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Subset representations and eigenvalues of the universal intertwining matrix

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ABSTRACT

We solve a combinatorial question concerning eigenvalues of the universal intertwining endomorphism of a subset representation.

1. Introduction

The symmetric group S_n acts on the set $\mathcal{C}_k^n$ of subsets of k elements of a set of n elements. This defines a representation, known as a subset representation. Subset representations are related to Young tableaux with 2 rows. We consider the universal intertwining matrix $B_{(n-k,k)}$ for a subset representation and use Schur's Lemma and Young's rule to show that the eigenvalues are $\mathbb{Z}$-linear in the natural parameters of the intertwining matrix (Proposition 1). Next we compute the eigenvalues (Theorem 1).

In the terminology that is customary in algebraic combinatorics, what we are doing is recomputing the "eigenmatrix P of a Johnson scheme". This eigenmatrix was determined much earlier by Delsarte [1], but his answer is different and does not help us with determining the signature in the last section. (Note that

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Fig. 1. $B_{3,3}$ and $B_{4,4}$; coloured by b_p.

[1] is a publication of Philips Research. Around this time Philips was developing the compact disc. Error correcting codes are crucial for compact discs.)

In the last section Theorem 1 is applied to justify the evaluation in [6] of the Eisenbud-Levine-Khimshiashvili (ELK) signature formula for the gradient index at a degenerate star. For this we also need the package MultiSum [7], in order to perform a summation of complicated hypergeometric terms.

2. Subset representations, intertwiners, Young's rule

2.1. The question

Let n be a positive integer and $0 \leq k \leq \lfloor n/2 \rfloor$. Consider combinations $\mathcal{C}_k^n$ of k elements (unordered) out of a set of n elements. Take an arbitrary tuple of complex numbers $b_0, \cdots, b_k$. We constitute a matrix $B = B_{(n-k,k)}$, where the rows and columns are indexed (in lexicographic order) by elements of $\mathcal{C}_k^n$. The matrix elements of B are defined as follows: If $\langle \sigma, \tau \rangle$ denotes the entry with row index σ and column index τ, then

$$\langle \sigma, \tau \rangle = b_p \text{ if } \sigma \cap \tau \text{ has } p \text{ elements } (0 \leq p \leq k).$$

We want to compute the eigenvectors and eigenvalues of B. (Fig. 1.) The result is needed in [6, §3] for the computation of a 'gradient index'.

2.2. Intertwining

The symmetric group S_n acts on $\mathcal{C}_k^n$ and thus defines a permutation representation known as $M^{(n-k,k)}$ (see for instance [3, p. 86]). In Prasad [4, §2.5] the representation is called a subset representation $\mathbb{C}[X_k]$, where X_k is our $\mathcal{C}_k^n$.

A linear map $N : M^{(n-k,k)} \to M^{(n-k,k)}$ is called *intertwining* if $gN = Ng$ for all $g \in S_n$. The next result shows that B is the 'universal intertwining matrix'.

Lemma 1. *Let $N \in \mathrm{Hom}_{\mathbb{C}}(M^{(n-k,k)}, M^{(n-k,k)})$. Use the standard basis of $M^{(n-k,k)}$ to view N as a matrix. Then N is intertwining if and only if the matrix elements N_{σ_1,τ_1} and N_{σ_2,τ_2} are equal as soon as $\sigma_1 \cap \tau_1$ and $\sigma_1 \cap \tau_1$ have the same cardinality.*

Proof. This is easy and essentially Theorem 2.51 in Prasad [4]. $\square$

2.3. Specht modules

With n, k as above, let ν be the two-part partition $(n-k,k)$ of n. We define T_k to be the maximal standard tableau [3, pp. 84–85] of shape ν. That is, one puts 1 through $n-k$ in the first row, in that order, and similarly $n-k+1$ through n in the second row. One could call its numbering lexicographic. See Fig. 2 for examples.

$$\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 6\\\cline{1-1}\end{array}\;,\qquad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\;,\qquad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}\;.$$

Fig. 2. T_1, T_2, T_3 if $n = 6$.

Recall that the *row subgroup* $R(T_k)$ of T_k is the subgroup of S_n which consists of those permutations that permute the entries of each row among themselves. Similarly the *column subgroup* $C(T_k)$ of T_k is the subgroup of S_n which consists of those permutations that permute the entries of each column among themselves. One now puts

$$a_k = \sum_{p\in R(T_k)} p\ , \qquad b_k = \sum_{q\in C(T_k)} \mathrm{sgn}(q)q\ , \qquad c_k = b_k a_k\ ,$$

where the product and sums are taken in the group ring $\mathbb{C}[S_n]$. The c_k are known as *Young symmetrizers*. The *Specht module* S^ν may now be defined as the image of the endomorphism of $\mathbb{C}[S_n]$ that is right multiplication by c_k [3, p. 119]. The Specht module S^ν is an irreducible S_n module of dimension

$$f^\nu = \frac{n!(n-2k+1)}{k!(n-k+1)!} = \tbinom{n}{k} - \tbinom{n}{k-1}$$

[4, Exercise 2.5.4], [3, p. 88]. The S^ν for distinct ν's are non-isomorphic.

Proposition 1 *(Young's rule).* [3, p. 92], [4, Thm. 3.3.1, Exercise 3.3.5]. *Let* $0 \le m \le \lfloor n/2 \rfloor$. *Then*

$$M^{(n-m,m)} \cong \bigoplus_{k=0}^{m} S^{(n-k,k)}. \quad \square$$

3. Schur's lemma and eigenvalues

3.1. Diagonalizing

Let $0 \le m \le \lfloor n/2 \rfloor$. Choose a basis $d = d_1, \cdots, d_{\binom{n}{m}}$ of $M^{(n-m,m)}$ which is the union of bases of the $m+1$ irreducible submodules. Then the basis d diagonalizes all intertwining maps $M^{(n-m,m)} \to M^{(n-m,m)}$ simultaneously, by Schur's Lemma. In particular, $B_{(n-m,m)}$ transforms to a diagonal matrix D with linear combinations of the b_i on the diagonal. As recalled after Lemma 5 below, the d_i may be chosen in the $\mathbb{Q}$-span of the standard basis. Then $B_{(n-m,m)}$ actually transforms to a diagonal matrix D with $\mathbb{Q}$-linear combinations of the b_i on the diagonal. If one specializes $b_i = 1$ and puts the other b_j equal to zero, then the eigenvalues become algebraic integers because they are roots of the characteristic polynomial of a matrix with integer entries. We have proved:

Proposition 2. *The matrix* $B_{(n-m,m)}$ *has the properties:*

- *The eigenspaces are independent of a (generic) choice of* $b_0, \cdots b_m$,
- *The eigenvalues are* $\mathbb{Z}$*-linear combinations of* $b_0, \cdots b_m$. $\square$

3.2. Mapping Specht modules to a subset representation

Let Ω be the last element of $\mathcal{C}^n_m$, that is, $\Omega = [\![n-m+1, n]\!]$, the set of integers in $[n-m+1, n]$. We define an S_n-linear map $\pi : \mathbb{C}[S_n] \to M^{(n-m,m)}$ by

$$\pi(p) = p(\Omega).$$

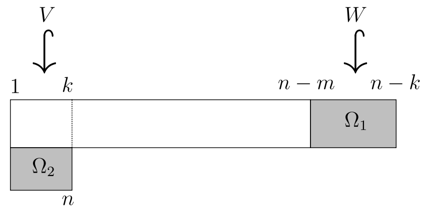

Fig. 3. Positions of V and W in Young diagram of shape $(n - k, k)$.

Our strategy is now as follows. We know already the eigenspaces of $B = B_{(n-m,m)}$ and want to compute eigenvalues. Below we take the eigenvector $\pi(c_k)$ and compare it with its image $B\pi(c_k)$. It is sufficient to consider only one of the coordinates, in fact the Ω coordinate will do. We will also refer to the Ω coordinate as the last coordinate.

Note that c_k is a double sum of signed products $\pm qp$. We first look at the effect of π on each term.

Let $0 \leq k \leq m$. We will focus on the sets $\Omega \cap \pi(qp)$, where $p \in R(T_k)$, $q \in C(T_k)$. Notice that p, q permute the elements of $[\![1, n]\!]$, not the boxes in a tableau. Nevertheless a diagram-figure makes it easier to follow the actions of p and q. See Fig. 3. We write $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = [\![n - m + 1, n - k]\!]$ and $\Omega_2 = [\![n - k + 1, n]\!]$.

We put

$$V = \{\ i \mid q(i) \neq i \leq k\ \} \quad \text{and} \tag{1}$$

$$W = \{\ i \in \Omega \mid p(i) \notin \Omega \cup V\ \} \subseteq \Omega_1. \tag{2}$$

Notice that $\operatorname{sgn}(q) = (-1)^{\#V}$, where $\#X$ denotes the cardinality of a set X.

Lemma 2. *With these q, p, V, W, the cardinality of $\Omega \cap \pi(qp)$ equals $m - \#V - \#W$.* $\square$

Lemma 3. *Given $V \subseteq [\![1, k]\!]$, there is a unique $q \in C(T_k)$ such that equation (1) holds.* $\square$

Lemma 4. *Given $V \subseteq [\![1, k]\!]$, $W \subseteq \Omega_1$, there are*

$$\binom{n - m - \#V}{\#W}(\#W)!\binom{m - k + \#V}{m - k - \#W}(m - k - \#W)!k!(n - m)!$$

elements p in $R(T_k)$ such that equation (2) holds.

Proof. There are $\binom{n-m-\#V}{\#W}(\#W)!$ possibilities for the restriction of p to W. There are $\binom{m-k+\#V}{m-k-\#W}(m - k - \#W)!$ possibilities for the restriction of p to $\{\ i \in \Omega_1 \backslash W\ \}$. Given both restrictions of p there are still $k!$ possibilities for the restriction to Ω_2 and then $(n - m)!$ possibilities for the restriction to the remainder. $\square$

Lemma 5. *The last coordinate of $\pi(c_k)$ is $(m - k)!k!(n - m)!$.*

Proof. We must take V and W empty. $\square$

In particular, the last coordinate of $\pi(c_k)$ is nonzero. This means that π maps $S^{(n-k,k)}$ isomorphically into $M^{(n-m,m)}$. By Schur's Lemma $\pi(c_k)$ is an eigenvector of our universal intertwining matrix $B_{(n-m,m)}$. Notice that the S_n orbits of the $\pi(c_k)$, $k = 0, \cdots, m$, together span all of $M^{(n-m,m)}$, because $M^{(n-m,m)}$ is the sum of the images of the $S^{(n-k,k)}$ (Proposition 1). So we may assume our diagonalizing basis d is contained in the union of these orbits. Then the d_i are $\mathbb{Q}$-linear combinations of the standard basis.

Lemma 6. *The last coordinate of $B_{(n-m,m)}\pi(c_k)$ is*

$$\sum_{v=0}^{k}\sum_{w=0}^{m-k}(-1)^v\binom{k}{v}\binom{m-k}{w}\binom{n-m-v}{w}w!\binom{m-k+v}{m-k-w}(m-k-w)!k!(n-m)!b_{m-v-w}.$$

Proof. Multiply the last row of the matrix $B_{(n-m,m)}$ by $\pi(c_k)$. The result is the sum over all choices of $V\subseteq [\![1,k]\!]$, $W\subseteq \Omega_1$, where $v=\#V$ and $w=\#W$. $\square$

Theorem 1. *The eigenvalue associated with the eigenvector $\pi(c_k)$ of $B_{n-m,m}$ is*

$$\lambda_k=\sum_{j=0}^{k}\sum_{p=0}^{m-k}(-1)^{k-j}\binom{k}{j}\binom{m-j}{p}\binom{n-m-k+j}{m-k-p}b_{j+p}$$

and its multiplicity is $f^{(n-k,k)}=\frac{n!(n-2k+1)}{k!(n-k+1)!}$.

Proof. Divide the last coordinate of $B_{(n-m,m)}\pi(c_k)$ by the last coordinate of $\pi(c_k)$. Then rewrite, using the substitutions $v\mapsto k-j$, $w\mapsto m-k-p$. $\square$

Example 1. $B_{3,3}$ has eigenvalues:

- $b_0+9b_1+9b_2+b_3$ with multiplicity 1,
- $-b_0-3b_1+3b_2+b_3$ with multiplicity 5,
- $b_0-b_1-b_2+b_3$ with multiplicity 9,
- $-b_0+3b_1-3b_2+b_3$ with multiplicity 5.

3.3. Eberlein polynomials

The Eberlein polynomial E_k is defined [1, (4.33)] as

$$E_k(u)=\sum_{j=0}^{k}(-1)^{k-j}\binom{m-j}{k-j}\binom{m-u}{j}\binom{n-m+j-u}{j},$$

where $0\le k\le m\le \lfloor n/2\rfloor$ as above. It is of degree k in the variable $u(n+1-u)$.

There are several more descriptions of E_k in [2]. By comparing Theorem 1 with [1, Thm 4.6] we now get:

Corollary 1. *With k,m,n as above, one has*

$$E_t(k)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{m-j}{m-t-j}\binom{n-m-k+j}{n-m-t}$$

for $0\le t\le m$.

Proof. By [1, Thm 4.6] one may view $E_t(k)$ as the coefficient of b_{m-t} in λ_k of our Theorem 1. $\square$

3.4. The Eisenbud-Levine-Khimshiashvili index computation

We now turn to the problem that motivated the present work.

Recall that $(-1)!!=0!!=1$ and $n!!=n((n-2)!!)$ for $n\ge 1$.

Proposition 3. ([6, Prop. 4]) *Substitute*

$$b_i = (-1)^i \frac{(2m-2i-1)!!(2i)!!}{(2m-1)!!}$$

into $B_{m,m}$. *The eigenvalues are*

$$\lambda_k = (-1)^m \frac{2m+1}{2m-2k+1}$$

for $0 \le k \le m$, *with multiplicity* $f^{(2m-k,k)} = \frac{(2m)!(2m-2k+1)}{k!(2m-k+1)!}$.

Proof. Plugging these values of b_i into the formula in Theorem 1, one ends up with a multisum with a complicated hypergeometric summand. We need to evaluate this multisum. Numerical experiments suggested the answer. We now use the computer algebra package Multisum [7] that aims to give hints for proving a guessed answer. The appendix to [6] gives further details. Or see the Mathematica notebook that we attach to the arXiv-version of this paper. The book [5] describes some computer algebra in the background. □

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