TISCHLER GRAPHS OF CRITICALLY FIXED RATIONAL MAPS AND THEIR APPLICATIONS

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Abstract. A rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere $\widehat{\mathbb{C}}$ is called critically fixed if each critical point of f is fixed under f. In this article we study properties of a combinatorial invariant, called Tischler graph, associated with such a map f. More precisely, we show that the Tischler graph of a critically fixed rational map is always connected, establishing a conjecture made by Kevin Pilgrim. We also discuss the relevance of this result for classical open problems in holomorphic dynamics, such as combinatorial classification problem and global curve attractor problem.

1. Introduction

Fix an integer $d \geq 2$, and let $\mathrm{Rat}_d[\mathbb{C}]$ be the space of all rational maps of degree d with complex coefficients. Each function $f \in \mathrm{Rat}_d[\mathbb{C}]$ may be viewed as a self-map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the Riemann sphere $\widehat{\mathbb{C}}$. For $n \in \mathbb{N}$, we define $f^n = f \circ f \circ \cdots \circ f$ to be the n-fold composition of f with itself, called the *n-th iterate of f*. The iterates of f yield a holomorphic dynamical system on $\widehat{\mathbb{C}}$. One of the most important open problems in holomorphic dynamics is to describe (or distinguish) the different maps within a particular family of rational maps in combinatorial terms with the ultimate goal of better understanding the structure of the space $\mathrm{Rat}_d[\mathbb{C}]$.

For an $f \in \mathrm{Rat}_d[\mathbb{C}]$, we denote by C_f the set of critical points of f, that is, points $z \in \widehat{\mathbb{C}}$ at which f is not locally injective. By the Riemann-Hurwitz formula, f has $2d-2$ critical points when counted with multiplicity. We also denote by $P_f := \bigcup_{n=1}^{\infty} f^n(C_f)$ the *postcritical set* of f. The rational map f is said to be *postcritically-finite* if $\#P_f < \infty$, that is, each critical point has finite orbit. Over 100 years ago, Fatou and Julia established the fact that the global dynamical behavior of f is controlled by the forward orbits of the critical points of f, see for instance [Mil06]. The set of fixed points of f is denoted by $\mathrm{fix}(f)$. From the holomorphic fixed point formula, $\#\,\mathrm{fix}(f) = d+1$, if counted with multiplicity.

In this article, we study properties of *critically fixed rational maps*, that is, rational maps $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for which $C_f \subset \mathrm{fix}(f)$, so that each critical point of f is also a fixed-point of f.

The class of critically fixed rational maps is very special: for every degree d there are only finitely many critically fixed rational maps of degree d up to conformal conjugation. Also, they are obviously postcritically-finite. At the same time, exceptional properties of critically fixed rational maps allow to elegantly answer many open dynamical questions for them. For instance, Tischler provided a complete combinatorial classification of critically fixed polynomials in terms of certain planar embedded connected trees in [Tis89, Theorem 4.2]. Recently, Cordwell et. al. attempted to extend Tischler's considerations to the case of general critically fixed rational maps in [CGN$^+$15]. However, their was not complete. In this article, we provide the main missing ingredient, which addresses the properties of a certain graph naturally associated to each critically fixed rational map, complete the classification, and discuss its applications for various open problems in holomorphic dynamics. The article is based on the PhD thesis of the author [Hlu17]. Before giving more details about the main results we introduce some definitions.

Tischler graphs. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a critically fixed rational map and $c \in C_f$ be a fixed critical point of f. The *basin of attraction of c* is the set

$$B_c := \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = c\}.$$

Date: April 10, 2019.

The connected component of B_c containing the point c is called the *immediate basin* of c and denoted by Ω_c. It follows from [Mil06, Theorem 9.3] that Ω_c is a simply connected open set. Moreover, there exists a conformal map $\tau_c : \mathbb{D} \to \Omega_c$ and a number $d_c \in \mathbb{N}$ such that

$$(\tau_c \circ f \circ \tau_c^{-1})(z) = z^{d_c}$$

for all z in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The number d_c is unique and is equal to the local degree of f at c, that is, $d_c - 1$ is the multiplicity of the critical point c. Furthermore, the conformal map τ_c extends to a continuous and surjective map $\tau_c : \overline{\mathbb{D}} \to \overline{\Omega_c}$ between the closures. We call the conformal map $\tau_c : \mathbb{D} \to \Omega_c$ (and its extension $\tau_c : \overline{\mathbb{D}} \to \overline{\Omega_c}$) the *Böttcher map* of the immediate basin Ω_c. An *internal ray of angle* $\theta \in [0, 2\pi)$ is the image of the radial arc $r(\theta) := \{te^{i\theta} : t \in [0, 1]\}$ under the Böttcher map τ_c. The point $\tau_c(e^{i\theta}) \in \partial\Omega_c$ is called the *landing point* of the internal ray of angle θ. Note that the internal ray of angle θ is fixed under f if and only if $\theta = 2\pi \frac{j}{d_c - 1}$ for some $j \in \{0, \dots, d_c - 2\}$.

The *Tischler graph* of a critically fixed rational map f is the planar embedded graph $\mathrm{Tisch}(f)$ whose edge set consists of the fixed internal rays in the immediate basins of all critical points of f and vertex set consists of the endpoints of these rays. That is, as a subset of $\widehat{\mathbb{C}}$, $\mathrm{Tisch}(f)$ is the union of all fixed internal rays constructed in the previous paragraph.

Now, we are ready to formulate one of the main results of this article.

Theorem 1. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a critically fixed rational map with* $\deg(f) \geq 2$. *Then the Tischler graph* $\mathrm{Tisch}(f)$ *is connected.*

The above statement was conjectured by Kevin Pilgrim and is crucial for combinatorial classification of critically fixed rational maps and has other important applications that we discuss below.

Combinatorial classification problem. One of the main open problems in holomorphic dynamics asks to find a combinatorial description of all postcritically-finite rational maps in terms of finite data. More precisely, given a postcritically-finite rational map, one first wants to assign a certain certificate or model to it. Then, one wants to describe the arising models, and determine whether there is a one-to-one correspondence between a class of postcritically-finite rational maps and a catalog of models, that is, provide a *classification* of the maps from the class.

First combinatorial models, given by finite invariant graphs, were constructed for polynomial maps by Douady and Hubbard in the 1980's [DH84]. Later these models were used to classify all postcritically-finite polynomials [BFH92, Poi10]. At the moment, there are very few classification results and typically they are quite tedious. Besides the polynomial case mentioned earlier, see also various classification results for Newton maps in [LMS15, DMRS18, Hea88].

Completing the work started in [CGN$^+$15], we prove the following result.

Theorem 2. *There is a bijection between the conformal conjugacy classes of critically fixed rational maps and the isomorphism classes of planar embedded connected graphs without loops.*

Global curve attractor problem. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically-finite rational map. We denote by $\mathscr{C}(f)$ the set of all simple closed curves in $\widehat{\mathbb{C}} \setminus P_f$. The map f defines a natural *pullback operation* on the elements of $\mathscr{C}(f)$: a *pullback* of a curve $\gamma \in \mathscr{C}(f)$ under f is a connected component of $f^{-1}(\gamma)$.

The *global curve attractor problem* asks the following question: Given a postcritically-finite rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, is there a finite set $\mathscr{A}(f)$ of simple closed curves in $\widehat{\mathbb{C}} \setminus P_f$ such that, for every curve $\gamma \in \mathscr{C}(f)$, each pullback δ of γ under f^n is contained in $\mathscr{A}(f)$ up to an isotopy relative to P_f, for every sufficiently large $n \in \mathbb{N}$? The minimal set $\mathscr{A}(f)$ of such curves, if it exists, is called the *global curve attractor* of f.

It is conjectured that a finite global curve attractor exists for all postcritically-finite rational maps with a *hyperbolic orbifold*.[1] Pilgrim provides a sufficient condition for existence of a finite

[1]A generic postcritically-finite rational map f has hyperbolic orbifold, for instance, if $\#P_f \geq 5$. The postcritically-finite rational maps that do not have hyperbolic orbifold, that is, they have *parabolic* orbifold, are very special and well-understood, see a discussion in [DH93].

global curve attractor in [Pil12, Theorem 1.4]. In the same paper, he suggests an analytic method to prove existence of a finite global curve attractor by looking at contracting properties of a map on moduli space. In particular, Pilgrim shows that quadratic polynomials whose finite critical point is periodic have a finite global curve attractor [Pil12, Corollary 7.2].

In this article we provide a positive answer to the global curve attractor problem for critically fixed rational maps.

Theorem 3. *Each critically fixed rational map has a finite global curve attractor.*

To the best of our knowledge, besides quadratic polynomials whose finite critical point is periodic and some specific examples of rational maps (see [Pil12] and [Lod13]), critically fixed rational maps provide the only explicit family of rational maps (of arbitrary degree) for which the global curve attractor problem is solved.

We also note that lifting properties of the essential simple closed curves play a crucial role in understanding the properties of the *Thurston pullback map*, see [Lod13] for the details.

Further applications. Since each edge of the Tischler graph $\mathrm{Tisch}(f)$ of a critically fixed rational map f is invariant under f, an arbitrary spanning tree in $\mathrm{Tisch}(f)$ is an f-invariant planar embedded tree. Consequently, we have the following result.

Corollary 4. *Let f be a critically fixed rational map. Then there exists a finite planar embedded tree with the vertex set containing $P_f = C_f$ that is invariant under f.*

The invariant tree as above allows one to deeply understand the mapping properties of a critically fixed rational map f. In particular, it defines a *one-tile subdivision rule* for the dynamics of f in the sense of Cannon-Floyd-Parry [CFP01]. Furthermore, it appears to be very useful for computation of the *iterated monodromy group* as discussed in [Hlu17]. Iterated monodromy groups were introduced by Nekrashevych in 2001 as groups that are naturally associated to certain dynamical systems, in particular, to a rational maps whose critical points have finite orbits, see a precise definition in [Nek05]. The iterated monodromy group contains all the essential information about the dynamics: one can reconstruct from it the action of the map on its Julia set. An invariant tree as in Corollary 4 drastically simplifies the computations of iterated monodromy group action of critically fixed rational maps carried in [CGN+15]. Furthermore, it allows to show that the iterated monodromy groups of critically fixed rational maps have quite "exotic" properties from the point of view of classical group theory. More precisely, the following result is proven in [Hlu17, Corollary 5.6.6 and Corollary 6.4.3].

Theorem 5. *The iterated monodromy group of a critically fixed rational map f with $\#C_f \geq 3$ is an amenable group of exponential growth.*

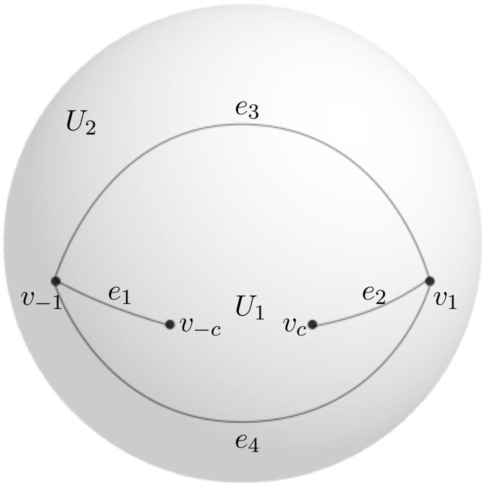
Very little is known about properties of iterated monodromy groups of general rational maps. In this way, critically fixed rational maps form a quite large class of maps for which we have a good understanding of the properties of iterated monodromy groups, see a discussion of the subject in [Hlu17].

Structure of the article. First, we review some graph theoretical notions that we use in this article in Section 2. Then, we provide an example of a critically fixed rational map and its Tischler graph in Section 3. The connectivity of Tischler graphs, that is, Theorem 1, is proven in Section 4. We discuss the combinatorial classification of critically fixed rational maps and prove Theorem 2 in Section 5. Finally, we show Theorem 3 that answers the global curve attractor problem for critically fixed rational maps in Section 6.

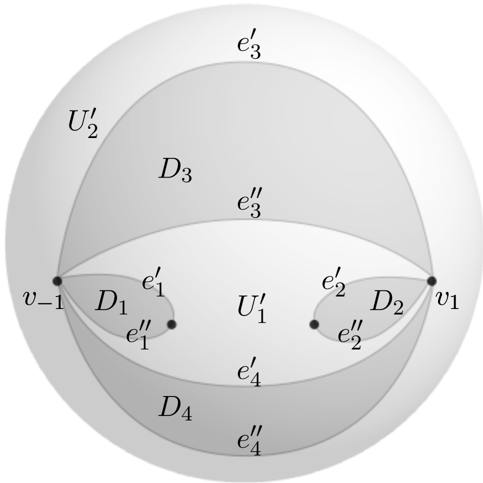
Notation. The cardinality of a set S is denoted by $\#S$.

The Riemann sphere is denoted by $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. For a subset $U \subset \widehat{\mathbb{C}}$, we denote by $\overline{U}$, $\mathrm{int}(U)$, and ∂U the topological closure, the interior, and the boundary of U, respectively.

Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. We denote by $\deg(f)$ the degree of f. The n-th iterate of f is denoted by f^n, for $n \in \{1, 2, 3, \dots\}$. The sets of fixed points, critical points, and postcritical points of f are denoted by $\mathrm{fix}(f)$, C_f, and P_f, respectively. Suppose a subset $U \subset \widehat{\mathbb{C}}$ is given.



(a)



(b)

FIGURE 1. A planar embedded graph G (left) and its blow up graph G' (right).

We denote by $f^{-n}(U)$ the preimage of U under f^n, that is, $f^{-n}(U) = \{z \in \widehat{\mathbb{C}} : f^n(z) \in U\}$. For simplicity, we denote $f^{-n}(z) :- f^{-n}(\{z\})$ for $z \in \widehat{\mathbb{C}}$. Also we denote the restriction of f to U by $f|U$.

2. Planar embedded graphs

The goal of this section is to define planar embedded graphs, introduce related constructions, and setup the notation.

Formally, a *planar embedded graph* G (without loops) is a pair (V, E), where V is a finite set of points in the Riemann sphere $\widehat{\mathbb{C}}$ and E is a finite set of Jordan arcs in $\widehat{\mathbb{C}}$, such that

(1) for each $e \in E$, both endpoints of e are in V;
(2) for each $e, e' \in E$ with $e \neq e'$, the interiors $\operatorname{int}(e)$ and $\operatorname{int}(e')$ are disjoint.

The sets V and E are called the *vertex* and *edge* sets of G, respectively. Note that our notion of a planar embedded graph allows *multiple edges*, that is, distinct edges that connect the same pair of vertices.

Let $\mathsf{G} = (V, E)$ and $\mathsf{G}' = (V', E')$ be two planar embedded graphs. We say that G is isomorphic to G' if there exists an orientation-preserving homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that it maps the vertices and edges of G to the vertices and edges of G', that is, $\phi(v) \in V'$ and $\phi(e) \in E'$, for all $v \in V$ and $e \in E$. Clearly, isomorphisms induce an equivalence relation on the set of planar embedded graphs. An equivalence class of this relation is called an *isomorphism class of planar embedded graphs*.

In the following, we assume that $\mathsf{G} = (V, E)$ is a planar embedded graph.

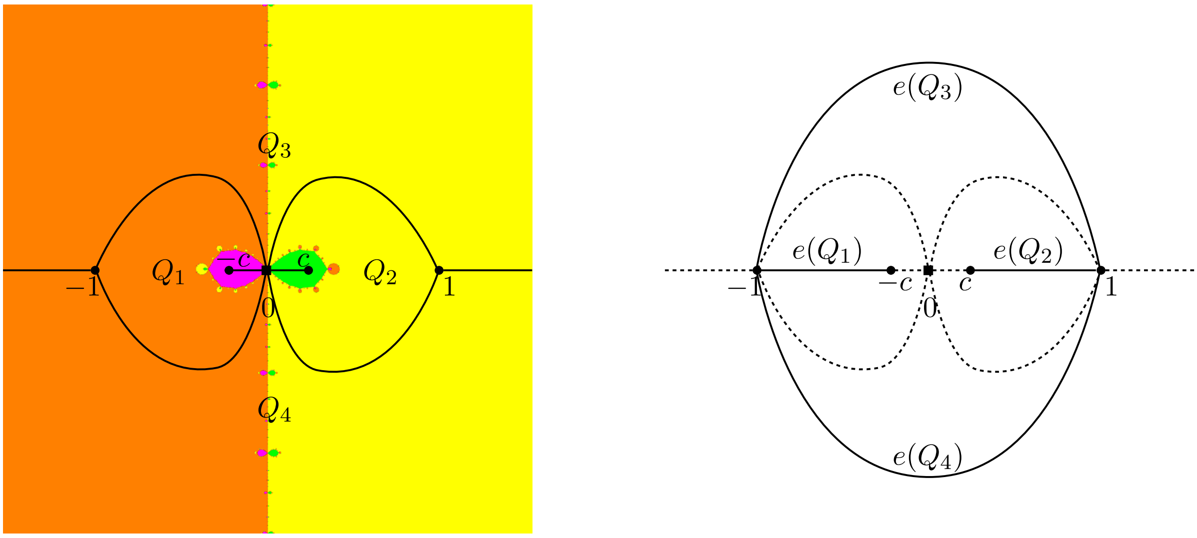
The *degree* of a vertex v in G, denoted $\deg_{\mathsf{G}}(v)$, is the number of edges of G incident to v. Note that $2\#E = \sum_{v \in V} \deg_{\mathsf{G}}(v)$.

The graph G is called *bipartite* if the vertices of G can be partitioned into two disjoint sets V_1 and V_2 so that every edge of G connects a vertex in V_1 to a vertex in V_2. In this case, the vertex subsets V_1 and V_2 are called the *parts* of the graph G.

A *path between vertices v and v'* in G is a sequence $v_0 := v, e_0, v_1, e_1, \ldots, e_{n-1}, v_n := v'$, where e_j is an edge incident to the vertices v_j and v_{j+1} for each $j = 0, \ldots, n-1$. A path $v_0, e_0, v_1, e_1, \ldots, e_{n-1}, v_n$ with $v_0 = v_n$ and $n \geq 1$ is called an *edge cycle of length* n in G and is denoted by $(e_0, e_1, \ldots, e_{n-1})$. Such a cycle is called *simple* if all vertices v_j, $j = 0, \ldots, n-1$, are distinct.

A *subgraph* of G is a planar embedded graph $\mathsf{G}' = (V', E')$ with $V' \subset V$ and $E' \subset E$. A *connected component* of G is a subgraph $\mathsf{G}' = (V', E')$ such that

(1) for all $v', v'' \in V'$, there exists a path between v' and v'' in G;



(a) (b)

FIGURE 2. The Tischler graph Tisch(f) (left) and a charge graph Charge(f) (right) of the map f from Section 3.

(2) for all $v' \in V'$ and $v \in V \setminus V'$, there is no path between v and v' in G;
(3) E' consists of all edges of G with both endpoints in V'.

The graph G is called *connected* if it has a unique connected component. We say that G is a *tree* if it is connected and has no simple cycles.

The subset $\mathcal{G} := \bigcup_{e\in E} e$ of $\widehat{\mathbb{C}}$ equal to the union of edges of G is called the *realization* of G. A *face* of G is a connected component of $\widehat{\mathbb{C}} \setminus \mathcal{G}$.

As follows from [Die05, Lemma 4.2.2], the topological boundary ∂U of each face U of G may be viewed as the realization of a subgraph of G. Furthermore, a walk around a connected component of the boundary ∂U traces an edge cycle $(e_0, e_1, \ldots, e_{n-1})$ in G, such that each edge of G appears zero, one, or two times in the sequence $e_0, e_1, \ldots, e_{n-1}$. We will say that the cycle $(e_0, e_1, \ldots, e_{n-1})$ *bounds* the face U or *traces* (a connected component of) the boundary ∂U.

Example. Consider the planar embedded graph G shown in Figure 1a. It has four vertices, denoted by v_1, v_{-1}, v_c, v_{-c}, and four edges, denoted by e_1, e_2, e_3, e_4. Clearly, G is connected. It has two faces, denoted by U_1 and U_2, such that U_1 is bounded by the edge cycle $(e_1, e_3, e_2, e_2, e_4, e_1)$ and U_2 is bounded by the edge cycle (e_3, e_4).

3. An example of critically fixed rational map

In this section we introduce a specific critically fixed rational map that we will use throughout the article to illustrate various constructions and phenomena. More explicit examples of critically fixed rational maps can be found in [CGN+15, §11]

Consider the rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by

$$f(z) = \frac{z}{2}\,\frac{(5-\sqrt{5})z^4 + 10(\sqrt{5}-1)z^2 - 5(7-3\sqrt{5})}{5z^4 + 10(\sqrt{5}-2)z^2 + (2\sqrt{5}-5)}, \quad z \in \widehat{\mathbb{C}}. \tag{3.1}$$

The sets of critical and fixed points of f are

$$C_f = \{-1, -c, c, 1\} \text{ and } \operatorname{fix}(f) = \{-1, -c, 0, c, 1, \infty\},$$

respectively, where $c = \sqrt{5}-2$ and $\{\infty\} = \widehat{\mathbb{C}}\setminus\mathbb{C}$. The critical points -1 and 1 have multiplicity 3, while the critical points $-c$ and c have multiplicity 1.

The Tischler graph of f is shown in Figure 2a (note that Tisch(f) has a vertex at ∞ connected to the critical points -1 and 1). The basins of attraction of the critical points -1, $-c$, c, and 1 are drawn in orange, purple, green, and yellow color, respectively. Note that Tisch(f) is connected and bipartite (with parts $\{-1, -c, c, 1\}$ and $\{0, \infty\}$); its vertex set coincides with $\mathrm{fix}(f)$; and each of its faces, denoted by Q_1, Q_2, Q_3, Q_4, is bounded by an edge cycle of length 4. The faces Q_1 and Q_2 are bigons with a sticker inside, while the faces Q_3 and Q_4 are quadrilaterals.

4. Connectivity of Tischler graphs

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a critically fixed rational map of degree $d \geq 2$. Suppose that $\mathsf{T} := \mathrm{Tisch}(f)$ is the Tischler graph of f (see the definition in Section 1) with the vertex set V and the edge set E. In this section we show that T is a connected graph, that is, prove Theorem 1.

It follows from the definition that T is a bipartite graph with parts C_f and R_f, where R_f is the set of landing points of the fixed internal rays. Since each point $v \in R_f$ is fixed under f, we have $V = C_f \cup R_f \subset \mathrm{fix}(f)$ and

$$\#V \leq \#\,\mathrm{fix}(f) = d + 1. \tag{4.1}$$

Note that the degree $\deg_{\mathsf{T}}(c)$ of a critical point $c \in C_f$ in the graph T coincides with the multiplicity of c. So, $\deg_{\mathsf{T}}(c) = d_c - 1$, where d_c denotes the local degree of f at the critical point c. Now, the Riemann-Hurwitz formula implies that

$$2d - 2 = \sum_{c \in C_f} (d_c - 1) = \sum_{c \in C_f} \deg_{\mathsf{T}}(c) = \#E. \tag{4.2}$$

Claim 1. If a pair of distinct edges $e_1, e_2 \in E$ forms an edge cycle of length 2 in T (that is, $e_1 \cup e_2$ is a closed Jordan curve in $\widehat{\mathbb{C}}$), then each of the two complementary components of $e_1 \cup e_2$ in $\widehat{\mathbb{C}}$ contains a critical point of f.

Proof. Let $e_1, e_2 \in E$ be two distinct edges that join a critical fixed point $c \in C_f$ and a (repelling) fixed point $r \in R_f$. Suppose also that e_1 and e_2 correspond to the fixed internal rays of angles θ_1 and θ_2 in the immediate basin Ω_c of c. Let U be a complementary component of the closed Jordan curve $e_1 \cup e_2$. Assume that the statement is false, that is, $U \cap C_f = \emptyset$.

Since U is a Jordan domain and, by assumption, $\overline{U} \cap f(C_f) = \overline{U} \cap C_f = \{c\}$, it follows that each connected component U' of $f^{-1}(U)$ is a Jordan domain and $f|\overline{U'} : \overline{U'} \to \overline{U}$ is a homeomorphism [Pil96, Proposition 2.8]. Since f is injective in a neighborhood of r and the edges e_1 and e_2 are fixed under f, it follows that U is a connected component of $f^{-1}(U)$. However, this is not possible, because $f|\overline{U}$ is not injective. Indeed, one of the two internal rays of angles $\theta_1 \pm 2\pi\frac{1}{d_c}$ in Ω_c belongs to $\overline{U}$. At the same time, both of these rays and the edge e_1 are mapped by f to e_1, so $f|\overline{U}$ is not injective. This gives the desired contradiction. □

Claim 2. Let F be the set of faces of T. Then $2\#E \geq 4\#F$.

Proof. Let Q be an arbitrary face of the planar embedded graph T. Since T is bipartite, each connected component of the boundary ∂Q is traced by an edge cycle $(e_0, e_1, \ldots, e_{n-1})$ of even length $n \geq 2$. Claim 1 implies that Q cannot be bounded just by one edge cycle of length 2. Consequently, the total length of edge cycles that bound an arbitrary face is at least 4. Since each edge $e \in E$ appears exactly twice among the edge cycles tracing the boundary components of faces of T, the statement of the claim now follows from a double counting argument. □

Denote by ℓ the number of connected components of the graph T. The Euler formula implies that

$$\#V - \#E + \#F = \ell + 1. \tag{4.3}$$

Using Claim 2 and substituting (4.1) and (4.2) into (4.3), we get

$$\ell + 1 = \#V - \#E + \#F \leq \#V - \frac{1}{2}\#E \leq (d+1) - \frac{1}{2}(2d-2) = 2. \tag{4.4}$$

Consequently, $\ell = 1$, that is, the Tischler graph of f is connected.

5. Combinatorial classification of critically fixed rational maps

The goal of this section is to prove Theorem 2, that is, we aim to construct a bijection

$$\Phi : \mathrm{ConPlanGr} \to \mathrm{CrFixRat}$$

between the set ConPlanGr of isomorphism classes of planar embedded connected graphs and the set CrFixRat of conformal conjugacy classes of critically fixed rational maps. Before we can describe the map Φ, we have to introduce some preliminary constructions.

From graphs to maps. Let $\mathsf{G} = (V, E)$ be a planar embedded graph without isolated vertices, that is, $\deg_{\mathsf{G}}(v) \geq 1$ for all $v \in V$. We construct a branched covering map $\widetilde{f}_{\mathsf{G}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by *blowing up* each edge of the graph G in the sense of Pilgrim-Tan Lei [PL98, §2.5] (starting with the identity map on $\widehat{\mathbb{C}}$). Namely, we cut the sphere $\widehat{\mathbb{C}}$ open along the interior of each edge and glue in a closed Jordan region "patch" inside each slit along the boundaries. Each complementary component of the union of the patches is then mapped by $\widetilde{f}_{\mathsf{G}}$ homeomorphically to the corresponding face of G. At the same time, $\widetilde{f}_{\mathsf{G}}$ maps the interior of each Jordan region patch homeomorphically onto the complement of the respective edge. We present a formal construction below.

Suppose that $E = \{e_1, \ldots, e_n\}$ and $F = \{U_1, \ldots, U_m\}$ are the edge and face sets of the graph G, respectively. For each edge $e_j \in E$, we choose a closed Jordan region $D_j \subset \widehat{\mathbb{C}}$ that satisfies the following properties.

(B1) The endpoints of e_j lie on ∂D_j, that is, they split ∂D_j into two Jordan arcs, which we denote by e'_j and e''_j.
(B2) The arcs e'_j and e''_j are isotopic to e_j relative to the vertex set V.
(B3) Distinct Jordan regions D_{j_1} and D_{j_2} intersect only at the common vertices (if any) of the edges e_{j_1} and e_{j_2}, that is, $D_{j_1} \cap D_{j_2} = e_{j_1} \cap e_{j_2}$ for $j_1 \neq j_2$.
(B4) For each vertex $v \in V$, the cyclic order of edges around v agrees with the cyclic order of the chosen Jordan regions around v. That is, if $d := \deg_{\mathsf{G}}(v)$ and $e_{x_1}, \ldots, e_{x_d}$ are all edges of G incident to v, then the cyclic order of these edges around v in $\widehat{\mathbb{C}}$ agrees with the cyclic order of the regions $D_{x_1}, \ldots, D_{x_d}$ around v in $\widehat{\mathbb{C}}$.

Set $E' := \bigcup_{j=1}^{n} \{e'_j, e''_j\}$. Clearly, $\mathsf{G}' := (V, E')$ is a planar embedded graph, which we call a *blow up* of G. Let F' be the set of faces of G'. Note that G and G' have the same number of connected components, which we denote by ℓ. The Euler formula implies that

$$\#F' = (\ell + 1) - \#V + \#E' = (\#V - \#E + \#F) - \#V + 2\#E = \#E + \#F = n + m.$$

By construction, $\mathrm{int}(D_j)$ is a face of G', for each $j = 1, \ldots, n$. The remaining m faces of G', which we denote by $U'_1, \ldots, U'_m$, are in natural correspondence with the faces of G. Namely, if a face U_k, $k = 1, \ldots, m$, of G is bounded by an edge cycle $(e_{x_1}, \ldots, e_{x_p})$ in G, then the face U'_k of G' is "surrounded" by the Jordan regions $D_{x_1}, \ldots, D_{x_p}$. Note that

$$\widehat{\mathbb{C}} = \bigcup_{j=1}^{n} D_j \cup \bigcup_{k=1}^{m} U'_k.$$

Example. A blow up of the graph G from Figure 1a is shown in Figure 1b. The regions in dark gray color in Figure 1b correspond to the Jordan region patches.

First, we define the map $\widetilde{f}_{\mathsf{G}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the realization $\mathcal{G}'$ of G' so that

(B5) $\widetilde{f}_{\mathsf{G}}(v) = v$, for each $v \in V$;
(B6) $\widetilde{f}_{\mathsf{G}}|e'_j : e'_j \to e_j$ and $\widetilde{f}_{\mathsf{G}}|e''_j : e''_j \to e_j$ are homeomorphisms, for each $j = 1, \ldots, n$.

Evidently, the image $\widetilde{f}_{\mathsf{G}}(\mathcal{G}')$ is the realization $\mathcal{G}$ of G. Then, we continuously extend $\widetilde{f}_{\mathsf{G}}$ to the whole sphere $\widehat{\mathbb{C}}$ so that

(B7) $\widetilde{f}_{\mathsf{G}}|U'_k : U'_k \to U_k$ is a homeomorphism, for each $k = 1, \ldots, m$;
(B8) $\widetilde{f}_{\mathsf{G}}|\,\mathrm{int}(D_j) : \mathrm{int}(D_j) \to \widehat{\mathbb{C}} \setminus e_j$ is a homeomorphism, for each $j = 1, \ldots, n$.

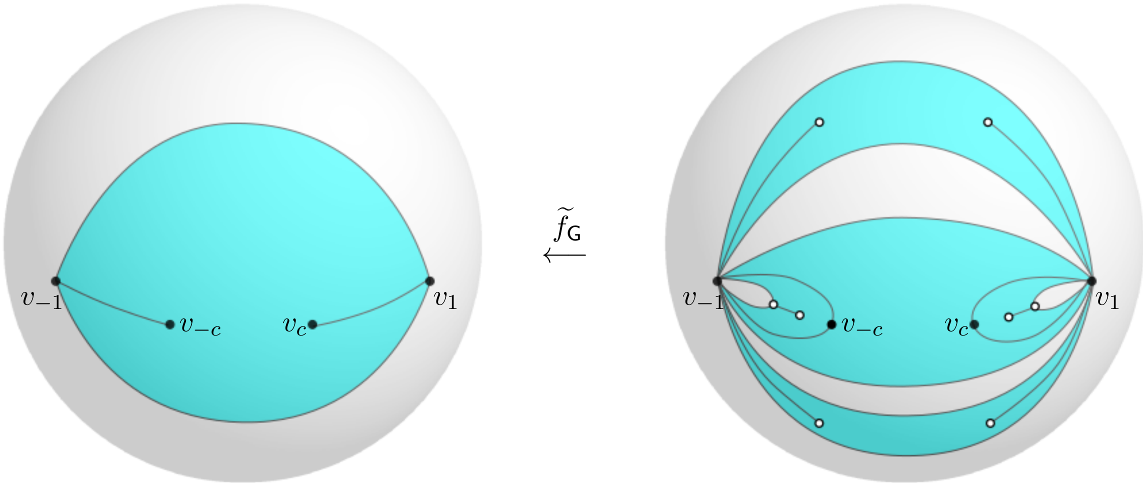


FIGURE 3. The map $\widetilde{f}_{\mathsf{G}}$.

Existence of such an extension can be deduced, for example, from the Jordan-Schönflies theorem.

One can check that the obtained continuous map $\widetilde{f}_{\mathsf{G}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched cover (this easily follows from [BM17, Corollary A.14]). Note that the topological degree of $\widetilde{f}_{\mathsf{G}}$ equals $\#E + 1 = n + 1$ and $\widetilde{f}_{\mathsf{G}}$ is a local homeomorphism outside V. Furthermore, the local degree of $\widetilde{f}_{\mathsf{G}}$ at a vertex $v \in V$ is given by $\deg_{\mathsf{G}}(v) + 1$, and thus the set $C_{\widetilde{f}_{\mathsf{G}}}$ of critical points of $\widetilde{f}_{\mathsf{G}}$ equals V. Condition (B5) implies that each critical point of $\widetilde{f}_{\mathsf{G}}$ is fixed, consequently, the postcritical set $P_{\widetilde{f}_{\mathsf{G}}} := \bigcup_{k=1}^{\infty} f^k(C_{\widetilde{f}_{\mathsf{G}}})$ of $\widetilde{f}_{\mathsf{G}}$ is equal to V and $\widetilde{f}_{\mathsf{G}}$ is postcritically-finite.

Example. The map $\widetilde{f}_{\mathsf{G}}$ obtained by blowing up edges of the graph G from Figure 1a is illustrated in Figure 3. The right picture shows the preimage $\widetilde{f}_{\mathsf{G}}^{-1}(\mathcal{G})$ of the realization $\mathcal{G}$ of G. The small white discs correspond to the points from $\widetilde{f}_{\mathsf{G}}^{-1}(V) \setminus V$, where $V := \{v_{-1}, v_{-c}, v_c, v_1\}$ is the vertex set of G. Each gray and blue (open) domain on the right picture is mapped by $\widetilde{f}_{\mathsf{G}}$ homeomorphically to the face of G of the corresponding color on the left picture.

As follows from the construction, the map $\widetilde{f}_{\mathsf{G}}$ is not uniquely defined. However, it is uniquely determined up to certain natural equivalence relation. We say that two postcritically-finite branched covering maps $f, g : S^2 \to S^2$ on a topological 2-sphere S^2 are *combinatorially* (or *Thurston*) *equivalent* if they commute up to an isotopy relative to the postcritical set[2]. That is, there exist homeomorphisms $h_0, h_1 : S^2 \to S^2$ that are isotopic relative to P_f and satisfy $h_0 \circ f = g \circ h_1$. It follows from [PL98, Proposition 2] that the combinatorial equivalence class of $\widetilde{f}_{\mathsf{G}}$ depends only on the isomorphism class of G. In the case when G is connected, [PL98, Corollary 3] implies that $\widetilde{f}_{\mathsf{G}}$ is combinatorially equivalent to a rational map $f_{\mathsf{G}} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, which is uniquely defined up to conformal conjugacy. In fact, the converse is also true: if $\widetilde{f}_{\mathsf{G}}$ is combinatorially equivalent to a rational map, then G is connected.

The preceding discussion allows to define a map Φ : ConPlanGr $\to$ CrFixRat using the "blowing up" construction: Φ sends the isomorphism class of a planar embedded connected graph G to the conformal conjugacy class of the rational map f_{G}. The injectivity of the map Φ follows from [CGN+15, Theorem 1.3]. Thus, to prove Theorem 2, we are only left to show that Φ is surjective. We do this in the remainder of the section.

From maps to graphs. A careful look at the last stage of the proof of Theorem 1 gives that all inequalities in Equation (4.4) are, in fact, equalities. This implies the following statement.

[2]A celebrated theorem due to William Thurston characterizes those postcritically-finite branched covering maps on S^2 that are combinatorially equivalent to a rational map on $\widehat{\mathbb{C}}$ [DH93].

Corollary 6. *Let f be a critically fixed rational map with $\deg(f) \geq 2$. Then the following is true.*

1. *The vertex set of the Tischler graph* $\mathrm{Tisch}(f)$ *coincides with the set* $\mathrm{fix}(f)$ *of fixed points of f. Consequently, each fixed point lies on a fixed internal ray.*
2. *The boundary ∂U of each face U of* $\mathrm{Tisch}(f)$ *is traced by an edge cycle of length* 4. *That is, each face is either a quadrilateral or a bigon with a sticker inside.*
3. *The number of faces of* $\mathrm{Tisch}(f)$ *equals* $\deg(f) - 1$.

In the following, we assume that $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a critically fixed rational map with degree $d \geq 2$. Also, let $\mathsf{T} := \mathrm{Tisch}(f)$ be the Tischler graph of f with the vertex, edge, and face sets denoted by V_T, E_T, and F_T, respectively. As earlier, R_f is the set of landing points of the fixed internal rays of f, so that $V_\mathsf{T} = C_f \cup R_f = \mathrm{fix}(f)$.

For each face Q of T, we select a Jordan arc $e(Q)$ joining the (only) two critical points on ∂Q so that $\mathrm{int}(e(Q)) \subset Q$. The planar embedded graph with the vertex set C_f and the edge set $\{e(Q) : Q \in F_\mathsf{T}\}$ is called a *charge graph* of f and is denoted by $\mathrm{Charge}(f)$. Note that $e(Q)$ splits each face $Q \in F_\mathsf{T}$ into two triangular domains with three fixed points on the boundary (one from R_f and two from C_f). Let H_T be the set of all such domains, which we call *half-faces* of T, that is, H_T is the set of complementary components of the union $\mathcal{G} \cup \mathcal{T}$ of the realizations of G and T.

Example. A charge graph $\mathrm{Charge}(f)$ of the map f from Section 3 is shown in Figure 2b, where the dashed arcs represent the edges of the Tischler graph of f from Figure 2a. Note that $\mathrm{Charge}(f)$ is isomorphic to the graph G from Figure 1a.

Proposition 7. *Each critically fixed rational map f with $\deg(f) \geq 2$ is obtained from the charge graph* $\mathrm{Charge}(f)$ *by blowing up its edges. That is, if $V := C_f$ and $E := \{e_1, \dots, e_n\}$ are the vertex and edge sets of* $\mathsf{G} := \mathrm{Charge}(f)$, *respectively, then there exist closed Jordan regions $D_1, \dots, D_n$ that satisfy Conditions* (B1)–(B8) *with $\widetilde{f}_\mathsf{G} := f$.*

Remark. Since the charge graph of the map f from Section 3 is isomorphic to the graph G from Figure 1a, Proposition 7 would imply that the map f and the map $\widetilde{f}_\mathsf{G}$ from Figure 3, obtained by blowing up the edges of G, are combinatorially equivalent.

Before we prove Proposition 7, we need the following lemma.

Lemma 8. *Each face of G contains exactly one point from R_f. Furthermore, if U_r is the face of G containing a point $r \in R_f$, then*

$$U_r = \bigcup_{\substack{W \in H_\mathsf{T},\\ s.t.\ r \in \partial W}} \left(\overline{W} \setminus e(W)\right), \tag{5.1}$$

where $e(W)$ denotes the unique edge of G which belongs to the boundary ∂W of the half-face W.

Proof. Let $e_0, e_1, \dots, e_{n-1}$ be the edges of T incident to a vertex $r \in R_f$ and labeled in a cyclic order, and $Q_0, \dots, Q_{n-1}$ be the sequence of faces of T surrounding the vertex r so that Q_j lies in between the edges e_j and e_{j+1}, for $j = 0, \dots, n-1$ (where $e_n = e_0$). Also, let W_j be the half-face of T in between the edges e_j and e_{j+1}. Then ∂W_j is a Jordan domain with $\partial W_j = e_j \cup e_{j+1} \cup e(Q_j)$, $j = 0, \dots, n-1$. It follows that $(e(U_0), e(Q_1), \dots, e(Q_{n-1}))$ is an edge cycle in G that traces the boundary of the face U_r. In fact,

$$U_r = \bigcup_{j=0}^{n-1} \left(\overline{W_j} \setminus e(Q_j)\right),$$

which completes the proof of the lemma. □

Remark. Lemma 8 implies that the charge graph G is connected. Indeed, let $c, c' \in C_f$ be arbitrary two vertices of G. By Theorem 1, the Tischler graph T is connected, so there exists a path $c_0 = c, r_0, c_1, r_1, \dots, r_{n-1}, c_n = c'$ in T connecting c to c', where $c_j \in C_f$ and $r_j \in R_f$, for each $j = 0, \dots, n-1$. The proof of Lemma 8 implies that c_j and c_{j+1} are connected by

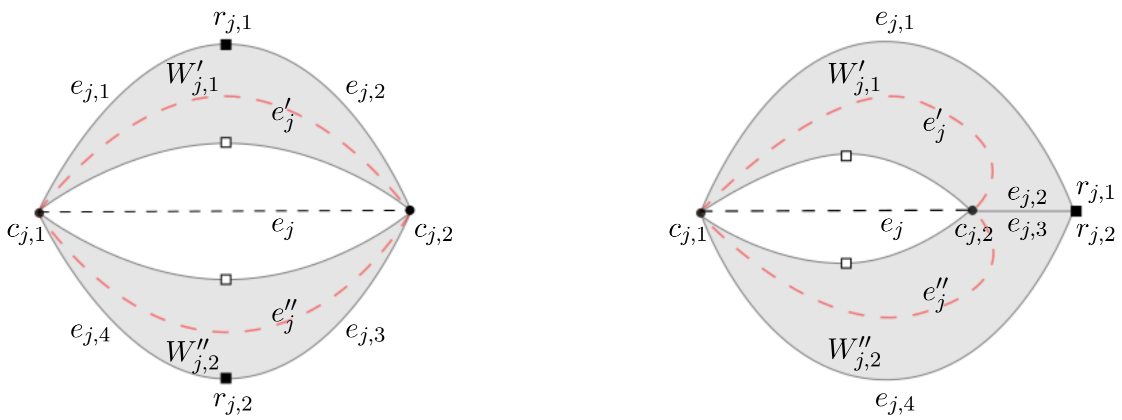


FIGURE 4. Constructing the Jordan region D_j inside a face Q_j, if Q_j is a quadrilateral (left) and a bigon with a sticker inside (right).

a path in G for each $j = 0, \dots, n-1$, so $c = c_0$ and $c' = c_n$ are connected by a path in G as well. Since $c, c' \in C_f$ are arbitrary, it follows that G is connected. This would also follow from Proposition 7, because $\widetilde{f}_{\mathsf{G}}$ is combinatorially equivalent to a rational map only if G is connected.

Proof of Proposition 7. Let $V = C_f$, $E = \{e_1, \dots, e_n\}$, and $F = \{U_1, \dots, U_m\}$ be the vertex, edge, and face sets of the charge graph $\mathsf{G} := \mathrm{Charge}(f)$, respectively. Up to relabeling, by Lemma 8, we may assume that $R_f = \{r_1, \dots, r_m\}$ and $r_k \in U_k$, for each $k = 1, \dots, m$. Similarly, we assume that $F_{\mathsf{T}} = \{Q_1, \dots, Q_n\}$ so that $e_j = e(Q_j)$,for each $j = 1, \dots, n$.

First, we define the closed Jordan regions $D_1, \dots, D_n$. Let Q_j, $j = 1, \dots, n$, be an arbitrary face of the Tischler graph T. By Corollary 6, ∂Q_j is traced by an edge cycle

$$c_{j,1},\ e_{j,1},\ r_{j,1},\ e_{j,2},\ c_{j,2},\ e_{j,3},\ r_{j,2},\ e_{j,4},\ c_{j,1},$$

where $e_{j,1}, e_{j,2}, e_{j,3}, e_{j,4} \in E_{\mathsf{T}}$, $c_{j,1}, c_{j,2} \in C_f$, and $r_{j,1}, r_{j,2} \in R_f$. Since the edges $e_{j,1}, e_{j,2}, e_{j,3}, e_{j,4}$ are fixed under f and f is locally injective at $r_{j,1}$ and $r_{j,2}$, there are exactly two connected components Q'_j and Q''_j of $f^{-1}(Q_j)$ such that $e_{j,1}, e_{j,2} \subset \partial Q'_j$ and $e_{j,3}, e_{j,4} \subset \partial Q''_j$. Note that both components Q'_j and Q''_j belong to Q_j. We set

$$e'_j := \overline{Q'_j \cap f^{-1}(e_j)} \text{ and } e''_j := \overline{Q'_j \cap f^{-1}(e(_j))}.$$

Since $Q_j \cap C_f = \emptyset$, $f|Q'_j : Q'_j \to Q_j$ and $f|Q''_j : Q''_j \to Q_j$ are homeomorphisms. Thus, e'_j and e''_j are Jordan arcs connecting $c_{j,1}$ and $c_{j,2}$. Let D_j be the closed Jordan region bounded by the Jordan curve $e'_j \cup e''_j$ such that $\mathrm{int}(D_j) \subset Q_j$, see Figure 4. The figure shows the face Q_j, where the components Q'_j and Q''_j of $f^{-1}(Q_j)$ are shown in gray color, the preimages of $r_{j,1}$ and $r_{j,2}$ inside Q_j are represented by small white squares, and the dashed red curve bounds the constructed Jordan region D_j .

We claim that $D_1, \dots, D_n$ satisfy Conditions (B1)–(B8) with $\widetilde{f}_{\mathsf{G}} := f$, so that f is obtained from G by blowing up its edges.

Conditions (B1)–(B6) follow immediately from the definition of $D_1, \dots, D_n$.

Let $\mathsf{G}' := (V, E')$ be the planar embedded graph with the vertex set $V = C_f$ and the edge set $E' := \bigcup_{j=1}^{n} \{e'_j, e''_j\}$. For each $j = 1, \dots, n$, let $W_{j,1}$ and $W_{j,2}$ be the two half-faces of T inside the face Q_j such that

$$\partial W_{j,1} = e_{j,1} \cup e_{j,2} \cup e_j \text{ and } \partial W_{j,2} = e_{j,3} \cup e_{j,4} \cup e_j.$$

Set $W'_{j,1} := (f|Q'_j)^{-1}(W_{j,1})$ and $W''_{j,2} := (f|Q''_j)^{-1}(W_{j,2})$, then

$$Q_j = W'_{j,1} \sqcup \mathrm{int}(e'_j) \sqcup \mathrm{int}\, D_j \sqcup \mathrm{int}(e''_j) \sqcup W''_{j,2},$$

see Figure 4. For $k = 1, \dots, m$, we set

$$U_k' := \bigcup_{\substack{W \in H_{\mathsf{T}}', \\ \text{s.t. } r_k \in \partial W}} \left(\overline{W} \setminus e(W)\right), \tag{5.2}$$

where $H_T' := \bigcup_{j=1}^{n} \{W_{j,1}', W_{j,2}'\}$ and $e(W)$ denotes the unique edge of G' which belongs to the boundary ∂W. An argument similar to the proof of Lemma 9 implies that U_k' is a face of G' containing r_k, for each $k = 1, \dots, m$. Consequently, the set F' of faces of G' is given by

$$F' = \bigcup_{j=1}^{n} \operatorname{int}(D_j) \cup \bigcup_{k=1}^{m} U_k'.$$

Since $f|W_{j,1}' : W_{j,1}' \to W_{j,1}$ and $f|W_{j,2}'' : W_{j,2}'' \to W_{j,2}$ are homeomorphisms, for each $j = 1, \dots, n$, and f is locally injective at r_k, for each $k = 1, \dots, m$, Equations (5.1) and (5.2) imply that $f|U_k' : U_k' \to U_k$ is a homeomorphism, for each $k = 1, \dots, m$. So, (B7) is satisfied.

Conditions (B1) and (B6), together with $\operatorname{int}(D_j) \cap C_f = \emptyset$, imply that $f|D_j : D_j \to \widehat{\mathbb{C}} \setminus e_j$ is a covering map, for each $j = 1, \dots, n$. Let us choose a point in the complement of the realization of G, say $r_1 \in R_f$. From the considerations above, all the preimages of r_1 lie in $U_1', D_1, \dots, D_n$. Thus,

$$\deg(f) = \#f^{-1}(r_1) = 1 + \sum_{j=1}^{n} \deg(f|D_j).$$

By Corollary 6, $n = \#F_{\mathsf{T}} = \deg(f) - 1$, thus $\deg(f|D_j) = 1$, for each $j = 1, \dots, n$. Consequently, $f|D_j : D_j \to \widehat{\mathbb{C}} \setminus e_j$ is a homeomorphism and Condition (B8) follows. Alternatively, one can deduce (B8) from the fact that e_1^j and e_4^j are consecutive fixed internal rays at the critical point c_1^j.

We checked all the necessary conditions to conclude that f is obtained from G by blowing up its edges, so Proposition 7 is proven. □

Clearly, Proposition 7 implies that $\Phi : \text{ConPlanGr} \to \text{CrFixRat}$ is surjective, which completes the proof of Theorem 2.

6. Global curve attractor problem

Here, we provide a positive answer to the global curve attractor problem, that is, prove Theorem 3. First, we set up some notation.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a critically fixed rational map of degree d and $\text{Charge}(f)$ be a charge graph of f defined in the previous section, so that f is obtained by blowing up the edges of $\text{Charge}(f)$. Choose a *spanning tree* S of the charge graph of f, that is, a subgraph of $\text{Charge}(f)$ with the vertex set C_f that is also a tree. Let $E_{\mathsf{S}} := \{e_1, \dots, e_n\}$ be the edge set of S, so that $n = \#E_{\mathsf{S}} = \#C_f - 1$.

Suppose that $\mathscr{C}(f)$ is the set of all simple closed curves in $\widehat{\mathbb{C}} \setminus C_f = \widehat{\mathbb{C}} \setminus P_f$. Recall that a pullback of a curve $\gamma \in \mathscr{C}(f)$ is a connected component of the preimage $f^{-1}(\gamma)$. We denote the isotopy class relative to C_f of a curve $\gamma \in \mathscr{C}(f)$ by $[\gamma]$. Given a curve $\gamma \in \mathscr{C}(f)$ and a Jordan arc e in $\widehat{\mathbb{C}}$ with endpoints in C_f, we define the *intersection number between γ and e*, denoted $\gamma \cdot e$, by

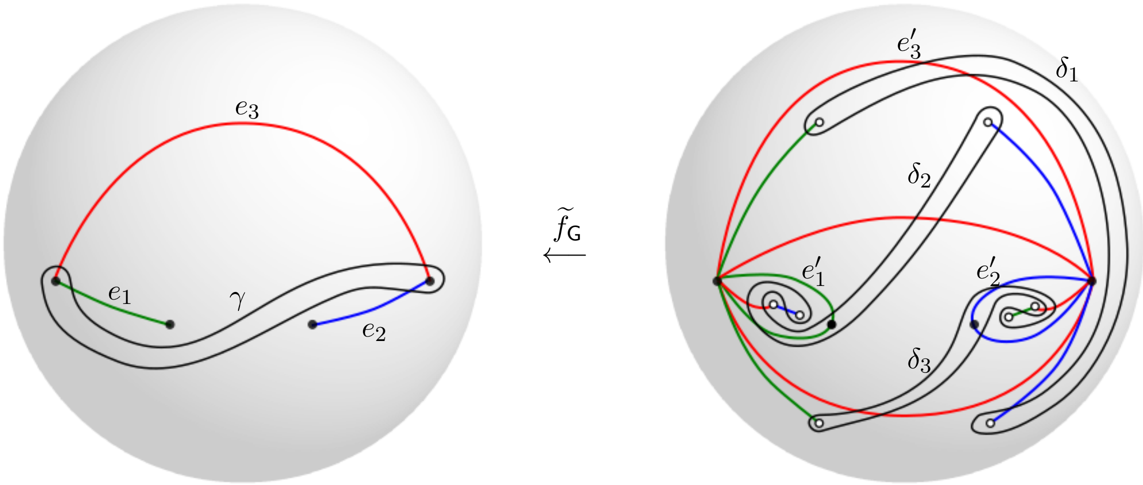
$$\gamma \cdot e := \min_{\gamma' \in [\gamma]} \#(\gamma' \cap e).$$

Similarly, we define the *intersection number between γ and the spanning tree S*, denoted $\gamma \cdot \mathsf{S}$, by

$$\gamma \cdot \mathsf{S} := \min_{\gamma' \in [\gamma]} \#(\gamma' \cap \mathcal{S}),$$

where $\mathcal{S}$ is the realization of S. Note that

$$\gamma \cdot \mathsf{S} = \sum_{j=1}^{n} \gamma \cdot e_j.$$



(a) (b)

FIGURE 5. The pullbacks of a curve γ under the map $\widetilde{f}_{\mathsf{G}}$.

We will refer to the intersection number $\gamma \cdot \mathsf{S}$ as the *complexity* of the curve γ (relative to S).

The next key lemma relates the complexities of a curve $\gamma \in \mathscr{C}(f)$ and its pullbacks.

Lemma 9. *Let $\gamma \in \mathscr{C}(f)$ be arbitrary and $\Delta(\gamma)$ be the set of pullbacks of γ under f. Then*

$$\sum_{\delta \in \Delta(\gamma)} \delta \cdot \mathsf{S} \leq \gamma \cdot \mathsf{S}, \tag{6.1}$$

that is, the total complexity of the pullbacks of γ does not exceed the complexity of γ.

Furthermore, if $\gamma \cdot e > 1$ for some edge $e \in E_{\mathsf{T}}$ then

$$\sum_{\delta \in \Delta(\gamma)} \delta \cdot \mathsf{S} < \gamma \cdot \mathsf{S}. \tag{6.2}$$

Example. Instead of the map f from Section 3, let us consider the map $\widetilde{f}_{\mathsf{G}}$ obtained by blowing up the edges of the graph G from Figure 1a, which is combinatorially equivalent to f. Consider the spanning tree S of G with the edge set $\{e_1, e_2, e_3\}$ and the curve γ as in Figure 5a. Note that $\gamma \cdot e_1 = \gamma \cdot e_2 = 1$ and $\gamma \cdot e_3 = 2$, so $\gamma \cdot \mathsf{S} = 4$. The colored arcs in Figure 5 correspond to the edges of G and their preimages: the arcs colored red, blue, and green in Figure 5b are mapped by $\widetilde{f}_{\mathsf{G}}$ homeomorphically onto the red, blue, and green edge in Figure 5a, respectively. The curve γ has three pullbacks, denoted by $\delta_1, \delta_2, \delta_3$ in Figure 5b, such that $\delta_1 \cdot \mathsf{S} = 0$ and $\delta_2 \cdot \mathsf{S} = \delta_3 \cdot \mathsf{S} = 1$. Thus,

$$\delta_1 \cdot \mathsf{S} + \delta_2 \cdot \mathsf{S} + \delta_3 \cdot \mathsf{S} < \gamma \cdot \mathsf{S},$$

which agrees with Lemma 9 since $\gamma \cdot e_3 = 2$.

Proof of Lemma 9. Without loss of generality, let us assume that the curve $\gamma \in \mathscr{C}(f)$ has the smallest number of intersections with S within its isotopy class $[\gamma]$, that is, $\gamma \cdot \mathsf{S} = \#(\gamma \cap \mathcal{S}) = \sum_{j=1}^{n} \#(\gamma \cap e_j)$.

Let D_j, e_j', and e_j'', $j = 1, \dots, n$, be as in the proof of Proposition 7. By (B6), f maps the Jordan arc e_j' homeomorphically onto e_j, thus $\#(\gamma \cap e_j) = \#(f^{-1}(\gamma) \cap e_j')$. At the same time, by (B4), e_j' is isotopic to e_j relative to C_f, so $\delta \cdot e_j = \delta \cdot e_j'$ for all pullbacks $\delta \in \Delta(\gamma)$. So,

$$\begin{aligned} \sum_{\delta \in \Delta(\gamma)} \delta \cdot \mathsf{S} &= \sum_{\delta \in \Delta(\gamma)} \sum_{j=1}^{n} \delta \cdot e_j = \sum_{j=1}^{n} \sum_{\delta \in \Delta(\gamma)} \delta \cdot e_j' \\ &\leq \sum_{j=1}^{n} \sum_{\delta \in \Delta(\gamma)} \#(\delta \cap e_j') = \sum_{j=1}^{n} \#(f^{-1}(\gamma) \cap e_j') = \sum_{j=1}^{n} \#(\gamma \cap e_j) = \gamma \cdot \mathsf{S}, \end{aligned} \tag{6.3}$$

which proves Equation 6.1.

To prove Equation 6.2, we need to provide a more careful estimate for the total complexity of the pullbacks of γ. For this, let us fix a basepoint t on the curve γ outside of the realization $\mathcal{T}$ of T. For each edge $e_j \in E_{\mathsf{T}}$, we choose a simple closed oriented loop γ_j at t that intersects the realization $\mathcal{T}$ only once in the interior of the edge e_j. Clearly, the (homotopy classes of) curves γ_j, $j = 1, \ldots, n$, define a generating set of the fundamental group $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$.

Choose an orientation of the curve γ. Then γ is homotopic in $\widehat{\mathbb{C}} \setminus C_f$ to the concatenated curve $\beta_1\beta_2 \ldots \beta_K$, where $m = \gamma \cdot \mathsf{S}$, $\beta_k \in \{\gamma_j, \gamma_j^{-1} : j = 1, \ldots, n\}$, for each $k = 1, \ldots, K$, and γ_j^{-1} denotes the loop γ_j with the reversed orientation.

Without loss of generality, assume that $\gamma \cdot e_n > 1$. Since the curve γ is simple, the loop γ_n and the reversed loop γ_n^{-1} both appear in the word $\beta_1\beta_2 \ldots \beta_K$ at least once. Up to taking a cyclic shift, we may assume that $\beta_k \in \{\gamma_j, \gamma_j^{-1} : j = 1, \ldots, n-1\}$ for $k = 1, \ldots, M$, $\beta_{M+1} = \gamma_n^{-1}$, and $\beta_K = \gamma_n$.

Consider the unique preimage t_n of t that belongs to the Jordan region D_n, that is, $t_n := D_n \cap f^{-1}(t)$. There exists a unique pullback δ of γ that passes through t_n. Note that the lift of the concatenated path $\beta_1 \ldots \beta_M$ under f starting at t_n stays inside $\mathrm{int}(D_n)$. Consequently, (a part of) the pullback δ enters D_n through one of the two edges e_n' and e_n'', say e_n' up to relabeling, and then leaves it through the same edge. Then, since e_n and e_n' are isotopic relative to C_f,

$$\delta \cdot e_n = \delta \cdot e_n' \leq \#(\delta \cap e_n') - 2.$$

Thus, the inequality in (6.3) is strict, which completes the proof of Equation 6.2. □

Now, let $\mathcal{A}(f)$ be the set of isotopy classes of curves that intersect each edge of T at most once, that is,

$$\mathcal{A}(f) = \{[\gamma] : \gamma \in \mathscr{C}(f), \text{ such that } \gamma \cdot e \leq 1 \text{ for all } e \in E_{\mathsf{S}}\}.$$

Lemma 9 and an induction on the curve complexity imply the following result.

Proposition 10. *For every curve $\gamma \in \mathscr{C}(f)$, each pullback δ of γ under f^n satisfies $[\delta] \in \mathcal{A}(f)$ for all sufficiently large n.*

Since $\mathcal{A}(f)$ is a finite set, Proposition 10 immediately implies existence of a finite global curve attractor for f and completes the proof of Theorem 3.

7. Acknowledgments

The author gratefully acknowledges the support of the Studienstiftung des Deutschen Volkes. He would like to express sincere gratitude to his PhD advisers, Dierk Schleicher and Daniel Meyer, for their guidance and encouragement during the graduate studies. The author is grateful for insightful discussions with Mario Bonk, Dzmitry Dudko, Russell Lodge, Kevin Pilgrim, and Palina Salanevich, and especially thankful to Kevin Pilgrim for introducing him to the problem of connectivity of Tischler graphs. He also thanks Indiana University Bloomington and the University of California, Los Angeles for their hospitality where a part of this work has been done during research visits of the author.

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