

Dipole Deformations of $\mathcal{N} = 1$ SYM and Supergravity backgrounds with $U(1) \times U(1)$ global symmetry

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ABSTRACT

We study $SL(3, R)$ deformations of a type IIB background based on $D5$ branes that is conjectured to be dual to $\mathcal{N} = 1$ SYM. We argue that this deformation of the geometry correspond to turning on a dipole deformation in the field theory on the $D5$ branes. We give evidence that this deformation only affects the KK-sector of the dual field theory and helps decoupling the KK dynamics from the pure gauge dynamics. Similar deformations of the geometry that is dual to $\mathcal{N} = 2$ SYM are studied. Finally, we also study a deformation that leaves us with a possible candidate for a dual to $N = 0$ YM theory.

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1 Introduction

The AdS/CFT conjecture [1] [2] [3] is one of the most powerful analytic tools for studying strong coupling effects in gauge theories. There are many examples that go beyond the initially conjectured duality. First steps in generalizing the original duality to non-conformal examples were taken in [4]. Later, very interesting developments led to the construction of the gauge-string duality in phenomenologically more relevant theories *i.e.* minimally or non-supersymmetric gauge theories [5]. New geometries that realize various different aspects of gauge theories allowed us to deepen our knowledge on the duality.

Conceptually, the most clear set up for less symmetric theories is obtained by breaking the conformality and partial supersymmetry by deforming $\mathcal{N} = 4$ SYM with relevant operators or VEV's. The models put forward by Polchinski and Strassler [6] and Klebanov and Strassler [7] belong to this class. Many authors have contributed to the understanding of this class of set-ups. The review [8] provides a nice summary of the procedure to compute correlation functions and various observables.

On the other hand, a different set of models, that are less conventional regarding the UV completion of the field theory have been developed. The idea here is to start from a set of Dp-branes (usually with $p > 3$), that wrap a q-dimensional compact manifold in a way such that two conditions are satisfied: One imposes that the low energy description of the system is $(p - q)$ dimensional, that is, the size of the q-manifold is small and is not observable at low energies. Secondly, it gives technical control over the theory to require that a minimal amount of SUSY is preserved. For example, the resolution of the the Einstein eqs. is eased. It turns out that, in this second class of phenomenologically interesting dualities, the UV completion of the field theory is a higher dimensional field theory.

There are several models that belong to this latter class, and in this paper we are interested in those that are dual to $N = 1$ and $N = 2$ SYM. The model dual to $N = 1$ SYM [9] builds on a geometry that was originally found in 4-d gauged supergravity in [10]. The model that

is dual to $N = 2$ SYM was later found in [11]. It must be noted that all of the models in this category are afflicted by the following problem: they are not dual to “pure” field theory of interest, but instead, the field theory degrees of freedom are entangled with the KK modes on the q -manifold in a way that depends on the energy scale of the field theory. The KK modes enter the theory at an energy scale which is inversely proportional to the size of the q -manifold and the main problem is that this size is comparable to the scale that one wants to study non-perturbative phenomena such as confinement, spontaneous breaking of chiral symmetry, etc. Nevertheless, this limitation can be seen as an artifact of the supergravity approximation and will hopefully be avoided once the formulations of the string sigma model on these RR backgrounds becomes available. Many articles have studied different aspects of these models. Instead of revisiting the main results here, we refer the interested reader to the review articles [13].

Very recently, an interesting development took place by the paper of Lunin and Maldacena [14]. The authors considered a general background of IIB SG ¹ that possess two shift isometries, hence includes a torus as a part of the geometry. This $U(1) \times U(1)$ isometry allows one to generate new SG solutions by performing an $SL(2, R)$ transformation on the τ parameter of the torus:

$$\tau = B_{12} + i\sqrt{\det[g]}, \quad (1.1)$$

where the real and imaginary parts are the component of the NS two-form along the torus and the volume of the torus respectively. This transformation combined with the usual $SL(2, R)$ symmetry of the IIB theory (that acts on the axion and dilaton) closes onto a larger group $SL(3, R)$. A two-parameter subgroup of this general symmetry is singled out by the requirement of regularity in the transformed solutions. This specific subgroup is referred to as the β -transformation. A natural question from the point of view of the gauge-gravity duality concerns the dual of the β transformations on the field theory side. In other words, if we consider a geometry that is associated to a known field theory, what deformation in the field theory does the transformed solution produce? The answer to this question that was proposed by Lunin and Maldacena [14] is quite interesting: Associated with the $U(1) \times U(1)$ isometry of the geometry, there are two separate shift transformations that act on the component fields of the field theory. If we denote the charges of two canonical fields ϕ_1 and ϕ_2 in the field theory as q_i^1 and q_i^2 under these transformations, then, the effect of the β -transformation in the dual field theory can be viewed as modifying the product of fields in the Lagrangian according to the following rule:

$$\phi^1(x)\phi^2(x) \rightarrow \phi^1(x) * \phi^2(x) = e^{i\pi\beta \det(q_i^j)} \phi^1(x)\phi^2(x), \quad (1.2)$$

Therefore this deformation is very much in the spirit of non-commutative deformations of field theories [15].

¹See [14] for discussion also in cases of IIA and 11D SG.

The reason behind this result lies in the consideration of associated D-brane picture. One considers the geometry produced by a number of D-branes. Then the general idea in [14] is that, depending on the different locations of the torus in the full geometry, one introduces various different type of β -deformations on the gauge theory that lives on the D-branes. For example, the choice of the torus in the directions transverse to the D-branes yields a deformation where the two symmetry transformations in (1.2) are two global $U(1)$ symmetries of the field theory. Lunin and Maldacena gave a specific Leigh-Strassler deformation [16] of $\mathcal{N} = 4$ SYM as an example of this case. On the other hand, when the torus is along the D-brane coordinates then the associated deformation of the field theory is precisely the standard non-commutative deformation of the field theory along the torus directions. In this case, the two charges in (1.2) are the momenta $q_{x,y}^i = p_{x,y}^i$ of ϕ^i along the torus. Finally, another interesting case that we have more to say in this paper is the case where one of the torus directions is along the branes and the other along one of the transverse directions. In this case the β -transformation of the original geometry corresponds to the so-called “dipole deformation” of the field theory [17].

1.1 General idea of this paper

In this work, we consider the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ geometries of [9] and [11] and study the effects of various β -deformations. From the general arguments of [14] that we repeated above, we expect that, if one considers a toroidal isometry that is transverse to the field theory directions, and one makes the β -deformation along these directions, this will modify *only* the fields that are charged under these transverse directions. In other words, in the field theory dual to this particular β -transformed theory, the dynamics of the KK-modes will be modified, whereas the gauge theory dynamics—that we are ultimately interested in—will not be affected. Then, one may ask, whether or not the change that one produces in the KK-sector of the field theory cures the problem of entanglement of these unwanted modes with the “pure” gauge theory dynamics. In this paper we present evidence that the answer is in the affirmative. We present our discussion mainly for the case of $\mathcal{N} = 1$ theory, but the same considerations apply in the case of $\mathcal{N} = 2$.

Specifically, we consider the geometry that is presented in [9] and apply a *real* β transformation that also keeps the supersymmetry of the theory intact. Then we repeat the computations of the VEV of Wilson loops, θ_{YM} and β_{YM} -function of the theory from the deformed geometry. We show that these results are independent of the deformation parameter. The case of imaginary β deformation is also interesting in that it changes these results, however we show that in that case the dual geometry is singular, hence the gravity computations of the field theory quantities cannot be trusted. In order to investigate whether the real β transformation affects the KK sector of the theory, we compute the masses of a particular kind of KK modes in the deformed geometry. This computation is done both as a dipole

field theory computation and as a supergravity computation. As a field theory calculation it is easy to show that the particular dipole deformations that we consider when reduced to 4D yield shifts in the KK masses. In the supergravity side, we compute the volume of S^3 in the deformed solution and show that indeed the volume becomes smaller as one turns on β . Therefore these KK-modes indeed begin to decouple from the pure gauge theory dynamics when one turns on the deformation parameter. We also consider the pp-wave limits of our deformed geometries with the same goal in mind. Namely, to analyze the effects of deformation on the dynamics of the KK-sector. The analysis results in a β correction to the original pp-wave that was obtained from the original geometry of [9]. Therefore the pp-wave analysis also confirms the claim that the dynamics of the KK-modes are modified in a non-trivial way. Similar observations hold for the case of β -deformed $\mathcal{N} = 2$ geometry.

We also consider another interesting geometry that is obtained from the *singular* “UV” solution that was found in [9]. The singular solutions dual to minimally supersymmetric gauge theories are interesting in their own right. Indeed, historically first the singular dual geometries have been found [12][9] and then the resolution of the singularities have been discovered [7][9]. The singular solutions generally believed to encode information on the UV behavior of $\mathcal{N} = 1$ SYM. The singular solution in [9] preserves an additional $U(1)$ symmetry that is associated with the chiral R-symmetry of the $\mathcal{N} = 1$ SYM in the UV. It is sometimes denoted as the ψ isometry of the solution. We consider choosing the torus such that one leg is along the direction of ψ . Then we perform the β transformation to generate new solutions. As this $SL(3, R)$ transformation does not commute with the R-symmetry the resulting theory is non-supersymmetric. Therefore by this method, we generate a geometry that would-be a candidate for a dual of non-supersymmetric pure YM, once the singularity at $r = 0$ is resolved. We also observe that in the case of β -transforming along the R-symmetry direction both the real and imaginary parts of the transformation is allowed: One does not generate any irregularities that were present in the previous case of supersymmetry preserving transformation. This may rise some hope that in a particular regime of the relatively larger parameter space of the theory (now consists of both real and the imaginary parts of β) one may be able to find a resolution to the singularity at the origin. We leave this question for future work.

In the next section, we move on to a presentation of the β deformations of [14]. Rather than repeating the discussion in [14], we introduce the basic idea and the technical aspects of the method in a simple “warm-up” example: the flat $D5$ geometry. We stress the discussion of the irregularities associated with the imaginary β transformations.

Section three reviews the dual of $\mathcal{N} = 1$. We review the geometry and present a detailed discussion of the mixing between the KK and pure gauge dynamics in the original theory. Specifically, we review the “twisting” procedure and we make a comparison of confinement and KK scales. In this section we also present our field theory argument for the change in the masses of the KK modes. We show that the dipole deformation of the 6D field theory

when reduced to 4D indeed realizes the idea of improving the KK entanglement problem.

In section 4, we present the β -transformation of the non-singular solution where the β transformation is chosen in such a way to preserve supersymmetry. In this section, we also obtain and discuss the aforementioned non-supersymmetric solution that explicitly breaks the ψ -isometry of singular $\mathcal{N} = 1$ solution.

The main results of our work are presented in section 5, that is devoted to the discussion of the β deformation of the non-singular $\mathcal{N} = 1$ geometry. There we introduce the transformed solution and discuss its properties. Finally we compute the field theory observables of interest. In particular, we show that the expectation value of the Wilson loop, θ_{YM} and the beta function β_{YM} are independent of the deformation parameter and we present the change in the mass of the KK-modes on S^3 ; we also study domain walls and their tensions.

Section 6 discusses pp-wave limits of some of our solutions. We first make some general observations about how to perform the pp-wave limits in general β -deformed geometries. In particular we argue that β should be scaled to zero along with $R \rightarrow \infty$. This fact was already observed in [14].² Then we apply this general method of taking the pp-waves to three geometries in the following order: The flat β -transformed $D5$ geometry, the transformed geometry obtained from the singular geometry of $D5$ wrapped on S^2 and finally the non-singular β deformed $\mathcal{N} = 1$ geometry.

Section 7 includes our results for the deformations of the $\mathcal{N} = 2$ geometry. We give a summary and discuss various open directions in our work in the final section 8. Various appendices contain the details of our computations.

Here is a brief summary of the very recent literature on the subject. After the paper [14] presented the idea described above, together with some checks, Ref. [18] studied the proposal from the view point of comparing semi-classical strings moving in these backgrounds with anomalous dimensions of gauge theory operators. Also aspects of integrability of the spin chain system that is associated with the Leigh-Strassler deformation are studied. Ref. [19], presented a nice way of understanding (the bosonic NS part of) the $SL(3, R)$ transformations in terms of T dualities; also made a connection with Lax pairs and found new non-supersymmetric deformations.

A final note on the notation: As we discuss in the next section, we will mainly be concerned with the *real* β transformations in this paper —unless specified otherwise— because of the concerns about irregularity of the imaginary β transformations. These specific real transformations were denoted as γ -transformations in [14]. Therefore the term γ -transformation will be used instead of β in what follows.

²Note a misprint in [14]: β goes to zero instead of ∞ .

2 Warm-up: Transformations of the Flat D5 Brane Solution

In this section, we outline the solution generating technique in a simple warm-up example. This allows us to introduce the basic idea and the necessary notation that will be used in the following sections. More importantly, the examples we discuss here shall serve as an illustration of when and how the $SL(3, R)$ transformations lead to irregular solutions. The criteria for regularity of the transformation was outlined in [14] and a specific transformation of the NS5 (or D5) brane was mentioned as an example of irregular behavior. Here, we discuss this point in detail.

2.1 The flat D5 brane

Let us consider N D5 branes in flat 10D space-time. The solution in the string frame is,

$$ds^2 = e^\phi \left[dx_{1,5}^2 + \alpha' g_s N (dr^2 + \frac{1}{4} \sum_{i=1}^3 w_i^2) \right], \quad F_{(3)} = \frac{\alpha' N}{4} \omega_1 \wedge w_2 \wedge w_3, \quad e^\phi = \frac{\alpha' g_s (2\pi)^{\frac{3}{2}}}{\sqrt{N}} e^r. \quad (2.3)$$

We define the $su(2)$ left-invariant one forms as,

$$\begin{aligned} w_1 &= \cos \psi d\tilde{\theta} + \sin \psi \sin \tilde{\theta} d\tilde{\varphi}, \\ w_2 &= -\sin \psi d\tilde{\theta} + \cos \psi \sin \tilde{\theta} d\tilde{\varphi}, \\ w_3 &= d\psi + \cos \tilde{\theta} d\tilde{\varphi}. \end{aligned} \quad (2.4)$$

Ranges of the three angles are $0 \leq \tilde{\varphi} < 2\pi$, $0 \leq \tilde{\theta} \leq \pi$ and $0 \leq \psi < 4\pi$.

One easily sees that this solution includes a torus that is parametrized by ψ and $\tilde{\varphi}$ as part of its isometries. In order to perform the $SL(3, R)$ transformation that was specified in [14], one writes (2.3) in the form that separates the torus part from the 8D part that is transverse to the torus. The 8D part is left invariant (up to an overall factor) under the transformation. We shall use the notation introduced in [14] in order to avoid confusion (see our Appendix A for a summary). The D5 metric, (2.3) can be written as,

$$ds^2 = \frac{F}{\sqrt{\Delta}} (D\varphi_1 - CD\varphi_2)^2 + F\sqrt{\Delta} (D\varphi_2)^2 + g_{\mu\nu} dx^\mu dx^\nu. \quad (2.5)$$

In this particular case, the torus is given by,

$$D\varphi_1 = d\psi + \mathcal{A}^{(1)}, \quad D\varphi_2 = d\tilde{\varphi} + \mathcal{A}^{(2)}. \quad (2.6)$$

The connection one-forms, $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ that are required to put a generic metric in the above form, vanish in this simple case:

$$\mathcal{A}^{(1)} = \mathcal{A}^{(2)} = 0. \quad (2.7)$$

The 8D part of the metric in (2.3) is,

$$g_{\mu\nu}dx^\mu dx^\nu = dx_{1,5}^2 + N(dr^2 + \frac{1}{4}d\tilde{\theta}^2). \quad (2.8)$$

We introduced the following metric functions,

$$F = \frac{\alpha' g_s N}{4} e^\phi \sin \tilde{\theta}, \quad \Delta = \sin^2 \tilde{\theta}, \quad C = -\cos \tilde{\theta}. \quad (2.9)$$

Similarly, one separates the torus and the transverse part in the RR two form in a generic way as follows [14] and Appendix A:

$$C_2 = C_{12} D\varphi_1 \wedge D\varphi_2 + C^{(1)} \wedge D\varphi_1 + C^{(2)} \wedge D\varphi_2 - \frac{1}{2}(\mathcal{A}^{(a)} \wedge C^{(a)} - \tilde{c}). \quad (2.10)$$

$C^{(1)}$ and $C^{(2)}$ are one-forms and \tilde{c} is a two-form on the 8D transverse part. In this particular example, one can take the RR form as follows:

$$C_2 = \frac{\alpha' N}{4} \psi w_1 \wedge w_2. \quad (2.11)$$

Then, various objects that appear in the general formula (2.10) become,

$$C^{(2)} = \frac{\alpha' N}{4} \psi \sin \tilde{\theta} d\tilde{\theta}, \quad C_{12} = C^{(1)} = \tilde{c} = 0. \quad (2.12)$$

2.2 The General Transformation

Now, we consider the transformation of a general solution of IIB SG under the following two-parameter subgroup of $SL(3, R)$ ³:

$$\Lambda = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & \sigma & 1 \end{pmatrix}$$

In the next subsection, we work out the details of the transformation in the simple example of (2.3).

³This specific subgroup - actually a larger one which also includes the ordinary $SL(2, R)$ of IIB in part - was specified in [14] as a necessary condition for the regularity of the transformed geometry. Although necessary, it is not sufficient for the regularity, as we discuss further below.

Consider a general solution to IIB Supergravity. Various components of the RR two-form, NSNS two-form, the four-form and the \mathcal{A} -vectors that are defined in (2.5) are grouped into the following combinations that transform as vectors under $SL(3, R)$ ([14] and Appendix A):

$$\begin{aligned} V_\mu^{(i)} &= (-\epsilon^{ij} B_\mu^{(j)}, A_\mu^{(i)}, \epsilon^{ij} C_\mu^{(j)}), \\ W_{\mu\nu} &= (\tilde{c}_{\mu\nu}, \tilde{d}_{\mu\nu}, \tilde{b}_{\mu\nu}). \end{aligned} \quad (2.13)$$

Here $B^{(1)}$, $B^{(2)}$, \tilde{b} and \tilde{d} are components of the NS form and the four-form that follows from the separation of the torus and the transverse part in complete analogy with (2.10)[14]. Their explicit transformation under (2.2) is given as,

$$\begin{aligned} (V^{(i)})' &= (-\epsilon^{ij} B^{(j)}, A^{(i)} + \gamma \epsilon^{ij} B^{(j)} - \sigma \epsilon^{ij} C^{(j)}, \epsilon^{ij} C^{(j)}) \\ W' &= (\tilde{c} + \gamma \tilde{d}, \tilde{d}, \tilde{b} + \sigma \tilde{d}) \end{aligned} \quad (2.14)$$

Transformation of various scalar fields that we defined above is obtained as follows (see our Appendix A for a summary of these results). One constructs a 3×3 matrix, g^T , from the scalar F , the dilaton, and the torus components of the RR and NS forms, C_{12} and B_{12} . The components of the initial g^T are ⁴,

$$g_{1,1}^T = (e^\phi F)^{-1/3}, \quad g_{2,2}^T = e^{-\phi/3} F^{2/3}, \quad g_{3,2}^T = -C_{12}(F)^{-1/3} e^{2/3\phi}, \quad g_{3,3}^T = e^{2/3\phi} (F)^{-1/3} \quad (2.15)$$

and all of the other components are zero.

Now, a few words about how to obtain the transformed matrix: The object that transforms as a matrix under $SL(3, R)$ is the following one:

$$M = gg^T, \quad M \rightarrow \Lambda M \Lambda^T. \quad (2.16)$$

Thus, one can read off the transformation of g^T as,

$$g^T \rightarrow \eta^T g^T \Lambda^T, \quad (2.17)$$

where η is an $SO(3)$ matrix that can be parametrized by three Euler angles. These angles are determined by demanding that the transformed g^T has the specific structure given in [14], namely its (1, 3), (2, 1) and (2, 3) components vanish. We obtain the new values for $F, e^\phi, C_{12}, \chi, B_{12}$ as,

$$F' = FG\sqrt{H}, \quad e^{\phi'} = e^\phi H\sqrt{G}, \quad B'_{12} = \gamma F^2 G, \quad \chi' = \gamma \frac{J}{H}, \quad C'_{12} = -JG \quad (2.18)$$

where we defined,

$$\begin{aligned} H &= (1 - C_{12}\sigma)^2 + F^2\sigma^2 e^{-2\phi}, \\ G &= ((1 - C_{12}\sigma)^2 + F^2\sigma^2 e^{-2\phi} + \gamma^2 F^2)^{-1}, \\ J &= \sigma F^2 e^{-2\phi} - C_{12}(1 - C_{12}\sigma). \end{aligned} \quad (2.19)$$

⁴Here, we consider the special case of $B_{12} = 0$. The most general case is further discussed in the Appendix A.

Using these transformation properties one obtains the new metric as follows:

$$ds^2 = \frac{F'}{\sqrt{D}} (D\varphi'_1 - CD\varphi'_2)^2 + F'\sqrt{D}(D\varphi'_2)^2 + U_{st}g_{\mu\nu}dx^\mu dx^\nu. \quad (2.20)$$

Here $D\varphi'_i$ include the transformed \mathcal{A} forms, *i.e.* $D\varphi'_i = d\varphi_i + \mathcal{A}'^{(i)}$. The volume ratio that appears in front of the 8D transverse part is,

$$U_{st} = \frac{e^{\frac{2}{3}(\phi' - \phi)}}{\left(\frac{F'}{F}\right)^{\frac{1}{3}}} = H^{\frac{1}{2}}. \quad (2.21)$$

where we used (2.18) in the last line. The expression for H in (2.19) tells us the important fact that *the 8D volume ratio is different than 1 only for non-zero σ transformations*. In particular an arbitrary γ transformation (with $\sigma = 0$), leaves the 8D volume invariant:

$$U_{st}(\sigma = 0) = 1. \quad (2.22)$$

To extract useful information about the field theory that is dual to the transformed geometry, one often needs the analogous expression in the Einstein frame. Making a Weyl transformation to the Einstein frame, one obtains the following volume ratio of the 8D part in the Einstein frame:

$$U_E = e^{-\frac{1}{2}(\phi' - \phi)} U_{st} = G^{-\frac{1}{4}}. \quad (2.23)$$

This ratio is generally different than one for any transformation.

2.3 Regularity of Transformed D5

Now, let us apply this procedure to the particular solution given in (2.9) and (2.12). From (2.14) we read off the new values of the \mathcal{A} forms, and the vector components of the RR form as,

$$\begin{aligned} \mathcal{A}^{(1)'} &= -\sigma C^{(2)} = -\sigma \frac{\alpha' N}{4} \psi \sin \tilde{\theta} d\tilde{\theta}, \\ C^{(2)'} &= C^{(2)} = -\frac{\alpha' N}{4} \psi \sin \tilde{\theta} d\tilde{\theta}, \quad \mathcal{A}^{(2)'} = C^{(1)'} = 0 \end{aligned} \quad (2.24)$$

Various scalar fields in the new solution are obtained from (2.18). The new metric is, using (2.24) in (2.20),

$$ds^2 = \frac{F'}{\sqrt{\Delta}} \left(d\psi - \sigma \frac{\alpha' N}{4} \psi \sin \tilde{\theta} d\tilde{\theta} + \cos \tilde{\theta} d\tilde{\varphi}^2 \right)^2 + F'\sqrt{\Delta} d\tilde{\varphi}^2 + U_{st}g_{\mu\nu}dx^\mu dx^\nu. \quad (2.25)$$

The particular appearance of ψ above makes the transformed metric irregular. This is because, ψ originally was defined as a periodic variable with period 4π . However the transformed metric is no longer periodic in ψ . One can easily track the origin of this irregularity:

It is coming from the contribution of the σ transformation to the \mathcal{A} one-forms in (2.24). In this case where the torus is chosen in the directions transverse to the D5 brane, the RR two form has the particular form in (2.12) with a bare dependence on ψ and this bare dependence directly carries on to the metric under the σ transformation. Notice that this irregular behavior does not happen for a one parameter γ -transformation where one sets $\sigma = 0$. In that case one obtains a new regular solution to IIB SG! (or at least one in which we do not generate *new* singularities).

One may wonder if this irregularity is due to our particular choice of (2.11): C_2 is defined only up to a gauge transformation and one can try to use a gauge-equivalent expression where ψ does not appear in this form. For example a gauge equivalent choice of C_2 is given by,

$$C_2 = \frac{\alpha' N}{4} d\psi \wedge w_3 = \frac{\alpha' N}{4} \cos \tilde{\theta} d\psi \wedge d\tilde{\varphi}. \quad (2.26)$$

Comparison with the general expression, (2.10) shows that,

$$C_{12} = \frac{\alpha' N}{4} \cos \tilde{\theta}, \quad (2.27)$$

and the rest of the components in (2.10) vanish. In particular, the vector $C^{(1)}$ vanish and one does not generate any irregular behavior in the metric, as in (2.25).

However, as described in [14], for the regularity of the full solution, one should also make sure that the RR two-form in the original solution go to the same integer at various potentially dangerous points where the volume of the torus shrinks to zero size.⁵ One sees from (2.3) that these singular points where the volume of the ψ - $\tilde{\varphi}$ torus shrinks to zero are given by $\tilde{\theta} = 0$ and π . Equation, (2.27) tells us that at these points, C_2 goes over to $-N$ and $+N$, respectively. As $\tilde{\theta}$ is a periodic variable with period π , a discrete jump of $2N$ in the flux as one completes the period is unacceptable and generates an infinite field strength. Thus we conclude that, in case where the torus is chosen in the transverse directions to the D5 brane, σ transformation is sick, independently of the gauge choice for C_2 . This does not happen for the γ transformation. With a completely analogous argument, one shows that ⁶ the reverse phenomena happens in case of the NS5 brane solution. In that case there is a non-trivial B_2 form and this time the γ transformation exhibits the same irregular behavior, whereas the σ transformation is free of irregularity.

Finally, let us recall that the Ricci scalar of the original flat D5 brane (2.3) geometry is bounded for large values of the radial coordinate, where the dilaton becomes large. The divergence in the dilaton indicates the need to passing to an S-dual description that is

⁵In [14], this argument was made for B_2 under the γ transformation. Same argument applies to C_2 in the case of σ transformation.

⁶One can use the formulae given in the previous subsection to work this case out. Our formula (2.18) is applicable only to the case $B_{12} = 0$ (but we wrote general formulas in Appendix A) and one can choose a gauge in the NS5 solution such that this happens.

given in terms of NS5 branes. At small values of the radial coordinate, the Ricci scalar diverges instead, thus indicating that we are in the regime where the 6D SYM theory is the weakly coupled description of the system. In the case of the transformed metric (2.20) things are more interesting. One can check that, for large values of the radial coordinate, the transformed dilaton in (2.18) does not diverge (except for the values $\theta = 0$, $\theta = \pi$) and the Ricci scalar has an expression that depends on the transformation parameter,

$$R_{eff} \approx e^{-r} \left(\frac{-384 + 26\eta^2\gamma^2 e^{2r} - \eta^2\gamma^4 e^{4r} + \eta^2\gamma^2 \cos(2\theta) e^{2r} (54 + e^{2r}\gamma^2)}{\eta^2(16 + \gamma^2 e^{2r} \sin^2 \theta)} \right), \eta = \sqrt{g_{YM}^2 N (2\pi)^{-3}}, \quad (2.28)$$

where $\eta = \sqrt{g_{YM}^2 N (2\pi)^{-3}}$. So, if we fix θ as $n\pi$, for large values of the radial coordinate, we should pass to a dual description. In this case, doing a T-duality seems appropriate. On the other hand, for values of the angle θ different from $n\pi$ we see that the Ricci scalar (which is non-zero) and the dilaton are bounded for large values of the radial coordinates, in contrast with the usual $D5$ case. For this case, it does not seem necessary to pass to a NS5 description.

For values of the radial coordinate near negative infinity, the Ricci scalar diverges and the good description is in terms of a 6D gauge theory. The main point of the discussion above is the fact that the transformed system has in principle a more complicated ‘phase space’ than the original flat D5-branes [4].

2.4 The Gauge Theory

Let us briefly comment on the gauge theory dual of the transformed $D5$ background. First, let us recall that the gauge theory on flat D5 branes is a 6D maximally supersymmetric Yang-Mills theory. The bosonic part of the Lagrangian can be obtained by reducing ten-dimensional SYM on a four-torus. Schematically, it is given as

$$S = Tr \int d^6x \left(F_{\mu\nu}^2 + (D_\mu \Phi)^2 + V([\Phi, \Phi]) \right) \quad (2.29)$$

According to the prescription of [14] the new background in (2.20) is dual to (2.29) but with the potential replaced by a γ deformed one obtained by replacing the products of fields in $V[\dots]$ by the deformed product of (1.2). We would like to remark that as the torus of transformation that is formed by ψ and $\tilde{\varphi}$ is transverse to the D5 branes this will only introduce some phases in the scalar potential.

One should also note that as the transformation that we perform mixes angles that correspond to the $SU(2)_L \times SU(2)_R$ R-symmetry of the 6D SYM theory, this transformation will break supersymmetry. We shall not dwell on this issue for this warm-up exercise but it is discussed in more detail for the model of our interest in the next section.

2.5 Dual of the Dipole Theory

We would like to end this warm-up section by a discussion of the dipole theories that shall interest us in the following sections. Our main interest in these exotic quantum theories lies in the fact that the transformations studied in this section when applied to a supergravity dual to $\mathcal{N} = 1$ SYM leads to a dipole theory for the KK section of the field theory. This is discussed in section 3.2 in detail. For the literature on the dipole theories, see [17]. Here we only summarize some features that will interest us.

A dipole theory is a field theory where the locality is lost due to the fact that some of the fields come equipped with a “dipolar moment” L_μ such that the product of fields (Φ_1, Φ_2) with moments \vec{L}_1, \vec{L}_2 is given by

$$\Phi_1(x)\Phi_2(x) = \Phi_1(x - L_2/2)\Phi_2(x + L_1/2) \quad (2.30)$$

The non-locality of the interaction is manifest. Also, the presence of dipole moments break Lorentz invariance. We are interested in the particular case where all of the dipole moments point in the same direction.

If we have a field theory with many fields Φ_i , each one of them with a global conserved charge q_i , we can pick an arbitrary vector L_μ and the dipole moment of each field is taken as $q_i L_\mu$. More generally, if for each field Φ_i in a collection of n fields, there exist a set of conserved charges q_1^i, \dots, q_k^i , then one can pick an $n \times k$ matrix (that we call S) and assign each field a dipole moment of the form $\vec{L} = S\vec{q}$. In this general case the arbitrary choice of S , breaks the Lorentz invariance.

The way to construct the dipole Lagrangian is clearly explained in [17]. Basically, the idea follows the approach of [15], only that in this case, we allow for a non-constant NS field, in addition to the RR forms. Briefly, the proposal is that, any term in the original Lagrangian that is of the following form,

$$L_{int} = \Phi_1(x) \dots \Phi_n(x) \quad (2.31)$$

is replaced (in momentum space) by

$$L'_{int} = e^{\sum_{i < j} p_i L_j} \Phi_1(x) \dots \Phi_n(x) \quad (2.32)$$

If we follow the proposal of Lunin and Maldacena [14] that was reviewed above, this is equivalent to performing an $SL(3, R)$ transformation on a torus that has one direction on the brane and one external to the brane.

To improve and clarify this discussion, let us now construct the background dual to a dipole 6D field theory. Once again, we do not worry about preservation of supersymmetry in this section. If one performs the γ transformation on the torus with one direction along the D5 brane and one transverse to it, one produces a background dual to a six dimensional dipole theory as described above. Obviously the Lorentz invariance $SO(1, 5)$ is explicitly

broken by the choice of one coordinate along the brane. The background dual to this field theory is given by

$$ds^2 = \frac{F'}{\sqrt{\Delta}}(d\psi + \cos\theta d\varphi)^2 + F'\sqrt{\Delta}dz^2 + U_{st}(dx_{1,4}^2 + \alpha'g_s N(dr^2 + \frac{1}{4}(d\theta^2 + \sin^2\theta d\varphi^2))) \quad (2.33)$$

where we labeled by z, ψ the directions of the two torus. Following the notation in eq.(2.5), the functions are defined as

$$F^2 = \frac{\alpha'g_s N e^{2\phi}}{4}, \quad \Delta = \frac{4}{\alpha'g_s N}, \quad F' = \frac{F}{1 + \gamma^2 F^2}, \quad \mathcal{A}^{(1)} = \cos\theta d\varphi, \mathcal{A}^2 = C^i = C = 0 \quad (2.34)$$

and U_{st} is given by (2.21). There is also a dilaton and NS and RR forms that transform according to eq.(2.18). The original RR two form only has the following component,

$$\tilde{c} = 2\psi\alpha'N \sin\theta d\theta \wedge d\varphi \quad (2.35)$$

and it transforms as we indicated in (2.14).

Let us briefly comment on the geometry (2.33). The transformed dilaton is bounded above for large values of the radial coordinate (and for any values of the angles) unlike the case of the flat D5 brane analyzed in the previous subsection. The effective curvature is given by,

$$R \approx 2e^{-r} \left(\frac{-96 + 8\eta^2\gamma^2 e^{2r} + \eta^4\gamma^4 e^{4r}}{\eta^2(4 + \gamma^2 e^{2r})^2} \right), \quad \eta = \sqrt{g_{YM}^2 N (2\pi)^{-3}}. \quad (2.36)$$

Again, we find no problem for the large- r regime, neither for the Ricci scalar, nor for the dilaton. It is peculiar that it is possible to find a value of the radial coordinate where the Ricci scalar vanishes.

3 $\mathcal{N} = 1$ SYM and the KK-Mixing Problem

3.1 Review of the Geometry Dual to $\mathcal{N} = 1$ SYM

We work with the model presented in [9] (the solution was first found in a 4d context in [10]) and described and studied in more detail in [22]. Let us briefly describe the main points of this supergravity dual to $N = 1$ SYM and its UV completion.

Suppose that we start with N D5 branes. The field theory that lives on them is 6D SYM with 16 supercharges. Then, suppose that we wrap two directions of the D5 branes on a curved two manifold that can be chosen as a sphere. In order to preserve SUSY a twisting procedure has to be implemented and actually, there are two ways of doing it. The one we

will be interested in this section, deals with a twisting that preserves four supercharges. In this case the two-cycle mentioned above lives inside a CY3 fold. On the other hand, if the twist that preserves eight supercharges is performed, the two cycle lives inside a CY2-fold [11]. This second case will be analyzed in section 7. We note that this supergravity solution is dual to a four dimensional field theory, only for low energies (small values of the radial coordinate). Indeed, at high energies, the modes of the gauge theory begins to fluctuate also on the two-cycle and as the energy is increased further, the theory first becomes $\mathcal{N} = 1$ SYM in six dimensions and then, blowing-up of the dilaton forces one to S-dualize. Therefore the UV completion of the model is given by the little string theory.

The supergravity solution that interests us in this section, preserves four supercharges and has the topology $R^{1,3} \times R \times S^2 \times S^3$. There is a fibration of the two spheres in such a way that that $\mathcal{N} = 1$ supersymmetry is preserved. By going near $r = 0$ it can be seen that the topology is $R^{1,6} \times S^3$. The full solution and Killing spinors are written in detail in [22]. The metric in the Einstein frame reads,

$$ds_{10}^2 = \alpha' g_s N e^{\frac{\phi}{2}} \left[\frac{1}{\alpha' g_s N} dx_{1,3}^2 + e^{2h} (d\theta^2 + \sin^2 \theta d\varphi^2) + dr^2 + \frac{1}{4} (w^i - A^i)^2 \right], \quad (3.37)$$

where ϕ is the dilaton. The angles $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ parametrize a two-sphere. This sphere is fibered in the ten dimensional metric by the one-forms A^i ($i = 1, 2, 3$). Their are given in terms of a function $a(r)$ and the angles (θ, φ) as follows:

$$A^1 = -a(r)d\theta, \quad A^2 = a(r)\sin\theta d\varphi, \quad A^3 = -\cos\theta d\varphi. \quad (3.38)$$

The w^i one-forms are defined in (2.4).

The geometry in (3.37) preserves supersymmetry when the functions $a(r)$, $h(r)$ and the dilaton ϕ are:

$$\begin{aligned} a(r) &= \frac{2r}{\sinh 2r}, \\ e^{2h} &= r \coth 2r - \frac{r^2}{\sinh^2 2r} - \frac{1}{4}, \\ e^{-2\phi} &= e^{-2\phi_0} \frac{2e^h}{\sinh 2r}, \end{aligned} \quad (3.39)$$

where ϕ_0 is the value of the dilaton at $r = 0$. Near the origin $r = 0$ the function e^{2h} behaves as $e^{2h} \sim r^2$ and the metric is non-singular. The solution of the type IIB supergravity includes a RR three-form $F_{(3)}$ that is given by

$$\frac{1}{\alpha' N} F_{(3)} = -\frac{1}{4} (w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \frac{1}{4} \sum_a F^a \wedge (w^a - A^a), \quad (3.40)$$

where F^a is the field strength of the $\text{su}(2)$ gauge field A^a , defined as:

$$F^a = dA^a + \frac{1}{2}\epsilon_{abc} A^b \wedge A^c . \quad (3.41)$$

Different components of F^a read,

$$F^1 = -a' dr \wedge d\theta , \quad F^2 = a' \sin \theta dr \wedge d\varphi , \quad F^3 = (1 - a^2) \sin \theta d\theta \wedge d\varphi , \quad (3.42)$$

where the prime denotes derivative with respect to r . Since $dF_{(3)} = 0$, one can represent $F_{(3)}$ in terms of a two-form potential $C_{(2)}$ as $F_{(3)} = dC_{(2)}$. Actually, it is not difficult to verify that $C_{(2)}$ can be taken as:

$$\begin{aligned} \frac{C_{(2)}}{\alpha' N} = & \frac{1}{4} \left[\psi (\sin \theta d\theta \wedge d\varphi - \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi}) - \cos \theta \cos \tilde{\theta} d\varphi \wedge d\tilde{\varphi} - \right. \\ & \left. - a (d\theta \wedge w^1 - \sin \theta d\varphi \wedge w^2) \right] . \end{aligned} \quad (3.43)$$

The equation of motion of $F_{(3)}$ in the Einstein frame is $d(e^\phi * F_{(3)}) = 0$, where $*$ denotes Hodge duality. Let us stress that the configuration presented above is non-singular. Finally, let us mention that the BPS equations also admit a solution in which the function $a(r)$ vanishes, *i.e.* in which the one-form A^i has only one non-vanishing component, namely A^3 . We will refer to this solution as the abelian $\mathcal{N} = 1$ background. Its explicit form can easily be obtained by taking the $r \rightarrow \infty$ limit of the functions given in eq. (3.39). Notice that, indeed $a(r) \rightarrow 0$ as $r \rightarrow \infty$ in eq. (3.39). Neglecting exponentially suppressed terms, one obtains

$$e^{2h} = r - \frac{1}{4} , \quad (a = 0) , \quad (3.44)$$

while ϕ can be obtained from the last equation in (3.39). The metric of the abelian background is singular at $r = 1/4$ (the position of the singularity can be moved to $r = 0$ by a redefinition of the radial coordinate). This IR singularity of the abelian background is removed in the non-abelian metric by switching on the A^1, A^2 components of the one-form (3.38).

3.2 Dual Field Theory and the Dipole Deformation of the KK Sector

Let us first summarize some aspects of the field theory that is dual to the geometry above. In [9] this solution was argued to be dual to $\mathcal{N} = 1$ SYM.

The 4D field theory is obtained by reduction of N $D5$ branes on S^2 with a twist that we explain below. Therefore, as the energy scale of the 4D field theory becomes comparable to the inverse volume of S^2 , the KK modes begin to enter the spectrum.

To analyze the spectrum in more detail, we briefly review the twisting procedure. In order to have a supersymmetric theory on a curved manifold like the S^2 here, one needs globally defined spinors. A way to achieve this was introduced in [24]. In our case the argument goes as follows. As D5 branes wrap the two sphere, the Lorentz symmetry along the branes decompose as $SO(1, 3) \times SO(2)$. There is also an $SU(2)_L \times SU(2)_R$ symmetry that rotates the transverse coordinates. This symmetry corresponds to the R-symmetry of the supercharges on the field theory of D5 branes. One can properly define $\mathcal{N} = 1$ supersymmetry on the curved space that is obtained by wrapping the D5 branes on the two-cycle, by identifying a $U(1)$ subgroup of either $SU(2)_L$ or $SU(2)_R$ R-symmetry with the $SO(2)$ of the two-sphere.⁷ To fix the notation, let us choose the $U(1)$ in $SU(2)_L$. Having done the identification with $SO(2)$ of the sphere, we denote this twisted $U(1)$ as $U(1)_T$.

After this twisting procedure is performed, the fields in the theory are labeled by the quantum numbers of $SO(1, 3) \times U(1)_T \times SU(2)_R$. The bosonic fields are,

$$A_\mu^a = (4, 0, 1), \quad \Phi^a = (1, \pm, 1), \quad \xi^a = (1, \pm, 2). \quad (3.45)$$

Respectively they are the gluon, two massive scalars that are coming from the reduction of the original 6D gauge field on S^2 , (explicitly from the A_φ and A_θ components) and finally four other massive scalars (that originally represented the positions of the D5 branes in the transverse R^4). As a general rule, all the fields that transform under $U(1)_T$ —the second entry in the above charge designation—are massive. For the fermions one has,

$$\lambda^a = (2, 0, 1), (\bar{2}, 0, 1), \quad \Psi^a = (2, ++, 1), (\bar{2}, --, 1), \quad \psi^a = (2, +, 2), (\bar{2}, -, 2). \quad (3.46)$$

These fields are the gluino plus some massive fermions whose $U(1)_T$ quantum number is non-zero. The KK modes in the 4D theory are obtained by the harmonic decomposition of the massive modes, Φ, ξ, Ψ and ψ that are shown above. Their mass is of the order of $M_{KK}^2 = (Vol_{S^2})^{-1} \propto \frac{1}{g_s \alpha' N}$. A very important point to notice here is that these KK modes are charged under $U(1)_T \times U(1)_R$ where the second $U(1)$ is a subgroup of the $SU(2)_R$ that is left untouched in the twisting procedure. On the other hand, the pure gauge fields gluon and the gluino are not charged under either of the $U(1)$'s.

The dynamics of these KK modes mixes with the dynamics of confinement in this model because the strong coupling scale of the theory is of the order of the KK mass. One way to evade the mixing problem would be to work instead with the full string solution, namely the world-sheet sigma model on this background (or in the S-dual NS5 background) which would give us control over the duality to all orders in α' , hence we would be able to decouple the dynamics of KK-modes from the gauge dynamics. This direction is unfortunately not yet available.

The dynamics of these KK modes have not been studied in much detail in the literature. In [20] a very interesting object—the *annulon*—was introduced. It is composed out of a

⁷The choice of a diagonal $U(1)$ inside $SU(2)_L \times SU(2)_R$ leads to an $N = 2$ field theory instead, see [11].

condensate of many KK modes. More interesting studies on the annulons in this and related models—some being non-supersymmetric—are in [21].

We would like to argue in this paper that the proposal of Lunin and Maldacena [14] opens a new path to approach the KK mixing problem in a controlled way. Let us consider the torus of the β transformations as $U(1)_T \times U(1)_R$ —that is in fact the only torus in the non-singular solution (3.37) and it is given by shifts along φ and $\tilde{\varphi}$. Then the proposal of LM implies that,

The β deformation of (3.37) generates a dipole deformation in the dual field theory, but only in the KK sector that is the only sector which is charged under $U(1)_T \times U(1)_R$.

We propose that the deformation does not affect the 4D field theory modes namely the gluon and the gluino, encouraged by the fact that they are not charged under this symmetry, as explicitly shown in (3.45) and (3.46).

Let us sketch the general features of this dipole field theory and then present an explicit argument showing an example of how the dynamics of the KK sector of the theory is affected under the dipole deformation. In particular, we would like to show that the masses of the KK modes are shifted under the dipole deformation.

One can schematically write a Lagrangian for these fields as follows:

$$L = -Tr[\frac{1}{4}F_{\mu\nu}^2 + i\lambda D\lambda - (D_\mu\Phi_i)^2 - (D_\mu\xi_k)^2 + \Psi(iD - M)\Psi + M_{KK}^2(\xi_k^2 + \Phi_i^2) + V[\xi, \Phi, \Psi]] \quad (3.47)$$

The potential typically contains the scalar potential for the bosons, Yukawa type interactions and more. This expression is schematic because of (at least) two reasons. First of all, the potential presumably contains very complicated interactions involving the KK and massless fields. Secondly, there is mixing between the infinite tower of spherical harmonics that are obtained by reduction on S^2 and S^3 . (We were being schematic in the definitions of (3.45) and (3.46). For example a precise designation for Φ should involve the spherical harmonic quantum numbers (l, m) on S^2 .)

Let us now discuss the effect of the β deformation in more detail. The $U(1)_T$ corresponds to shift isometries along φ and $U(1)_R$ corresponds to shift along $\tilde{\varphi}$ in (3.37). From the D5 brane point of view, the former is a dipole charge and the latter is a global phase on the 6D fields. Here we only focus on the γ transformation because as we discussed in the previous section, only the real part of the β deformation gives rise to a regular dual geometry. In this case, the prescription of Lunin and Maldacena [14] tells us to deform the product of two fields in the *superpotential* as follows:

$$X[\vec{x}_6]Y[\vec{x}_6] \rightarrow e^{i\gamma(\hat{Q}_X\hat{L}_Y - \hat{Q}_Y\hat{L}_X)}X[\vec{x}_6]Y[\vec{x}_6], \quad (3.48)$$

where X and Y are either of the fields that appear in (3.47) and Q and L denotes the charges of the indicated fields under $U(1)_R$ and $U(1)_T$ respectively. Note that the action of \hat{L} on a field that is charged under $U(1)_T$ is a dipole deformation.

As a simple exercise, let us investigate the implications of the γ deformation in (3.48) for the masses of the KK modes. Mass terms in (3.47) are coming from the quadratic terms in the superpotential. Let us consider the following term in original (undeformed) superpotential,

$$W \sim M_{KK} \Phi_0^+[\vec{x}_6] \xi_+^+[\vec{x}_6] + \dots \quad (3.49)$$

Here we take two fields of the types in (3.45) with the denoted charges under $U(1)_T$ and $U(1)_R$: Φ has charge +1 under $U(1)_T$ and uncharged under $U(1)_R$. ξ carries +1 under both. Under the deformation (3.48) this term becomes,

$$\begin{aligned} W &\rightarrow W_\gamma \sim M_{KK} e^{i\gamma(\hat{Q}_\Phi \hat{L}_\xi - \hat{Q}_\xi \hat{L}_\Phi)} \Phi_0^+[\vec{x}_6] \xi_+^+[\vec{x}_6] = M_{KK} e^{-i\gamma R_3 R_2 \hat{L}_\Phi} \Phi_0^+[\vec{x}_6] \xi_+^+[\vec{x}_6] \\ &\approx M_{KK} \Phi_0^+[\vec{x}_6] (1 + \gamma (R_2 R_3) \partial_\varphi) \xi_+^+[\vec{x}_6]. \end{aligned} \quad (3.50)$$

In the last step in (3.50) we used the fact that the dipole generator \hat{L} corresponds to shifts in φ

Here R_2 and R_3 are length scales that are associated with the two-cycle and the three-cycle in the geometry.⁸ In general these length scales depend on the radial coordinate r , hence the charges Q and L will depend on the energy scale of the field theory. In the far IR region—that we are ultimately interested in—they are both proportional to $\sqrt{\alpha' g_s \mathcal{N}}$. Indeed, we support this claim by our supergravity computations in section 5.3.

Now, consider the spherical decomposition of ξ_+^+ on S^2 :

$$\xi_+^+[\vec{x}_6] = \sum_{l,m} \xi_+^+[\vec{x}_4] Y_{l,m}[\theta, \varphi]. \quad (3.51)$$

The spherical harmonics satisfy⁹,

$$-i\partial_\varphi Y_{l,m}(\theta, \varphi) = m Y_{l,m}(\theta, \varphi). \quad (3.52)$$

Then, consider a specific KK mode of ξ_+^+ with the quantum numbers l, m in eq.(3.50). Expanding (3.50) for small γ and using (3.52) we get the following particular term in the deformed superpotential,

$$W_\gamma \sim M_{KK} \Phi_0^+[\vec{x}_6] \xi_+^+[\vec{x}_4; l, m] (1 + i\gamma R_2 R_3 m). \quad (3.53)$$

There is a mass term in the scalar potential of the theory that derives from

$$\left| \frac{\partial W_\gamma}{\partial \Phi_0^+} \right|^2$$

⁸Note that γ has dimensions of α'^{-1} therefore in order to make the exponential dimensionless the charges Q and L should have dimensions of length.

⁹for a detailed study of the KK modes spectrum see [25]

and gives the following mass for this specific KK mode:

$$(M_{KK}^\gamma)^2 = \left(1 + (\gamma R_2 R_3 m)^2\right) M_{KK}^2. \quad (3.54)$$

Of course, our computation should be viewed as schematic because in order to obtain the true mass eigenstates one should diagonalize the mass matrix on infinite dimensional space of KK harmonics. Nevertheless, it is enough to support our main claim that the dipole deformation of the $\mathcal{N} = 1$ theory changes the dynamics of the KK modes in a very specific and controlled way. It is also enough to show that the change in the masses of the KK modes are always in the positive direction, they increase. We shall verify the mass shift in (3.54) by explicitly computing the volumes of the two and three cycles in the IR geometry in section 5.3. There we obtain analytical expressions—as functions of γ —for volume ratios of deformed over undeformed theories for the S^2 and S^3 cycles. It is encouraging to see that the mass ratio is greater than one. This fact is quite tempting to believe that the β deformation of [9] is a positive step in curing the main problem associated with this model and similar ones, namely mixing with the KK modes. Our computations therefore suggest that for large values of γ , one may be able to decouple the KK modes from the IR dynamics; but the value of γ is to be restricted from above by the condition of small Ricci scalar that is necessary for the validity of supergravity approximation. Therefore we seem to have a finite window for the γ parameter where we can make an improvement of the model by pushing the KK modes up.

Taking very large values of γ might conflict with having a bounded Ricci scalar to stay in the Supergravity approximation.

Another encouraging fact is directly seen when one considers applying the deformation in (3.48) in case either of X or Y is the vector multiplet. As the vector multiplet is uncharged under either of the Q or L we see that the γ transformation of the the $\mathcal{N} = 1$ theory indeed does not affect the pure gauge dynamics. It acts on the gauge and KK sectors of the theory (but it may translate into the massless sector through interaction terms). Motivated by the mass calculation that we presented above, one can consider more elaborate computations regarding the interaction terms in the potential in (3.47). An interesting question to ask is how is the IR theory “renormalized” under the γ transformations.

4 Deformations of the Singular $\mathcal{N} = 1$ Theory

4.1 The Singular Solution $a(r)=0$

In order to apply the $SL(2, R)$ transformation we shall put the metric in (3.37) in the form explained in [14] (See also Appendix A in this paper). Like before, we will stick with the definitions given in this paper to avoid confusion. It is very useful to first consider the transformation of the simpler but singular case of $a(r) = 0$. This provides us with some

intuition and physical insight. The metric of eq.(3.37), written in string frame, in the case $a(r)$ set to zero reads as follows:

$$ds^2 = F \left(\frac{1}{\sqrt{\Delta}} (d\varphi - Cd\tilde{\varphi} + \mathcal{A}^{(1)} - C\mathcal{A}^{(2)})^2 + \sqrt{\Delta} (d\tilde{\varphi} + \mathcal{A}^{(2)})^2 \right) + \frac{e^{2/3\phi}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu \quad (4.55)$$

where we write the eight dimensional part of the metric that is transverse to the torus as,

$$g_{\mu\nu} dx^\mu dx^\nu = e^{-2/3\phi} F^{1/3} [Dd\psi^2 + e^\phi (dx_{1,3}^2 + g_s N \alpha' (e^{2h} d\theta^2 + dr^2 + \frac{1}{4} d\tilde{\theta}^2))], \quad (4.56)$$

we have defined the following vectors,

$$\mathcal{A}^{(1)} = \alpha d\psi, \quad \mathcal{A}^{(2)} = \beta d\psi, \quad (4.57)$$

and the functions $F, D, \Delta, C, \alpha, \beta$ that appear above are:

$$\begin{aligned} F &= \frac{\alpha' g_s N e^\phi}{4} (4e^{2h} \sin^2 \theta + \cos^2 \theta \sin^2 \tilde{\theta})^{\frac{1}{2}}, \quad \Delta = \frac{4e^{2h} \sin^2 \theta + \sin^2 \tilde{\theta} \cos^2 \theta}{(4e^{2h} \sin^2 \theta + \cos^2 \theta)^2}, \\ C &= -\frac{\cos \theta \cos \tilde{\theta}}{4e^{2h} \sin^2 \theta + \cos^2 \theta}, \quad D = \frac{\alpha' g_s N e^{2h+\phi} \sin^2 \theta \sin^2 \tilde{\theta}}{4e^{2h} \sin^2 \theta + \sin^2 \tilde{\theta} \cos^2 \theta}, \\ \alpha &= \frac{\cos \theta \sin^2 \tilde{\theta}}{\cos^2 \theta \sin^2 \tilde{\theta} + 4e^{2h} \sin^2 \theta}, \quad \beta = \frac{4e^{2h} \sin^2 \theta \cos \tilde{\theta}}{4e^{2h} \sin^2 \theta + \cos^2 \theta \sin^2 \tilde{\theta}} \end{aligned} \quad (4.58)$$

The RR two form given in (3.43) for $a(r) = 0$ can be written as,

$$\begin{aligned} C_2 &= C_{12} (d\varphi + \mathcal{A}^{(1)}) \wedge (d\tilde{\varphi} + \mathcal{A}^{(2)}) + C_\mu^{(1)} (d\varphi + \mathcal{A}^{(1)}) \wedge dx^\mu + C_\mu^{(2)} (d\tilde{\varphi} + \mathcal{A}^{(2)}) \wedge dx^\mu \\ &\quad - \frac{1}{2} (\mathcal{A}_\mu^{(a)} C_\nu^{(a)} - \tilde{c}_{\mu\nu}) dx^\mu \wedge dx^\nu. \end{aligned} \quad (4.59)$$

Here various components and one-forms read as follows:

$$\begin{aligned} C_{12} &= -\frac{1}{4} \cos \theta \cos \tilde{\theta}, \quad C^{(1)} = -C_{12} \beta d\psi - \frac{\psi}{4} \sin \theta d\theta, \quad C^{(2)} = C_{12} \alpha d\psi + \frac{\psi}{4} \sin \tilde{\theta} d\tilde{\theta}, \\ \tilde{c}_2 &= \frac{\psi}{4} (\alpha \sin \theta d\psi \wedge d\tilde{\theta} + \beta \sin \tilde{\theta} d\tilde{\theta} \wedge d\psi) \end{aligned} \quad (4.60)$$

Like in the previous section, various components of the two-form and the A -vectors above are grouped in combinations that transform under $SL(2, R)$ as vectors:

$$V_\mu^{(i)} = (-\epsilon^{ij} B_\mu^{(j)}, A_\mu^{(i)}, \epsilon^{ij} C_\mu^{(j)}) = (0, A_\mu^{(i)}, \epsilon^{ij} C_\mu^{(j)}), \quad W_{\mu\nu} = (\tilde{c}_{\mu\nu}, \tilde{d}_{\mu\nu}, \tilde{b}_{\mu\nu}) = (\tilde{c}_{\mu\nu}, 0, 0) \quad (4.61)$$

that transform explicitly as,

$$(V^{(i)})' = (-\epsilon^{ij} B^{(j)}, A^{(i)} + \gamma \epsilon^{ij} B^{(j)} - \sigma \epsilon^{ij} C^{(j)}, \epsilon^{ij} C^{(j)}) \quad W' = (\tilde{c} + \gamma \tilde{d}, \tilde{d}, \tilde{b} + \sigma \tilde{d}) \quad (4.62)$$

We will concentrate on the γ -transformation. In this case, the vectors and tensors in (2.14) transform as $(V^{(i)})' = (0, A^{(i)}, \epsilon^{ij}C^{(j)})$, $W' = (\tilde{c}, 0, 0)$. Notice that due to the transformations above, the differentials $D\varphi, D\tilde{\varphi}$ do not change under $SL(3, R)$. Following what we explained in the previous section, we can transform the functions appearing in (4.55). We obtain the new values for $F, e^\phi, C_{12}, \chi, B_{12}$ as given by eq.(2.18). For the $D5$ branes, the γ -transformation leads to nonsingular spaces according to the discussion of the previous section, we will concentrate in the following on transformations that have $\sigma = 0$. Then the new fields become,

$$F' = \frac{F}{(F^2\gamma^2 + 1)}, \quad B'_{12} = \frac{F^2\gamma}{(F^2\gamma^2 + 1)}, \quad e^{2\phi'} = \frac{e^{2\phi}}{(F^2\gamma^2 + 1)}, \quad \chi' = \gamma C_{12}, \quad C'_{12} = \frac{C_{12}}{(F^2\gamma^2 + 1)}. \quad (4.63)$$

Putting all of the things together, we can get the new configuration, that consists in this case of metric, dilaton, axion, RR and NS two forms and four form. The new metric, in string frame is given by,

$$ds^2 = \frac{F'}{\sqrt{\Delta}}(D\varphi - CD\tilde{\varphi})^2 + F'\sqrt{\Delta}(D\tilde{\varphi})^2 + \left(\frac{e^{2\phi'}}{F'}e^{-2\phi}F\right)^{1/3}[Dd\psi^2 + e^\phi(dx_{1,3}^2 + \alpha'g_s N(e^{2h}d\theta^2 + dr^2 + \frac{1}{4}d\tilde{\theta}^2))] \quad (4.64)$$

the NS gauge field is $B_{\varphi, \tilde{\varphi}} = B'_{12}D\varphi' \wedge D\tilde{\varphi}'$, with B'_{12} given in (4.63). The new dilaton and axion are $e^{2\phi'} = \frac{e^{2\phi}}{(1+\gamma^2F^2)}$, $\chi' = -\frac{\gamma}{4}\cos\theta\cos\tilde{\theta}$, the new RR two-form,

$$C^{(2)'} = C'_{12}D\varphi \wedge D\tilde{\varphi} - C^{(1)} \wedge D\varphi - C^{(2)} \wedge D\tilde{\varphi} - \frac{1}{2}(A^{(1)} \wedge C^{(1)} + A^{(2)} \wedge C^{(2)} - \tilde{c}), \quad (4.65)$$

with C'_{12} given in (4.63). Finally, the RR four form is given by

$$(C_4)' = -\frac{1}{2}B'_{12}(\tilde{c} - (A^{(1)} \wedge C^{(1)} + A^{(2)} \wedge C^{(2)})) \wedge D\varphi \wedge D\tilde{\varphi}. \quad (4.66)$$

For completeness, let us write here the expressions for the gauge field strengths H'_3, F'_3, F'_5

$$H'_3 = dB'_{12} \wedge (d\varphi \wedge d\tilde{\varphi} + \alpha d\psi \wedge d\tilde{\varphi} + \beta d\varphi \wedge d\psi) + B'_{12}(d\tilde{\varphi} \wedge d\alpha + d\beta \wedge d\varphi) \wedge d\psi \quad (4.67)$$

$$\begin{aligned} F'_3 &= dC_2 - \chi' H'_3 = \\ &\frac{1}{4}d\psi \wedge (\sin\theta d\theta \wedge d\varphi - \sin\tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi}) + d\varphi \wedge d\tilde{\varphi} \wedge \left(\frac{1}{1+\gamma^2F^2}dC_{12} + C_{12}d\left(\frac{1-\gamma^2F^2}{1+\gamma^2F^2}\right)\right) + \\ &(-\alpha d\tilde{\varphi} + \beta d\varphi) \wedge d\psi \wedge \frac{\gamma^2F^2}{1+\gamma^2F^2}dC_{12} \end{aligned} \quad (4.68)$$

and the RR four form that reads

$$-2C'_4 = B'_{12}(\tilde{c} - \mathcal{A}^{(i)} \wedge C^{(i)}) \wedge D\varphi \wedge D\tilde{\varphi} \quad (4.69)$$

and finally

$$F'_5 = dC'_4 - C'_2 \wedge H'_3 \quad (4.70)$$

What can we learn from this transformed background? One first thing that comes to mind is related to chiral symmetry breaking. Indeed, as shown in [26], it is enough to work with the singular background and the respective RR fields to see explicitly the phenomena of χ SB as a Higgs mechanism for a gauge field used to gauge the isometry corresponding to chiral symmetry (translations in the angle ψ in this case). It should be interesting to do again this computation in this transformed background. The new ingredients to take into account are the new NS and RR fields as shown above. We believe that the anomaly will be the same because, as an anomaly, it can only be affected by the mass less fields. In our case, the transformed background only takes into account changes in the dynamics of the massive KK modes. It should be interesting to see this argument working explicitly.

4.2 Transformations in the R-symmetry Directions

In the previous section, we performed rotations that commuted with the R symmetry of the field theory. Indeed, the R -symmetry of N=1 SYM is represented in the background studied in in the previous subsection by changes in the angle ψ . The rotations done above, preserve SUSY, since they do not involve the angle ψ . In this section, we will concentrate on rotations taking the torus, to be composed of $\psi, \tilde{\varphi}$; this will break SUSY. Notice that in this case, the dual field theory to the transformed solution will not be a dipole theory, but a theory where phases has been added to the interaction terms. Also, notice that we can do this in the singular solution only, because in the desingularized solution, we do not have the invariance $\psi \rightarrow \psi + \epsilon$.

So, to remind the reader, let us write explicitly the 10 metric (3.37) in the singular case (string frame is used)

$$ds_{10}^2 = \alpha' g_s N e^\phi \left[\frac{1}{\alpha' g_s N} dx_{1,3}^2 + e^{2h} (d\theta^2 + \sin^2 \theta d\varphi^2) + dr^2 + \frac{1}{4} (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2 + (d\psi + \cos \theta d\varphi + \cos \tilde{\theta} d\tilde{\varphi})^2) \right], \quad (4.71)$$

We notice that this metric is already written in the form that we want it. Indeed, comparing we can see that

$$F = \frac{\alpha' g_s N}{4} e^\phi \sin \tilde{\theta}, \quad \Delta = \sin^2 \tilde{\theta}, \quad D\varphi_1 = D\psi = d\psi + \cos \theta d\varphi, \quad D\varphi_2 = D\tilde{\varphi} = d\tilde{\varphi}, \quad C = -\cos \tilde{\theta}. \quad (4.72)$$

and

$$g_{\mu\nu} dx^\mu dx^\nu = e^{-2\phi/3} F^{1/3} [\alpha' g_s N e^\phi (\frac{1}{\alpha' g_s N} dx_{1,3}^2 + e^{2h} (d\theta^2 + \sin^2 \theta d\varphi^2) + dr^2 + \frac{d\tilde{\theta}^2}{4})] \quad (4.73)$$

After doing some computations, see Appendix D, we can see that the transformed string metric reads,

$$(ds_{string}^2)' = e^{2(\phi'-\phi)/3} \left(\frac{F}{F'}\right)^{1/3} \alpha' g_s N e^\phi \left[\frac{1}{\alpha' g_s N} dx_{1,3}^2 + e^{2h} (d\theta^2 + \sin^2 \theta d\varphi^2) + dr^2 + \frac{1}{4} d\tilde{\theta}^2 \right] + (F' \sqrt{\Delta} (D\tilde{\varphi})^2 + \frac{F'}{\sqrt{\Delta}} (D\psi + \cos \tilde{\theta} D\tilde{\varphi})^2) \quad (4.74)$$

The new RR and NS fields are explicitly written in the Appendix D. The reader can check that there is no five form generated. Indeed, there is no C_4 generated by the $SL(3, \mathbb{R})$ rotation and besides the term $C_2 \wedge H_3$ does not contribute.

This new solution is expected to be non-SUSY, and an occasional resolution of the singularity at $r = 0$ could give a dual to YM theory. One might think about, for example, a resolution by turning on a black hole, dual to YM at finite temperature. This should be very interesting to solve.

Like in the examples of the D5 brane, doing this transformation is changing quantities like for example the Ricci scalar, but the place where $\alpha' R_{eff}$ diverges ($r \rightarrow 0$ with a divergence of the form $\alpha' R_{eff} \approx r^{-7/4}$) are the same before and after the transformation. The divergent structure does not seem to become worse by the effect of the gamma transformation.

4.3 The General β Transformation

Transformations of the singular solution *i.e.* (3.37) for $a = 0$ where the torus is chosen transverse to the D5 brane have the interesting property of being regular *both for real and imaginary parts of β* . Therefore it is tempting to perform the general $\beta = \gamma + i\sigma$ transformation. Here we, show the regularity of the general transformed solution explicitly, by presenting the resulting geometry.

The only new feature that one introduces to the results of section 4.2 is that, turning on σ changes the connections $\mathcal{A}^{(i)}$ according to (2.14). The transformed connections are:

$$\mathcal{A}^{(1)'} = \mathcal{A}^{(1)} = \cos \tilde{\theta} d\varphi, \quad \mathcal{A}^{(2)'} = \mathcal{A}^{(2)} + \sigma C^{(1)} = \frac{\alpha' N}{4} \sigma \cos \theta \varphi, \quad (4.75)$$

Details are given in Appendix D. Using (2.18), we find the the string frame metric as,

$$ds_{string}^2 = U_{st} \alpha' g_s N e^\phi \left[\frac{1}{\alpha' g_s N} dx_{1,3}^2 + e^{2h} (d\theta^2 + \sin^2 \theta d\varphi^2) + dr^2 + \frac{1}{4} d\tilde{\theta}^2 \right] \quad (4.76)$$

$$+ F' \sqrt{\Delta} (d\tilde{\varphi} + \sigma \frac{\alpha' N}{4} \cos \theta d\varphi)^2 + \frac{F'}{\sqrt{\Delta}} (d\psi + \cos \theta d\varphi + \cos \tilde{\theta} d\tilde{\varphi} + \sigma \frac{\alpha' N}{4} \cos \tilde{\theta} \cos \theta d\varphi)^2 \quad (4.77)$$

Here the new metric functions are given by (2.18) and (2.21) and they generally depend on both γ and σ . Eqs. (2.18) also determine how the RR two-form and the dilaton transforms and shows the new fields that are generated by the general transformation.

It should be interesting to study the fate of the symmetry $\psi \rightarrow \psi + \epsilon$ after this transformation. This symmetry, in the case of $N = 1$ SYM was associated with chiral rotations [26]. In this case, our theory does not have massless fermions. So, the symmetry corresponds to some kind of flavor symmetry within the KK sector. It should be interesting to see if this symmetry gets broken and the way studied in [26].

Now that we gained some insight with this type of configurations, let us study the $U(1) \times U(1)$ transformation in the case of the non-singular background.

5 Deformation of the Non-singular $\mathcal{N} = 1$ Theory

Now, let us discuss the non-singular case, $a(r) \neq 0$. The type of problems we have in mind that might be tackled, are related to previous computations of non-perturbative field theory aspects from a supergravity perspective. Indeed, in some cases, it was not clear if the result of this computation was afflicted by the presence of the massive KK modes. So, finding the transformed background and re-doing the computations with it, might help us improve this situation, since as we remarked above, both backgrounds do differ in the fact that their dual theories have different dynamics for the KK modes, results that depend on the transformation parameter γ will indicate the presence of effects of the KK modes.

We first write the full metric in (3.37) in the following appropriate form for the $SL(3, R)$ rotation in which the torus and the transverse parts have been separated as follows:

$$\begin{aligned}
ds_{string}^2 = & e^\phi [dx_{1,3}^2 + \alpha' g_s N dr^2] + D_1 d\psi^2 + D_2 d\theta^2 + D_3 d\tilde{\theta}^2 + E_1 d\theta d\tilde{\theta} + \\
& E_2 d\theta d\psi + E_3 d\tilde{\theta} d\psi + \frac{F}{\sqrt{\Delta}} [d\varphi + (\alpha_1 - C\beta_1)d\theta + (\alpha_2 - C\beta_2)d\tilde{\theta} + (\alpha_3 - C\beta_3)d\psi - Cd\tilde{\varphi}]^2 + \\
& F\sqrt{\Delta}(d\tilde{\varphi} + \beta_1 d\theta + \beta_2 d\tilde{\theta} + \beta_3 d\psi)^2
\end{aligned} \tag{5.78}$$

To simplify the notation let us define,

$$f = 4e^{2h} \sin^2 \theta + \cos^2 \theta + a^2 \sin^2 \theta, \quad g = a \sin \theta \sin \tilde{\theta} \cos \psi - \cos \theta \cos \tilde{\theta}. \tag{5.79}$$

Then, various functions in (5.78) are given as,

$$\begin{aligned}
F &= \frac{\alpha' g_s N e^\phi}{4} \sqrt{f - g^2}, \quad \Delta = \frac{f - g^2}{f^2}, \quad C = \frac{g}{f}, \\
\beta_1 &= \frac{f}{f - g^2} a \sin \psi \sin \tilde{\theta}, \quad \beta_2 = \frac{g}{f - g^2} a \sin \psi \sin \theta, \quad \beta_3 = \frac{f \cos \tilde{\theta} + g \cos \theta}{f - g^2} \\
\alpha_1 &= \frac{a \sin \tilde{\theta} \sin \psi g}{f - g^2}, \quad \alpha_2 = \frac{a \sin \theta \sin \psi}{f - g^2}, \quad \alpha_3 = \frac{\cos \theta + g \cos \tilde{\theta}}{f - g^2}
\end{aligned}$$

$$\begin{aligned}
D_1 &= \frac{\alpha' g_s N e^\phi}{4(f-g^2)} (f \sin^2 \tilde{\theta} - g^2 - \cos^2 \theta - 2g \cos \theta \cos \tilde{\theta}), \\
D_2 &= \frac{\alpha' g_s N e^\phi}{4} (a^2 + 4e^{2h} - \frac{f}{f-g^2} a^2 \sin^2 \psi \sin^2 \tilde{\theta}) \\
D_3 &= \frac{\alpha' g_s N e^\phi}{4} (1 - \frac{a^2 \sin^2 \theta \sin^2 \psi}{f-g^2}), \quad E_1 = \frac{a}{2} \alpha' g_s N e^\phi (\cos \psi - \frac{g}{f-g^2} a \sin^2 \psi \sin \theta \sin \tilde{\theta}) \\
E_2 &= -\frac{\alpha' g_s N e^\phi a \sin \psi \sin \tilde{\theta} (f \cos \tilde{\theta} + g \cos \theta)}{2(f-g^2)}, \quad E_3 = -\frac{\alpha' g_s N e^\phi a \sin \theta \sin \psi}{2(f-g^2)} (\cos \theta + g \cos \tilde{\theta}).
\end{aligned} \tag{5.80}$$

Now, let us focus on the RR two form. Like before, it is useful to define four one forms as $\mathcal{A}^{(i)}, \mathcal{C}^{(i)}$, $i = 1, 2$

$$\mathcal{A}^{(1)} = \alpha_1 d\theta + \alpha_2 d\tilde{\theta} + \alpha_3 d\psi, \quad \mathcal{A}^{(2)} = \beta_1 d\theta + \beta_2 d\tilde{\theta} + \beta_3 d\psi, \tag{5.81}$$

and

$$\mathcal{C}^{(1)} = C_\theta^{(1)} d\theta + C_{\tilde{\theta}}^{(1)} d\tilde{\theta} + C_\psi^{(1)} d\psi, \quad \mathcal{C}^{(2)} = C_\theta^{(2)} d\theta + C_{\tilde{\theta}}^{(2)} d\tilde{\theta} + C_\psi^{(2)} d\psi, \tag{5.82}$$

and the two form $\tilde{c}_{\mu\nu} dx^\mu \wedge dx^\nu$, with components $\tilde{c}_{\theta\tilde{\theta}}, \tilde{c}_{\theta\psi}$, and all others being zero. In order to reproduce the RR two-form potential given in (3.43) in the following form,

$$C^{(2)} = C_{12} D\varphi \wedge D\tilde{\varphi} - C^{(1)} \wedge D\varphi - C^{(2)} \wedge D\tilde{\varphi} - \frac{1}{2} (A^{(1)} \wedge C^{(1)} + A^{(2)} \wedge C^{(2)} - \tilde{c}); \tag{5.83}$$

the quantities defined above are determined as,

$$\begin{aligned}
C_{12} &= \frac{1}{4} (a(r) \sin \theta \sin \tilde{\theta} \cos \psi - \cos \theta \cos \tilde{\theta}), \\
\mathcal{C}^{(1)} &= (-\frac{\psi}{4} \sin \theta - C_{12} \beta_1) d\theta - (\frac{1}{4} a(r) \sin \theta \sin \psi + C_{12} \beta_2) d\tilde{\theta} - \beta_3 C_{12} d\psi, \\
\mathcal{C}^{(2)} &= (\frac{1}{4} a(r) \sin \tilde{\theta} \sin \psi + C_{12} \alpha_1) d\theta + (\frac{\psi}{4} \sin \tilde{\theta} - C_{12} \alpha_2) d\tilde{\theta} + \alpha_3 C_{12} d\psi, \\
\tilde{c} &= \tilde{c}_{\theta\tilde{\theta}} d\theta \wedge d\tilde{\theta} + \tilde{c}_{\theta\psi} d\theta \wedge d\psi + \tilde{c}_{\tilde{\theta}\psi} d\tilde{\theta} \wedge d\psi
\end{aligned} \tag{5.84}$$

with

$$\begin{aligned}
\tilde{c}_{\theta\tilde{\theta}} &= \alpha_1 \alpha_2 + \beta_1 \beta_2 + C_{12} (\alpha_1 \beta_2 - \alpha_2 \beta_1) - \frac{1}{4} \psi \sin \tilde{\theta} \beta_1 + \frac{1}{4} a \sin \theta \sin \psi \alpha_1 \\
&\quad - \frac{1}{4} a \cos \psi + \frac{1}{4} a \beta_2 \sin \psi \sin \tilde{\theta}, \\
\tilde{c}_{\theta\psi} &= \alpha_1 \alpha_3 + \beta_1 \beta_3 - C_{12} (\alpha_1 \beta_3 - \alpha_3 \beta_1) - \frac{1}{4} \psi \sin \theta \alpha_3 + \frac{1}{4} a \sin \psi \sin \tilde{\theta} \beta_3, \\
\tilde{c}_{\tilde{\theta}\psi} &= \alpha_2 \alpha_3 + \beta_2 \beta_3 - C_{12} (\alpha_2 \beta_3 - \alpha_3 \beta_2) + \frac{1}{4} \psi \sin \tilde{\theta} \beta_3 - \frac{1}{4} a \sin \psi \sin \theta \alpha_3.
\end{aligned} \tag{5.85}$$

One can check that in the limit $a(r) \rightarrow 0$ one recovers the result in (4.55).

So, after the $SL(3, R)$ rotation we obtain a metric that reads

$$\begin{aligned}
(ds_{string}^2)' &= \left(\frac{e^{2(\phi'-\phi)}F}{F'}\right)^{1/3} \left(e^\phi [dx_{1,3}^2 + \alpha' g_s N dr^2] + D_1 d\psi^2 + D_2 d\theta^2 + D_3 d\tilde{\theta}^2 + E_1 d\theta d\tilde{\theta} + \right. \\
&E_2 d\theta d\psi + E_3 d\tilde{\theta} d\psi \left. \right) + \frac{F'}{\sqrt{\Delta}} [d\varphi + (\alpha_1 - C\beta_1)d\theta + (\alpha_2 - C\beta_2)d\tilde{\theta} + (\alpha_3 - C\beta_3)d\psi - Cd\tilde{\varphi}]^2 \\
&+ F'\sqrt{\Delta} (d\tilde{\varphi} + \beta_1 d\theta + \beta_2 d\tilde{\theta} + \beta_3 d\psi)^2
\end{aligned} \tag{5.86}$$

The RR and NS fields will transform according to the rules discussed in previous sections. Notice that as happened before, the γ transformation leaves the one forms $\mathcal{A}^{(i)}$, $C^{(i)}$ invariant. This is a good point to notice that the torus given by the cycle of constant $\theta, \tilde{\theta}, \psi, r, x$, has volume $V_{T^2} = F' = \frac{F}{1+\gamma^2 F^2}$ after the transformation and that only vanished if F vanishes or becomes infinite. So, if the original geometry is nonsingular, the transformed one also is. In the points where the volume of the torus shrinks, it happens that it does in a way $B'_{12} = Re[\tau] \rightarrow 0$ when $\sqrt{g'} = Im[\tau] \rightarrow 0$, thus the metric satisfies the criteria in [14] for nonsingularities.

5.1 Confinement

The first check that this new solution should pass is the confinement. As clearly explained for example in [27], the expectation value of the Wilson loop can be computed by calculating the Nambu-Goto action for a string that is connected to a probe brane at infinity and explores the IR region ($r=0$) of the background. The criteria for confinement is that if the following combination (that can be intuitively associated with the tension of the confining string)

$$T_s = \sqrt{g_{tt}g_{xx}} \Big|_{r=0} \tag{5.87}$$

is non-vanishing then there is linear confinement. The ‘‘QCD string’’ tension, T_s defined above, is non vanishing. In the original background, the value of $T_s = e^{\phi(0)}$. After the transformation, we can see that the string tension is given by

$$T'_s = \sqrt{g'_{tt}g'_{xx}} = U_{st} \sqrt{g_{tt}g_{xx}} = T_s \tag{5.88}$$

where U_{st} is defined in (2.21). We note that the tension would change if we perform a σ -transformation. It should be interesting to understand the effects of the sigma transformation in more detail, since they seem to alter important aspects of the gauge theory.

5.2 The Beta Function

Let us compute the beta function, following the lines of [28], [29]. We first revise their steps and add some comments that might make the derivation more clear. In order to compute

beta functions, one needs to define a SUSY cycle where to wrap a $D5$ brane at finite distance from the origin. It was found in [22] that such cycle exists for an infinite value of the radial constant. The cycle is given by the following identifications ¹⁰

$$\theta = \tilde{\theta}, \quad \varphi = 2\pi - \tilde{\varphi}, \quad \psi = \pi, 3\pi \quad (5.89)$$

Notice that, since this is a calibrated (SUSY) cycle for large values of the radial coordinate, we should be considering the “abelian version” ($a(r) = 0$) of the background. We proceed with the full nonsingular solution, but one should keep in mind that this is a good approximation only for $r \rightarrow \infty$. Then, one needs to introduce a relation between the radial coordinate r , and the energy scale of the theory. We will take the following relation [30], [28], that identifies the gaugino condensate with the function $a(r)$.

$$a(r) = \frac{\Lambda^3}{\mu^3} = \langle \lambda\lambda \rangle \quad (5.90)$$

This relation has an intuitive explanation since the gaugino condensate, like the turning-on of the function $a(r)$ are phenomena that occur in the IR, that is at small values of the energy/radial coordinate. This identification (5.90), is indeed the reason for doing the calculation in the non-singular background, even when the cycle we use is supersymmetric only for large values of r .

Using (5.89) one defines the coupling constant as

$$\frac{1}{g_{YM}^2} = \frac{1}{(2\pi)^2 g_s \alpha'} \int_0^{2\pi} d\varphi \int_0^\pi d\theta e^{-\phi} (g^{xx})^2 \sqrt{\det G_6}. \quad (5.91)$$

This gives,

$$\frac{\pi^2}{4g_{YM}^2 N} = (4e^{2h} + (a(r)^2 - 1)^2) \quad (5.92)$$

Therefore one obtains the beta function as

$$\beta = \frac{dg_{YM}}{d\log(\mu/\Lambda)} = -\frac{g_{YM}^3 N}{8\pi^2} \frac{d(\frac{\pi}{4g^2 N})}{dr} \left(\frac{d(-1/3\text{Log}(\mu/\Lambda))}{dr} \right)^{-1}. \quad (5.93)$$

Expanding the result for large values of the radial coordinate and using an expansion for large r of (5.92) one gets,

$$\beta = -\frac{3g_{YM}^3 N}{8\pi} \frac{1}{(1 - \frac{g_{YM}^2 N}{8\pi})} \quad (5.94)$$

If one keeps higher orders in the r expansion, one gets extra terms that in [28] were attributed to fractional instantons.

¹⁰notice that also the cycle $\theta = \pi - \tilde{\theta}$, $\varphi = \tilde{\varphi}$, $\psi = 0, 2\pi$ is SUSY at large values of r

The reader may wonder what is the origin of these extra effects. On one hand, one might think that they are a pure $\mathcal{N} = 1$ SYM effects. Or might argue that they are an effects due to the KK modes of the field theory to which this supergravity solution is dual. Apart from these two possibilities, one can think that they are spurious effects coming from the fact that for small values of r in the expansion of the quantities above, the supersymmetry is broken, since the cycle (5.89) is no longer SUSY.

In order to discard some of these alternatives, we compute the beta function in the γ deformed non-singular background. If these extra terms that seem to modify the original NSVZ result are effects of the KK modes, then by our general philosophy in this paper, the beta function should be different in the deformed theory.

Let us take the same cycle in the transformed solution (5.86) ¹¹ The relevant six dimensional part of the metric is

$$(ds_{6D}^2)' = [e^\phi(dx_{1,3}^2 + g_s N \alpha' dr^2) + (D_2 + D_3 + E_1)d\theta^2] + \frac{F'}{\sqrt{\Delta}}((1 + C)d\varphi + (\alpha_2 - C\beta_2)d\theta)^2 + F'\sqrt{\Delta}(-d\varphi + (\beta_1 + \beta_2)d\theta)^2 \quad (5.95)$$

We note that at the special cycle the metric components reduce to the original undeformed components except than the change from F to F' . This gives a factor of F'/F which exactly cancels out the change in the dilaton in (5.91). So, computing the determinant and defining the coupling as before,

$$\frac{1}{g_{YM}^2} = \frac{1}{(2\pi)^2 g_s \alpha'} \int_0^{2\pi} d\varphi \int_0^\pi d\theta e^{-\phi'} (g'^{xx})^2 \sqrt{\det G_6}. \quad (5.96)$$

Using the fact that

$$e^{\phi-\phi'} = (1 + \gamma^2 F^2)^{1/2} \quad (5.97)$$

the coupling reads

$$\frac{\pi^2}{4g_{YM}^2 N} = [4e^{2h} + (a^2 - 1)^2]. \quad (5.98)$$

This is precisely the same as (5.92). Then, one repeats the procedure in eq. (5.93) [28], [29] and obtains the same result for the beta function in the deformed theory.

One can repeat the same computation for the theta parameter of Yang-Mills. The essentials of the computation do not change. θ_{YM} is the same as in the undeformed theory.

We learn that the computation for beta function seems robust under the deformations. Indeed, the beta function is a field theory result, that should be independent of the KK modes dynamics. So, this transformed solution that only changes the KK sector of the dual field theory, does not produce any effect in the result (5.93). On the other hand, the fact

¹¹We believe that by general properties of the γ transformation this cycle is also supersymmetric in the deformed geometry. It should be interesting to verify this by an explicit BPS calculation.

that we got the same result is perhaps indicating that the result should only be taken to first order in the large r expansion thus implying that there are no “fractional instanton” corrections. Or may be, the “fractional instantons” are a genuine field theory effects; but the result of these computations above, make sure that the KK modes have no relation to them whatsoever. Finally, we note that it should of course be of great interest to find the SUSY cycle that computes the beta function in the IR.

5.3 KK Modes and the Domain Wall Tension

There are two types of KK modes, those that are proportional to the volume of an S^2 in the geometry and those that are proportional to the volume of a three cycle, the first type are the ‘gauge’ KK modes. On the other hand, the volume of the non-vanishing three cycle at the IR (at $r = 0$) is inversely related to the masses of the “geometric” KK modes in the theory. Therefore it is very interesting to see whether or not these volumes are changed by the transformation. If the volume decreases under the transformation, it would be a non-trivial improvement of the model as the undesired KK degrees of freedom would have better decoupling. Let us start with the ‘geometric’ KK modes. The three cycle in the original theory is given by,

$$\theta = \varphi = r = \vec{x} = 0. \quad (5.99)$$

In the original metric (3.37), this volume is, (in the string frame),

$$V = 2\pi^2(\alpha'g_sNe^{\phi(0)})^{\frac{3}{2}}. \quad (5.100)$$

Now, we consider the same cycle in the transformed geometry (5.86). Using eqs. (5.81) the reader can see that one obtains

$$\begin{aligned} Vol(S^{3'}) &= \pi^2(\alpha'g_sNe^{\phi_0})^{3/2} \int_0^\pi d\tilde{\theta} \frac{\sin \tilde{\theta}}{1 + \mu^2 \sin^2 \tilde{\theta}} \\ &= \pi^2(\alpha'g_sNe^{\phi_0})^{3/2} \left(\frac{2\text{arctanh}(\mu/\sqrt{1 + \mu^2})}{\mu\sqrt{1 + \mu^2}} \right) \\ &\propto \pi^2(\alpha'g_sNe^{\phi_0})^{3/2} \left(2 - \frac{4}{3}\mu^2 + \dots \right), \end{aligned}$$

where $16\mu^2 = \gamma^2(\alpha'g_sNe^{\phi_0})^2$. It is useful to consider the ratio,

$$\left(\frac{M'_{KK}}{M_{KK}} \right)^2 = \left(\frac{VolS^3}{VolS^{3'}} \right)^{2/3} = \left(\frac{\mu\sqrt{1 + \mu^2}}{2\text{arctanh}[\mu/\sqrt{1 + \mu^2}]} \right)^{2/3}. \quad (5.101)$$

So, we see that the mass of these ‘geometric’ KK modes indeed increase improving the decoupling between the KK and the pure SYM sector.

Now, let us analyze the mass of the ‘gauge’ KK modes, that are inversely proportional to the volume of some S^2 defined in the geometry. Let us define the two cycle as,

$$\tilde{\theta} = \tilde{\varphi} = \psi = x = r = 0, (\theta, \varphi) \quad (5.102)$$

Computing the line element of this two cycle one gets (once again, we concentrate on the γ transformation)

$$ds^2 = (D_2 + \beta_1^2 F' \sqrt{\Delta}) d\theta^2 + \frac{F'}{\sqrt{\Delta}} d\varphi^2 \quad (5.103)$$

so, the volume of this two cycle near $r = 0$ is

$$\begin{aligned} Vol(S^{2'}) &= \frac{\pi \alpha' g_s N e^{\phi_0}}{2} \int_0^\pi \frac{d\theta}{\sqrt{1 + \frac{1}{16} (\gamma \alpha' g_s N e^{\phi_0} \sin \theta)^2}} \rightarrow \\ Vol(S^{2'}) &\approx \frac{\pi \alpha' g_s N e^{\phi_0}}{2} \left(1 - \frac{\gamma^2 \alpha' (g_s N)^2 e^{2\phi_0}}{64}\right) \end{aligned} \quad (5.104)$$

again, it is convenient to present the quotient,

$$\left(\frac{m'_{kk}}{m_{kk}}\right)^2 = \left(\int_0^\pi \frac{d\theta}{\sqrt{1 + \frac{1}{16} (\gamma \alpha' g_s N e^{\phi_0} \sin \theta)^2}}\right)^{-1} \approx 1 + \frac{1}{16} (\gamma \alpha' g_s N e^{\phi_0})^2 \quad (5.105)$$

So, we see that the mass of the ‘gauge’ KK modes also increases when this transformation is performed. We would like to stress again that under the σ transformation, the volumes above indeed change. However, as the σ transformation produces irregularities, the role of the σ modification of the original model is unclear ¹²

Another interesting quantity to compute is the tension of a Domain Wall. A domain wall is thought as a $D5$ brane that wraps the three cycle (5.99) and extends in $(2 + 1)$ directions of spacetime. The way of computing the tension of this wall is by computing the coefficient in front of the Born-Infeld action for a $D5$ as indicated above.

$$S_{wall} = \int d^3x \left(\int d\Omega_3 e^{-\phi'} \sqrt{\det G_6} \right) + CS \quad (5.106)$$

from the term in parentheses we read the tension of the domain wall, to be

$$T_{wall} = \int_0^{4\pi} \int_0^{2\pi} \int_0^\pi d\tilde{\theta} d\tilde{\varphi} d\psi e^{-\phi' + 2\phi} (e^{2(\phi' - \phi)} \frac{F}{F'} \alpha' g_s N)^{1/2} \frac{F}{(1 + \gamma^2 F^2)} \Big|_{r=0} \quad (5.107)$$

Using (5.97) we see that the result changes and the chane is explicitly given as,

$$\frac{T'}{T} = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{4e^{-\phi_0}}{\gamma \alpha' g_s N}\right). \quad (5.108)$$

¹²We are grateful to Juan Maldacena for suggesting that redefinitions of the periodicities of the original cycles may improve the situation.

One should perhaps interpret this as an effect of the KK modes in field theory observables because the cycle we are choosing (5.99) is not the appropriate one for computing domain walls. This point deserves more study.

In this line, it would be nice to analyze what happens to the computation of baryonic strings as D3 branes wrapping the previous 3 cycle done in the paper [31].

The result of the last two subsections is basically that under the γ transformations many field theory aspects of N=1 SYM are not changed when computed from the transformed background, while the KK modes change their masses. This, on one hand is reassuring of the conjectured duality, because as we stressed many times, the transformed background only differs in the dynamics of the KK modes. It is of interest to see an example where some changes in the dynamics are indeed expected to happen. This example is provided by pp-waves. The objects called annulons and studied in [20], [21] are constructed to be heavy composites of many KK modes (this composite is called ‘hadron’ in the papers mentioned above). These objects are studied in a plane wave approximations of the geometry. So, in order to see real changes in dynamics of KK modes, in the next section, we will study these composites in the transformed backgrounds.

6 PP-waves of the Transformed Solutions

PP-wave limits of various transformed solutions are quite interesting. The first example of this kind was discussed by Lunin and Maldacena [14] where they considered a particular PP-wave limit of the β transformed $AdS_5 \times S^5$ geometry; it is quite amusing to observe that with different arguments and objectives Niarchos and Prezas found the same plane wave some years ago [23]. In most cases, one can indeed obtain PP-waves that lead to quantizable string actions. Here, we would like to first make some observations about the general properties that are satisfied by the PP-waves of the transformed geometries. Then, we exemplify these properties by studying two simple cases, those of flat the $D5$ brane and singular $D5$ wrapped on S^2 . Finally we discuss the PP-wave associated to the transformed non-singular geometry that we obtained in section 4.2.

We restrict our attention to the pp-wave limits that are obtained by the scaling of the *same* coordinates, before and after the transformation. The motivation for this is to study the effects of the $SL(3, R)$ transformation to the pp-wave limits. In general one may also ask whether or not the transformed geometry admits interesting pp-wave limits other than the pp-waves of the original geometry. A separate issue is to apply the $SL(3, R)$ transformation to the pp-wave itself and seek for new quantizable geometries. We note that generally the pp-wave limit does not commute with the $SL(3, R)$ transformation. We do not investigate these interesting issues further in this paper.

6.1 General Properties

For all of the geometries that we consider, the original metric is proportional to e^ϕ (in the string frame) and the metric before the $SL(3, R)$ transformation can generally be written in the following form:

$$ds^2 = \frac{F}{\sqrt{\Delta}} (D\varphi_1 - CD\varphi_2)^2 + F\sqrt{\Delta}(D\varphi'_2)^2 + e^\phi (-dt^2 + d\vec{x}^2 + p(r)dr^2 + q(r)d\Omega^2) \quad (6.109)$$

where r is a radial coordinate p and q are some given functions and Ω is a compact space. It is important to notice that, the function F is proportional to e^ϕ . We allow for singular geometries, because it is well-known that singularities are usually smoothed out in the pp-wave limit.

Suppose that the radial coordinate runs from r_0 to infinity. We define $e^{\phi(r_0)} = R^2$. A general class of pp-waves are obtained by scaling some of the coordinates as follows:

$$r \rightarrow r_0 + \frac{r}{R}, \quad \vec{x} \rightarrow \frac{\vec{x}}{R}, \quad \phi_i \rightarrow \frac{\phi_i}{R}, \quad R \rightarrow \infty,$$

where ϕ_i are some angular coordinates on Ω or the along torus.

Since $F \sim e^\phi$, one observes that, in this limit

$$F \sim R^2 f\left[\frac{r}{R}, \frac{\phi_i}{R}, \dots\right]$$

where $f[\dots]$ is a function of the indicated variables. We assume that this function behaves like $\mathcal{O}(1)$ or $\mathcal{O}(R^{-1})$. Therefore the function F blows up in the limit as $F \sim R^2$ or $F \sim R$. Also, in many cases, $\sqrt{\Delta} \sim \mathcal{O}(1)$ or $\mathcal{O}(R^{-1})$ in this limit. Thus, we have,

$$F \sim R^t, \quad t \in \{1, 2\}, \quad \sqrt{\Delta} \sim R^{-p}, \quad p \in \{0, 1\}. \quad (6.110)$$

Now, consider applying the same pp-wave limit to the geometry after the transformation, *i.e.* to the new geometry that is given by (2.20). We limit our attention only to γ transformations for simplicity. In this case, $U_{st} = 1$ and F in (6.109) is replaced by,

$$F' = \frac{F}{1 + \gamma^2 F^2}.$$

From the above scaling behaviour of the function F , we see that the torus in the transformed geometry shrinks to zero size, hence yields a singular pp-wave limit *unless one also scales γ appropriately*. Consider the following scaling:

$$\gamma R^s = \tilde{\gamma} = \text{const.}, \quad s > 0$$

If in addition to regularity of the limit, one asks for a linear pp-wave that has the potential of being quantizable, one would also like the denominator in F' be expanded in powers of $\tilde{\gamma}$.

Then, a glance at (6.109) and (6.110) leads us to consider the following, most appropriate scaling:

$$s = \frac{1}{2}(3t + p). \quad (6.111)$$

Let us denote the pp-wave limit of the original geometry by ds_{pp}^2 and the same limit after the $SL(3, R)$ transformation by $ds_{pp}^{\prime 2}$. Then, if we fix s as above, in the $R \rightarrow \infty$ limit we obtain the following general result:

$$ds_{pp}^{\prime 2} = ds_{pp}^2 - \tilde{\gamma}^2 ds_{pp\text{-torus}}^2. \quad (6.112)$$

Here the second term is the pp-wave limit of only the torus part of the geometry. There is no guarantee that (6.111) will allow for a linear pp-wave, however this is—in some sense—the best one can do in order to avoid non-linearity in the pp-wave. In addition—as we assume that the original metric has a nice, linear pp-wave limit—we are able to isolate the non-linearity—if any—in the second term in (6.112).

6.2 PP-wave of the Transformed Flat $D5$

Let us consider the flat $D5$ geometry, (2.3), as our first example. A smooth linear pp-wave limit can be obtained by performing the following coordinate transformations, (we define $\eta^2 = \alpha' g_s N$)

$$r \rightarrow r_0 + \frac{r}{R}, \quad \vec{x}_5 \rightarrow \frac{\vec{x}_5}{R}, \quad \tilde{\theta} = \frac{\tilde{\theta}}{R} \quad (6.113)$$

We also define

$$dt = dx^+ + \frac{dx^-}{L^2}, \quad d\psi + d\tilde{\varphi} = \frac{2}{\eta}(dx^+ - \frac{dx^-}{L^2}), \quad e^{\phi(r_0)} = R^2. \quad (6.114)$$

The limit $R \rightarrow \infty$ yields the following geometry,

$$ds_{pp1}^2 = -4dx^+ dx^- + d\vec{x}_5^2 + \eta^2 dr^2 + \frac{\eta^2}{4}(d\tilde{\theta}^2 + \tilde{\theta}^2 d\tilde{\varphi}^2) - \frac{\eta^2}{4}\tilde{\theta}^2 dx^+ d\tilde{\varphi}. \quad (6.115)$$

Now let us consider applying the same pp-wave limit, (6.113), to the case of the transformed $D5$ brane that we discussed in section 3.3. We have the geometry, given by (2.24) and (2.25). We consider the regular case, *i.e.* $\sigma = 0$. In this case we see from (2.9) that

$$F \rightarrow \frac{\eta^2}{4} R \tilde{\theta}, \quad \sqrt{\Delta} = \frac{\tilde{\theta}}{R}. \quad (6.116)$$

Therefore $t = p = 1$ in (6.110). From (6.111) we see that the appropriate scaling of γ as $s = 2$ *i.e.* $\tilde{\gamma} = \gamma R^2 = \text{fixed}$. Then from (6.112) one obtains the pp-wave that results from the transformed $D5$ solution as follows:

$$ds_{pp1}^{\prime 2} = ds_{pp1}^2 - \tilde{\gamma}^2 \left(\frac{\eta^2}{4}\right)^3 \tilde{\theta}^2 dx^{+2}, \quad (6.117)$$

where ds_{pp1}^2 is given in (6.115). We see that the $SL(3, R)$ transformation produced a simple γ - correction in the original pp-wave geometry. This fact was observed in [14] in case of the deformed $AdS_5 \times S^5$. The string theory is quadratic and easily quantizable and one observes that the bosonic $\tilde{\theta}$ field receives a correction to its mass that is proportional to $\tilde{\gamma}^2$.

6.3 PP-wave of the Transformed D5 on S2

Let us now apply our general discussion to a slightly more complicated example: The singular geometry of $D5$ brane wrapping an S^2 . The metric is,

$$ds^2 = e^\phi \left[dx_{1,3}^2 + \eta^2 (dr^2 + r(d\theta^2 + \sin^2 \theta d\varphi^2)) + \frac{1}{4} (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2 + (d\psi + \cos \theta d\varphi + \cos \tilde{\theta} d\tilde{\varphi})^2) \right] \quad (6.118)$$

This geometry is supplied with a dilaton $e^{2\phi} = \frac{e^{2\phi_0+2r}}{\sqrt{r}}$ and an RR three form. We take a similar geodesic as the one above (6.113):

$$r \rightarrow r_0 + \frac{r}{R}, \quad \theta \rightarrow \frac{\theta}{R}, \quad \tilde{\theta} \rightarrow \frac{\tilde{\theta}}{R}, \quad \vec{x}_3 \rightarrow \frac{\vec{x}_3}{R}, \quad dt = dx^+ + \frac{dx^-}{R^2}, \quad d\psi + d\varphi + d\tilde{\varphi} = \frac{2}{\eta} (dx^+ - \frac{dx^-}{R^2}) \quad (6.119)$$

where R is again defined by $e^{\phi(r_0)} = R^2$. The following linear pp-wave geometry follows from the $R \rightarrow \infty$ limit:

$$ds_{pp2}^2 = -4dx^+ dx^- + d\vec{x}_3^2 + \eta^2 dr^2 + \eta^2 r_0 (d\theta^2 + \theta^2 d\varphi^2) + \frac{\eta^2}{4} (d\tilde{\theta}^2 + \tilde{\theta}^2 d\tilde{\varphi}^2) - (\theta^2 d\varphi + \tilde{\theta}^2 d\tilde{\varphi}) dx^+ \quad (6.120)$$

Now we consider the γ transformation where we take the same torus as in section 5.2. F and $\sqrt{\Delta}$ behaves exactly the same as in (6.116). Therefore, we also fix $s = 2$ and we get the following new pp-wave from (6.112) by focusing on the same geodesic as in (6.119):

$$ds_{pp2}'^2 = ds_{pp2}^2 - \tilde{\gamma}^2 \left(\frac{\eta^2}{4} \right)^3 \tilde{\theta}^2 dx^{+2} \quad (6.121)$$

where ds_{pp2}^2 is given in (6.120).

6.4 PP-wave Limit of the Transformed Non-singular Solution

We finally consider the physically most interesting case of the transformed non-singular solution. It has proved somewhat tricky to obtain a regular pp-wave limit of this solution even before the $SL(3, R)$ transformation. This limit is discussed in [20]. Here we first review the argument of [20] and we revise it slightly so that it can be applied also for the case of γ deformed non-singular background.

In order to explore the gauge dynamics at IR one is interested in a geodesic near $r = 0$. With this purpose in mind, the authors of [20] considered the non-singular $\mathcal{N} = 1$ geometry

in (3.37) and picked up a null geodesic that is on the S^3 at the origin. It is tricky to find the suitable coordinates for this geodesic essentially because of the following fact: the one-forms w^i of (2.4) that involve the angular coordinates of S^3 are fibered by the S^2 coordinates and this fibration is given by the connection one form A^i . As one scales $r \rightarrow r/R$ and the angles on S^3 by $\phi_i \rightarrow \phi/R$ together, one does not obtain a metric that is suitable for the pp-wave limit. This is because the one-forms are $\mathcal{O}(1)$ near $r \sim 0$ (the function $a(r)$ approaches to 1 as $r \rightarrow 0$). Therefore this part of the metric blows up as $R \rightarrow \infty$.

This only indicates that one should use a better set of coordinates that is more suitable for the pp-wave limit of this geometry. [20] solved this problem as follows. The geometry when Scherk-Schwarz reduced on the S^3 produces an $SU(2)$ gauged supergravity in 7D [10]. On the other hand it is well-known that the gauge field A —that was the cause of the problem that we mentioned above— is pure gauge at the origin, up to $\mathcal{O}(r^2)$ corrections:

$$A = -idhh^{-1} + \mathcal{O}(r^2), \quad h = e^{-i\sigma^1 \frac{\theta}{2}} e^{-i\sigma^3 \frac{\varphi}{2}}. \quad (6.122)$$

Here, we defined $A = A^i \frac{\sigma^i}{2}$. Therefore [20] simply gauged the $\mathcal{O}(1)$ part of A away by the following gauge transformation,

$$A \rightarrow h^1 A h + ih^{-1} dh, \quad (6.123)$$

and then took the appropriate pp-wave limit as

$$r \rightarrow \frac{r}{R}, \quad \tilde{\theta} \rightarrow \frac{\tilde{\theta}}{R}, \quad \tilde{x}_3 = \frac{\tilde{x}_3}{R}, \quad dt = dx^+ + \frac{dx^-}{R^2}, \quad d\psi + d\tilde{\varphi} = \frac{2}{\eta} (dx^+ - \frac{dx^-}{R^2}). \quad (6.124)$$

This limit produces the linear, quantizable pp-wave that is given in [20]. Here, we would like to consider the same pp-wave limit that is given in (6.124) in the case of the γ deformed solution (5.86). As the S^3 of the undeformed solution is distorted by the γ transformation, one cannot simply make a “gauge transformation” in A . Below, we explain the appropriate way to put the metric in a suitable form.

The gauge transformation (6.123) is nothing else but a coordinate transformation on the angular variables $\tilde{\theta}, \tilde{\varphi}, \psi$ from the 10D point of view. Explicitly, it is equivalent to changing the coordinates as,

$$w \rightarrow h w h^{-1} - idh h^{-1}, \quad (6.125)$$

where we defined $w = w^i \frac{\sigma^i}{2}$. Noting that the one-forms, (2.4) on the $SU(2)$ group manifold are given as follows,

$$w = -idg g^{-1}, \quad g = e^{i\sigma^3 \frac{\psi}{2}} e^{i\sigma^1 \frac{\tilde{\theta}}{2}} e^{i\sigma^3 \frac{\tilde{\varphi}}{2}}$$

one sees that the transformation in (6.125) is equivalent to

$$g \rightarrow h g. \quad (6.126)$$

where h is given in (6.122). We give the explicit form of this coordinate transformation in Appendix E. Now we apply this coordinate transformation to the γ deformed metric in (5.86). Using the explicit expressions in Appendix E one can work out the full coordinate transformation of (5.86). However, it is sufficient for us to observe the following. At the end of section 6.1 we argued that, if one knows the pp-wave limit in the undeformed solution one can obtain the new pp-wave of the deformed solution from (6.112). Having performed the coordinate transformation (6.126), we put the first term in (6.112) in the appropriate form to perform the pp-wave limit in (6.124). Therefore the limit of this part will be explicitly given as the pp-wave in [20], that is

$$ds^2 = -2dx^+ dx^- - \frac{1}{g_s N \alpha'} \left(\frac{1}{9} \bar{u}^2 + \bar{v}^2 \right) (dx^+)^2 + d\vec{x}_3^2 + dz^2 + d\bar{u}^2 + d\bar{v}^2 \quad (6.127)$$

Here u and v are two-planes and z is a line. There is also an expression for the three form. The second part in (6.112), then produces some non-linear corrections to this pp-wave that is proportional to $\tilde{\gamma}$. In order to obtain a meaningful limit we found that the scaling should be defined as follows:

$$\gamma R^4 = \tilde{\gamma} = \text{const.}$$

We explicitly see that the γ deformation produces a non-linear correction to the pp-wave of the original solution and this deformation should be dual to the complicated dynamics that affects the KK-sector of the $\mathcal{N} = 1$ theory.

7 Deformations of the $\mathcal{N} = 2$ Theory

With the fields $(A_\mu^a, \phi^a, \psi^a)$ mentioned in section 2, one can perform a different twisting from the one explained there. One can choose the second $U(1)$ to twist, inside the diagonal combination of the $SU(2)$'s. That is $U(1)_D$ inside $\text{diag}(SU(2)_L \times SU(2)_R)$. If this is done, we can see that the massless spectrum can be put in correspondence with a vector multiplet of $\mathcal{N} = 2$ in $d=(3+1)$. The corresponding solution, preserves eight supercharges as was found in [11]

$$ds^2 = e^\phi \left(dx_{1,3}^2 + \frac{z}{\lambda} \left(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2 \right) \right) + \frac{e^{-\phi}}{\lambda} \left(d\rho^2 + \rho^2 d\varphi_2^2 \right) \quad (7.128)$$

$$+ \frac{e^{-\phi}}{\lambda z} \left(d\tilde{\sigma}^2 + \tilde{\sigma}^2 \left(d\varphi_1 + \cos \tilde{\theta} d\tilde{\varphi} \right)^2 \right), \quad (7.129)$$

where $\lambda = (g_s N \alpha')^{-1}$. The dilaton is,

$$e^{2\phi} = e^{2z} \left(1 - \sin^2 \theta \frac{1 + ce^{-2z}}{2z} \right). \quad (7.130)$$

Here z and θ are given in terms of the radial functions that appear in (7.128) as

$$\rho = \sin \theta e^z, \quad \tilde{\sigma} = \sqrt{z} \cos \theta e^{z-x} \quad (7.131)$$

and x is the following radial function:

$$e^{-2x} = 1 - \frac{1 + ce^{-2z}}{2z}. \quad (7.132)$$

c is an integration constant. The RR two-form field reads,

$$C^{(2)} = g_s N \alpha' \varphi_2 d \left(\xi (d\varphi_1 + \cos \tilde{\theta} d\tilde{\varphi}) \right), \quad (7.133)$$

where

$$\xi = (1 + e^{2x} \cot^2 \theta)^{-1}. \quad (7.134)$$

It is well-known that the R-symmetry of the theory is dual to the isometry along the φ_2 direction. This chiral symmetry is anomalous because of the φ_2 dependence of $C^{(2)}$.

7.1 Transformation along a non-R-symmetry Direction

We shall first discuss the simpler case of $SL(2, R)$ transformations along the torus that is composed of φ_1 and $\tilde{\varphi}$. The metric is already in the desired form given in eq. (4.55)

$$ds^2 = F \left(\frac{1}{\sqrt{\Delta}} (d\varphi_1 - C d\tilde{\varphi} + \mathcal{A}^{(1)} - C \mathcal{A}^{(2)})^2 + \sqrt{\Delta} (d\tilde{\varphi} + \mathcal{A}^{(2)})^2 \right) + g_{\mu\nu} dx^\mu dx^\nu \quad (7.135)$$

with,

$$F = \frac{\tilde{\sigma} \sin \tilde{\theta}}{\lambda}, \quad \Delta = \frac{e^{2\phi} z^2 \sin^2 \tilde{\theta}}{\tilde{\sigma}^2}, \quad C = -\cos \tilde{\theta}, \quad \mathcal{A}^{(1)} = \mathcal{A}^{(2)} = 0 \quad (7.136)$$

and

$$g_{\mu\nu} dx^\mu dx^\nu = e^\phi (dx_{1,3}^2 + \frac{z}{\lambda} d\tilde{\theta}^2) + \frac{e^{-\phi}}{\lambda z} d\tilde{\sigma}^2 + \frac{e^{-\phi}}{\lambda} (d\rho^2 + \rho^2 d\varphi_2^2). \quad (7.137)$$

The RR two form is put in the following form:

$$\begin{aligned} C_2 &= C_{12} (d\varphi_1 + \mathcal{A}^{(1)}) \wedge (d\tilde{\varphi} + \mathcal{A}^{(2)}) + C_\mu^{(1)} (d\varphi_1 + \mathcal{A}^{(1)}) \wedge dx^\mu + C_\mu^{(2)} (d\tilde{\varphi} + \mathcal{A}^{(2)}) \wedge dx^\mu \\ &\quad - \frac{1}{2} (\mathcal{A}_\mu^{(a)} C_\nu^{(a)} - \tilde{c}_{\mu\nu}) dx^\mu \wedge dx^\nu. \end{aligned} \quad (7.138)$$

Comparison of (7.133) with (7.138) yields the following components of the RR two-form:

$$\begin{aligned} C_{12} &= 0, \quad C_\rho^{(1)} = -\frac{\varphi_2}{\lambda} \frac{\partial \xi}{\partial \rho}, \quad C_{\tilde{\sigma}}^{(1)} = -\frac{\varphi_2}{\lambda} \frac{\partial \xi}{\partial \tilde{\sigma}}, \\ C_\rho^{(2)} &= -\frac{\varphi_2}{\lambda} \frac{\partial \xi}{\partial \rho} \cos \tilde{\theta}, \quad C_{\tilde{\sigma}}^{(2)} = -\frac{\varphi_2}{\lambda} \frac{\partial \xi}{\partial \tilde{\sigma}} \cos \tilde{\theta}, \quad C_\theta^{(2)} = \frac{\varphi_2}{\lambda} \xi \sin \tilde{\theta}, \quad \tilde{c}_{\mu\nu} = 0. \end{aligned} \quad (7.139)$$

The transformation yields the following results.

$$\mathcal{A}'^{(1)} = -\tilde{\sigma}C^{(2)}, \mathcal{A}'^{(2)} = \tilde{\sigma}C^{(1)}, C'^{(1)} = C^{(1)}, C'^{(2)} = C^{(2)}, \tilde{c}'_{\mu\nu} = 0. \quad (7.140)$$

New metric functions are given by the general formula, eq. (2.18) with $C_{12} = 0$. In the simpler case of $\sigma = 0$ transformation one obtains,

$$C'_{12} = 0, F' = \frac{F}{1 + \gamma^2 F^2}, B'_{12} = \frac{\gamma F^2}{1 + \gamma^2 F^2}, e^{2\phi'} = \frac{e^{2\phi}}{1 + \gamma^2 F^2}, \chi' = 0. \quad (7.141)$$

Now, let us see whether or not the gravity dual computations of the β -function and the chiral anomaly are affected by the $SL(2, R)$ transformation. We shall consider the general transformation that depends on both σ and γ in what follows. Gravity duals of the theta angle and the YM coupling constant can be obtained by placing a probe D5-brane on the supersymmetric cycle $\tilde{\sigma} = 0$ in the geometry [28]. From the DBI action on the probe brane one can read off the information after reducing on the S^2 . Essentially, the coupling constant is determined by the volume of S^2 that the D5 branes wrap and the theta angle is determined by the flux of the RR form on this sphere. One gets the following results (before the transformation):

$$\frac{1}{g^2} = \frac{1}{2(2\pi)^3 g_s \alpha'} \int_0^{2\pi} d\tilde{\varphi} \int_0^\pi d\tilde{\theta} e^{-\phi} Vol(S^2) = \frac{N}{4\pi^2} \log \rho. \quad (7.142)$$

$$\theta_{YM} = \frac{1}{2\pi g_s \alpha'} \int_0^{2\pi} d\tilde{\varphi} \int_0^\pi d\tilde{\theta} C_{\tilde{\theta}\tilde{\varphi}}^{(2)} = -2N\varphi_2. \quad (7.143)$$

We consider the transformed solution determined by the transformation rules in (2.18) and (7.140). One obtains the following result for the coupling constant:

$$\frac{1}{g'^2} = \frac{1}{2(2\pi)^3 g_s \alpha'} \int_0^{2\pi} d\tilde{\varphi} \int_0^\pi d\tilde{\theta} e^{-\phi'} Vol(S^2)' = \frac{1}{2(2\pi)^3 g_s \alpha'} \int_0^{2\pi} d\tilde{\varphi} \int_0^\pi d\tilde{\theta} H^{-\frac{1}{2}} e^{-\phi} Vol(S^2). \quad (7.144)$$

Here $Vol(S^2)'$ denotes the volume of S^2 in the transformed geometry and the function H is given by,

$$H = 1 + \sigma^2 \frac{\tilde{\sigma}^2}{\lambda} \sin^2 \tilde{\theta} e^{-2\phi}. \quad (7.145)$$

However one should remember to place the probe brane on the supersymmetric cycle. From the general transformation rules under $SL(2, R)$ it is not hard to see that $\tilde{\sigma} = 0$ maps to $\tilde{\sigma} = 0$ hence we have the same supersymmetric cycle. On this cycle one learns from (7.145) that $H = 1$. Therefore we refer that the coupling function is unaffected by the transformation.

For the theta angle, one has the following new expression:

$$\theta'_{YM} = \frac{1}{2\pi g_s \alpha'} \int_0^{2\pi} d\tilde{\varphi} \int_0^\pi d\tilde{\theta} C_{\tilde{\theta}\tilde{\varphi}}^{(2)'} = \frac{1}{2\pi g_s \alpha'} \int_0^{2\pi} d\tilde{\varphi} \int_0^\pi d\tilde{\theta} (1 - JG\sigma) C_{\tilde{\theta}\tilde{\varphi}}^{(2)}. \quad (7.146)$$

From (2.19) we see that $J = 0$ at the supersymmetric cycle $\tilde{\sigma} = 0$ (Remember that when $C_{12} = 0$). Therefore we learn that θ_{YM} is also unaffected by the transformation. This result is indeed expected because the particular $SL(3, R)$ transformation that we make here is not breaking the supersymmetry. Then, this result can be viewed as a consistency check for the whole setup.

7.2 Transformation along the R-symmetry Directions

Now, let us work out a more complicated case of the transformation involving the torus on the φ_2 and $\tilde{\varphi}$ angles. As φ_2 was identified with the R-symmetry we expect that this transformation breaks supersymmetry. We rewrite the metric again in the form (4.55) that makes the torus explicit.

$$ds^2 = F \left(\frac{1}{\sqrt{\Delta}} (d\varphi_2 - Cd\tilde{\varphi} + \mathcal{A}^{(1)} - C\mathcal{A}^{(2)})^2 + \sqrt{\Delta} (d\tilde{\varphi} + \mathcal{A}^{(2)})^2 \right) + \frac{e^{2/3\phi}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu \quad (7.147)$$

with,

$$g_{\mu\nu} dx^\mu dx^\nu = e^{-2\phi/3} F^{1/3} \left(e^\phi (dx_{1,3}^2 + \frac{z}{\lambda} d\tilde{\theta}^2) + \frac{e^{-\phi}}{\lambda z} d\tilde{\sigma}^2 + \frac{e^{-\phi}}{\lambda} d\rho^2 + Dd\varphi_1^2 \right). \quad (7.148)$$

Various functions that appear in the metric is determined as,

$$\begin{aligned} \frac{F}{\sqrt{\Delta}} &= e^{-\phi} \frac{\tilde{\sigma}^2}{\lambda}, \quad F\sqrt{\Delta} = e^\phi \sin^2 \tilde{\theta} \frac{z}{\lambda} + e^{-\phi} \cos^2 \tilde{\theta} \frac{\tilde{\sigma}^2}{z\lambda}, \\ D &= e^{-\phi} \frac{\tilde{\sigma}^2}{z\lambda} - F\sqrt{\Delta} \mathcal{A}_{\varphi_1}^{(2)2}, \quad \mathcal{A}^{(2)} = e^{-\varphi} \frac{\tilde{\sigma}^2 \cos \tilde{\theta}}{z\lambda F\sqrt{\Delta}} d\varphi_1, \\ \mathcal{A}^{(1)} &= C = 0. \end{aligned} \quad (7.149)$$

$$\mathcal{A}^{(1)} = C = 0. \quad (7.150)$$

The RR two-form is written,

$$\begin{aligned} C_2 &= C_{12} (d\varphi_2 + \mathcal{A}^{(1)}) \wedge (d\tilde{\varphi} + \mathcal{A}^{(2)}) + C_\mu^{(1)} (d\varphi_2 + \mathcal{A}^{(1)}) \wedge dx^\mu + C_\mu^{(2)} (d\tilde{\varphi} + \mathcal{A}^{(2)}) \wedge dx^\mu \\ &\quad - \frac{1}{2} (\mathcal{A}_\mu^{(a)} C_\nu^{(a)} - \tilde{c}_{\mu\nu}) dx^\mu \wedge dx^\nu. \end{aligned} \quad (7.151)$$

Comparison of (7.133) with (7.151) yields the following components of the RR two-form:

$$\begin{aligned} C_{12} &= 0, \quad C_\mu^{(1)} = 0, \quad C_\rho^{(2)} = -\frac{\varphi_2}{\lambda} \frac{\partial \xi}{\partial \rho} \cos \tilde{\theta}, \\ C_{\tilde{\sigma}}^{(2)} &= -\frac{\varphi_2}{\lambda} \frac{\partial \xi}{\partial \tilde{\sigma}} \cos \tilde{\theta}, \quad C_{\tilde{\theta}}^{(2)} = \frac{\varphi_2}{\lambda} \xi \sin \tilde{\theta}, \quad \tilde{c}_{\mu\nu} = 0, \quad \tilde{c}_{\varphi_1 \tilde{\sigma}} = \frac{\varphi_2}{\lambda} \frac{\partial \xi}{\partial \tilde{\sigma}} \left(\frac{1}{2} \cos \tilde{\theta} - 1 \right), \\ \tilde{c}_{\varphi_1 \rho} &= \frac{\varphi_2}{\lambda} \frac{\partial \xi}{\partial \rho} \left(\frac{1}{2} \cos \tilde{\theta} - 1 \right), \quad \tilde{c}_{\varphi_1 \tilde{\theta}} = -\frac{\varphi_2}{\lambda} \xi \sin \tilde{\theta}. \end{aligned}$$

The transformation yields the following results.

$$\mathcal{A}'^{(1)} = -\tilde{\sigma}C^{(2)}, \mathcal{A}'^{(2)} = \mathcal{A}^{(2)}, C'^{(1)} = 0, C'^{(2)} = C^{(2)}, \tilde{c}'_{\mu\nu} = \tilde{c}_{\mu\nu}. \quad (7.152)$$

New metric functions are given by the general formula, eq. (2.18) with $C_{12} = 0$.

For the new coupling constant and the new theta angle it is not hard to see that one would get exactly the same results had one allowed to put the probe brane at the would-be susy cycle $\tilde{\sigma} = 0$. However as the transformation breaks supersymmetry, $\tilde{\sigma} = 0$ is not a supersymmetric cycle anymore. Therefore one does not trust the computations of the beta function and the theta angle in this deformed solution. It will be interesting to study these issues further.

8 Summary and Conclusions

In this paper we studied in detail the proposal of Lunin and Maldacena[14], that is to interpret particular $SL(3, R)$ transformations of supergravity backgrounds as duals to field theories where the product between fields have been modified in a particular way.

We considered the quantum theories that correspond to the β deformations of D5 branes and analyzed the regularity of the transformed configuration.

More importantly, we applied this to models that are argued to be dual to the strong coupling regime of $\mathcal{N} = 1, 2$ SYM. In this case, β transformations affect the so called KK modes of the theory, that can be thought as ‘contaminating’ the pure SYM theory in the supergravity approximation. We argued by a simple dipole field theory computation that the γ deformations increase the mass of the KK modes. However they do not directly affect the pure gauge dynamics. We confirmed these expectations from the supergravity side by computing some field theory observables in the transformed supergravity backgrounds. When these quantities were purely of field theory origin, no modifications were observed, thus providing a check for the original proposal. We also showed that, as a consequence of the transformations, there is an increase in the mass of the KK modes, hence the decoupling is improved. We would also like to stress here the fact that in these transformed geometries, the dilaton does not blow at infinity. This fact was already observed in the non-commutative case and here we extend it for all β deformations.

We have studied pp-waves of the configurations that are mentioned above. In this case, these pp waves are dual to some condensates of KK modes called annulons. We have studied changes in the dynamics of these annulons.

As a check of consistency, we also computed the dual to $N = 1$ non-commutative SYM with the methods explained here and obtained the solution previously studied by [33]. This is in Appendix B.

Let us now comment on possible further work. It should be interesting to study the

physical effects of these transformations on observables of the field theory. For example, by considering Wilson loops that wrap the transformed directions one could get some information that depends on the deformation parameters. Also, the study of strings rotating in the internal space could yield additional information. Another field theory quantity that is interesting to study, and that is not expected to change, is the law obeyed by the tensions of confining strings. These models usually obey a sine law behaviour [32].

We did not check explicitly the SUSY of the deformed backgrounds, but we expect that, unless where we indicated, the same spinors will be preserved. In this line, it should be interesting to study the SUSY holomorphic two cycles to place probe D5 branes in the deformed geometry. This is interesting for the quenched SQCD and meson dynamics as studied in [22]. It is possible that some of the holomorphic two cycles found in [22] do indeed change. These changes should be reflected in the dynamics and spectrum of mesons.

It will also be interesting to study the glueballs of the transformed models. Indeed, since one of the problems of studying glueballs from a supergravity perspective is that we cannot disentangle those glueballs that belong to the original field theory from those that are coming from KK modes, the study of this spectrum for the transformed models is interesting because in the comparison we might be able to identify what belongs to the field theory we are interested in and what is an artifact of the supergravity approximation. In this line, one can study the possibility of finding massless glueballs, following what is done in [36]. In this case, the asymptotic behavior of the dilaton in the transformed solution is different, thus it might be possible to find a massless glueball. However, this should be analyzed in detail. A study of the transformed family of metrics developed in [37] may also be interesting.

We also note that a more detailed study of the annulons in these confining models is required. The annulons are precisely made of the part of the spectrum that is modified by the $SL(3, R)$ transformations.

It should also be of interest to study the G_2 holonomy manifolds in M theory. These backgrounds can be thought of as coming from D6 branes wrapping a three manifold inside a $CY3$ -fold. There are at least three $U(1)$'s that can be identified (see [34], for a summary on these geometries and the explicit expression of the $U(1)$'s of interest). It should also be of interest to study different geometries, like the ones found by Pando-Zayas and Tseytlin [35]. As well as other geometries in massive IIA; for example the duals of CFT's in five and three dimensions that are constructed in [38] also present all the necessary characteristics for the $SL(3, R)$ transformations to be applied. Computing observables in these transformed geometries could be a way of understanding better the dual field theories.

To end this paper, we want to stress that the differences and similarities between the original confining model and the transformed one leads us, by comparison, to a better understanding of the Physics underlying these models.

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A Appendix: Solution Generating Technique

In this appendix we summarize the technique which was used to generate supergravity solutions presented in the paper [14].

Let us write the most general configuration in IIB as

$$\begin{aligned}
ds_{IIB}^2 &= F \left[\frac{1}{\sqrt{\Delta}} (D\varphi_1 - C(D\varphi^2))^2 + \sqrt{\Delta} (D\varphi_2)^2 \right] + \frac{e^{2\phi/3}}{F^{1/3}} g_{\mu\nu} dx^\mu dx^\nu, \\
B &= B_{12}(D\varphi^1) \wedge (D\varphi^2) + \{B_{1\mu}(D\varphi^1) + B_{2\mu}(D\varphi^2)\} \wedge dx^\mu \\
&\quad - \frac{1}{2} A_\mu^m B_{m\nu} dx^\mu \wedge dx^\nu + \frac{1}{2} \tilde{b}_{\mu\nu} dx^\mu \wedge dx^\nu \\
e^{2\Phi} &= e^{2\phi}, \quad C^{(0)} = \chi \\
C^{(2)} &= C_{12}(D\varphi^1) \wedge (D\varphi^2) + \{C_{1\mu}(D\varphi^1) + C_{2\mu}(D\varphi^2)\} \wedge dx^\mu \\
&\quad - \frac{1}{2} A_\mu^m C_{m\nu} dx^\mu \wedge dx^\nu + \frac{1}{2} \tilde{c}_{\mu\nu} dx^\mu \wedge dx^\nu \\
C^{(4)} &= -\frac{1}{2} (\tilde{d}_{\mu\nu} + B_{12} \tilde{c}_{\mu\nu} - \epsilon^{mn} B_{m\mu} C_{n\nu} - B_{12} A_\mu^m C_{m\nu}) dx^\mu dx^\nu D\varphi^1 D\varphi^2 \\
&\quad + \frac{1}{6} (C_{\mu\nu\lambda} + 3(\tilde{b}_{\mu\nu} + A_\mu^1 B_{1\nu} - A_\mu^2 B_{2\nu}) C_{1\lambda}) dx^\mu dx^\nu dx^\lambda D\varphi^1 + \\
&\quad + d_{\mu_1\mu_2\mu_3\mu_4} dx^{\mu_1} dx^{\mu_2} dx^{\mu_3} dx^{\mu_4} + \hat{d}_{\mu_1\mu_2\mu_3} dx^{\mu_1} dx^{\mu_2} dx^{\mu_3} D\varphi^2
\end{aligned} \tag{A.1}$$

where as defined in the main text of this paper

$$D\varphi^2 = d\varphi^2 + A^2, \quad D\varphi^1 = d\varphi^1 + A^1 \tag{A.2}$$

We have three objects which transform as vectors (see also (2.13)-(2.14) in the main text):

$$V_\mu^{(1)} = \begin{pmatrix} -B_{2\mu} \\ A_\mu^1 \\ C_{2\mu} \end{pmatrix}, \quad V_\mu^{(2)} = \begin{pmatrix} B_{1\mu} \\ A_\mu^2 \\ -C_{1\mu} \end{pmatrix}; \quad V_\mu^{(i)} \rightarrow (\Lambda^T)^{-1} V_\mu^{(i)}; \tag{A.3}$$

$$W_{\mu\nu} = \begin{pmatrix} \tilde{c}_{\mu\nu} \\ \tilde{d}_{\mu\nu} \\ \tilde{b}_{\mu\nu} \end{pmatrix} \rightarrow \Lambda W_{\mu\nu} \tag{A.4}$$

and one matrix

$$M = gg^T, \quad g^T = \begin{pmatrix} e^{-\phi/3}F^{-1/3} & 0 & 0 \\ 0 & e^{-\phi/3}F^{2/3} & 0 \\ 0 & 0 & e^{2\phi/3}F^{-1/3} \end{pmatrix} \begin{pmatrix} 1 & B_{12} & 0 \\ 0 & 1 & 0 \\ \chi & -C_{12} + \chi B_{12} & 1 \end{pmatrix} \quad (\text{A.5})$$

which transforms as

$$M \rightarrow \lambda M \Lambda^T \quad (\text{A.6})$$

The scalars Δ , C as well as the three form $C_{\mu\nu\lambda}$ stay invariant under these $SL(3, R)$ transformations. Finally, let us write the elements of the transformed matrix $(g^T)'$ for the most general transformation, with parameters (σ, γ) nonzero and find the expression for the transformed fields $B'_{12}, C'_{12}, \chi', e^{\phi'}, F'$

$$\begin{aligned} g_{11}^T &= \frac{e^{-\phi/3}\kappa}{\mu}, \quad g_{12}^T = \frac{e^{5/3\phi}}{\mu\kappa}(B_{12} + \gamma B_{12}^2 - B_{12}C_{12}\sigma + F^2(\gamma - \chi\sigma)), \quad g_{2,2}^T = \frac{(e^{2\phi}F^2)^{1/3}}{\kappa}, \\ g_{32}^T &= \frac{e^{-\phi/3}}{\mu}(B_{12}\chi e^{2\phi} + C_{12}^2\sigma e^{2\phi} + B_{12}^2\sigma(1 + \chi^2 e^{2\phi}) + F^2\sigma - C_{12}e^{2\phi}(1 + 2B_{12}\chi\sigma)) \\ g_{31}^T &= \frac{e^{-\phi/3}}{\mu}(-C_{12}\gamma e^{2\phi} + C_{12}^2\gamma\sigma e^{2\phi} + B_{12}\chi^2 e^{2\phi}\sigma(1 + B_{12}\gamma) + \sigma(B_{12} + \\ & B_{12}^2\gamma + F^2\gamma) - \chi e^{2\phi}(-1 + C_{12}\sigma + B_{12}\gamma(2C_{12}\sigma - 1))) \\ g_{3,3}^T &= \left(\frac{e^{-\phi}}{F}\right)^{1/3}\sqrt{(B_{12}^2 + F^2)\sigma^2 + e^{2\phi}(1 - C_{12}\sigma + B_{12}\chi\sigma)^2}, \\ \mu &= F^{1/3}\sqrt{(B_{12}^2 + F^2)\sigma^2 + e^{2\phi}(1 - C_{12}\sigma + B_{12}\chi\sigma)^2} \\ \kappa^2 &= F^2\sigma^2 + e^{2\phi}((B_{12}\gamma)^2 - 2B_{12}\gamma(C_{12}\sigma - 1) + (C_{12}\sigma - 1)^2 + F^2(\gamma - \sigma\chi)^2) \end{aligned} \quad (\text{A.7})$$

all the other components are zero.¹³ The transformed fields are

$$\begin{aligned} B'_{12} &= \frac{g_{12}^T}{g_{11}^T}, \quad e^{\phi'} = \frac{g_{33}^T}{g_{11}^T}, \quad \chi' = \left(\frac{g_{22}^T g_{11}^T}{g_{33}^T}\right)^{1/3} g_{31}^T \\ C'_{12} &= \chi' B'_{12} - g_{32}^T g_{22}^T g_{11}^T \end{aligned} \quad (\text{A.8})$$

Some colleagues might find useful the complete expression of the transformed fields in the most general situation, with $C_{12}, B_{12}, \chi, \phi$ turned on, that can be obtained from the previous eq.(A.8)

B Appendix : The Non-Commutative $\mathcal{N} = 1$ SYM Solution

We might try the methods developed in the the core of the paper, to make a transformation on the directions where the gauge theory lives. Let us pick these two $U(1)$'s to be the ones

¹³We thank Changhyun Ahn for pointing out typos in a previous version of this appendix

labeled by the compactified coordinates x_1, x_2 . the reader can check that in this case, the original configuration can be written as

$$ds_{10}^2 = F \left(\frac{(Dx_1 - CDx_2)^2}{\sqrt{\Delta}} + \sqrt{\Delta} Dx_2^2 \right) + \left(\frac{e^{2/3\phi}}{F} \right) (e^{-2/3\phi} F) \alpha' g_s N e^\phi \left[\frac{1}{\alpha' g_s N} dx_{1,1}^2 + e^{2h} (d\theta^2 + \sin^2 \theta d\varphi^2) + dr^2 + \frac{1}{4} (w^i - A^i)^2 \right], \quad (\text{B.1})$$

With $F = e^\phi$, $\Delta = 1$, $C = 0$, $\mathcal{A}^{(i)} = 0$, and with our eight dimensional metric given by

$$g_{\mu\nu} dx^\mu dx^\nu = (e^{-2/3\phi} F) \alpha' g_s N e^\phi \left[\frac{1}{\alpha' g_s N} dx_{1,1}^2 + e^{2h} (d\theta^2 + \sin^2 \theta d\varphi^2) + dr^2 + \frac{1}{4} (w^i - A^i)^2 \right], \quad (\text{B.2})$$

and the RR two form

$$C_{(2)} = \frac{1}{4} \left[\psi (\sin \theta d\theta \wedge d\varphi - \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi}) - \cos \theta \cos \tilde{\theta} d\varphi \wedge d\tilde{\varphi} - a (d\theta \wedge w^1 - \sin \theta d\varphi \wedge w^2) \right] = \tilde{c}_{\mu\nu} dx^\mu \wedge dx^\nu \quad (\text{B.3})$$

and the rest of the fields

$$\mathcal{A}^{(i)} = C^{(i)} = C_{12} = B_{12} = B^{(i)} = \tilde{b} = \tilde{d} = 0 \quad (\text{B.4})$$

so, upon performing the transformation in the two torus (x_1, x_2) , we get a new metric

$$ds_{10}^2 = F' (dx_1^2 + dx_2^2) + \left(\frac{e^{2/3\phi'}}{F'} \right) (e^{-2/3\phi} F) \alpha' g_s N e^\phi \left[\frac{1}{\alpha' g_s N} dx_{1,1}^2 + e^{2h} (d\theta^2 + \sin^2 \theta d\varphi^2) + dr^2 + \frac{1}{4} (w^i - A^i)^2 \right], \quad (\text{B.5})$$

with F' and the new matter fields given in (2.18).

So, we can see that doing the rotation in this case, has generated a NS two form, via the term B'_{12} , and a new C'_{12} (for nonzero σ !). If we concentrate on the $\sigma = 0$ transformation, we see that we have to add to the metric in (B.5) the NS two form and the RR four forms

$$B'_2 = B'_{12} dx^1 \wedge dx^2, \quad C'_2 = \tilde{c}, \quad 2C_4 = -B'_{12} \tilde{c} \wedge dx^1 \wedge dx^2, \quad e^{\phi'} = \frac{e^\phi}{1 + \gamma^2 e^{2\phi}} \quad (\text{B.6})$$

this is precisely the configuration found by Mateos, Pons and Talavera in [33]. It is quite a nice check of the method and also a check of the way in which we are thinking about this transformed configurations. Indeed, the authors in [33] have checked that many observables do not change compared to the commutative background.

C Appendix: The Non-Commutative KK theory

In this appendix, we will write the expression for the metric in the case in which we pick up the two torus to be in one of the directions of the field theory (that we label by z) and the angle in the two sphere, labeled before as φ , notice that this gives a NC six dimensional field theory for the KK modes. We could also choose the second angle to be $\tilde{\varphi}$ and in this case, like in previous sections the background would have been dual to a dipole for the KK field theory.

The metric reads

$$\begin{aligned}
 ds_{string}^2 = & e^\phi [dx_{1,2}^2 + \alpha' g_s N dr^2] + G d\psi^2 + D d\theta^2 + E d\tilde{\theta}^2 + H d\theta d\tilde{\theta} + K d\theta d\psi + P d\tilde{\theta} d\psi + \\
 & N d\tilde{\varphi}^2 + Q d\tilde{\theta} d\tilde{\varphi} + R d\psi d\tilde{\varphi} + S d\theta d\tilde{\varphi} + \frac{F}{\sqrt{\Delta}} [d\varphi + C dz + \alpha_1 d\theta + \alpha_2 d\tilde{\theta} + \alpha_3 d\psi + \alpha_4 d\tilde{\varphi}]^2 + \\
 & F \sqrt{\Delta} dz^2
 \end{aligned} \tag{C.1}$$

with

$$\begin{aligned}
 F^2 = \frac{e^{2\phi}}{\Delta} = \frac{\alpha' g_s N e^{2\phi}}{4} (\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta), \quad \alpha_1 = K = C = 0, \quad H = \frac{\alpha' g_s N e^{2\phi}}{2} \cos \psi \\
 D = \frac{\alpha' g_s N e^{2\phi}}{4} (\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta), \quad S = \frac{\alpha' g_s N e^{2\phi}}{2} a \sin \psi \sin \tilde{\theta}, \\
 E = \frac{\alpha' g_s N e^\phi}{4} \left(\frac{\cos^2 \theta + (4e^{2h} + a^2 \cos^2 \psi) \sin^2 \theta}{\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta} \right), \quad G = \frac{\alpha' g_s N e^\phi}{4} \left(\frac{(4e^{2h} + a^2) \sin^2 \theta}{\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta} \right), \\
 N = \frac{\alpha' g_s N e^\phi}{4} \frac{4e^{2h} \sin^2 \theta + 2a \cos \psi \cos \theta \sin \theta \cos \tilde{\theta} \sin \tilde{\theta} + \cos^2 \theta \sin^2 \tilde{\theta} + a^2 \sin^2 \theta (1 - \sin^2 \tilde{\theta} \cos^2 \psi)}{\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta}, \\
 Q = \frac{\alpha' g_s N e^\phi}{2} \sin \psi \sin \theta \left(\frac{a \cos \psi \sin \theta \sin \tilde{\theta} - \cos \theta \cos \tilde{\theta}}{\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta} \right), \\
 R = \frac{\alpha' g_s N e^\phi}{2} \sin \theta \left(\frac{a \cos \psi \cos \theta \sin \tilde{\theta} + \sin \theta \cos \tilde{\theta} (4e^{2h} + a^2)}{\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta} \right), \\
 \alpha_2 = \frac{a \sin \psi \sin \theta}{\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta}, \quad \alpha_3 = \frac{\cos \theta}{\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta}, \\
 \alpha_4 = \frac{\cos \theta \cos \tilde{\theta} - a \cos \psi \sin \theta \sin \tilde{\theta}}{\cos^2 \theta + (4e^{2h} + a^2) \sin^2 \theta},
 \end{aligned} \tag{C.2}$$

the gauge and tensor fields, following the notation adopted in the main text are given by

$$\begin{aligned}
 \mathcal{A}^{(1)} = & \alpha_2 d\tilde{\theta} + \alpha_3 d\psi + \alpha_4 d\tilde{\varphi}, \quad \mathcal{A}^{(1)} = 0, \\
 C^{(1)} = & \frac{a}{4} \sin \theta \omega_2 - \frac{\psi}{4} \sin \theta d\theta - \frac{1}{4} \cos \theta \cos \tilde{\theta} d\tilde{\varphi}, \quad C^{(2)} = 0 \\
 \tilde{c} = & C^{(1)} \wedge \mathcal{A}^{(1)} - \frac{1}{4} (\psi \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\varphi} + a d\theta \wedge \omega_1), \quad C_{12} = 0.
 \end{aligned} \tag{C.3}$$

the laws of transformation are those in (2.13)-(2.20) and the reader can easily obtain the background corresponding to this NC KK fields theory.

D Appendix: Details on Rotations in R-symmetry Direction

Here we give details of the computations of section 4.2. Components of the RR two-form and the gauge connections are,

$$C_{12} = -\frac{\cos \tilde{\theta}}{4}, \quad C^{(1)} = \frac{\cos \theta}{4} d\varphi, \quad A^{(1)} = \cos \theta d\varphi, \quad C^{(2)} = A^{(2)} = \tilde{c}_2 = 0 \quad (\text{D.1})$$

The RR gauge two form can be written as

$$C_2 = \frac{1}{4} [d\psi \wedge (\cos \theta d\varphi - \cos \tilde{\theta} d\tilde{\varphi}) - \cos \theta \cos \tilde{\theta} d\varphi \wedge d\tilde{\varphi}] = C_{12} D\psi \wedge D\tilde{\varphi} - C^{(1)} \wedge D\psi - \frac{1}{2} A^{(1)} \wedge C^{(1)}. \quad (\text{D.2})$$

After the transformation we will have a metric that reads as in (4.74). For completeness let us write the transformed vielbein in Einstein frame defined by

$$\begin{aligned} e^{x_i} &= U dx_i, \quad e^r = \eta U dr, \quad e^\theta = \eta U e^h d\theta, \\ e^\varphi &= \eta U e^h \sin \theta d\varphi, \quad e^{\tilde{\theta}} = \frac{\eta}{2} U d\tilde{\theta}, \quad e^{\tilde{\varphi}} = e^{-\phi'/4} \sqrt{F' \sqrt{\Delta}} d\tilde{\varphi}, \\ e^\psi &= e^{-\phi'/4} \sqrt{\frac{F'}{\sqrt{\Delta}}} (d\psi + \cos \theta d\varphi + \cos \tilde{\theta} d\tilde{\varphi}) \end{aligned} \quad (\text{D.3})$$

where $\eta = \sqrt{g_s N \alpha'}$ and we have defined, $U = e^{(\phi'+2\phi)/12} (\frac{F}{F'})^{1/6}$ while the matter fields transform according to the rules in (2.18). When NS flux is absent, the γ transformation never changes the connection one-forms (C one-forms do not change for any transformation):

$$(C^{(1)})' = C^{(1)} = \frac{\cos \theta}{4} d\varphi, \quad (A^{(1)})' = A^{(1)} = \cos \theta d\varphi. \quad (\text{D.4})$$

So, the transformed RR field is

$$(C_2)' = \frac{\cos \theta}{4} d\psi \wedge d\varphi + C'_{12} [d\psi + \cos \theta d\varphi] \wedge d\tilde{\varphi}, \quad C'_{12} = \frac{C_{12}}{(F^2 \gamma^2 + 1)}, \quad (\text{D.5})$$

and the new NS 2 form becomes

$$(B_{2,NS})' = B'_{12} [d\psi + \cos \theta d\varphi] \wedge d\tilde{\varphi}, \quad B'_{12} = \frac{F^2 \gamma}{F^2 \gamma^2 + 1} \quad (\text{D.6})$$

The new dilaton and the axion fields are, $e^{2\phi'} = \frac{e^{2\phi}}{F^2\gamma^2+1}$, $\chi' = \gamma C_{12}$. Let us also present the RR field strength,

$$H'_3 = dB_2 = \frac{e^{-2h+\phi'/4}}{\eta^2 U^2 \sqrt{F'} \sqrt{\Delta}} B'_{12} e^\varphi \wedge e^\theta \wedge e^{\tilde{\varphi}} + \frac{e^{\phi'/2}}{\eta F' U} (\partial_r B'_{12} e^r + 2\partial_{\tilde{\theta}} B'_{12} e^{\tilde{\theta}}) \wedge e^\psi \wedge e^{\tilde{\varphi}} \quad (\text{D.7})$$

$$\begin{aligned} F'_3 = dC'_2 - \chi' H'_3 = & \\ & \frac{e^{-2h+\phi'/4}}{\eta^2 U^2 \sqrt{F'} \sqrt{\Delta}} (C'_{12} + \frac{\cos \tilde{\theta}}{4} - \chi' B'_{12}) e^\varphi \wedge e^\theta \wedge e^{\tilde{\varphi}} + \frac{e^{-2h+\phi'/4}}{4\eta^2 U^2 \sqrt{\frac{F'}{\Delta}}} e^\theta \wedge e^\varphi \wedge e^\psi \\ & + \frac{e^{\phi'/2}}{F' \eta U} ([\partial_r C'_{12} - \chi \partial_r B'_{12}] e^r + 2[\partial_{\tilde{\theta}} C'_{12} - \chi \partial_{\tilde{\theta}} B'_{12}] e^{\tilde{\theta}}) \wedge e^\psi \wedge e^{\tilde{\varphi}}. \end{aligned} \quad (\text{D.8})$$

E Appendix: Explicit Form of the coordinate transformation in Section 6.4

Defining the new coordinates as $\bar{\theta}, \bar{\psi}, \bar{\varphi}$, one shows that (6.126) is given by:

$$\begin{aligned} \cos \bar{\theta} &= \cos \tilde{\theta} \cos \theta - \sin \tau \sin \tilde{\theta} \cos(\psi + \varphi), \\ \cot \bar{\psi} &= \cos \theta \cos(\psi + \varphi), \\ \cos \bar{\varphi} \sin \bar{\theta} &= \cos \tilde{\varphi} \sin \tilde{\theta} \cos \theta + \sin \theta \cos \tilde{\theta} \cos \tilde{\varphi} \cos(\psi + \varphi) - \sin \theta \sin \tilde{\varphi} \sin(\psi + \varphi). \end{aligned} \quad (\text{F.1})$$

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