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Expressive voting, graded interests and participation

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Abstract
I assume that voters mark ballots exclusively to express their true preferences among parties, leaving aside any considerations about an election's possible outcome. The paper then analyzes the resulting voting behavior.In particular, it studies how effective different voting systems such as plurality rule, approval voting, and range voting are in fostering high turnout rates of such expressive voters.

Keywords Expressive voting · Voting theory · Approval voting · Range voting · Issue voting · Spatial voting · Directional voting · Proximity voting

1 Introduction

In reasoning about elections, much analysis builds on a simple question: why do people vote the way they do? A major received answer to that question invokes the election's outcome. Briefly put, the instrumental account of voting stipulates that rational voters cast their votes in order to render the election's expected outcome as preferable as possible. Such instrumental analysis of voting has been employed successfully to explain a number of observed patterns, for instance concerning the expected number of parties in a given political system (Duverger 1959). However, the account also has some shortcomings. In particular, as Downs (1957) observed, it cannot satisfactorily answer why voters would participate in large scale elections at all.[1]

A second framework fills that lacuna. Often, voting behavior is not guided by the election's possible outcome. Rather, we may choose a certain option because we judge it ethically correct, attractive, fair, in line with our general political convictions, or simply to cohere with our values and preferences. In short, in the perspective of *expressive voting*, utility is derived directly from the act of expressing one's preferences, rather than from any

[1] The argument, briefly, goes as follows. The chances of making a difference in the outcome of a large scale election are miniscule. Hence, the expected benefit of casting one's vote will be smaller than that of many other thing one could do at the same time.

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considerations about possible outcomes (Brams and Fishburn 1978, 2007; Brennan and Lomasky 1993).

In the present paper, we explore some consequences of the expressive take on voting, mainly in relation to the phenomenon of abstentions. Within expressive voting, abstentions are not explained by the cost of voting. Rather, abstentions occur if an empty ballot is the best available expression of a voter's preferences. We compare three voting systems, plurality rule, approval voting, and range voting, with respect to their propensity for creating high voter turnout.

In what follows, we will build a formal model of expressive voting, within which we assume that voters' preferences range over an agenda of issues. Parties then will be evaluated by their attitudes towards the different agenda items. A similar, topic-based perspective on voting has been assumed in two recent papers by Aragones et al. (2011) and Dean and Parikh (2011). Within our discussion, we make explicit how those frameworks fit within the current approach.

The first of them, the paper by Aragones et al. (2011), is related to our theme of voter participation. It addresses the propensities for various voting systems to promote high turnouts among expressive voters. We are sympathetic to their general approach and adopt a similar underlying framework. However, we raise various criticisms of their conceptualization of approval voting and the resulting comparison of voting systems. Responding to that criticism, we offer an alternative conceptualization of approval voting and then use it to compare expected voter participation within different voting systems.

The remainder of this paper is structured as follows. In Sect. 2, we introduce our general model of expressive voting over an agenda consisting of various items. In Sect. 3, we then present Aragones et al. (2011)'s analysis of approval voting, before raising a criticism of that framework in Sect. 4. We then proceed to introduce our own conceptualization of voters' behavior under three voting methods, plurality, approval and range voting (Sect. 5), and compare them with respect to expected voter turnouts (Sect. 6). All proofs and calculations are provided in the "Appendix".

2 The model

In this section we present our basic electoral model. The central object of concern is a finite agenda of topics or issues for the upcoming election, denoted by $A = \{1 \ldots n\}$. We assume the agenda items to be propositions that the individual parties and candidates can either endorse or oppose. Additionaly, we fix a set C of parties or candidates.

Each voter is represented by a vector $\mathbf{v} \in [-1, 1]^n$, representing her positions on the various topics. The intended reading is that $v_i \in [-1, 1]$, the i-th entry of $\mathbf{v}$, is the degree to which voter $\mathbf{v}$ supports the proposition underlying issue i, where $+1$ stands for total support and -1 for total opposition to the statement in question. Notably, we allow voters to have positions anywhere in $[-1, 1]$ in order to allow for uncertainty about the right course of action, or to mirror graded degrees of interest in the different topics. The only case we exclude are universally disinterested voters. Thus, we assume that $\mathbf{v} \neq \mathbf{0}$.

As with voters, each party is characterized by its positions on the various issues of concern $A = \{1, \ldots, n\}$. Unlike voters, though, we assume parties to have extremal positions on each topic. That is, they are represented by vectors in $\{-1, 1\}^n$. Briefly, that assumption has two different justifications. Firstly, parties are identified with the policies they would implement if elected. We assumed the individual agenda items to be propositional, thus they only can be implemented or not implemented. Of course, a party could break some of its promises and act differently to what they claimed prior to Election Day. Yet, any policy choice will either implement some agenda item i or not. Thus, no space for graded judgments is available, but parties eventually will have to decide for or against implementing any particular item on the agenda. Secondly, Aragones et al. (2011) argue that political discourse moves parties to extreme positions. In an attempt to position themselves on the political scale and to stand out from their opponents, they ultimately will have to take a clear position on each topic.

For notational convenience, we will use the letter $\mathbf{p}$ in different forms to denote parties or candidates, whereas $\mathbf{v}$ and all of its variants denotes voters. Bold letters always refer to vectors, while their entries are denoted by italics, for example $\mathbf{v} = \langle v_1 \dots v_n \rangle$.

Finally, we define the three voting systems we study: plurality rule, approval voting and range voting. In line with the underlying intuitions of expressive voting, our main emphasis lies on the possible ballots a voter can choose amongst, rather than the outcome of an election.

Under *plurality rule*, each voter can vote for a single candidate $\mathbf{p} \in C$. The candidate with the most votes then wins the election. The set of possible ballots hence are

$$F^M := \{\{\mathbf{p}\} \mid \mathbf{p} \in C\} \cup \{\emptyset\}.$$

Under *approval voting*, each voter selects any subset of candidates of which they approve. Again, the candidate receiving the most approval wins the election. Thus, the set of ballots are

$$F^A := \{J \mid J \subseteq C\}.$$

Range voting, finally, refers to a family of related procedures, sometimes also going by the name of score voting or majority judgment (Fishkin 1997; Balinski and Laraki 2010). In range voting, a fixed set of grades $g_1, \dots, g_k \in \mathbb{R}$ is provided that the voters use to assess candidates.[2] Depending on the exact formulation, the candidate with the highest average or median grade then wins the election. The set of admissible ballots thus is

$$F^R := \{f \mid f : C \to \{g_1, \dots, g_k\}\}.$$

Clearly, the set of ballots available in approval voting is a superset of that available under plurality rule. Hence, within expressive voting, abstentions under approval voting should be less frequent that under plurality rule.

To determine voting behavior, we need to specify how an expressive voter $\mathbf{v}$ chooses among her available ballots. That is, we need to explicate the utility $\mathbf{v}$ gains from the different ballots, which depends on her own standpoint as well as the various parties' positions. However, we will not provide any specific utility-function $u : F^* \to \mathbb{R}$ for $* \in \{M, A, R\}$ here. Rather, we state a condition that every reasonable payoff function should satisfy and that is sufficient to determine the voter's choice. We present the condition used by Aragones et al. (2011), before introducing our own framework.

[2] Unlike ranked voting methods such as Borda Count, no structural requirements limit how often each vote can be assigned.

3 Voting decisions in Aragones et al.

Aragones et al. (2011) offer an analysis of the first two voting rules introduced above, plurality rule and approval voting. The core principle within their approach is that a voter will prefer a party that is closer to her own position to a party that is further away. To make this precise, we measure distances in $[-1,1]^n$ in the Euclidean distance: $dist(x,y)=\sqrt{\sum(x_i-y_i)^2}$. Aragones, Gilboa and Weiss then define the following choice rules:

Rule (Aragones et al.-plurality) *Under plurality rule, voter* $\mathbf{v}$ *votes for the candidate that is closest to her and abstains if the empty ballot* $\mathbf{0}$ *is closer than any of the candidates. In other words,* $\mathbf{v}$ *chooses the ballot* $\mathbf{x}_m \in F^M$ *that is closest to her own standpoint in the Euclidean distance. Formally speaking,*

$$\mathbf{x}_m = argmin_{\mathbf{x}\in F^M}\ dist(\mathbf{v},\mathbf{x}).$$

For approval voting, the above rule must be extended to the set of approval ballots. To that end, Aragones et al. represent every approval set $\mathbf{x}^J$ with its arithmetic mean[3] $\frac{1}{|J|}\sum_{j\in J}\mathbf{p}^j$, leading to the following decision rule:

Rule (Aragones et al.-approval) *Under approval voting, voter* $\mathbf{v}$ *chooses the ballot* $\mathbf{x}_a \in F^A$ *that is closest to her own standpoint in the Euclidean distance. Formally,*

$$\mathbf{x}_a = argmin_{\mathbf{x}^J\in F^A}\ dist\left(\mathbf{v},\frac{1}{|J|}\sum_{\mathbf{p}\in J}\mathbf{p}\right).$$

Building on that framework, Aragones et al. show two results, both related to the question of when voters participate in an election. Note that within approval voting, a voter abstains only if the corresponding position vector $\mathbf{0}$ is closer to her than any other possible ballot. That is, abstentions are not caused by an external cost, as in Downs' analysis, but by the fact that the voter fails to find any alternative that is more appealing. The central results of Aragones et al. compare the two voting systems with respect to their potential for generating a high level of electoral involvement, measured by the number of abstentions. The first result studies a best-case scenario, while the second analyzes a case wherein parties' positions are absolutely uncorrelated. Both results confine themselves to situations when voters are fully opinionated, i.e., $\mathbf{v}\in\{-1,1\}^n$.

Theorem 1 (Theorem 1 of Aragones et al. 2011)

(i) *Under approval voting, four strategically positioned parties are sufficient to ensure that no fully opinionated voter abstains*
(ii) *Under plurality vote, the number of parties necessary to ensure that no fully opinionated voter abstains is exponential in the number n of agenda items.*

Theorem 2 (Theorem 2 of Aragones et al. 2011)

Assume that the agenda consists of n topics and we place n parties randomly on that agenda (i.e., for every party we have a fair lottery over the 2^n *possible positions). As*

[3] To facilitate our presentation, we set $\frac{1}{|\emptyset|}\sum_{j\in\emptyset}p^j := \mathbf{0}$. Thus, the empty approval set is represented by the zero-vector.

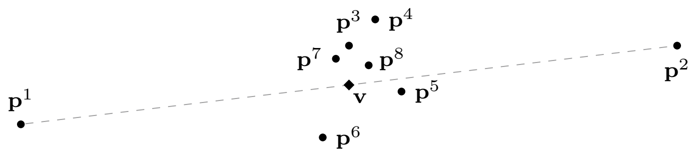

Fig. 1 Voter $\mathbf{v}$'s position is (exactly) the arithmetic mean of the two most extreme parties

$n \to \infty$, *the probability that a fully opinionated voter abstains in such a setting goes to 1 under plurality rule and to 0 under approval voting*.

Hence, approval voting is judged categorically better than plurality voting in preventing voter abstentions. Later, we analyze the same cases within our framework, arriving at partially conflicting findings.

4 A shortcoming of Aragones et al.'s take on approval voting

While we are sympathetic to Aragones et al.'s general approach and their treatment of plurality rule, we identify a conceptual shortcoming within their treatment of approval voting. That shortcoming is then taken to motivate our alternative account, which is presented in the following section.

By its underlying assumptions, expressive voting is blind to any possible outcome of an election, as voters obtain their utility straight from the act of submitting their ballots. However, we maintain that some mild consistency requirement between expressed consent and electoral outcomes is in order. We take the following to be an uncontroversial desideratum.

Unanimity desideratum In elections wherein all voters share exactly the same preferences, and thus submit the same ballots, any single voter should approve of the resulting outcome.

As we will show, Aragones et al.'s approach can violate the desideratum maximally, at least if the election is assumed to have a single winner. In short, we will show that a voter may end up approving exactly of those parties she prefers least. Accordingly, if everybody voted the same way, the winner would be a party the voter maximally dislikes, thus violating the unanimity desideratum.

Example 1 (below) constructs a situation wherein that conclusion is true. The gist of the example is that a moderate voter $\mathbf{v}$'s position may happen to be exactly the average of two extremist parties—even though every moderate party is closer to her than each of the extremists; see Fig. 1 for an illustration. Under the above semantics of approval voting, $\mathbf{v}$'s approval set would consist of precisely the two extremist parties. Now if every voter had the same preference as $\mathbf{v}$, all votes would go to the two extremist parties and, thus, assuming a single winner election, one of the two would be voted into office, clearly producing the outcome $\mathbf{v}$ dislikes most.

Since we represent parties by their fully opinionated positions on a vector of topics, rather than by degrees of extremism, we cannot straightforwardly translate Fig. 1 into a formal counterexample. The following example, though, shares the relevant characteristics with our informal story.

Example 1 Assume that the agenda consists of nine issues $t_1 \dots t_9$. The first four items concern the economy, taxes, environmental issues, and the social system, four issues about which $\mathbf{v}$ has some mild opinions. She assigns positions $\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}$, respectively, to $t_1 \dots t_4$. The other five topics concern difficult decisions in foreign policy on which $\mathbf{v}$ finds it hard to choose sides, so she assigns them weights of zero. The two extremist parties are $\mathbf{e}_+$, assigning 1 to every topic, and $\mathbf{e}_-$, assigning -1 to every topic. Every other party $\mathbf{p}_i$ assigns weights $1, -1, 1, -1$, respectively, to the first four topics and 1 to all remaining items. Then the setup is as claimed above, i.e., all moderate parties $\mathbf{p}_i$ are closer to $\mathbf{v}$ than both $\mathbf{e}_+$ and $\mathbf{e}_-$, but $\{\mathbf{e}_+, \mathbf{e}-\}$ is the approval set chosen by $\mathbf{v}$. We detail the relevant calculations in the "Appendix".

5 Our model

In this section, we offer an alternative framework for approval voting that squares naturally with Aragones et al.'s decision rule for plurality voting, while also satisfying the unanimity desideratum identified in the previous section.

Crucially, plurality and approval voting invoke different choice strategies. The former requires the voter to optimize, that is, identify the *best* among the parties and either vote for that party or else abstain. The latter, in contrast, is built around the notion of satisficing. The central task a voter faces under approval voting is to identify some threshold quality requirement to impose on candidates. She will then approve of every party that meets or exceeds her threshold. In this section, we identify two different ways in which voters could formulate their threshold requirements, one in terms of expected utility, the other as geometrical proximity, and show that both lead to the same choices. We also show that a similar formalism applies to range voting.

To introduce the framework, recall that a voter's position on some topic i is given by a number $v_i \in [-1, 1]$, where -1 stands for maximal opposition and 1 for consent with maximal possible weight. We can decompose that attitude into:

$$v_i = sign(v_i) \cdot |v_i|$$

where $sign(v_i)$[4] indicates whether $\mathbf{v}$ is inclined in favor of or against t_i, while the absolute value $|v_i|$ measures the degree of commitment[5] $\mathbf{v}$ attaches to topic i. We assume commitment $|v_i|$ to be related to the payoff $\mathbf{v}$ can obtain on the agenda item. More specifically, we assume that, by voting for some party $\mathbf{p}$, a voter $\mathbf{v}$ gets a payoff $|v_i|$ on item i if $\mathbf{v}$ and $\mathbf{p}$ agree on whether or not i is a good thing to do, i.e., about the sign of i. Otherwise, $\mathbf{v}$ receives a payoff of $-|v_i|$. Since we have assumed that $p_i \in \{-1;1\}$, that payoff can be expressed as

$$v_i \cdot p_i.$$

Thus, the total payoff $u(\mathbf{v}, \mathbf{p})$ a voter $\mathbf{v}$ receives by voting for party $\mathbf{p}$, i.e., the sum of his or her individual payoffs is:

[4] $sign(x)$ is 1 if $x \geq 0$ and -1 else.

[5] Here, commitment may again reflect the importance $\mathbf{v}$ attaches to that topic as well as her uncertainty about the right course of action.

$$u(\mathbf{v},\mathbf{p}) = \mathbf{v}\cdot\mathbf{p} = \sum v_i p_i.$$

Thus, exploiting again that $p \in \{-1;1\}$, the maximal payoff a voter can get is:

$$|\mathbf{v}| := \sum_i |v_i|.$$

In order to state our decision rule for approval voting, we also need the voter's approval threshold, stating how much deviation from her optimal position she is prepared to accept. The threshold is given by an *approval coefficient* $k \in [-1, 1]$, where a smaller coefficient stands for greater tolerance. As such, we can formulate our decision rule for approval voting .

Rule (approval voting) *Let* $\mathbf{v}$ *be a voter with approval coefficient* $k \in [-1, 1]$. *Then* $\mathbf{v}$ *approves of all parties* $\mathbf{p}$ *that satisfy* $\mathbf{p}\cdot\mathbf{v} = \sum p_i v_i \geq k \cdot \sum |v_i|$, or, equivalently:

$$\frac{\mathbf{p}\cdot\mathbf{v}}{|\mathbf{v}|} \geq k. \tag{1}$$

Note the subtle dependency on the approval coefficient k. For the extreme value of $k = 1$, the voter will approve only of an optimal party coinciding with her on the inclination of every topic. If no such party exists, the voter will submit an empty approval set, i.e., abstain. Conversely, a voter with an approval coefficient of -1 will approve indiscriminately of every party, no matter what that party claims, wants, or does. Finally, a middle value of $k = 0$ corresponds to a fairly tolerant voter, approving of every party that agrees with her more often than it disagrees. For most of the following applications we will thus assume that $k \geq 0$.

Next, we examine two alternative intuitions relating to how a voter could choose parties of which to approve. As it turns out, both of the alternatives are equivalent to our choice rule. We take that equivalence as an argument for the naturalness of our definition.

The first alternative choice rule is given in terms of percental agreement. An agent chooses a percental threshold $t \in [0, 100]$ and approves of every party that agrees with her on at least t percent of the topics. Since the voter has different degrees of commitment to the various agenda items, the percental agreement needs to be weighted by the agent's commitments $|v_i|$. Thus, the corresponding rule is:

Rule (approval voting: 1st alternative) *Let* $\mathbf{v}$ *be a voter with percental threshold* $t \in [0, 100]$. *Then*, $\mathbf{v}$ *approves of all parties* $\mathbf{p}$ *that satisfy*

$$\frac{1}{\sum |v_i|} \sum_{\{i:p_i v_i > 0\}} |v_i| \geq \frac{t}{100}.$$

That decision rule is equivalent to our original rule, as expressed by the following lemma.

Lemma 1 *A voter* $\mathbf{v}$ *approves of some party* $\mathbf{p}$ *with approval coefficient* $k \in [-1, 1]$ *if and only if she approves of* $\mathbf{p}$ *in the alternative definition with percental threshold* $t = 100 \cdot \frac{1+k}{2}$.

The second alternative rule is of a geometric nature. Recall that we represent voters and parties by their positions on the agenda items, that is, as a vector in $\mathbb{R}^n$. So why not define the voter's approval decision by geometric proximity? Arguably, an adequate measure of proximity is the angle between two position vectors, showing how far the two diverge in

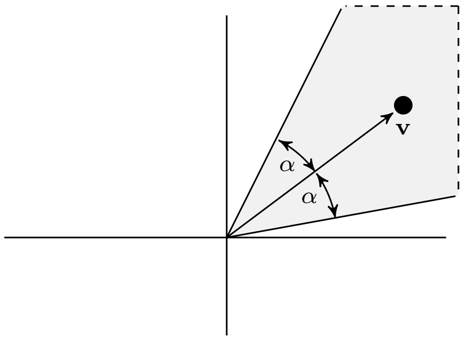

Fig. 2 The approval cone of voter **v** (shaded)

their political opinions.[6] The maximal angle of 180° between a voter **v** and some party **p** implies that $p_i \cdot v_i \leq 0$ for every i, that is, **v** and **p** disagree about every single topic. Conversely, a relatively small angle between a party and a voter corresponds to a high degree of agreement between the voter's inclination and the party's position; see Fig. 2. Again, we need to fix a threshold angle α for formulating the corresponding decision rule. For some given threshold angle α, let $\mathcal{C}(\mathbf{v}, \alpha)$ be the cone of all vectors **y** in $\mathbb{R}^n \setminus \{0\}$ such that the angle between **v** and **y** is at most α.

Rule (approval voting: 2nd alternative) *Let* **v** *be a voter with threshold angle* $\alpha \in [0, 180]$. *Then* **v** *approves of all parties* **p** *that satisfy:*

$$\mathbf{p} \in \mathcal{C}(\mathbf{v}, \alpha).$$

Again, the alternative is related closely to the original decision rule. This time, though, the exact relationship between the approval coefficient k and threshold angle α depends upon the exact position of voter **v**. The correspondence is:

Lemma 2 *Let* **v** *be a voter with approval coefficient k. Then some angle α depending upon n, k and* **v** *exists such that* **v** *approves of some party* **p** *exactly if* $\mathbf{p} \in C(\mathbf{v}, \alpha)$. *Furthermore, the angle α satisfies* $\arccos(k) \leq \alpha \leq \arccos(\frac{k}{\sqrt{n}})$.

Thus, for any possible voter **v**, the three different possible interpretations of approval thresholds are equivalent to one another. Before proceeding to some general results, we will return quickly to the unanimity desideratum that was at the heart of our argument against Aragones et al.'s (2011) approach. Briefly, the desideratum demanded that, assuming a single-winner election, if all voters submitted the same ballot, each should approve of the electoral result. That is indeed the case. The approval set of an agent contains only those parties of which the voter approves *individually*. If every voter happened to submit the same approval set as **v**, the winner would be some member of that approval set and, thus, a party of which **v** approves.

[6] To elaborate a bit further on why we take the angle between two vectors and not, for instance, their length, recall that a change in the length of some vector **v**, that is, replacing **v** by $\lambda\mathbf{v}$ for some $\lambda > 0$, simply denotes a change in political commitment while leaving the general position intact. Conversely, a non-zero angle between two voters **v** ad $\mathbf{v}'$ implies that the two disagree about the relative importance attributed to the various topics or even about the right course of action about some agenda item i.

Finally, we show that our interpretation of approval voting extends readily to range voting. To do so, we fix a set of grades, $g_1, \dots, g_k$, with g_1 being the worst grade and g_k the best. Also, fix a set of *grade requirements* $-1 = t_1 \leq \dots \leq t_k \in [-1, 1]$, which generalize on our approval coefficients:

Rule (range voting) *Let* $\mathbf{v}$ *be a voter and let her grade requirements be* $-1 = t_1 \leq \dots \leq t_k \in [-1, 1]$. *Then the grade some candidate* $\mathbf{p}$ *receives is given by:*

$$grade(\mathbf{v}, \mathbf{p}) := \max \left\{ g_i \middle| \frac{\mathbf{v} \cdot \mathbf{p}}{|\mathbf{v}|} \geq t_i \right\}$$

Note that approval voting is a special case of range voting with only two possible grades, *approval* and *disapproval*. Thus, the set of candidates some voter approves of under approval voting is exactly the set that she grades with *approval*, the higher of the two possible grades. The following straightforward lemma shows that the analogy is compatible with our formal decision rules for range voting and approval voting:

Lemma 3 *Assume that two grades,* g_1 *and* g_2, *are possible and let* $\mathbf{v}$ *be a voter with grade requirements* $-1 = t_1 < t_2$. *Then* $\mathbf{v}$ *evaluates a candidate* $\mathbf{p}$ *with the maximum grade* g_2 *if and only if she approves of* $\mathbf{v}$ *with approval coefficient* $k = t_2$.

Finally, note that the two alternative interpretations of approval voting above, percental agreement and geometrical proximity, can be extended to range voting by using analoga of Lemmas 1 and 2, respectively. In the first case, percental approval, that interpretation moves us even closer to grading as known from school contexts. For each grade, a certain percentage of agreement between a voter and a party is necessary. In other words, to reach a given grade, the party needs to score a certain number of points on the political agreement scale of that voter.

The second alternative account, geometric proximity, translates grade requirements into geometrical objects. Instead of a single approval cone, each set of grade requirements $-1 = t_1 \leq \dots \leq t_k$ translates into a sequence of ever narrower cones around the voter $\mathbf{v}$:

$$\mathcal{C}(\mathbf{v}, \alpha_1) \supseteq \dots \supseteq \mathcal{C}(\mathbf{v}, \alpha_k).$$

Here, the cones' indices stand for different grades, that is, $\mathcal{C}(\mathbf{v}, \alpha_i)$ depicts the area a party needs to fall into so as to receive *at least* grade g_i. The actual grade g_i some party receives thus is determined by the index of the *narrowest* cone within which it is contained.

To end this section, we relate the various takes on approval voting to a more general debate on issue-based voting, the distinction between *distance* and *proximity voting* (Rabinowitz and Macdonald 1989; Lewis and King 1999). In a nutshell, proximity voting assesses parties by their exact positions in $[-1, 1]^n$, while in directional voting it is only the direction of the party's vector $\mathbf{p}$, i.e., $\frac{\mathbf{p}}{|\mathbf{p}|}$, that matters. Correspondingly, proximity voting assumes that a voter will support the party that is closest to her in Euclidean distance, while in directional voting judgments are made by comparing the voter's and parties' directions.

At first sight, the current framework sides with the directional approach, while Aragones et al. (2011) is situated in the proximity camp. As will turn out, though, behavioral differences between both frameworks are rather small. Since parties are assumed to be fully opinionated on all topics, distance and directional voting yield the same recommendations under plurality rule, save for abstentions, cf. Lemma 4 below. Accordingly, differences in behavior occur mainly within approval voting. However, the chief source of divergence there is not the difference between direction and proximity, but the fact that voters in our framework decide whether or not to approve of each party individually, rather than evaluating approval sets by fictive coalitions of their members, as Aragones et al. do.

6 Results

In this section, we will explore plurality rule, approval voting and range voting with respect to their propensity for fostering large electoral turnouts. We begin our discussion by clarifying the relationship between approval and plurality voting. Recall that the admissible ballots for plurality voting are a subset of those available in approval voting. More specifically, every approval ballot that supports at most one candidate also is a legal ballot under plurality rule. Hence, a natural question to ask is whether or not both systems are compatible, i.e., whether or not an agent who decides to approve of at most one candidate in approval voting would submit the same ballot under plurality voting. Within Aragones et al.'s framework, that result is seen immediately to be true, as their rules for approval and plurality voting employ the same distance-based approach. The following lemma shows the result also to hold for the present interpretation of approval voting, save for the possibility of abstentions:

Lemma 4 *Let* $\mathbf{v}$ *be a voter. Assume that under approval voting* $\mathbf{v}$ *approves of the set* $\{\tilde{\mathbf{p}}\}$, *while under plurality vote she votes for* $\mathbf{p}'$. *Then* $\mathbf{p}' = \tilde{\mathbf{p}}$.

We now return to the initial topic of determining when people vote. More precisely, within approval voting, electoral participation boils down to the question of when some voter can find a ballot that she prefers over abstaining. Our results will fall into two groups, roughly corresponding to Theorems 1 and 2 of Aragones et al.. The first class of results asks about the minimal number of parties needed to avoid abstentions, assuming that those parties are positioned optimally. For plurality voting, the question has been answered by Theorem 1 of Aragones et al. (2011), cited above. The number of parties necessary to ensure that no voter abstains under plurality rule is exponential in the number n of agenda items.

In the case of approval voting, the answer to the same question will depend on the pickiness of voters. Naturally, the stricter voters are, that is, the higher their approval coefficient, the more candidates are needed in order to ensure that every voter finds a suitable candidate of which to approve:

Theorem 3

(i) *If* $k \leq 0$, *two parties are enough to ensure that every voter approves of at least one of them.*
(ii) *If* $k > 0$, *the number of parties needed to ensure that no (possible) voter abstains grows exponentially in the number of agenda items.*
(iii) *Assume that the agenda contains at least three items. If voters are infinitesimally more demanding than* $k = 0$, *approving only of those parties that, given the voters' weights, share strictly more than half of their position, i.e.,* $\frac{\mathbf{p}\cdot\mathbf{v}}{|\mathbf{v}|} > 0$, *then exactly* $n + 1$ *parties are needed to ensure that every voter approves of at least one party.*

Next, we consider the opposite extreme at which parties are not placed strategically in relation to the positions of others, but each party is positioned randomly. No matter how many candidates are available, we cannot *guarantee* that every voter will find some candidate that is more attractive than abstaining. However, we can provide probabilistic estimates of how likely it is for a random voter to be attracted by at least one of the candidates. As the number of topics on the agenda grows, it becomes more and more probable that some group of people will be dissatisfied with all of the existing candidates and, hence, decide to form their own party. We thus will assume that an election with n different topics on the agenda attracts n such randomly distributed parties.[7] Under those conditions, let $P(n)$ denote the probability that some voter does *not* abstain in an election based on an n-topic agenda. Again, the case of plurality voting has been analyzed by Aragones et al. (2011) in Theorem 2, cited above: Under plurality voting, we have $\lim_{n\to\infty} P(n) = 0$.

Regarding approval voting, the chance of some randomly chosen party appealing to some generic voter $\mathbf{v}$ will depend on $\mathbf{v}$'s approval coefficient k. Naturally, a voter with a high threshold k is more likely to abstain in such a situation than somebody with lower standards of approval. Thus, we extend our definition of $P(n)$ above to $P(n, k)$, denoting the probability that a random voter with approval coefficient k does not abstain in an n-topic situation. We obtain

Theorem 4 *For* $k \leq 0$, *we have* $\lim_{n\to\infty} P(n,k) = 1$. *If* $k \in (0;1]$; *however, the converse holds*: $\lim_{n\to\infty} P(n,k) = 0$.

Thus, if voters are modestly demanding, that is, $k > 0$, approval voting is not categorically better than plurality rule. That conclusion stands in stark contrast to Aragones et al. (2011)'s original results, wherein the former is judged categorically better; cf. their Theorem 2 cited above. For our result, though, a word of caution is warranted. The skeptical result that a sufficiently demanding voter will almost certainly abstain if the agenda is large enough, that is, $\lim_{n\to\infty} P(n,k) = 0$, is a worst case result only, depending on the voter's exact interest. Naturally, a universally interested voter, having opinions on most of the agenda items, is harder to satisfy accidentally than somebody who is interested only in a small section of the agenda. In the most extreme case, a voter is focused primarily on a single topic, that is, $\mathbf{v} \approx \pm e_i$. Such a voter will almost certainly find some party she approves of, regardless of her approval coefficient k, i.e., $\lim_{n\to\infty} P(n,k) = 1$ for all $k \in [-1;1)$.

Finally, we return to our last voting mechanism, range voting, In that protocol, voters are asked to grade all parties within a given grading scale. Thus, nothing such as abstaining or submitting an empty ballot arises and, hence, we need to reformulate our original question. So let us assume that a voter is motivated to engage in range voting only if she finds some relevant *differences* between candidates that she could express. That is, we ask for the conditions under which our voter finds two parties to which she assigns different grades. As in the case of approval voting, that possibility will depend on her exact grade requirements. For the case of strategically positioned parties, we obtain:

Theorem 5 *Assume that every voter has some* i *with* $t_i = 0$ *and that at least three items are on the agenda. Then,* $n + 1$ *parties are enough to ensure that every* (*possible*) *voter finds two parties she grades differently. Conversely, if no index* i *with* $t_i = 0$ *exists, the*

[7] That is, we draw each party's position from a uniform distribution over the 2^n possible positions.

number of parties needed to ensure that every possible voter finds two parties she wishes to grade differently grows exponentially in n.

Next, we consider again the case of randomly distributed parties. We will rely on the same setting as for approval voting. That is, we consider the case of n randomly distributed parties in an election ranging over an agenda with n topics. As above, our results will depend on the exact grade requirements adopted by the voter. For any set $\mathbf{t} = (t_1, \dots, t_k)$ of grade requirements, let $P(n, \mathbf{t})$ denote the probability that, given n randomly distributed parties on an n-topic agenda, a voter with grade requirements $\mathbf{t}$ will find two parties that she wishes to grade differently:

Theorem 6 *If* $t_i = 0$ *for some* i, *then* $\lim_{n\to\infty} P(n, \mathbf{t}) = 1$. *If, however, no such* i *exists, then* $\lim_{n\to\infty} P(n, \mathbf{t}) = 0$.

In other words, the probability that $\mathbf{v}$ finds two random parties she wishes to grade differently depends crucially on whether or not she gives different grades to those parties coinciding with her opinion on at least half of the topics and those that do not. Notably, the same word of caution as for approval voting applies here. Theorem 6 studies a worst case scenario that applies mainly to broadly interested voters. A less universal voter, primarily interested in one or two agenda items will, again, satisfy $\lim_{n\to\infty} P(n, \mathbf{t}) = 1$ almost irrespective of which grade requirements she uses.

7 Discussion and outlook

The theory of expressive voting is a potent counterpart to the instrumental analysis of voting behavior, not least because expressive accounts offer a solution to some paradoxes of the instrumental account, such as the question of why people vote in the first place. Of course, neither alternative alone can provide a satisfactory account for much of the voting behavior we observe. Both theories are highly idealized, studying the behavior of ideal types of voters. They do, though, have some descriptive backing. Brennan and Lomasky (1993, pp. 40–46) show that actual voting behavior is best explained by a superpositioning of instrumental and expressive considerations, wherein the weights given to the two accounts depend on various factors, such as, for instance, the stakes involved or how close the election is expected to be.

Crucially, the two accounts may yield divergent advice about whom to vote for, but also about whether or not to participate in the election at all. Given that voters will rely on to both types of considerations, expressive and instrumental, a suitable approach for analyzing voting situations is to start by studying them from both standpoints in order to compare and combine these findings later.

In line with such reasoning, our paper has presented a formal account of expressive voting based on the underlying political agenda of an election. In doing so, it focussed on the propensity of voting systems to create large voter turnouts. In particular, we employed an expressive account of voting to compare three different voting systems, plurality rule, approval voting and range voting. We find significant differences between those voting systems regarding their ability to create high voter turnouts. Of course, the perspective offered here is idealized. Within realistic voting scenarios, correlations between individual voter positions as well as between various agenda items will be found. Here, the current framework may form a starting point for empirical approaches addressing the numbers and positions of parties within a political system and how they relate to the distributions of voter preferences.

The positioning of parties may, though, also be approached on analytic grounds. Paralleling seminal results in instrumental voting, one may analyze the emergence of new parties, the positioning of existing parties or the complexities of political campaigning from an expressive viewpoint. The first results in that direction suggest that strategic campaigning can be a very complex endeavor. Successful campaigning may, for instance, require detailed information about the voters' attitudes to uncertainty (Dean and Parikh 2011) and the relative importance they attribute to various items on the policy agenda, but also about competitors' campaigning behavior (Klein and Pacuit 2017; Klein 2015).

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Appendix: Proofs

We start by showing that Example 1 satisfies all of the properties claimed. In particular, we have to show that (i) $\operatorname{dist}(\mathbf{v}, \mathbf{p}) < \operatorname{dist}(\mathbf{v}, \mathbf{e}_{\pm})$ and (ii) that $\{\mathbf{e}_+, \mathbf{e}_-\}$ is the coalition approved by $\mathbf{v}$.

For (i), observe that

$$\operatorname{dist}(\mathbf{v}, \mathbf{p}) = \sqrt{4 \cdot \left(\frac{2}{3}\right)^2 + 5} = \sqrt{\frac{16}{9} + 5} \quad \text{and}$$
$$\operatorname{dist}(\mathbf{v}, \mathbf{e}_*) = \sqrt{2 \cdot \left(\frac{2}{3}\right)^1 + 2 \cdot \left(\frac{4}{3}\right)^2 + 5} = \sqrt{\frac{40}{9} + 5} \text{ for } * \in \{+, -\}$$

Thus, $\mathbf{e}_{\pm}$ are furthest away from $\mathbf{v}$'s preferences.

For (ii), observe that

$$\operatorname{dist}\left(\frac{1}{2}(\mathbf{e}_+ + \mathbf{e}_-), v\right) = \sqrt{4 \cdot \left(\frac{1}{3}\right)^2} = \frac{2}{3}.$$

To see that $\{\mathbf{e}_+, \mathbf{e}_-\}$ is the closest coalition, we first show that any coalition C containing three or more members has a distance of at least $\frac{\sqrt{5}}{3}$ from $\mathbf{v}$. For any such coalition, the last five entries of C are all at least $\frac{1}{3}$ (with the minimum reached if C consists of exactly three entries, one of them being $\mathbf{e}_{-}$). Thus, $\operatorname{dist}(C, \mathbf{v}) \geq \frac{\sqrt{5}}{3}$. A similar argument shows that $\operatorname{dist}(C', \mathbf{v}) \geq \sqrt{5}$ holds for $C' = \{\mathbf{e}_{+}, \mathbf{p}\}$. For the coalition $C'' = \{\mathbf{p}_i, \mathbf{p}_j\}$, we have that $\operatorname{dist}(C'', \mathbf{v}) = \operatorname{dist}(\mathbf{p}_i, \mathbf{v}) = \sqrt{\frac{16}{9} + 5}$. Finally, for the coalition $\mathbf{C}'''\{\mathbf{e}_{-}, \mathbf{p}\}$ we have

$$\operatorname{dist}(C''', \mathbf{v}) = \sqrt{2 \cdot \left(\frac{1}{3}\right)^2 + 2 \cdot \left(\frac{2}{3}\right)^2} = \frac{\sqrt{10}}{3}$$

finishing the proof.

Proof of Lemma 2 For $x, y \in \mathbb{R}^n$, the angle α between x and y is described by the following well-known equation

$$\frac{x \cdot y}{|x|_2 |y|_2} = \cos \alpha \tag{2}$$

where $|\mathbf{x}|_2 = \sqrt{\sum x_i^2}$ denotes the Euclidean length. Moreover, the rule for approval voting (Equation 1) can be transformed into

$$\frac{\mathbf{v} \cdot \mathbf{p}}{|\mathbf{v}|} \geq k \Leftrightarrow \frac{\mathbf{v} \cdot \mathbf{p}}{|\mathbf{v}|_2 \sqrt{n}} \geq \frac{k}{\sqrt{n}} \frac{|\mathbf{v}|}{|\mathbf{v}|_2} \Leftrightarrow \frac{\mathbf{v} \cdot \mathbf{p}}{|\mathbf{v}|_2 |\mathbf{p}|_n} \geq \frac{k}{\sqrt{n}} \frac{|\mathbf{v}|}{|\mathbf{v}|_2},$$

where the last equivalence exploits that $|\mathbf{p}|_2 = \sqrt{\sum_i 1} = \sqrt{n}$. By Eq. (2), this is equivalent to $\mathbf{p} \in \mathcal{C}(\mathbf{v}, \alpha)$ for

$$\alpha = \arccos\left(\frac{k}{\sqrt{n}} \frac{|\mathbf{v}|}{|\mathbf{v}|_2}\right).$$

The last claim follows from the inequality $|x|_2 \leq |x| \leq \sqrt{n}|x|_2$ for all $x \in \mathbb{R}^n$.

Proof of Lemma 4 Recall that under approval voting, $\mathbf{v}$ approves of $\mathbf{p}$ iff $\frac{\mathbf{v} \cdot \mathbf{p}}{|\mathbf{v}|} \geq k$, where k is $\mathbf{v}$'s approval coefficient. Since $\mathbf{p}'$ is the only party $\mathbf{v}$ approves of, we get

$$\frac{\sum v_i p_i'}{\sum |v_i|} = \max_{p \in C} \frac{\sum v_i p_i}{\sum |v_i|}.$$

On the other hand, the fact that $\tilde{\mathbf{p}}$ is the winner under plurality vote is expressed by the equation

$$\operatorname{dist}(\tilde{\mathbf{p}}, \mathbf{v}) = \min_{\mathbf{p} \in C} \operatorname{dist}(\mathbf{p}, \mathbf{v}).$$

Thus, we have to show the following condition to hold for every party $\mathbf{p}^*$

$$\operatorname{dist}(\mathbf{p}^*, \mathbf{v}) = \min_{\mathbf{p} \in C} \operatorname{dist}(\mathbf{p}, \mathbf{v}) \Leftrightarrow \frac{\sum v_i p_i^*}{\sum |v_i|} = \max_{p \in C} \frac{\sum v_i p_i}{\sum |v_i|}.$$

Recall that $p_i \in \{-1;1\}$ for each topic $i \in A$. Fix a voter $\mathbf{v}$. For any party $\mathbf{p}$, let $U_{\mathbf{p}} \subseteq \{1 \ldots n\}$ be defined by:

$$i \in U_{\mathbf{p}} \Leftrightarrow v_i \cdot p_i < 0.$$

Thus $U_{\mathbf{p}}$ is the set of indices where the signs of $\mathbf{v}$ and $\mathbf{p}$ disagree. Now we have

$$\begin{aligned}\operatorname{dist}(\mathbf{v},\mathbf{p}) &= \sqrt{\sum_i (v_i - p_i)^2} = \sqrt{n + \sum_i v_i^2 - 2\sum_i v_i p_i} \\ &= \sqrt{n + \sum_i v_i^2 - 2\sum_i |v_i| + 4\sum_{i\in U_{\mathbf{p}}} |v_i|}.\end{aligned}$$

Only the last term depends on $\mathbf{p}$. Thus for any $\mathbf{p}, \mathbf{p}' \in C$:

$$\operatorname{dist}(\mathbf{p},\mathbf{v}) \le \operatorname{dist}(\mathbf{p}',\mathbf{v}) \Leftrightarrow \sum_{i\in U_{\mathbf{p}}} |v_i| \le \sum_{i\in U_{\mathbf{p}'}} |v_i|.$$

On the other hand we have:

$$\sum_i v_i p_i = \sum_i |v_i| - 2\sum_{i\in U_{\mathbf{p}}} |v_i|,$$

where, again, the first term is independent of $\mathbf{p}$. Thus also

$$\frac{\sum v_i p_i}{\sum |v_i|} \ge \frac{\sum v_i p_i'}{\sum |v_i'|} \Leftrightarrow \sum_{i\in U_{\mathbf{p}}} |v_i| \le \sum_{i\in U_{\mathbf{p}}'} |v_i|.$$

Before we can prove Theorems 3 and 4 we need the following lemma:

Lemma 5 *Let $m \in \mathbb{N}\backslash\{0\}$. Then we have for any natural number n*

$$\frac{\sum_{k=\lceil n(\frac{1}{2}+\frac{1}{m})\rceil}^{n} \binom{n}{k}}{2^n} \le \frac{1}{(1+\frac{1}{m})^{\lceil \frac{n}{m}\rceil} - 1}. \tag{3}$$

Proof We make a case distinction between n even and odd. We show the formula for n even. For n odd, the proof is similar. First we show that for any natural number $i \in [0, \frac{n}{m}]$ we have that

$$\binom{n}{\frac{n}{2}+i} \ge \left(1+\frac{1}{m}\right)^{\lceil \frac{n}{m}\rceil} \binom{n}{\frac{n}{2}+\lceil\frac{n}{m}\rceil+i}. \tag{4}$$

To this end observe that

$$\begin{aligned}\frac{\binom{n}{\frac{n}{2}+i}}{\binom{n}{\frac{n}{2}+\lceil\frac{n}{m}\rceil+i}} &= \frac{(\frac{n}{2}+\lceil\frac{n}{m}\rceil+i)!(\frac{n}{2}-\lceil\frac{n}{m}\rceil-i)!}{(\frac{n}{2}-i)!(\frac{n}{2}+i)!} \\ =&\frac{(\frac{n}{2}+i+1)\cdot(\frac{n}{2}+i+2)\cdot\ldots\cdot(\frac{n}{2}+\lceil\frac{n}{m}\rceil+i)}{(\frac{n}{2}-\lceil\frac{n}{m}\rceil-i+1)\cdot(\frac{n}{2}-\lceil\frac{n}{m}\rceil-i+2)\cdot\ldots\cdot(\frac{n}{2}-i)} \\ =&\frac{\frac{n}{2}+1+i}{(\frac{n}{2}-\lceil\frac{n}{m}\rceil-i+1)}\cdot\ldots\cdot\frac{\frac{n}{2}+\lceil\frac{n}{m}\rceil+i}{\frac{n}{2}-i}.\end{aligned}$$

Now, since $\frac{k+i}{l-i} \geq \frac{k}{l}$ for any $k, l, i > 0$, every quotient in the last formula is at least as large as the right-most quotient $\frac{\frac{n}{2}+\lceil\frac{n}{m}\rceil+i}{\frac{n}{2}-i}$. Moreover, this quotient satisfies

$$\frac{\frac{n}{2}+\lceil\frac{n}{m}\rceil+i}{\frac{n}{2}-i} = 1 + \frac{\lceil\frac{n}{m}\rceil+2i}{\frac{n}{2}-i} \geq 1 + \frac{\lceil\frac{n}{m}\rceil}{\frac{n}{2}} \geq 1 + \frac{\frac{n}{m}}{\frac{n}{2}} > 1 + \frac{1}{m},$$

as $m, n, i \geq 0$. As the product $\frac{\frac{n}{2}+1+i}{(\frac{n}{2}-\lceil\frac{n}{m}\rceil-i+1)} \cdot \ldots \cdot \frac{\frac{n}{2}+\lceil\frac{n}{m}\rceil+i}{\frac{n}{2}-i}$ contains $\lceil\frac{n}{m}\rceil$ factors, it is thus larger than $(1+\frac{1}{m})^{\lceil\frac{n}{m}\rceil}$ and (4) holds. In the following, let $\alpha := \left((1+\frac{1}{m})^{-1}\right)^{\lceil\frac{n}{m}\rceil}$.

Repeatedly applying (4) gives us for all natural numbers j with $0 \leq j < \lceil\frac{n}{m}\rceil$

$$\sum_{i=1}^{m} \binom{n}{\frac{n}{2}+j+i\lceil\frac{n}{m}\rceil} \leq \sum_{i=1}^{m} \alpha^i \binom{n}{\frac{n}{2}+j} \leq \frac{\alpha}{1-\alpha} \binom{n}{\frac{n}{2}+j}$$

Using that $\binom{n}{k} = 0$ whenever $k > n$, this implies

$$\sum_{k=\lceil n(\frac{1}{2}+\frac{1}{m})\rceil}^{n} \binom{n}{k} = \sum_{i=1}^{m} \sum_{j=0}^{\lceil\frac{n}{m}\rceil-1} \binom{n}{\frac{n}{2}+j+i\lceil\frac{n}{m}\rceil}$$

$$\leq \frac{\alpha}{1-\alpha} \sum_{j=0}^{\lceil\frac{n}{m}\rceil-1} \binom{n}{\frac{n}{2}+j} < \frac{\alpha}{1-\alpha} \sum_{j=0}^{n} \binom{n}{j}.$$

Resubstituting $\alpha = (1+\frac{1}{m})^{-\lceil\frac{n}{m}\rceil}$ and exploiting $\sum_{j=0}^{n} \binom{n}{j} = 2^n$ gives us

$$\frac{\sum_{k=\lceil n(\frac{1}{2}+\frac{1}{m})\rceil}^{n} \binom{n}{k}}{2^n} \leq \frac{(1+\frac{1}{m})^{-\lceil\frac{n}{m}\rceil}}{1-(1+\frac{1}{m})^{-\lceil\frac{n}{m}\rceil}} = \frac{1}{(1+\frac{1}{m})^{\lceil\frac{n}{m}\rceil}-1}.$$

Proof of Theorem 3 For (i) observe that candidates $\mathbf{p}_1 := (1, 1, \ldots 1)$ and $\mathbf{p}_2 := -\mathbf{p}_1$ have the property that for any voter $\mathbf{v}$ at least one of the two statements $\mathbf{p}_1 \cdot \mathbf{v} \geq 0$ and $\mathbf{p}_2 \cdot \mathbf{v} \geq 0$ holds. Thus each voter approves of at least one of these two parties.

(ii) Assume $k > 0$ and let $\mathbb{V} := \{-1;1\}^n$ be the set of voters who have extreme positions on every single topic. We will show that the number of parties needed to ensure that all members of $\mathbb{V}$ vote is exponential in n. Fix some natural number m such that $\frac{1}{m} \leq k$. Since the number of parties some voter $\mathbf{v}$ approves of is decreasing in k it suffices to show the theorem with $k = \frac{1}{m}$. Observe that for any party $\mathbf{p}$ and any voter $\mathbf{v} \in \mathbb{V}$ holds:

$$\frac{\mathbf{v}\cdot\mathbf{p}}{|\mathbf{v}|} \geq \frac{1}{m} \Leftrightarrow |\{i|v_i = p_i\}| \geq \frac{n}{2} + \frac{n}{2m}.$$

Since for any party $\mathbf{p}$ and any $l \in \mathbb{N}$

$$\left|\left\{\mathbf{v} \in \mathbb{V} \middle| \{i : v_i = p_i\}| = l\right\}\right| = \binom{n}{l},$$

each party $\mathbf{p}$ can be be approved by at most $\sum_{k=\lceil n(\frac{1}{2}+\frac{1}{2m})\rceil}^{n} \binom{n}{k}$ many members of $\mathbb{V}$. Since $|\mathbb{V}| = 2^n$ this implies that the number of parties needed to make sure that no member of $\mathbb{V}$ abstains is at least

$$\frac{2^n}{\sum_{k=\lceil n(\frac{1}{2}+\frac{1}{2m})\rceil}^{n} \binom{n}{k}}.$$

By Lemma 5 this quotient is at least $(1+\frac{1}{m})^{\lceil \frac{n}{m} \rceil} - 1$ and thus also at least $(1+\frac{1}{m})^{\frac{n}{m}} - 1$. In particular it is at least exponential in n. Since 2^n parties are enough to ensure that everybody votes, the number of parties needed cannot be worse than exponential.

The proof of *iii*) consists of two parts. First, we show that *at least* $n+1$ parties are needed in order to ensure that every voter finds a party she approves of. Assume to the contrary that $\mathbf{p}_1 \dots \mathbf{p}_n$ are enough to attract every possible voter. Recall that, by the voting rule used for *iii*), a voter $\mathbf{v}$ approves of a party $\mathbf{p}$ iff $\mathbf{v} \cdot \mathbf{p} > 0$. For $i < n$ define X_i to be the $n-1$ dimensional hypersurface defined by

$$X_i = \{\mathbf{x} \in [-1, 1]^n | \mathbf{x} \cdot \mathbf{p}_i = 0\}.$$

Thus $X := X_1 \cap \dots \cap X_{n-1}$ is a vector space of dimension at least 1 and thus $Y = X \cap \{\mathbf{x} \in [-1, 1]^n | \mathbf{x} \cdot \mathbf{p}_n \leq 0\} \neq \{\mathbf{0}\}$. Pick some non-zero $\mathbf{v} \in Y$. Then $\mathbf{v} \cdot \mathbf{p_i} = 0$ for $i < n$ and $\mathbf{v} \cdot \mathbf{p}_n \leq 0$, thus the voter $\mathbf{v}$ would abstain in an election with candidates $\mathbf{p}_1 \dots \mathbf{p}_n$, contradicting our assumption.

Next, we show that $n+1$ parties are sufficient to attract all voters if there are at least $n \geq 3$ topics. To this end, let $\mathbf{1}$ be the vector $(1, \dots, 1)$ and for $i \leq n$ let $\mathbf{e}_i$ be the vector with 1 at the i-th position and -1 on all others. Moreover, let P be the set $\{\mathbf{1}, \mathbf{e}_1, \dots, \mathbf{e}_n\}$. We will show that every voter $\mathbf{v}$ approves of at least one party in P. We do so by case distinction. The first case is that $\mathbf{v} \cdot \mathbf{e}_i > 0$ for some i. In this case, $\frac{\mathbf{v}\cdot\mathbf{e}_i}{|\mathbf{v}|} > 0$ and $\mathbf{v}$ approves of party $\mathbf{e}_i$. The second case is that $\mathbf{v} \cdot \mathbf{e}_i \leq 0$ for all $i \leq n$. First, note that $\mathbf{e}_1, \dots, \mathbf{e}_n$ form a basis of $\mathbb{R}^n$. Thus, $\mathbf{v} \cdot \mathbf{e}_i = 0$ for all i would imply that $\mathbf{v} = \mathbf{0}$. Since we have excluded voters from assuming position $\mathbf{0}$, we can infer that there is some j with $\mathbf{v} \cdot \mathbf{e}_j < 0$. Now, note that $\mathbf{1} = -\frac{1}{n-2} \sum_{i \leq n} \mathbf{e}_i$. We hence get

$$\mathbf{v} \cdot \mathbf{1} = \mathbf{v} \cdot \left(-\frac{1}{n-2} \sum_{i \leq n} \mathbf{e}_i \right) = -\frac{1}{n-2} \sum_{i \leq n} \mathbf{v} \cdot \mathbf{e}_i$$

Since $\mathbf{v} \cdot \mathbf{e}_i \leq 0$ for all i and $\mathbf{v} \cdot \mathbf{e}_j < 0$, we obtain that $-\frac{1}{n-2} \sum_{i \leq n} \mathbf{v} \cdot \mathbf{e}_i > 0$. Hence $\frac{\mathbf{v}\cdot\mathbf{1}}{|\mathbf{v}|} > 0$, showing that $\mathbf{v}$ approves of party $\mathbf{1}$. □

Proof of Theorem 4 Fix a voter $\mathbf{v}$. Observe that for $k = 0$ and any party $\mathbf{p}$ at least one of the following two holds: $\frac{\mathbf{v}\cdot\mathbf{p}}{|\mathbf{v}|} \leq 0$ or $\frac{\mathbf{v}\cdot\mathbf{p}}{|\mathbf{v}|} \geq 0$. Let $\mathcal{P} = \{-1;1\}^n$ be the set of all possible parties. Since for any $\mathbf{p} \in \mathcal{P}$ also $-\mathbf{p} \in \mathcal{P}$ and $\frac{\mathbf{v}\cdot\mathbf{p}}{|\mathbf{v}|} \geq 0$ iff $\frac{\mathbf{v}\cdot-\mathbf{p}}{|\mathbf{v}|} \leq 0$, we get that

$$\frac{|\{\mathbf{p} \in \mathcal{P} | \mathbf{p} \cdot \mathbf{v} \geq 0\}|}{|\mathcal{P}|} \geq \frac{1}{2}.$$

Since picking a random party is the same as randomly drawing a party from $\mathcal{P}$, the chance that a random party $\mathbf{p}$ satisfies $\mathbf{p} \cdot \mathbf{x} \geq 0$ is, thus, at least one half. Thus the chance that

$\mathbf{v}$ approves of *none* of n random parties is at most $\frac{1}{2}^n$, thus $P(n, n, 0) \to 1$. As approval is monotonous in k, this implies $P(n, k) \to 1$ for any $k \leq 0$.

Since $P(n, k)$ is monotonous in k, and for every $k > 0$ there is some $m \in \mathbb{N}$ with $\frac{1}{m} \leq k$, it suffices to show that $P(n, n, \frac{1}{m}) \to 0$ for any natural number m. Let $\mathbf{v} = (1, 1, \ldots)$ be a voter who fully approves of all topics and let $m \in \mathbb{N}$. Observe that for any party $\mathbf{p}$ holds:

$$\mathbf{v} \cdot \mathbf{p} \geq \frac{1}{m}|\mathbf{v}|_1 \Leftrightarrow |\{i | p_i = 1\}| \geq \frac{n}{2} + \frac{n}{2m}.$$

Thus for the uniform distribution $\mathbb{P}$ over $\mathcal{P}$ we have

$$\mathbb{P}\left(\mathbf{v} \cdot \mathbf{p} \geq \frac{1}{m}|\mathbf{v}|_1\right) = \frac{\sum_{k=\lceil n(\frac{1}{2}+\frac{1}{2m})\rceil}^{n} \binom{n}{k}}{2^n}.$$

As above Lemma 5 yields that

$$\mathbb{P}\left(\mathbf{v} \cdot \mathbf{p} \geq \frac{1}{m}|\mathbf{v}|_1\right) \leq \frac{1}{(1+\frac{1}{m})^{\lceil \frac{n}{m} \rceil} - 1}.$$

Thus $P(n, n, \frac{1}{m}) \leq 1 - \left(1 - \frac{1}{(1+\frac{1}{m})^{\lceil \frac{n}{m} \rceil} - 1}\right)^n$.

Note that for n large enough,

$$\left(1 - \frac{1}{(1+\frac{1}{m})^{\frac{n}{m+1}}}\right)^n \leq \left(1 - \frac{1}{(1+\frac{1}{m})^{\lceil \frac{n}{m} \rceil} - 1}\right)^n \leq \left(1 - \frac{1}{(1+\frac{1}{m})^n}\right)^n$$

It is a general fact that $(1 - x^n)^n \to 1$ for any $x \in (0, 1)$. As both the left- and rightmost member of the above inequality are of this form, we obtain $P(n, n, \frac{1}{m}) \to 0$ as claimed.

Proof of Theorem 5 Fix a voter $\mathbf{v}$ and let i such that $t_i = 0$. The third part of Theorem 3 applied to voter $-\mathbf{v}$ shows that $n + 1$ parties are enough to guarantee that some party gets graded at most g_{i-1}. Equally, the same theorem applied to $\mathbf{v}$ herself shows that the same $n + 1$ parties also guarantee that some candidate gets grade g_i or higher. Finally assume that there is no i with $t_i = 0$ and let j be maximal such that $t_{j-1} < 0$. Then the second part of Theorem 3 applied to $\mathbf{v}$ and $-\mathbf{v}$ shows that exponentially many parties are needed in order to ensure that some party gets a grade unequal to g_{j-1}.

Proof of Theorem 6 First assume that there is some i with $t_i = 0$. Then, by Theorem 4, the probability that at least one out of n random parties gets grade at least g_i goes to 1. Applying Theorem 4 to $-\mathbf{v}$ we see that also the probability that a party gets grade at most g_{i-1} goes to 1. In particular, the probability for two parties receiving different grade assignments goes to 1, thus proving the first part. For the second part assume that there is no such i. Let i_0 be such that $t_i < 0$ for all $i \leq i_0$ and $t_i > 0$ for all $i > i_0$. Then applying Theorem 4 with $k = t_{i_0+1}$ (if defined) yields that the probability that some party gets grade larger than g_{i_0} goes towards 0. Applying 4 to $-\mathbf{v}$ yields that the probability for parties getting a grade below g_{i_0} also goes to zero. Hence the probability of all parties getting the same grade g_{i_0} goes towards 1.

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