel to the sides and diagonals of the square. The text presents what is probably an approximate construction.

d. Eight divisions of a square are drawn with either no accompanying text or so little that it is not clear what exactly was intended by the diagrams. In many cases, Bulatov has a reconstruction (for references, see the endnotes below).

Fol. 188r [29]:[59] a complicated division of a square. The diagram also reveals some intermediate steps in the construction by means of two dotted lines and two dotted semicircles, but the precise position of many of the points and lines is unclear.

Fol. 192v [39]:[60] a division of a square into 17 pieces, including two kites, as subdivided in figure 12, without text.

Fol. 192v [40]:[61] a division of a square into 16 pieces, including four subdivided kites, as in figure 12, without text.

The following five divisions of a square are found on the same page of the manuscript: fol. 195r [49],[62] fol. 195r [50],[63] fol. 195r [51],[64] fol. 195r [52] (a construction without text),[65] and fol. 195r [53].[66]

Our examples will be fol. 188v [30] and fol. 189r [31], which are illustrated by the same figure in the manuscript (i.e., fol. 188v [30], fig. 11), although the patterns are mathematically different. Figure 11 includes transcriptions of all the letters that are used to label points in the figure in the manuscript. We note that the letters F and O are used as labels for more than one point, in contrast to the procedures of theoretically trained mathematicians, who, if there were more points than could be accommodated by the letters of the alphabet (the *abjad* system), used two-letter combinations beginning with the letter *lām*, such as *lām-alif*, *lām-bā'*, etc., to designate the extra points.

In fol. 188v [30], the large square with the broken lines as sides is given. One has to construct a concentric intermediate square *KIML*, and divide it by combinations of line segments *FCQRJ*, *XhtJ*, etc., in such a way that the intermediate square is divided into four kites, that is to say, a quadrilateral with two pairs of equal sides and two equal angles, which are right angles in this case; a small square in the middle (with side *RJ*) and four non-convex hexagons *RJthK*, etc. The lines *Xht*, *FCQ*, etc., should be perpendicular to the sides *KhL*, *KCI*, etc., of the intermediate square. All of this should be done in such a way that $FC = CQ = QR = RJ = Jt = th = hX$. In other words, the thickness of the "band" between the big and the intermediate square ($CF = Xh$) should be equal to the side of the small square (*JR*) and also equal to the short sides (*QR*, *CQ,* etc.) of the kites.

The text presents the solution as follows: the sides of the intermediate square (*KL*) and of the big square (*OF*) should be "two units plus the square root of 5" and "four units plus the square root of 5," so if one of them is known, the other can be found. It is also specified that line *Qt* (broken line in the figure) should be "the square root of 5." The text assumes (but does not specify) that $FC = CQ = QR = RJ = Jt = th = Xh$ are equal to 1. Then triangle *JTQ* is a right-angled triangle with sides $Jt = 1$ and $QJ = 2$, so the hypotenuse *Qt* is $\sqrt{5}$ by the Pythagorean theorem. The triangle should be drawn in such a way that its hypotenuse *Qt* is parallel to the side *KI* of the intermediate square.

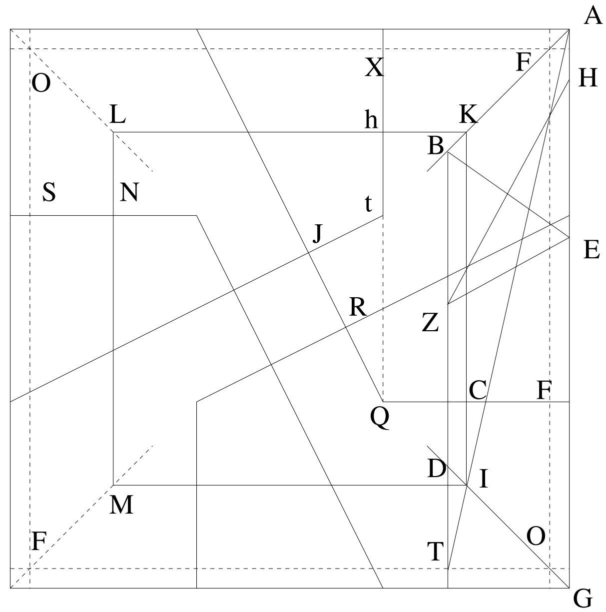

Fig. 11.

The same figure 11 is then used in the manuscript for the pattern textually described in fol. 189r [31]. Through points *A* and *G* we draw bisectors *AB*, *GD*, and a parallel *BD*. Line *AB* is called a "line of four," meaning that one of the angles with the sides of the square is 1/4 times two right angles. Then the text constructs an equilateral triangle *BZE* such that the base *BZ* is part of segment *BD* and point *E* is on *AG*. One then constructs *H* such that *EH* is equal to the side of the equilateral triangle (so *BEH* is another equilateral triangle [fig. 11 is not drawn to scale]). Then the text constructs *T* on *BZ*, extended such that $TZ = HZ$. Line *AT* is joined to intersect *GD* at point *I*, and line *IK* is drawn parallel to *AG* to meet *AB* in *K*. Then *IK* is taken as the side of an intermediate square. Finally, the text constructs in the intermediate square *IKLM* four "kites 6" (Persian: *turunj-6*). Apparently, a "kite 6" is a combination of two congruent *gūnyā-6* triangles, which produce a kite with angles of 60°, 90°, 90°, and 120°. The text does not say how to find these kites: this can be done by dividing the sides of the intermediate square in the ratio $1{:}\sqrt{3}$. Apparently the lines *FO, Qt*, etc., are no longer needed.

The manuscript uses the same figure for two different patterns only in the case of fol. 188v [30] and fol. 189r

[31]. One wonders whether these two constructions and the figure originated from a different source than the rest of the *Anonymous Compendium*.

VI. DIFFICULT KITES AND THEIR PATTERNS

The two mathematically most interesting patterns in the manuscript are related to two special kites (*turunj*). See *ABCD* in figure 12, where $AD = AB$, $CD = CB$, angles *D* and *B* are equal, and $AB \neq BC$; almost always, angles *B* and *D* are right angles.

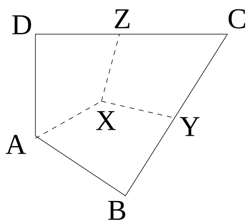

Fig. 12.

In Isfahan, one often sees decorations such as the pattern in the Hakim Mosque (fig. 13), where a big square is divided into four congruent kites and a small square; such an ornament is called a "four kite" (Persian: *chahār-turunj*).[67] The small squares and the four kites are sometimes inscribed with calligraphy. The size of the square in the middle of the *chahār-turunj* depends on the shape of the kite, which in turn is completely determined by its smallest angle.

Fig. 13.

In some of the diagrams in the manuscript, the kites are subdivided into three quadrilateral pieces as in figure 12 by choosing points *Y* and *Z* on the longer sides *BC* and *DC* and point *X* on the diagonal *AC* in such a way that $BY = BA = DA = DZ$ and $AX = XY = XZ$. The division is indicated by dotted lines in figure 12 but will mostly be omitted in the following figures to minimize the clutter. Kites subdivided as in figure 12 occur in fol. 189r [32], fol. 192v [39], fol. 192v [40], and fol. 192v [42], which have been mentioned in the previous sections.

Two special types of kites in the *Anonymous Compendium* are more difficult than the others because they cannot be constructed by ruler and compass in the manner of Euclid's *Elements*. Mathematically, the construction of these two types of kites is equivalent to the solution of a cubic equation, as we will see below. The two special kites occur in patterns that I call the "twelve kite-pattern" and the "variant pattern" (my terminology). The twelve-kite pattern was also studied by the famous mathematicians Ibn al-Haytham (965–ca. 1040) and ʿUmar Khayyam (1048–1131).[68] Before explaining the patterns in detail I will list the relevant constructions in the *Anonymous Compendium*:

a. Twelve-kite pattern

Fol. 188r [28]:[69] three approximate constructions of the twelve-kite pattern, called "circular," "linear," and "another type." It turns out that the three are not equivalent and that the "linear" one is the best; the error in the angle is only a few minutes (if angles are measured in degrees); the others have errors in the order of half a degree.

Fol. 189v [33]:[70] another approximate construction of the twelve-kite pattern, mathematically equivalent to the "linear" construction in fol. 188r [28].

Fols. 191r–191v [36]:[71] a trial-and-error "construction" of the twelve-kite pattern by means of a T-shaped instrument, which may have been the invention of someone called "Kātib."[72] The diagram does not display the complete twelve-kite pattern but only the part corresponding to the intermediate square *EFGH* in figure 14 below, which is not found in the manuscript. The manuscript also tells us that the pattern can only be found by means of conic sections, and that finding the pattern is equivalent to the construction of a right-angled triangle, such that the length of the shortest side plus the length of the perpendicular drawn on the hypotenuse are equal to the length of the hypotenuse itself (see below). According to the manuscript, Ibn al-Haytham wrote a treatise on this triangle in which he constructed it by means of two conic sections, namely, a parabola and hyperbola. His treatise and construction are now lost. The second part of the construction is a description of the T-shaped instrument, which is compared to the alidade of the "boat" astrolabe (*usṭurlāb-i zawraqī*). [73]

b. Variant pattern

The variant pattern occurs in the following five constructions:

Fol. 185r [16]:[74] a construction by trial and error of a certain right-angled triangle, which can be used to construct the variant pattern. The triangle is defined below, at the end of this section. The construction is executed by means of a moving ruler and a compass.

Fol. 185v [18]:[75] an approximate construction of the variant kite pattern. It has an error of approximately a quarter of a degree.

Fol. 187v [26]:[76] an approximate construction of the variant pattern, based on a regular octagon. In the middle of the page is an annotation stating that the ratio between the length and the width of the pattern is 7:6; this is another good approximation. Then the construction based on the octagon is described again in general terms.

Fol. 187v [27]:[77] This seems to be an approximate construction of the variant kite pattern, which is based on the same idea as the approximate construction of fol. 187v [26], but the two constructions differ in some details. The diagram is labeled with letters but there is no corresponding text in the manuscript. The text of fol. 187v [27] begins with another very easy approximation, to the effect that the angles in triangle *ABK* in figure 17 below are in the ratio 4:5:9. The error in this approximation of angles *A* and *B* is approximately half a degree.

Fol. 190r [34]:[78] another construction by trial and error of the variant kite pattern, using a T-shaped ruler.

It is likely that the following construction relates to the variant pattern:

Fol. 185v [17]:[79] a figure with a right-angled triangle in which, according to the text, a kite has to be inscribed. It is stated that "[o]btaining a triangle of this sort is difficult. It falls outside Euclid's *Elements* and concerns the science of conics. The construction is achieved by the motion of a ruler, if the magnitude of the perpendicular is assumed." Bulatov does not translate the text but his interpretation of the figure is plausible. In Bulatov's interpretation (followed by Özdural), the ratio of the hypotenuse of the triangle to its intermediate side is 3:2. It follows that the angles of the triangle differ only a little more than one degree from the corresponding ones in the variant pattern. The placement of this construction on fol. 185v between nos. 16 and 18 suggests a relationship to the variant kite, although the altitude of the right-angled triangle (drawn as a broken line in the manuscript) only occurs in the triangle related to the twelve-kite pattern, which triangle was constructed by Ibn al-Haytham and ʿUmar Khayyam by means of conic sections.

We now turn to the details. The pattern that I call the "twelve-kite pattern" consists of a big square with a small, rotated square in the middle, surrounded by a number of kites. A precise description now follows.

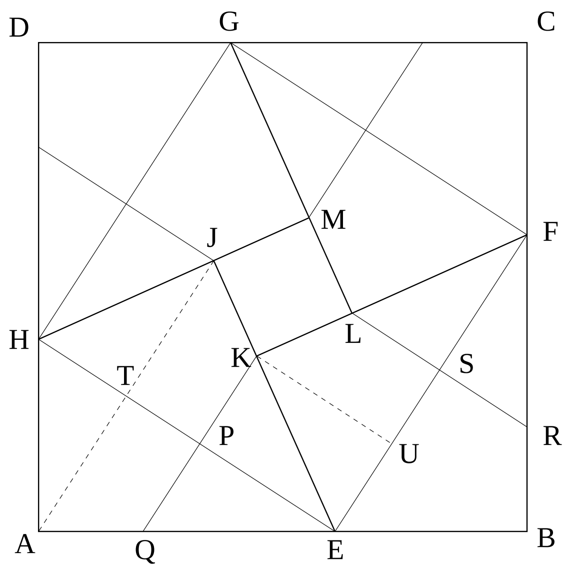

Fig. 14.

In figure 14 we consider a (large) square *ABCD* and choose a point *E* on side *AB* such that *AE* > *EB*. We also choose points *F*, *G*, and *H* on the three other sides of the square such that *AE* = *BF* = *CG* = *DH*, so also *EB* = *FC* = *GD* = *HA*. Then figure *EFGH* will also be an (intermediate) square. Let points *J*, *K*, *L*, and *M* be the mirror images of points *A*, *B*, *C*, and *D* in sides *HE*, *EF*, *FG*, and *GH*. Then figures *AHJE*, *BEKF*, *CFLG*, and *DGMH* are four (big) kites and *JKLM* is a (small) square. Draw perpendicular *KPQ* to *HE* to meet *HE* at *P* and *AE* at *Q*, and draw perpendicular *LSR* to *EF* to meet *EF* at *S* and *BF* at *R*.

In the general case, *JK* and *KP* need not be equal but there is one particular choice of point *E* for which *JK* = *KP*. Since also *HK* = *HK* and the two angles *J* and *P* are right angles, the two triangles *JKH* and *PKH* are congruent; so *JH* = *HP* and *JKPH* is also a (small) kite. Similarly, *AQPH* must also be a kite because it is the mirror image of *JKPH* in side *HE*. In the same way, it can be shown that there are six more (small) kites *KLSE*, *BRSE*, etc.; hence there is a total of twelve kites in the big square. Therefore I call the resulting pattern the "twelve-kite pattern."

How should point *E* be chosen so as to produce this nice arrangement? This question can be resolved if we know the ratio *AE* : *EB*; since *AH* = *EB*, the ratio determines (and is determined by) the shape of the right triangle *HAE*. For the next argument we draw two dotted lines in figure 14: diagonal *ATJ*, which is perpendicular to *HE*, and perpendicular *KU* onto *EF*.

We now concentrate on this triangle *HAE* and note that segment *AT* is the altitude of this triangle. By symmetry, *AT* = *KU* = *PE* and also *AH* = *PH* in the kite *AQPH*; so *AT* + *AH* = *PE* + *PH* = *EH*. Thus triangle *AEH* is a right-angled triangle such that the shortest side (*AH*) plus the altitude (*AT*) is equal to the hypotenuse (*EH*). On fol. 191r–191v [36], we learn that Ibn al-Haytham wrote a treatise in which he constructed this triangle by means of a parabola and a hyperbola. ʿUmar Khayyam knew that constructing the triangle is equivalent to solving a cubic equation.[80]

The manuscript contains the following intriguing approximate construction of the twelve-kite pattern in fol. 189v [33].

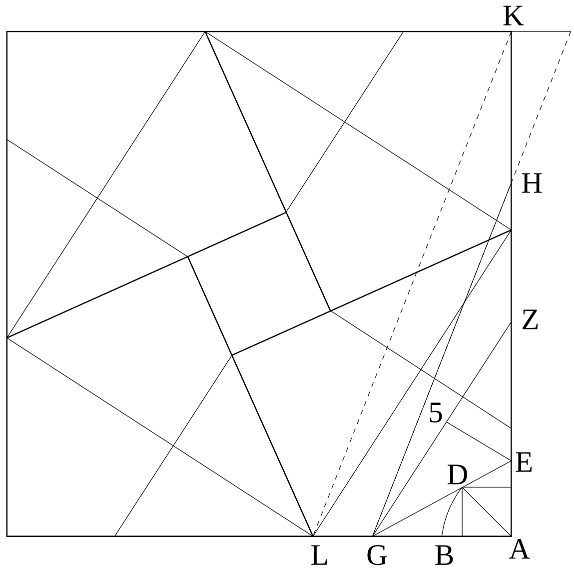

Fig. 15.

On vertex *A* of the big square we construct a little square of arbitrary size, with diagonal *AD*. Find point *G* on one of the sides through point *A* of the big square such that *AG* = 2*AD*; draw *GD* and extend it to meet the other side *AK* at point *E*. Then find point *H* on *AE* extended such that *EH* = 2*AG* and draw *GH*. Through vertex *K* draw *KL* parallel to *GH*. Point L is one of the vertices of the intermediate square. Now the rest of the pattern can easily be found. The construction is approximate but the error is slight; if side *AK* is one meter, the error in the position of L is only a few millimeters.[81]

In order to introduce the "variant pattern," we reconsider a kite from the standard pattern: figure 16 displays a right-angled triangle such that the altitude (not drawn in figure 16) plus the shorter side *BK* is equal to the hypotenuse *AB*. If we add the dotted lines of the "decorative" division of the smaller kite as in figure 12 above, we have *DG* = *GE* = *EK* = *KL*; it is also possible to show that *AE*, *BK*, and *BG* are equal.[82]

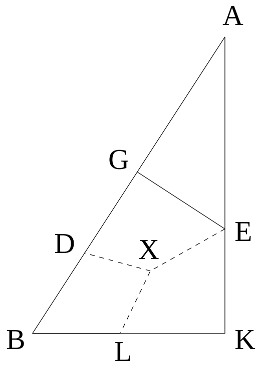

Fig. 16.

By playing with figure 16, the designers may have considered the variant triangle of figure 17. There, we have $BG = BK \neq AE$, but $BD = AG$. In this way, two triangles with congruent kites *BGEK* and *ADPQ* can be placed in a rectangle *BKAQ*; the result is what I call the variant pattern. The construction of this pattern is also equivalent to a cubic equation.[83] Again, this kite cannot be constructed by ruler and compass.

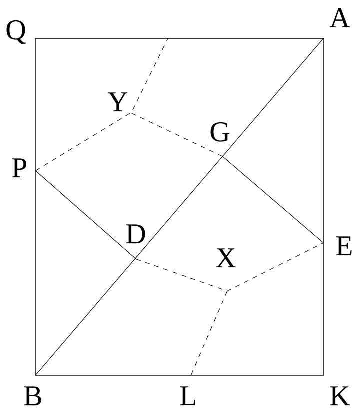

Fig. 17.

The text in fol. 185v [17] suggests that at least some scholars in the medieval Islamic tradition knew that an exact construction of the variant pattern is possible by means of conic sections but not by means of a ruler and a compass. It seems to me that the pattern was discovered as a variant of the twelve-kite pattern, in the way suggested above. Another indication is the fact that just like the twelve-kite pattern, the "variant pattern" was related to the problem of constructing a right-angled triangle, which is mentioned in fol. 185r [16]. In this case, the triangle is less natural and more complicated; the text in fol. 185r [16] is corrupted, but it should have been "a right-angled triangle in which the ratio of the difference between the shortest side and the hypotenuse to the difference between the [former] difference and the shortest side [is the same as the ratio of the intermediate side to the shortest side]." In figure 17 this property corresponds to $AG : GE = AK : KB$; we note that $AG = AB - BG = AB - BK$ and $GE = KL = BK - BL = BK - AG$. Although the "variant pattern" may never have received the same mathematical prestige as the "twelve-kite pattern," the craftsman or craftsmen who were the author(s) of the *Anonymous Compendium* nevertheless came up with a number of intriguing approximate constructions.

VII. *MUQARNAS*

Three diagrams in the manuscript, fol. 184v [15],[84] fol. 194r [46],[85] and fol. 194r [47],[86] are in all probability horizontal projections of decorative stalactite-like construction in Islamic vaults, i.e., a *muqarnas*. These simple diagrams are furnished with numbers but not accompanied by instructions in the text. The numbers refer to the successive altitudes of layers of the *muqarnas*-structure.[87]

CONCLUSION

For a historian of mathematics, the *Anonymous Compendium* is interesting because it is different from the standard literature on mathematics in the medieval Islamic tradition. The constructions of the *Anonymous Compendium* are witness to types of mathematical expertise and training few traces of which have come down to us thus far. The material in the *Anonymous Compendium* is heterogeneous and in rather random order, and may well date back to different authors who may have lived centuries apart. Because the *Anonymous Compendium* is one of a kind it is difficult to draw general conclusions about it. It may well be that more manuscripts of the same type existed in the Middle Ages, but were used up until they were worn out and discarded. Nevertheless, some more manuscripts of this type may well have survived the ravages of time in Iran and neighboring countries and await discovery by scholars. If such manuscripts are found, we will be able to say more with certainty about the contents and context of the mathematical material in the *Anonymous Compendium*.

Department of Mathematics
University of Utrecht

NOTES

Author's note: I wish to thank Karen A. Leal, managing editor of *Muqarnas*, for her very careful editing of this paper.

1. For a discussion of the possible title of the work, "On Similar and Complementary Interlocking Figures," see, in the present volume, the introduction ("In Memory of Alpay Özdural and His Unrealized Book Project") by Gülru Necipoğlu, and chapter 2, "An Early Tradition in Practical Geometry: The Telling Lines of Unique Arabic and Persian Sources" by Elaheh Kheirandish.
2. See Richard 1989, 183–87, where the manuscript is tentatively dated to the sixteenth century; cf. also the older catalogue Blochet 1912, 41–47, who dated the Paris Codex to the seventeenth century. See Alpay Özdural's "Preliminaries" (from his unpublished book manuscript), published as chapter 4 in this volume, for the fifteenth-century date proposed by Priscilla Soucek. For further information on the manuscript codex and the other treatises in it, with references to the catalogues, see the chapter by Necipoğlu ("Ornamental Geometries: A Persian Compendium at the Intersection of the Visual Arts and Mathematical Sciences") and the one by Kheirandish in the present volume, and Brentjes 1994, 55–56.
3. Geometry was considered a prerequisite for studying astronomy, as can be seen from the numerous extant manuscripts of the "Middle Books" (Arabic: *mutawassiṭāt*), which were meant to be studied between the *Elements* of Euclid and the *Almagest* of Ptolemy: see Steinschneider 1865, 456–98; see also Kheirandish 2006, 135–54.
4. Complete references to editions and translations of Abu'l-Wafa' al-Buzjani's book are found in Kheirandish's chapter in this volume. See also Necipoğlu's chapter for a discussion of Abu'l-Wafa''s *Geometrical Constructions* in relation to the *Anonymous Compendium*'s contents.
5. These discussions were published in Özdural 2000, 171–201.
6. On this issue, see Cromwell 2015: http://link.springer.com/article/10.1007/s00283-015-9538-9.
7. See Bulatov 1988, which contains an abbreviated Russian translation of the *Anonymous Compendium* by A. B. Vildanova; and Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, appendix 1, pp. 73–95. On some of the problems with the latter work, see Kheirandish's chapter in the present volume.
8. Bulatov 1988, 318, fig. 8; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 74, fig. between no. 2 and no. 3.
9. Bulatov 1988, 319, fig. 13; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 76, no. 10.
10. Bulatov 1988, 323, fig. 25; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 80, no. 24.
11. Bulatov 1988, 339, fig. 60, where the isosceles triangle is omitted; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 92, no. 63.
12. Bulatov 1988, 339, fig. 58; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 93, no. 64.
13. Bulatov 1988, 339, fig. 59; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 92, no. 62.
14. Bulatov 1988, 316, fig. 1; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 73, no. 1. The construction is mentioned in Chorbachi 1989b, 751–89, at 774. See also Özdural 2000, 171–201, esp. 187–90.
15. Bulatov 1988, 317, fig. 7; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 74, no. 3.
16. Bulatov 1988, 319, fig. 11ab; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 75, no. 8.
17. Bulatov 1988, 319, fig. 12; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 76, no. 9. See also Özdural 2000, 185–86.
18. See Heath 1956, vol. 1, p. 205; and Özdural 2000, 185–86.
19. Bulatov 1988, 320, fig. 14; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 76–77, nos. 11–12.
20. Bulatov 1988, 318, fig. 9ab; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 74, nos. 4–5.
21. Heath 1956, vol. 2, 54–56.
22. Bulatov 1988, 318, fig. 9cd; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 75, no. 6.
23. Bulatov 1988, 319, fig. 10; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 75, no. 7.
24. Bulatov 1988, 320, fig. 15; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 77, no. 13.
25. Bulatov 1988, 321, fig. 16; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 77–78, nos. 14–17. See Necipoğlu's chapter in the present volume for the relationship to a work by Ibn al-Haytham on the regular heptagon.
26. This construction, which consists of a figure only, is omitted in Bulatov 1988, and also in Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī.
27. Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 87, no. 44. The construction is omitted in Bulatov 1988.
28. See Chorbachi 1989a; see also Bulatov 1988, 322–23, and Özdural's drawing in pl. 14.1 in the present volume.
29. Bulatov 1988, 321, fig. 17; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 78, no. 18.
30. Bulatov 1988, 321, fig. 21; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 79, no. 21.
31. Bulatov 1988, 323, fig. 24; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 80, nos. 22–23.
32. Bulatov 1988, 322, fig. 22; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 81, at the beginning of no. 25.

33. Bulatov 1988, 323, fig. 23; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 81, at the end of no. 25.
34. Bulatov 1988, 324, fig. 26; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 81, no. 26. On Abū Bakr [b.] al-Khalīl al-Tājir al-Raṣadī, see the extensive discussions in Necipoğlu's chapter as well as in Kheirandish's chapter.
35. The letter *B* does not occur in the figure but has been supplied following Bulatov 1988, 321, fig. 17; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 78, no. 18; and Özdural, on the basis of the information in the text.
36. Suppose that the radius of the circle is 1, and drop a perpendicular *ZP* onto *AG*. Then we have $ZA = 1$; $\angle ZAP = 30^o$; $ZP = 1/2$; $AP = (\sqrt{3})/2$; $GP = 2 - (\sqrt{3})/2$; $\angle ZGP = \arctan(ZP/GP) \approx 23.8^o$. Because $\angle DGP = 60^o$, $\angle ZAH = \angle ZGD \approx 36.2^o$.
37. Suter 1922, 102–4, and Woepcke 1855, 328.
38. Heath 1956, vol. 1, p. 176.
39. Bulatov 1988, 327, fig. 35; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 85, nos. 38–39.
40. Bulatov 1988, 329, figs. 38–39; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 86–87, nos. 41–43.
41. Bulatov 1988, 330, fig. 41; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 88, no. 47.
42. Bulatov 1988, 332, figs. 42–44; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 88, no. 48.
43. Bulatov 1988, 332, fig. 45; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 88-89, nos. 49–51.
44. Bulatov 1988, 337, fig. 53; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 91, part of no. 59.
45. Bulatov 1988, 338, figs. 56–57; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 92, no. 61.
46. Bulatov 1988, 327, fig. 32; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 84, no. 34.
47. Heath 1956, vol. 1, 402–3.
48. Bulatov 1988, 338, fig. 54; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 91, part of no. 59.
49. The figure is in Bulatov 1988, 339, fig. 61, and in Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 95, and also on the front page.
50. See Hogendijk 2008, 121, for the discovery of this pattern on the North Dome. For a further discussion of this pattern and the related *girih* tiles, see also Necipoğlu's chapter in the present volume.
51. Bulatov 1988, 329, 332–33.
52. Bulatov 1988, 316–17, figs. 2–6; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 73, no. 2. See also Özdural 2000, 190–92.
53. Bulatov 1988, 333–34, figs. 46 and 47; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 89, no. 52.
54. Bulatov 1988, 335, figs. 50 and 51; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 90, nos. 53 and 54.
55. Bulatov 1988, 325, fig. 30; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 83, between no. 32 and no. 33.
56. Bulatov 1988, 326, fig. 31; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 83, no. 33.
57. Bulatov 1988, 330, fig. 40; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 87, no. 46. See also Chorbachi 1989b, 777–85, and Kappraff 1991, 206–7.
58. Bulatov 1988, 338, fig. 55; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 91, no. 60.
59. Bulatov 1988, 325, fig. 29; omitted in Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī.
60. Both Bulatov 1988 and Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, omit this construction, but it is mentioned in Chorbachi 1989b, 777.
61. Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 87, no. 45; see also Chorbachi 1989b, 777. The construction is omitted in Bulatov 1988.
62. Bulatov 1988, 336 fig. 52, upper left; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 90, no. 55.
63. Bulatov 1988, 336, fig. 52, upper right; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 90, no. 56.
64. Bulatov 1988, 336, fig. 52, lower left; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 90, no. 57.
65. Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 91, part of no. 58. The construction is omitted in Bulatov 1988.
66. Bulatov 1988, 336, fig. 52, lower right; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 91, part of no. 58.
67. See Necipoğlu's chapter in the present volume, and also Cromwell and Beltrami 2011, 84–93.
68. See Özdural 1996, 191–211, esp. 197, and Özdural 1995, 54–71, esp. 56–64, where attention is drawn to an extant treatise on algebra by the mathematician and poet ʿUmar Khayyam. In this treatise, a fundamental triangle in the twelve-kite pattern is constructed by means of conic sections, but Khayyam does not inform the reader that the triangle is related to decorative patterns.
69. Bulatov 1988, 325, fig. 28; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 82–83, nos. 29–32. See Özdural 1995, 67, fig. 12.
70. Bulatov 1988, 327, fig. 33; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 84, no. 35. See Özdural 1995, 66, fig. 11.
71. Bulatov 1988, 328, figs. 36 and 37; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 85–86, no. 40. On this construction, see Özdural 1996, 197, and Özdural 1995, 64–65, but note that fol. 191r–191v [36] is not a "verging construction" in the sense of ancient Greek geometry. For a further discussion of the relationship with Ibn al-Haytham and the T-shaped instrument, see Necipoğlu's chapter in the present volume.
72. I interpret the Persian *istinbāṭ* as "discovery," and not as "deduction," as in the English translation in this volume. In my opinion, the discovery in question is the mathematical insight that many constructions by conic sections can be realized by trial-and-error constructions by the T-shaped instrument. Therefore, I find it implausible to assume that the following word "*kātib*" refers to the scribe of the present manuscript, who was not trained in mathematics. In Özdural 1996, 196, Özdural (who based his interpretation on that of his Persian translator Zaka Siddiqi) interprets *istinbāṭ* as "inference," and concludes that "*kātib*" is "more likely the compiler himself, who could hardly be a mathematician, than the copyist or the translator." In her chapter, Necipoglu interprets "*kātib*" to mean "scribe," and thus has

the word been rendered in the English translation in this volume.

73. On the second part of fol. 191r–191v [36], see Özdural 1996, 196; on the "boat astrolabe" (*usṭurlāb-i zawraqī*), see Frank 1920, 18–21.

74. Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 78, no. 19. Bulatov 1988, 323, states that the "text of this construction is not understood," and the corresponding figure in Bulatov 1988, 321, fig. 18, is incorrect.

75. Bulatov 1988, 321 fig. 20; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 79, second part of no. 20. On this construction, see Özdural 1996, 201, fig. 6

76. Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 81, no. 27; omitted in Bulatov 1988. On this construction, see Özdural 1996, 200, fig. 4.

77. Bulatov 1988, 324, fig. 27; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 82, no. 28. See Özdural 1996, 200, fig. 5.

78. Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 84, no. 36-37. The construction was misinterpreted in Bulatov 1988, 329–30, and the corresponding figure in Bulatov 1988, 327, fig. 34, is incorrect. On this construction, see Özdural 1996, 198–99.

79. Bulatov 1988, 321, fig. 19; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 79, beginning of no. 20.

80. See Özdural 1996, 197; Özdural 1995, 56–64; and Hogendijk 2012, 41, for more information and references. Here is an easy method to find a cubic equation in a simple form. In figure 14 we no longer assume that the big square is known, but we choose HA as our point of departure. Put $HA = BE = 1$, $AE = BF = x$, then $HE = EF = \sqrt{1+x^2}$ by the Pythagorean theorem. By similar triangles $HA : AT = HE : EA$, so $1 : AT = \sqrt{1+x^2} : x$ whence $AT = x/\sqrt{1+x^2}$. Because $AH + AT = EH$ we have $1 + x/\sqrt{1+x^2} = \sqrt{1+x^2}$, which boils down to $x^3 - 2x^2 + 2x - 2 = 0$ with positive real root $x = 1.54369$ .... Hence, the larger acute angle in the right-angled triangle is $\angle AHE = \angle BEF = \arctan(x) = 57.0649$ ... degrees. Özdural 1995, 64, argues that the scribe of the manuscript was confused, and that his reference to Ibn al-Haytham was incorrect and should have been a reference to ʿUmar Khayyam. I disagree with Özdural's argument on this point. Ibn al-Haytham was one of the foremost experts in conic sections in the Islamic tradition, much more so than ʿUmar Khayyam; the fact that a treatise on the relevant triangle did not occur in any of Ibn al-Haytham's lists of his own works is unimportant because other extant works by Ibn al-Haytham do not occur in the same lists either. See also Özdural 1996, 205n45.

81. If in figure 15 we make the side of the little square equal to 1, we have $AG = 2\sqrt{2}$, and by similar triangles $AE/AG = 1/(2\sqrt{2} - 1)$. Since $EZ = AG$, it follows that $AZ/AG = 1 + 1/(2\sqrt{2}-1) = 2\sqrt{2}/(2\sqrt{2}-1) = (8 + 2\sqrt{2})/7$; therefore $\angle ZGA = \arctan(8 + 2\sqrt{2})/7 = 57.1195...^\circ$, which is only 0.056 degrees more than the exact value.

82. In figure 14, by symmetry $EQ = EK$, and since the triangles EKF and AHE are congruent, we have $EK = AH$, and so $EQ = AH$ in figure 14; thus $BK = AE$ in fig. 16.

83. If we put in fig. 17 $BK = 1$ and $AK = x$, and also $DG = GE = EK = KL = y$, we have $BD = AG = 1 - y$. By similar triangles $AG : GE = AK : KB$ or $(1 - y) : y = x : 1$ or $y = 1/(1 + x)$ and $AB = 2 - y = (2x + 1)/(x + 1)$. Then, by the Pythagorean theorem $(2x+1)^2/(x+1)^2 = 1+x^2$, which equation boils down to $x^3+2x^2-2x-2 = 0$. This equation has a positive real root $x = 1.170086$ .... which produces $\angle ABK = \arctan(x) = 49.481 ...^\circ$ and $\angle BAK = 40.518 ...^\circ$. In fol. 185v [17], one of the angles of the right-angled triangle is $\arcsin(2/3) = 41.81 ...^\circ$, which may be interpreted as an approximation to $\angle BAK$ in fig. 14.

84. This figure is omitted in Bulatov 1988, and also in Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī.

85. Bulatov 1988, 334, fig. 48; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 89, figure between nos. 52 and 53.

86. See Bulatov 1988, 334, fig. 49; Abū'l-Wafā' al-Būzjānī 1990–91 (1369), ed. Jaẕbī, 89, figure between nos. 52 and 53.

87. These three patterns were first identified as *muqarnas* projections by Özdural: see his chapter ("Preliminaries") in this volume, and his plates reconstructing the *muqarnas* tiers (pls. 15, 46, and 47). On *muqarnas*, see Necipoğlu 1995, 4–27, 44–50, 159–60, 179, 350, 353, 359, and 173, as well as al-Asad's appendix on "The Muqarnas: A Geometric Analysis." I thank Professor Necipoğlu for providing me with these detailed references. See also Dold-Samplonius 1992, 193–242.

BIBLIOGRAPHY

Abū'l-Wafā' al-Būzjānī. 1990–91 (1369). *Handasa-i Īrānī: Kitāb-i Tijārat: Fī mā yaḥtāju ilayhi al-ʿummāl wa-al-ṣunnā' min al-ashkāl al-handasiyya, yā, Kārburd-i handasa dar ʿamal.* Edited and adapted (to modern Persian) by Sayyid ʿAlī Riżā Jaẕbī. Tehran: Surūsh.

Al-Asad, Mohammad. 1995. "The Muqarnas: A Geometric Analysis." In *The Topkapı Scroll: Geometry and Ornament in Islamic Architecture; Topkapı Palace Museum Library MS H. 1956* by Gülru Necipoğlu, 349–59. Santa Monica, Calif.: Getty Center for the History of Art and the Humanities.

Blochet, Edgar. 1905–34. *Catalogue des manuscripts persans.* 4 vols. Paris: Imprimerie nationale.

Brentjes, Sonja. 1994. "Textzeugen und Hypothesen zum arabischen Euklid in der Überlieferung von al-Ḥaǧǧāǧ b. Yūsuf b. Maṭar (zwischen 786 und 833)." *Archive for History of Exact Sciences* 47, no. 1: 53–92.

Bulatov, Mitkhat S. 1988. *Geometricheskaia garmonizatsiia v arkhitekture Srednei Azii IX–XV vv*. Rev. ed. Moscow: "Nauka." Original edition Moscow, 1978.

Chorbachi, Wasma'a K. 1989a. "Beyond the Symmetries of Islamic Geometric Patterns: The Science of Practical Geometry and the Process of Islamic Design." PhD diss., Harvard University.

———. 1989b. "In the Tower of Babel: Beyond Symmetry in Islamic Design." *Computers and Mathematics, with Applications* 17:751–89.

Cromwell, Peter R. 2015. "Cognitive Bias and Claims of Quasiperiodicity in Traditional Islamic Patterns." *The Mathematical Intelligencer* (June 20, 2015). doi: 10.1007/s00283-015-9538-9.

Cromwell, Peter R., and Elisabetta Beltrami. 2011. "The Whirling Kites of Isfahan: Geometric Variations on a Theme." *The Mathematical Intelligencer* 33, no. 3: 84–93.

Dold-Samplonius, Yvonne. 1992. "Practical Arabic Mathematics: Measuring the Muqarnas by al-Kāshī." *Centaurus* 35:193–242.

Frank, Josef. 1920. *Zur Geschichte des Astrolabs*. Erlangen: Universitäts-Buchdruckerei von Junge & Sohn.

Heath, Thomas L., ed. and trans. 1956. *The Thirteen Books of Euclid's Elements*. Translated from the text of Heiberg. 2nd ed. Cambridge: Cambridge University Press, 1956.

Hogendijk, Jan P. 2012. "Mathematics and Geometric Ornamentation in the Medieval Islamic World." *Newsletter of the European Mathematical Society* 86:37–43.

———. 2008. "Ancient and Modern Secrets of Isfahan." *Nieuw Archief voor Wiskunde*, 5th ser., 9, no. 2: 121.

Kappraff, Jay. 1991. *Connections: The Geometric Bridge between Art and Science.* New York: McGraw-Hill Pub. Co.

Kheirandish, Elaheh. 2006. "Organizing Scientific Knowledge: The 'Mixed' Sciences in Early Classifications." In *Organizing Knowledge: Encyclopaedic Activities in the Pre-Eighteenth Century Islamic World*, edited by Gerhard Endress, 135–54. Leiden: Brill.

Necipoğlu, Gülru. 1995. *The Topkapı Scroll: Geometry and Ornament in Islamic Architecture; Topkapı Palace Museum Library MS H. 1956.* Santa Monica, Calif.: Getty Center for the History of Art and the Humanities.

Özdural, Alpay. 2000. "Mathematics and Arts: Connections between Theory and Practice in the Medieval Islamic World." *Historia Mathematica* 27, no. 2: 171–201.

———. 1996. "On Interlocking Similar or Corresponding Figures and Ornamental Patterns of Cubic Equations." *Muqarnas* 13:191–211.

———. 1995. "Omar Khayyam, Mathematicians and *Conversazioni* with Artisans." *Journal of the Society of Architectural Historians* 54, no. 1: 54–71.

Richard, Francis. 1989. *Catalogue des manuscrits persans*. Vol. 1, *Ancien fonds*. Paris: Bibliothèque nationale de France.

Steinschneider, Moritz. 1865. "Die 'mittleren' Bücher der Araber und ihre Bearbeiter." *Zeitschrift für Mathematik und Physik* 10:456–98.

Suter, Heinrich. 1922. "Das Buch der geometrischen Konstruktionen des Abu'l Wefā'." *Abhandlungen zur Geschichte der Naturwissenschaften und Medizin* 4:94–109.

Woepcke, Franz. 1855. "Recherches sur l'histoire des sciences mathématiques chez les Orientaux d'après des traités inédits arabes et persans. Deuxième article. Analyse et extrait d'un recueil de constructions géométriques par Aboūl Wefā." *Journal Asiatique*, ser. 5, no. 5: 218–56, 309–59.