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A note on the geodetic number and the Steiner number of AT-free graphs

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ABSTRACT

We study two graph parameters, namely the geodetic number and the Steiner number, which are related to the concept of convexity. We show that, in asteroidal triple-free graphs, the Steiner number is greater than or equal to the geodetic number. This answers a question posed by Hernando, Jiang, Mora, Pelayo, and Seara in 2005. Besides, we show that the gap between the two parameters can be arbitrarily large even in unit-interval graphs, a proper subclass of AT-free graphs.

1. Introduction

The geodetic number of a graph was introduced by Buckley, Harary and Quintas [1] (see also [5] for a recent survey). It is defined as follows. A *geodesic* in a graph is a shortest path between two vertices — that is — a path that connects the two vertices with the fewest number of edges. Let $G = (V, E)$ be a graph. For a set $S \subseteq V$ let $\mathrm{I}(S) = \{\, z \mid \exists\, x, y \in S \quad z$ lies on a x, y-geodesic $\}$. A set S is *geodetic* if $\mathrm{I}(S) = V$. The geodetic number $g(G)$ of G is defined as the cardinality of a minimum geodetic set.

Let $G = (V, E)$ be a graph and $W \subseteq V$. A *Steiner W-tree* is a connected subgraph T of G with the least number of edges that contains all vertices of W. Any vertex in $V(T) \setminus W$ is called a Steiner vertex. The Steiner interval $S(W)$ is the set of all vertices such that each of them is in some Steiner W-tree. If $S(W) = V$, then W is a Steiner set. The Steiner number $s(G)$ is defined as the cardinality of a minimum Steiner set [5]. Fig. 1 gives an example showing the two parameters.

For graphs in general there is no order relation between the Steiner number and the geodetic number [6]. For distance-hereditary and interval graphs Hernando, Jiang, Mora, Pelayo, and Seara [3] showed that every Steiner set is geodetic — that is — $g(G) \le s(G)$. In their paper the authors posed the question whether the same holds true for AT-free graphs. We answer the question positively in Section 2.

Let G be a graph. For any subgraph H of G, we use $V(H)$ to denote the set of vertices of H. An edge with endpoints u and v is denoted by $u \to v$, and u is a *neighbor* of v. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of neighbors of v. The *closed neighborhood* of a vertex v, denoted by $N[v]$, is $N(v) \cup \{v\}$. A vertex v *attaches to* a subgraph H of

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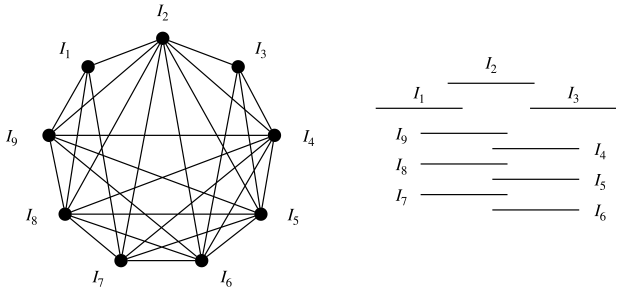

Fig. 1. A unit interval graph with geodetic number 4 and Steiner number 5. The right side is the intersection model of the graph on the left. The set $\{I_1, I_3, I_4, I_7\}$ is a geodetic set, and $\{I_1, I_3, I_4, I_5, I_6\}$ is a Steiner set. It is shown in Theorem 2 that both sets are minimum.

G if $N(v) \cap V(H) \neq \varnothing$. For $U \subseteq V(G)$, the subgraph induced by U is denoted by $G[U]$. We use $u \rightsquigarrow v$ to denote a u, v-path in G. For a path P and two vertices x and y on P, we denote the subpath that runs between x and y by $x \underset{P}{\rightsquigarrow} y$. The *distance* between x and y, denoted by $d(x, y)$, is the number of edges on an x, y-geodesic.

2. Steiner sets in AT-free graphs

Asteroidal triples were introduced by Lekkerkerker and Boland to identify those chordal graphs that are interval graphs [4]. An *asteroidal triple*, AT for short, is a set of three vertices $\{x, y, z\}$ such that for every pair of them there is a path between them that avoids the closed neighborhood of the third. A graph is AT-*free* if it has no asteroidal triple. Well-known examples of AT-free graphs are cocomparability graphs. However, AT-free graphs need not be perfect; for example, C_5 is AT-free.

A *dominating set* is a subset D of vertices such that the intersection of D and the closed neighborhood of any vertex is nonempty. Two vertices constitute a *dominating pair* if every path between them induces a dominating set. The following result appears in [2].

Lemma 1 *(See [2]). Every connected* AT*-free graph has a dominating pair.*

A tree is a caterpillar if the removal of all leaves results in a path. The path is called the *backbone* of the caterpillar. From Lemma 1 we obtain the following immediately.

Lemma 2. *Given an* AT*-free graph G, let $W \subseteq V(G)$, and let T be a Steiner W-tree. Then there exists a Steiner W-tree T' such that $V(T) = V(T')$ and T' is a caterpillar.*

Proof. Let T be a Steiner W-tree and let $X = V(T)$. Then the graph $G[X]$ is connected and AT-free since the class of AT-free graphs is hereditary. By Lemma 1, $G[X]$ has a dominating pair, say $\{x, y\}$. Consider any path P in $G[X]$ with endpoints x and y. Then $V(P)$ is a dominating set of $G[X]$. To obtain a caterpillar T' that spans X, connect each vertex of $X \setminus V(P)$ to a neighbor in P. □

Remark 1. In an AT-free graph, due to the existence of a dominating pair, every Steiner-W tree can be converted to a caterpillar T with a chordless backbone. Such a Steiner-W tree is called *canonical.*

Let W be a Steiner set of an AT-free graph G. To prove that every Steiner set W of G is geodetic, we consider the nontrivial case in which $W \neq V$. Let T be a canonical Steiner W-tree, and let P be the backbone of T. On P there is a Steiner vertex z. Let x' and y' be the pair of vertices on P such that the path $x' \underset{P}{\rightsquigarrow} y'$ contains z and no vertex in W except x' and y'. The path $x' \underset{P}{\rightsquigarrow} y'$ is called the *critical path* corresponding to z. Note that if for every z the corresponding critical path is a geodesic, then W is geodetic. However, if this is not the case (e.g. Fig. 2), we show that any critical path is "short enough" such that the AT-freeness ensures that some geodesic, connecting two vertices in W and containing z, overlaps with it.

In a graph G with Steiner set W, let $u \in V(G)$, and let $X \subseteq Y \subseteq V(G)$. The vertex u is a *private neighbor* of X with respect to Y if $N[u] \cap X \neq \varnothing$ and $N[u] \cap (Y \setminus X) = \varnothing$. A leaf x of a Steiner-W tree is private to X with respect to Y if in G the vertex x is a private neighbor of X with respect to Y.

Lemma 3. *Let T be a canonical Steiner-W tree. If there is a Steiner vertex, then the difference between the length of the corresponding critical path $x' \underset{T}{\rightsquigarrow} y'$ and that of an x', y'-geodesic is at most* 2.

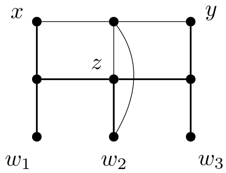

Fig. 2. An AT-free graph with $W = \{x, y, w_1, w_2, w_3\}$ being a Steiner set. Thick edges form a Steiner-W tree, which contains a Steiner vertex z. The x, y-path in the tree is critical corresponding to z, but it is not a geodesic.

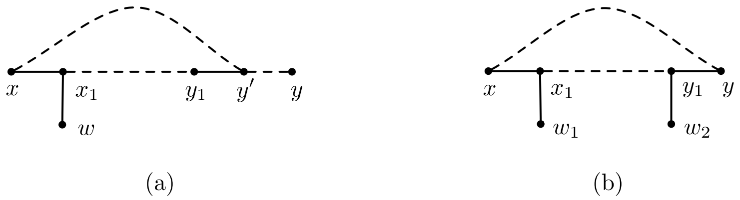

Fig. 3. Cases when z not on a x', y'-geodesic. (a) $|\{x', y'\} \cap \{x, y\}| = 1$. Assume $x = x'$. Vertex z lies on a w, y'-geodesic. (b) $|\{x', y'\} \cap \{x, y\}| = 2$. Vertex z lies on a w_1, w_2-geodesic.

Proof. Let $Q = x' \underset{T}{\rightsquigarrow} y'$ and Q' be an x', y'-geodesic, with q and q' being the lengths, respectively. Suppose to the contrary that $q > q' + 2$. Then $q \geq 4$, and therefore there exist $x_1 \neq y_1$ such that $N(x') \cap V(Q) = \{x_1\}$ and $N(y') \cap V(Q) = \{y_1\}$. Let L be the set of leaves private to $V(Q)$ with respect to $V(T)$, and let $\tilde{L}$ be the set of leaves private to $\{x_1, y_1\}$ with respect to $V(Q)$. We claim that

$$\forall\, u \in L \setminus \tilde{L} \quad N[u] \cap V(Q') \neq \varnothing.$$

To see that, assume that there exists a vertex u belonging to $L \setminus \tilde{L}$, and $N[u] \cap V(Q') = \varnothing$. Clearly, there is an x', y'-path that avoids $N[u]$ via Q'. Furthermore, the backbone of T is chordless, so we have a u, x'-path that avoids $N[y']$ via Q and a u, y'-path that avoids $N[x']$ via Q. This results in an asteroidal triple $\{x', y', u\}$. Therefore, the claim holds, and Q can be substituted for $x_1 \to x' \underset{Q'}{\rightsquigarrow} y' \to y_1$, with every leaf in $L \setminus \tilde{L}$ attaching to this path, forming a smaller tree containing W. $\square$

From the proof of Lemma 3 one can see that when substituting a subpath from the backbone for another, the AT-freeness ensures the adjacency for some leaves to the new path. Similar operations are applied in the proof of the main result (Theorem 1). A general statement is given in Lemma 4.

Lemma 4. *Let G be an AT-free graph and W a Steiner set of G. Let T be a canonical Steiner-W tree, and P be the backbone of T. For any subpaths $P' = x' \underset{P}{\rightsquigarrow} y'$ of P and $P'' = x'' \underset{P}{\rightsquigarrow} y''$ of P', if the distance between an endpoint of P and one of P' is at least 2, i.e. $\min\{d(x'', x'), d(x'', y'), d(y'', x'), d(y'', y')\} \geq 2$, then any leaf private to $V(P'')$ with respect to $V(P')$ attaches to all x', y'-paths in G.*

Theorem 1. *Let G be* AT*-free. Every Steiner set of G is geodetic.*

Proof. Let W be a Steiner set of G and let $z \in V(G) \setminus W$. We show that z is on a geodesic between two vertices in W. Let T be a canonical Steiner W-tree containing z, P be the backbone of T, and $Q = x' \underset{P}{\rightsquigarrow} y'$ be the critical path corresponding to z.

Assume that Q is not an x', y'-geodesic. Then by Lemma 4 there is a nonempty set of leaves $\tilde{L}$ private to $\{x_1, y_1\}$ with respect to $V(P)$, where x_1 and y_1 are the neighbors of x' and y' on Q, respectively. In particular, let L_x be the set of leaves private to x_1 with respect to $V(P)$ and L_y the set of leaves private to y_1 with respect to $V(P)$. Observe that $|\{x', y'\} \cap \{x, y\}| > 0$ since otherwise by Lemma 4 Q can be substituted for an x', y'-geodesic, forming a smaller tree containing W. Let x and y be the endpoints of P, it suffices to consider the two cases (see Fig. 3): $|\{x', y'\} \cap \{x, y\}| = 1$ and $|\{x', y'\} \cap \{x, y\}| = 2$. We show that

(i) If $|\{x', y'\} \cap \{x, y\}| = 1$, then z is on a geodesic between a leaf to either x' or y'.
(ii) If $|\{x', y'\} \cap \{x, y\}| = 2$, then z is on a geodesic between a leaf to either x' or y', or on a geodesic between two leaves.

In the following, let q and q' be the length of Q and that of an x', y'-geodesic Q', respectively.

For $|\{x', y'\} \cap \{x, y\}| = 1$, assume without loss of generality that $x' = x$. Then $q = q' + 1$ and there is a leaf w in L_x such that $N[w] \cap V(Q') = \varnothing$, since by Lemma 4 every leaf non-private to x_1 with respect to $V(Q)$ attaches to an x, y-path. We show that $w \to x_1 \underset{P}{\rightsquigarrow} y'$ is a w, y'-geodesic.

Claim: Let u be a neighbor of w on a w, y'-geodesic. If $d(w, y') < q$, then $u \in N(x')$.

Proof Any w, y'-geodesic Q'_w contains a neighbor of x' since otherwise $\{w, x', y'\}$ is an asteroidal triple. Then, Q'_w is of the form

$$w \rightsquigarrow u_x \rightsquigarrow y'$$

where $u_x \in N(x')$. With $d(x', y') = q'$, we have

$$q' = d(x', y') \le 1 + d(u_x, y') \le d(w, y') < q = q' + 1.$$

Thus, $d(w, u_x) = 1$ and $x' \to u_x \underset{Q'_w}{\rightsquigarrow} y'$ is an x', y'-geodesic. ■

Claim: If $d(w, y') < q$, then there is a w, y'-geodesic Q'_w such that every leaf in L_x attaches to Q'_w.

Proof Let L_x be the set of leaves private to x_1 with respect to $V(P)$. If no such geodesic exists, then there are distinct elements $u_1, u_2 \in L_x$ such that $N[u_1] \cap V(Q'_{u_2}) = \varnothing$ and $N[u_2] \cap V(Q'_{u_1}) = \varnothing$, where Q'_{u_1} and Q'_{u_2} are a u_1, y'-geodesic and a u_2, y'-geodesic, respectively. It follows that $x_1 \notin V(Q'_{u_1}) \cup V(Q'_{u_2})$, and $\{u_1, u_2, y'\}$ is an asteroidal triple. ■

The two claims given above show that $d(w, y') \ge q$ and z is on a w, y'-geodesic.

For $|\{x', y'\} \cap \{x, y\}| = 2$, if $q = q' + 1$, then the same argument as in the previous case applies. By Lemma 3 it remains to consider $q = q' + 2$. For any x', y'-geodesic Q', there exist leaves w_1 and w_2 private to x_1 and y_1, respectively, such that $N(w_1) \cap V(Q') = \varnothing$ and $N(w_2) \cap V(Q') = \varnothing$. We show that $w_1 \to x_1 \underset{P}{\rightsquigarrow} y_1 \to w_2$ is a w_1, w_2-geodesic. Suppose to the contrary that

$$d(w_1, w_2) < q.$$

Then every w_1, w_2-path contains both a neighbor u of x' and a neighbor v of y' since otherwise either $\{w_1, x', y'\}$ or $\{w_2, y', x'\}$ is an asteroidal triple. Therefore,

$$q' = d(x', y') \le 2 + d(u, v) \le d(w_1, w_2) < q = q' + 2. \tag{1}$$

It follows that

$$q' - 2 \le d(u, v) \le q' - 1, \tag{2}$$

and at least one of w_1 and w_2 is adjacent to u or v — that is —

$$N(w_1) \cap \{u, v\} \neq \varnothing \quad \text{or} \quad N(w_2) \cap \{u, v\} \neq \varnothing.$$

Assume without loss of generality that $u \in N(w_1)$.

Claim: Let L_x be the leaves private to x_1 with respect to $V(P)$ and L_y the leaves private to y_1 with respect to $V(P)$. There exist $w_1 \in L_x$ and $w_2 \in L_y$ such that every vertex in $\tilde{L}$ attaches to a w_1, w_2-geodesic.

Proof Similar to the claim in the previous case, for every $w_1 \in L_x$, there exists $w_2 \in L_y$ such that every vertex in L_y attaches to a w_1, w_2-geodesic. If there is no geodesic as requested in the claim, then there exists $w'_1 \in L_x$ such that $N[w'_1] \cap V(Q'_{w_1}) = \varnothing$ and $N[w_1] \cap V(Q'_{w'_1}) = \varnothing$, where Q'_{w_1} and $Q'_{w'_1}$ are a w_1, w_2-geodesic and a w'_1, w_2-geodesic, respectively. It follows that $x_1 \notin V(Q'_{w_1}) \cup V(Q'_{w'_1})$, and $\{w_1, w'_1, w_2\}$ is an asteroidal triple. ■

Let R be the w_1, w_2-geodesic specified in the claim above. By (2) the path $R' = x' \to u \underset{R}{\rightsquigarrow} v \to y'$ is an x', y'-path shorter than Q. By (1) we have either $d(w_1, w_2) = d(u, v) + 2$ or $d(w_1, w_2) > d(u, v) + 2$. For the former case every leaf in $\tilde{L}$ attaches to R', and Q can be substituted for R' to form a smaller tree containing W. For the latter, the w_1, w_2-geodesic is of the form

$$w_1 \to u \underset{R}{\rightsquigarrow} v \to v' \to w_2,$$

where v' is the neighbor of w on the w_1, w_2-geodesic. Every leaf in $\tilde{L}$ attaches to the subtree T' that consists of the path $x' \to u \underset{R}{\rightsquigarrow} v \to y'$ and the edge $v \to v'$. We can obtain a subgraph of G containing W with fewer edges by substituting Q for T'. This leads to a contradiction, and $w_1 \to x_1 \underset{P}{\rightsquigarrow} y_1 \to w_2$ is a w_1, w_2-geodesic containing z.

As a result, in either $|\{x', y'\} \cap \{x, y\}| = 1$ or $|\{x', y'\} \cap \{x, y\}| = 2$, there is a requested geodesic, and the theorem is proved. □

The following is an immediate consequence of Theorem 1.

Corollary 1. *Let G be* AT-*free. It holds that* $g(G) \le s(G)$.

Although $g(G) \le s(G)$ when G is AT-free, the equality is not guaranteed to hold even for subclasses like unit-interval graphs, as shown in Theorem 2.

Theorem 2. *For a unit-interval graph, the geodetic number and the Steiner number are, in general, not equal. Moreover, the difference between the two numbers can be arbitrarily large.*

Proof. Consider the unit interval graph given in Fig. 1. We show that the geodetic number is 4 and the Steiner number is 5.

It is shown that there is a geodetic set of size 4. We claim that there is no geodetic set of size 3. Since both I_1 and I_3 are simplicial— that is — each of the neighborhoods is a clique, a geodesic contains neither of them. Furthermore, I_2 is adjacent to all the other vertices. No geodesic has I_2 as an endpoint. By symmetry, $\{I_1, I_3, I_4\}$ has to be a geodetic set if there exists one of size 3. However, neither I_5 nor I_6 is on the geodesics with endpoints in $\{I_1, I_3, I_4\}$. It follows that the geodetic number of the graph is 4.

For the Steiner number, a Steiner set of size 5 is given in Fig. 1. We show that the size of a Steiner set of the graph is at least 5. Note that the Steiner set containing I_2 consists of all the nine vertices so we consider the Steiner sets that do not contain I_2. Since the neighborhoods of I_1 and I_3 are cliques, both of them have to be in a Steiner set. The minimal connected subgraph containing I_1 and I_3 has no vertices in $X = \{I_i \mid 4 \le i \le 9\}$. Thus, for a Steiner set of size less than 9 at least one element of X has to be in. Because of symmetry, assume that I_4 is in the Steiner set. Since a minimal connected subgraph containing I_1, I_3, and I_4 contains neither I_5 nor I_6, it follows that a Steiner set contains $\{I_4, I_5, I_6\}$ or, by symmetry, $\{I_7, I_8, I_9\}$. Along with I_1 and I_3, we have that the Steiner number of the graph is at least 5.

To complete the proof, we modify the graph by adding $2n - 2$ vertices, where $n - 1$ of them correspond to the interval identical to I_4 and the other $n - 1$ correspond to I_7. Then the gap between the Steiner number and the geodetic number becomes n. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] F. Buckley, F. Harary, L. Quintas, Extremal results on the geodetic number of a graph, Scientia, Ser. A 2 (1988) 17–26.
[2] D.G. Corneil, S. Olariu, L. Stewart, Asteroidal triple-free graphs, SIAM J. Discrete Math. 10 (1997) 399–430.
[3] C. Hernando, T. Jiang, M. Mora, P. Pelayo, C. Seara, On the Steiner, geodetic and hull numbers of graphs, Discrete Math. 293 (2005) 139–154.
[4] C. Lekkerkerker, J. Boland, Representation of a finite graph by a set of intervals on the real line, Fundam. Math. 51 (1962) 45–64.
[5] I. Pelayo, Geodesic Convexity in Graphs, Springer Briefs in Mathematics, Springer Science+Business Media, New York, 2013.
[6] I. Pelayo, Comment on "The Steiner number of a graph" by G. Chartrand and P. Zhang: [Discrete Math. 242 (2002) 41–54], Discrete Math. 280 (2004) 259–263.