Ordinal Completeness of Bimodal Provability Logic GLB

Dedicated to Leo Esakia on the occasion of his 75-th birthday

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Abstract. Bimodal provability logic **GLB**, introduced by G. Japaridze, currently plays an important role in the applications of provability logic to proof-theoretic analysis. Its topological semantics interprets diamond modalities as derived set operators on a scattered bitopological space. We study the question of completeness of this logic w.r.t. the most natural space of this kind, that is, w.r.t. an ordinal α equipped with the interval topology and with the so-called club topology. We show that, assuming the axiom of constructibility, **GLB** is complete for any $\alpha \geq \aleph_\omega$. On the other hand, from the results of A. Blass it follows that, assuming the consistency of "there is a Mahlo cardinal," it is consistent with ZFC that **GLB** is incomplete w.r.t. any such space. Thus, the question of completeness of **GLB** w.r.t. natural ordinal spaces turns out to be independent of ZFC.

1 Introduction

This paper links together two topics in the study of provability logic both of which originated in Georgia. The first one is the interpretation of modal $\diamond$ of the standard provability logic **GL** in topological terms as the derived set operator on a scattered topological space. The idea of treating topological derivative as a modality goes back to McKinsey and Tarski [22], however it was Leo Esakia who linked scattered spaces with the axioms of the logic of provability (see [15, 16]). In particular, Esakia proved that **GL** was complete under this interpretation w.r.t. the class of all scattered spaces. Merab Abashidze [1] and later independently Andreas Blass [11] established the completeness of **GL** w.r.t. some natural individual scattered spaces. Thus, Abashdze and Blass showed the completeness of **GL** w.r.t. the standard interval topology on any ordinal $\alpha \geq \omega^\omega$. In addition, by an interesting combinatorial construction, Blass also showed **GL** to be complete w.r.t. another natural topology on an ordinal α, the so-called *club topology*, provided $\alpha \geq \aleph_\omega$. However, this latter result could only be proved assuming the set-theoretic axiom of constructibility (or Jensen's square principle). In fact, assuming the consistency of "there is a Mahlo cardinal," Blass also showed that

it is consistent with the axioms of ZFC that **GL** is *in*complete w.r.t. the club topology on any ordinal. Both the club topology and Blass's theorem will play a prominent role in this paper.

The second topic that emerged in Georgia is the extension of the language of **GL** by an infinite sequence of new diamond modalities $\langle 0 \rangle$, $\langle 1 \rangle$, $\langle 2 \rangle$, ... corresponding to n-consistency predicates in first-order arithmetic. (Under this interpretation, $\langle 0 \rangle$ means the same as the usual $\diamond$, i.e., consistency over a given theory.) This leads to a much more expressive polymodal provability logic **GLP**, which was first formulated and proved arithmetically complete by Giorgi Japaridze [19]. His study has later been simplified and extended by Konstantin Ignatiev [18] and George Boolos [13, 12]. More recently, **GLP** has found interesting applications in proof-theoretic analysis of arithmetic [3, 4, 2, 5] which stimulated some further interest in the study of modal-logical properties of **GLP** [14, 9, 17, 6, 7].

Unlike **GL**, which is complete w.r.t. its Kripke semantics, that is, w.r.t. the class of finite irreflexive trees, it is well-known that even the bimodal fragment of **GLP**, denoted **GLB** by Boolos, is not complete w.r.t. any class of Kripke frames. Therefore, the problem of finding a tractable complete semantics for this logic becomes more urgent. Topological semantics can be considered as a generalization of Kripke semantics, therefore it is natural to ask if there is a complete topological semantics for **GLP**.

Esakia-style topological models for **GLP** have been introduced in [8] under the name **GLP**-*spaces*. These are polytopological spaces of the form $(X; \tau_0, \tau_1, \dots)$, where modality $\langle n \rangle$ is interpreted as a derived set operator d_n on X w.r.t. topology τ_n. To satisfy all the axioms of **GLP** we must require, for each n, that

- τ_n is a scattered topology on X;
- $\tau_n \subseteq \tau_{n+1}$;
- for all $A \subseteq X$, $d_n(A)$ is τ_{n+1}-open.

GLB-*spaces* are structures $(X; \tau_0, \tau_1)$ of the same kind with only the first two topologies present. The simplest nontrivial example of a **GLB**-space is an ordinal space of the form $(\alpha; \tau_0, \tau_1)$ where τ_0 is the interval topology on an ordinal α, and τ_1 is the above mentioned club topology. In fact, τ_1 happens to be the coarsest topology such that $(\alpha; \tau_0, \tau_1)$ is a **GLB**-space, which can be considered as a definition of τ_1 (see [8]). Thus, the general notion of **GLB**-space links the two prominent ordinal topologies: the interval topology and the club topology.[1] We call such spaces *natural ordinal* **GLB**-*spaces*.

In [8], it was shown that **GLB** is complete w.r.t. the class of all **GLB**-spaces, thus, a bimodal analogue of Esakia theorem was established. However,

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[1] Blass treated the club topology in terms of its (punctured) neighborhood filters — the so-called *club filters*. His interest in this semantics was motivated by the importance of the concept of club filter in set theory. In contrast, in our situation the club topology is more or less forced on us by the general concept of **GLB**-space, that is, by the axioms of **GLB**. The fact that we come in this way to an important set-theoretic notion additionally speaks for the naturality of the concept of **GLB**-space.

two further questions were left open. Firstly, the question whether this result can be extended to the full language of **GLP**. Secondly, the question whether **GLB** is complete w.r.t. any *natural* (ordinal) **GLB**-space, that is, if an analog of Abashidze–Blass theorem holds for **GLB**. (A similar but more difficult question also makes sense for the full **GLP**.)

In this paper we answer the second question for the case of **GLB** in essentially the same sense as Blass. Namely, under the assumption of the axiom of constructibility, we show that **GLB** is complete w.r.t. any natural ordinal **GLB**-space of the form $(\alpha; \tau_0, \tau_1)$ where $\alpha \geq \aleph_\omega$. This result can be considered as an extension of both topological completeness theorems of Abashidze and Blass. In fact, the proof of our theorem relies on Blass's construction in an essential way.

The question of general topological completeness of **GLP** has recently been affirmatively answered jointly by David Gabelaia and the author of this paper. This result is based on some additional techniques and will be published separately.

2 Preliminaries and statement of main result

2.1 Axioms of GLB

We consider the language of propositional logic enriched by two modalities $[0]$ and $[1]$. Dual modalities are denoted $\langle 0 \rangle$ and $\langle 1 \rangle$ and treated as abbreviations for $\neg[0]\neg$ and $\neg[1]\neg$ respectively. The system **GLB** is given by the following axiom schemata and inference rules:

Axioms:
- (i) Boolean tautologies;
- (ii) $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$ (for $n = 0, 1$);
- (iii) $[n]([n]\varphi \to \varphi) \to [n]\varphi$ (for $n = 0, 1$);
- (iv) $[0]\varphi \to [1]\varphi$;
- (v) $\langle 0 \rangle \varphi \to [1]\langle 0 \rangle \varphi$.

Rules:
- (i) $\vdash \varphi,\ \vdash \varphi \to \psi\ \Rightarrow\ \vdash \psi$ (modus ponens);
- (ii) $\vdash \varphi\ \Rightarrow\ \vdash [n]\varphi$, for $n = 0, 1$ (necessitation).

2.2 Topological semantics

Let $(X; \theta_0, \theta_1)$ be a bitopological space. For $n = 0, 1$ let $d_n(A)$ denote the set of limit points of $A \subseteq X$ w.r.t. θ_n, that is, $x \in d_n(A)$ iff A intersects any neighborhood U of x at a point $y \neq x$. Thus, d_0 and d_1 denote the derived set operators associated with the topologies θ_0 and θ_1.

When X is fixed and $A \subseteq X$ we write $-A$ for $X - A$. We also denote $\tilde{d}_n(A) = -d_n(-A)$. The operator $\tilde{d}_n$ will interpret $[n]$, whereas d_n will interpret $\langle n \rangle$.

A *valuation on X* is a map $v : \mathrm{Form} \to \mathcal{P}(X)$ of propositional formulas to subsets of X such that

- $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$, $v(\neg \varphi) = -v(\varphi)$, etc.;
- $v(\langle n \rangle \varphi) = d_n(v(\varphi))$, $v([n]\varphi) = \tilde{d}_n(v(\varphi))$, for $n = 0, 1$.

A formula φ is *valid in* X if $v(\varphi) = X$, for any valuation v on X. This fact will be denoted $(X; \theta_0, \theta_1) \vDash \varphi$. The *logic of* $(X; \theta_0, \theta_1)$ is the set of all formulas valid in X:

$$L(X; \theta_0, \theta_1) = \{\varphi : (X; \theta_0, \theta_1) \vDash \varphi\}.$$

2.3 Ordinal GLB-spaces

We consider bitopological spaces of the form $(\kappa; \tau_0, \tau_1)$, where κ is an ordinal, τ_0 is the interval topology on κ, and τ_1 is the club topology. As usual, with the topologies τ_0 and τ_1 we associate derived set operators d_0 and d_1.

Recall that τ_0 is generated by $\{0\}$ and intervals of the form $(\alpha, \beta]$, for all $\alpha < \beta < \kappa$. Notice that $d_0(A)$ is often denoted A'. We have: $\alpha \in d_0(A)$ iff (α is a limit ordinal and $A \cap \alpha$ is unbounded in α). Topology τ_0 is *scattered*, that is, every non-empty subspace $A \subseteq \kappa$ has an isolated point; as such a point one can take the minimum of A.

By definition, the *club topology* τ_1 on κ is generated by τ_0 and all sets of the form $d_0(A)$, where $A \subseteq \kappa$. Since $\tau_0 \subseteq \tau_1$ we observe that τ_1 is also scattered.

A subset $C \subseteq \alpha$ is called a *club in* α, if C is τ_0-closed in the relative topology of α and unbounded in α. Notice that if $\alpha \in d_0(A)$ then $d_0(A) \cap \alpha$ is a club in α. If α is a limit ordinal of uncountable cofinality $\mathrm{cf}(\alpha)$, the intersection of finitely many clubs in α is again a club. It is easy to infer from this property that the limit points of τ_1 (that is, points of the set $d_1(\kappa)$) are exactly the ordinals of uncountable cofinality below κ.

Similarly, we can characterize τ_1 in terms of neighborhoods: A is a τ_1-neighborhood of α iff $\alpha \in A$ and either $\mathrm{cf}(\alpha) \leq \omega$ or A contains a club in α. As a consequence, we obtain the following characterization of derivative d_1: $\alpha \in d_1(A)$ iff ($\mathrm{cf}(\alpha) > \omega$ and $A \cap \alpha$ intersects every club C in α). A subset $A \subseteq \alpha$ is called *stationary in* α if A intersects every club in α. Thus, $\alpha \in d_1(A)$ iff ($\mathrm{cf}(\alpha) > \omega$ and $A \cap \alpha$ is stationary in α). In set theory, one usually calls d_1 *Mahlo operation* (see [20]).

Let $D_\kappa^0 = \kappa$ and D_κ^n denote $\{\alpha < \kappa : \mathrm{cf}(\alpha) \geq \aleph_n\}$ for $n > 0$. The following simple lemma is well-known and will be useful below.

Lemma 1. *For any n, $D_\kappa^n = d_1^n(\kappa)$.*

Proof. By induction, it is sufficient to show that

$$d_1(D_\kappa^n) = D_\kappa^{n+1}.$$

Suppose $\lambda \in D_\kappa^{n+1}$ and C is a club in λ. Let α be the $\aleph_n$-th element of C. Since $\aleph_n$ is regular, $\mathrm{cf}(\alpha) \geq \aleph_n$, hence $\alpha \in D_\kappa^n$. Therefore, $d_1(D_\kappa^n) \supseteq D_\kappa^{n+1}$.

Conversely, consider an ordinal $\lambda \notin D_\kappa^{n+1}$. If $\mathrm{cf}(\lambda) \leq \omega$, clearly λ is not a limit point of D_κ^n. Otherwise, $\mathrm{cf}(\lambda) < \aleph_{n+1}$, hence there is a club C in λ whose order type $\mathrm{ot}(C)$ satisfies $\aleph_1 \leq \mathrm{ot}(C) \leq \aleph_n$. Note that $d_0(C) \cap \lambda$ is a club in λ and every $\alpha \in d_0(C) \cap \lambda$ satisfies $\mathrm{cf}(\alpha) < \mathrm{ot}(C)$, hence $\mathrm{cf}(\alpha) < \aleph_n$. Therefore, $D_\kappa^n \cap d_0(C)$ is empty and λ is not a limit point of D_κ^n. $\square$

2.4 Main theorem

We are interested in characterizing the logic $L(\kappa; \tau_0, \tau_1)$ for natural ordinal **GLB**-spaces $(\kappa; \tau_0, \tau_1)$. Since τ_0 is scattered, $\tau_0 \subseteq \tau_1$ and every set of the form $d_0(A)$ is τ_1-open, it is easy to see that $(\kappa; \tau_0, \tau_1)$ is a **GLB**-space, in particular, all theorems of **GLB** are valid in $(\kappa; \tau_0, \tau_1)$.

We show that under some standard set-theoretic assumptions, for ordinals $\kappa \geq \aleph_\omega$, the space $(\kappa; \tau_0, \tau_1)$ is, in fact, complete for **GLB**. Since the completeness of **GLB** implies the completeness of **GL** w.r.t. the club topology on κ, it follows from the results of Blass [11] that some set-theoretic assumptions beyond ZFC are necessary for the completeness proof.

For any infinite cardinal κ, *Jensen's Principle* $\square_\kappa$ is the assertion that there exists a sequence of sets C_α, for limit ordinals $\alpha < \kappa^+$, with the following properties:

(i) C_α is a club in α;
(ii) If $\mathrm{cf}(\alpha) < \kappa$ then $|C_\alpha| < \kappa$;
(iii) If $\beta \in d_0(C_\alpha)$ then $C_\beta = \beta \cap C_\alpha$.

It is a well-known result of Jensen that the axiom of constructibility $V = L$ implies $\square_\kappa$, for all κ. (For $\kappa = \aleph_0$ the assertion $\square_\kappa$ trivializes and is provable in ZFC. See Jech [20] for the details on any specific set-theoretic notions and facts mentioned in this paper.)

Under the assumption of $\square_{\aleph_n}$, for every $n < \omega$, we prove the following completeness theorem.

Theorem 1 $(\square_{\aleph_n}, \textbf{ for } n < \omega)$. *For any $\kappa \geq \aleph_\omega$, $L(\kappa; \tau_0, \tau_1) = \textbf{GLB}$.*

A proof of this theorem will be obtained by combining several ingredients. We will use a reduction of **GLB** to a subsystem **J** isolated in [6], which is complete w.r.t. a nice class of finite Kripke frames called *treelike J-frames*. Then we will state a general Embedding lemma which allows to 'embed' such treelike J-frames into ordinal spaces. Thus, given a formula φ such that $\textbf{GLB} \nvdash \varphi$ we will be able to produce a valuation on $([1, \aleph_n]; \tau_0, \tau_1)$ falsifying φ. Finally, all such valuations will be merged into a single one on the space $(\aleph_\omega; \tau_0, \tau_1)$.

In Section 4 we state the Embedding lemma and infer from it Theorem 1. A proof of the Embedding lemma is postponed until Section 5. In Section 3 we formulate a lemma summarizing the key properties of Blass's construction necessary for the statement and the proof of the Embedding lemma.

3 Blass's construction

The proof of Theorem 1 will heavily rely on the previous results of Andreas Blass. The following lemma can be extracted from his paper [11].

Lemma 2 $(\square_{\aleph_n}, \textbf{ for } n < \omega)$. *For any finite tree $(T, \prec)$ of height n there is a map S associating with every $x \in T$ a nonempty subset $S_x \subseteq [1, \aleph_n]$ with the following properties:*

(i) $\{S_x : x \in T\}$ *is a partition of* $[1, \aleph_n]$;

(ii) $S_r = \{\aleph_n\}$, *if* $r \in T$ *is the root of* T;

(iii) *If* $x \prec y$ *then* $S_x \subseteq d_1(S_y)$; *in other words, if* $x \prec y$ *and* $\lambda \in S_x$ *then* $\mathrm{cf}(\lambda) > \omega$ *and* $S_y \cap \lambda$ *is stationary in* λ;

(iv) $S_x \subseteq \tilde{d}_1(\bigcup_{y \succ x} S_y)$; *in other words, if* $\lambda \in S_x$ *and* $\mathrm{cf}(\lambda) > \omega$, *then* $\bigcup_{y \succ x} S_y$ *contains a club in* λ.

Rather than being literally stated in Blass's paper, this lemma is obtained by some adaptation of the construction in his proof. Before discussing the relationship in more detail a few general remarks are in order.

Firstly, we notice that the conditions of this lemma are quite similar to those of the Solovay construction in provability logic. Therefore, in a similar manner, it is easy to infer from it the completeness of **GL** w.r.t. the topological space $(\aleph_\omega, \tau_1)$. This is one of the two main results of Blass's paper; the inference of completeness from an analogue of this lemma is essentially the content of his Theorem 3.

Secondly, with the map $S : x \longmapsto S_x$ we can associate a surjective function $f : [1, \aleph_n] \to T$ mapping every point of S_x to x. Let $(T, \prec)$ be equipped with the Alexandroff topology whose open sets are exactly the upwards closed ones w.r.t. the ordering $\prec$. Conditions (iii) and (iv) are then equivalent to f being continuous, open and pointwise discrete (the latter means that $f^{-1}\{a\}$ is discrete, for each $a \in T$). Such functions f have been introduced in [10] under the name *d-maps*. They play for our kind of topological semantics a role similar to the one of p-morphism for Kripke frames. Thus, Lemma 2 can be restated as follows.

Corollary 1 $(\square_{\aleph_n}$, **for** $n < \omega)$. *For every finite tree* $(T, \prec)$ *of depth* n *there is a surjective d-map* $f : [1, \aleph_n] \to T$.

If $f : X \to Y$ is a *d*-map and v is a valuation on Y, we can associate with it a valuation v' on X by defining, for each variable p,

$$v'(p) := \{x \in X : f(x) \in v(p)\}.$$

Then it is easy to see that $v'(\varphi) = f^{-1}(v(\varphi))$, for each formula φ. Hence, the logic of X is contained in the logic of Y. This is another way of looking at the inference of the completeness of **GL** from Lemma 2.[2]

Thus, for the applications of Lemma 2 only Conditions (i), (iii) and (iv) are substantial. Condition (ii) plays a purely technical role in Blass's paper — to make the inductive construction of the *d*-map work. In fact, from the next lemma we can conclude that Condition (ii) actually *follows* from the other three conditions.

Let $\mathrm{dp}(x)$ denote the depth of a node $x \in T$ w.r.t. the ordering $\prec$. The height of T is the depth of its root.

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[2] We can also look at the same situation algebraically. A surjective *d*-map $f : X \to Y$ induces a dual map $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by $f^*(A) := f^{-1}(A)$, which happens to be a homomorphic embedding of the modal algebra $(\mathcal{P}(Y), d_Y)$ into $(\mathcal{P}(X), d_X)$. In particular, this justifies the name 'Embedding lemma' for the results of this kind.

Lemma 3. *Let $f : [1, \aleph_n] \to (T, \prec)$ be a surjective d-map. Then, for any $\alpha \in [1, \aleph_n]$ and $k \leq n$, $\mathrm{cf}(\alpha) = \omega_k$ iff $\mathrm{dp}(f(\alpha)) = k$.*

Proof. Let d denote the derivative operation w.r.t. the Alexandroff topology of the tree $(T, \prec)$. Thus,

$$d(A) = \{x \in T : \exists y \in A \ x \prec y\}.$$

We obviously have, by the definition of depth:

$$\mathrm{dp}(x) \geq n \iff x \in d^n(T).$$

Since f is a surjective d-map, it satisfies $f^{-1}(dA) = d_1(f^{-1}(A))$, for each $A \subseteq T$. Hence, $f^{-1}(d^k T) = d_1^k[1, \aleph_n]$. In turn, from Lemma 1 for $\alpha \in [1, \aleph_n]$, we conclude that $\alpha \in d_1^k[1, \aleph_n]$ iff $\mathrm{cf}(\alpha) \geq \aleph_k$, whence the claim. $\qquad\square$

Corollary 2. *Condition (ii) follows from (i), (iii) and (iv).*

Proof. The root is the only point of T of maximal depth. $\qquad\square$

Recall that the ordinal function $f(\alpha) = \omega\alpha$ is monotone and continuous w.r.t. τ_0. It enumerates 0 and all limit ordinals in the increasing order, so it maps $X = [1, \aleph_n]$ onto $Y = [1, \aleph_n] \cap \mathrm{Lim}$. We consider $Y \subseteq X$ equipped with the topologies inherited from X. The following lemma is straightforward.

Lemma 4. $f : X \to Y$ *is a homeomorphism w.r.t. both τ_0 and τ_1.*

Proof. f is obviously bijective and continuous w.r.t. τ_0. Since f is monotone, $f(\alpha, \beta] = (f(\alpha), f(\beta)] \cap Y$. Hence, $f(\alpha, \beta]$ is open in Y. Therefore, f is open and hence a homeomorphism w.r.t. τ_0.

Since f is a homeomorphism and an order isomorphism, C is a club in α if and only if $f(C)$ is a club in $f(\alpha)$, for any $C \subseteq X$. We show that f is continuous and open w.r.t. τ_1.

Suppose U is open in Y and $f(\alpha) \in U$. We show that there is a neighborhood V of α such that $f(V) \subseteq U$. We consider two cases. If $\mathrm{cf}(f(\alpha)) = \omega$, then $\mathrm{cf}(\alpha) = \omega$, hence $\{\alpha\}$ is open and $f\{\alpha\} \subseteq U$.

If $\mathrm{cf}(f(\alpha)) > \omega$, then $U = U_0 \cap Y$ where U_0 contains a club C in $f(\alpha)$. Obviously, $C' = d_0(C) \subseteq C$ is a club in $f(\alpha)$ contained in Y. Then $f^{-1}(C')$ is a club in α, hence $V = \{\alpha\} \cup f^{-1}(C')$ is a neighborhood of α in X and $f(V) \subseteq U$.

Suppose now U is a neighborhood of $\alpha \in X$. If $\mathrm{cf}(\alpha) = \omega$, α is isolated in X and $f(\alpha)$ is isolated in Y, as required. If $\mathrm{cf}(\alpha) > \omega$ we find a club $C \subseteq U$ in α. Then $f(C)$ is a club in $f(\alpha)$, hence $f(U)$ contains a neighborhood of $f(\alpha)$ (both in X and in Y). $\qquad\square$

Remark 1. In view of the previous lemma, Lemma 2 is equivalent to the one where Condition (i) is replaced by the requirement that $\{S_x : x \in T\}$ be a partition of the set $[1, \aleph_n] \cap \mathrm{Lim}$. In fact, Blass's original construction is formulated in this way. In our proof we will need both versions of Lemma 2.

Remark 2. Another formal difference between Lemma 2 and [11] is that, rather than constructing a d-map from $[1, \aleph_n] \cap \text{Lim}$ to an arbitrary finite tree of depth n, Blass uses some explicit *universal tree* K_n of depth n and constructs a d-map from $[1, \aleph_n] \cap \text{Lim}$ to K_n. Universality here simply means that there is a surjective d-map from K_n to any finite tree T of depth n. Trees K_n are infinitely branching and somewhat cumbersome to handle, and from the point of view of the proof of Theorem 1 nothing is gained by working with K_n, so we prefer to deal with the somewhat weaker statement of Lemma 2.

Remark 3. Since the spaces $[1, \kappa]$ and $[0, \kappa]$ are obviously isomorphic for infinite ordinals κ, the reader may wonder why we prefer to begin with 1 in the statement of Lemma 2. The answer is that this innocent choice greatly simplifies some formulas in the proof of the Embedding lemma.

4 Embedding lemma for *J*-frames

4.1 *J*-frames

Similarly to [8], our proof of topological completeness will make use of a subsystem of **GLB** introduced in [6] and denoted **J**. This logic is defined by weakening axiom (iv) of **GLB** to the following axioms (vi) and (vii), both of which are theorems of **GLB**:

(vi) $[0]\varphi \rightarrow [1][0]\varphi$;
(vii) $[0]\varphi \rightarrow [0][1]\varphi$.

J is the logic of a simple class of frames, which is established by standard methods [6, Theorem 1].

Lemma 5. J *is sound and complete with respect to the class of (finite) frames* $(W; R_1, R_2)$ *such that, for all* $x, y, z \in W$,

1. R_0 *and* R_1 *are transitive and dually well-founded binary relations;*
2. *If* xR_1y, *then* xR_0z *iff* yR_0z;
3. xR_0y *and* yR_1z *imply* xR_0z.

   If we let $\overline{R_1}$ denote the reflexive, symmetric, transitive closure of R_1, then we call each $\overline{R_1}$ equivalence class a *1-sheet*. By *2.*, all points in a 1-sheet are R_0 incomparable. But R_0 defines a natural ordering on the set of 1-sheets: if α and β are 1-sheets, then $\alpha R_0 \beta$, iff $\exists x \in \alpha$, $\exists y \in \beta$, xR_0y. By standard techniques, one can improve on Lemma 5 to show that **J** is complete for such frames, in which each 1-sheet is a tree under R_1, and if $\alpha R_0 \beta$ then xR_0y for all $x \in \alpha$, $y \in \beta$ (see [6, Theorem 2 and Corollary 3.3]). Thus, models of **J** can be seen as R_0-orders (and even tree-like orders), in which the nodes are 1-sheets that are themselves R_1-trees. We call such frames *tree-like J-frames*. The R_1-*height* of such a frame is the maximum height of its 1-sheets.

As shown in [6], **GLB** is reducible to **J** in the following sense. Let

$$M(\varphi) := \bigwedge_{i<s} ([0]\varphi_i \to [1]\varphi_i),$$

where $[0]\varphi_i$, $i < s$, are all subformulas of φ of the form $[0]\psi$. Also, let

$$M^+(\varphi) := M(\varphi) \wedge [0]M(\varphi) \wedge [1]M(\varphi).$$

We have: **GLB** $\vdash \varphi$ iff **J** $\vdash M^+(\varphi) \to \varphi$. In fact, below we only use the obvious part of this reduction, the implication from right to left.

4.2   Embedding lemma

For a binary relation R on X let $R(x)$ denote the set $\{y \in X : xRy\}$.

Lemma 6 (Embedding). *Let $\mathcal{W} = (W; R_0, R_1)$ be a finite tree-like J-frame with the root r and R_1-height n. There is an ordinal $\kappa < \aleph_{n+1}$ and a map S associating with every $x \in W$ a nonempty subset $S_x \subseteq [1, \kappa]$ with the following properties:*

1. *$\{S_x : x \in T\}$ is a partition of $[1, \kappa]$; $S_r = \{\kappa\}$;*
2. *If xR_1y then $S_x \subseteq d_1(S_y)$; in other words, if xR_1y and $\lambda \in S_x$ then $\mathrm{cf}(\lambda) > \omega$ and $S_y \cap \lambda$ is stationary in λ;*
3. *If xR_0y then $S_x \subseteq d_0(S_y)$; in other words, if xR_0y and $\lambda \in S_x$ then $\lambda \in \mathrm{Lim}$ and $S_y \cap \lambda$ is unbounded in λ;*
4. *$S_x \subseteq \tilde{d}_1(\bigcup_{y \in R_1(x)} S_y)$; in other words, if $\lambda \in S_x$ and $\mathrm{cf}(\lambda) > \omega$, then $\bigcup_{y \in R_1(x)} S_y$ contains a club in λ;*
5. *$S_x \subseteq \tilde{d}_0(\bigcup_{y \in \widetilde{R}_0(x)} S_y)$; in other words, if $\lambda \in S_x$ and $\lambda \in \mathrm{Lim}$, then $\bigcup_{y \in \widetilde{R}_0(x)} S_y$ contains an end-segment of λ, where $\widetilde{R}_0(x) := R_0(x) \cup R_1(x_1)$ and x_1 denotes the root of the 1-sheet of x.*

This lemma gives us a surjective function $f : [1, \kappa] \to W$ defined by $f^{-1}(x) = S_x$. Such a function is a d-map w.r.t. τ_1, open w.r.t. τ_0, but is not, in general, τ_0-continuous. Nonetheless, these conditions turn out to be sufficient for the completeness proof. We shall call such functions *weak d-maps*. A proof of this lemma will be given in Section 5.

Now we are ready to infer Theorem 1 from the Embedding lemma.

Proof. Suppose **GLB** $\nvdash \varphi$. Then, obviously, **J** $\nvdash M^+(\varphi) \to \varphi$. Let $\mathcal{W}$ be a finite tree-like J-model with the root r such that $\mathcal{W} \vDash M^+(\varphi)$ and $\mathcal{W} \nvDash \varphi$. Let S be a map given by Lemma 6. Define a valuation v on $[1, \kappa]$ by $v(p) := \bigcup_{x \vDash p} S_x$, for each propositional variable p.

Lemma 7. *For all subformulas θ of φ we have:*

$$v(\theta) = \bigcup_{x \vDash \theta} S_x.$$

Proof. Induction on the build-up of θ. If θ is a variable, we refer to the definition of v. The cases $\theta = \psi_1 \wedge \psi_2$ and $\theta = \neg\psi$ follow from the induction hypothesis and the fact that S_x partition $[1, \kappa]$.

Let $\theta = [1]\psi$. If $\mathcal{W}, x \vDash \theta$ then $\forall y \in R_1(x) \ \mathcal{W}, y \vDash \psi$. Hence, by the induction hypothesis, $\forall y \in R_1(x) \ S_y \subseteq v(\psi)$, that is, $\bigcup_{y \in R_1(x)} S_y \subseteq v(\psi)$. By Lemma 6, Part 4,

$$S_x \subseteq \tilde{d}_1(\textstyle\bigcup_{y \in R_1(x)} S_y) \subseteq \tilde{d}_1(v(\psi)).$$

This shows $\bigcup_{x \vDash \theta} S_x \subseteq v(\theta)$.

If $\mathcal{W}, x \nvDash \theta$ then $\exists y \in R_1(x) \ \mathcal{W}, y \nvDash \psi$, and by the induction hypothesis $\exists y \in R_1(x) \ S_y \subseteq -v(\psi)$. By Lemma 6, Part 2,

$$S_x \subseteq d_1(S_y) \subseteq d_1(-v(\psi)) = -v(\theta).$$

This holds for all x such that $\mathcal{W}, x \nvDash \theta$, hence $v(\theta) \subseteq \bigcup_{x \vDash \theta} S_x$.

Let $\theta = [0]\psi$. If $\mathcal{W}, x \nvDash \theta$ then $\exists y \in R_0(x) \ \mathcal{W}, y \nvDash \psi$, and by the induction hypothesis $\exists y \in R_0(x) \ S_y \subseteq -v(\psi)$. By Lemma 6, Part 3,

$$S_x \subseteq d_0(S_y) \subseteq d_0(-v(\psi)) = -v(\theta).$$

This holds for all x such that $\mathcal{W}, x \nvDash \theta$, hence $v(\theta) \subseteq \bigcup_{x \vDash \theta} S_x$.

If $\mathcal{W}, x \vDash \theta$ then $\forall y \in R_0(x) \ \mathcal{W}, y \vDash \psi$. Moreover, since $\mathcal{W}$ is a J-frame, we also have $\mathcal{W}, x_1 \vDash [0]\psi$ (this is because $R_0(x) = R_0(x_1)$ in a J-frame). Since $\mathcal{W} \vDash M^+(\varphi)$ we have, in particular, $\mathcal{W}, x_1 \vDash [0]\psi \to [1]\psi$ and hence $\mathcal{W}, x_1 \vDash [1]\psi$. Therefore, $\forall y \in R_1(x_1) \ \mathcal{W}, y \vDash \psi$. So, we have $\forall y \in \widetilde{R_0}(x) \ \mathcal{W}, y \vDash \psi$ and by the induction hypothesis $\bigcup_{y \in \widetilde{R_0}(x)} S_y \subseteq v(\psi)$. By Lemma 6, Part 5,

$$S_x \subseteq \tilde{d}_0(\textstyle\bigcup_{y \in \widetilde{R_0}(x)} S_y) \subseteq \tilde{d}_0(v(\psi)) = v(\theta).$$

This holds for all x such that $\mathcal{W}, x \vDash \theta$, hence $\bigcup_{x \vDash \theta} S_x \subseteq v(\theta)$. $\square$

Since $\mathcal{W} \nvDash \varphi$, from the above lemma we conclude that $S_x \nsubseteq v(\varphi)$, hence $v(\varphi) \neq [1, \kappa]$. This shows that **GLB** is the intersection of logics of all **GLB**-spaces $([1, \kappa], \tau_0, \tau_1)$, for $\kappa < \aleph_\omega$. To infer from this the stronger claim of the theorem, we need two additional general lemmas.

Let $(X_i)_{i \in I}$ be a family of **GLB**-spaces. The disjoint union $\bigsqcup_{i \in I} X_i$ of this family is defined in a natural way: τ_n-open sets $A \subseteq \bigsqcup_{i \in I} X_i$ are those for which $A \cap X_i$ is τ_n-open, for each $i \in I$. The following lemma is obvious.

Lemma 8. *Suppose $f_i : X_i \to \mathcal{W}_i$ is a family of weak d-maps. Let $f : \bigsqcup_{i \in I} X_i \to \bigsqcup_{i \in I} \mathcal{W}_i$ be such that $f \restriction X_i = f_i$, for each $i \in I$. Then f is a weak d-map.*

Lemma 9. *Let $(\kappa_\alpha)_{\alpha < \lambda}$ be a family of non-zero ordinals and let $\kappa = \sum_{\alpha < \lambda} \kappa_\alpha$. Let τ_0 and τ_1 be the interval and the club topology on κ, respectively. Then **GLB**-space $([1, \kappa), \tau_0, \tau_1)$ is isomorphic to $\bigsqcup_{\alpha < \lambda}([1, \kappa_\alpha], \tau_0, \tau_1)$.*

Proof. This is a consequence of two facts: 1) An interval $(\alpha, \alpha + \beta]$ is isomorphic to $[1, \beta]$ both w.r.t. the order and the club topology; 2) The intervals $[1, \kappa_0]$ and $(\kappa_\alpha, \kappa_{\alpha+1}]$, for all $\alpha < \lambda$, are clopen w.r.t. both τ_0 and τ_1, and they partition $[1, \kappa)$. □

Consider now some enumeration of all finite tree-like J-models $(\mathcal{W}_i)_{i < \omega}$. Let $f_i : [1, \kappa_i] \to \mathcal{W}_i$ be some surjective weak d-maps given by the Embedding Lemma. Applying the previous two lemmas, we obtain a surjective weak d-map $f : [1, \kappa) \to \bigsqcup_{i < \omega} \mathcal{W}_i$, where $\kappa := \sum_{i < \omega} \kappa_i$. Hence, every formula falsified on some $\mathcal{W}_i$ is falsifiable on $[1, \kappa)$. Since $\kappa_i < \aleph_\omega$, for each i, we have $\kappa \leq \aleph_\omega$. From Lemma 3 and the fact that the sequence $\mathcal{W}_i$ contains J-frames of arbitrary finite R_1-depth one can infer that $\kappa \geq \aleph_\omega$, hence $\kappa = \aleph_\omega$. It follows that **GLB** is complete for $\aleph_\omega$ and therefore for any $\kappa \geq \aleph_\omega$. □

Thus, we have demonstrated that Theorem 1 follows from Lemma 6. This proof actually delivers slightly more than is stated in Theorem 1.

Recall that the *free* ***GLB****-algebra* on a set of generators V can be defined as the Lindenbaum algebra of **GLB** in the vocabulary V, that is, the modal boolean algebra of all formulas in the language of **GLB** (in variables V) modulo provable equivalence in **GLB** (see [8]). We have the following result, which is already of some interest when V is countable or even finite.

Theorem 2 ($\square_{\aleph_n}$, **for** $n < \omega$)**.** *Suppose* $|V| < \aleph_\omega$. *Then the free* ***GLB****-algebra on generators* V *is embeddable into the algebra of the* ***GLB****-space* $(\kappa; \tau_0, \tau_1)$, *for any* $\kappa \geq \aleph_\omega$.

Proof. We only sketch an easy proof. Let $\lambda = \max(\aleph_0, |V|)$. Enumerate all finite J-models in the vocabulary V of R_1-height n in a sequence of type λ and denote their disjoint union $\mathcal{W}_n$. As above, construct a surjective weak d-map

$$f_n : [1, \kappa_n] \to \mathcal{W}_n,$$

where $\kappa_n \leq \aleph_{n+1} \cdot \lambda$ (ordinal multiplication). Then, using Lemma 9, join all these maps f_n for all $n < \omega$ into a surjective weak d-map $f : [1, \aleph_\omega) \to \mathcal{W}$, where $\mathcal{W}$ is the union of all finite J-models in the vocabulary V. To define the embedding of algebras, map any formula φ to the set $\{\alpha \in [1, \aleph_\omega) : \mathcal{W}, f(\alpha) \vDash \varphi\}$. The analog of Lemma 7 shows that this is, indeed, an embedding. □

Notice that this theorem is an analog of the so-called uniform Solovay theorem in provability logic (see [2]). The rest of the paper is devoted to the proof of Lemma 6.

5   Proof of Embedding Lemma

We argue by induction on the R_0-height of $\mathcal{W}$.

BASIS. If R_0-height of $\mathcal{W}$ is 0 then $\mathcal{W}$ consists of a single 1-sheet ordered by R_1 with a root r. By Lemma 2 there is a κ and a surjective d-map $([1, \kappa], \tau_1) \to \mathcal{W}$.

For the associated map S conditions 1, 2, 4 hold by definition of d-map. Condition 3 holds trivially (its premise is always false). Condition 5 holds, because $S_r = \{\kappa\}$ and hence

$$\bigcup_{y \in \widetilde{R}_0(x)} S_y = \bigcup_{y \in R_1(r)} S_y = [1, \kappa),$$

which contains an end-segment of κ and of any limit $\lambda < \kappa$.

INDUCTION STEP. Since $\mathcal{W}$ is tree-like, it consists of the root 1-sheet $\mathcal{W}_0$ (with its R_1-root r_0) and of the immediate subframes $\mathcal{W}_1$, ..., $\mathcal{W}_n$. By the induction hypothesis we have embeddings S^i of $\mathcal{W}_i$ to $[1, \kappa_i]$, for $i = 1, \ldots, n$. By Lemma 2 we also obtain a d-map $f_0 : ([1, \kappa_0] \cap \mathrm{Lim}, \tau_1) \to \mathcal{W}_0$ and the associated embedding S^0. We set:

$$\kappa := (\kappa_1 + \cdots + \kappa_n) \cdot \kappa_0.$$

Define a partition $\{S_x : x \in \mathcal{W}\}$ as follows:

$$S_x := \{(\kappa_1 + \cdots + \kappa_n) \cdot \alpha : \alpha \in S_x^0\}, \quad \text{if } x \in \mathcal{W}_0; \tag{1}$$

$$S_x := \{(\kappa_1 + \cdots + \kappa_n) \cdot \alpha + \kappa_1 + \cdots + \kappa_i + \beta : \alpha < \kappa_0, \beta \in S_x^{i+1}\}, \tag{2}$$
$$\text{if } x \in \mathcal{W}_{i+1} \text{ and } 0 \le i < n.$$

Denote:

$$\delta := \kappa_1 + \cdots + \kappa_n,$$
$$\delta_i := \kappa_1 + \cdots + \kappa_i.$$

Obviously, sets of the form (1) partition the set $\{\delta\alpha : \alpha \in \mathrm{Lim} \cap [1, \kappa_0]\}$, moreover $S_{r_0} = \{\kappa\}$. Sets of the form (2) partition

$$\{\delta\alpha + \delta_i + \beta : \alpha < \kappa_0, \beta \in [1, \kappa_{i+1}]\}.$$

For different i these sets do not intersect and their union is

$$\{\delta\alpha + \delta : \alpha < \kappa_0\} = \{\delta\alpha : \alpha \in \mathrm{Suc} \cap [1, \kappa_0]\}.$$

This yields Condition 1.

Condition 2. Suppose $\lambda \in S_x$, xR_1y. If $x \in \mathcal{W}_{i+1}$ then $\lambda = \delta\alpha + \delta_i + \lambda'$, where $\lambda' \in S_x^{i+1}$. Since S_y^{i+1} is stationary in λ' the set $\{\delta\alpha + \delta_i + \beta : \beta \in S_y^{i+1}\}$ is stationary in λ. Then S_y is also stationary in λ. For the same reason $\mathrm{cf}(\lambda) = \mathrm{cf}(\lambda') > \omega$.

If $x \in \mathcal{W}_0$ then $\lambda = \delta\lambda'$ with $\lambda' \in S_x^0$. We claim that $\{\delta\alpha : \alpha \in S_y^0\}$ is stationary in λ.

Consider any club C in λ. Since $D := \{\delta\alpha : \alpha < \lambda'\}$ is a club in λ, $C \cup D$ is also a club. Hence, $C_1 := \{\alpha : \delta\alpha \in C\}$ is a club in λ'. Therefore, $C_1 \cap S_y^0 \ne \varnothing$, so there is an $\alpha \in S_y^0$ such that $\delta\alpha \in C$.

Condition 3. Suppose $\lambda \in S_x$, xR_0y. If $x \in \mathcal{W}_{i+1}$ then $\lambda = \delta\alpha + \delta_i + \lambda'$, where $\lambda' \in S_x^{i+1}$ and $y \in \mathcal{W}_{i+1}$. Then, clearly $\{\delta\alpha + \delta_i + \beta : \beta \in S_y^{i+1}\}$ is unbounded in λ.

If $x \in \mathcal{W}_0$ and $y \in \mathcal{W}_{i+1}$, then $\lambda = \delta\lambda'$ with $\lambda' \in S_x^0$, and therefore $\lambda' \in \mathrm{Lim}$. Select a $\beta_0 \in S_y^{i+1}$. The set $\{\delta\alpha + \delta_i + \beta_0 : \alpha < \lambda'\}$ is unbounded in λ and is contained in S_y.

Condition 4. Suppose $\lambda \in S_x$, $\mathrm{cf}(\lambda) > \omega$. If $x \in \mathcal{W}_{i+1}$ then $\lambda = \delta\alpha + \delta_i + \lambda'$, where $\lambda' \in S_x^{i+1}$. The set $\bigcup_{y \in R_1(x)} S_y^{i+1}$ contains a club in λ'. Then

$$\{\delta\alpha + \delta_i + \beta : \beta \in \bigcup_{y \in R_1(x)} S_y^{i+1}\}$$

contains a club in λ. This set equals

$$\bigcup_{y \in R_1(x)} \{\delta\alpha + \delta_i + \beta : \beta \in S_y^{i+1}\},$$

which is contained in $\bigcup_{y \in R_1(x)} S_y$.

If $x \in \mathcal{W}_0$ then $\lambda = \delta\lambda'$ with $\lambda' \in S_x^0$. The set $\bigcup_{y \in R_1(x)} S_y^0$ contains a club in λ'. Then $\{\delta\beta : \beta \in \bigcup_{y \in R_1(x)} S_y^0\}$ contains a club in λ. But this set equals

$$\bigcup_{y \in R_1(x)} \{\delta\beta : \beta \in S_y^0\} = \bigcup_{y \in R_1(x)} S_y.$$

Condition 5. Suppose $\lambda \in S_x$, $\lambda \in \mathrm{Lim}$. If $x \in \mathcal{W}_0$ then $\lambda = \delta\lambda'$ with $\lambda' \in S_x^0$. Since $\widetilde{R_0}(x) = \mathcal{W} - \{r_0\}$ we have

$$\bigcup_{y \in \widetilde{R_0}(x)} S_y = [1, \kappa).$$

Since $\lambda \leq \kappa$ the set $\bigcup_{y \in \widetilde{R_0}(x)} S_y$ contains an end-segment of λ.

If $x \in \mathcal{W}_{i+1}$ then $\lambda = \delta\alpha + \delta_i + \lambda'$, where $\lambda' \in S_x^{i+1}$. We have that $\bigcup_{y \in \widetilde{R_0}(x)} S_y^{i+1}$ contains an end-segment of λ' and $\widetilde{R_0}(x) \subseteq \mathcal{W}_{i+1}$. Then

$$\{\delta\alpha + \delta_i + \beta : \beta \in \bigcup_{y \in \widetilde{R_0}(x)} S_y^{i+1}\}$$

contains an end-segment of λ. But this set is contained in $\bigcup_{y \in \widetilde{R_0}(x)} S_y$. □

6 Concluding remarks

We have shown that the completeness of **GLB** is consistent with the axioms of ZFC (provided ZFC itself is consistent). The fact that the *in*completeness of

GLB is consistent with ZFC directly follows from the results of Blass (which, in turn, rely on some deep results of Harrington and Shelah concerning stationary reflection). We do not have to go deeply into set theory here, as it is well-known that **GLB** is conservative over the axioms of **GL** for the language restricted to any individual modality. Hence, the completeness of **GLB** implies the completeness of **GL** under the interpretation w.r.t. the club topology. Assuming that there are models of ZFC with Mahlo cardinals, Blass has shown that there is a model of ZFC in which **GL** is incomplete w.r.t. the club topology. Hence, the same model demonstrates the incompleteness of **GLB** w.r.t. the considered interpretation.

In the other direction, following [11, Theorem 6], we remark that by a result of Jensen [21] the negation of $\Box_\kappa$ implies that κ^+ is Mahlo in L. Hence, by Theorem 1, if **GLB** is incomplete there are Mahlo cardinals in L. Thus, we obtain the second part of the following corollary.

Corollary 3. (i) *The statement that **GLB** is complete w.r.t. the class of spaces of the form $(\kappa; \tau_0, \tau_1)$ is equiconsistent with* ZFC*;*

(ii) *The statement that **GLB** is incomplete w.r.t. the class of spaces of the form $(\kappa; \tau_0, \tau_1)$ is equiconsistent with the existence of Mahlo cardinals.*

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