Evaluation of argument strength in attack graphs: Foundations and semantics

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A B S T R A C T

An argumentation framework is a pair made of a *graph* and a *semantics*. The nodes and the edges of the graph represent respectively *arguments* and *relations* (e.g., attacks, supports) between arguments while the semantics evaluates the *strength* of each argument of the graph. This paper investigates *gradual semantics* dealing with *weighted graphs*, a family of graphs where each argument has an initial weight and may be attacked by other arguments. It contains four contributions. The first consists of laying the foundations of gradual semantics by proposing key principles on which evaluation of argument strength may be based. Foundations are important not only for a better understanding of the evaluation process in general, but also for clarifying the basic assumptions underlying semantics, for comparing different (families of) semantics, and for identifying families of semantics that have not been explored yet. The second contribution consists of providing a formal analysis and a comprehensive comparison of the semantics that have been defined in the literature for evaluating arguments in weighted graphs. As a third contribution, the paper proposes three novel semantics and shows which principles they satisfy. The last contribution is the implementation and empirical evaluation of the three novel semantics. We show that the three semantics are very efficient in that they compute the strengths of arguments in less than 20 iterations and in a very short time. This holds even for very large graphs, meaning that the three semantics scale very well.

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1. Introduction

Argumentation is a reasoning approach based on the justification of claims by arguments, which are reasons for accepting claims. It has received great interest from the Artificial Intelligence community since late 1980s, mainly as a unifying approach for nonmonotonic reasoning [1]. It was later used for solving other problems including reasoning with inconsistent information [2], reasoning with defeasible information [3], decision making [4], classification [5], etc. It was also used in various practical applications, namely in legal and medical domains [6].

Whatever the application, an argumentation-based model usually follows a four-step process: support claims by arguments, identify relations (e.g. attack or support) between the generated arguments, evaluate the strength of each argument, and define the output of the model (e.g. the set of formulas to be inferred from a knowledge base). Arguments and their

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relations are represented by a graph called *flat* when arguments are not assigned initial (or basic) weights and *weighted* otherwise. The initial weight of an argument may represent various issues like probability of believing the argument [7,8], certainty degree of the argument's reasons [9], votes provided by users [10], importance degree of a value promoted by the argument [11], trustworthiness of the argument's source [12].

The last step of an argumentation process depends broadly on arguments' strengths. For instance, a decision system would recommend to users options that are supported by strong arguments. Consequently, a plethora of evaluation methods, called *semantics*, have been proposed in the literature. There exist several approaches according to the nature of their outcomes, such as *extension* semantics, *labelling* semantics, *gradual* semantics and *ranking* semantics. Extension semantics have been introduced for the first time by Dung [13]. They look for arguments that can be *accepted* by a rational agent, and more precisely those that can be jointly accepted. Examples of such semantics are stable and preferred from [13], and those based on the SCC-recursive schema [14]. More generally, labelling semantics [15] are closely related to extension semantics. Each labelling assigns three possible values to the elements of a graph: *in*, *out* and *undecided*, and a set of arguments labelled *in* corresponds to an extension.

Gradual semantics, initiated by Cayrol and Lagasquie in [16], quantify argument *strength*. They are defined as functions which assign a unique numerical or qualitative value to each argument. In [17], the author discussed how to identify acceptable arguments from the values of their strengths. For instance, one may accept any argument whose value is beyond a given threshold. Examples of gradual semantics are Trust-based [12], social simple product [10], and (Discontinuity-Free)-QuAD [18,19].

Ranking semantics have been proposed by Amgoud and Ben-Naim in [20]. They return a (total or partial) preordering on arguments, thus ranking them from the strongest to the weakest ones. Obviously, any gradual semantics may be transformed into a ranking one, but the converse is not necessarily true. Indeed, *pure* ranking semantics may be defined without assigning values to arguments. Examples of such semantics are Burden-based and Discussion-based from [20], the propagation-based ones from [21,22] and those based on subgraphs analysis [23]. It has been argued in [17] that the choice of the family of semantics depends broadly on the problem to solve.

Given the importance of semantics in argumentation, it is important to have a good understanding of the rules that control how evaluation of arguments is performed by a semantics, in other words a good understanding of the *principles* underlying a semantics. In the argumentation literature, several works have been devoted to the definition of principles for each of the families of semantics. A principle is a formal property that a semantics may satisfy. In [24], Baroni and Giacomin proposed some principles that extension semantics should satisfy. The list was later extended in [25] and used to theoretically analyze and compare the existing extension semantics that deal with flat graphs. Amgoud and Ben-Naim proposed in [20] another set of principles for ranking semantics. Each principle expresses a property that a ranking of arguments should satisfy. That set was used in [26] for comparing some existing gradual/ranking semantics among those devoted to flat graphs. Regarding gradual semantics, some principles were proposed by Amgoud and Ben-Naim in [27] for flat graphs and extended to weighted graphs in [28] and bipolar ones in [29,30]. Each principle describes an elementary property of argument strength.

Focusing on gradual semantics that deal with weighted graphs, this paper presents four contributions. The first consists of simplifying the principles presented in [28] and proposing five novel ones. The principles describe very *elementary properties*, which then serve as basic building blocks for proving higher-level ones. Some of them are mandatory while others represent *reasonable choices* for some applications but not for others. Indeed, argumentation is a rich theory that may be applied for solving a large variety of problems, thus the requirements may vary from one application to another. The second contribution of the paper consists of providing the first theoretical analysis of each of the ten semantics that were proposed in the literature for evaluating arguments in weighted graphs. We studied the four semantics that follow the contraction-based approach for dealing with preferences between arguments [11,31,32], the four semantics that follow the change-based approach [33], and the two gradual semantics proposed in [12,34]. This study allowed the first thorough comparison of i) semantics of different families (extension vs gradual), ii) the two approaches that deal with preferences in argumentation (contraction vs change), and iii) semantics of the same family, e.g. Trust-based and Iterative Schema (IS). The results show that the ten semantics are different in that they made different design choices. The study also revealed the kind of semantics that are missing in the literature. For instance, there is no semantics that satisfies all the compatible principles, there is no semantics that privileges the quantity of attackers over their quality when it faces a dilemma between the two criteria, and there is no semantics that favors the quality criterion and at the same time satisfies all the remaining principles. The third contribution of the paper fills the previous gaps by introducing three novel semantics, one for each of the three cases. The fourth contribution of the paper consists of implementing algorithms for computing the strengths of arguments using the three novel semantics, and running several experiments on a publicly available benchmark proposed in [35,36]. The results show that our semantics are very efficient as the strengths of arguments can be calculated quickly and in less than 20 iterations whatever the size and typology of the graph.

The paper is organized as follows: Section 2 introduces some basic concepts. Section 3 discusses the notion of argument strength. Section 4 introduces the list of principles whose properties are investigated in Section 5. Section 6 recalls existing semantics from the literature, and analyzes them against the set of principles. Section 7 presents the three novel semantics and investigates their formal properties, and Section 8 analyzes them empirically. Section 9 discusses related work, and the last section concludes and presents some perspectives.

Remark. The paper extensively develops the content of the conference paper [28]. It simplifies some principles presented in [28], proposes five novel ones, presents a more detailed analysis and comparison of existing semantics, and presents for the first time an experimental analysis of the performances of the three new semantics.

2. Background

Throughout the paper, a *weighted argumentation graph* (WAG) is a graph whose nodes are *arguments* and edges represent *attacks* between them. Each argument has an initial or basic weight (called *basic score* in [19]) from the interval [0, 1]. The smaller the weight of an argument, the weaker the argument. The basic weight of an argument may represent different issues like certainty degree of its premises [9], degree of trust in its source [12], an aggregation of votes provided by users [10], etc. For the sake of generality, the origin of weights and arguments is left unspecified. Similarly, arguments and attacks are considered as abstract notions. Before formally introducing WAGs, let us define the useful notion of *weighting*.

Definition 1 (*Weighting*). A *weighting* on a set X is a function from X to the interval [0, 1].

Weighted argumentation graphs (or weighted graphs for short) are defined as follows.

Definition 2 (*WAG*). A *weighted argumentation graph* is a triple $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A}$ is a non-empty finite set of arguments, w is a weighting on $\mathcal{A}$, and $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$. We denote by WAG the class of all weighted argumentation graphs.

Intuitively, for $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in$ WAG, $a, b \in \mathcal{A}$, $w(a)$ represents the basic weight of argument a, and $(a, b) \in \mathcal{R}$ (or equivalently $a\mathcal{R}b$) means argument a attacks argument b. Let $\langle a_0, \ldots, a_{2n+1} \rangle$ be a non-empty sequence of arguments of $\mathcal{A}$ such that $n \in \mathbb{N}$, for every $0 \le j < 2n + 1$, $a_j \mathcal{R} a_{j+1}$, $a_0 = a$, and $a_{2n+1} = b$. If $n = 0$, then a is a *direct attacker* of b; it is an *indirect attacker* of b if $n \ge 1$.

Definition 3 (*Isomorphism*). Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle, \mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in$ WAG. An *isomorphism* from $\mathbf{G}$ to $\mathbf{G}'$ is a bijective function f from $\mathcal{A}$ to $\mathcal{A}'$ such that:

- $\forall a \in \mathcal{A}$, $w(a) = w'(f(a))$,
- $\forall a, b \in \mathcal{A}$, $a\mathcal{R}b$ iff $f(a)\mathcal{R}'f(b)$.

Let us recall the notion of path between two nodes in a graph.

Definition 4 (*Path*). Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in$ WAG, and $a, b \in \mathcal{A}$. A *path* from b to a is a finite non-empty sequence $\langle x_1, \ldots, x_n \rangle$ such that $x_1 = b$, $x_n = a$, and $\forall 1 \le i < n$, $x_i \mathcal{R} x_{i+1}$.

Note that, according to the previous definition, a sequence of length one is also considered to be a path. We present next the list of *all* notations used in the paper.

Notations. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in$ WAG and $a \in \mathcal{A}$. We denote by $\text{Att}_{\mathbf{G}}(a)$ the set of all *attackers* of a in $\mathbf{G}$, i.e., $\text{Att}_{\mathbf{G}}(a) = \{b \in \mathcal{A} \mid b\mathcal{R}a\}$. Let $\mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in$ WAG such that $\mathcal{A} \cap \mathcal{A}' = \emptyset$. We denote by $\mathbf{G} \oplus \mathbf{G}'$ the element $\langle \mathcal{A} \cup \mathcal{A}', w'', \mathcal{R} \cup \mathcal{R}' \rangle$ of WAG such that for any $x \in \mathcal{A}$ (resp. $x \in \mathcal{A}'$), $w''(x) = w(x)$ (resp. $w''(x) = w'(x)$).

3. Strength of arguments

In most approaches in the literature [37–40], an argument is a set of premises that serve as reasons for accepting a claim.[1] However, unlike a mathematical demonstration, it does not necessarily guarantee the truth of the claim. Its *strength* may range from very weak to strong depending on the plausibility of the premises, the strength of the link between the premises and the claim and its interaction with other arguments.

In the literature, evaluation of arguments is conducted by formal methods, called *semantics*. Their key idea is to predict whether an argument can be accepted by a rational agent so that its claim can safely be used for drawing conclusions, making decisions, etc. The very first semantics, proposed by Dung in his seminal paper [13], focused on *extensions*. For a given argument graph, each of Dung's semantics returns several extensions, where each extension is a set of arguments representing an individually reasonable position. In this setting, acceptability status of an argument in a graph is defined as follows [14,16,42,38,43]: an argument is *sceptically accepted* if it belongs to all extensions, it is *credulously accepted* it belongs to some but not all extensions, and it is *rejected* if it does not belong to any extension.

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[1] There are some approaches where an argument contains more components, e.g. the Toulmin's model of argument [41].

Gradual semantics [10,12,16,18,19,44] take a view from another perspective. Instead of calculating possible coherent points of view (i.e. extensions) they assign a unique numerical or qualitative value to each argument, representing its strength in a graph. Indeed, having more than three acceptability statuses (skeptically / credulously accepted and rejected) might be beneficial in some applications. Take, for instance, multiple criteria decision making (MCD), where the main objective is to define mathematical models that are able to compare different alternatives on the basis of how they perform regarding a set of criteria. The more discriminating a model between alternatives, the more efficient it is. In argument-based MCD models, an argument in favor of an alternative expresses how the latter satisfies a criterion (see [45,46]). Thus, it is not sufficient to identify sceptically/credulously acceptable arguments (as alternatives may all be supported by acceptable arguments), arguments strengths are crucial for fine-grained comparisons of alternatives.

In this paper, we focus on argument strength. We start the analysis with reference to some basic questions:

- Does an argument have a *unique* strength?
- What is the *form* of argument strength?
- What are the *factors* that may impact argument strength?
- Is there a unique way of evaluating argument strength?
- What is a good evaluation method of argument strength?

Concerning the first question, in this paper we focus on a single-status notion of strength. As pointed out by Leite and Martins [10], even if assigning multiple strength values may be interesting from a theoretical point of view, most users (for instance, in the domain of e-democracy / on-line debates) would be turned away by a system which is based on such a semantics.

Concerning the second question, argument strength may take two forms, as *absolute* value or *relative* one compared to other arguments. Obviously, a ranking of arguments can be constructed from absolute argument strengths. However, the converse is not always true. In the argumentation literature, gradual semantics initiated in [16] assign a qualitative or numerical value to each argument, while ranking semantics introduced in [20] rank-order arguments from the strongest to the weakest ones without necessarily computing the exact strength of arguments. Given that gradual semantics can also be used to induce a ranking of arguments, both classes of semantics are suitable when a comparison of arguments is needed. This is particularly the case in decision problems where the goal is to rank-order different alternatives (e.g. candidates for a research position) on the basis of the strength of their supporting arguments. In addition, gradual semantics are also able to go beyond qualitative comparison, namely they can say *to what extent* one argument is stronger than another one.

Regarding the third question, argument strength depends on two main elements: i) the (typology of the) *weighted argumentation graph* to which the argument belongs, and ii) the *application* that gave birth to the weighted argumentation graph. The strength of the same argument may change from one graph to another due to the following key factors influencing it within a fixed graph:

- *Basic weights* of arguments
- *Quantity* of attackers
- *Strengths* of attackers

Besides, argumentation theory has a very broad and diversified range of practical applications (e.g. critical debate, committees, trial) and theoretical ones (e.g. inconsistency handling in knowledge bases [39], defeasible reasoning [3], decision making [4], classification [5], negotiation [47]). All these disparate applications may require different types of arguments (deductive, abductive, analogical, etc), each of which may impose particular *constraints* on the evaluation of argument strength. Some of them may concern the use of the previous factors. For example, assume an application where arguments may be *analogical* in nature, i.e., their reasoning is based on perceived similarities between two objects. Dissimilarities between objects might decrease the strength of the analogy [48–51]. In the framework discussed by Amgoud [52], every attack raises a novel case of dissimilarity between the two objects. Hence, in such a setting, the more an analogical argument is attacked, the weaker it may be. This means that the quantity of attackers of an argument is important. However, this factor may be less desirable when handling inconsistency in logical knowledge bases, since a single attack is generally lethal for its target. Hence, the above factors may be considered in some applications and not in others, and semantics need not capture all of them. Applications may require other constraints, which are not related to the above factors. An example of such a constraint is the impact of *worthless* attackers on their targets. A worthless attacker is an argument whose strength is extremely weak (say 0 when argument strength is evaluated on a scale [0,1]). In a scientific debate about the safety of a recently developed vaccine, such arguments should not have any impact on their targets. However, it is argued that worthless arguments may have an impact in political debates [53].

From the above discussion, it follows that there is no unique way of evaluating arguments, which answers the fourth question. A reasonable semantics is one that takes into account the factors that are suitable for the application under study, and the peculiarities of the application.
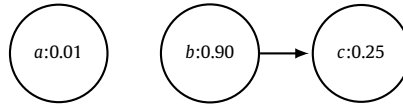
Fig. 1. Weighted graph G_1.

4. Foundational principles for semantics

Throughout the paper, we focus on semantics that evaluate argument strength in the context of weighted argumentation graphs. We are particularly interested in gradual semantics, which ascribe to each argument a single absolute value taken from a totally ordered scale with the convention that the greater the value, the stronger the argument. The choice of the exact scale is not crucial for the definition of principles, the only requirement is that it should have minimum V_{min} and maximum V_{max} values. The former specifies *worthless* arguments and the latter refers to *perfect* ones. For the sake of illustration, in this paper we consider the unit interval [0, 1], and call a non-worthless argument *alive*.

Definition 5 *(Semantics)*. A *semantics* is a function **S** transforming any weighted argumentation graph $G = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ into a weighting Deg_G^S on $\mathcal{A}$ (i.e., $\text{Deg}_G^S : \mathcal{A} \to [0, 1]$). For any $a \in \mathcal{A}$, $\text{Deg}_G^S(a)$ represents the *strength* of a.

The previous definition introduces the notion of semantics in general terms, without specifying whether and how it depends on the two main elements analyzed in the previous section: the (typology of the) weighted argumentation graph, and the application that gave birth to the graph. In this paper, we abstract away from applications, and investigate evaluations of arguments *exclusively* on the basis of a weighted argumentation graph. For that purpose, we propose a set of 21 *principles* that describe basic properties of semantics and factors that may be taken into account in the evaluation of strength. These principles will play three roles:

1. clarifying foundations of argument evaluation,
2. theoretically analyzing each existing semantics. This would clarify the choices made by those semantics,
3. theoretically comparing the plethora of semantics that exist in the literature.

The principles are *elementary* properties that are free of implicit assumptions. Each of them describes a unique idea, some of their combinations lead to higher-level properties like those proposed in [20], and four principles follow from the others. We considered them in the paper for an in-depth analysis of certain existing semantics. Finally, it is worth mentioning that some principles extend properties proposed by Amgoud and Ben-Naim in [27] for flat graphs. The new versions take into account basic weights of arguments. Other principles are simplified versions of those proposed in [28]. Five properties (Weakening, Reinforcement, Proportionality, Invariance and Strict Invariance) are novel and have no counterparts in [27,28]. Let us now introduce the principles.

The first principle, called *anonymity*, states that arguments should not be evaluated on the basis of their names exactly as the evaluation of students' work should not depend on their identity. This property can be found in many axiomatic studies including those in the domain of cooperative games [54]. In the argumentation literature, it is called Anonymity in [27], *abstraction* in [20], and *language independence* in [24]. This principle expresses some rationality of a semantics, it is thus mandatory and any semantics should satisfy it.

Principle 1 *(Anonymity)*. A semantics **S** satisfies anonymity iff, for all $G = \langle \mathcal{A}, w, \mathcal{R} \rangle$, $G' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in \text{WAG}$, for any isomorphism f from G to G', the following property holds:

$$\forall a \in \mathcal{A}, \text{Deg}_G^S(a) = \text{Deg}_{G'}^S(f(a)).$$

The second principle, called *independence*, is about the essence of strength which expresses to what extent an argument is robust against attacks. It states that the strength of an argument should be independent of any argument that is not connected to it by a path. It delimits thus the sub-graph of a weighted argumentation graph that *may* have an impact on the strength of an argument. This principle is crucial to avoid strength being biased by irrelevant information. Consider the argumentation graph depicted in Fig. 1 and which is extracted from an online debate platform.

a: Señor Taco is the best restaurant in Toulouse, since they have very good guacamole,
b: School meals are generally not healthy,
c: Schools should provide breakfast to pupils at the start of each day because schools are the best place to ensure good nutrition.

Obviously, the strength of *c* should not depend on that of *a* since their topics are completely unrelated. Finally, it is worth mentioning that this principle generalizes the independence axiom from [27].

Principle 2 (*Independence*). *A semantics* **S** *satisfies* independence *iff, for all* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, $\mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in \mathtt{WAG}$ *such that* $\mathcal{A} \cap \mathcal{A}' = \emptyset$, *the following property holds*:

$$\forall a \in \mathcal{A}, \; \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G} \oplus \mathbf{G}'}(a).$$

Example 1. Consider the weighted argumentation graph $\mathbf{G}_1$ depicted in Fig. 1, where the numerical values represent basic weights. The strength of a should be independent from that of c since there is no path from c to a.

The third principle, called *directionality*, states that an attacker may influence the strength of its target but the converse is not allowed. In other words, attacking other arguments cannot be beneficial or harmful for an attacker. Recall that argument strength expresses the plausibility of premises, prior acceptance of the argument's claim and the solidity of the link. These three parameters can only be affected by incoming attacks showing their weaknesses, the fact that an argument attacks another argument does neither improve nor worsen their strength. Like the two previous principles, we regard Directionality as mandatory for each semantics. Formally, Directionality states that if one adds an attack from an argument a to another argument b in a given graph, then this additional attack may impact the strength of b, but not that of any other argument c which is not related to b by a *path*. This definition is more general than the *Circumscription* axiom presented in [27] even when the arguments have the same basic weights. Indeed, the formal definition of Circumscription assumes the addition of an attack towards an argument b which does not attack any argument while in our case this constraint is relaxed. The principle was also considered in [24] but at the level of extensions of flat argumentation graphs.

Principle 3 (*Directionality*). *A semantics* **S** *satisfies* directionality *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b \in \mathcal{A}$, *for any* $\mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in \mathtt{WAG}$ *such that* $\mathcal{A}' = \mathcal{A}$, $w' = w$, *and* $\mathcal{R}' = \mathcal{R} \cup \{(a, b)\}$, *the following holds: for any* $x \in \mathcal{A}$, *if there is no path from b to x, then* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) = \mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}'}(x)$.

Example 1 (Cont) Let $\mathbf{G}'_1$ be the graph $\mathbf{G}_1$ augmented with a self-attack on c (i.e., c attacks itself in $\mathbf{G}'_1$). The strength of b should be the same in the two graphs $\mathbf{G}_1$ and $\mathbf{G}'_1$.

Remark 1. Example 1 shows some difference between Independence and Directionality. The former is defined at the level of two or more graphs (the two strongly connected components of $\mathbf{G}_1$) while the latter is at the level of one graph ($\mathbf{G}'_1$). It is worth noticing that Independence is silent about whether b should have the same strength in $\mathbf{G}_1$ and $\mathbf{G}'_1$.

Any argument in a weighted graph has a basic weight. Hence, an argument should not be considered strong just because it is not attacked; its strength should also depend on its basic weight. A non-attacked argument can be deemed as weak if its basic weight is low, otherwise the argument is *overvalued*. Consider the weighted argumentation graph $\langle \{A, B\}, w, \mathcal{R} = \emptyset \rangle$ where A, B are as follows:

> *Most people are right-handed. Therefore, Pat is right-handed.*   (***A***)
> *All people have DNA. Therefore, Pat has DNA.*   (***B***)

In possibilistic logic [55], the premises of A and B are encoded in propositional logic as implications $X \rightarrow Y$ and $X \rightarrow Z$, where X, Y, Z stand respectively for "being a person", "being right-handed" and "having DNA". Since the rule $X \rightarrow Y$ has exceptions, it is ascribed a *necessity* or *certainty degree* [55] from the scale $[0, 1]$ that is less than the maximal value 1. The second rule $X \rightarrow Z$ is certain, thus it is ascribed value 1. Consequently, in [9] a basic weight is assigned to each argument. It is the certainty degree of the least certain premise of an argument. Note that both A and B have the hidden certain premise: Pat is a person. Thus, $w(A) < 1$ and $w(B) = 1$. Unlike B, the argument A does not guarantee its conclusion since it is based on uncertain premises. Hence, even if both arguments are not attacked, B can be deemed stronger than A. For an accurate evaluation of arguments and since the scales of basic weight and strength are commensurate (both are the interval $[0, 1]$), our next principle, called *Maximality*, states that the strength of a non-attacked argument is equal to the basic weight of the argument ($w(A) < 1$ for A and $w(B) = 1$ for B). In [27], all the arguments of an argumentation graph are assumed to have a basic weight equal to 1; that is why non-attacked arguments get value 1.

Principle 4 (*Maximality*). *A semantics* **S** *satisfies* maximality *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for any* $a \in \mathcal{A}$, *if* $\mathtt{Att}_{\mathbf{G}}(a) = \emptyset$, *then* $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = w(a)$.

Example 1 (Cont) Consider again the weighted argumentation graph $\mathbf{G}_1$. Maximality principle ensures that $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a) = 0.01$ and $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(b) = 0.90$. Note that b is stronger than a even if both arguments are unattacked.

The next two principles, *Weakening* and *Strict Weakening*, are about the role of attacks. The latter being negative relations that highlight arguments' weaknesses (e.g., false premises/claim, inapplicable rules), they have negative impact on targets'

strengths. Hence, Weakening states that attacks may weaken, but never strengthen, an argument when they come from *alive* arguments. This principle leaves room for ineffective attacks that may exist in applications, namely in the legal domain. Consider the case of a judge who decides to ignore a given argument during a trial. Even if the argument is alive (for instance it has some basic weight and is not attacked), it has no effect on the arguments it attacks. Strict Weakening is more demanding as it ensures that argument strength should decrease when the argument has at least one alive attacker. This principle is desirable for normative systems, where evaluation of arguments is done objectively on the basis of weighted graphs only. Any semantics should satisfy at least Weakening since it defines the role of attacks without being too demanding. Note that the Trust-based semantics [12] satisfies Weakening but violates its strict version. It is worth mentioning that in [27,28], we only defined the strict version of weakening, and called it Weakening. In what follows, we call Weakening the non-strict version and Strict Weakening the strict one.

Principle 5 *(Weakening). A semantics* **S** *satisfies* weakening *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$*, for any* $a \in \mathcal{A}$*, if* $\exists b \in \mathtt{Att}_{\mathbf{G}}(a)$ *such that* $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) > 0$*, then* $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \leq w(a)$*.*

In addition to existence of alive attackers, Strict Weakening checks whether an argument can lose weight.

Principle 6 *(Strict Weakening). A semantics* **S** *satisfies* strict weakening *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$*, for any* $a \in \mathcal{A}$*, if*

- $w(a) > 0$,
- $\exists b \in \mathtt{Att}_{\mathbf{G}}(a)$ *such that* $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) > 0$,

then $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) < w(a)$*.*

Example 1 (Cont) If a given semantics **S** satisfies Maximality, then $\mathtt{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b) = 0.90$. If in addition **S** satisfies Weakening, then $\mathtt{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(c) \leq 0.25$ while if it satisfies Strict Weakening, then $\mathtt{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(c) < 0.25$.

Remark 2. It is important to underline that Strict Weakening does not formally imply Weakening. Namely, the strict variant does not impose any constraint on a target whose initial weight is zero. More generally, we will present three more pairs of principles which have "strict" and "non-strict" versions (Proportionality, Reinforcement and Invariance), and it is essential to point out that the notion "strict" does not indicate that those versions imply "non-strict" versions. The terminology originates from the type of inequality (strict / non strict) used in the conclusion of the principle.

In some approaches in the argumentation literature, namely in [31,32], if an attacker is weaker than its target, the attack fails. In the context of weighted argumentation graphs, this means that if the basic weight of an attacker is weaker than that of its target, the attack has no effect on the target. Note that this is no longer the case if the semantics satisfies Strict Weakening. Indeed, each attack coming from an alive argument b weakens its target a whatever the values of basic weights, i.e., regardless of whether $w(a) > w(b)$.

The next principle, called *Weakening Soundness*, goes further than the two previous ones by ensuring that attacks are the *only* source of strength loss. Indeed, if an argument loses weight, then it is certainly attacked by at least one alive attacker. This principle is suitable when evaluation of strength is done solely on the basis of weighted argumentation graphs, i.e., there is no extra information that is considered. The following definition simplifies the one from [28] by removing the condition on basic weight, since it follows naturally from the remaining one.

Principle 7 *(Weakening Soundness). A semantics* **S** *satisfies* weakening soundness *iff for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$*, for any* $a \in \mathcal{A}$*, if* $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) < w(a)$*, then* $\exists b \in \mathtt{Att}_{\mathbf{G}}(a)$ *such that* $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) > 0$*.*

We have seen that an alive attacker may weaken its target. An important question is: to what extent that attacker may be harmful? More precisely, can it make the target lose its entire basic weight? *Resilience* principle answers this question negatively. It states that an attack cannot make its targeted argument worthless. More formally, an argument whose basic weight is positive cannot get strength equal to 0 due to attacks. Considering this principle for a semantics depends largely on the application at hand. Resilience makes perfect sense, for instance, in debates deprived of formal rules, like those on societal issues (e.g. capital punishment, abortion) where people express their opinion. Assume the following dialogue between Carla and Paul:

Carla Let us go to Señor Taco because it has the best Mexican food in Toulouse. (A)
Paul: I don't agree that Señor Taco has the best Mexican food in the city. The food at La Sandia is better. (B)

Assume that Carla knows La Sandia restaurant. The argument B attacks A since the conclusion of B contradicts the premise of A. However, B does not make A worthless since it is simply based on Paul's personal opinion. Resilience is, however, not
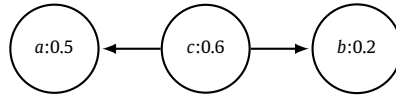
Fig. 2. Weighted graph $\mathbf{G}_2$.

suitable for reasoning with inconsistent propositional knowledge bases (e.g. [39,56]). In this case, an argument is deductive in nature, and it is valid if its premises are true. Note that when the premises are true, the conclusion of the argument is also true. An attack amounts to showing that one or more premises of an argument are false. Thus, it is lethal for the validity of the targeted argument. Note that Resilience generalizes the Resilience axiom from [27]. Furthermore, it was shown in [27] that in the case of flat argumentation graphs, Resilience is one of the main principles that distinguishes extension semantics [13] from those proposed in [20,39]. Indeed, the former violate the principle while the latter satisfy it.

Principle 8 (*Resilience*). *A semantics* **S** *satisfies* resilience *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in$ WAG*, for any* $a \in \mathcal{A}$*, if* $w(a) > 0$*, then* $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > 0$.

Example 1 (Cont) Consider the graph $\mathbf{G}_1$. If a semantics **S** satisfies Resilience, then $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(c) > 0$.

The two following principles concern the impact of *basic weights* on strengths of arguments. Proportionality states that if the basic weight of an argument increases, then its strength might increase as well, but it surely cannot decrease. Similarly, Strict Proportionality ensures that increasing the basic weight of an argument necessarily leads to an increase of the strength of that argument. Trust-based semantics [12] satisfies the non-strict version but violates the strict one. The non-strict variant of the principle (which we simply call Proportionality) is more suitable in applications where basic weights do not play an important role. Consider an application where the goal is to analyze whether users of an online debate platform evaluate arguments in a rational way. For that purpose, the application evaluates arguments using some semantics solely based on attacks, and then compares the results with votes provided by users. In this case, basic weights of arguments were not used by the semantics. In another application like decision making [19], basic weights are important and thus one may need Strict Proportionality. Note that in [28], we introduced only the strict version and called it Proportionality. For the sake of coherence, in what follows, this word will refer to the non-strict version.

Principle 9 (*Proportionality*). *A semantics* **S** *satisfies* proportionality *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in$ WAG*, for all* $a, b \in \mathcal{A}$ *such that*

- $w(a) \geq w(b)$,
- $\mathrm{Att}_{\mathbf{G}}(a) = \mathrm{Att}_{\mathbf{G}}(b)$,

then $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) \geq \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

The strict version of Proportionality is given below.

Principle 10 (*Strict Proportionality*). *A semantics* **S** *satisfies* strict proportionality *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in$ WAG*, for all* $a, b \in \mathcal{A}$*, if*

- $w(a) > w(b)$,
- $\mathrm{Att}_{\mathbf{G}}(a) = \mathrm{Att}_{\mathbf{G}}(b)$,
- $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > 0$,

then $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

Example 2. Let $\mathbf{G}_2$ be the weighted argumentation graph depicted in Fig. 2. If a semantics **S** satisfies Proportionality, then $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_2}(a) \geq \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_2}(b)$. Assume that **S** satisfies Resilience. Then, $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_2}(a) > 0$. If **S** satisfies Strict Proportionality, then $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_2}(a) > \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_2}(b)$. Note that Strict Proportionality is compatible with non-Resilience. Assume that **S** violates Resilience, and that, like extension semantics, an attack may make its target worthless. Hence, it is possible to have $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_2}(a) = \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_2}(b) = 0$. In this case, Principle 10 is not applicable since its third condition is not fulfilled.

The next principle, called *Monotony*, concerns the *quantity of attackers* of an argument. It states that the more an argument is attacked, the weaker it may be. This principle is desirable in various applications, such as decision making and analogical reasoning. For instance, it has been shown recently in [52], that the number of attackers plays a crucial role in the evaluation of analogical arguments. The latter are inductive arguments that rely on analogies for drawing conclusions. They cite accepted similarities between two items in support of the conclusion that some further similarity exists

between the items. Attacks amount at highlighting dissimilarities between the items. The more dissimilarities are pointed out, the weaker the analogy and thus the weaker the analogical argument. This principle extends the one proposed in [27] by accounting for basic weights.

Principle 11 *(Monotony). A semantics* **S** *satisfies* monotony *iff, for any* **G** $= \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b \in \mathcal{A}$, *if*

- $w(a) = w(b)$,
- $\mathtt{Att}_{\mathbf{G}}(a) \subseteq \mathtt{Att}_{\mathbf{G}}(b)$,

then $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

The next three principles concern the *strength of individual attackers. Neutrality* states that any worthless attacker (attacker whose strength is 0) has no impact on its target. In other words, being attacked by such an attacker is similar to not being attacked at all. This principle was initially proposed in [27] for flat argumentation graphs, then generalized for weighted graphs in [28]. Both versions make two implicit assumptions: *symmetry* and *independence*. Symmetry states that a set of attackers has the same effect on arguments having the same basic weight. The second assumption states that the attackers of an argument are independent from each other. We propose below an elementary version of Neutrality which does not implicitly suppose those assumptions. Regarding the suitability of Neutrality, in [53], the authors discussed some specificities of political debates and argued that in such debates worthless attackers may have an impact on their targets. In other applications like scientific debates, Neutrality is certainly suitable since arguments should only be rejected based on substantial grounds.

Principle 12 *(Neutrality). A semantics* **S** *satisfies* neutrality *iff, for any* **G** $= \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b, x \in \mathcal{A}$, *if*

- $w(a) = w(b)$,
- $\mathtt{Att}_{\mathbf{G}}(a) = \emptyset$,
- $\mathtt{Att}_{\mathbf{G}}(b) = \{x\}$ *with* $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$,

then $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

The two principles *Reinforcement* and *Strict Reinforcement* ensure that the strength of alive attackers is taken into account in the evaluation of argument strength. They respectively state that increasing the strength of an alive attacker may lead to a decrease in the strength of its target. In other words, the stronger an attacker, the more harmful it may be.

Under the name Reinforcement, the Strict Reinforcement property was proposed in [27] for flat graphs and extended in [28] for weighted ones. Furthermore, in both papers the formal definition of the property makes implicitly the two previous assumptions of symmetry and independence. In what follows, we simplify the principle from [28], call it Strict Reinforcement, and propose its non-strict version, called here Reinforcement. Both (strict and non-strict) versions are elementary in that they do not make the two assumptions. We will see later that the old versions of Reinforcement (from [27,28]) follow from our Strict Reinforcement principle together with some other principles. The non-strict version of Reinforcement may be suitable for neglecting small differences in strengths of attackers. In the literature, Trust-based semantics [12] satisfies the non-strict version of the above property and violates its strict version.

Principle 13 *(Reinforcement). A semantics* **S** *satisfies* reinforcement *iff, for any* **G** $= \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b, x, y \in \mathcal{A}$, *if*
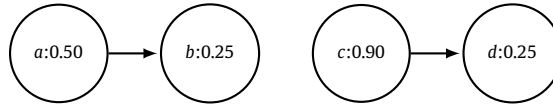
- $w(a) = w(b)$,
- $\mathtt{Att}_{\mathbf{G}}(a) = \{x\}$,
- $\mathtt{Att}_{\mathbf{G}}(b) = \{y\}$,
- $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) \geq \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$,

then $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

The fourth condition excludes worthless attackers since the latter are treated by Neutrality principle.

Principle 14 *(Strict Reinforcement). A semantics* **S** *satisfies* strict reinforcement *iff, for any* **G** $= \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b, x, y \in \mathcal{A}$, *if*

- $w(a) = w(b)$,
- $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$,
- $\mathtt{Att}_{\mathbf{G}}(a) = \{x\}$,
- $\mathtt{Att}_{\mathbf{G}}(b) = \{y\}$,

Fig. 3. Weighted graph $\mathbf{G}_3$.

- $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$,

then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Example 3. Consider the weighted argumentation graph $\mathbf{G}_3$ depicted in Fig. 3. Let $\mathbf{S}$ be a semantics which satisfies Maximality. Thus, $\mathrm{Deg}_{\mathbf{G}_3}^{\mathbf{S}}(a) = 0.5$ and $\mathrm{Deg}_{\mathbf{G}_3}^{\mathbf{S}}(c) = 0.9$. Reinforcement ensures that $\mathrm{Deg}_{\mathbf{G}_3}^{\mathbf{S}}(b) \geq \mathrm{Deg}_{\mathbf{G}_3}^{\mathbf{S}}(d)$ since the attacker of b is weaker than the attacker of d. If $\mathbf{S}$ satisfies Resilience, then $\mathrm{Deg}_{\mathbf{G}_3}^{\mathbf{S}}(b) > 0$ and thus, Strict Reinforcement leads to $\mathrm{Deg}_{\mathbf{G}_3}^{\mathbf{S}}(b) > \mathrm{Deg}_{\mathbf{G}_3}^{\mathbf{S}}(d)$.

The next four principles, *Symmetry, Equivalence, Invariance* and *Strict Invariance*, are about the strength of groups of attackers. They express the general idea that two equally strong groups of attackers should have the same impact on an argument. The four versions (from the simplest to the richest one) were proposed for an *accurate* analysis of existing semantics and a deeper comparison of pairs of them. For example, Dung's semantics violate *Invariance*, but satisfy its simplest form *Symmetry*. Trust-based semantics satisfies the three first versions but violates the strongest one (Strict Invariance). This finer-grained analysis is important for a better understanding of those semantics and their foundations.

The simplest principle, called *Symmetry*, states that the same group of attackers should have the same impact on all arguments having the same basic weights. It captures the symmetry assumption underlying the Neutrality and Reinforcement principles in [28].

Principle 15 *(Symmetry). A semantics* $\mathbf{S}$ *satisfies* symmetry *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b \in \mathcal{A}$, *if*

- $w(a) = w(b)$,
- $\mathrm{Att}_{\mathbf{G}}(a) = \mathrm{Att}_{\mathbf{G}}(b)$,

then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Equivalence is more demanding. It states that two groups of equally strong attackers should have the same impact on arguments having the same basic weights. This means also that the strength of an argument depends *only* on the basic weight of the argument and the strengths of its (direct and indirect) attackers.

Principle 16 *(Equivalence). A semantics* $\mathbf{S}$ *satisfies* equivalence *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b \in \mathcal{A}$, *if*

- $w(a) = w(b)$,
- *there exists a bijective function* f *from* $\mathrm{Att}_{\mathbf{G}}(a)$ *to* $\mathrm{Att}_{\mathbf{G}}(b)$ *such that* $\forall x \in \mathrm{Att}_{\mathbf{G}}(a)$, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$,

then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

The next two principles are new and have no counterparts in [27,28]. They capture the independence assumption that was implicit in Neutrality and Reinforcement principles from [28]. *Invariance* states that the strength of a group cannot decrease when a new attacker joins the group.

Principle 17 *(Invariance). A semantics* $\mathbf{S}$ *satisfies* invariance *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b, a', b', x, y \in \mathcal{A}$ *such that*

- $w(a) = w(a') = w(b) = w(b')$,
- $\mathrm{Att}_{\mathbf{G}}(a') = \mathrm{Att}_{\mathbf{G}}(a) \cup \{x\}$ *with* $x \notin \mathrm{Att}_{\mathbf{G}}(a)$,
- $\mathrm{Att}_{\mathbf{G}}(b') = \mathrm{Att}_{\mathbf{G}}(b) \cup \{y\}$ *with* $y \notin \mathrm{Att}_{\mathbf{G}}(b)$,
- $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(y)$,

the following holds: if $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$, *then* $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') \geq \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$.

The next principle defines the strict version of Invariance. This property is strong since, as we will see in Section 5, it enforces a semantics to consider all the attackers of an argument. In other words, it ensures that each alive attacker will have an impact on its target. The principle is thus suitable in applications where each attacker matters like the case
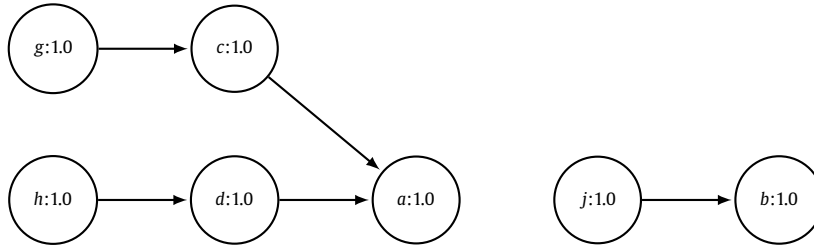
Fig. 4. Weighted graph $\mathbf{G}_4$.

of analogical arguments. Amgoud [52] proposed a framework for reasoning about analogical arguments. It is based on perceived similarities between two objects for inferring some further similarity that has yet to be observed. Its attackers point out cases of dissimilarity between the two objects. Hence, attackers are pairwise different and the greater the number of attackers, the greater the dissimilarity between the objects - and the weaker the target argument. This is in accordance with the claim of some philosophers [48–51] that every distinct dissimilarity decreases the strength of an analogy.

Principle 18 (*Strict Invariance*). *A semantics* **S** *satisfies* strict invariance *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b, a', b', x, y \in \mathcal{A}$ *such that:*

- $w(a) = w(a') = w(b) = w(b')$,
- $\mathtt{Att}_{\mathbf{G}}(a') = \mathtt{Att}_{\mathbf{G}}(a) \cup \{x\}$ *with* $x \notin \mathtt{Att}_{\mathbf{G}}(a)$,
- $\mathtt{Att}_{\mathbf{G}}(b') = \mathtt{Att}_{\mathbf{G}}(b) \cup \{y\}$ *with* $y \notin \mathtt{Att}_{\mathbf{G}}(b)$,
- $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(y)$,
- $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') > 0$,

the following holds: if $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$ *then* $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') > \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$.

The three last principles have no counterparts in [27]. They concern possible strategies that a semantics may follow when it faces a conflict between the *strength* and the *quantity* of attackers as shown by the following example.

Example 4. Consider the weighted argumentation graph $\mathbf{G}_4$ in Fig. 4. The argument a has two weak attackers (each attacker is attacked). The argument b has only one but strong attacker. The question is which of a and b is stronger?

The answer to the previous question depends on which of quantity and quality is more important. *Cardinality precedence* states that a great number of attackers has more effect on an argument than just a few. This strategy makes sense in some applications like debates, where the number of participants attacking a point of view is important.

Principle 19 (*Cardinality Precedence*). *A semantics* **S** *satisfies* cardinality precedence *(CP)* *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b \in \mathcal{A}$, *if*

- $w(a) = w(b)$,
- $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$,
- $|\{x \in \mathtt{Att}_{\mathbf{G}}(a) \mid \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0\}| < |\{y \in \mathtt{Att}_{\mathbf{G}}(b) \mid \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > 0\}|$,

then $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Quality precedence principle gives more importance to the strength of attackers.[2] This strategy is important, for instance, in debates requiring expertise. If a Fields medalist says P, whilst three students say $\neg P$, we probably believe P. It is worth mentioning that this principle is similar to the *Pessimistic* rule in decision making under uncertainty [57].

Principle 20 (*Quality Precedence*). *A semantics* **S** *satisfies* quality precedence *(QP)* *iff, for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b \in \mathcal{A}$, *if*

- $w(a) = w(b)$,
- $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$,

---

[2] We use the term quality exclusively to emphasize that the strength of an attacker influences the quality of the corresponding attack.

- $\exists y \in \text{Att}_{\mathbf{G}}(b)$ *such that* $\text{Deg}^{\mathbf{S}}_{\mathbf{G}}(y) > 0$ *and* $\forall x \in \text{Att}_{\mathbf{G}}(a)$, $\text{Deg}^{\mathbf{S}}_{\mathbf{G}}(y) > \text{Deg}^{\mathbf{S}}_{\mathbf{G}}(x)$,

then $\text{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > \text{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.

Compensation says that several weak attackers might, in some situations, compensate one or more strong attackers. For instance, in the graph $\mathbf{G}_4$, the two weak attackers of a may compensate the strong attacker of b, and a is declared as strong as b. This strategy is very common in multiple criteria decision making where several weak criteria compensate a strong one [58].

Principle 21 *(Compensation). A semantics* $\mathbf{S}$ *satisfies* compensation *iff it violates both CP and QP.*

Note how weak the compensation principle is. Namely, it is sufficient to find one argumentation graph that violates CP and one that violates QP, in order to satisfy this principle.

Example 4 (Cont) Assume a semantics $\mathbf{S}$ which satisfies Resilience. Thus by definition, $\text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(x) > 0$ for any $x \in \{a, b, c, d, g, h, j\}$. Assume also that $\mathbf{S}$ satisfies Maximality and Strict Weakening, then $\text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(j) = 1$, $\text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(c) < 1$, and $\text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(d) < 1$. Hence, $\text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(j) > \text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(c), \text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(d)$. If $\mathbf{S}$ satisfies (CP), then $\text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(a) < \text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(b)$. However, if $\mathbf{S}$ satisfies (QP), then $\text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(a) > \text{Deg}^{\mathbf{S}}_{\mathbf{G}_4}(b)$.

It is worth mentioning that two axioms, similar to (CP) and (QP), were proposed for the first time by Amgoud and Ben-Naim [20] for flat graphs and ranking semantics. Recall that the latter do not quantify strengths of arguments, but rather define a preference relation between arguments. Thus, the equivalent axiom of (QP) uses that preference relation while the one corresponding to (CP) simply counts the number of attackers, even the worthless ones. Our principles are finer since they do not consider worthless attackers. The three previous principles (CP, QP, Compensation) were also investigated by the same authors in [59] for support argumentation graphs, i.e., graphs where arguments may only *support* each other.

At the end of this section, let us discuss our choice to use the interval [0, 1] as the co-domain of gradual semantics. Note that all our principles are concerned with comparing arguments' scores, without essentially using the fact that the co-domain is equal to [0, 1]. They only use the ordering on the arguments' strengths, so they are also applicable to semantics that use other total orders with a minimal value. In case of semantics that are defined on a co-domain without a minimal value, Neutrality is non-applicable, and other principles can be easily modified by dropping the conditions which require that the degrees of arguments are strictly positive. For example, if a semantics would take values from a scale without a minimal value (e.g. (0, 1]), we could modify the principles as follows. For instance, Strict Reinforcement would be changed by dropping the condition $\text{Deg}(a) > 0$ and simply transforming the condition $\text{Deg}(y) > \text{Deg}(x) > 0$ into $\text{Deg}(y) > \text{Deg}(x)$. To take another example, Weakening would be changed by transforming the condition "there exists an argument b attacking a such that $\text{Deg}(b) > 0$" into "there exists an argument b attacking a".

Let us also note that almost all existing gradual/ranking semantics use either an abstract co-domain or the unit interval [0, 1] (for an overview see [30]). A notable exception is α-BBS [44], whose co-domain is the interval [1, +∞). Under this semantics, the value of an argument represents its burden (so the bigger the score the weaker the argument), but it was recently noted by Amgoud and Doder [60] that α-BBS is equivalent to a generalization of h-categorizer [39]. More precisely, the function which assigns reciprocals of α-BBS values to arguments has the co-domain [0, 1] and satisfies the principles.

5. Formal properties

This section presents three kinds of formal properties. The first concerns compatibilities and links between principles. The second concerns generalizations of the elementary principles. The third concerns features of semantics that satisfy subsets of the proposed principles.

5.1. Links and compatibility results

The three principles (CP, QP, Compensation) are incompatible, that is there is no semantics that can satisfy more than one of them. Any semantics should choose one of the three strategies for evaluating arguments in any graph, and thus violates the principles corresponding to the two others. Quality Precedence is also incompatible with another subset of principles.

Proposition 1. *The two following properties hold.*

1. *Cardinality Precedence, Quality Precedence, Maximality and Resilience are incompatible.*
2. *Resilience, Strict Reinforcement, Maximality, Strict Weakening, Strict Invariance and Quality Precedence are incompatible.*

The following result summarizes the various links that exist between the principles.

Proposition 2. *Let* **S** *be a semantics.*

- *If* **S** *satisfies Equivalence, then* **S** *satisfies Symmetry.*
- *If* **S** *satisfies Independence, Directionality, Invariance, and Maximality, then:*
  - *–* **S** *satisfies Equivalence.*
  - *– If* **S** *satisfies Neutrality, then* **S** *satisfies Weakening Soundness.*
  - *– If* **S** *satisfies Weakening, then* **S** *satisfies Monotony.*

Remark: Note that despite the fact that Weakening Soundness, Equivalence, Monotony, and Symmetry follow from other subsets of principles, they belong to our set of principles since they allow to understand the behavior of some existing semantics. As we will see in Section 6, some semantics may satisfy these basic properties while violating the ones implying them. For instance, Stable semantics [13] violates Invariance but satisfies Symmetry. This shows that this semantics does not violate all the cases covered by Invariance.

5.2. Generalizations of principles

The principles are presented in very basic forms for two reasons: i) such forms make them easy to grasp, and ii) their general versions follow from subsets of principles. In what follows, we provide the subsets which lead to the general versions of Directionality, Neutrality, Strict Reinforcement, Invariance and Strict Invariance.

Before presenting the generalization of Directionality, we start by introducing a new notion and a notation.

Given $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, a set $U \subseteq \mathcal{A}$ is *unattacked* in $\mathbf{G}$ if and only if for all $a \in \mathcal{A} \setminus U$ and $b \in U$, $(a, b) \notin \mathcal{R}$.

Notation. For a weighted argumentation graph $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $\mathcal{A}' \subseteq \mathcal{A}$, we denote by $\mathbf{G}|_{\mathcal{A}'}$ the element of $\mathtt{WAG}$ such that:

$$\mathbf{G}|_{\mathcal{A}'} = \langle \mathcal{A}', w|_{\mathcal{A}'}, \mathcal{R} \cap (\mathcal{A}' \times \mathcal{A}') \rangle$$

where $w|_{\mathcal{A}'}$ is the restriction of the function w to the set $\mathcal{A}'$.

The following result presents generalized Directionality. It is worth mentioning that it is an adaptation of the principle with the same name from [24].

Proposition 3. *If a semantics* **S** *satisfies Independence and Directionality, then for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ *and* $U \subseteq \mathcal{A}$ *which is unattacked in* $\mathbf{G}$*, for any* $a \in U$*, the following holds:*

$$\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}|_U}^{\mathbf{S}}(a).$$

The following two results generalize respectively Proportionality and Strict proportionality.

Proposition 4. *Let* **S** *be a semantics which satisfies Independence, Directionality, Equivalence and Proportionality. For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$*, for all* $a, b \in \mathcal{A}$*, if*

- $w(a) \geq w(b)$*,*
- *there exists a bijective function* f *from* $\mathtt{Att}_{\mathbf{G}}(a)$ *to* $\mathtt{Att}_{\mathbf{G}}(b)$ *such that* $\forall x \in \mathtt{Att}_{\mathbf{G}}(a), \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$*,*

then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$*.*

Proposition 5. *Let* **S** *be a semantics which satisfies Independence, Directionality, Equivalence and Strict Proportionality. For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$*, for all* $a, b \in \mathcal{A}$*, if*

- $w(a) > w(b)$*,*
- *there exists a bijective function* f *from* $\mathtt{Att}_{\mathbf{G}}(a)$ *to* $\mathtt{Att}_{\mathbf{G}}(b)$ *such that* $\forall x \in \mathtt{Att}_{\mathbf{G}}(a), \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$*,*
- $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$*,*

then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$*.*

Regarding Neutrality, the idea is that any worthless attacker will not have effect on its target. This is particularly the case for semantics satisfying Independence, Directionality, Invariance, and Neutrality.

Proposition 6. *Let* **S** *be a semantics which satisfies Independence, Directionality, Invariance, and Neutrality. For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$*, for all* $a, b \in \mathcal{A}$*, for any* $X \subseteq \mathcal{A} \setminus \mathtt{Att}_{\mathbf{G}}(a)$*, if*

- $w(a) = w(b)$,
- $\text{Att}_{\mathbf{G}}(b) = \text{Att}_{\mathbf{G}}(a) \cup X$ such that $X \neq \emptyset$ and for any $x \in X$, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$,

then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

The general version of Reinforcement follows from its basic form as well as Independence, Directionality, and Invariance.

Proposition 7. *Let* **S** *be a semantics which satisfies Independence, Directionality, Invariance, and Reinforcement. For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$, *for all* $a, b \in \mathcal{A}$, *If*

- $w(a) = w(b)$,
- $\text{Att}_{\mathbf{G}}(a) \setminus \text{Att}_{\mathbf{G}}(b) = \{x\}$,
- $\text{Att}_{\mathbf{G}}(b) \setminus \text{Att}_{\mathbf{G}}(a) = \{y\}$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$,

then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

The general version of Strict Reinforcement as defined in [28] follows from its basic form as well as Independence, Directionality, Maximality, Invariance, and Strict Invariance.

Proposition 8. *Let* **S** *be a semantics which satisfies Independence, Directionality, Maximality, Weakening, Invariance, Strict Invariance and Strict Reinforcement. For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$, *for all* $a, b \in \mathcal{A}$, *If*

- $w(a) = w(b)$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$,
- $\text{Att}_{\mathbf{G}}(a) \setminus \text{Att}_{\mathbf{G}}(b) = \{x\}$,
- $\text{Att}_{\mathbf{G}}(b) \setminus \text{Att}_{\mathbf{G}}(a) = \{y\}$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$,

then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

The generalized Invariance follows from its basic form as well as Independence and Directionality.

Proposition 9. *Let* **S** *be a semantics which satisfies Independence, Directionality, and Invariance. For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$, *for all* $a, b, a', b' \in \mathcal{A}$, *for all* $X, Y \in \mathcal{P}(\mathcal{A}) \setminus \emptyset$, *if*

- $w(a) = w(a')$,
- $w(b) = w(b')$,
- $\text{Att}_{\mathbf{G}}(a') = \text{Att}_{\mathbf{G}}(a) \cup X$,
- $\text{Att}_{\mathbf{G}}(b') = \text{Att}_{\mathbf{G}}(b) \cup Y$,
- *there exists a bijective function* f *from* X *to* Y *such that for any* $x \in X$, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$

then the following holds: if $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$ then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$.

Similarly, the generalized Strict Invariance follows from its basic form as well as Independence and Directionality.

Proposition 10. *Let* **S** *be a semantics which satisfies Independence, Directionality, and Strict Invariance. For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$, *for all* $a, b, a', b' \in \mathcal{A}$, *for all* $X, Y \in \mathcal{P}(\mathcal{A}) \setminus \emptyset$, *if*

- $w(a) = w(a')$,
- $w(b) = w(b')$,
- $\text{Att}_{\mathbf{G}}(a') = \text{Att}_{\mathbf{G}}(a) \cup X$,
- $\text{Att}_{\mathbf{G}}(b') = \text{Att}_{\mathbf{G}}(b) \cup Y$,
- *there exists a bijective function* f *from* X *to* Y *such that for any* $x \in X$, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') > 0$,

then the following holds: if $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$ then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$.

5.3. Consequences of principles

This section investigates properties of semantics that satisfy some subsets of principles. The first result states that under some principles, an argument that is attacked only by worthless attackers does not lose weight.

Proposition 11. *If a semantics **S** satisfies Independence, Directionality, Invariance, Neutrality, and Maximality, then for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for any* $a \in \mathcal{A}$ *such that* $\mathtt{Att}_{\mathbf{G}}(a) \neq \emptyset$, *if for every* $x \in \mathtt{Att}_{\mathbf{G}}(a)$, $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$, *then* $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = w(a)$.

In [27,28], the strict version of Monotony, called *Counting*, is considered as a principle. It ensures that each attacker impacts its target as soon as it has a strictly positive strength.

Definition 6 *(Counting).* A semantics **S** satisfies Counting iff for any $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, for all $a, b, x \in \mathcal{A}$, if

- $w(a) = w(b)$,
- $\mathtt{Att}_{\mathbf{G}}(b) = \mathtt{Att}_{\mathbf{G}}(a) \cup \{x\}$ with $x \notin \mathtt{Att}_{\mathbf{G}}(a)$ and $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$,
- $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$,

then $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Counting follows from more elementary principles, namely Independence, Directionality, Neutrality, Maximality, Strict Weakening, Invariance, and Strict Invariance.

Proposition 12. *If a semantics **S** satisfies Independence, Directionality, Neutrality, Maximality, Strict Weakening, Invariance, and Strict Invariance, then it satisfies Counting.*

We show next that, roughly speaking, an argument which loses its entire basic weight cannot become better off if it is further attacked.

Proposition 13. *If a semantics **S** satisfies Anonymity, Independence, Directionality, Neutrality, Monotony, Invariance, and Reinforcement, then for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for any* $a, b \in \mathcal{A}$ *such that*

- $w(a) = w(b)$,
- $\mathtt{Att}_{\mathbf{G}}(a) \setminus \mathtt{Att}_{\mathbf{G}}(b) = \{x\}$,
- $\mathtt{Att}_{\mathbf{G}}(b) \setminus \mathtt{Att}_{\mathbf{G}}(a) = \{y\}$,
- $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) \geq \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$,

if $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0$, *then* $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.
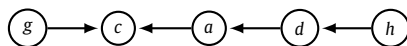
A semantics satisfying Independence, Directionality, Invariance, Neutrality, Maximality, and Weakening, assigns to each argument a value between 0 and its basic weight.

Theorem 1. *If a semantics **S** satisfies Independence, Directionality, Invariance, Neutrality, Maximality, and Weakening, then for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for any* $a \in \mathcal{A}$, $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \in [0, w(a)]$.

The next result delimits the subset of arguments in a weighted argumentation graph which may impact the strength of a given argument. Before introducing the formal result, let us first introduce the notion of *attack structure* of an argument.

Definition 7 *(Attack Structure).* For any $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, for any $a \in \mathcal{A}$, the *attack structure* of a in $\mathbf{G}$ is $\mathtt{Str}_{\mathbf{G}}(a) = \{a\} \cup \{x \in \mathcal{A} \mid$ there is a path from x to $a\}$.

Example 5. Consider the weighted argumentation graph $\mathbf{G}_5$ depicted below, where each argument has basic weight equal to 1. The attack structure of a is $\mathtt{Str}_{\mathbf{G}_5}(a) = \{a, d, h\}$.

Theorem 2. *If a semantics **S** satisfies Independence and Directionality, then for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for any* $a \in \mathcal{A}$, *the following holds:*

$$\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathtt{Deg}_{\mathbf{G}_{|\mathtt{Str}_{\mathbf{G}}(a)}}^{\mathbf{S}}(a).$$

Example 5 (Cont) The previous theorem ensures that only h and d are taken into account in the evaluation of a by any semantics satisfying Independence and Directionality.

We also show that any pair of arguments which have similar attack structures get the same strengths.

Theorem 3. *Let* **S** *be a semantics which satisfies Anonymity, Independence and Directionality. For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b \in \mathcal{A}$, *if there exists an isomorphism* $f : \mathbf{G}|_{\mathtt{Str}_{\mathbf{G}}(a)} \to \mathbf{G}|_{\mathtt{Str}_{\mathbf{G}}(b)}$ *such that* $f(a) = b$, *then*

$$\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b).$$

Another property which follows from a subset of principles is *Counter-Transitivity*, which was introduced in [20] for ranking semantics in the case of non-weighted graphs. It states that: "if the attackers of an argument b are at least as numerous and strong as those of an argument a, then a is at least as strong as b".

Theorem 4. *If a semantics* **S** *satisfies Independence, Directionality, Invariance, Reinforcement, Maximality, Neutrality, and Weakening, then for any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ *and any* $a, b \in \mathcal{A}$, *if*

- $w(a) = w(b)$,
- *there exists an injective function* f *from* $\mathtt{Att}_{\mathbf{G}}(a)$ *to* $\mathtt{Att}_{\mathbf{G}}(b)$ *such that* $\forall x \in \mathtt{Att}_{\mathbf{G}}(a), \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) \leq \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$,

then $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

In case of flat argumentation graphs, it was shown in [27] that any semantics which satisfies Anonymity, assigns the same strength to any argument of an *elementary cycle*. This result still holds in the case of weighted argumentation graphs, namely when the arguments of the elementary cycle all have the same basic weights. Indeed, suppose that we are given an elementary cycle $a_1, \ldots, a_n$ and consider the bijection f such that $f(a_1) = a_2, \ldots, f(a_{n-1}) = a_n, f(a_n) = a_1$. Observe that f is an isomorphism. Then from Anonymity, we obtain that for every i, j, $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a_i) = \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a_j)$.

6. Analysis of some existing semantics

While semantics dealing with flat graphs have already been analyzed in [21,27], in this section we present the first analysis and comparison of existing semantics devoted to weighted argumentation graphs. We focus on semantics that can deal with graphs with cycles[3] and we present our results in Table 1.

We distinguish two families of semantics: The first family extends Dung's semantics with *preferences* between arguments (e.g. [11,31–33,61]), or with *weights* on attacks (e.g. [62–64]). In case of preferences, an argumentation framework takes as input a finite set $\mathcal{A}$ of arguments, a binary attack relation $\mathcal{R}$ between them, and a preference relation $\succeq$ between arguments. The relation $\succeq$ is a (partial or total) preorder, and $\succ$ is its strict version. For two arguments a, b, the notation $a \succeq b$ means that the argument a is at least as preferred as b. In [31–33], the preference relation is abstract and may capture different issues like differences in certainty degrees of information involved in arguments, differences in trustworthiness of arguments' sources, etc. In [11], the preference relation captures differences of importance of values promoted by arguments. Finally, in [61], it captures priorities between rules used in arguments. There are two approaches for dealing with preferences: *contraction-based* and *change-based*.

The contraction-based approach is followed in [11,31,32] and its basic idea is shrinking attacks by getting rid of those whose source is weaker than the target before computing extensions. Indeed, from an input $\langle \mathcal{A}, \mathcal{R}, \succeq \rangle$, a new *flat* argumentation graph $\langle \mathcal{A}, \mathcal{R}_c \rangle$ is computed as follows:

$$\mathcal{R}_c = \{(a, b) \mid (a, b) \in \mathcal{R} \text{ and } b \not\succ a\}.$$

Change-based approach was introduced for the first time in [33], then used in [61], for solving a drawback of the contraction-based approach. Indeed, when the original attack relation $\mathcal{R}$ is not symmetric, the extensions of the graph $\langle \mathcal{A}, \mathcal{R}_c \rangle$ may be conflicting, which is incompatible with the essence of extensions. Amgoud and Vesic [33] then proposed a set of axioms that an extension-based semantics should satisfy. They have then shown that such a semantics amounts to reversing the direction of each attack whose target is stronger than the source. Thus, from an input $\langle \mathcal{A}, \mathcal{R}, \succeq \rangle$, a new *flat* argumentation graph $\langle \mathcal{A}, \mathcal{R}_r \rangle$ is computed as follows:

$$\mathcal{R}_r = \{(a, b) \mid (a, b) \in \mathcal{R} \text{ and } b \not\succ a\} \cup \{(b, a) \mid (a, b) \in \mathcal{R} \text{ and } b \succ a\}.$$

---

[3] We also include Counting (introduced in Definition 6) in the table since it is considered to be a principle in several papers in the literature [27,28]. The goal is to allow the reader to have a full overview in the same table.

Table 1
The symbol ● (resp. ×) stands for satisfied (resp. violated).

Family of semantics	Extension semantics								Gradual semantics				
	Contraction-based				Change-based								
	Grd	Stb	Prf	Com	Grd	Stb	Prf	Com	TB	IS	Mbs	Cbs	Hbs
Anonymity	●	●	●	●	●	●	●	●	●	●	●	●	●
Independence	●	×	●	●	●	×	●	●	●	●	●	●	●
Directionality	●	×	●	●	×	×	×	×	●	●	●	●	●
Maximality	×	×	×	×	×	×	×	×	●	×	●	●	●
Weakening	×	×	×	×	×	×	×	×	●	×	●	●	●
Strict Weakening	×	×	×	×	×	×	×	×	×	×	●	●	●
Weakening Soundness	●	×	×	×	●	×	×	×	●	●	●	●	●
Resilience	×	×	×	×	×	×	×	×	×	×	●	●	●
Proportionality	●	●	●	●	×	×	×	×	●	●	●	●	●
Strict Proportionality	×	×	×	×	×	×	×	×	×	×	●	●	●
Monotony	●	●	●	●	×	×	×	×	×	●	●	●	●
Neutrality	●	●	×	×	×	×	×	×	●	●	●	●	●
Reinforcement	×	×	×	×	×	×	×	×	●	●	●	●	●
Strict Reinforcement	×	×	×	×	●	●	×	×	×	●	●	●	●
Symmetry	●	●	●	●	×	×	×	×	●	●	●	●	●
Equivalence	×	×	×	×	×	×	×	×	●	●	●	●	●
Invariance	×	×	×	×	×	×	×	×	●	●	●	●	●
Strict invariance	×	×	×	×	×	×	×	×	×	×	×	●	●
Cardinality Precedence	×	×	×	×	×	×	×	×	×	×	×	●	×
Quality Precedence	×	×	×	×	×	×	×	×	×	×	●	×	×
Compensation	●	●	●	●	●	●	●	●	●	×	×	×	●
Counting	×	×	×	×	×	×	×	×	×	×	×	●	●

In both approaches, the classical semantics of Dung [13] are applied to the new graphs ($\langle \mathcal{A}, \mathcal{R}_c \rangle$, $\langle \mathcal{A}, \mathcal{R}_r \rangle$). Those semantics are based on two key concepts: *conflict-freeness* and *defense*. Let $x \in \{c, r\}$.

- $\mathcal{E} \subseteq \mathcal{A}$ is *conflict-free* iff $\nexists a, b \in \mathcal{E}$ such that $(a, b) \in \mathcal{R}_x$.
- $\mathcal{E} \subseteq \mathcal{A}$ *defends* $a \in \mathcal{A}$ iff $\forall b \in \mathcal{A}$, if $(b, a) \in \mathcal{R}_x$, then $\exists c \in \mathcal{E}$ such that $(c, b) \in \mathcal{R}_x$.

The four semantics (grounded, complete, stable, preferred) we investigate in this paper are recalled below. Let $\mathcal{E} \subseteq \mathcal{A}$.

- $\mathcal{E}$ is an *admissible extension* iff it is conflict-free and defends all its elements.
- $\mathcal{E}$ is a *complete extension* iff it is admissible and contains all the arguments it defends.
- $\mathcal{E}$ is a *grounded extension* iff it is the minimal (wrt set inclusion) complete extension.
- $\mathcal{E}$ is a *preferred extension* iff it is a maximal (wrt set inclusion) admissible extension.
- $\mathcal{E}$ is a *stable extension* iff it is conflict-free and $\forall a \in \mathcal{A} \setminus \mathcal{E}$, $\exists b \in \mathcal{E}$ such that $(b, a) \in \mathcal{R}_x$.

Given a set of extensions, the most common way to assign a (qualitative) strength to an argument $a \in \mathcal{A}$ (see [14,16,42,38, 43]) is as follows:

- a is *sceptically accepted* if it belongs to all extensions,
- a is *credulously accepted* if it belongs to some but not all extensions,
- a is *rejected* if it does not belong to any extension.

Let us now analyze how this approach works when the input consists of a weighted graph $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$. The question of how to define a preference relation $\succeq$ between arguments has already been answered in [31]. The idea is simply to privilege arguments with the highest basic weight. Indeed, for $a, b \in \mathcal{A}$, a is preferred to b (i.e., $a \succeq b$) iff $w(a) \geq w(b)$. A new *flat* graph $\langle \mathcal{A}, \mathcal{R}_x \rangle$ (with $x \in \{c, r\}$) is then generated as explained above and Dung's semantics are applied to $\langle \mathcal{A}, \mathcal{R}_x \rangle$. The last issue to solve is transforming the three-valued qualitative scale of strengths of arguments into a numerical one. Amgoud and Ben-Naim [27] proposed the following translation: for any $a \in \mathcal{A}$, if $\langle \mathcal{A}, \mathcal{R}_x \rangle$ has no extensions, then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0.3$. Otherwise,

- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 1$ iff a belongs to all extensions.
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0.5$ iff a is in some but not all extensions.
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0.3$ iff a does not belong to any extension and is not attacked by any extension.
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0$ iff a does not belong to any extension and is attacked by at least one extension.

Note that the above definition distinguishes between two types of rejected arguments: those that are attacked by at least one extension, and those that are not attacked by any extension. We generalize the previous definition by replacing

the "magic numbers" 0.3 and 0.5 with β and α respectively such that $0 < \beta < \alpha < 1$. The results of our analysis (see Table 1) are independent of the exact values of the two parameters.

So far, we have shown how strengths of arguments can be computed in weighted graphs with existing preference-based argumentation frameworks. We are thus ready to analyze the proposed semantics against the set of principles described in this paper. Table 1 summarizes the properties of grounded, complete, preferred and stable semantics. It shows that most principles, including Strict Weakening, are violated by the four semantics. Remember that Strict Weakening principle defines formally the role of attacks. It states that each attacked argument should lose weight. The four semantics violate this principle, and an argument may not lose weight even when attacked by an alive attacker. Maximality is also violated since those semantics manipulate the preference relation issued from basic weights rather than the basic weights themselves. The table also shows that in the case of a dilemma between the quality and the quantity of attackers, the four semantics allow compensation. Finally, while preferred and complete semantics satisfy and violate the same principles, preferred and stable do not behave in the same way regarding Independence, Directionality and Neutrality. The grounded semantics satisfies two more principles than preferred semantics: Neutrality and Weakening Soundness.

In [62–64], the input is a finite set $\mathcal{A}$ of arguments, an attack relation $\mathcal{R}$ between arguments, and weights from the unit interval $[0, 1]$ assigned to attacks. It was shown in [63] that when all attacks are assigned weight 1, the framework coincides with Dung's one [13], namely a simple flat argumentation graph whose arguments are evaluated by extension semantics. Furthermore, a flat argumentation graph can be seen as a weighted graph whose arguments are all assigned a basic weight equal to 1. The properties of grounded, complete, stable, preferred semantics in the case of flat graphs were already investigated in [27]. Obviously, every principle violated by a semantics in the flat case is also violated by the same semantics in the weighted case and in the settings of [62–64]. Among the violated principles we mention Independence by stable semantics, Equivalence and Weakening Soundness by complete, stable and preferred semantics, Strict Proportionality and Resilience by the four semantics. It was shown in [27] that under the four semantics, attacks either completely kill their targets (the targets get degree 0) or have no effect.

The second family of semantics deals with basic weights, defines functions assigning a numerical degree to each argument, computes arguments' degrees in an iterative way, and does not resort to the intermediate step of computing extensions. The first semantics of this family is Trust-based (TB) which has been proposed by Da Costa et al. [12]. TB takes as input a weighted argumentation graph $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$ where $w(.)$ expresses the degree of trustworthiness of argument's source. It assigns to each argument $a \in \mathcal{A}$ a strength, which is the limit reached by the scoring function f defined as follows:

$$\text{Deg}_{\mathbf{G}}^{TB}(a) = \lim_{i \to +\infty} f_i(a), \text{ where}$$

$$f_i(a) = \frac{1}{2} f_{i-1}(a) + \frac{1}{2} \min[w(a), 1 - \max_{b\mathcal{R}a} f_{i-1}(b)] \tag{1}$$

It is worth mentioning that the strengths of arguments satisfy equation (1), namely

$$\text{Deg}_{\mathbf{G}}^{TB}(a) = \frac{1}{2} \text{Deg}_{\mathbf{G}}^{TB}(a) + \frac{1}{2} \min[w(a), 1 - \max_{b\mathcal{R}a} \text{Deg}_{\mathbf{G}}^{TB}(b)],$$

hence

$$\text{Deg}_{\mathbf{G}}^{TB}(a) = \min[w(a), 1 - \max_{b\mathcal{R}a} \text{Deg}_{\mathbf{G}}^{TB}(b)]. \tag{2}$$

However, the equation (2) may have more than one solution meaning that it is not a characterization of TB. To illustrate, consider the weighted argumentation graph below with both arguments having basic weight 1:
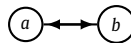
TB returns a single value to each argument, namely $\text{Deg}_{\mathbf{G}}^{TB}(a) = \text{Deg}_{\mathbf{G}}^{TB}(b) = 0.5$. However, it is easy to check that equation (2), simplified into $\text{Deg}_{\mathbf{G}}^{TB}(a) = 1 - \max_{b\mathcal{R}a} \text{Deg}_{\mathbf{G}}^{TB}(b)$, has several solutions including:

- $\text{Deg}_{\mathbf{G}}^{TB}(a) = \text{Deg}_{\mathbf{G}}^{TB}(b) = 0.5$,
- $\text{Deg}_{\mathbf{G}}^{TB}(a) = 1$ and $\text{Deg}_{\mathbf{G}}^{TB}(b) = 0$,
- $\text{Deg}_{\mathbf{G}}^{TB}(a) = 0$ and $\text{Deg}_{\mathbf{G}}^{TB}(b) = 1$,
- ...

Like extension semantics, Table 1 shows that TB follows the compensation strategy in the case of a dilemma between quality and quantity of attackers. However, it satisfies more principles than the four previous semantics. It violates the key principle of *Strict Weakening* meaning that an argument may not lose weight even if it is seriously attacked. Similarly, it

violates Strict Invariance showing that the number of attackers does not necessarily impact the strengths of arguments. This is not surprising since TB extends the labelling approach of extension semantics with weights on arguments.

Gabbay and Rodrigues [34] developed a semantics, called *Iterative Schema* (IS), for evaluating arguments in weighted argumentation graphs ($\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$), where basic weights of arguments may represent different issues. Like TB, this semantics aims at extending the labelling approach of extension-based semantics by taking labels from the unit interval $[0, 1]$ rather than $\{\mathtt{in}, \mathtt{out}, \mathtt{und}\}$. Like TB, IS returns a single labelling for every graph. In this labelling, the value of each argument is the limit reached by iterative applications of a scoring function g. At the initial step, this function assigns to each argument a its basic weight ($w(a)$), and at each step, it recomputes the value of a on the basis of its value and those of its attackers at the previous step.

$$g_i(a) = (1 - g_{i-1}(a)) \min\{\frac{1}{2}, 1 - \max_{b\mathcal{R}a} g_{i-1}(b)\} + g_{i-1}(a) \max\{\frac{1}{2}, 1 - \max_{b\mathcal{R}a} g_{i-1}(b)\}$$

Once the single labelling is computed, IS returns a single extension which contains all the arguments that get value 1. Table 1 shows that IS and Mbs (which will be introduced in the next section) are the only semantics that satisfy Quality Precedence. However, IS violates key principles like Maximality, and thus may return counter-intuitive results. Assume a graph made of a single argument a, which is not attacked and whose basic weight is 0. IS returns a single extension, $\{a\}$, thus declaring a as accepted.

Another semantics of the second family is Simple Product (SP), which was proposed by Leite and Martins [10]. It takes as input a set of arguments, positive and/or negative votes on each argument, and an attack relation between arguments. The basic weight $w(.)$ can be seen as an aggregation of the votes on an argument. Like TB, the semantics uses a scoring function which assigns values to arguments in an iterative way. In their paper, Leite and Martins conjecture that the function converges and assigns a single value to each argument. A counter-example was unfortunately found recently in [65] showing that the semantics may assign more than one value to an argument. Hence, the semantics is not compatible with Definition 5 and will not be investigated here. Table 1 summarizes the results concerning semantics that deal with cyclic weighted argumentation graphs.

Theorem 5. *The properties of Table 1 hold.*

Baroni et al. [18] focused on *bipolar acyclic* graphs, i.e., graphs where arguments may be attacked and supported but without forming cycles. Those graphs are weighted since each argument has a basic weight, which may represent different issues. The authors developed a semantics called QuAD, which was later extended to DF-QuAD by Rago et al. [19]. The two semantics coincide when the support relation is empty, i.e., in the case of weighted argumentation graphs as studied in our paper. In what follows, we study the properties of DF-QuAD in the particular case of a weighted argumentation graph $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$. The strength of any $a \in \mathcal{A}$ is defined as follows:

$$\text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) = w(a) \times \prod_{b\mathcal{R}a} (1 - \text{Deg}_{\mathbf{G}}^{DF-QuAD}(b)). \tag{3}$$

If an argument a has no attackers, then $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(b) = 0$, and hence $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) = w(a)$. It is worth noticing that (DF-)QuAD is not applicable for graphs containing cycles since it does not guarantee uniqueness of strength for each argument. Consider the previous two-length cycle. The degrees of the two arguments a and b are as follows:

$$\begin{cases} \text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) = 1 - \text{Deg}_{\mathbf{G}}^{DF-QuAD}(b) \\ \text{Deg}_{\mathbf{G}}^{DF-QuAD}(b) = 1 - \text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) \end{cases}$$

Solving the two equations amounts to solving $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) + \text{Deg}_{\mathbf{G}}^{DF-QuAD}(b) = 1$. The latter has several solutions including $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) = 1$, $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(b) = 0$ and $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) = 0$, $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(b) = 1$. The following result summarizes the list of principles satisfied/violated by DF-QuAD.

Theorem 6. *(DF-)QuAD violates Strict Invariance, Strict Proportionality, Resilience, QP and CP. It satisfies all the remaining ones.*

Like all the semantics reviewed so far, (DF-)QuAD follows the compensation strategy but it satisfies more principles. Note also that with (DF-)QuAD, an argument may lose its entire strength (i.e., get strength 0).

7. Three novel semantics

We have previously seen that all existing semantics violate the resilience principle. This section introduces three novel semantics that satisfy the principle: one for each of the three incompatible principles (QP, CP, Compensation). *Weighted max-based* semantics satisfies Quality Precedence, *Weighted cardinality-based* semantics satisfies Cardinality Precedence, and *weighted h-Categorizer* satisfies Compensation.

7.1. Weighted max-based semantics (Mbs)

The first semantics satisfies quality precedence, thus it favors the quality of attackers over their cardinality. It is based on a scoring function which follows a multiple step process. At each step, the function assigns a score to each argument. In the initial step, the score of an argument is its basic weight. Then, in each step, the score is recomputed on the basis of the basic weight as well as the score of the *strongest* attacker of the argument at the previous step.

Definition 8 (f_m). Let $G = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$. We define the *weighted max-based function* f_m from $\mathcal{A}$ to $[0, 1]$ as follows: for any argument $a \in \mathcal{A}$, for $i \in \{0, 1, 2, \ldots\}$,

$$f_m^i(a) = \begin{cases} w(a) & \text{if } i = 0 \\ \dfrac{w(a)}{1 + \max\limits_{b \in \text{Att}_G(a)} f_m^{i-1}(b)} & \text{otherwise} \end{cases}$$

By convention, $\max\limits_{b \in \text{Att}_G(a)} f_m^i(b) = 0$ if $\text{Att}_G(a) = \emptyset$.

The value $f_m^i(a)$ is the score of the argument a at step i. This value may change at each step, however, it converges to a unique value as shown in the next theorem.

Theorem 7. *The function f_m^i converges as i approaches infinity.*

Weighted max-based semantics is based on the previous scoring function. The strength of each argument is the *limit* reached using the scoring function f_m.

Definition 9 (Mbs). The *weighted max-based* semantics is a function Mbs transforming any $G = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ into a weighting $\text{Deg}_G^{\text{Mbs}}$ on $\mathcal{A}$ such that for any $a \in \mathcal{A}$, $\text{Deg}_G^{\text{Mbs}}(a) = \lim\limits_{i \to +\infty} f_m^i(a)$.

In what follows, the values are rounded to two decimal places.

Example 1 (Cont) The strengths of the three arguments of the weighted argumentation graph G_1 according to Mbs are as follows: $\text{Deg}_{G_1}^{\text{Mbs}}(a) = 0.01$, $\text{Deg}_{G_1}^{\text{Mbs}}(b) = 0.90$, $\text{Deg}_{G_1}^{\text{Mbs}}(c) = 0.13$. Thus, $\text{Deg}_{G_1}^{\text{Mbs}}(b) > \text{Deg}_{G_1}^{\text{Mbs}}(c) > \text{Deg}_{G_1}^{\text{Mbs}}(a)$. Note that even if a is not attacked, it is weaker than c because of its weak basic weight.

Example 2 (Cont) The strengths of the three arguments of the weighted argumentation graph G_2 according to Mbs are as follows: $\text{Deg}_{G_2}^{\text{Mbs}}(a) = 0.31$, $\text{Deg}_{G_2}^{\text{Mbs}}(b) = 0.12$, $\text{Deg}_{G_2}^{\text{Mbs}}(c) = 0.60$.

Example 3 (Cont) The strengths of the four arguments of the weighted argumentation graph G_3 according to Mbs are as follows: $\text{Deg}_{G_3}^{\text{Mbs}}(a) = 0.50$, $\text{Deg}_{G_3}^{\text{Mbs}}(b) = 0.17$, $\text{Deg}_{G_3}^{\text{Mbs}}(c) = 0.90$, $\text{Deg}_{G_3}^{\text{Mbs}}(d) = 0.13$.

Example 4 (Cont) Consider the graph G_4. Since the graph is acyclic, the strengths can be calculated in two steps as follows:

i	a	b	c	d	g	h	j
0	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.50	0.50	0.50	0.50	1.00	1.00	1.00
2	0.67	0.50	0.50	0.50	1.00	1.00	1.00

The strengths of the seven arguments of the weighted argumentation graph G_4 according to Mbs are as follows: $\text{Deg}_{G_4}^{\text{Mbs}}(g) = 1.00$, $\text{Deg}_{G_4}^{\text{Mbs}}(h) = 1.00$, $\text{Deg}_{G_4}^{\text{Mbs}}(c) = 0.50$, $\text{Deg}_{G_4}^{\text{Mbs}}(d) = 0.50$, $\text{Deg}_{G_4}^{\text{Mbs}}(a) = 0.67$, $\text{Deg}_{G_4}^{\text{Mbs}}(j) = 1.00$, $\text{Deg}_{G_4}^{\text{Mbs}}(b) = 0.50$. Note that $\text{Deg}_{G_4}^{\text{Mbs}}(a) > \text{Deg}_{G_4}^{\text{Mbs}}(b)$.

We show next that the limit scores of arguments satisfy the equation of Definition 8.

Theorem 8. *For any $G = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$, for any $a \in \mathcal{A}$,*

$$\text{Deg}_G^{\text{Mbs}}(a) = \frac{w(a)}{1 + \max\limits_{b \in \text{Att}_G(a)} \text{Deg}_G^{\text{Mbs}}(b)}. \tag{4}$$

The next result states that equation (4) is not just a property of weighted max-based semantics, but also its *characterization*. Indeed, it is the *only* function satisfying the equation. Due to this characterization, equation (4) represents an alternative definition of weighted max-based semantics (Definition 9).

Theorem 9. *Let* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ *and* $D : \mathcal{A} \to [0, 1]$. *If* $D(a) = \frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} D(b)}$, *for all* $a \in \mathcal{A}$, *then* $D \equiv \mathtt{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}$.

Weighted max-based semantics satisfies Quality Precedence as well as *all* the principles which are compatible with it. It violates Strict Invariance since by definition, this semantics focuses only on the strongest attacker of an argument, and neglects the remaining ones.

Theorem 10. *Weighted max-based semantics satisfies all the principles except Strict Invariance, CP and Compensation. It also violates Counting.*

From the previous result, weighted max-based semantics satisfies Resilience, thus an argument cannot lose its entire basic strength. The next result goes further by showing that an argument cannot lose more than half of its basic weight with this semantics.

Theorem 11. *For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for any* $a \in \mathcal{A}$, $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) \in [\frac{w(a)}{2}, w(a)]$.

From the previous result it follows that an argument gets value 0 iff its basic weight is already 0.

Corollary 1. *Let* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ *and* $a \in \mathcal{A}$. *For any* $a \in \mathcal{A}$, $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) = 0$ *iff* $w(a) = 0$.

7.2. Weighted card-based semantics (Cbs)

The second semantics, called *weighted card-based*, favors the number of attackers over their strength. It considers only *founded* arguments, i.e., arguments with a strictly positive basic weight. This restriction is based on the consideration that unfounded arguments are worthless and their attacks are ineffective.

Definition 10 (*Founded Argument*). Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a \in \mathcal{A}$. The argument a is *founded* iff $w(a) > 0$. It is called *unfounded* otherwise. Let $\mathtt{AttF}_{\mathbf{G}}(a)$ denote the set of founded attackers of a.

Weighted card-based semantics is based on a recursive function $\mathtt{f}_c$, which assigns a score to each argument on the basis of its basic weight, the number of its founded attackers, and their scores. The latter (i.e., scores) are considered in order to ensure the Reinforcement principle, that is to take into account the strength of attackers when it is not in conflict with their number.

Definition 11 ($\mathtt{f}_c$). Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$. We define the *weighted card-based function* $\mathtt{f}_c$ from $\mathcal{A}$ to $[0, 1]$ as follows: for any argument $a \in \mathcal{A}$, for $i \in \{0, 1, 2, \ldots\}$,

$$\mathtt{f}_c^i(a) = \begin{cases} w(a) & \text{if } i = 0, \\ \dfrac{w(a)}{1 + |\mathtt{AttF}_{\mathbf{G}}(a)| + \dfrac{\sum\limits_{b \in \mathtt{AttF}_{\mathbf{G}}(a)} \mathtt{f}_c^{i-1}(b)}{|\mathtt{AttF}_{\mathbf{G}}(a)|}} & \text{otherwise.} \end{cases}$$

By convention, $\sum\limits_{b \in \mathtt{AttF}_{\mathbf{G}}(a)} \mathtt{f}_c^{i-1}(b) = 0$ if $\mathtt{AttF}_{\mathbf{G}}(a) = \emptyset$.

The value $\mathtt{f}_c^i(a)$ is the score of the argument a at step i. This value converges to a unique value as i becomes high.

Theorem 12. *The function* $\mathtt{f}_c^i$ *converges as* i *approaches infinity.*

The strength of each argument is the limit reached using the scoring function $\mathtt{f}_c$.

Definition 12 (Cbs). The *weighted card-based* semantics is a function Cbs transforming any $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ into a weighting $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}$ on $\mathcal{A}$ such that for any $a \in \mathcal{A}$, $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(a) = \lim\limits_{i \to +\infty} \mathtt{f}_c^i(a)$.

Example 1 (Cont) Consider the weighted argumentation graph $\mathbf{G}_1$. The strengths of the three arguments according to Cbs are as follows: $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_1}(a) = 0.01$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_1}(b) = 0.90$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_1}(c) = 0.08$.

Example 2 (Cont) Consider the weighted argumentation graph $\mathbf{G}_2$. The strengths of the three arguments according to Cbs are as follows: $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_2}(a) = 0.19$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_2}(b) = 0.07$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_2}(c) = 0.60$. Note that $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_2}(a) > \mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_2}(b)$ even if the two arguments are attacked by the same argument. As we will show next, this semantics satisfies (Strict) Proportionality, that is the intensity of an attack depends on the basic weight of the target. The stronger the target, the more resistant it is to attacks.

Example 3 (Cont) Consider now the weighted argumentation graph $\mathbf{G}_3$. The strengths of the four arguments according to Cbs are as follows: $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_3}(a) = 0.50$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_3}(b) = 0.10$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_3}(c) = 0.90$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_3}(d) = 0.08$. Note that b and d have the same basic weight but d loses more weight since its attacker is stronger than the attacker of b. This shows, as we will see, that the semantics satisfies (Strict) Reinforcement.

Example 4 (Cont) Consider the weighted argumentation graph $\mathbf{G}_4$. Since the graph is acyclic, the strengths can be calculated in two steps as follows:

i	a	b	c	d	g	h	j
0	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.25	0.33	0.33	0.33	1.00	1.00	1.00
2	0.30	0.33	0.33	0.33	1.00	1.00	1.00

The strengths of the arguments according to Cbs are as follows: $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_4}(g) = 1.00$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_4}(h) = 1.00$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_4}(c) = 0.33$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_4}(d) = 0.33$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_4}(a) = 0.30$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_4}(j) = 1.00$, $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_4}(b) = 0.33$. Note that $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_4}(b) > \mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}_4}(a)$.

We show next that the limit scores of arguments satisfy the equation of Definition 11.

Theorem 13. *For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathrm{WAG}$*, for any* $a \in \mathcal{A}$*,*

$$\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}}(a) = \frac{w(a)}{1 + |\mathrm{AttF}_{\mathbf{G}}(a)| + \frac{\sum_{b \in \mathrm{AttF}_{\mathbf{G}}(a)} \mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}}(b)}{|\mathrm{AttF}_{\mathbf{G}}(a)|}}. \tag{5}$$

We also show that equation (5) represents an alternative definition of weighted card-based semantics, i.e., $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}}$ is the only function which satisfies the equation (2).

Theorem 14. *Let* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$ *be a finite WAG, and let* $D : \mathcal{A} \to [0, 1]$*. If*

$$D(a) = \frac{w(a)}{1 + |\mathrm{AttF}_{\mathbf{G}}(a)| + \frac{\sum_{b \in \mathrm{AttF}_{\mathbf{G}}(a)} D(b)}{|\mathrm{AttF}_{\mathbf{G}}(a)|}},$$

for all $a \in \mathcal{A}$*, then* $D \equiv \mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}}$*.*

As shown next, the weighted card-based semantics satisfies Cardinality Precedence as well as all the principles that are compatible with it. Table 1 also shows that Cbs is the *first* semantics in the literature that satisfies CP and deals with weighted argumentation graphs. In [20], two ranking semantics (Burden-based semantics (Bbs) and Discussion-based semantics (Dbs)) that satisfy CP were proposed but for flat (i.e., non-weighted) graphs.

Theorem 15. *Weighted card-based semantics satisfies all the principles except Quality Precedence and Compensation.*

The next result follows from the two Theorems 13 and 1. It shows the lower and upper bounds of the strength of an argument, as obtained with Cbs.

Theorem 16. *For any WAG* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$*, for any* $a \in \mathcal{A}$*,* $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}}(a) \in [\frac{w(a)}{2 + |\mathrm{AttF}_{\mathbf{G}}(a)|}, w(a)]$*.*

As with the max-based semantics, an argument gets value 0 iff its basic weight is already 0.

Corollary 2. *Let* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathrm{WAG}$ *and* $a \in \mathcal{A}$*. For any* $a \in \mathcal{A}$*,* $\mathrm{Deg}^{\mathrm{Cbs}}_{\mathbf{G}}(a) = 0$ *iff* $w(a) = 0$*.*

7.3. Weighted h-categorizer semantics (Hbs)

Weighted *h*-Categorizer semantics is based on *h*-Categorizer, initially proposed by Besnard and Hunter [39] for *non-weighted* and *acyclic* graphs. It extends the definition of *h*-Categorizer to account for varying degrees of basic weights, and any graph structure. Like the two previous semantics (Mbs and Cbs), Hbs follows a multiple step process. In the initial step, it assigns to every argument its basic weight. Then, in each step, all the scores are simultaneously recomputed on the basis of the attackers' scores in the previous step.

Definition 13 *(*f_h*).* Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$. We define the *weighted h-Categorizer function* f_h from $\mathcal{A}$ to $[0, 1]$ as follows: for any $a \in \mathcal{A}$, for $i \in \{0, 1, 2, \ldots\}$,

$$f_h^i(a) = \begin{cases} w(a) & \text{if } i = 0; \\ \dfrac{w(a)}{1 + \sum\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} f_h^{i-1}(b)} & \text{otherwise.} \end{cases}$$

By convention, if $\mathtt{Att}_{\mathbf{G}}(a) = \emptyset$, $\sum\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} f_h^{i-1}(b) = 0$.

Like the two previous scoring functions, the function f_h^i converges.

Theorem 17. *The function* f_h^i *converges as i approaches infinity.*

According to weighted *h*-Categorizer, the strength of each argument in a weighted argumentation graph is the limit reached using the function f_h.

Definition 14 *(*Hbs*).* The weighted *h*-Categorizer semantics is a function Hbs transforming any $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ into a weighting $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Hbs}}$ on $\mathcal{A}$ such that for any $a \in \mathcal{A}$, $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Hbs}}(a) = \lim\limits_{i \to +\infty} f_h^i(a)$.

In weighted argumentation graphs where each argument is attacked by at most one argument, the two semantics Mbs and Hbs coincide, that is they return the same values for the arguments.

Proposition 14. *Let* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ *be a weighted argumentation graph such that for any* $a \in \mathcal{A}$, $|\mathtt{Att}_{\mathbf{G}}(a)| \leq 1$. *For any* $a \in \mathcal{A}$, $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Hbs}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a)$.

From the previous result, it follows that Hbs assigns the same strengths as Mbs to the arguments of the three weighted argumentation graphs $\mathbf{G}_1, \mathbf{G}_2$ and $\mathbf{G}_3$. However, the two semantics assign different values to argument a of the graph $\mathbf{G}_4$.

Example 4 (Cont) Consider the weighted argumentation graph $\mathbf{G}_4$. Since the graph is acyclic, the strengths can be calculated in two steps as follows:

i	a	b	c	d	g	h	j
0	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1	0.33	0.50	0.50	0.50	1.00	1.00	1.00
2	0.50	0.50	0.50	0.50	1.00	1.00	1.00

The strengths of the arguments according to Hbs are as follows: $\mathrm{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(g) = 1.00$, $\mathrm{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(h) = 1.00$, $\mathrm{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(c) = 0.50$, $\mathrm{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(d) = 0.50$, $\mathrm{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(a) = 0.50$, $\mathrm{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(j) = 1.00$, $\mathrm{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(b) = 0.50$. Note that $\mathrm{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(a) = \mathrm{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(b)$, thus Hbs follows a compensation strategy.

We now show that the limit scores of arguments (thus their strengths) satisfy the equation of Definition 13.

Theorem 18. *For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for any* $a \in \mathcal{A}$,

$$\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Hbs}}(a) = \frac{w(a)}{1 + \sum\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Hbs}}(b)}. \tag{6}$$

The following theorem states that equation (6) is a characterization of weighted *h*-Categorizer semantics, i.e., it is the only function satisfying the equations (3) (one equation per argument).

Theorem 19. *Let* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$*, and* $D : \mathcal{A} \to [0, 1]$*. If* $D(a) = \frac{w(a)}{1 + \sum\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} D(b)}$*, for all* $a \in \mathcal{A}$*, then* $D \equiv \mathtt{Deg}_{\mathbf{G}}^{\mathtt{Hbs}}$*.*

The weighted *h*-Categorizer semantics satisfies compensation as well as *all* the principles that are compatible with it. Table 1 shows that Hbs is the only semantics in the literature that satisfies all the principles compatible with compensation, and for any structure of graphs. In the particular case of acyclic graphs, the (DF-)QuAD semantics satisfies the same principles except Resilience. Indeed with DF-QuAD, an argument may lose its entire strength (i.e., get strength 0) while this is not possible with Hbs unless the basic weight of the argument is 0.

Theorem 20. *Weighted h-Categorizer semantics satisfies all the principles except CP and QP.*

The lower and upper bounds of the strength of each argument are identified.

Theorem 21. *Let* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$*. For any* $a \in \mathcal{A}$*,* $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Hbs}}(a) \in [\frac{w(a)}{1 + |\mathtt{Att}_{\mathbf{G}}(a)|}, w(a)]$*.*

Like the two previous semantics, an argument gets value 0 with Hbs iff its basic weight is already 0.

Corollary 3. *Let* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ *and* $a \in \mathcal{A}$*. For any* $a \in \mathcal{A}$*,* $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Hbs}}(a) = 0$ *iff* $w(a) = 0$*.*

8. Experimental analysis

In order to check the performance of the three novel semantics (Mbs, Cbs, Hbs), we experimentally investigated, for each semantics, the number of iterations and the time needed to calculate the strengths of arguments. We used Algorithm 1 to calculate the result.

Input: a weighted graph, precision ϵ, maximal number of steps *maxSteps*
Result: the score of each argument
set the number of iterations to 0;
initialize the current scores of arguments by setting the score of each argument to its initial weight;
calculate new scores by using the corresponding formula (from equation (4), (5) or (6));
while *there exists an argument a_i s.t. the difference between the current and new score of a_i is greater than ϵ* **do**
    number of iterations = number of iterations + 1;
    current scores = new scores;
    update new scores by using the corresponding formula (from equation (4), (5) or (6));
    if *the number of iterations is equal to maxSteps* **then**
        | break;
end
if *number of iterations < maxSteps* **then**
    we successfully calculated the scores with precision ϵ;
else
    we reached the maximal number of iterations before reaching the desired precision;
end

Algorithm 1: Algorithm used to calculate arguments' scores.

An important step consists of choosing the set of benchmarks, i.e., the set of argumentation graphs on which the three semantics will be applied. There exist two families of benchmarks in the argumentation literature. The first is used for testing the performance of solvers on Dung's semantics in the International Competition on Computational Models of Argumentation [66]. The second family of benchmarks in [35] is used for testing the performance of gradual semantics. In our work, we use the latter since our semantics are gradual. More precisely, we consider four datasets: Erdős-Rényi dataset, Barabási-Albert dataset, Kleinberg dataset, and Sophia Antipolis dataset. The first three are well-known random graph models with respectively 100 to 300 arguments, 30 to 20000 arguments, and 9 to 900 arguments. The concrete graphs we used were generated by Bistarelli [36] and are freely downloadable.[4] The fourth dataset was generated by Pereira et al. [35] by combining different argumentation patterns from the literature. It can also be found online.[5] Its graphs contain between 5000 and 100000 arguments. A common feature of the graphs of the four datasets is that they are flat, i.e., arguments do not have basic weights.

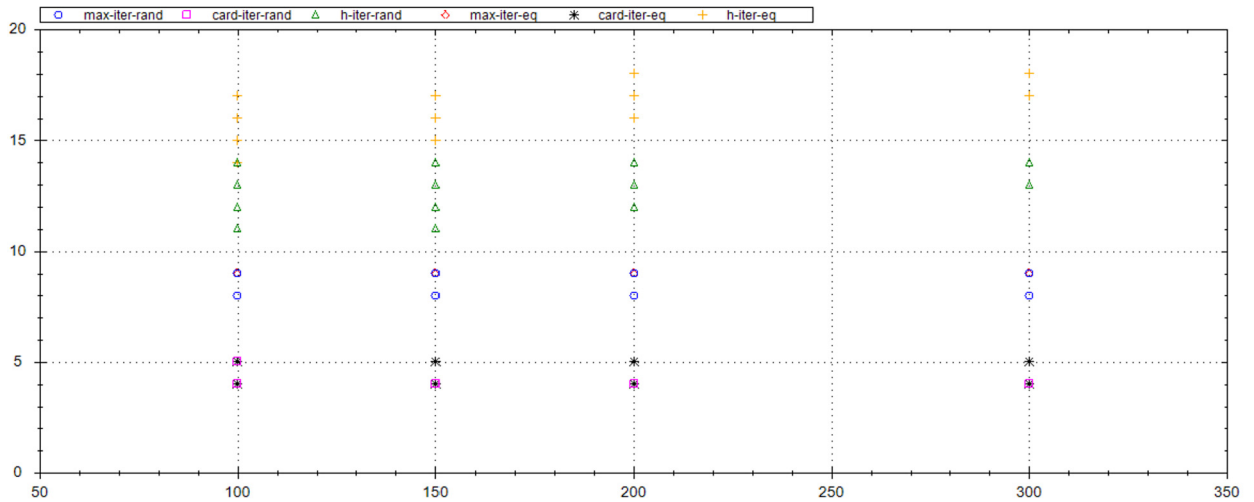[4] http://www.dmi.unipg.it/conarg/dwl/networks.tgz.
[5] https://goo.gl/pN1M9r.

Fig. 5. Number of iterations on Erdős-Rényi dataset. The *x* axis shows the number of arguments. The *y* axis shows the number of iterations.

We implemented the three semantics in Java, and the solvers ran on a cluster of identical computers with two processors Intel XEON E5-2643 - 4 cores - 3.3 GHz running CentOS 6.0 x86 64 with 32 GB of memory, using the tool runsolver [67].

It is worth mentioning that our goal was not to produce the fastest possible solver but to check two aspects: the average number of iterations needed to get the strengths of arguments, and the general tendency in time when the number of arguments increases. For that purpose, we run the experiments twice: once we attributed random basic weights to arguments, and once we attributed the basic weight 1 to all arguments. Furthermore, the iterative algorithms stop when the difference in scores between two successive iterations is less than 0.001 for each argument. The results are shown in Figs. 5-16. Note that we used the following abbreviations: max (for weighted max-based semantics), card (for weighted card-based semantics), h (for weighted *h*-Categorizer semantics), iter (for number of iterations needed to calculate the scores), time (for time needed to calculate the scores), rand (for arguments are given random initial weights), eq (for each argument's basic weight is set to 1). For example, max-iter-rand stands for the number of iterations needed to calculate the scores when using max-based semantics and when arguments are assigned random basic weights.

Considering the Erdős-Rényi dataset, Fig. 5 shows that the number of iterations is limited to 18. The least number of iterations is needed for card-based semantics, max-based semantics is in the middle and *h*-Categorizer semantics needs the greatest number of iterations in order to calculate the scores. Using random basic weights results in slightly smaller number of iterations. Fig. 6 shows the time in milliseconds needed to calculate the result when arguments are attributed random initial weights. The fastest semantics is the card-based one, *h*-Categorizer is in the middle, whereas the max-based semantics is the slowest one. Fig. 7 shows the time in milliseconds needed to calculate the result when arguments are attributed equal initial weights. Max-based and card-based semantics still show quasi linear behavior, but there are six graphs with 150 arguments where *h*-Categorizer took more time than would be expected.

Considering the Barabási-Albert dataset, Fig. 8 shows that the number of iterations is limited to 8. All three semantics give similar results. Figs. 9 and 10 show the time (in thousands of seconds) needed to calculate the result in function of the number of arguments (in thousands). The three semantics have similar performances that do not change a lot in function of the initial weights (random or equal).

Considering the Kleinberg dataset, Fig. 11 shows that the number of iterations is limited to 18. The least number of iterations is needed for card-based semantics, max-based semantics is in the middle and *h*-Categorizer semantics needs the greatest number of iterations in order to calculate the scores. Using random initial weights results in slightly smaller number of iterations. Figs. 12 and 13 show the time in milliseconds needed to calculate the result. The convergence is very fast. We see that *h*-Categorizer is clearly the slowest semantics in the case of equal initial weights, whereas this cannot be said for the case with random initial weights. However, all the semantics converge very fast and we do not think this difference is significant.

Considering the Sophia Antipolis dataset, Fig. 14 shows that the number of iterations is limited to 12. The least number of iterations is needed for card-based semantics, max-based semantics is in the middle and *h*-Categorizer semantics needs the greatest number of iterations in order to calculate the scores. Using random initial weights results in slightly smaller number of iterations. Figs. 15 and 16 show the time (in thousands of seconds) needed to calculate the result in function of number of arguments (in thousands). We see that *h*-Categorizer is clearly the slowest semantics in both cases, whereas the card-based semantics is the fastest.

To sum up, the number of iterations is always less than 20, even for graphs with 100000 arguments. It seems that the number of iterations is constant with respect to the graph size, which shows that our algorithms scale well. The time needed is low, and what is especially important, all graph topologies are treatable even for big graph sizes. Namely, no
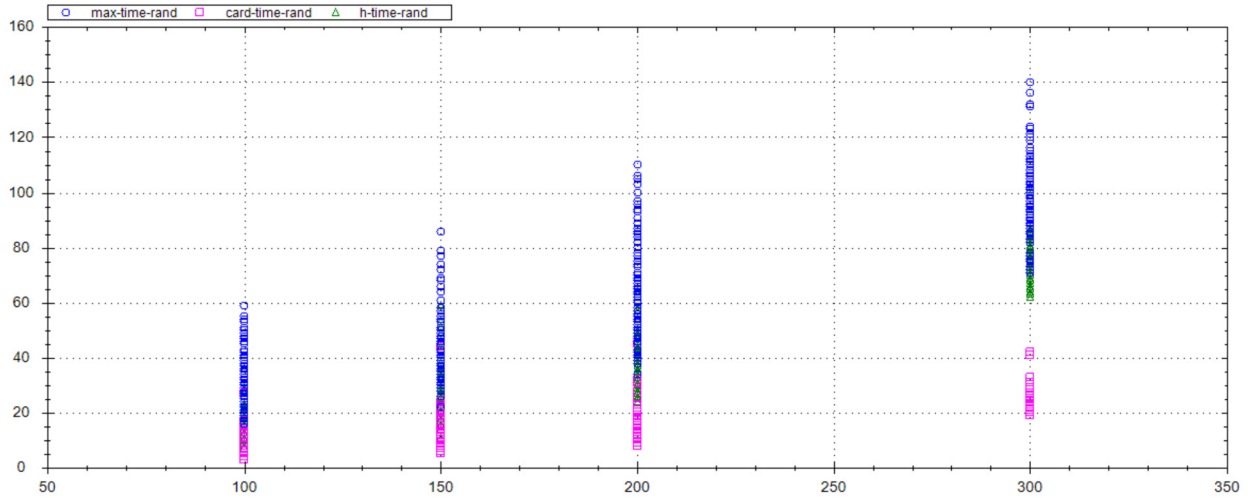
Fig. 6. Time on Erdős-Rényi dataset when arguments are attributed random initial weights. The *x* axis shows the number of arguments. The *y* axis shows the time in milliseconds.
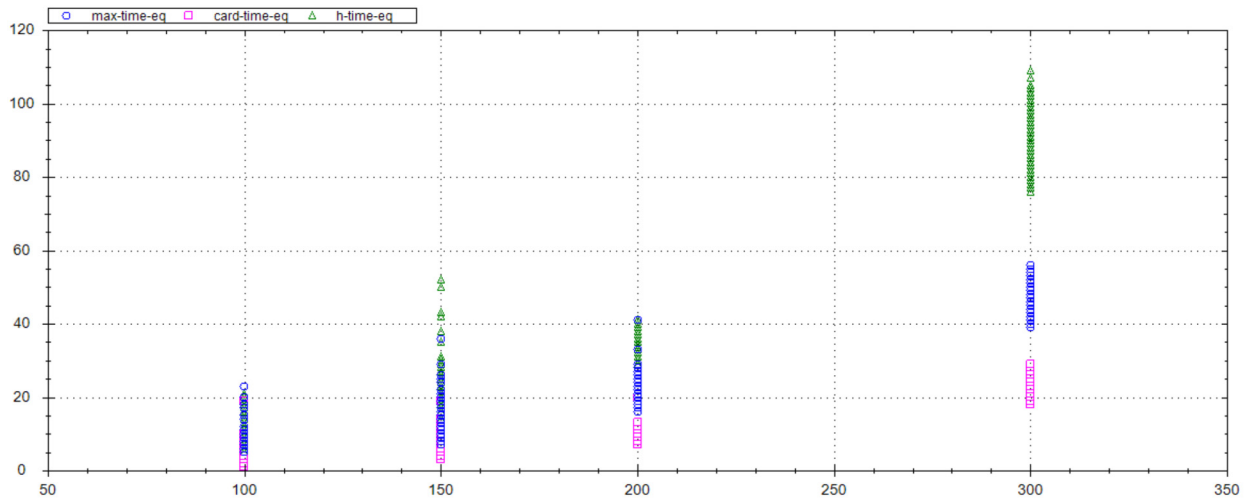

Fig. 7. Time on Erdős-Rényi dataset when arguments' initial weights are set to 1. The *x* axis shows the number of arguments. The *y* axis shows the time in milliseconds.

instances timed-out or crashed. Our algorithms always converge. The worst run time (i.e. *h*-Categorizer on the graphs with 100000 arguments) was around 2250 seconds. There is a tendency for all three semantics to converge slightly faster when initial weights are attributed in a random way. Note that the difference is not very big (around 10%).

9. Related work

There are some notable works in the literature that have been concerned by the definition and/or analysis of principles.

Baroni et al. [30] discussed principles for weighted bipolar graphs. They focused on principles that were suggested in different papers in the literature, namely in [27,59,28]. In those papers, we defined principles as *elementary* properties that cannot be further decomposed into more elementary ones. Some of those principles were generalized in [30] for dealing with other scales of argument strength, and others were grouped into meta-level properties. In the present paper, we still prefer elementary principles to meta-level ones because they allow a clear understanding of the set of rules underlying semantics, and a better comparison of semantics. Assume for instance a meta-level principle P which implies two elementary ones P1 and P2. Assume also two semantics S1 and S2 that violate P but S1 satisfies P1 and S2 instead satisfies P2. Note that the two semantics cannot be distinguished by P. However, the two elementary properties clarify the rules underlying each semantics and specify the difference between S1 and S2. To summarize, while our paper is more concerned with proposing concrete principles and studying whether they are satisfied by semantics, the goal in [30] is not necessarily to define novel principles but rather to show how to represent in a compact way existing ones within a generalized framework.
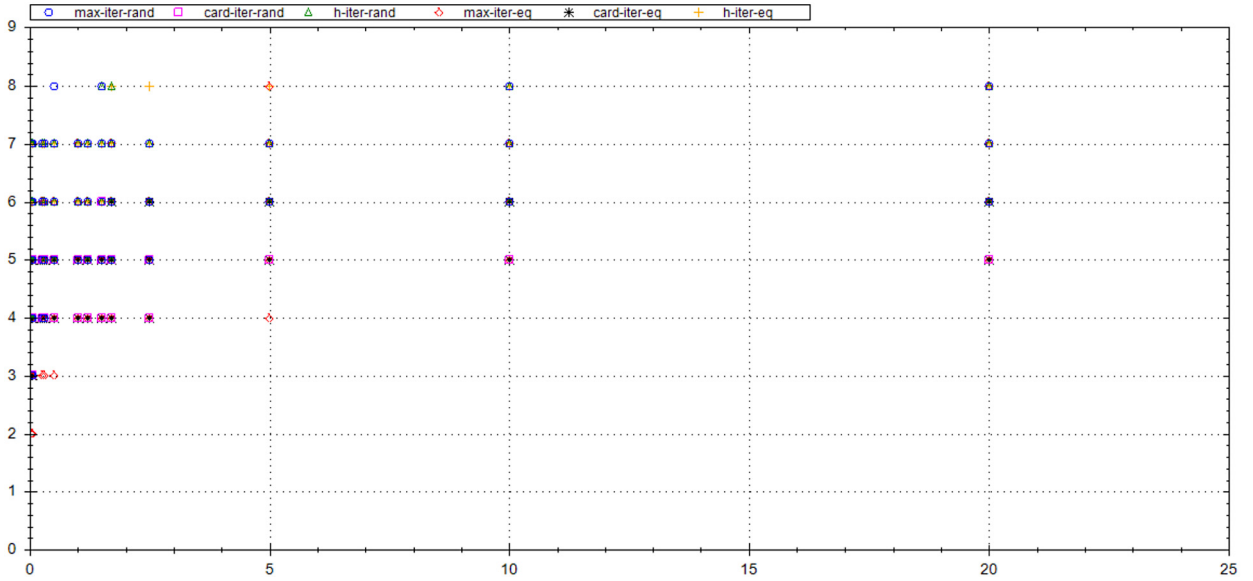
Fig. 8. Number of iterations on Barabási-Albert dataset. The *x* axis shows the number of arguments in <u>thousands</u>. The *y* axis shows the number of iterations.
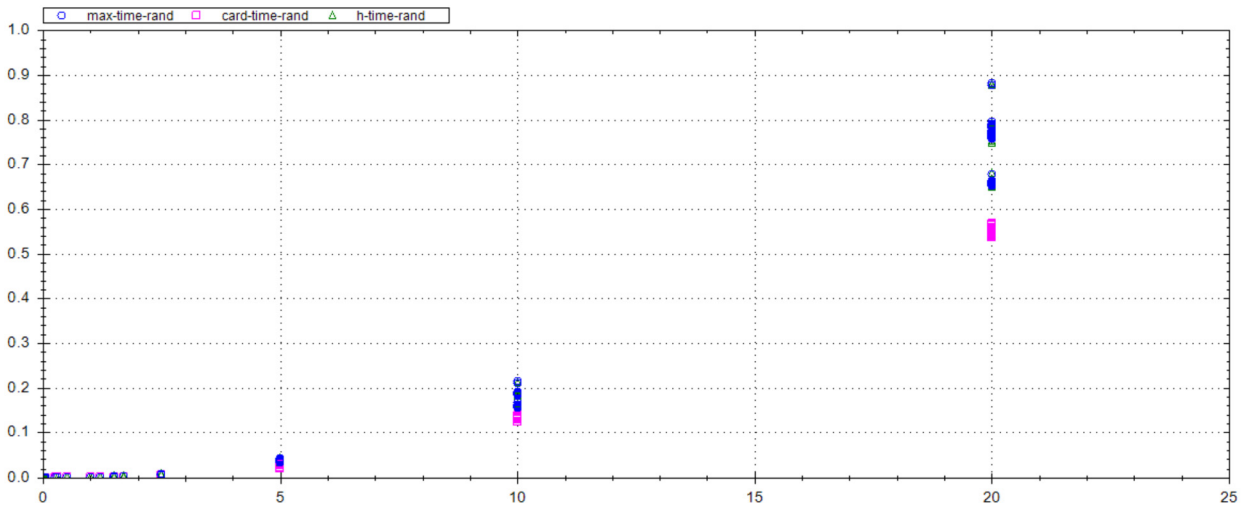

Fig. 9. Time on Barabási-Albert dataset when arguments are attributed random initial weights. The *x* axis shows the number of arguments in <u>thousands</u>. The *y* axis shows the time in <u>thousands</u> of seconds.

Bonzon et al. [68] introduced a novel principle for ranking semantics, which states that the longer a line of defense of an argument, the less it has impact on the argument. They argued that this principle is suitable in persuasion dialogues. In [21], they used the principles proposed in [20,16,69] for comparing existing *pure ranking semantics*, i.e., ranking semantics that do not compute numerical/qualitative strengths of arguments. In our paper, we rather focused on gradual semantics, i.e. those that compute strengths. The second main difference with [21] lies in the fact that most principles considered in [21] are high-level properties. For instance, they use counter-transitivity (CT) from [20] which states that the more an argument is attacked and the stronger its attackers, the weaker it is. We have shown that CT is a consequence of a couple of elementary principles. As a third difference, in [21] the authors considered flat graphs, i.e., arguments do not have basic weights while we focused on weighted graphs.

In [60], the authors extended the principles we presented in [28] for dealing with argumentation graphs whose arguments and attacks are weighted. They also introduced a novel one that captures sensitivity of a semantics to weights of attack relations. Finally, they extended in various ways our three novel semantics (Hbs, Mbs, Cbs).

More recently, the discussion around similarity in argumentation is gaining more and more interest (e.g. [70–73]). This is mainly due to noteworthy presence of similar arguments in online debate platforms and also between logical arguments [74]. Consequently, the authors in [75,76] investigated gradual semantics that take as input a similarity measure. They
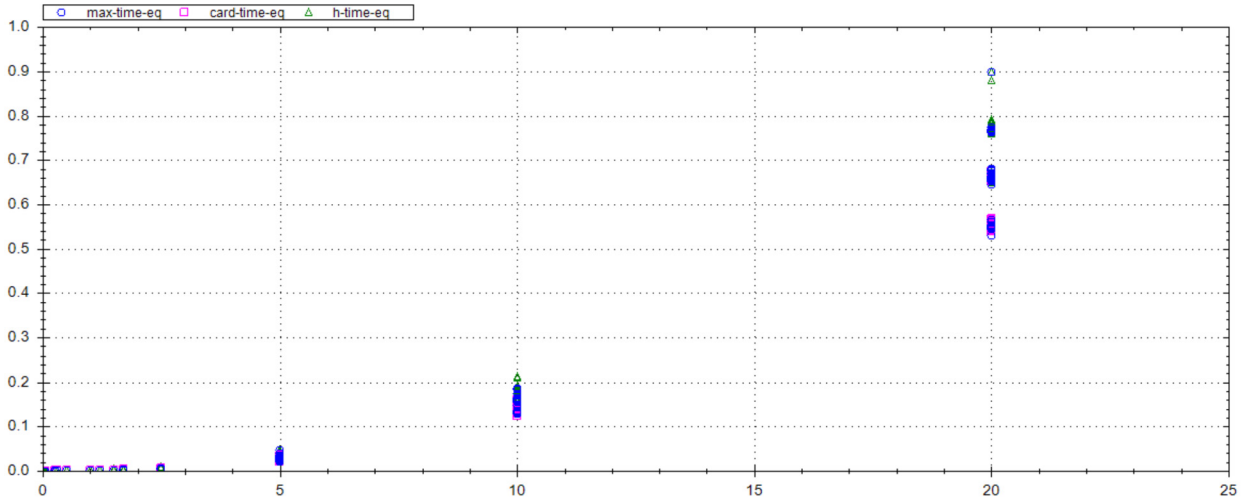
Fig. 10. Time on Barabási-Albert dataset when arguments' initial weights are set to 1. The *x* axis shows the number of arguments in <u>thousands</u>. The *y* axis shows the time in <u>thousands</u> of seconds.
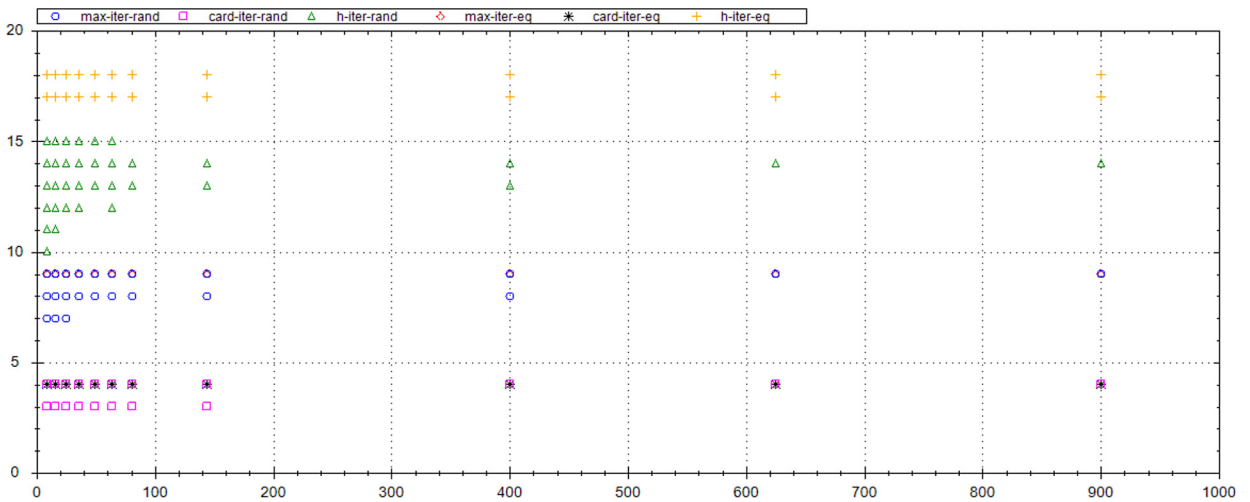

Fig. 11. Number of iterations on Kleinberg dataset. The *x* axis shows the number of arguments. The *y* axis shows the number of iterations.

extended some of our principles (like Reinforcement and Monotony) and defined novel ones that are specific to sensitivity to similarity. In our paper, we assume that arguments are pairwise independent.

In the last few years, there has been an increasing interest in probabilistic argumentation whose aim is handling uncertainty in an argumentation context. Two approaches are distinguished in [7]: the *constellation* approach and the *epistemic* one. The former handles uncertainty about the topology of an argumentation graph [77,78]. For instance in [78], the input is an argumentation graph and two functions assigning respectively for each argument and each attack the *likelihood* of its *existence*. These probabilities are used for generating probabilities over sub-graphs of the initial graph before computing its extensions (à la Dung [13]). When all arguments and all attacks have probability 1, we get the original Dung's framework. This work is different from what we investigated in this paper. Indeed, in our setting the argumentation graph is *fixed*, and thus its *arguments and attacks exist*. The value $w(.)$ expresses the intrinsic strength of each argument and not the probability of its existence. Thus, our principles are not suitable in this probabilistic setting.

In the epistemic approach, Thimm [79] considered as input a *flat* argumentation graph, and then generalized Dung's semantics [13] with probabilities. The idea is to assign a probability to each possible extension, and consequently to each argument. Thus, unlike our work, the strength of an argument is a probability of membership to extensions. The motivation and spirit of this work are very similar to the *equational approach* by Gabbay [80], where the labelling of a graph is made by more than the three classical values: in, out, und. Both works generalize Dung's semantics, which were already investigated in [27].

Hunter [7] discussed another epistemic approach for probabilistic argumentation. The input is a fixed *weighted* argumentation graph $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $w(.)$ expresses a probability of *believing* an argument. As argued in [[7], page 7],

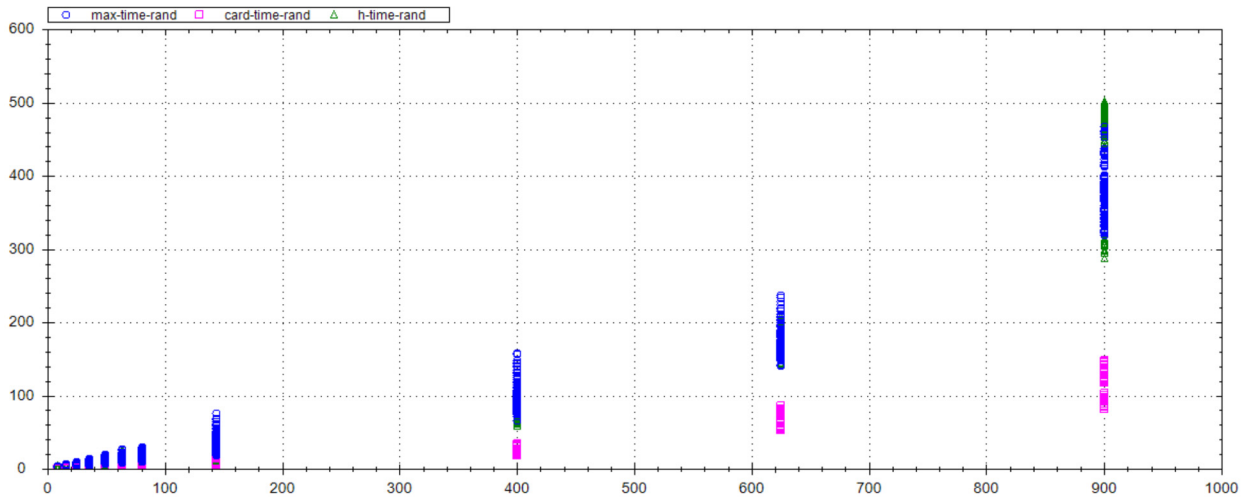**Fig. 12.** Time on Kleinberg dataset when arguments are attributed random initial weights. The *x* axis shows the number of arguments. The *y* axis shows the time in milliseconds.
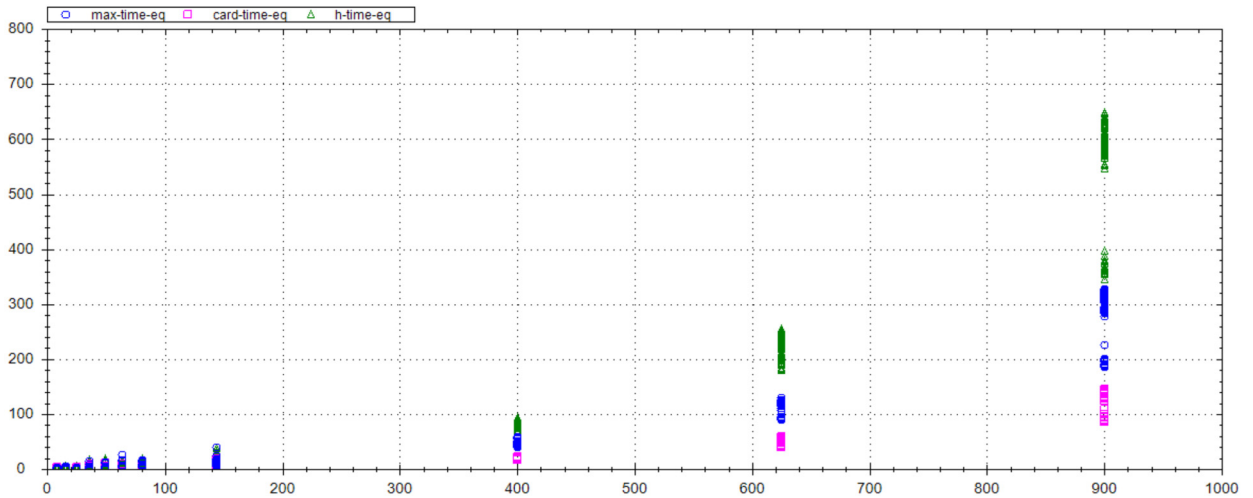

Fig. 13. Time on Kleinberg dataset when arguments' initial weights are set to 1. The *x* axis shows the number of arguments. The *y* axis shows the time in milliseconds.

this probability is seen as an extra meta-level information about the quality of an argument's components (premises, link). The function w satisfies the following "rationality" condition: if an argument a attacks another argument b and $w(a) > 0.5$, then $w(b) \leq 0.5$. In terms of arguments' evaluation, an *epistemic extension* $\mathcal{E}$ is built for each graph such that $\mathcal{E} = \{a \in \mathcal{A} \mid w(a) > 0.5\}$. Finally, a two-valued qualitative status is assigned to each argument as follows: an argument is accepted if it belongs to $\mathcal{E}$ and rejected otherwise. This approach is very different from ours. First, basic weights of arguments depend on attacks, while in our approach the two are completely independent. Second, the approach generalizes Dung's semantics, in that if an argument is not attacked, then it is assigned probability 1 and thus accepted. In our setting, such arguments get their basic weights.

More recently, Hunter and Thimm [81] presented a new probabilistic framework where they combined the ideas of their two papers [7,79]. The input is a fixed flat argumentation graph and they assign a probability to each argument representing the *degree of belief that the argument is acceptable*. The authors presented some examples of what is an acceptable argument, and basically the idea is that the premises of the argument are true and its logical link is correct. The probability of an argument captures the uncertainty inherent to the argument. In our paper, such information is captured by the function w, i.e., the basic weight of the argument and not by the strength. Furthermore, in all the provided examples, the probability is given as an extra information, thus it cannot be the output of a semantics.
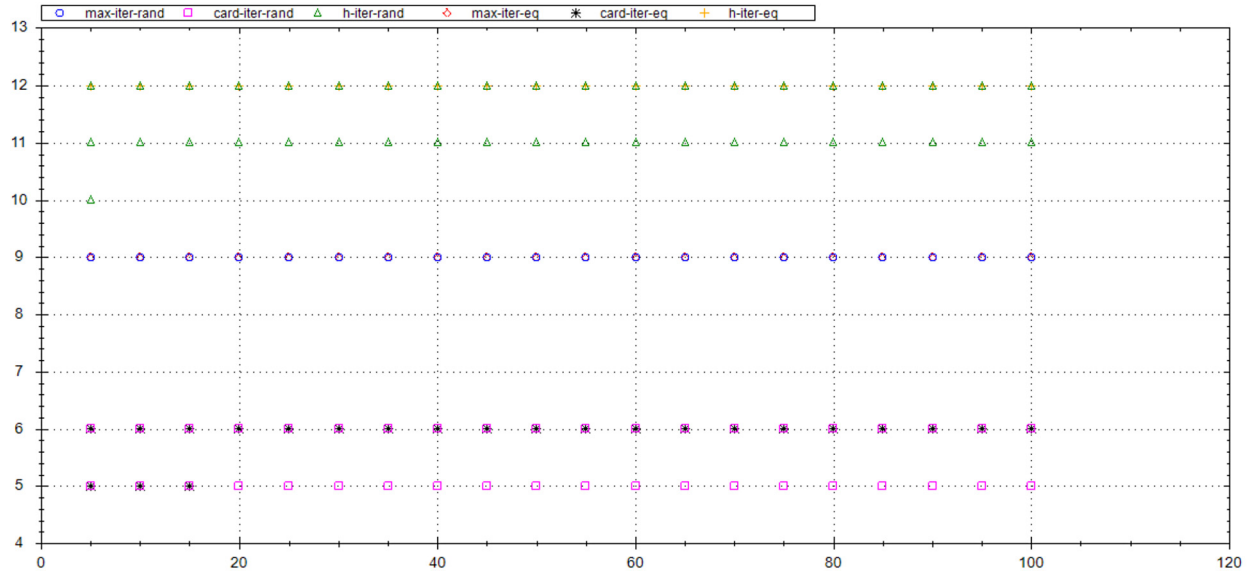
Fig. 14. Number of iterations on Sophia Antipolis dataset. The *x* axis shows the number of arguments in <u>thousands</u>. The *y* axis shows the number of iterations.
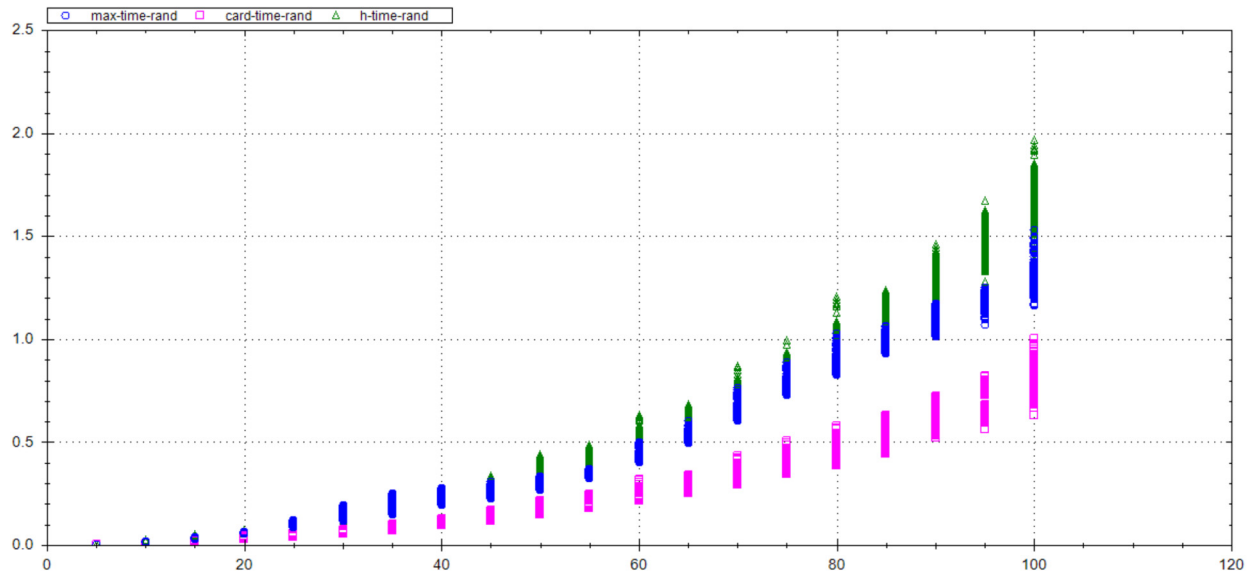

Fig. 15. Time on Sophia Antipolis dataset when arguments are attributed random basic weights. The *x* axis shows the number of arguments in <u>thousands</u>. The *y* axis shows the time in <u>thousands</u> of seconds.

10. Conclusion

The paper investigated the issue of evaluating arguments in weighted argumentation graphs, i.e., graphs where arguments have initial weights. It proposed a set of principles that serve as guidelines for defining, theoretically analyzing and comparing semantics. Some principles like Anonymity are mandatory while others, like Strict Invariance and Resilience, are optional as they are suitable in some applications and not in others. Finally, the last three principles (Quality Precedence, Cardinality Precedence, Compensation) define three incompatible strategies that a semantics may follow when it encounters a dilemma between the quality and the quantity of attackers.

The second contribution of the paper consists of using the principles for comparing ten semantics that were proposed in the literature for evaluating arguments in weighted graphs. We studied 8 extension semantics and 2 gradual semantics. We compared i) semantics of different families (extension vs gradual), ii) the two approaches that deal with preferences in argumentation (contraction vs change), and iii) semantics of the same family (e.g., Trust-based and Iterative Schema). Table 1 summarizes the current landscape, and shows that the ten semantics are different in that they made different
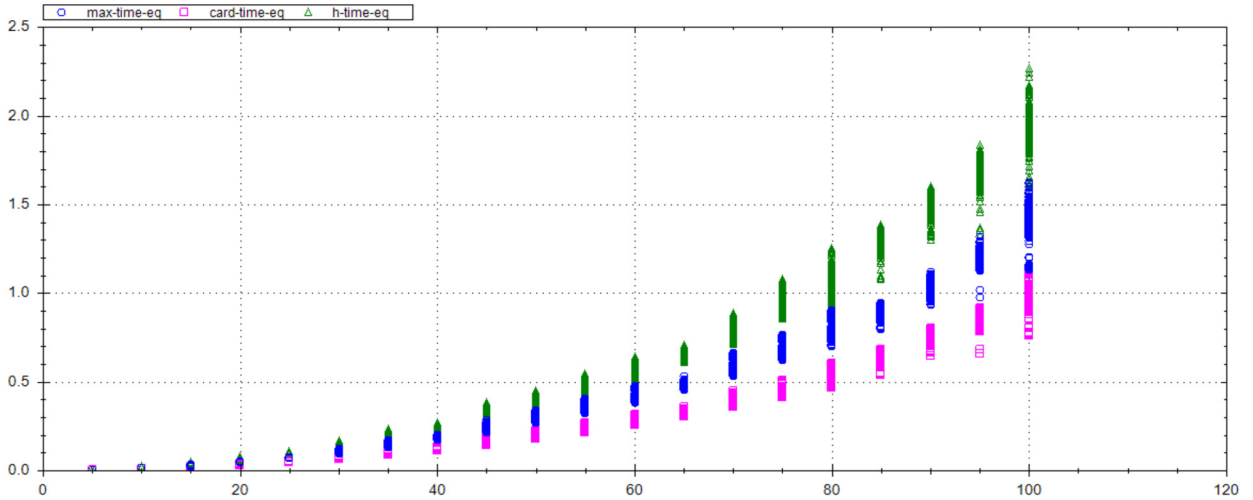
Fig. 16. Time on Sophia Antipolis dataset when arguments' basic weights are set to 1. The x axis shows the number of arguments in <u>thousands</u>. The y axis shows the time in <u>thousands</u> of seconds.

design choices. It reveals also the kind of semantics that are missing in the literature. For instance, there is no semantics that satisfies all the compatible principles as well as compensation, there is no semantics that satisfies Cardinality Precedence, and there is no semantics that satisfies all the principles which are compatible with Quality Precedence. This led to our third contribution, which is the introduction of three novel semantics Mbs, Cbs, Hbs, one for each of the three previous strategies (QP, CP, and compensation). Like IS, Mbs satisfies Quality Precedence. However, unlike IS, it satisfies all the principles which are compatible with QP. Cbs is the sole semantics that satisfies Cardinality Precedence. Hbs is the sole semantics that satisfies compensation as well as all the other compatible principles, and that deals with any graph structure. The three semantics were also analyzed experimentally. The study revealed that the three semantics are very efficient. Indeed, they compute the strengths of arguments in few iterations, and in very short time. This is true even for big graphs with 100000 arguments, which means that the semantics scale well.

Future work consists of characterizing families of semantics that satisfy all or subsets of the proposed principles. Another line of research consists of applying the new semantics in different contexts, namely for defining argument-based paraconsistent logics, and argument-based decision systems.

Declaration of competing interest

Acknowledgements

Appendix A

This appendix contains the proofs of the results presented in the paper. Note that the order in which we prove our results is not exactly the same as the order in which we present them. Namely, we first need to prove Lemma 1, Propositions 1, 3, 6, 9, 10 and 11, and then Theorem 1 and Proposition 12. Later, we prove the other results, by the order of their appearance.

Proof of Proposition 1. Let us prove the two items.

1. Suppose that there exists a semantics **S**, which satisfies Cardinality Precedence, Quality Precedence, Resilience and Maximality. Consider the argumentation graph $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$ with $\mathcal{A} = \{x, y, z, a, b\}$, $w(x) = w(y) = 0.5$, $w(z) = w(a) = w(b) = 1$ and $\mathcal{R} = \{(x, a), (y, a), (z, b)\}$. From Maximality we obtain $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = 0.5$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(z) = 1$. By Resilience, both $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) > 0$. Now, using Cardinality Precedence we conclude that $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a)$, while Quality Precedence implies $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$, contradiction.

2. Let **S** be a semantics which satisfies Resilience, Strict Reinforcement, Maximality, Strict Weakening, Strict Invariance and Quality Precedence. Consider the graph **G** depicted below, and assume that the basic weight of each argument is equal to 1.
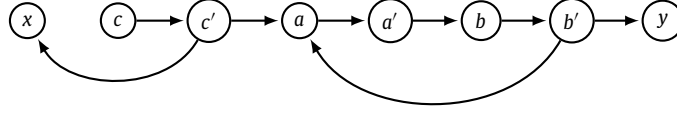
From Resilience, the eight arguments have strictly positive degree $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(.) > 0$. From Maximality, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(c) = 1$ and from Strict Weakening, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) < 1$. Thus, by Strict Reinforcement, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(c')$.

Case $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$**:** From Quality Precedence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. Thus, from Strict Reinforcement $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b') > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a')$, which is impossible.

Case $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b') > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a')$**:** From Strict Reinforcement, we obtain $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y)$. Note that
- $\text{Att}_{\mathbf{G}}(y) = \text{Att}_{\mathbf{G}}(c) \cup \{b'\}$ and
- $\text{Att}_{\mathbf{G}}(a) = \text{Att}_{\mathbf{G}}(x) \cup \{b'\}$.

Since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(c) = 1$ (by Maximality), $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) < 1$ (by Strict Weakening), we obtain $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) < \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(c)$. Thus, from Strict Invariance, obtain $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) < \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y)$. Consequently, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) < \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. Thus, from Strict Reinforcement $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$ which is impossible.

Case $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$**:** Note that
- $\text{Att}_{\mathbf{G}}(b) = \text{Att}_{\mathbf{G}}(c) \cup \{a'\}$ and
- $\text{Att}_{\mathbf{G}}(a) = \text{Att}_{\mathbf{G}}(x) \cup \{b'\}$.

Since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) < \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(c)$ (by Maximality and Strict Weakening), from Strict Invariance we obtain $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) < \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. Thus, from Strict Reinforcement $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$ which is impossible. $\square$

Proof of Proposition 2. Let **S** be a semantics which satisfies Equivalence, and let us show that it also satisfies Symmetry. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a, b \in \mathcal{A}$ such that:

- $w(a) = w(b)$, and
- $\text{Att}_{\mathbf{G}}(a) = \text{Att}_{\mathbf{G}}(b)$.

Thus, there exists a bijective function f from $\text{Att}_{\mathbf{G}}(a)$ to $\text{Att}_{\mathbf{G}}(b)$ such that $\forall x \in \text{Att}_{\mathbf{G}}(a)$, $f(x) = x$, thus $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$. From Equivalence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Let **S** satisfy Independence, Directionality, Invariance, and Maximality. Let us show that **S** satisfies Equivalence. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a, b \in \mathcal{A}$ such that:

- $w(a) = w(b)$, and
- there exists a bijective function f from $\text{Att}_{\mathbf{G}}(a)$ to $\text{Att}_{\mathbf{G}}(b)$ such that $\forall x \in \text{Att}_{\mathbf{G}}(a)$, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$,

Assume a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \text{WAG}$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{a', b'\}$, for any $x \in \mathcal{A}$, $w_1(x) = w(x)$, $w_1(a') = w_1(b') = w(a)$, and $\mathcal{R}_1 = \mathcal{R}$. Note that $\text{Att}_{\mathbf{G}_1}(a') = \text{Att}_{\mathbf{G}_1}(b') = \emptyset$. From Maximality, it holds that $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a') = w_1(a')$ and $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b') = w_1(b')$. So,

$$\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a') = \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b').$$

From Proposition 9, it holds that $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b)$. Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain that for any $x \in \mathcal{A}$, $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$. Thus, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Assume that **S** satisfies in addition Neutrality, and let us prove that it satisfies Weakening Soundness. Suppose a semantics **S** which satisfies the five principles. Assume also a weighted argumentation graph $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$. Let $a \in \mathcal{A}$. From Proposition 11, if for every $b \in \text{Att}_{\mathbf{G}}(a)$, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$, then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = w(a)$. Weakening Soundness follows from the previous property by contraposition.

Let **S** be a semantics which satisfies Independence, Directionality, Invariance, Maximality and Weakening. Let us show that it satisfies Monotony. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a, b \in \mathcal{A}$ such that:

- $w(a) = w(b)$,
- $\text{Att}_{\mathbf{G}}(b) = \text{Att}_{\mathbf{G}}(a) \cup X$.

Note that according to what we already proved in this proposition, Independence, Directionality, Invariance and Maximality imply Equivalence, which implies Symmetry. Thus, $\mathbf{S}$ satisfies Symmetry. If $X = \emptyset$, $\mathrm{Att}_{\mathbf{G}}(b) = \mathrm{Att}_{\mathbf{G}}(a)$, and so from Symmetry, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. Assume now that $X \neq \emptyset$, and consider a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \mathtt{WAG}$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{a', b'\}$, for any $x \in \mathcal{A}$, $w_1(x) = w(x)$, $w_1(a') = w_1(b') = w(a) = w(b)$, and $\mathcal{R}_1 = \mathcal{R} \cup \{(x, b') \mid x \in X\}$. Note that $\mathrm{Att}_{\mathbf{G}_1}(a') = \emptyset$ and $\mathrm{Att}_{\mathbf{G}_1}(b') = X$. From Maximality, it holds that $\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a') = w(a)$. From Weakening, it holds that $\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b') \leq w(b')$. Since $w_1(a') = w_1(b')$, then

$$\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a') \geq \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b').$$

From Proposition 9, it follows that

$$\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) \geq \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b).$$

Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain that for any $x \in \mathcal{A}$

$$\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x).$$

Thus, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. $\quad\square$

Proof of Proposition 3. Suppose that $\mathcal{A} \setminus U = \{b_1, \ldots, b_n\}$, for some $n \geq 1$ (the case when $U = \mathcal{A}$ is trivial). Let us denote the graph $\mathbf{G}|_U$ by $\mathbf{G}_0$. Next we define $\mathbf{G}_1, \ldots, \mathbf{G}_n$ such that every $\mathbf{G}_k$ is obtained by adding the argument b_k to $\mathbf{G}_{k-1}$ together with its weight. In other words, for every $k \in \{1, \ldots, n\}$, if $\mathbf{G}_{k-1} = \langle \mathcal{A}_{k-1}, w_{k-1}, \mathcal{R}_{k-1} \rangle$, then $\mathbf{G}_k = \langle \mathcal{A}_k, w_k, \mathcal{R}_k \rangle$ is such that

- $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \{b_k\}$
- $w_k(a) = w(a)$ for every $a \in \mathcal{A}_k$
- $\mathcal{R}_k = \mathcal{R}_{k-1}$.

Using Independence, we obtain that $\mathrm{Deg}_{\mathbf{G}_k}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_{k-1}}^{\mathbf{S}}(a)$ for every $a \in U$ and every $k \in \{1, \ldots, n\}$. Consequently,

$$\mathrm{Deg}_{\mathbf{G}_n}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}|_U}^{\mathbf{S}}(a)$$

for every $a \in U$. Note that $\mathbf{G}_n = \langle \mathcal{A}, w, \mathcal{R} \cap (U \times U)) \rangle$.

Now we consider the attacks from $\mathbf{G}$ that are not from $\mathbf{G}|_U$. If $\mathcal{R} \setminus (\mathcal{R} \cap (U \times U)) = \emptyset$, then $\mathbf{G}_n = \mathbf{G}$, which completes the proof. Let $\mathcal{R} \setminus (\mathcal{R} \cap (U \times U)) \neq \emptyset$. Suppose that $\mathcal{R} \setminus (\mathcal{R} \cap (U \times U)) = \{(c_1, d_1), \ldots, (c_m, d_m)\}$, for some $m \geq 1$. Then we define the graphs $\mathbf{G}_{n+1}, \ldots, \mathbf{G}_{n+m}$ such that every $\mathbf{G}_{n+k}$ is obtained by adding the attack (c_k, d_k) to $\mathbf{G}_{n+k-1}$. Formally, for every $k \in \{1, \ldots, m\}$, if $\mathbf{G}_{n+k-1} = \langle \mathcal{A}, w, \mathcal{R}_{k-1} \rangle$, then $\mathbf{G}_{n+k} = \langle \mathcal{A}, w, \mathcal{R}_{k-1} \cup \{(c_k, d_k)\} \rangle$. Note that $\mathbf{G}_{n+m} = \mathbf{G}$.

Next we prove that $\mathrm{Deg}_{\mathbf{G}_{n+k}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_{n+k-1}}^{\mathbf{S}}(a)$ for every $a \in U$ and every $k \in \{1, \ldots, m\}$. Let us chose an arbitrary $a \in U$. For an arbitrary attack $(c_k, d_k) \in \mathcal{R} \setminus (\mathcal{R} \cap (U \times U))$, from the fact that it is impossible that both c_k and d_k belong to U, and the assumption that U is unattacked, we obtain that $d_k \notin U$ (otherwise, if $d_k \in U$, then $c_k \notin U$, so U would not be unattacked since $(c_k, d_k) \in \mathcal{R}$). Then, from the fact that U is unattacked we obtain that there is no path from d_k to a. Using Directionality, we obtain that $\mathrm{Deg}_{\mathbf{G}_{n+k}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_{n+k-1}}^{\mathbf{S}}(a)$ for every $k \in \{1, \ldots, m\}$. Consequently,

$$\mathrm{Deg}_{\mathbf{G}_{n+m}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_n}^{\mathbf{S}}(a)$$

for every $a \in U$. Since we proved that $\mathrm{Deg}_{\mathbf{G}_n}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}|_U}^{\mathbf{S}}(a)$ for every $a \in U$, from $\mathbf{G}_{n+m} = \mathbf{G}$ we obtain

$$\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}|_U}^{\mathbf{S}}(a)$$

for every $a \in U$. $\quad\square$

Proof of Proposition 4. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and let $a, b \in \mathcal{A}$ be two arguments such that $w(a) \geq w(b)$, and there exists a bijective function f from $\mathrm{Att}_{\mathbf{G}}(a)$ to $\mathrm{Att}_{\mathbf{G}}(b)$ such that $\forall x \in \mathrm{Att}_{\mathbf{G}}(a)$, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$.

Assume a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \mathtt{WAG}$ such that

- $\mathcal{A}_1 = \mathcal{A} \cup \{c\}$,
- for any $x \in \mathcal{A}$, $w_1(x) = w(x)$, $w_1(c) = w(a)$,
- $\mathcal{R}_1 = \mathcal{R} \cup \{(x, c) \mid x \in \mathrm{Att}_{\mathbf{G}}(b)\}\}$.

Using Equivalence we obtain $\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(c)$, and from Proportionality we obtain $\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(c) \geq \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b)$. Therefore, we have that

$$\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(c) \geq \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b).$$

Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a)$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b)$. Consequently, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. $\quad\square$

Proof of Proposition 5. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and let $a, b \in \mathcal{A}$ be two arguments such that $w(a) > w(b)$, there exists a bijective function f from $\mathtt{Att}_{\mathbf{G}}(a)$ to $\mathtt{Att}_{\mathbf{G}}(b)$ such that $\forall x \in \mathtt{Att}_{\mathbf{G}}(a)$, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$, and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$.

Assume a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \mathtt{WAG}$ such that

- $\mathcal{A}_1 = \mathcal{A} \cup \{c\}$,
- for any $x \in \mathcal{A}$, $w_1(x) = w(x)$, $w_1(c) = w(a)$,
- $\mathcal{R}_1 = \mathcal{R} \cup \{(x, c) \mid x \in \mathtt{Att}_{\mathbf{G}}(b)\}\}$.

Using Equivalence we obtain $\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(c)$, and from Strict Proportionality we obtain $\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(c) > \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b)$. Therefore, we have that

$$\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) > \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b).$$

Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a)$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b)$. Consequently, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. $\square$

Lemma 1. *Let* $\mathbf{S}$ *be a semantics which satisfies Independence, Directionality, Invariance, and Neutrality. For any* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, *for all* $a, b \in \mathcal{A}$, *for any* $x \in \mathcal{A} \setminus \mathtt{Att}_{\mathbf{G}}(a)$, *if*

- $w(a) = w(b)$,
- $\mathtt{Att}_{\mathbf{G}}(b) = \mathtt{Att}_{\mathbf{G}}(a) \cup \{x\}$ *with* $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$,

then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Proof. Let $\mathbf{S}$ be a semantics which satisfies Independence, Directionality, Invariance, and Neutrality. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b \in \mathcal{A}$ such that:

- $w(a) = w(b)$,
- $\mathtt{Att}_{\mathbf{G}}(b) = \mathtt{Att}_{\mathbf{G}}(a) \cup \{x\}$ with $x \in \mathcal{A} \setminus \mathtt{Att}_{\mathbf{G}}(a)$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$,

Assume a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \mathtt{WAG}$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{a', b'\}$, for any $x \in \mathcal{A}$, $w_1(x) = w(x)$, $w_1(a') = w_1(b') = w(a)$, and $\mathcal{R}_1 = \mathcal{R} \cup \{(x, b')\}$. Note that $\mathtt{Att}_{\mathbf{G}_1}(a') = \emptyset$ and $\mathtt{Att}_{\mathbf{G}_1}(b') = \{x\}$. From Neutrality, it holds that

$$\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a') = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b').$$

From Proposition 9, it follows that

$$\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b).$$

Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain

$$\forall x \in \mathcal{A}, \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x).$$

Hence, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. $\square$

Proof of Proposition 6. Let $\mathbf{S}$ be a semantics which satisfies Independence, Directionality, Invariance, and Neutrality. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, and let $a, b \in \mathcal{A}$, such that $X \subseteq \mathcal{A} \setminus \mathtt{Att}_{\mathbf{G}}(a)$, if

- $w(a) = w(b)$,
- $\mathtt{Att}_{\mathbf{G}}(b) = \mathtt{Att}_{\mathbf{G}}(a) \cup X$ such that $X \neq \emptyset$ and for any $x \in X$, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$,

Let $X = \{x_1, \ldots, x_n\}$. Assume a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \mathtt{WAG}$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{a_1, \ldots, a_{n-1}\}$, for any $x \in \mathcal{A}$, $w_1(x) = w(x)$, for any $i \in \{1, \ldots, n-1\}$, $w_1(a_i) = w(a)$, and

$$\mathcal{R}_1 = \mathcal{R} \cup \bigcup_{i=1}^{n} \{(x, a_i) \mid x \in \mathtt{Att}_{\mathbf{G}}(a)\} \cup \bigcup_{i=1}^{n} \{(x_j, a_i) \mid x_j \in X, j \leq i\}.$$

Note that $\mathtt{Att}_{\mathbf{G}_1}(a_1) = \mathtt{Att}_{\mathbf{G}}(a) \cup \{x_1\}$ and for $1 < i < n$, $\mathtt{Att}_{\mathbf{G}_1}(a_i) = \mathtt{Att}_{\mathbf{G}_1}(a_{i-1}) \cup \{x_i\}$ and $\mathtt{Att}_{\mathbf{G}_1}(a_n) = \mathtt{Att}_{\mathbf{G}}(b) \cup \{x_n\}$. By applying several times Lemma 1, we get

$$\mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a_1) = \ldots \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a_{n-1}) = \mathrm{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b).$$

Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a) = \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a)$ and $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(b) = \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$. Hence, $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) = \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$ □

Proof of Proposition 7. Let $\mathbf{S}$ be a semantics which satisfies Independence, Directionality, Invariance, and Reinforcement. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b, x, y \in \mathcal{A}$ such that:

- $w(a) = w(b)$,
- $\mathrm{Att}_{\mathbf{G}}(a) \setminus \mathrm{Att}_{\mathbf{G}}(b) = \{x\}$,
- $\mathrm{Att}_{\mathbf{G}}(b) \setminus \mathrm{Att}_{\mathbf{G}}(a) = \{y\}$, and
- $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(y) \geq \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) > 0$,

Assume a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \mathtt{WAG}$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{a', b'\}$, $w_1(z) = w(z)$ for every $z \in \mathcal{A}$, $w_1(a') = w_1(b') = w(a)$, and $\mathcal{R}_1 = \mathcal{R} \cup \{(x, a'), (y, b')\}$. Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(z) = \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(z)$ for every $z \in \mathcal{A}$. From Reinforcement, it holds that

$$\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a') \geq \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(b').$$

From Proposition 9, we get

$$\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a) \geq \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(b).$$

Consequently, $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) \geq \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$. □

Proof of Proposition 8. Let $\mathbf{S}$ be a semantics that satisfies Independence, Directionality, Maximality, Weakening, Invariance, Strict Invariance and Strict Reinforcement. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b, x, y \in \mathcal{A}$ such that:

- $w(a) = w(b)$,
- $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > 0$,
- $\mathrm{Att}_{\mathbf{G}}(a) \setminus \mathrm{Att}_{\mathbf{G}}(b) = \{x\}$,
- $\mathrm{Att}_{\mathbf{G}}(b) \setminus \mathrm{Att}_{\mathbf{G}}(a) = \{y\}$, and
- $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(y) > \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) > 0$,

Assume a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \mathtt{WAG}$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{a', b'\}$, $w_1(z) = w(z)$ for every $z \in \mathcal{A}$, $w_1(a') = w_1(b') = w(a)$, and $\mathcal{R}_1 = \mathcal{R} \cup \{(x, a'), (y, b')\}$. Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(z) = \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(z)$ for every $z \in \mathcal{A}$. Since $\mathrm{Att}_{\mathbf{G}_1}(a') \subseteq \mathrm{Att}_{\mathbf{G}_1}(a)$, then from Proposition 2 it holds that $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a') \geq \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a) = \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a)$. From the condition $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > 0$, we obtain $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a') > 0$. From Strict Reinforcement, it holds that

$$\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a') > \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(b').$$

From Proposition 10, we get

$$\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a) > \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(b).$$

Consequently, $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) > \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$. □

Proof of Proposition 9. Let $\mathbf{S}$ be a semantics which satisfies Independence, Directionality, and Invariance. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, $a, b, a', b' \in \mathcal{A}$, $X, Y \in \mathcal{P}(\mathcal{A}) \setminus \emptyset$ such that:

- $w(a) = w(b)$,
- $w(a') = w(b')$,
- $\mathrm{Att}_{\mathbf{G}}(a') = \mathrm{Att}_{\mathbf{G}}(a) \cup X$,
- $\mathrm{Att}_{\mathbf{G}}(b') = \mathrm{Att}_{\mathbf{G}}(b) \cup Y$,
- there exists a bijective function f from X to Y such that $\forall x \in X$, $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(x) = \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(f(x))$,

Let $X = \{x_1, \ldots, x_n\}$.
  Assume that $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(a) \geq \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}}(b)$.
  Assume a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \mathtt{WAG}$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}\}$, for any $x \in \mathcal{A}$, $w_1(x) = w(x)$, for any $i \in \{1, \ldots, n-1\}$, $w_1(a_i) = w(a)$, $w_1(b_i) = w(b)$, and $\mathcal{R}_1 = \mathcal{R} \cup \{(x, a_i) \mid x \in \mathrm{Att}_{\mathbf{G}}(a)\} \cup \{(x_j, a_i) \mid x_j \in X, j \leq i\} \cup \{(x, b_i) \mid x \in \mathrm{Att}_{\mathbf{G}}(b)\} \cup \{(f(x_j), b_i) \mid x_j \in Y, j \leq i\}$. Note that for $a_i, b_i \in \mathcal{A}_1$, $\mathrm{Att}_{\mathbf{G}_1}(a_i) = \mathrm{Att}_{\mathbf{G}}(a) \cup \{x_1, \ldots, x_i\}$ and $\mathrm{Att}_{\mathbf{G}_1}(b_i) = \mathrm{Att}_{\mathbf{G}}(b) \cup \{f(x_1), \ldots, f(x_i)\}$. By applying several times Invariance, we get $\mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(a_1) \geq \mathrm{Deg}^{\mathbf{S}}_{\mathbf{G}_1}(b_1)$, $\ldots$,

$\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a_{n-1}) \geq \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b_{n-1})$, and $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a') \geq \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b')$. Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain that for any $x \in \mathcal{A}$, $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$. Hence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$. $\quad\square$

Proof of Proposition 10. Similar as the proof of Proposition 9. $\quad\square$

Proof of Proposition 11. Suppose a semantics **S** which satisfies Independence, Directionality, Invariance, Neutrality and Maximality. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$ be a weighted argumentation graph and $a \in \mathcal{A}$ such that $\text{Att}_{\mathbf{G}}(a) \neq \emptyset$ and $\forall x \in \text{Att}_{\mathbf{G}}(a)$, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$. Let $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle$ be another weighted argumentation graph such that $\mathcal{A}_1 = \mathcal{A} \cup \{b\}$ ($b \notin \mathcal{A}$), for any $x \in \mathcal{A}$, $w_1(x) = w(x)$, $w_1(b) = w(a)$, and $\mathcal{R}_1 = \mathcal{R}$. Note that $\text{Att}_{\mathbf{G}_1}(b) = \emptyset$. Hence, from Proposition 6 it holds that $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b)$. From Independence, $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a)$. Hence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b)$. From Maximality it holds that $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b) = w(a)$. So, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = w(a)$. $\quad\square$

Proof of Proposition 12. Let **S** be a semantics satisfying Independence, Directionality, Neutrality, Maximality, Strict Weakening, Invariance, and Strict Invariance. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$, and $a, b, x \in \mathcal{A}$, such that

- $w(a) = w(b)$,
- $\text{Att}_{\mathbf{G}}(b) = \text{Att}_{\mathbf{G}}(a) \cup \{x\}$ with $x \notin \text{Att}_{\mathbf{G}}(a)$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$

Assume a new graph $\mathbf{G}_1 = \langle \mathcal{A}_1, w_1, \mathcal{R}_1 \rangle \in \text{WAG}$ such that $\mathcal{A}_1 = \mathcal{A} \cup \{a', b', c, y\}$, $w_1(z) = w(z)$ for every $z \in \mathcal{A}$, $w_1(a') = w_1(b') = w_1(c) = w(a)$, $w_1(y) = 0$, and $\mathcal{R}_1 = \mathcal{R} \cup \{(x, b'), (y, a')\}$. Note that $\text{Att}_{\mathbf{G}_1}(a') = \{y\}$, $\text{Att}_{\mathbf{G}_1}(b') = \{x\}$ and $\text{Att}_{\mathbf{G}_1}(y) = \text{Att}_{\mathbf{G}_1}(c) = \emptyset$. Note that $\mathcal{A}$ is unattacked in $\mathbf{G}_1$, and that $(\mathbf{G}_1)|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain that for every $z \in \mathcal{A}$, it holds that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(z) = \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(z)$. Then, $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a)$ and $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Maximality ensures $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(y) = w(y) = 0$ and $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(c) = w(a)$. Neutrality ensures $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(c) = \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a') = w(a)$.

From Theorem 1, it follows that $0 \leq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \leq w(a)$. Since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$, then $w(a) > 0$. Since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$ and $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$, then $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(x) > 0$, and Strict Weakening leads to $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b') < w(a)$. Then, $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a') > \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b')$. From Proposition 10, it follows that $\text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(a) > \text{Deg}_{\mathbf{G}_1}^{\mathbf{S}}(b)$. Consequently, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. $\quad\square$

Proof of Proposition 13. Let **S** be a semantics which satisfies Anonymity, Independence, Directionality, Neutrality, Monotony, Invariance, and Reinforcement. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a, b \in \mathcal{A}$ such that:

- $w(a) = w(b)$,
- $\text{Att}_{\mathbf{G}}(a) \setminus \text{Att}_{\mathbf{G}}(b) = \{x\}$,
- $\text{Att}_{\mathbf{G}}(b) \setminus \text{Att}_{\mathbf{G}}(a) = \{y\}$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0$.

Let us show that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$. There are two cases:

- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$. From Proposition 7, it follows that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. Since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) \in [0, 1]$ by definition of a semantics, then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$. This case has two sub-cases.
  - Let $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$. $\mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in \text{WAG}$ such that $\mathcal{A}' = \mathcal{A} \cup \{b_1\}$, for all $t \in \mathcal{A}$, $w'(t) = w(t)$, $w'(b_1) = w(b)$, $\mathcal{R}' = \mathcal{R} \cup \{(t, b_1) \mid t \in X\}$, where $X = \text{Att}_{\mathbf{G}}(a) \setminus \{x\}$. Note that $\mathcal{A}$ is unattacked in $\mathbf{G}'$, and that $(\mathbf{G}')|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain that for each $t \in \mathcal{A}$, $\text{Deg}_{\mathbf{G}'}^{\mathbf{S}}(t) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(t)$. From Lemma 1, $\text{Deg}_{\mathbf{G}'}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}'}^{\mathbf{S}}(b_1)$. From Monotony, $\text{Deg}_{\mathbf{G}'}^{\mathbf{S}}(b) \leq \text{Deg}_{\mathbf{G}'}^{\mathbf{S}}(b_1)$. Consequently

    $$\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \text{Deg}_{\mathbf{G}'}^{\mathbf{S}}(b) \leq \text{Deg}_{\mathbf{G}'}^{\mathbf{S}}(b_1) = \text{Deg}_{\mathbf{G}'}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0.$$

  - Let $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$. Let $\mathbf{G}'' = \langle \mathcal{A}'', w'', \mathcal{R}'' \rangle \in \text{WAG}$ such that $\mathcal{A}'' = \mathcal{A} \cup \{a_2, b_2\}$, for all $t \in \mathcal{A}$ $w''(t) = w(t)$, $w''(a_2) = w''(b_2) = w(a) = w(b)$, $\mathcal{R}'' = \mathcal{R} \cup \{(t, a_2) \mid t \in X\} \cup \{(t, b_2) \mid t \in X\}$, where $X = \text{Att}_{\mathbf{G}}(a) \setminus \{x\}$. Since a_2 and b_2 have the same attackers in $\mathbf{G}''$, from Anonymity, $\text{Deg}_{\mathbf{G}''}^{\mathbf{S}}(a_2) = \text{Deg}_{\mathbf{G}''}^{\mathbf{S}}(b_2)$. From Invariance, $\text{Deg}_{\mathbf{G}''}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}''}^{\mathbf{S}}(b)$. Note that $\mathcal{A}$ is unattacked in $\mathbf{G}''$, and that $(\mathbf{G}'')|_{\mathcal{A}} = \mathbf{G}$. From Proposition 3 we obtain that for each $t \in \mathcal{A}$, $\text{Deg}_{\mathbf{G}''}^{\mathbf{S}}(t) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(t)$. Thus, $\text{Deg}_{\mathbf{G}''}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a)$ and $\text{Deg}_{\mathbf{G}''}^{\mathbf{S}}(b) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. Hence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0$. $\quad\square$

Proof of Theorem 1. Let **S** be a semantics which satisfies Independence, Directionality, Neutrality, Invariance, Weakening and Maximality. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$ be a weighted argumentation graph and $a \in \mathcal{A}$. There are two cases:

Case 1. $\text{Att}_{\mathbf{G}}(a) = \emptyset$. From Maximality, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = w(a)$.

Case 2. $\text{Att}_\mathbf{G}(a) \neq \emptyset$. There are again two sub-cases:

> **Case 2.1.** $\forall x \in \text{Att}_\mathbf{G}(a)$, $\text{Deg}_\mathbf{G}^\mathbf{S}(x) = 0$. From Proposition 11, $\text{Deg}_\mathbf{G}^\mathbf{S}(a) = w(a)$.
> **Case 2.2.** $\exists x \in \text{Att}_\mathbf{G}(a)$ such that $\text{Deg}_\mathbf{G}^\mathbf{S}(x) > 0$. From Weakening, it follows that $\text{Deg}_\mathbf{G}^\mathbf{S}(a) \leq w(a)$. $\square$

Proof of Theorem 2. Let $\mathbf{S}$ be a semantics which satisfies Independence and Directionality, let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a \in \mathcal{A}$. Let

$$\mathbf{G}_0 = \langle \mathcal{A}_0, w_0, \mathcal{R}_0 \rangle = \mathbf{G}|_{\text{Str}_\mathbf{G}(a)} \oplus \mathbf{G}|_{\mathcal{A} \setminus \text{Str}_\mathbf{G}(a)}.$$

By Independence, $\text{Deg}_{\mathbf{G}_0}^\mathbf{S}(a) = \text{Deg}_{\mathbf{G}|_{\text{Str}_\mathbf{G}(a)}}^\mathbf{S}(a)$. Note that $\mathcal{A}_0 = \mathcal{A}$ and $w_0 = w$, while $\mathcal{R}_0 \subseteq \mathcal{R}$. If $\mathcal{R}_0 = \mathcal{R}$, then $\mathbf{G}_0 = \mathbf{G}$ and $\text{Deg}_\mathbf{G}^\mathbf{S}(a) = \text{Deg}_{\mathbf{G}|_{\text{Str}_\mathbf{G}(a)}}^\mathbf{S}(a)$. If $\mathcal{R} \setminus \mathcal{R}_0 \neq \emptyset$, then for every $(x, y) \in \mathcal{R} \setminus \mathcal{R}_0$ exactly one argument from $\{x, y\}$ belongs to $\text{Str}_\mathbf{G}(a)$. If we suppose that $y \in \text{Str}_\mathbf{G}(a)$ and $x \in \mathcal{A} \setminus \text{Str}_\mathbf{G}(a)$, then there exists a path π from y to a, and the concatenation of the path (x, y) and π would be a path from x to a, which is impossible since $x \notin \text{Str}_\mathbf{G}(a)$. Consequently, letting $\mathcal{R} \setminus \mathcal{R}_0 = \{(x_1, y_1), \ldots, (x_n, y_n)\}$, it holds that $x_i \in \text{Str}_\mathbf{G}(a)$ and $y_i \in \mathcal{A} \setminus \text{Str}_\mathbf{G}(a)$ for every $i \in \{1, \ldots, n\}$. If $\mathbf{G}_i = \langle \mathcal{A}, w, \mathcal{R}_i \rangle$, where $\mathcal{R}_i = \mathcal{R}_0 \cup \{(x_1, y_1), \ldots, (x_i, y_i)\}$ for every $i \in \{1, \ldots, n\}$, then by Directionality

$$\text{Deg}_{\mathbf{G}_0}^\mathbf{S}(a) = \text{Deg}_{\mathbf{G}_1}^\mathbf{S}(a) = \cdots = \text{Deg}_{\mathbf{G}_n}^\mathbf{S}(a).$$

Note that $\mathbf{G}_n = \mathbf{G}$. Now the claim follows from $\text{Deg}_{\mathbf{G}_0}^\mathbf{S}(a) = \text{Deg}_{\mathbf{G}|_{\text{Str}_\mathbf{G}(a)}}^\mathbf{S}(a)$. $\square$

Proof of Theorem 3. Let $\mathbf{S}$ be a semantics which satisfies Anonymity, Independence and Directionality, let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a, b \in \mathcal{A}$ such that there exists an isomorphism $f : \mathbf{G}|_{\text{Str}_\mathbf{G}(a)} \to \mathbf{G}|_{\text{Str}_\mathbf{G}(b)}$ with $f(a) = b$. By Anonymity, $\text{Deg}_{\mathbf{G}|_{\text{Str}_\mathbf{G}(a)}}^\mathbf{S}(a) = \text{Deg}_{\mathbf{G}|_{\text{Str}_\mathbf{G}(b)}}^\mathbf{S}(b)$. From Theorem 2 we obtain both $\text{Deg}_\mathbf{G}^\mathbf{S}(a) = \text{Deg}_{\mathbf{G}|_{\text{Str}_\mathbf{G}(a)}}^\mathbf{S}(a)$ and $\text{Deg}_\mathbf{G}^\mathbf{S}(a) = \text{Deg}_{\mathbf{G}|_{\text{Str}_\mathbf{G}(b)}}^\mathbf{S}(b)$. Consequently, $\text{Deg}_\mathbf{G}^\mathbf{S}(a) = \text{Deg}_\mathbf{G}^\mathbf{S}(b)$. $\square$

Proof of Theorem 4. Let $\mathbf{S}$ be a semantics which satisfies Independence, Directionality, Invariance, Reinforcement, Maximality, Neutrality, and Weakening. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$ be a weighted argumentation graph and $a, b \in \mathcal{A}$ such that $w(a) = w(b)$ and there exists an injective function f from $\text{Att}_\mathbf{G}(a)$ to $\text{Att}_\mathbf{G}(b)$ such that $\forall x \in \text{Att}_\mathbf{G}(a)$, $\text{Deg}_\mathbf{G}^\mathbf{S}(x) \leq \text{Deg}_\mathbf{G}^\mathbf{S}(f(x))$. Let us show that $\text{Deg}_\mathbf{G}^\mathbf{S}(a) \geq \text{Deg}_\mathbf{G}^\mathbf{S}(b)$.

The case $\text{Deg}_\mathbf{G}^\mathbf{S}(b) = 0$ is trivial; this is why in the remainder of the proof we suppose $\text{Deg}_\mathbf{G}^\mathbf{S}(b) > 0$.

Let $\mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle$ be a weighted argumentation graph such that $\mathcal{A}' = \mathcal{A} \cup \{b_0\}$, for all $t \in \mathcal{A}$, $w'(t) = w(t)$, $w'(b_0) = w(b)$ and $\mathcal{R}' = \mathcal{R} \cup \{(f(t), b_0) \mid t \in \text{Att}_\mathbf{G}(a)\}$. Note that $\mathcal{A}$ is unattacked in $\mathbf{G}'$, and that $(\mathbf{G}')|_\mathcal{A} = \mathbf{G}$. From Proposition 3 we obtain that for all $t \in \mathcal{A}$, $\text{Deg}_\mathbf{G}^\mathbf{S}(t) = \text{Deg}_{\mathbf{G}'}^\mathbf{S}(t)$. From Proposition 2 we have that $\mathbf{S}$ satisfies Monotony, so $\text{Deg}_{\mathbf{G}'}^\mathbf{S}(b_0) \geq \text{Deg}_{\mathbf{G}'}^\mathbf{S}(b)$. Thus, it remains to prove that $\text{Deg}_{\mathbf{G}'}^\mathbf{S}(a) \geq \text{Deg}_{\mathbf{G}'}^\mathbf{S}(b_0)$.

If $\text{Att}_\mathbf{G}(a) = \emptyset$, then $\text{Att}_{\mathbf{G}'}(a) = \text{Att}_{\mathbf{G}'}(b_0) = \emptyset$. From Maximality and $w'(b_0) = w'(a)$ we obtain $\text{Deg}_{\mathbf{G}'}^\mathbf{S}(a) \geq \text{Deg}_{\mathbf{G}'}^\mathbf{S}(b_0)$.

If $\text{Att}_\mathbf{G}(a) \neq \emptyset$, let $\text{Att}_\mathbf{G}(a) = \{x_1, \ldots, x_n\}$, for some $n \geq 1$. Let $\mathbf{G}^{(1)} = \langle \mathcal{A}^{(1)}, w^{(1)}, \mathcal{R}^{(1)} \rangle$ be such that $\mathcal{A}^{(1)} = \mathcal{A}' \cup \{b_1, \ldots, b_n\}$, for all $t \in \mathcal{A}'$, $w^{(1)}(t) = w'(t)$, $w^{(1)}(b_1) = \cdots = w^{(1)}(b_n) = w(b)$ and $\mathcal{R}^{(1)}$ is such that

- for all $t \in \mathcal{A}'$, $\text{Att}_{\mathbf{G}^{(1)}}(t) = \text{Att}_{\mathbf{G}'}(t)$
- for every $i \in \{1, \ldots, n\}$, $\text{Att}_{\mathbf{G}^{(1)}}(b_i) = (\text{Att}_{\mathbf{G}^{(1)}}(b_{i-1}) \setminus \{f(x_i)\}) \cup \{x_i\}$.

Note that $\mathcal{A}'$ is unattacked in $\mathbf{G}^{(1)}$, and that $(\mathbf{G}^{(1)})|_{\mathcal{A}'} = \mathbf{G}'$. From Proposition 3 we obtain that for all $t \in \mathcal{A}'$, $\text{Deg}_{\mathbf{G}^{(1)}}^\mathbf{S}(t) = \text{Deg}_{\mathbf{G}'}^\mathbf{S}(t)$. Thus, in order to prove the proposition, it is sufficient to show that

$$\text{Deg}_{\mathbf{G}^{(1)}}^\mathbf{S}(b_i) \geq \text{Deg}_{\mathbf{G}^{(1)}}^\mathbf{S}(b_{i-1})$$

for every $i \in \{1, \ldots, n\}$. For each i we consider two possible cases:

- If $\text{Deg}_\mathbf{G}^\mathbf{S}(x_i) \neq 0$, then using Proposition 7 we obtain $\text{Deg}_{\mathbf{G}^{(1)}}^\mathbf{S}(b_i) \geq \text{Deg}_{\mathbf{G}^{(1)}}^\mathbf{S}(b_{i-1})$.
- If $\text{Deg}_\mathbf{G}^\mathbf{S}(x_i) = 0$, let $S_i = \text{Att}_{\mathbf{G}^{(1)}}(b_i) \cap \text{Att}_{\mathbf{G}^{(1)}}(b_{i-1})$. Note that $\text{Att}_{\mathbf{G}^{(1)}}(b_i) = S_1 \cup \{x_i\}$ and $\text{Att}_{\mathbf{G}^{(1)}}(b_{i-1}) = S_1 \cup \{f(x_i)\}$. Let $\mathbf{G}^{(i)} = \langle \mathcal{A}^{(i)}, w^{(i)}, \mathcal{R}^{(i)} \rangle$ be such that $\mathcal{A}^{(i)} = \mathcal{A}^{(1)} \cup \{c_i\}$, for all $t \in \mathcal{A}^{(1)}$, $w^{(i)}(t) = w^{(1)}(t)$, $w^{(i)}(c_i) = w(b)$ and $\mathcal{R}^{(i)} = \mathcal{R}^{(1)} \cup \{(t, c_i) \mid t \in S_i\}$. From Proposition 6 we obtain $\text{Deg}_{\mathbf{G}^{(i)}}^\mathbf{S}(b_i) = \text{Deg}_{\mathbf{G}^{(i)}}^\mathbf{S}(c_i)$, and from Proposition 2 we have $\text{Deg}_{\mathbf{G}^{(i)}}^\mathbf{S}(c_i) \geq \text{Deg}_{\mathbf{G}^{(i)}}^\mathbf{S}(b_{i-1})$. Thus, $\text{Deg}_{\mathbf{G}^{(i)}}^\mathbf{S}(b_i) \geq \text{Deg}_{\mathbf{G}^{(i)}}^\mathbf{S}(b_{i-1})$. Finally, note that $\mathcal{A}^{(1)}$ is unattacked in $\mathbf{G}^{(i)}$, and that $(\mathbf{G}^{(i)})|_{\mathcal{A}^{(1)}} = \mathbf{G}^{(1)}$. From Proposition 3 we obtain $\text{Deg}_{\mathbf{G}^{(1)}}^\mathbf{S}(b_i) \geq \text{Deg}_{\mathbf{G}^{(1)}}^\mathbf{S}(b_{i-1})$. $\square$

Proof of Theorem 5. Contraction-based approach: grounded, stable, preferred and complete semantics
    Anonymity is satisfied by the four semantics. This follows straightforwardly from the definitions of the semantics.

Independence is satisfied by grounded, preferred and complete semantics. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle, \mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in \mathtt{WAG}$ be such that $\mathcal{A} \cap \mathcal{A}' = \emptyset$. Since grounded, complete and preferred semantics satisfy Directionality as defined by Baroni and Giacomin [24], we have that for any $x \in \{g, c, p\}$, $\mathtt{Ext}_x(\mathbf{G}) = \{\mathcal{E} \cap \mathcal{A} \mid \mathcal{E} \in \mathtt{Ext}_x(\mathbf{G} \oplus \mathbf{G}')\}$. Hence, for any $a \in \mathcal{A}$, $\mathtt{Deg}^x_{\mathbf{G}}(a) = \mathtt{Deg}^x_{\mathbf{G} \oplus \mathbf{G}'}(a)$.

Stable semantics violates independence. Consider the two weighted argumentation graphs $\mathbf{G}$ and $\mathbf{G}'$ depicted below. Note that despite the fact that the two graphs do not share arguments, $\mathtt{Deg}^{st}_{\mathbf{G}}(b) = 1$ while $\mathtt{Deg}^{st}_{\mathbf{G} \oplus \mathbf{G}'}(b) = \beta$ since the graph $\mathbf{G}$ has one stable extension $\{b\}$ while the graph $\mathbf{G} \oplus \mathbf{G}'$ has no extension.
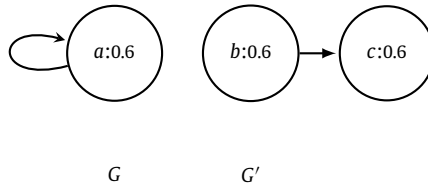

G $\qquad\qquad$ G'

Directionality is satisfied by grounded, preferred and complete semantics. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $\mathbf{G}' = \langle \mathcal{A}, w, \mathcal{R}' \rangle \in \mathtt{WAG}$ be such that $\mathcal{R}' = \mathcal{R} \cup \{(a, b)\}$. We denote by $\mathcal{A}' = \{x \in \mathcal{A} \mid$ there is a path from b to x with respect to $\mathcal{R}\}$. Let $\mathcal{A}'' = \mathcal{A} \setminus \mathcal{A}'$. Note that $\mathcal{A}'$ does not attack $\mathcal{A}''$ with respect to $\mathcal{R}$. A fortiori, $\mathcal{A}'$ does not attack $\mathcal{A}''$ with respect to the revised attack relation. Since grounded, preferred and complete semantics satisfy Directionality as defined by Baroni and Giacomin [24], the status of x is the same in $\mathbf{G}$ and $\mathbf{G}'$.

To see that (our) Directionality principle is not satisfied by stable semantics, consider the graph $\mathbf{G}$ depicted below. This graph has one stable extension $\{a, b\}$, hence $\mathtt{Deg}^{st}_{\mathbf{G}}(b) = 1$. If now we add an attack from a to itself (thus a attacks itself), the new graph has no stable extension, thus $\mathtt{Deg}^{st}_{\mathbf{G}}(b) = \beta$ even if there is no path from a to b.
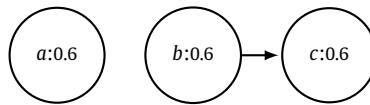

Maximality is violated by the four semantics. Consider the argument a of Example 1. The four semantics assign the value 1 to this argument while its basic weight is 0.01.

Weakening and **Strict Weakening** are violated by the four semantics. Consider the graph depicted below. The preference-based approaches based on Dung's semantics [11,31,32] remove the attack from a to b since $w(a) < w(b)$. The new graph has one grounded, complete, stable and preferred extension $\{a, b\}$. Hence, both a and b get degree 1. Note that $\mathtt{Deg}^{\mathbf{S}}_{\mathbf{G}}(b) > w(b)$.
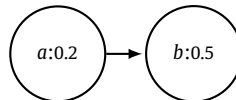
Weakening Soundness is violated by stable, preferred and complete semantics. To see why preferred and complete semantics violate Weakening Soundness, consider the graph depicted below. Both arguments b and c have degree 0 since they are rejected and each of them is attacked by an extension. The argument a has a basic weight equal to 1 but its strength is equal to $\beta < 1$.
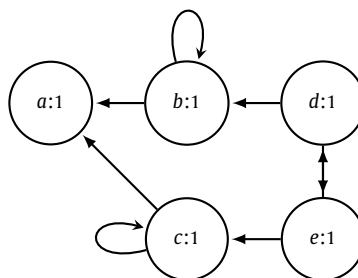
To see that stable semantics violates Weakening Soundness, it is sufficient to consider a graph containing two arguments a and b such that $w(a) = w(b) = 1$ and $\mathcal{R} = \{(b, b)\}$. Since there is no stable extension, the degree of a is β.

Let us now show that grounded semantics satisfies weakening soundness. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, $\mathbf{G}' = \langle \mathcal{A}, \mathcal{R}' \rangle$ is its revised graph, and $\mathtt{GE}(\mathbf{G}')$ the grounded extension of $\mathbf{G}'$. Let $a \in \mathcal{A}$ such that $w(a) > 0$. Assume also that $\mathtt{Deg}_{\mathbf{G}}^{g}(a) < w(a)$. Thus, $\mathtt{Deg}_{\mathbf{G}}^{g}(a) < 1$. Consequently, either $\mathtt{Deg}_{\mathbf{G}}^{g}(a) = 0$ or $\mathtt{Deg}_{\mathbf{G}}^{g}(a) = \beta$. Assume that $\mathtt{Deg}_{\mathbf{G}}^{g}(a) = 0$, then by definition, there exists $x \in \mathtt{GE}(\mathbf{G}')$ such that $x\mathcal{R}'a$, hence $x\mathcal{R}a$. Furthermore, $\mathtt{Deg}_{\mathbf{G}}^{g}(x) = 1$. Assume now that $\mathtt{Deg}_{\mathbf{G}}^{g}(a) = \beta$. Clearly $\mathtt{Att}_{\mathbf{G}'}(a) \neq \emptyset$ (otherwise a would belong to the grounded extension of $\mathbf{G}'$). If $\forall x \in \mathtt{Att}(a)$, $\mathtt{Deg}_{\mathbf{G}}^{g}(x) = 0$, then by definition of grounded extension, a should belong to the extension. Hence, there exists at least one attacker whose degree is greater than 0.

Neutrality is satisfied by grounded and stable semantics. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b \in \mathcal{A}$ be such that:

- $w(a) = w(b)$,
- $\mathtt{Att}_{\mathbf{G}}(a) = \emptyset$,
- $\mathtt{Att}_{\mathbf{G}}(b) = \{x\}$ with $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$.

Let us first present the proof for grounded semantics. Since a is not attacked, then a belongs to the grounded extension and thus $\mathtt{Deg}_{\mathbf{G}}^{g}(a) = 1$. From $\mathtt{Deg}_{\mathbf{G}}^{g}(x) = 0$, the argument x is attacked by the grounded extension. Since all the attackers of b are attacked by the extension, b belongs to the extension. Thus $\mathtt{Deg}_{\mathbf{G}}^{g}(b) = 1$.

Let us show that stable semantics satisfies Neutrality. If there are no stable extensions, $\mathtt{Deg}_{\mathbf{G}}^{st}(a) = \mathtt{Deg}_{\mathbf{G}}^{st}(b) = \beta$. Else, suppose that there exists at least one stable extension. Since a is not attacked, it belongs to all extensions and then $\mathtt{Deg}_{\mathbf{G}}^{st}(a) = 1$. Since $\mathtt{Deg}_{\mathbf{G}}^{st}(x) = 0$, then x does not belong to any extension. From the definition of stable semantics, this means that x is attacked by all the extensions. Since x is the only attacker of b, this means that b is not attacked by any extension. Hence, b belongs to all extensions. Consequently, $\mathtt{Deg}_{\mathbf{G}}^{st}(b) = 1$.

Neutrality is violated by complete and preferred semantics. Consider the graph below. Namely, the degree of a is equal to 1 whereas the degree of b is α, even if the degree of x is 0.
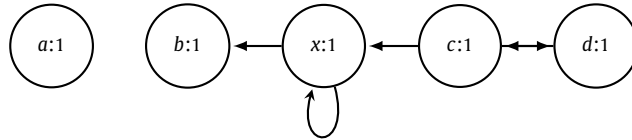

Symmetry is satisfied by the four semantics. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $\mathbf{G}' = \mathbf{G} = \langle \mathcal{A}, \mathcal{R}_c \rangle$ its revised version. Let us show that for any $a, b \in \mathcal{A}$, such that $w(a) = w(b)$ and $\mathtt{Att}_{\mathbf{G}}(a) = \mathtt{Att}_{\mathbf{G}}(b)$, it holds that $\mathtt{Att}_{\mathbf{G}'}(a) = \mathtt{Att}_{\mathbf{G}'}(b)$. Let $(x, a) \in \mathcal{R}$. There are two cases:

Case $w(x) \geq w(a)$ By definition of $\mathcal{R}_c$, it holds that $(x, a) \in \mathcal{R}_c$. Since $w(a) = w(b)$, then $w(x) \geq w(b)$. Since $(x, b) \in \mathcal{R}$, then $(x, b) \in \mathcal{R}_c$.

Case $w(x) < w(a)$ By definition of $\mathcal{R}_c$, it holds that $(x, a) \notin \mathcal{R}_c$. Since $w(a) = w(b)$, then $w(x) < w(b)$. Since $\mathtt{Att}_{\mathbf{G}}(a) = \mathtt{Att}_{\mathbf{G}}(b)$, then $(x, b) \notin \mathcal{R}_c$.

Hence, $\mathtt{Att}_{\mathbf{G}'}(a) = \mathtt{Att}_{\mathbf{G}'}(b)$. For each extension E, we have $a \in E$ if and only if $b \in E$. Hence, a and b have the same degrees.

Equivalence is violated by the four semantics. Consider the graph depicted below. Note that $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(c) = \mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(e) = 1$ while $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0$ and $\mathtt{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 1$ for any $\mathbf{S} \in \{s, g, c, p\}$.
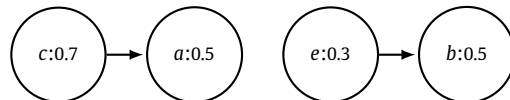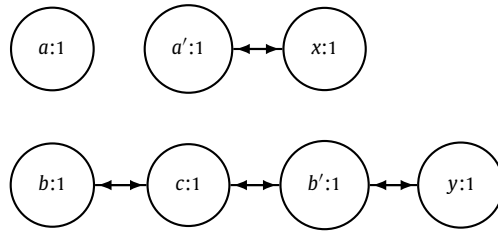

Invariance is violated by all the four semantics. Consider the graph depicted below.
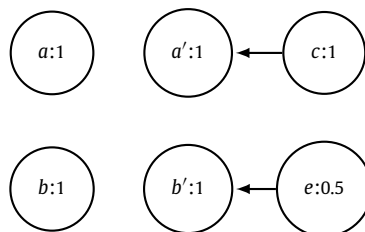
Since b' is strictly stronger than e, we obtain the following graph which has a unique stable / preferred / complete / grounded extension: $\{a, c, b, b', e\}$. Note that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 1$ while $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') = 0 < \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b') = 1$.
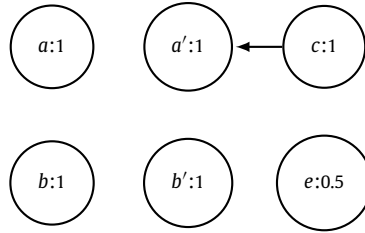
***Strict Invariance*** is violated by the four semantics. Consider the following graph.

For all four semantics, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 1$ whereas all the other arguments get the degree α. Thus $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$. Consequently, Strict Invariance is violated.

Monotony is satisfied by all the four semantics. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a, b \in \mathcal{A}$ be such that $w(a) = w(b)$ and $\text{Att}_{\mathbf{G}}(a) \subseteq \text{Att}_{\mathbf{G}}(b)$. Let us prove that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

- Case $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 1$. This means that b is in all extensions. Let E be an extension, hence $b \in E$. Observe that E defends b, hence E defends a. Moreover, $E \cup \{a\}$ is conflict-free. Hence $a \in E$. This means that a belongs to all the extensions and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 1$.
- Case $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \alpha$. Thus, b belongs to at least one extension E. Like in the previous item, we have $a \in E$. Hence a belongs to at least one extension and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \alpha$.
- Case $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \beta$. Thus, b is not attacked by any extension. Consequently, a is not attacked by any extension. Hence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \beta$.
- Case $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$ is trivial.

Counting is violated by the four semantics. Consider the graph depicted below. In case of complete, preferred or stable semantics, all arguments have degree α. In case of grounded semantics, all arguments have degree β. For every of the four semantics, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$ but $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. Thus, Counting is violated.
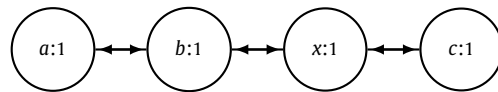
Reinforcement is violated by the four semantics. Consider the argumentation graph depicted below.
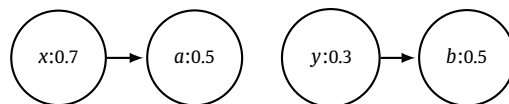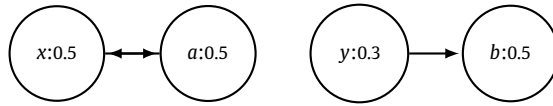

When preferences are taken into account, the attack from y to b is ignored and we obtain the following graph:
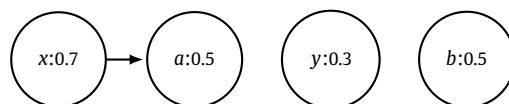
There is a unique stable / preferred / complete / grounded extension: $\{x, y, b\}$. Thus $\text{Deg}^{\textbf{S}}_{\textbf{G}}(x) = \text{Deg}^{\textbf{S}}_{\textbf{G}}(y)$ and $\text{Deg}^{\textbf{S}}_{\textbf{G}}(a) <$ $\text{Deg}^{\textbf{S}}_{\textbf{G}}(b)$, which means that Reinforcement is violated by the four semantics.

Strict Reinforcement is violated by all the four semantics. Consider the counter-example below.

When preferences are taken into account, the attack from y to b is ignored and we obtain the following graph:
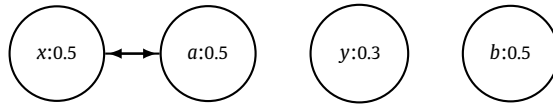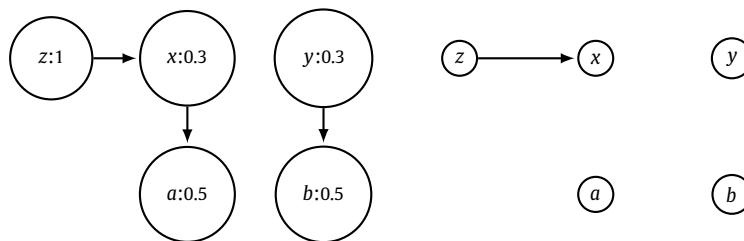

With respect to all semantics, $\text{Deg}^{\textbf{S}}_{\textbf{G}}(y) = \text{Deg}^{\textbf{S}}_{\textbf{G}}(b) = 1$. The two other arguments have score α with respect to stable, preferred and complete semantics; they have score β with respect to grounded semantics. In all cases, $\text{Deg}^{\textbf{S}}_{\textbf{G}}(a) > 0$ and $\text{Deg}^{\textbf{S}}_{\textbf{G}}(y) > \text{Deg}^{\textbf{S}}_{\textbf{G}}(x) > 0$. However, $\text{Deg}^{\textbf{S}}_{\textbf{G}}(a) < \text{Deg}^{\textbf{S}}_{\textbf{G}}(b)$. Thus, Strict Reinforcement is not satisfied.

Proportionality is satisfied by all the four semantics. In both cases, in the "revised" graph, i.e. the graph where some attacks might be deleted, we have $\text{Att}(a) \subseteq \text{Att}(b)$. Thus, $\text{Deg}^{\textbf{S}}_{\textbf{G}}(a) \geq \text{Deg}^{\textbf{S}}_{\textbf{G}}(b)$ (the proof is the same as the proof of Monotony).

Strict Proportionality is violated by all four semantics. It is sufficient to consider a graph with two arguments a and b with an empty attack relation and $w(a) = 0.7$, $w(b) = 0.5$. We have $\text{Deg}^{\textbf{S}}_{\textbf{G}}(a) = \text{Deg}^{\textbf{S}}_{\textbf{G}}(b) = 1$.

Resilience is violated by the four semantics. It is sufficient to consider the graph consisting of two arguments, a and x and one attack (x, a), where both arguments have the same weight 1. We have $w(a) = 1$ and $\text{Deg}^{\textbf{S}}_{\textbf{G}}(a) = 0$.

QP is violated by the four semantics as shown with the graph depicted below on the left-side. Its revised version, on the right-side, has one stable/preferred/complete/grounded extension $\{z, y, a, b\}$. Hence, $\text{Deg}^{\textbf{S}}_{\textbf{G}}(y) > \text{Deg}^{\textbf{S}}_{\textbf{G}}(x)$ while $\text{Deg}^{\textbf{S}}_{\textbf{G}}(a) = \text{Deg}^{\textbf{S}}_{\textbf{G}}(b) = 1$.

CP is violated by the four semantics as shown with the graph depicted below. Note that this graph is identical to its revised version. Under preferred, complete and stable semantics, all the arguments get degree α while they all get degree β under grounded semantics.
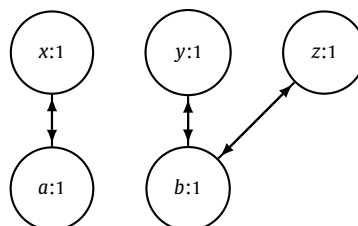
Compensation is satisfied by the four semantics since they all violate CP and QP.

Change-based approach: Grounded, Stable, Preferred and Complete semantics

Anonymity is satisfied by the four semantics (from the definitions of the four semantics).

Independence: the proof is the same as for the contraction-based approach. The counter-example for stable semantics is also the same.

Directionality is violated by the four semantics. Indeed, let **S** be any of the four change-based semantics, and let us consider the graph $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$ where $\mathcal{A} = \{a, b\}$, $w(a) = 0.2$, $w(b) = 0.4$ and $\mathcal{R} = \emptyset$. Since $\mathbf{G}$ does not contain any attack, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 1$. Let $\mathcal{R}' = \mathcal{R} \cup \{(a, b)\}$ and let $\mathbf{G}' = \langle \mathcal{A}, w, \mathcal{R}' \rangle$. Since $w(a) < w(b)$, change-based semantics will invert the attack, so we obtain the graph $\langle \mathcal{A}, \{(b, a)\} \rangle$ which has the unique extension $\{b\}$. Thus, $\mathrm{Deg}_{\mathbf{G}'}^{\mathbf{S}}(a) = 0$. Hence, Directionality is violated.

Maximality is violated by the four semantics. As in the case of contraction-based approach, consider the argument a of Example 1. The four semantics assign value 1 to this argument while its basic weight is 0.01.

Weakening and ***Strict Weakening*** are violated by the four semantics. Consider the graph below. Both arguments get the degree α in the case of stable, preferred and complete semantics. They both get degree β in case of grounded semantics. In all cases, their degree is strictly greater than their initial weight.
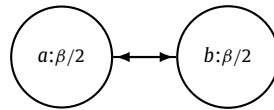
Weakening Soundness is violated by stable, preferred and complete semantics. It is sufficient to consider the same counter-example as for contraction-based semantics. Weakening Soundness is satisfied by grounded semantics and the proof is similar to the one for contraction-based approach.

Neutrality is violated by the four semantics. Consider the graph below.
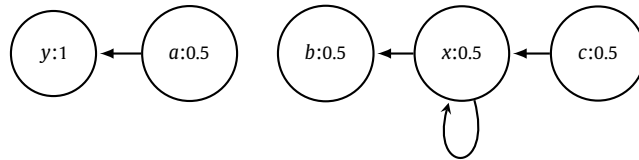
Its revised version is the graph below. Note that $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$ for the four semantics while $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0$, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 1$.
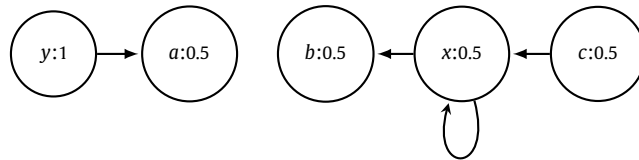
Symmetry is violated by the four semantics. Let us consider the weighted graph $\mathbf{G}$ depicted below on the left-side. Note that $w(a) = w(b)$ and $\mathrm{Att}_{\mathbf{G}}(a) = \mathrm{Att}_{\mathbf{G}}(b)$. Note also that b is weaker than its target y. Thus, the revised graph $\mathbf{G}'$ (depicted on the right-side) contains the inverted arrow from y to b. Thus, $\mathrm{Att}_{\mathbf{G}'}(a) \neq \mathrm{Att}_{\mathbf{G}'}(b)$. It is easy to check that $\mathbf{G}'$ has two complete/preferred/stable extensions: $\mathcal{E}_1 = \{x\}$ and $\mathcal{E}_2 = \{a, y\}$. Thus, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \beta$ while $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.
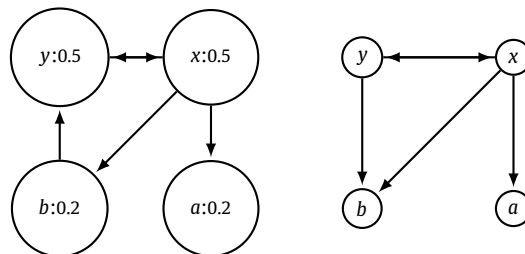
Let us now present a counter-example for grounded semantics. Let us consider the argumentation weighted graph **G** depicted below on the left-side. Note that $w(a) = w(b)$ and $\text{Att}_{\textbf{G}}(a) = \text{Att}_{\textbf{G}}(b)$. The revised graph **G**$'$ is depicted on the right-side. Its grounded extension is $\mathcal{E} = \{y\}$, thus $\text{Deg}_{\textbf{G}}^{\textbf{S}}(a) = \alpha$ while $\text{Deg}_{\textbf{G}}^{\textbf{S}}(b) = 0$.
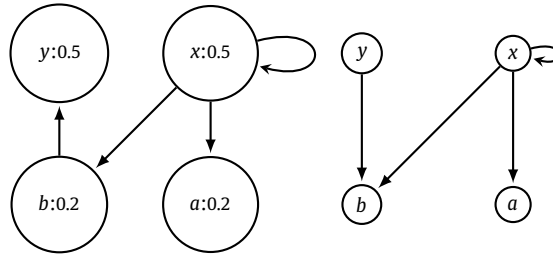
Equivalence is violated by the four semantics. This can be seen on the two counter-examples given for symmetry.

Invariance is violated by the four semantics. Consider the following counter-example for grounded semantics. It can be checked that the grounded extension of its revised version is $\{a, b, z\}$. Thus, $\text{Deg}_{\textbf{G}}^{\textbf{S}}(a) = \text{Deg}_{\textbf{G}}^{\textbf{S}}(b) = 1$ while $\text{Deg}_{\textbf{G}}^{\textbf{S}}(a') = \alpha$ and $\text{Deg}_{\textbf{G}}^{\textbf{S}}(b') = 0$.
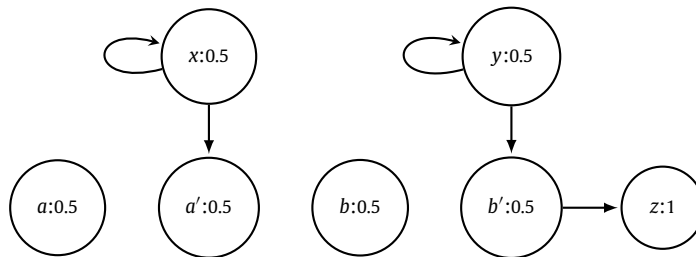
The following example shows that Invariance is violated by stable, preferred and complete semantics. Arguments a, b, b', x and y get the degree α whereas a' gets the degree 0.
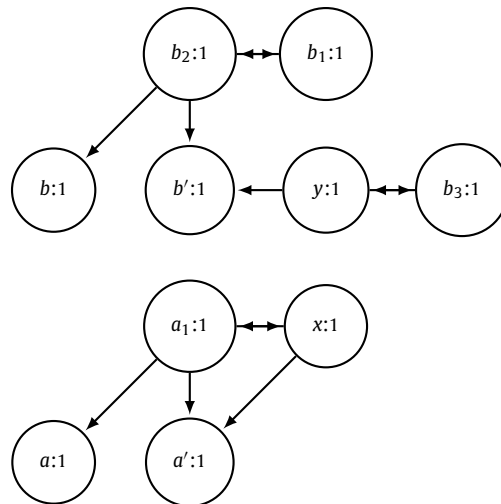
Strict Invariance is violated by the four semantics (consider the same counter-example as for the contraction-based approach).

Monotony is violated by all the four semantics (consider the same counter-examples as for symmetry).

Counting is violated by the four semantics (consider the same counter-example as for the contraction-based approach).

Reinforcement is violated by the four semantics. Consider the graph depicted below on the left-side and its revised version at the right-side. The grounded extension is $\{z\}$, then $\text{Deg}_{\textbf{G}}^{\textbf{S}}(x) = \text{Deg}_{\textbf{G}}^{\textbf{S}}(y) = \alpha > 0$ while $\text{Deg}_{\textbf{G}}^{\textbf{S}}(a) = \alpha$ and

$\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$. There are two stable/complete/preferred extensions: $\{x, z\}$ and $\{y, z, a\}$. Thus, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = \beta > 0$ while $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \beta$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.
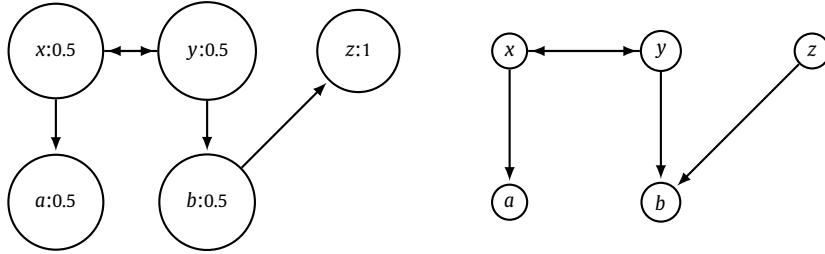
Strict Reinforcement is satisfied by stable and grounded semantics and violated by preferred and complete semantics. To see why preferred and complete semantics violate Strict Reinforcement, consider the following counter-example. Note that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$ (since y is credulously accepted and x is rejected). However, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$ since they are both credulously accepted (note that in the revised graph, z attacks a).
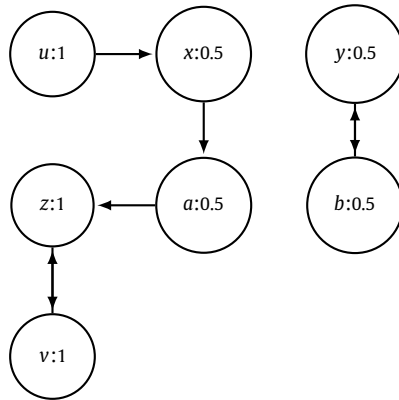
Let us now show that Strict Reinforcement is satisfied by stable and grounded semantics. Assume that $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in$ WAG, $a, b, x, y \in \mathcal{A}$ such that:

- $w(a) = w(b)$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$,
- $\text{Att}_{\mathbf{G}}(a) = \{x\}$,
- $\text{Att}_{\mathbf{G}}(b) = \{y\}$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$.

We denote by $\mathcal{R}'$ the attack relation of the revised graph. Let us prove the claim for stable semantics. Since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$ then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = 1$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \alpha$. It holds that $y\mathcal{R}'b$ or $b\mathcal{R}'y$. Since y is sceptically accepted, b is rejected. Thus $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.

As for grounded semantics, note that it must be $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \beta$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = 1$. It holds that $y\mathcal{R}'b$ or $b\mathcal{R}'y$. Case $y\mathcal{R}b$: obviously, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$. Case $b\mathcal{R}y$: then, since the grounded semantics is admissible, there exists z in the extension such that $z\mathcal{R}'b$. Thus, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.

Proportionality is violated by all four semantics. Consider the graph $\mathbf{G}$ depicted below on the left-side and its revised version $\mathbf{G}'$ on the right-side. Note that $w(b) > w(a)$ and $\text{Att}_{\mathbf{G}}(a) = \text{Att}_{\mathbf{G}}(b)$. However, $\mathbf{G}'$ has a single preferred/complete/stable/grounded extension $\{a, d\}$. Thus, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 1 > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.
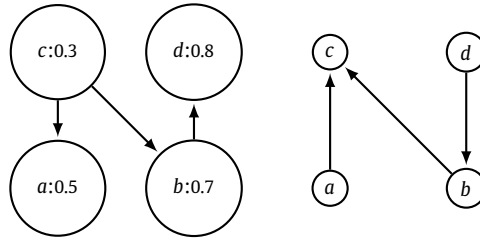
Strict Proportionality is violated by the four semantics. We can use the same counter-example as for contraction-based approach.

**Resilience** is violated by the four semantics. We can use the same counter-example as for contraction-based approach.

QP is violated by the four semantics as shown by the graph depicted below.
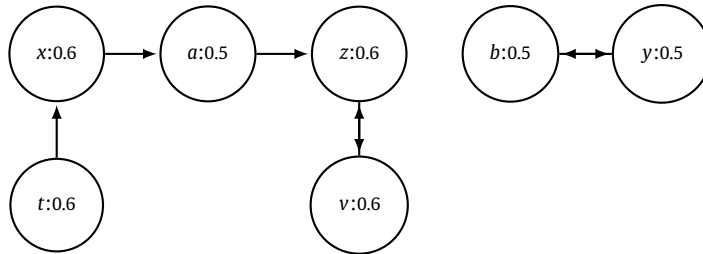
Its revised version is depicted below. It can be checked that under stable, preferred and complete semantics, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \alpha$ while $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = \alpha$. Under grounded semantics, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \beta$ while $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = \beta$.
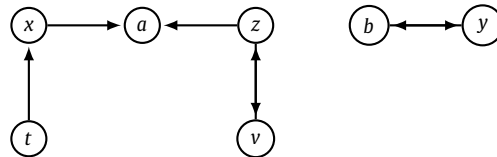
CP is violated by the four semantics. It is sufficient to consider the counter-example given for the contraction-based approach.

Compensation is satisfied by the four semantics since they all violate CP and QP.
TB semantics: Let us recall the definition of TB:

$$\text{Deg}_{\mathbf{G}}^{TB}(a) = \lim_{i \to +\infty} f_i(a), \text{ where} \tag{7}$$

$$f_i(a) = \frac{1}{2} f_{i-1}(a) + \frac{1}{2} \min[w(a), 1 - \max_{b\mathcal{R}a} f_{i-1}(b)] \tag{8}$$

Recall also that the following equation is satisfied:

$$\text{Deg}_{\mathbf{G}}^{TB}(a) = \min[w(a), 1 - \max_{b\mathcal{R}a} \text{Deg}_{\mathbf{G}}^{TB}(b)]. \tag{9}$$

Anonymity is satisfied since the above equations do not take into account arguments' names but only the topology of the graph.
Independence: Since equation (7) only takes into account direct attackers, independence is satisfied.
Directionality: We see from equation (7) that the strength of an argument is fully determined by the strengths of its ancestors (i.e., the parents, parents' parents and so on). Hence, adding the attack (a, b) does not impact the strength of argument x if there is no path from b to x.
Maximality is satisfied since from (9) we obtain

$$\text{Deg}_{\mathbf{G}}^{TB}(a) = \min[w(a), 1] = w(a). \tag{10}$$

Weakening follows directly from the same equation.

The other proofs follow immediately from the three equations above. We now present counter-examples for violated principles.

Strict Weakening: $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, b\}$, $w(a) = 0.5$, $w(b) = 0.1$, $\mathcal{R} = \{(b, a)\}$.

Strict Invariance: $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, a', b, b', c, d, x, y\}$, $w(a) = w(a') = w(b) = w(b') = 1$, $w(c) = 0.1$, $w(d) = 0.2$, $w(x) = 0.5$, $w(y) = 0.5$, $\mathcal{R} = \{(c, a), (c, a'), (x, a'), (d, b), (d, b'), (y, b')\}$.

Counting: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, b, x\}$, $w(a) = 0.5$, $w(b) = 0.5$, $w(x) = 0.1$, $\mathcal{R} = \{(x, b)\}$.

Strict Reinforcement: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, b, x, y\}$, $w(a) = 0.5$, $w(b) = 0.5$, $w(x) = 0.1$, $w(y) = 0.2$, $\mathcal{R} = \{(x, a), (y, b)\}$.

Strict Proportionality: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, b, x\}$, $w(a) = 0.7$, $w(b) = 0.3$, $w(x) = 0.9$, $\mathcal{R} = \{(x, a), (x, b)\}$.
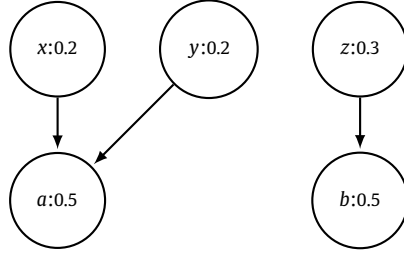
Resilience: $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, x\}$, $w(a) = 0.5$, $w(x) = 1$, $\mathcal{R} = \{(x, a)\}$.

CP and ***QP*** are violated by TB. Consider the following graph. It can be checked that $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0.5$.

Compensation is satisfied by TB since the latter violates both CP and QP.

Iterative Schema (IS) semantics:

Anonymity is obviously satisfied.

Independence is satisfied since the strength of an argument is a function of the strengths of its direct attackers.

Directionality is satisfied since the strength of an argument depends on its parents, grand-parents, and so on.

Maximality is violated. Consider a graph with only one argument a such that $w(a) = 0$ and the attack relation is empty. The strength of a is 1.

Neutrality is satisfied. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b \in \mathcal{A}$ such that $w(a) = w(b)$, $\mathtt{Att}_{\mathbf{G}}(a) = \emptyset$ and $\mathtt{Att}_{\mathbf{G}}(b) = \{x\}$, with $w(x) = 0$. The strength of a is 1 ($\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 1$). Furthermore, for every i, $g_i(b) = (1 - g_{i-1}(b)).0.5 + g_{i-1}(b)$. It is clear that $\lim_{i \to +\infty} g_i(b) = 1$.

Weakening and Strict Weakening are violated. Consider the graph depicted below. It can be checked that $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(c) = 0.5$.
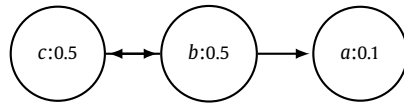
Weakening Soundness: is satisfied since non-attacked arguments get value 1 whatever their basic weights.

Resilience: is violated as shown in the following graph. Indeed, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 1$ while $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0$.
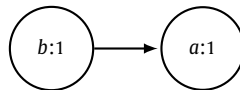
Monotony is satisfied since for any $a, b \in \mathcal{A}$ such that $w(a) = w(b)$ and $\mathtt{Att}_{\mathbf{G}}(a) \subseteq \mathtt{Att}_{\mathbf{G}}(b)$, it holds that $\max_{x \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) \leq \max_{x \in \mathtt{Att}_{\mathbf{G}}(b)} \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$. Hence, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Symmetry is satisfied. Indeed, for $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, for $a, b \in \mathcal{A}$ such that $w(a) = w(b)$ and $\mathtt{Att}_{\mathbf{G}}(a) = \mathtt{Att}_{\mathbf{G}}(b)$, for every i, $g_i(a) = g_i(b)$. Thus, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Equivalence is satisfied. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b \in \mathcal{A}$ such that $w(a) = w(b)$ and there exists a bijection f from $\mathtt{Att}_{\mathbf{G}}(a)$ to $\mathtt{Att}_{\mathbf{G}}(b)$ such that for each $x \in \mathtt{Att}_{\mathbf{G}}(x)$, it holds that $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(f(x))$. Let v be the degree of the strongest attacker of a. Corollary 2.5 by Gabbay and Rodrigues [34] shows that IS returns only three values (0, 0.5, 1). If $v = 1$, then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$. If $v = 0.5$, then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0.5$. If $v = 0$, then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 1$.

Invariance is satisfied. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, $a, b, a', b', x, y \in \mathcal{A}$ such that

- $w(a) = w(a') = w(b) = w(b')$,
- $\mathtt{Att}_{\mathbf{G}}(a') = \mathtt{Att}_{\mathbf{G}}(a) \cup \{x\}$ with $x \notin \mathtt{Att}_{\mathbf{G}}(a)$,

- $\text{Att}_{\mathbf{G}}(b') = \text{Att}_{\mathbf{G}}(b) \cup \{y\}$ with $y \notin \text{Att}_{\mathbf{G}}(b)$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y)$.

Since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$, then $\max_{x \in \text{Att}_{\mathbf{G}}(a)} \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) \leq \max_{x' \in \text{Att}_{\mathbf{G}}(b)} \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x')$. Since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y)$, then $\max_{x \in \text{Att}_{\mathbf{G}}(a')} \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) \leq \max_{x' \in \text{Att}_{\mathbf{G}}(b')} \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x')$. Thus, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$.

Strict Invariance is violated. Consider the graph depicted below. Clearly, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 1$, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0.5$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b') = 0$.

Counting is violated as shown on the following example. Indeed, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0.5$.
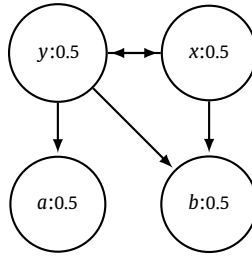

Reinforcement is satisfied. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a, b, x, y \in \mathcal{A}$ such that:

- $w(a) = w(b)$,
- $\text{Att}_{\mathbf{G}}(a) = \{x\}$,
- $\text{Att}_{\mathbf{G}}(b) = \{y\}$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$.

Since arguments get unique scores, we have:

$$\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = (1 - \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a)).A + \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a).B \tag{11}$$

$$\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = (1 - \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)).C + \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b).D \tag{12}$$

where $A = \min(\frac{1}{2}, 1 - \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x))$, $B = \max(\frac{1}{2}, 1 - \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x))$ $C = \min(\frac{1}{2}, 1 - \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y))$, $D = \max(\frac{1}{2}, 1 - \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y))$. From equation (11) (respectively (12)), we obtain:

$$\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \frac{A}{1 + A - B},$$

$$\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \frac{C}{1 + C - D}.$$

Since $A \geq C$ and $B \geq D$, we get $A(1 - D) \geq C(1 - D) \geq C(1 - B)$. Consequently, $A(1 + C - D) \geq C(1 + A - B)$. Hence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) \geq \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

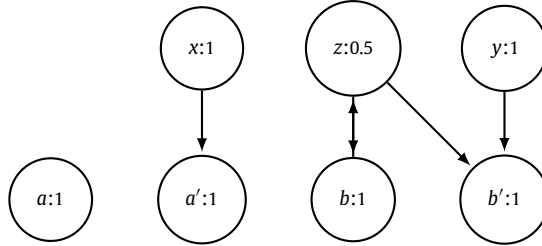**Strict Reinforcement** is satisfied. Assume that $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$, $a, b, x, y \in \mathcal{A}$ such that:

- $w(a) = w(b)$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$,
- $\text{Att}_{\mathbf{G}}(a) = \{x\}$,
- $\text{Att}_{\mathbf{G}}(b) = \{y\}$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$.

Corollary 2.5 by Gabbay and Rodrigues [34] shows that IS returns only three values (0, 0.5, 1). Hence, it must be that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = 1$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) = 0.5$. Furthermore, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = (1 - \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)).\min(\frac{1}{2}, 0) + \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)\max(\frac{1}{2}, 0)$. Hence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = \frac{1}{2}\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. Thus, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$. Since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$, then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

Proportionality is satisfied. Let us prove the property by induction. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a, b \in \mathcal{A}$ such that $w(a) \geq w(b)$ and $\text{Att}_{\mathbf{G}}(a) = \text{Att}_{\mathbf{G}}(b)$. Base case: $w(a) \geq w(b)$. Suppose now that $g_{i-1}(a) \geq g_{i-1}(b)$ and let us show that $g_i(a) \geq g_i(b)$. Let $A = \min(\frac{1}{2}, 1 - \max\limits_{x \in \text{Att}_{\mathbf{G}}(a)} g_{i-1}(x))$ and $B = \max(\frac{1}{2}, 1 - \max\limits_{x \in \text{Att}_{\mathbf{G}}(a)} g_{i-1}(x))$. Observe that $B \geq A$. Denote by $E = g_{i-1}(a) - g_{i-1}(b)$. It follows that $g_i(a) - g_i(b) = E.(B - A)$. Since $E \geq 0$ and $B - A \geq 0$, then $g_i(a) \geq g_i(b)$. By induction, for every i, $g_i(a) \geq g_i(b)$.

Strict Proportionality: is violated as shown on the simple graph depicted below. Indeed, note that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.

QP is satisfied. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a, b \in \mathcal{A}$ such that
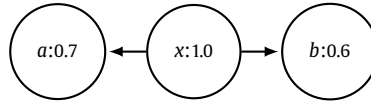
- $w(a) = w(b)$,
- $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$,
- $\exists y \in \text{Att}_{\mathbf{G}}(b)$ such that $\forall x \in \text{Att}_{\mathbf{G}}(a)$, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$.

Assume that $\exists y \in \text{Att}_{\mathbf{G}}(b)$ such that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = 1$. Thus, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$. Hence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$. Assume now that $\max\limits_{z \in \text{Att}_{\mathbf{G}}(b)} \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(z) = 0.5$ (1), then $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0.5$. From (1), we also conclude that $\max\limits_{z \in \text{Att}_{\mathbf{G}}(a)} \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(z) = 0$. Hence, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 1$. Consequently, $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$.

CP is violated as shown with the following graph.
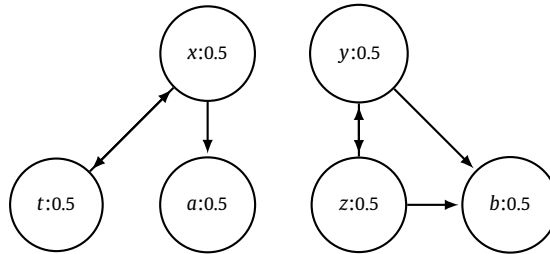
Compensation is violated since QP is satisfied. □

Proof of Theorem 6. Let us recall the definition of a degree of an argument with respect to DF-Quad, in the case of a graph without supports:

$$\text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) = w(a) \times \prod_{b \mathcal{R} a}(1 - \text{Deg}_{\mathbf{G}}^{DF-QuAD}(b)). \tag{13}$$

Anonymity follows from the fact that (13) does not use the names of the arguments.

Independence follows from the fact that the degree of an argument does not depend on the arguments that are not connected to it.

Directionality follows from the fact that (13) uses the degrees of parents, their parents, and so on but not the other arguments.

Maximality follows from (13) since if an argument a has no attackers, we obtain $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) = w(a) \times 1$.

Weakening and **Strict Weakening** follow from (13) since of an argument a has an attacker b such that $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(b) > 0$, we obtain $\text{Deg}_{\mathbf{G}}^{DF-QuAD}(a) = w(a) \times \alpha$, with $0 < \alpha < 1$.

The other proofs also follow directly from equation (13).

We now present the counter-examples.

Strict Invariance: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, a', b, b', c, d, x, y\}$, $w(a) = w(a') = w(b) = w(b') = 1$, $w(c) = 0.5$, $w(d) = 0.8$, $w(x) = 1$, $w(y) = 1$, $\mathcal{R} = \{(c, a), (c, a'), (x, a'), (d, b), (d, b'), (y, b')\}$. It is easy to check that $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0.5 > \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0.2$ while $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.

Strict Proportionality: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, b, x\}$, $w(a) = 0.5$, $w(b) = 0.3$, $w(x) = 1$, $\mathcal{R} = \{(x, a), (x, b)\}$. Strict Proportionality is not satisfied since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.

Resilience: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, x\}$, $w(a) = 0.5$, $w(x) = 1$, $\mathcal{R} = \{(x, a)\}$. Resilience is not satisfied since $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0$.

CP: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, b, x, y_1, y_2\}$, $w(a) = 1$, $w(b) = 1$, $w(x) = 0.9$, $w(y_1) = 0.1$, $w(y_2) = 0.1$, $\mathcal{R} = \{(x, a), (y_1, b), (y_2, b)\}$. We have $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0.1$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0.81$, thus CP is not satisfied.

QP: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$, where $\mathcal{A} = \{a, b, x_1, x_1, y\}$, $w(a) = 1$, $w(b) = 1$, $w(x_1) = 0.4$, $w(x_2) = 0.4$, $w(y) = 0.5$, $\mathcal{R} = \{(x_1, a), (x_2, a), (y, b)\}$. We obtain $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0.36$ and $\text{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0.5$, thus QP is not satisfied. $\quad\square$

Lemma 2. *Let* $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ *and* $a \in \mathcal{A}$. *For any* $i \in \{0, 1, \ldots\}$, $\mathtt{f}_{\mathtt{m}}^{i}(a) \leq w(a)$.

Proof. It is obvious from the definition that $\mathtt{f}_{\mathtt{m}}^{i}$ is nonnegative for each i, so $1 + \max\limits_{b \in \text{Att}_{\mathbf{G}}(a)} \mathtt{f}_{\mathtt{m}}^{i-1}(b) \geq 1$. $\quad\square$

Proof of Theorem 7. Let $\langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and assume an enumeration $\mathcal{A} = \{a_1, \ldots, a_n\}$ of the arguments. We denote by $\mathtt{f}_{\mathtt{m}}^{i}(\mathcal{A})$ the vector (n-tuple) $(\mathtt{f}_{\mathtt{m}}^{i}(a_1), \ldots, \mathtt{f}_{\mathtt{m}}^{i}(a_n))$, for every $i \in \mathbb{N}$. We need to prove that $\mathtt{f}_{\mathtt{m}}^{i}(\mathcal{A})$ converges in the vector space $\mathbb{R}^n$. First, note that if $w(a_i) = 0$, then $\mathtt{f}_{\mathtt{m}}^{k}(a_i) = 0$ for every $k \in \mathbb{N}$; otherwise $\mathtt{f}_{\mathtt{m}}^{k}(a_i) \neq 0$ for every $k \in \mathbb{N}$. Also note that an attack of an argument whose weight is zero doesn't affect the scoring value of attacked arguments. Thus, without any loss of generality, in this proof we can assume that $w(a_i) > 0$ for all $i \in \{1, \ldots, n\}$. Let us define the function $F : [0, 1]^n \to [0, 1]^n$ by $F((x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$, where

$$F_i(x_1, \ldots, x_n) = \frac{w(a_i)}{1 + \max\limits_{j : a_j \in \text{Att}(a_i)} x_j} \tag{14}$$

for every $i \in \{1, \ldots, n\}$. We also define the partial order $\leq$ on $\mathbb{R}^n$ in the following way: if $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, then $\mathbf{x} \leq \mathbf{y}$ iff for every $i \in \{1, \ldots, n\}$ the inequality $x_i \leq y_i$ holds. Then it is clear that F is a non-increasing function with respect to $\leq$, i.e., that $F(\mathbf{x}) \geq F(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$. Consequently, the function $G = F \circ F$ is non-decreasing, since $F(\mathbf{x}) \geq F(\mathbf{y})$ implies $F(F(\mathbf{x})) \leq F(F(\mathbf{y}))$.

Note that for every $i \in \mathbb{N}$ we have $\mathtt{f}_{\mathtt{m}}^{i+1}(\mathcal{A}) = F(\mathtt{f}_{\mathtt{m}}^{i}(\mathcal{A}))$. Since $\mathtt{f}_{\mathtt{m}}^{0}(\mathcal{A}) = (w(a_1), \ldots, w(a_n))$, by Lemma 2 we obtain both

$$\mathtt{f}_{\mathtt{m}}^{1}(\mathcal{A}) \leq \mathtt{f}_{\mathtt{m}}^{0}(\mathcal{A}) \tag{15}$$

and

$$\mathtt{f}_{\mathtt{m}}^{2}(\mathcal{A}) \leq \mathtt{f}_{\mathtt{m}}^{0}(\mathcal{A}). \tag{16}$$

Applying the non-increasing function F to the inequality (16), we also obtain $F(\mathtt{f}_{\mathtt{m}}^{2}(\mathcal{A})) \geq F(\mathtt{f}_{\mathtt{m}}^{0}(\mathcal{A}))$, i.e.,

$$\mathtt{f}_{\mathtt{m}}^{1}(\mathcal{A}) \leq \mathtt{f}_{\mathtt{m}}^{3}(\mathcal{A}). \tag{17}$$

Since $\mathtt{f}_{\mathtt{m}}^{i+2}(\mathcal{A}) = G(\mathtt{f}_{\mathtt{m}}^{i}(\mathcal{A}))$, applying G to the inequality (16) and using non-decreasingness of G we derive $\mathtt{f}_{\mathtt{m}}^{4}(\mathcal{A}) = G(\mathtt{f}_{\mathtt{m}}^{2}(\mathcal{A})) \leq G(\mathtt{f}_{\mathtt{m}}^{0}(\mathcal{A})) = \mathtt{f}_{\mathtt{m}}^{2}(\mathcal{A})$. Repeating the application of G, we obtain

$$\mathtt{f}_{\mathtt{m}}^{0}(\mathcal{A}) \geq \mathtt{f}_{\mathtt{m}}^{2}(\mathcal{A}) \geq \mathtt{f}_{\mathtt{m}}^{4}(\mathcal{A}) \geq \mathtt{f}_{\mathtt{m}}^{6}(\mathcal{A}) \geq \cdots \geq (0, \ldots, 0) \tag{18}$$

Thus, the sequence $\{\mathtt{f}_{\mathtt{m}}^{2i}(\mathcal{A})\}_{i \in \mathbb{N}}$ is monotonically non-increasing in $\mathbb{R}^n$ and bounded by $(0, \ldots, 0)$, so it has a limit $\overline{\mathtt{f}_{\mathtt{m}}}(\mathcal{A}) = \lim\limits_{i \to +\infty} \mathtt{f}_{\mathtt{m}}^{2i}(\mathcal{A})$. Similarly, applying G to the inequality (17), we conclude

$$\mathtt{f}_{\mathtt{m}}^{1}(\mathcal{A}) \leq \mathtt{f}_{\mathtt{m}}^{3}(\mathcal{A}) \leq \mathtt{f}_{\mathtt{m}}^{5}(\mathcal{A}) \leq \mathtt{f}_{\mathtt{m}}^{7}(\mathcal{A}) \leq \ldots \tag{19}$$

so the sequence $\{\mathtt{f}_{\mathtt{m}}^{2i+1}(\mathcal{A})\}_{i \in \mathbb{N}}$ is monotone (non-decreasing) in $\mathbb{R}^n$. Since it is bounded by $(w(a_1), \ldots, w(a_n))$ (by Lemma 2), it has a limit $\underline{\mathtt{f}_{\mathtt{m}}}(\mathcal{A}) = \lim\limits_{i \to +\infty} \mathtt{f}_{\mathtt{m}}^{2i+1}(\mathcal{A})$. It remains to prove that $\underline{\mathtt{f}_{\mathtt{m}}}(\mathcal{A}) = \overline{\mathtt{f}_{\mathtt{m}}}(\mathcal{A})$. Similarly as above, we can apply G to (15) and show that $\mathtt{f}_{\mathtt{m}}^{2i+1}(\mathcal{A}) \leq \mathtt{f}_{\mathtt{m}}^{2i}(\mathcal{A})$ for every $i \in \mathbb{N}$ and, consequently, that $\underline{\mathtt{f}_{\mathtt{m}}}(\mathcal{A}) \leq \overline{\mathtt{f}_{\mathtt{m}}}(\mathcal{A})$.

Let us prove that $\underline{\mathtt{f}_{\mathtt{m}}}(\mathcal{A}) \geq \overline{\mathtt{f}_{\mathtt{m}}}(\mathcal{A})$. Using the Archimedean property, we obtain that for every $k \in \mathbb{N}$ there exists $\alpha_k > 0$ such that

$$\mathtt{f}_{\mathtt{m}}^{2k+1}(\mathcal{A}) \geq \alpha_k \mathtt{f}_{\mathtt{m}}^{2k}(\mathcal{A}).$$

Let $\pi_k := \sup\{\alpha_k \mid \mathtt{f}_{\mathtt{m}}^{2k+1}(\mathcal{A}) \geq \alpha_k \mathtt{f}_{\mathtt{m}}^{2k}(\mathcal{A})\}$. Obviously $\pi_k \leq 1$ for every $k \in \mathbb{N}$, and the sequence $\{\pi_k\}_{k \in \mathbb{N}}$ is non-decreasing in $\mathbb{R}$. Then there exists the limit $\pi = \lim\limits_{k \to +\infty} \pi_k$, and $\pi \leq 1$. Note that, since $\mathtt{f}_{\mathtt{m}}^{2k+1}(\mathcal{A}) \geq \pi_k \mathtt{f}_{\mathtt{m}}^{2k}(\mathcal{A})$, we have $F(\mathtt{f}_{\mathtt{m}}^{2k+1}(\mathcal{A})) \leq F(\pi_k \mathtt{f}_{\mathtt{m}}^{2k}(\mathcal{A}))$, i.e.,

$$\mathtt{f}_{\mathtt{m}}^{2k+2}(\mathcal{A}) \leq F(\pi_k \mathtt{f}_{\mathtt{m}}^{2k}(\mathcal{A})). \tag{20}$$

For $i \in \{1, \ldots, n\}$, $\mathbf{x} = (x_1, \ldots, x_n)$ and $\alpha \in (0, 1]$ we have

$$F_i(\alpha \mathbf{x}) = \frac{w(a_i)}{1 + \max\limits_{j:a_j \in \text{Att}(a_i)} \alpha x_j} = \frac{w(a_i)}{1 + \alpha \max\limits_{j:a_j \in \text{Att}(a_i)} x_j}$$

$$= \frac{w(a_i)}{(\alpha - \alpha) + 1 + \alpha \max\limits_{j:a_j \in \text{Att}(a_i)} x_j}$$

$$= \frac{w(a_i)}{(1 - \alpha) + \alpha(1 + \max\limits_{j:a_j \in \text{Att}(a_i)} x_j)}$$

$$= \frac{w(a_i)}{(1 - \alpha) + \alpha \frac{w(a_i)}{F_i(\mathbf{x})}} \cdot \frac{F_i(\mathbf{x})}{F_i(\mathbf{x})}$$

It follows that

$$F_i(\alpha \mathbf{x}) = \frac{w(a_i)}{(1 - \alpha) F_i(\mathbf{x}) + \alpha w(a_i)} F_i(\mathbf{x}). \tag{21}$$

If we apply (21) to the inequality (20), we have that for every $i \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$

$$\mathtt{f}_{\mathrm{m}}^{2k+2}(a_i) \le \frac{w(a_i)}{(1 - \pi_k) F_i(\mathtt{f}_{\mathrm{m}}^{2k}(\mathcal{A})) + \pi_k w(a_i)} F_i(\mathtt{f}_{\mathrm{m}}^{2k}(\mathcal{A})),$$

or, equivalently,

$$\mathtt{f}_{\mathrm{m}}^{2k+2}(a_i) \le \frac{w(a_i) \mathtt{f}_{\mathrm{m}}^{2k+1}(a_i)}{(1 - \pi_k) \mathtt{f}_{\mathrm{m}}^{2k+1}(a_i) + \pi_k w(a_i)}. \tag{22}$$

Let us rewrite (22) as

$$\mathtt{f}_{\mathrm{m}}^{2k+3}(a_i) \ge$$
$$\frac{\mathtt{f}_{\mathrm{m}}^{2k+3}(a_i)[(1 - \pi_k) \mathtt{f}_{\mathrm{m}}^{2k+1}(a_i) + \pi_k w(a_i)]}{w(a_i) \mathtt{f}_{\mathrm{m}}^{2k+1}(a_i)} \mathtt{f}_{\mathrm{m}}^{2k+2}(a_i). \tag{23}$$

Since $\pi_{k+1} = \sup\{\alpha \mid \mathtt{f}_{\mathrm{m}}^{2k+3}(\mathcal{A}) \ge \alpha \mathtt{f}_{\mathrm{m}}^{2k+2}(\mathcal{A})\}$, we conclude that π_{k+1} is the maximal number β such that for every $i \in \{1, \ldots, n\}$ we have

$$\mathtt{f}_{\mathrm{m}}^{2k+3}(a_i) \ge \beta \cdot \mathtt{f}_{\mathrm{m}}^{2k+2}(a_i).$$

Combining this observation with (23) we conclude that for every $k \in \mathbb{N}$ there is $i(k) \in \{1, \ldots, n\}$ such that

$$\frac{\mathtt{f}_{\mathrm{m}}^{2k+3}(a_{i(k)})[(1 - \pi_k) \mathtt{f}_{\mathrm{m}}^{2k+1}(a_{i(k)}) + \pi_k w(a_{i(k)})]}{w(a_{i(k)}) \mathtt{f}_{\mathrm{m}}^{2k+1}(a_{i(k)})} \le \pi_{k+1}. \tag{24}$$

Here we wish to use the fact that all the indexes of the weighted max-based function $\mathtt{f}_{\mathrm{m}}$ on the left hand side of the inequality (24) are odd, and to apply $\lim\limits_{k \to +\infty}$ to the both sides of the inequality. The problem is that, although the sequence $\{\mathtt{f}_{\mathrm{m}}^{2k+1}(\mathcal{A})\}_{k \in \mathbb{N}}$ converges, the sequence $\{\mathtt{f}_{\mathrm{m}}^{2k+1}(a_{i(k)})\}_{k \in \mathbb{N}}$ doesn't necessarily converge since indexes $i(k)$ may take different values for different k. For that reason, note that the set $\{1, \ldots, n\}$ is finite and that there is at least one number from the set that appears infinitely many times in the sequence $\{i(k)\}_{k \in \mathbb{N}}$. Without any loss of generality, suppose that one such number is j. Denote by $\underline{\mathtt{f}}_{\mathrm{m}}^j$ the j-th projection of the vector $\underline{\mathtt{f}}_{\mathrm{m}}$. Using the fact that if a sequence converges, then its subsequences converge as well, we apply limit to the inequality (24) using the subsequence obtained by taking only those k for which $i(k) = j$. Then we obtain

$$\frac{\underline{\mathtt{f}}_{\mathrm{m}}^j[(1 - \pi) \underline{\mathtt{f}}_{\mathrm{m}}^j + \pi w(a_j)]}{w(a_j) \underline{\mathtt{f}}_{\mathrm{m}}^j} \le \pi. \tag{25}$$

We can rewrite (25) as

$$(1 - \pi) \underline{\mathtt{f}}_{\mathrm{m}}^j + \pi w(a_j) \le \pi w(a_j). \tag{26}$$

Note that from $w(a_j) > 0$ and $1 + \max_{b \in \text{Att}_{\mathbf{G}}(a_j)} \mathtt{f}_{\mathrm{m}}^{i-1}(b) \le 2$ we obtain that $\mathtt{f}_{\mathrm{m}}^k(a_j) \ge \frac{w(a_j)}{2}$, for all k, so $\underline{\mathtt{f}}_{\mathrm{m}}^j > 0$.

Since $\underline{f}_m^j \neq 0$, from (26) we obtain $1 - \pi \leq 0$, i.e., $\pi \geq 1$. Finally, from $\pi \leq 1$ we obtain $\pi = 1$.

We proved that for every $k \in \mathbb{N}$ we have $f_m^{2k+1}(\mathcal{A}) \leq f_m^{2k}(\mathcal{A})$. Together with the inequality $f_m^{2k+1}(\mathcal{A}) \geq \pi_k f_m^{2k}(\mathcal{A})$ it gives us

$$\pi_k f_m^{2k}(\mathcal{A}) \leq f_m^{2k+1}(\mathcal{A}) \leq f_m^{2k}(\mathcal{A}). \tag{27}$$

Now we let $k \to +\infty$ and obtain

$$1 \cdot \overline{f_m}(\mathcal{A}) \leq \underline{f_m}(\mathcal{A}) \leq \overline{f_m}(\mathcal{A}).$$

Thus, $\lim_{k \to +\infty} f_m^{2k}(\mathcal{A}) = \overline{f_m}(\mathcal{A}) = \underline{f_m}(\mathcal{A}) = \lim_{k \to +\infty} f_m^{2k+1}(\mathcal{A})$, so the sequence $\{f_m^k(\mathcal{A})\}_{k \in \mathbb{N}}$ converges. $\quad\square$

Proof of Theorem 8. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a \in \mathcal{A}$. Letting $i \to +\infty$ in the equality

$$f_m^{i+1}(a) = \frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} f_m^i(b)}$$

and using the fact that arithmetical operations and max are continuous functions, we obtain

$$\lim_{i \to +\infty} f_m^{i+1}(a) = \frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} \lim\limits_{i \to +\infty} f_m^i(b)}$$

i.e.

$$\mathrm{Deg}_{\mathbf{G}}^{\mathrm{Mbs}}(a) = \frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathrm{Mbs}}(b)}. \quad\square$$

Proof of Theorem 9. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and suppose that $D : \mathcal{A} \to [0, 1]$ is the function such that

$$D(a) = \frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}(a)} D(b)} \tag{28}$$

for every $a \in \mathcal{A}$. Since the function D is nonnegative, we obtain that

$$D(a) \leq \frac{w(a)}{1 + 0} = w(a), \qquad \forall a \in \mathcal{A}. \tag{29}$$

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ and let $F : [0, +\infty)^n \to [0, +\infty)^n$ be the function such that $F((x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$, where F_i's are defined by the equalities (14) in the proof of Theorem 8. From (28) it follows that

$$F(D(a_1), \ldots, D(a_n)) = (D(a_1), \ldots, D(a_n)). \tag{30}$$

Recall that F is a non-increasing function and $G = F \circ F$ is a non-decreasing function, and that

$$(f_m^{i+1}(a_1), \ldots, f_m^{i+1}(a_n)) = F(f_m^i(a_1), \ldots, f_m^i(a_n))$$

for every $i \in \mathbb{N}$.

Since $(f_m^0(a_1), \ldots, f_m^0(a_n)) = (w(a_1), \ldots, w(a_n))$, by the inequalities (29) we obtain

$$(f_m^0(a_1), \ldots, f_m^0(a_n)) \geq (D(a_1), \ldots, D(a_n)). \tag{31}$$

Now, applying F on (31), and using non-increasingness of F and (30), we obtain

$$(f_m^1(a_1), \ldots, f_m^1(a_n)) \leq (D(a_1), \ldots, D(a_n)). \tag{32}$$

Finally, applying the non-decreasing function $G = F \circ F$ on (31) and (32), we conclude that

$$(f_m^{2i}(a_1), \ldots, f_m^{2i}(a_n)) \geq (D(a_1), \ldots, D(a_n)) \tag{33}$$

and

$$(f_m^{2i+1}(a_1), \ldots, f_m^{2i+1}(a_n)) \leq (D(a_1), \ldots, D(a_n)), \tag{34}$$

for every $i \in \mathbb{N}$. Since $\lim\limits_{i \to +\infty}((f_m^i(a_1), \ldots, f_m^i(a_n)) = (\mathrm{Deg}_{\mathbf{G}}^{\mathrm{Mbs}}(a_1), \ldots, \mathrm{Deg}_{\mathbf{G}}^{\mathrm{Mbs}}(a_n))$, we let $i \to +\infty$ in (33) and (34), and obtain

$$(\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a_1), \ldots, \text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a_n)) \geq (D(a_1), \ldots, D(a_n))$$

and

$$(\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a_1), \ldots, \text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a_n)) \leq (D(a_1), \ldots, D(a_n)),$$

respectively. Thus, $\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a) = D(a)$ for all $a \in \mathcal{A}$. $\square$

Proof of Theorem 10. *Anonymity:* Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $\mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in \text{WAG}$ and f be an isomorphism from $\mathbf{G}$ to $\mathbf{G}'$. Let $\mathbf{f}_{\text{m}}^i$ and $\mathbf{g}_{\text{m}}^i$ be their corresponding weighted max-based functions, respectively. If $a \in \mathcal{A}$, using the induction on i, it is easy to show that $\mathbf{f}_{\text{m}}^i(a) = \mathbf{g}_{\text{m}}^i(f(a))$, for every $i \in \mathbb{N}$. Consequently, $\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a) = \text{Deg}_{\mathbf{G}'}^{\text{Mbs}}(f(a))$.

Independence: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $\mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in \text{WAG}$ such that $\mathcal{A} \cap \mathcal{A}' = \emptyset$. Let $\mathbf{G} \oplus \mathbf{G}' = \langle \mathcal{A} \cup \mathcal{A}', w \oplus w', \mathcal{R} \cup \mathcal{R}' \rangle$. Let $\mathbf{f}_{\text{m}}^i$ and $\mathbf{g}_{\text{m}}^i$ be the weighted max-based functions of $\mathbf{G}$ and $\mathbf{G} \oplus \mathbf{G}'$, respectively. We show by induction that $\forall i \in \{0, 1, \ldots\}$, the following holds:

$$P(i): \quad \forall a \in \mathcal{A}, \mathbf{f}_{\text{m}}^i(a) = \mathbf{g}_{\text{m}}^i(a)$$

Let $i \in \{0, 1, \ldots\}$ and suppose that $\forall j \in \{0, 1, \ldots, i-1\}$, $P(j)$ holds. We show that $P(i)$ holds. Let $a \in \mathcal{A}$.

- Case 0: $i = 0$.
  We have that $\forall a \in \mathcal{A}$, $\mathbf{f}_{\text{m}}^0(a) = \mathbf{g}_{\text{m}}^0(a) = w(a)$. Thus, $P(0)$ holds.
- Case 1: $i > 0$.

  Let $X = \text{Att}_{\mathbf{G}}(a)$ and $X' = \text{Att}_{\mathbf{G} \oplus \mathbf{G}'}(a)$. Then $\mathbf{f}_{\text{m}}^i(a) = \frac{w(a)}{1 + \max\limits_{b \in X} \mathbf{f}_{\text{m}}^{i-1}(b)}$ and $\mathbf{g}_{\text{m}}^i(a) = \frac{w(a)}{1 + \max\limits_{b \in X'} \mathbf{g}_{\text{m}}^{i-1}(b)}$. Since $X = X'$, we have $\mathbf{f}_{\text{m}}^i(a) = \frac{w(a)}{1 + \max\limits_{b \in X'} \mathbf{f}_{\text{m}}^{i-1}(b)}$. By $P(i-1)$, $\forall b \in X'$, $\mathbf{f}_{\text{m}}^i(b) = \mathbf{g}_{\text{m}}^i(b)$. Consequently, $\mathbf{f}_{\text{m}}^i(a) = \frac{w(a)}{1 + \max\limits_{b \in X'} \mathbf{g}_{\text{m}}^{i-1}(b)}$. So, $\mathbf{f}_{\text{m}}^i(a) = \mathbf{g}_{\text{m}}^i(a)$. Thus, $P(i)$ holds.

Consequently, $\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a) = \text{Deg}_{\mathbf{G} \oplus \mathbf{G}'}^{\text{Mbs}}(a)$.

Directionality: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $\mathbf{G}' = \langle \mathcal{A}', w', \mathcal{R}' \rangle \in \text{WAG}$ such that $\mathcal{A} = \mathcal{A}'$, $w' = w$, and $\mathcal{R}' = \mathcal{R} \cup \{(a, b)\}$. We denote the set $\{c \in \mathcal{A} \mid \text{there is a path from } b \text{ to } c\}$ with $p(b)$ (since $\mathcal{R}' = \mathcal{R} \cup \{(a, b)\}$, there is a path from b to c in $\mathbf{G}$ iff there is a path in $\mathbf{G}'$, so we use the same notation $p(b)$ for both graphs). Let $\mathbf{f}_{\text{m}}^i$ and $\mathbf{g}_{\text{m}}^i$ be the weighted max-based functions of $\mathbf{G}$ and $\mathbf{G}'$, respectively. We show by induction that $\forall i \in \{0, 1, \ldots\}$, the following holds:

$$P(i): \quad \forall x \notin p(b), \mathbf{f}_{\text{m}}^i(x) = \mathbf{g}_{\text{m}}^i(x).$$

Let $i \in \{0, 1, \ldots\}$ and suppose that $\forall j \in \{0, 1, \ldots, i-1\}$, $P(j)$ holds. We show that $P(i)$ holds.

- Case $i = 0$. We have that $\forall x \notin p(b)$, $\mathbf{f}_{\text{m}}^0(x) = \mathbf{g}_{\text{m}}^0(x) = w(x)$. Thus, $P(0)$ holds.
- Case $i > 0$. We have that

  $$\mathbf{f}_{\text{m}}^i(x) = \frac{w(x)}{1 + \max_{z \in \text{Att}_{\mathbf{G}}(x)} \mathbf{f}_{\text{m}}^{i-1}(z)}$$

  and

  $$\mathbf{g}_{\text{m}}^i(x) = \frac{w(x)}{1 + \max_{z \in \text{Att}_{\mathbf{G}'}(x)} \mathbf{g}_{\text{m}}^{i-1}(z)}.$$

  Since $\mathcal{R}' = \mathcal{R} \cup \{(a, b)\}$, we have that $\text{Att}_{\mathbf{G}}(x) = \text{Att}_{\mathbf{G}'}(x)$. From $x \notin p(b)$ we deduce that for every $z \in \text{Att}_{\mathbf{G}}(x)$ we have $z \notin p(b)$, so $\mathbf{f}_{\text{m}}^{i-1}(z) = \mathbf{g}_{\text{m}}^{i-1}(z)$ by $P(i-1)$. Thus we proved that

  $$\max_{z \in \text{Att}_{\mathbf{G}}(y)} \mathbf{f}_{\text{m}}^{i-1}(z) = \max_{z \in \text{Att}_{\mathbf{G}'}(y)} \mathbf{g}_{\text{m}}^{i-1}(z),$$

  so $\mathbf{f}_{\text{m}}^i(x) = \mathbf{g}_{\text{m}}^i(x)$.

Letting $i \to +\infty$, we obtain our result.

Maximality: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$. By definition, $\forall a \in \mathcal{A}$ such that $\text{Att}_{\mathbf{G}}(a) = \emptyset$, for every $i \in \mathbb{N}$ we have $\mathbf{f}_{\text{m}}^i(a) = w(a)$, so $\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a) = w(a)$.

Weakening: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \text{WAG}$ and $a \in \mathcal{A}$. If $w(a) = 0$, then $\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a) = 0$ by Theorem 8. If $w(a) > 0$, then $\text{Deg}_{\mathbf{G}}^{\text{Mbs}}(a) < w(a)$, since the semantics satisfies Strict Weakening (see below).

Strict Weakening: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b \in \mathcal{A}$ such that $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b) > 0$. By Theorem 8,

$$\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) = \frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)}.$$

Assume that $\exists b \in \mathtt{Att}_{\mathbf{G}}(a)$ such that $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b) > 0$. Thus, $\max\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b) > 0$. So, $1 + \max\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b) > 1$. Consequently, $\frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)} < w(a)$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) < w(a)$.

Weakening Soundness is satisfied by $\mathtt{Mbs}$. Indeed, we showed that it satisfies Directionality, Independence, Invariance, Maximality and Neutrality, so it must also satisfy Weakening Soundness, by Proposition 2.

Resilience: Follows directly from Theorem 11.

Proportionality: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b \in \mathcal{A}$ such that $w(a) \geq w(b)$ and $\mathtt{Att}_{\mathbf{G}}(a) = \mathtt{Att}_{\mathbf{G}}(b)$. If $w(a) = w(b)$, then $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)$, since $\mathtt{Mbs}$ satisfies Symmetry (see below). Suppose that $w(a) > w(b)$. Then $w(a) > 0$, so $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) > 0$, since $\mathtt{Mbs}$ satisfies Resilience. From the fact that $\mathtt{Mbs}$ also satisfies Strict Proportionality (see below), we obtain $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) > \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)$.

Strict Proportionality: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b \in \mathcal{A}$ such that i) $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) > 0$, ii) $w(a) > w(b)$, and iii) $\mathtt{Att}_{\mathbf{G}}(a) = \mathtt{Att}_{\mathbf{G}}(b)$. Since $\mathtt{Att}_{\mathbf{G}}(a) = \mathtt{Att}_{\mathbf{G}}(b)$, then $\frac{1}{1 + \max\limits_{a_i \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a_i)} = \frac{1}{1 + \max\limits_{b_i \in \mathtt{Att}_{\mathbf{G}}(b)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b_i)}$. From ii), $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) > \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)$.

Neutrality: Follows directly from Theorem 8.

Reinforcement: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b, x, y \in \mathcal{A}$ such that $w(a) = w(b)$, $\mathtt{Att}_{\mathbf{G}}(a) = \{x\}$, $\mathtt{Att}_{\mathbf{G}}(b) = \{y\}$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) \geq \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$. If $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$, then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$ since $\mathtt{Mbs}$ satisfies Equivalence (see below). Suppose that $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$. If $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$, then $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b)$ since $\mathtt{Mbs}$ satisfies Strict Reinforcement (see below). Otherwise, if $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = 0$, from Theorem 8 we obtain $w(a) = 0$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b) = 0$.

Strict Reinforcement: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$, and $a, b, x, y \in \mathcal{A}$ such that $w(a) = w(b)$, $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$, $\mathtt{Att}_{\mathbf{G}}(a) = \{x\}$, $\mathtt{Att}_{\mathbf{G}}(b) = \{y\}$, and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x) > 0$. From Theorem 8 and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) > 0$ we obtain $w(a) > 0$. From $w(a) = w(b)$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(y) > \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(x)$, using Theorem 8 we obtain

$$\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a) = \frac{w(a)}{1 + \max\limits_{x \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x)} > \frac{w(b)}{1 + \max\limits_{y \in \mathtt{Att}_{\mathbf{G}}(b)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y)} = \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b).$$

Symmetry: Since the semantics satisfies Equivalence (see below), it obviously satisfies Symmetry (from Proposition 2).

Equivalence: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle$ be a weighted argumentation graph and let $a, b \in \mathcal{A}$ be two arguments such that $w(a) = w(b)$ and there exists a bijective function f from $\mathtt{Att}_{\mathbf{G}}(a)$ to $\mathtt{Att}_{\mathbf{G}}(b)$ such that $\forall x \in \mathtt{Att}_{\mathbf{G}}(a)$, $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(f(x))$. Let us show that $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)$. If $\mathtt{Att}_{\mathbf{G}}(a) = \mathtt{Att}_{\mathbf{G}}(b) = \emptyset$, then obviously $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b) = w(a)$. If $\mathtt{Att}_{\mathbf{G}}(a) \neq \emptyset$, then from the assumption that f is a bijection from $\mathtt{Att}_{\mathbf{G}}(a)$ to $\mathtt{Att}_{\mathbf{G}}(b)$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(f(x))$ for every $x \in \mathtt{Att}_{\mathbf{G}}(a)$, we obtain $\max\limits_{x \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x) = \max\limits_{y \in \mathtt{Att}_{\mathbf{G}}(b)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y)$. Since also $w(a) = w(b)$, we conclude

$$\frac{w(a)}{1 + \max\limits_{x \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x)} = \frac{w(b)}{1 + \max\limits_{y \in \mathtt{Att}_{\mathbf{G}}(b)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y)},$$

so $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)$ by Theorem 8.

Invariance: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b, a', b', x, y \in \mathcal{A}$ are such that i) $w(a) = w(a') = w(b) = w(b')$, ii) $\mathtt{Att}_{\mathbf{G}}(a') = \mathtt{Att}_{\mathbf{G}}(a) \cup \{x\}$, iii) $\mathtt{Att}_{\mathbf{G}}(b') = \mathtt{Att}_{\mathbf{G}}(b) \cup \{y\}$, iv) $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y)$, and v) $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) \geq \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)$. Let us show that $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') \geq \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$. From conditions i) to v) we obtain, using Theorem 8, that $\max\limits_{z \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(z) \geq \max\limits_{t \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(t)$. By condition iv), $\max\limits_{z \in \mathtt{Att}_{\mathbf{G}}(b) \cup \{y\}} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(z) \geq \max\limits_{t \in \mathtt{Att}_{\mathbf{G}}(a) \cup \{x\}} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(t)$, i.e., $\max\limits_{z \in \mathtt{Att}_{\mathbf{G}}(b')} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(z) \geq \max\limits_{t \in \mathtt{Att}_{\mathbf{G}}(a')} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(t)$. Since $w(a') = w(b')$, using Theorem 8 we obtain $\mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(a') \geq \mathrm{Deg}_{\mathbf{G}}^{\mathbf{S}}(b')$.

Monotony: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b \in \mathcal{A}$ such that $w(a) = w(b)$ and $\mathtt{Att}_{\mathbf{G}}(a) \subseteq \mathtt{Att}_{\mathbf{G}}(b)$. From Theorem 8, $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) = \frac{w(a)}{1 + \max\limits_{x \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x)}$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b) = \frac{w(b)}{1 + \max\limits_{y \in \mathtt{Att}_{\mathbf{G}}(b)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y)}$. From $\mathtt{Att}_{\mathbf{G}}(a) \subseteq \mathtt{Att}_{\mathbf{G}}(b)$, we obtain $\max\limits_{x \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x) \leq \max\limits_{y \in \mathtt{Att}_{\mathbf{G}}(b)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y)$. Since $w(a) = w(b)$, we have $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) \geq \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)$.

Strict Invariance: Mbs violates Strict Invariance. Consider the graph $\mathbf{G}$ depicted below. $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y) = 1$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(z) = 0.5$, by Maximality. Also, using Theorem 8 we calculate $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b) = \frac{2}{3}$ and $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a') = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b') = 0.5$. Note that

1. $w(a) = w(a') = w(b) = w(b')$,
2. $\mathtt{Att}_{\mathbf{G}}(a') = \mathtt{Att}_{\mathbf{G}}(a) \cup \{x\}$,
3. $\mathtt{Att}_{\mathbf{G}}(b') = \mathtt{Att}_{\mathbf{G}}(b) \cup \{y\}$,
4. $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x) = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y)$, and
5. $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) > \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)$,

but $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a') = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b')$. Thus, Mbs does not satisfy Strict Invariance.
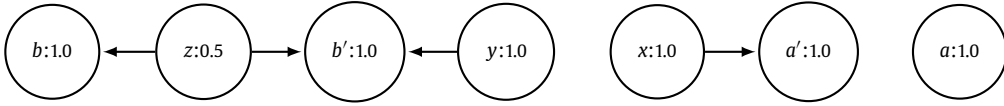
Counting: Mbs violates Counting (see previous example where $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a') = \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b')$).

Cardinality Precedence: Since Mbs satisfies Quality Precedence (see below), Resilience and Maximality, it does not satisfy Cardinality Precedence, by Proposition 1.

Quality Precedence: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and let $a, b \in \mathcal{A}$ be two arguments such that i) $w(a) = w(b)$, ii) $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) > 0$, iii) $\exists y \in \mathtt{Att}_{\mathbf{G}}(b)$ such that $\forall x \in \mathtt{Att}_{\mathbf{G}}(a)$, $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y) > \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x)$. From the last condition we obtain that $\max\limits_{y \in \mathtt{Att}_{\mathbf{G}}(b)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y) > \max\limits_{x \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x)$. From $w(a) = w(b)$ we obtain

$$\frac{w(a)}{1 + \max\limits_{x \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(x)} > \frac{w(b)}{1 + \max\limits_{y \in \mathtt{Att}_{\mathbf{G}}(b)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(y)},$$

thus $\mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) > \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b)$.

Compensation: Since Mbs satisfies Quality Precedence, then it violates Compensation. $\square$

Proof of Theorem 11. It is direct consequence of Theorem 8. Indeed, since the function Mbs maps arguments to unit interval of reals, for every $a \in \mathcal{A}$, we obtain $1 \leq 1 + \max_{b \in \mathtt{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(b) \leq 2$, and using the equation (1) we have

$$\frac{w(a)}{2} \leq \mathrm{Deg}_{\mathbf{G}}^{\mathtt{Mbs}}(a) \leq \frac{w(a)}{1}. \quad \square$$

Proof of Theorem 12. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$. Similarly as in the proof of Theorem 7, we assume an enumeration $\mathcal{A} = \{a_1, \ldots, a_n\}$ of the arguments and we denote by $\mathtt{f}_{\mathtt{c}}^i(\mathcal{A})$ the vector $(\mathtt{f}_{\mathtt{c}}^i(a_1), \ldots, \mathtt{f}_{\mathtt{c}}^i(a_n))$ (for every $i \in \mathbb{N}$). We can also use the same argument as in that proof to assume, without loss of generality, that all the arguments have positive basic weight. We define the function $F : [0, 1]^n \to [0, 1]^n$ by $F((x_1, \ldots, x_n)) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n))$, where

$$F_i(x_1, \ldots, x_n) = \frac{w(a_i)}{1 + |\mathtt{AttF}_{\mathbf{G}}(a_i)| + \frac{\sum\limits_{j : a_j \in \mathtt{AttF}_{\mathbf{G}}(a_i)} x_j}{|\mathtt{AttF}_{\mathbf{G}}(a_i)|}} \tag{35}$$

for every $i \in \{1, \ldots, n\}$. Then for every $i \in \mathbb{N}$ we have $\mathtt{f}_{\mathtt{c}}^{i+1}(\mathcal{A}) = F(\mathtt{f}_{\mathtt{c}}^i(\mathcal{A}))$. In the same way as in the proof of Theorem 7, we:

1. define the partial order $\leq$ on $\mathbb{R}^n$ and show that F is a non-increasing function with respect to $\leq$;
2. show that the sequence $\{\mathtt{f}_{\mathtt{c}}^{2i}(\mathcal{A})\}_{i \in \mathbb{N}}$ is monotonically non-increasing, and $\{\mathtt{f}_{\mathtt{c}}^{2i+1}(\mathcal{A})\}_{i \in \mathbb{N}}$ is monotonically non-decreasing in $\mathbb{R}^n$;
3. show that $\mathtt{f}_{\mathtt{c}}^{2i+1}(\mathcal{A}) \leq \mathtt{f}_{\mathtt{c}}^{2i}(\mathcal{A})$ for every $i \in \mathbb{N}$, so $\lim\limits_{i \to +\infty} \mathtt{f}_{\mathtt{c}}^{2i+1}(\mathcal{A}) \leq \lim\limits_{i \to +\infty} \mathtt{f}_{\mathtt{c}}^{2i}(\mathcal{A})$;

4. define $\pi_k := \sup\{\alpha_k \mid f_c^{2k+1}(\mathcal{A}) \geq \alpha_k f_c^{2k}(\mathcal{A})\}$ and we note that:
   (a) the sequence $\{\pi_k\}_{k\in\mathbb{N}}$ is non-decreasing in $\mathbb{R}$;
   (b) $\pi = \lim\limits_{k\to+\infty} \pi_k \leq 1$;
   (c) $f_m^{2k+2}(\mathcal{A}) \leq F(\pi_k f_m^{2k}(\mathcal{A}))$

Then for $i \in \{1,\ldots,n\}$, $\mathbf{x} = (x_1,\ldots,x_n)$ and $\alpha \in (0,1]$ we have

$$
\begin{aligned}
F_i(\alpha\mathbf{x}) &= \frac{w(a_i)}{1 + |\mathrm{AttF}_\mathbf{G}(a_i)| + \frac{\sum\limits_{j:a_j\in\mathrm{AttF}_\mathbf{G}(a_i)} \alpha x_j}{|\mathrm{AttF}_\mathbf{G}(a_i)|}} \\[2mm]
&= \frac{w(a_i)}{1 + |\mathrm{AttF}_\mathbf{G}(a_i)| + \alpha\frac{\sum\limits_{j:a_j\in\mathrm{AttF}_\mathbf{G}(a_i)} x_j}{|\mathrm{AttF}_\mathbf{G}(a_i)|}} \\[2mm]
&= \frac{w(a_i)}{1 + |\mathrm{AttF}_\mathbf{G}(a_i)| + \alpha(\frac{w(a_i)}{F_i(\mathbf{x})} - 1 - |\mathrm{AttF}_\mathbf{G}(a_i)|)} \\[2mm]
&= \frac{w(a_i)}{(1-\alpha)(1 + |\mathrm{AttF}_\mathbf{G}(a_i)|) + \alpha\frac{w(a_i)}{F_i(\mathbf{x})}} \\[2mm]
&= \frac{w(a_i)}{(1-\alpha)(1 + |\mathrm{AttF}_\mathbf{G}(a_i)|)F_i(\mathbf{x}) + \alpha w(a_i)} F_i(\mathbf{x})
\end{aligned}
$$

Now we combine this result with the inequality 4(c) and we obtain that for every $i \in \{1,\ldots,n\}$ and $k \in \mathbb{N}$

$$
f_c^{2k+2}(a_i) \leq
$$
$$
\frac{w(a_i)}{(1-\pi_k)(1 + |\mathrm{AttF}_\mathbf{G}(a_i)|)F_i(f_c^{2k}(\mathcal{A})) + \pi_k w(a_i)} F_i(f_c^{2k}(\mathcal{A})),
$$

so

$$
f_c^{2k+2}(a_i) \leq \frac{w(a_i)f_c^{2k+1}(a_i)}{(1-\pi_k)(1 + |\mathrm{AttF}_\mathbf{G}(a_i)|)f_c^{2k+1}(a_i) + \pi_k w(a_i)}.
$$

This inequality can be transformed to

$$
f_m^{2k+3}(a_i) \geq
$$
$$
\frac{f_m^{2k+3}(a_i)[(1-\pi_k)(1 + |\mathrm{AttF}_\mathbf{G}(a_i)|)f_m^{2k+1}(a_i) + \pi_k w(a_i)]}{w(a_i)f_m^{2k+1}(a_i)} \cdot \cdot f_m^{2k+2}(a_i).
$$

So, for every $k \in \mathbb{N}$ there is $j \in \{1,\ldots,n\}$ such that

$$
\frac{f_m^{2k+3}(a_j)[(1-\pi_k)(1 + |\mathrm{AttF}_\mathbf{G}(a_j)|)f_m^{2k+1}(a_j) + \pi_k w(a_j)]}{w(a_j)f_m^{2k+1}(a_j)}
$$
$$
\leq \pi_{k+1}.
$$

Similarly as in the proof of Theorem 7, we choose one $j \in \{1,\ldots,n\}$ such that previous inequality holds for infinitely many k's, and we apply $\lim\limits_{k\to+\infty}$ to all those inequalities in which j appears. If we denote by $(\underline{f_c^1},\ldots,\underline{f_c^n})$ the vector $\lim\limits_{k\to+\infty} f_c^{2k+1}(\mathcal{A})$, we obtain

$$
\frac{\underline{f_c^j}[(1-\pi)(1 + |\mathrm{AttF}_\mathbf{G}(a_j)|)\underline{f_c^j} + \pi w(a_j)]}{w(a_j)\underline{f_c^j}} \leq \pi.
$$

This is equivalent to

$$
(1-\pi)(1 + |\mathrm{AttF}_\mathbf{G}(a_j)|)\underline{f_c^j} + \pi w(a_j) \leq \pi w(a_j).
$$

Since $(1 + |\mathrm{AttF}_\mathbf{G}(a_j)|)\underline{f_c^j} > 0$, either $1 - \pi \leq 0$ or $\underline{f_c^j} = 0$. Since $w(a_j) > 0$ and

$$1 + |\text{AttF}_{\mathbf{G}}(a_j)| + \frac{\sum\limits_{b \in \text{AttF}_{\mathbf{G}}(a_j)} \mathtt{f}_{\mathtt{c}}^{i-1}(b)}{|\text{AttF}_{\mathbf{G}}(a_j)|} \leq |\text{AttF}_{\mathbf{G}}(a_j)| + 2$$

we obtain that

$$\mathtt{f}_{\mathtt{c}}^{k}(a_j) \geq \frac{w(a_j)}{|\text{AttF}_{\mathbf{G}}(a_j)| + 2},$$

for all k, so $\underline{\mathtt{f}_{\mathtt{c}}^{j}} > 0$. Since $\underline{\mathtt{f}_{\mathtt{c}}^{j}} \neq 0$, we obtain $\pi \geq 1$. Together with 4(b), it gives $\pi = 1$. We can use that fact to prove that $\lim\limits_{i \to +\infty} \mathtt{f}_{\mathtt{c}}^{2i+1}(\mathcal{A}) = \lim\limits_{i \to +\infty} \mathtt{f}_{\mathtt{c}}^{2i}(\mathcal{A})$, in the same way as we proved the analogous statement for max-based function in the proof of Theorem 7. □

Proof of Theorem 13. Analogous to the proof of Theorem 8. □

Proof of Theorem 14. Analogous to the proof of Theorem 9. □

Proof of Theorem 15. The proofs that $\mathtt{Cbs}$ satisfies *Anonymity, Independence, Directionality, Maximality, Weakening, Strict Weakening, Weakening Soundness, Proportionality, Strict Proportionality, Reinforcement, Strict Reinforcement, Symmetry, Equivalence* and *Invariance* are very similar to the corresponding ones in Theorem 10.

Resilience: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a \in \mathcal{A}$ such that $w(a) > 0$. Note that

$$\sum_{b \in \text{AttF}_{\mathbf{G}}(a)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(b) \leq |\text{AttF}_{\mathbf{G}}(a)|,$$

so

$$\frac{w(a)}{1 + |\text{AttF}_{\mathbf{G}}(a)| + \frac{\sum_{b \in \text{AttF}_{\mathbf{G}}(a)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(b)}{|\text{AttF}_{\mathbf{G}}(a)|}} \geq \frac{w(a)}{1 + |\text{AttF}_{\mathbf{G}}(a)| + 1}.$$

Thus, $\text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(a) \geq \frac{w(a)}{2 + |\text{AttF}_{\mathbf{G}}(a)|}$ (by Theorem 13), and, consequently, $\text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(a) > 0$.

Strict Invariance: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and suppose that $a, b, a', b', x, y \in \mathcal{A}$ are arguments such that:

1. $w(a) = w(a') = w(b) = w(b')$,
2. $\text{Att}_{\mathbf{G}}(a') = \text{Att}_{\mathbf{G}}(a) \cup \{x\}$,
3. $\text{Att}_{\mathbf{G}}(b') = \text{Att}_{\mathbf{G}}(b) \cup \{y\}$,
4. $\text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(x) = \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(y)$, and
5. $\text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(a) > \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(b)$.

We need to show that $\text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(a') > \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(b')$. Since $w(a') = w(b')$, by Theorem 13 it is sufficient to show

$$1 + |\text{AttF}_{\mathbf{G}}(a')| + \frac{\sum_{t \in \text{AttF}_{\mathbf{G}}(a')} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(t)}{|\text{AttF}_{\mathbf{G}}(a')|} < 1 + |\text{AttF}_{\mathbf{G}}(b')| + \frac{\sum_{z \in \text{AttF}_{\mathbf{G}}(b')} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(z)}{|\text{AttF}_{\mathbf{G}}(b')|}. \qquad (36)$$

From the conditions 1 and 5 we obtain, using Theorem 13, that $1 + |\text{AttF}_{\mathbf{G}}(a)| + \frac{\sum_{t \in \text{AttF}_{\mathbf{G}}(a)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(t)}{|\text{AttF}_{\mathbf{G}}(a)|} < 1 + |\text{AttF}_{\mathbf{G}}(b)| + \frac{\sum_{z \in \text{AttF}_{\mathbf{G}}(b)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(z)}{|\text{AttF}_{\mathbf{G}}(b)|}$. Note that this directly implies (36) if $\text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(x) = \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(y) = 0$. Thus, in the rest of the proof we assume $\text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(x) = \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(y) > 0$. Note that $|\text{AttF}_{\mathbf{G}}(a)| > |\text{AttF}_{\mathbf{G}}(b)|$ is not possible, since it would imply, together with

$$\frac{\sum_{z \in \text{AttF}_{\mathbf{G}}(b)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(z)}{|\text{AttF}_{\mathbf{G}}(b)|} \leq 1,$$

that $1 + |\text{AttF}_{\mathbf{G}}(a)| \geq 1 + |\text{AttF}_{\mathbf{G}}(b)| + \frac{\sum_{z \in \text{AttF}_{\mathbf{G}}(b)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(z)}{|\text{AttF}_{\mathbf{G}}(b)|}$. We distinguish two possible cases:

1. Suppose that $|\text{AttF}_{\mathbf{G}}(a)| = |\text{AttF}_{\mathbf{G}}(b)|$. Then $\frac{\sum_{t \in \text{AttF}_{\mathbf{G}}(a)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(t)}{|\text{AttF}_{\mathbf{G}}(a)|} < \frac{\sum_{z \in \text{AttF}_{\mathbf{G}}(b)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(z)}{|\text{AttF}_{\mathbf{G}}(b)|}$, and, consequently, $\sum_{t \in \text{AttF}_{\mathbf{G}}(a)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(t) < \sum_{z \in \text{AttF}_{\mathbf{G}}(b)} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(z)$. From the conditions 2–4, we obtain $\sum_{t \in \text{AttF}_{\mathbf{G}}(a')} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(t) < \sum_{z \in \text{AttF}_{\mathbf{G}}(b')} \text{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(z)$. From $|\text{AttF}_{\mathbf{G}}(a')| = |\text{AttF}_{\mathbf{G}}(a)| + 1 = |\text{AttF}_{\mathbf{G}}(b)| + 1 = |\text{AttF}_{\mathbf{G}}(b')|$, we obtain (36).

2. Suppose that $|\mathtt{AttF_G}(a)| < |\mathtt{AttF_G}(b)|$. Then $|\mathtt{AttF_G}(a)| + 1 \leq |\mathtt{AttF_G}(b)|$, so, since $\frac{\sum_{t \in \mathtt{AttF_G}(a')} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(t)}{|\mathtt{AttF_G}(a')|} \leq 1$, we have

$|\mathtt{AttF_G}(a)| + \frac{\sum_{t \in \mathtt{AttF_G}(a')} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(t)}{|\mathtt{AttF_G}(a')|} \leq |\mathtt{AttF_G}(b)|$. From $\frac{\sum_{z \in \mathtt{AttF_G}(b')} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(z)}{|\mathtt{AttF_G}(b')|} > 0$ we obtain $|\mathtt{AttF_G}(a)| + \frac{\sum_{t \in \mathtt{AttF_G}(a')} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(t)}{|\mathtt{AttF_G}(a')|}$

$< |\mathtt{AttF_G}(b)| + \frac{\sum_{z \in \mathtt{AttF_G}(b')} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(z)}{|\mathtt{AttF_G}(b')|}$. Finally, from $|\mathtt{AttF_G}(a')| = |\mathtt{AttF_G}(a)| + 1$ and $|\mathtt{AttF_G}(b')| = |\mathtt{AttF_G}(b)| + 1$, we obtain (36).

Monotony: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a, b \in \mathcal{A}$ such that $w(a) = w(b)$ and $\mathtt{Att_G}(a) \subseteq \mathtt{Att_G}(b)$. From Theorem 13, it holds that

$$\mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(a) = \frac{w(a)}{1 + |\mathtt{AttF_G}(a)| + \frac{\sum_{t \in \mathtt{AttF_G}(a)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(t)}{|\mathtt{AttF_G}(a)|}}$$

and

$$\mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(b) = \frac{w(b)}{1 + |\mathtt{AttF_G}(b)| + \frac{\sum_{y \in \mathtt{AttF_G}(y)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(y)}{|\mathtt{AttF_G}(b)|}}.$$

Since $\mathtt{Att_G}(a) \subseteq \mathtt{Att_G}(b)$, it holds that $\mathtt{AttF_G}(a) \subseteq \mathtt{AttF_G}(b)$. There are two cases: Case where $\mathtt{AttF_G}(a) = \mathtt{AttF_G}(b)$, then from condition $w(a) = w(b)$, it follows that $\mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(a) = \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(b)$.

Case where $\mathtt{AttF_G}(a) \subset \mathtt{AttF_G}(b)$. Hence, $|\mathtt{AttF_G}(a)| < |\mathtt{AttF_G}(b)|$ and $|\mathtt{AttF_G}(b)| = |\mathtt{AttF_G}(a)| + x$, with $x \geq 1$. Furthermore, $\frac{\sum_{t \in \mathtt{AttF_G}(a)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(t)}{|\mathtt{AttF_G}(a)|} \leq 1$ and $\frac{\sum_{y \in \mathtt{AttF_G}(b)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(y)}{|\mathtt{AttF_G}(b)|} \leq 1$. Hence, $1 + |\mathtt{AttF_G}(a)| + \frac{\sum_{t \in \mathtt{AttF_G}(a)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(t)}{|\mathtt{AttF_G}(a)|} < 1 + |\mathtt{AttF_G}(a)| + x + \frac{\sum_{t \in \mathtt{AttF_G}(b)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(t)}{|\mathtt{AttF_G}(b)|}$. Since $w(a) = w(b)$, then $\mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(a) \geq \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(b)$.

Counting: It follows from Proposition 12.

Cardinality Precedence: Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and let $a, b \in \mathcal{A}$ be two arguments such that

- $w(a) = w(b)$,
- $\mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(b) > 0$,
- $|\{x \in \mathtt{Att_G}(a) \mid \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(x) > 0\}| > |\{y \in \mathtt{Att_G}(b) \mid \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(y) > 0\}|$,

First, note that, by the third condition, there exists $x_1 \in \mathtt{Att_G}(a)$ such that $\mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(x_1) > 0$, so

$$\frac{\sum_{x \in \mathtt{AttF_G}(a)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(x)}{|\mathtt{AttF_G}(a)|} > 0. \tag{37}$$

From Theorem 13, we have that for any argument c, $\mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(c) > 0$ iff $w(c) > 0$. Thus, $\mathtt{AttF_G}(a) = \{x \in \mathtt{Att_G}(a) \mid \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(x) > 0\}$ and $\mathtt{AttF_G}(b) = \{y \in \mathtt{Att_G}(b) \mid \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(y) > 0\}$. Then, by the third condition, $|\mathtt{AttF_G}(a)| > |\mathtt{AttF_G}(b)|$, so $|\mathtt{AttF_G}(a)| \geq |\mathtt{AttF_G}(b)| + 1$. Since

$$\frac{\sum_{y \in \mathtt{AttF_G}(b)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(y)}{|\mathtt{AttF_G}(b)|} \leq 1,$$

we obtain

$$|\mathtt{AttF_G}(a)| \geq |\mathtt{AttF_G}(b)| + \frac{\sum_{y \in \mathtt{AttF_G}(b)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(y)}{|\mathtt{AttF_G}(b)|}. \tag{38}$$

From (37) and (38) we conclude

$$1 + |\mathtt{AttF_G}(a)| + \frac{\sum_{x \in \mathtt{AttF_G}(a)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(x)}{|\mathtt{AttF_G}(a)|} >$$
$$1 + |\mathtt{AttF_G}(b)| + \frac{\sum_{y \in \mathtt{AttF_G}(b)} \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(y)}{|\mathtt{AttF_G}(b)|}.$$

Then, by Theorem 13, $\mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(a) < \mathtt{Deg}_\mathbf{G}^{\mathtt{Cbs}}(b)$.

Finally, since $\mathtt{Cbs}$ satisfies CP, Maximality and Resilience, it does not satisfy ***QP***, by Proposition 1. It also does not satisfy ***Compensation***. □

Proof of Theorem 16. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathtt{WAG}$ and $a \in \mathcal{A}$. Let us show that $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(a) \in [\frac{w(a)}{2+|\mathtt{AttF}_{\mathbf{G}}(a)|}, w(a)]$. From Theorems 1 and 15, it follows that $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(a) \leq w(a)$. From Theorem 13,

$$\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(a) = \frac{w(a)}{1 + |\mathtt{AttF}_{\mathbf{G}}(a)| + \frac{\sum\limits_{b \in \mathtt{AttF}_{\mathbf{G}}(a)}}{|\mathtt{AttF}_{\mathbf{G}}(a)|}}.$$

For each $x \in \mathcal{A}$, it holds that $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(x) \in [0, 1]$. Hence, $0 \leq \frac{\sum\limits_{b \in \mathtt{AttF}_{\mathbf{G}}(a)}}{|\mathtt{AttF}_{\mathbf{G}}(a)|} \leq 1$. Thus,

$$1 + |\mathtt{AttF}_{\mathbf{G}}(a)| \leq 1 + |\mathtt{AttF}_{\mathbf{G}}(a)| + \frac{\sum\limits_{b \in \mathtt{AttF}_{\mathbf{G}}(a)}}{|\mathtt{AttF}_{\mathbf{G}}(a)|} \leq 2 + |\mathtt{AttF}_{\mathbf{G}}(a)|$$

and $\mathtt{Deg}_{\mathbf{G}}^{\mathtt{Cbs}}(a) \in [\frac{w(a)}{2+|\mathtt{AttF}_{\mathbf{G}}(a)|}, w(a)]$. $\square$

Proof of Theorem 17. The proof is straightforward modification of the proof of Theorem 7, obtained by replacing the function $\mathtt{f}_m^i$ with $\mathtt{f}_h^i$. We will only prove the equation (analogous to the equation (21))

$$F_i(\alpha\mathbf{x}) = \frac{w(a_i)}{(1 - \alpha)F_i(\mathbf{x}) + \alpha w(a_i)} F_i(\mathbf{x}),$$

where

$$F_i(\mathbf{x}) = \frac{w(a_i)}{1 + \sum\limits_{j: a_j \in \mathtt{Att}(a_i)} x_j},$$

$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha \in (0, 1]$.

$$\begin{aligned}
F_i(\alpha\mathbf{x}) &= \frac{w(a_i)}{1 + \sum\limits_{j: a_j \in \mathtt{Att}(a_i)} \alpha x_j} = \frac{w(a_i)}{1 + \alpha \sum\limits_{j: a_j \in \mathtt{Att}(a_i)} x_j} \\
&= \frac{w(a_i)}{(\alpha - \alpha) + 1 + \alpha \sum\limits_{j: a_j \in \mathtt{Att}(a_i)} x_j} \\
&= \frac{w(a_i)}{(1 - \alpha) + \alpha(1 + \sum\limits_{j: a_j \in \mathtt{Att}(a_i)} x_j)} \\
&= \frac{w(a_i)}{(1 - \alpha) + \alpha \frac{w(a_i)}{F_i(\mathbf{x})}} \cdot \frac{F_i(\mathbf{x})}{F_i(\mathbf{x})} \\
&= \frac{w(a_i)}{(1 - \alpha)F_i(\mathbf{x}) + \alpha w(a_i)} F_i(\mathbf{x}).
\end{aligned}$$

The rest of the proof is identical to the proof of Theorem 7. $\square$

Proof of Proposition 14. Obvious since each argument is attacked by at most one argument. Then, the strongest attacker of each argument is its single attacker. $\square$

Proof of Theorem 18. Analogous to the proof of Theorem 8. $\square$

Proof of Theorem 19. Analogous to the proof of Theorem 9. $\square$

Proof of Theorem 20. The proofs that weighted h-Categorizer semantics satisfies ***Anonymity, Independence, Directionality, Maximality, Weakening, Strict Weakening, Weakening Soundness, Proportionality, Strict Proportionality, Reinforcement, Strict Reinforcement, Symmetry, Equivalence, Counting*** and ***Invariance*** are very similar to the corresponding ones of Theorem 10. The proofs of ***Resilience, Strict Invariance*** and ***Monotony*** are similar to the corresponding ones in Theorem 15. Finally, we show that $\mathtt{Hbs}$ violates both ***Quality Precedence*** and ***Cardinality Precedence***. Consequently, it satisfies ***Compensation***. Consider the weighted argumentation graph $\mathbf{G}_4$ of Example 4. Using Theorem 18, we get $\mathtt{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(c) = \mathtt{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(d) = \frac{1}{2}$ and $\mathtt{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(j) = 1$. We also get $\mathtt{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(a) = \mathtt{Deg}_{\mathbf{G}_4}^{\mathtt{Hbs}}(b) = \frac{1}{2}$. Thus,

- $w(a) = w(b)$,
- $\mathrm{Deg}_{\mathbf{G_4}}^{\mathrm{Hbs}}(a) > 0$,
- $\mathrm{Deg}_{\mathbf{G_4}}^{\mathrm{Hbs}}(b) > 0$,
- $|\{x \in \mathrm{Att}_{\mathbf{G_4}}(a) \mid \mathrm{Deg}_{\mathbf{G_4}}^{\mathrm{Hbs}}(x) > 0\}| > |\{y \in \mathrm{Deg}_{\mathbf{G_4}}^{\mathrm{Hbs}}(b) \mid \mathrm{Deg}_{\mathbf{G_4}}^{\mathrm{Hbs}}(y) > 0\}|$,
- $\exists y \in \mathrm{Att}_{\mathbf{G_4}}(b)$ such that $\forall x \in \mathrm{Att}_{\mathbf{G_4}}(a)$, $\mathrm{Deg}_{\mathbf{G_4}}^{\mathrm{Hbs}}(y) > \mathrm{Deg}_{\mathbf{G_4}}^{\mathrm{Hbs}}(x)$

and $\mathrm{Deg}_{\mathbf{G_4}}^{\mathrm{Hbs}}(a) = \mathrm{Deg}_{\mathbf{G_4}}^{\mathrm{Hbs}}(b)$. Thus, neither Quality Precedence nor Cardinality Precedence is satisfied by Hbs. Consequently, Hbs satisfies Compensation. □

Proof of Theorem 21. Let $\mathbf{G} = \langle \mathcal{A}, w, \mathcal{R} \rangle \in \mathrm{WAG}$ and $a \in \mathcal{A}$. Let us show that $\mathrm{Deg}_{\mathbf{G}}^{\mathrm{Hbs}}(a) \in [\frac{w(a)}{1+|\mathrm{Att}_{\mathbf{G}}(a)|}, w(a)]$. From Theorems 1 and 20, it follows that $\mathrm{Deg}_{\mathbf{G}}^{\mathrm{Hbs}}(a) \leq w(a)$.
From Theorem 18,

$$\mathrm{Deg}_{\mathbf{G}}^{\mathrm{Hbs}}(a) = \frac{w(a)}{1 + \sum_{b \in \mathrm{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathrm{Hbs}}(b)}.$$

For each $x \in \mathcal{A}$, it holds that $\mathrm{Deg}_{\mathbf{G}}^{\mathrm{Hbs}}(x) \in [0, 1]$. Hence,

$$0 \leq \sum_{b \in \mathrm{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathrm{Hbs}}(b) \leq |\mathrm{Att}_{\mathbf{G}}(a)|$$

and

$$1 \leq 1 + \sum_{b \in \mathrm{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathrm{Hbs}}(b) \leq 1 + |\mathrm{Att}_{\mathbf{G}}(a)|.$$

We have also

$$1 \geq \frac{1}{1 + \sum_{b \in \mathrm{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathrm{Hbs}}(b)} \geq \frac{1}{1 + |\mathrm{Att}_{\mathbf{G}}(a)|}.$$

Finally,

$$w(a) \geq \frac{w(a)}{1 + \sum_{b \in \mathrm{Att}_{\mathbf{G}}(a)} \mathrm{Deg}_{\mathbf{G}}^{\mathrm{Hbs}}(b)} \geq \frac{w(a)}{1 + |\mathrm{Att}_{\mathbf{G}}(a)|}. \quad \square$$

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