Logics for reasoning about degrees of confirmation

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Abstract

In this paper, we present a first-order and a propositional logic for reasoning about degrees of confirmation. We define the appropriate formal languages and describe the corresponding classes of models. We provide infinitary axiomatizations for both logics and we prove that the axiomatizations are sound and strongly complete. We also show that our propositional logic is decidable. For some restrictions of the logics, we provide finitary axiomatic systems.

Keywords: probabilistic logic, measure of confirmation, completeness theorem, decidability

1 Introduction

In the past several decades, different tools have been developed for representing and reasoning with uncertain knowledge, including probability as a dominant representation of uncertainty. One particular line of research concerns the formalization in terms of probabilistic logic. The modern development in this field started with Keisler's seminal work on probabilistic quantifiers [24]. After Nilsson [27] rediscovered Boole's procedure for probabilistic entailment which, given probabilities of premises, calculates bounds on the probabilities of the conclusion, researchers from the areas of logic, computer science and artificial intelligence started investigations about formal systems for probabilistic reasoning and provided several languages, axiomatizations and decision procedures for various probabilistic logics [2, 5, 11, 13, 14, 16, 18–20, 22, 28, 29]. Those logics extend the classical (propositional or first-order) calculus with expressions that speak about probability, while formulas remain true or false. They allow one to formalize statements of the form 'the probability of α is at least a half.' The corresponding probability operators behave like modal operators and

[1]This paper is a revised and extended version of the conference paper [4] presented at the Third International Conference on Logic and Argumentation, in which we introduced a propositional logic for reasoning about degrees of confirmation, where nesting of the probabilistic and confirmation operators is not allowed. In this work, we develop an extension of the logic from [4] by allowing iterations of the operators and we axiomatize the logic using a similar technique as in [4]. However, the proof of decidability is completely different from the corresponding proof form [4]. In addition, in this work, we also present a first-order variant of the logic and the variants of the logic with finite ranges of probabilities, for which we propose finitary axiomatizations.

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the corresponding semantics consists of special types of Kripke models, with accessibility relations replaced with probability measures defined over the worlds.

This paper contributes to the field by proposing a logical formalization of the Bayesian measure of confirmation (or evidential support). Although contemporary Bayesian confirmation theorists investigated degrees of confirmation developing a variety of different probability-based measures, that field attracted little attention from the logical side, probably because of the complexity of a potential formal language that would be adequate to capture those measures. In Carnap's [3] book, one of the main tasks is 'the explication of certain concepts which are connected with the scientific procedure of confirming or disconfirming hypotheses with the help of observations and which we therefore will briefly call *concepts of confirmation*'. Carnap distinguished three different semantical concepts of confirmation: the classificatory concept ('a hypothesis A is confirmed by an evidence B'), the comparative concept ('A is confirmed by B at least as strongly as C is confirmed by D') and the quantitative concept of confirmation. The third one, one of the basic concepts of inductive logic, is formalized by a numerical function c which maps pairs of sentences to the reals, where $c(A,B)$ is the *degree of confirmation* of the hypothesis A on the basis of the evidence B.

Bayesian epistemology proposes various candidate functions for measuring the degree of confirmation $c(A,B)$, defined in terms of subjective probability. They all agree in the following qualitative way: $c(A,B) > 0$ iff the posterior probability of A on the evidence B is greater than the prior probability of A (i.e. $\mu(A|B) > \mu(A)$), which correspond to the classificatory concept ('A is confirmed by B') [15]. Up to now, only the classificatory concept of confirmation is logically formalized, in our previous work [7].

In this paper, we formalize the quantitative concept of confirmation, first within a propositional logical framework $\mathrm{LPP}_1^{\mathrm{conf}}$ and then using its first-order extension $\mathrm{LFOP}_1^{\mathrm{conf}}$. We focus on the most standard[2] measure of degree of confirmation, called *difference* measure:

$$c(A,B) = \mu(A|B) - \mu(A).$$

Our formal languages extend classical (propositional/first-order) logic with the unary probabilistic operators of the form $P_{\geq r}$ ($P_{\geq r}\alpha$ reads 'the probability of α is at least r'), where r ranges over the set of rational numbers from the unit interval [28], and the binary operators $c_{\geq r}$ and $c_{\leq r}$, which we semantically interpret using the difference measure. The corresponding semantics consists of a special type of Kripke models, with probability measures defined over the worlds.

Our main results are sound and strongly complete (every consistent set of formulas is satisfiable) axiomatizations for the logics. We prove completeness using a modification of Henkin's construction. Since the logics are not compact, in order to obtain the strong variant of completeness, we use infinitary inference rules. From the technical point of view, we modify some of our earlier methods presented in [8–10, 26, 30, 32, 34]. We point out that our formal languages are countable and all formulas are finite, while only proofs are allowed to be infinite. However, for some restrictions of the logics, we provide finitary axiomatic systems. We also prove that our propositional logic $\mathrm{LPP}_1^{\mathrm{conf}}$ is decidable, combining the method of filtration [23] and a reduction to a system of inequalities.

Many measures of confirmation have been proposed over the years. We point out that it is not our intention to pick sides among them. We simply chose the difference measure because of its popularity. However, we discuss in Section 10 that our axiomatization technique can be easily modified to incorporate other measures of confirmation.

[2] According to Eells and Fitelson [12].

The structure of this paper is as follows. In Section 2, we recall some basic notions of probability. In Section 3, we present the syntax and semantics of our propositional logic $\mathrm{LPP}_1^{\mathrm{conf}}$ in detail. In Section 4, we propose an axiomatization for $\mathrm{LPP}_1^{\mathrm{conf}}$ and we prove its soundness. In Section 5, we prove that the axiomatization is strongly complete with respect to the proposed semantics. In Section 6, we show that the satisfiability problem for $\mathrm{LPP}_1^{\mathrm{conf}}$ is decidable. In Section 7, we present the first-order extension $\mathrm{LFOP}_1^{\mathrm{conf}}$ of our logic $\mathrm{LPP}_1^{\mathrm{conf}}$, but we do not go into the details. In Section 8, we discuss the cases where the probabilities are restricted to a finite set and we propose finitary strongly complete axiomatizations for those logics. We conclude in Section 9.

2 Preliminaries

Let us introduce some basic probabilistic notions that will be used in this paper.

For a non-empty set $W \neq \emptyset$, we say that $H \subseteq 2^W$ is an *algebra of subsets* of W, if the following conditions hold:

1. $W \in H$;
2. if $A \in H$, then $W \setminus A \in H$; and
3. if $A, B \in H$, then $A \cup B \in H$.

For a given algebra H of subsets of W, a function $\mu : H \longrightarrow [0, 1]$ is a *finitely additive probability measure*, if it satisfies the following properties:

1. $\mu(W) = 1$;
2. $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$.

For W, H and μ described above, the triple $\langle W, H, \mu \rangle$ is called a finitely additive *probability space*. The elements of H are called *measurable sets*.

For a probability measure μ, the conditional probability is defined in the following way:

$$\mu(A|B) = \begin{cases} \frac{\mu(A\cap B)}{\mu(B)}, & \mu(B) > 0 \\ \text{undefined}, & \mu(B) = 0. \end{cases}$$

As we mentioned in the introduction, several functions for measuring the degree of confirmation, based on probabilities, are proposed in the literature (see, e.g. [33]). In this paper, we focus on the *difference measure*. For a given probability measure μ, the difference measure is defined in [12] as

$$c(A, B) = \mu(A|B) - \mu(A),$$

where the value $c(A, B)$ represents *the degree of confirmation* of the hypothesis A on the basis of evidence B. Note that according to the definition of conditional probability $c(A, B)$ is not defined when $\mu(B) = 0$.

It easy to see that if $\mu(A|B) > \mu(A)$, then $c(A, B) > 0$ and we say that B *confirms* A. If $\mu(A|B) < \mu(A)$, then $c(A, B) < 0$ and we say that B *disconfirms* A. Finally, if $\mu(A|B) = \mu(A)$, then $c(A, B) = 0$ and we say that A and B are *independent*.

In this paper, we always interpret the degree of confirmation as the value based on the difference measure. However, in the conclusion, we discuss how our results can be adapted to other measures of confirmation.

EXAMPLE 2.1
Consider a family that usually spends Sundays in picnics, going on picnics 9 out of 10 Sundays, but in the case of rain, they go once in 10 times. The probabilities are $\mu(Picnic) = 0.9$ and $\mu(Picnic|Rain) = 0.1$. Then, we can calculate

$$c(Picnic, Rain) = \mu(Picnic|Rain) - \mu(Picnic) = -0.8.$$

In this case, we can see that the evidence *Rain* disconfirms the hypotheses *Picnic* with degree -0.8.

Note that a probability measure μ does not assign a probability to all subsets of W but only to the subsets that belong to H. One way of assigning a value to every set is by considering two functions induced by μ: the *inner measure* μ_* and the *outer measure* μ^*. They are defined as

$$\mu_*(A) = \sup\{\mu(B) \mid B \subset A,\ B \in H\}, \quad \mu^*(A) = \inf\{\mu(B) \mid A \subset B,\ B \in H\},$$

where A is a subset of W and inf and sup denote the infimum and supremum functions, respectively. It is easy to see that μ_* and μ^* coincide on measurable sets, i.e. if A is a from H, then $\mu(A) = \mu_*(A) = \mu^*(A)$.

3 The logic $\mathrm{LPP}_1^{\mathrm{conf}}$: syntax and semantics

In this section, we introduce the set of formulas of the logic[3] $\mathrm{LPP}_1^{\mathrm{conf}}$ and the class of semantical structures in which those formulas are evaluated.

3.1 Syntax

Let $\mathcal{P} = \{p, q, r, \dots\}$ be a denumerable set of propositional letters. Let Q denote the set of all rational numbers. For given rational numbers a and b such that $a < b$, let $[a,b]_Q$ denote the set $[a,b] \cap Q$. The language of the logic $\mathrm{LPP}_1^{\mathrm{conf}}$ is built up from

- the elements of the set $\mathcal{P}$;
- the classical propositional connectives $\neg$ and $\wedge$;
- the list of unary probability operators of the form $P_{\geq r}$, for every $r \in [0,1]_Q$;
- the list of binary probability operators of the form $c_{\geq r}$, for every $r \in [-1,1]_Q$; and
- the list of binary probability operators of the form $c_{\leq r}$, for every $r \in [-1,1]_Q$.

Note that we use conjunction and negation as primitive connectives. The other propositional connectives, $\vee$, $\rightarrow$ and $\leftrightarrow$, are introduced as abbreviations, in the usual way.

DEFINITION 3.1 (LPP_1^{conf}-Formula).
The set $For_{LPP_1^{conf}}$ of all formulas of the logic $\mathrm{LPP}_1^{\mathrm{conf}}$ is the smallest set such that

1. $\mathcal{P} \subset For_{LPP_1^{conf}}$;
2. if $\alpha \in For_{LPP_1^{conf}}$ and $r \in [0,1]_Q$, then $P_{\geq r}\alpha \in For_{LPP_1^{conf}}$;
3. if α and β are LPP_1^{conf}-formulas and $r \in [-1,1]_Q$, then $c_{\geq r}(\alpha,\beta), c_{\leq r}(\alpha,\beta) \in For_{LPP_1^{conf}}$;
4. if α and β are LPP_1^{conf}-formulas, then $\neg\alpha, \alpha \wedge \beta \in For_{LPP_1^{conf}}$.

[3] As we will discuss later, our language extends the language of the logic LPP_1 [29], with adding confirmation operators to the syntax. In LPP_1, L stands for *logic*, P for *propositional* and the second P stands for *probability*. Following the notation form [29], we used the index 1 to denote that the logic allows nesting of probability operators.

We denote arbitrary formulas by $\alpha, \beta, \gamma, \ldots$, possibly with subscripts.

Intuitively, $P_{\geq r}\alpha$ means that the probability that α is true is greater or equal to r, while $c_{\geq r}(\alpha, \beta)$ ($c_{\leq r}(\alpha, \beta)$) means that the formula β confirms the formula α with the degree at least r (at most r, respectively). As we pointed out in the introduction, we focus on the most standard measure of degree of confirmation, difference measure, so the interpretation of the formula $c_{\geq r}(\alpha, \beta)$ is 'conditional probability of α given β minus probability of α'. The operators $P_{\geq r}$ are the standard probability operators [29] and the language of the logic $\mathrm{LPP}_1^{\mathrm{conf}}$ can be seen as an extension of the language of LPP_1 logic [29] with two types of binary confirmation operators, $c_{\geq r}$ and $c_{\leq r}$.

The other types of probabilistic operators are defined as follows: $P_{<r}\alpha$ is $\neg P_{\geq r}\alpha$, $P_{\leq r}\alpha$ is $P_{\geq 1-r}\neg\alpha$, $P_{>r}\alpha$ is $\neg P_{\leq r}\alpha$ and $P_{=r}\alpha$ is $P_{\geq r}\alpha \wedge P_{\leq r}\alpha$. We use the following abbreviations to introduce other types of confirmation operators:

- $c_{=r}(\alpha, \beta)$ is $c_{\geq r}(\alpha, \beta) \wedge c_{\leq r}(\alpha, \beta)$,
- $c_{>r}(\alpha, \beta)$ is $c_{\geq r}(\alpha, \beta) \wedge \neg c_{\leq r}(\alpha, \beta)$ and
- $c_{<r}(\alpha, \beta)$ is $c_{\leq r}(\alpha, \beta) \wedge \neg c_{\geq r}(\alpha, \beta)$.

Also, following the usual convention, we denote $\alpha \wedge \neg\alpha$ by $\perp$ and $\alpha \vee \neg\alpha$ by $\top$.

One might think that $c_{<r}(\alpha, \beta)$ could be defined simply as $\neg c_{\geq r}(\alpha, \beta)$, in an analogous way as $P_{<s}$ is introduced. However, we will see later that this does not hold under our satisfiability relation.

The following example illustrates the meaning of a confirmation operator.

EXAMPLE 3.2 (Continued).
The situation when the evidence *Rain* disconfirms the hypotheses *Picnic* with the degree -0.8 might be represented by the formula

$$c_{\leq -0.8}(p, r),$$

where the propositional letter p stands for *Picnic* and r for *Rain*.

3.2 *Semantics of* $\mathrm{LPP}_1^{\mathrm{conf}}$

Now, we introduce the semantics of our logic $\mathrm{LPP}_1^{\mathrm{conf}}$. We start by recalling the standard probability structure (Definition 3.3) for probabilistic logic [28]. Nevertheless, that is just a first step, since evaluation of the formulas from $For_{\mathrm{LPP}_1^{\mathrm{conf}}}$ imposes additional measurability constraints specific for this logic. For that reason, we introduce the concept of $\mathrm{LPP}_1^{\mathrm{conf}}$-*measurable structure* (Definition 3.6). In the following sections, we will prove strong completeness theorem with respect to the class of $\mathrm{LPP}_1^{\mathrm{conf}}$-measurable structures.

DEFINITION 3.3 (LPP_1-Structure [28]).
An LPP_1-structure is a tuple $(W, Prob, v)$ where

1. W is a non-empty set of objects called *worlds*;
2. $v : W \times \mathcal{P} \to \{true, false\}$ assigns to each world $w \in W$ a two-valued evaluation $v(w, \cdot)$ of propositional letters;
3. $Prob(w) = (W(w), H(w), \mu(w))$ is a triple where
   - $W(w)$ is a non-empty subset of W,
   - $H(w)$ is an algebra of subsets of $W(w)$,
   - $\mu(w) : H(w) \longrightarrow [0, 1]$ is a finitely additive measure.

Thus, this semantics of probability logic uses possible world structures, with a valuation assigned to each world. In standard modal logic, the truth of a formula is determined relative to a world w and its truth value may depend on what is true at other accessible worlds, which is formalized by choosing an accessibility relation. In our semantics, an accessibility relation is replaced with probability measures defined over the worlds. This modification allows us to measure, from each world w, probabilities of sets of worlds that are considered possible in w (i.e. they are within $W(w)$).

EXAMPLE 3.4
Let us consider a finite set of propositional letters $A = \{p, q, r\}$ and the following structure $M = (W, Prob, v)$ such that

– $W = \{w, t, u\}$;
– $Prob(w) = (W(w), H(w), \mu(w))$
  - $W(w) = W$;
  - $H(w)$ is the power set $\mathrm{P}(W)$;
  - $\mu(w)$ is characterized by $\mu(w)(\{w\}) = \mu(w)(\{t\}) = \frac{2}{5}$, $\mu(w)(\{u\}) = \frac{1}{5}$ (other values can be easily calculated using the properties of the measure $\mu(w)(\emptyset) = 0$, $\mu(w)(\{w, t\}) = \frac{4}{5}$, $\mu(w)(\{w, u\}) = \mu(w)(\{t, u\}) = \frac{3}{5}$ and $\mu(w)(W) = 1$).

– $Prob(t) = (W(t), H(t), \mu(t))$
  - $W(t) = W$;
  - $H(t)$ is the power set $\mathrm{P}(W)$;
  - $\mu(t)$ is characterized by $\mu(t)(\{w\}) = \mu(t)(\{u\}) = \frac{1}{4}$, $\mu(t)(\{t\}) = \frac{1}{2}$.

– $Prob(u) = (W(u), H(u), \mu(u))$
  - $W(u) = W$;
  - $H(u)$ is the power set $\mathrm{P}(W)$;
  - $\mu(u)$ is characterized by $\mu(u)(\{w\}) = \frac{1}{3}$,$\mu(u)(\{t\}) = \frac{1}{2}$, $\mu(u)(\{u\}) = \frac{1}{6}$.

– $v(w, p) = v(w, q) = v(w, \neg r) = true$, $v(t, p) = v(t, \neg q) = v(t, r) = true$ and $v(u, p) = v(u, q) = v(u, r) = true$.

Next, we define what does it mean that a formula is satisfied in a world of an LPP_1-structure. Intuitively, if we assume that satisfaction of a formula α is already determined in every world, then $P_{\geq r}\alpha$ should hold in the world w if the set S of all worlds from $W(w)$ in which α holds is such that $\mu(w)(S) \geq r$.

The problem with a direct formalization of this idea is that the set S might not be in $H(w)$, in which case we cannot apply the measure $\mu(w)$. For that reason, we follow the approach of Fagin and Halpern [13] and first define the satisfiability relation $\models$ using inner and outer measures. Then, we restrict our attention to the measurable structures, in which we know that formulas correspond to the measurable sets of worlds.

DEFINITION 3.5
Let M be an LPP_1-structure, and let w be some world from M. The *satisfiability relation* $\models$ is defined recursively as follows:

1. if $\alpha \in \mathcal{P}$ then $M, w \models \alpha$ iff $v(w, \alpha) = true$;
2. $M, w \models P_{\geq r}\alpha$ if $\mu_*(w)(\{w' \in W(w) \mid M, w' \models \alpha\}) \geq r$;
3. $M, w \models c_{\geq r}(\alpha, \beta)$ if $\mu^*(w)(\{w' \in W(w) \mid M, w' \models \beta\}) > 0$ and $\frac{\mu_*(w)(\{w' \in W(w) \mid M, w' \models \alpha \wedge \beta\})}{\mu^*(w)(\{w' \in W(w) \mid M, w' \models \beta\})} - \mu^*(w)(\{w' \in W(w) \mid M, w' \models \alpha\}) \geq r$;

4. $M, w \models c_{\leq r}(\alpha, \beta)$ if $\mu_(w)(\{w' \in W(w) \mid M, w' \models \beta\}) > 0$ and $\frac{\mu^*(w)(\{w' \in W(w) \mid M, w' \models \alpha \wedge \beta\})}{\mu_*(w)(\{w' \in W(w) \mid M, w' \models \beta\})} - \mu_*(w)(\{w' \in W(w) \mid M, w' \models \alpha\}) \leq r$;
5. $M, w \models \neg\alpha$ iff $M, w \not\models \alpha$;
6. $M, w \models \alpha \wedge \beta$ iff $M, w \models \alpha$ and $M, w \models \beta$.

In order to relax the notation, we denote by $[\alpha]_{M,w}$ the set of all worlds from $W(w)$ in which α holds, i.e.

$$[\alpha]_{M,w} = \{w' \in W(w) \mid M, w' \models \alpha\}.$$

We write $[\alpha]$ instead of $[\alpha]_{M,w}$ when M and w are clear from the context.

DEFINITION 3.6 ($\mathrm{LPP}_1^{\mathrm{conf}}$-Measurable structure).
An LPP_1-structure M is $\mathrm{LPP}_1^{\mathrm{conf}}$-measurable iff $[\alpha]_w \in H(w)$ for every world w from M and every formula $\alpha \in For_{\mathrm{LPP}_1^{\mathrm{conf}}}$. We denote the set of all $\mathrm{LPP}_1^{\mathrm{conf}}$-measurable LPP_1-structures with $\mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$.

REMARK 3.7
Restriction to the class of measurable structures is a standard approach in the field of probabilistic logic [14, 29]. Note that logics with different languages have different classes of measurable structures. For example, the set $S = [c_{\geq r}(\beta, \gamma)]$ is a measurable set in every world w of every structure $M \in \mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$. However, S is not necessarily a measurable set in an arbitrary measurable structure of the logic LPP_1, simply because $c_{\geq r}(\beta, \gamma)$ is not a formula of that logic.

In this paper, we focus on $\mathrm{LPP}_1^{\mathrm{conf}}$-measurable structures and we prove completeness and decidability results for this class of structures. Note that for the class of $\mathrm{LPP}_1^{\mathrm{conf}}$-measurable structures, we do not need inner and outer measures in the definition of $\models$. Indeed, since μ, μ^* and μ_* coincide on measurable sets, in Definition 3.5, we can replace Conditions 2–4 with

2* $M, w \models P_{\geq r}\alpha$ if $\mu(w)(\{w' \in W(w) \mid M, w' \models \alpha\}) \geq r$;
3* $M, w \models c_{\geq r}(\alpha, \beta)$ if $\mu(w)(\{w' \in W(w) \mid M, w' \models \beta\}) > 0$ and $\mu^(w)(\{w' \in W(w) \mid M, w' \models \alpha\} \mid \{w' \in W(w) \mid M, w' \models \beta\}) - \mu(w)(\{w' \in W(w) \mid M, w' \models \alpha\}) \geq r$;
4* $M, w \models c_{\leq r}(\alpha, \beta)$ if $\mu(w)(\{w' \in W(w) \mid M, w' \models \beta\}) > 0$ and $\mu(w)(\{w' \in W(w) \mid M, w' \models \alpha\} \mid \{w' \in W(w) \mid M, w' \models \beta\}) - \mu(w)(\{w' \in W(w) \mid M, w' \models \alpha\}) \leq r$.

EXAMPLE 3.8 (Continued).
Let us consider satisfiability of the formulas

$$c_{\leq -0.2}(p \wedge q, r) \text{ and } c_{\leq -0.2}(p \wedge q, P_{\geq 0.2}r)$$

in the world w of the model M.

First formula: we know that $M, w \models c_{\leq -0.2}(p \wedge q, r)$ if $\mu(w)([r]_w) > 0$ and $\mu(w)([p \wedge q]_w | [r]_w) - \mu(w)([p \wedge q]_w) \leq -0.2$.

For the world w, we have $[r]_w = \{t, u\}$, so it is $\mu(w)([r]_w) = \mu(w)(\{t, u\}) = \frac{3}{5} > 0$. Similarly, we have that $\mu(w)([p \wedge q \wedge r]_w) = \mu(w)(\{u\}) = \frac{1}{5}$ and $\mu(w)([p \wedge q]_w) = \mu(w)(\{w, u\}) = \frac{3}{5}$. In that case, we have $\mu(w)([p \wedge q]_w | [r]_w) - \mu(w)([p \wedge q]_w) = \frac{1}{3} - \frac{3}{5} = -\frac{4}{15} \leq -0.2$. So we get $M, w \models c_{\leq -0.2}(p \wedge q, r)$.

Second formula: we know that $M, w \models c_{\leq -0.2}(p \wedge q, P_{\geq 0.2}r)$ if $\mu(w)([P_{\geq 0.2}r]_w) > 0$ and $\mu(w)([p \wedge q]_w | [P_{\geq 0.2}r]_w) - \mu(w)([p \wedge q]_w) \leq -0.2$.

It easy to check that $[P_{\geq 0.2}r]_w = \{w, t, u\}$, i.e. $\mu(w)([P_{\geq 0.2}r]_w) = 1$. We get $\mu(w)([p \wedge q]_w | [P_{\geq 0.2}r]_w) - \mu(w)([p \wedge q]_w) = \frac{3}{5} - \frac{3}{5} = 0 > -0.2$. So, $M, w \not\models c_{\leq -0.2}(p \wedge q, P_{\geq 0.2}r)$.

Using the definition of satisfiability relation and properties of reals, it is easy to obtain satisfiability conditions for the other types of operators. For example,

4* $M, w \models c_{<r}(\alpha, \beta)$ iff $\mu(w)([\beta]) > 0$ and $\mu(w)([\alpha] | [\beta]) - \mu(w)([\alpha]) < r$.

Now, it is obvious that the operator $c_<$ is not equivalent to the 'negation of $c_\geq$', i.e. $M, w \not\models c_{\geq r}(\alpha, \beta)$ does not imply $M, w \models c_{<r}(\alpha, \beta)$, the reason is that $c([\alpha], [\beta])$ might simply be undefined in M (if $\mu(w)([\beta]_{M,w}) = 0$).

At the end of this section, we define some basic semantical notions. We start with the concept of a model.

DEFINITION 3.9 (Model).
For an $M = (W, Prob, v) \in \mathrm{LPP}^{\mathrm{conf}}_{1,\mathrm{Meas}}$, $w \in W$ and a set of formulas T, we say that M, w is a *model* (or *pointed model*) of T and write $M, w \models T$, iff $M, w \models \alpha$ for every $\alpha \in T$. The set T is $\mathrm{LPP}^{\mathrm{conf}}_{1,\mathrm{Meas}}$-*satisfiable*, if there is $M \in \mathrm{LPP}^{\mathrm{conf}}_{1,\mathrm{Meas}}$ and a world w from M such that $M, w \models T$. Formula α is valid if $\neg\alpha$ is not satisfiable.

In Sections 4– 6, we simply say that a set is *satisfiable* when we refer to $\mathrm{LPP}^{\mathrm{conf}}_{1,\mathrm{Meas}}$-*satisfiablity*, if it is clear from context.

Now, we define the notion of semantical entailment.

DEFINITION 3.10 (Entailment).
We say that a set of formulas T *entails* a formula α and write $T \models \alpha$, if for every $M = (W, Prob, v) \in \mathrm{LPP}^{\mathrm{conf}}_{1,\mathrm{Meas}}$ and every $w \in W$ if $M, w \models T$, then $M, w \models \alpha$.

4 Axiomatization of $\mathrm{LPP}^{\mathrm{conf}}_1$

In this section, we present an axiomatization of our logic, which we denote by $Ax(\mathrm{LPP}^{\mathrm{conf}}_1)$. The axiom system $Ax(\mathrm{LPP}^{\mathrm{conf}}_1)$ contains 10 axiom schemes and 5 inference rules. We implicitly assume that all formulas respect Definition 3.1. For example, we consider only those instances of A9 and A10 for which $s(r+t) \leq 1$.

Axiom schemes:

(A1) All instances of classical propositional tautologies.
(A2) $P_{\geq 0}\alpha$
(A3) $P_{\leqslant r}\alpha \to P_{<s}\alpha$ whenever $r < s$
(A4) $P_{<r}\alpha \to P_{\leqslant r}\alpha$
(A5) $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg\alpha \vee \neg\beta)) \to P_{\geq r+s}(\alpha \vee \beta)$
(A6) $(P_{\leqslant r}\alpha \wedge P_{<s}\beta) \to P_{<r+s}(\alpha \vee \beta)$
(A7) $c_{\geq r}(\alpha, \beta) \to P_{>0}\beta$
(A8) $c_{\leq r}(\alpha, \beta) \to P_{>0}\beta$
(A9) $(P_{\geq t}\alpha \wedge P_{\geq s}\beta \wedge c_{\geq r}(\alpha, \beta)) \to P_{\geq s(r+t)}(\alpha \wedge \beta)$
(A10) $(P_{\leq t}\alpha \wedge P_{\leq s}\beta \wedge c_{\leq r}(\alpha, \beta)) \to P_{\leq s(r+t)}(\alpha \wedge \beta)$

Inference rules:

(R1) From $\{\alpha, \alpha \to \beta\}$ infer β.
(R2) From α infer $P_{\geq 1}\alpha$.
(R3) From the set of premises $\{\gamma \to P_{\geq r-\frac{1}{k}}\alpha \mid k \in \mathbb{N}, k \geq \frac{1}{r}\}$ infer $\gamma \to P_{\geq r}\alpha$.
(R4) From the set of premises

$$\{\gamma \to P_{>0}\beta\} \cup \{\gamma \to ((P_{\geq t}\alpha \wedge P_{\geq s}\beta) \to P_{\geq s(r+t)}(\alpha \wedge \beta)) | t, s \in [0,1]_Q\}$$

infer $\gamma \to c_{\geq r}(\alpha, \beta)$.
(R5) From the set of premises

$$\{\gamma \to P_{>0}\beta\} \cup \{\gamma \to ((P_{\leq t}\alpha \wedge P_{\leq s}\beta) \to P_{\leq s(r+t)}(\alpha \wedge \beta)) \mid t, s \in [0,1]_Q\}$$

infer $\gamma \to c_{\leq r}(\alpha, \beta)$.

Let us briefly comment on the axiomatization $Ax(\mathrm{LPP}_1^{\mathrm{conf}})$. The axioms A1–A6 and the inference rules R1–R3 form the axiom system for the logic LPP_1 [29]. The rule R3 is the so-called Archimedean rule. It ensures that the ranges of probability measures do not take non-standard values (in the sense of non-standard analysis). Intuitively, it claims that if the probability of α is approximately close to r, then it must be r. The axioms A7 and A9, together with the rule R4 properly capture the third condition of Definition 3.5. Similarly, A8, A10 and R5 correspond to the fourth condition of Definition 3.5.

The rules R3–R5 are infinitary inference rules. The necessity of employing such rules comes form the non-compactness phenomena. Indeed, it is known that in a real-valued probabilistic logic, there exist unsatisfiable, but finitely satisfiable, sets of formulas. As pointed out in [36], one consequence of that fact is that any finitary axiomatization would not be strongly complete.

Let us now define some basic notions of proof theory.

DEFINITION 4.1 (Theorem, proof).
A formula α is a *theorem*, denoted by $\vdash_{Ax(\mathrm{LPP}_1^{\mathrm{conf}})} \alpha$, if there is a sequence of formulas $\alpha_0, \alpha_1, \ldots, \alpha_\lambda$ (λ is finite or countable ordinal), such that $\alpha_\lambda = \alpha$ and every α_i, $i < \lambda$ is an axiom, or it is derived from the preceding formulas by an inference rule.

A formula α is deducible from a set $T \subseteq For_{\mathrm{LPP}_1^{\mathrm{conf}}}$ ($T \vdash_{Ax(\mathrm{LPP}_1^{\mathrm{conf}})} \alpha$) if there is a sequence of formulas $\alpha_0, \alpha_1, \ldots, \alpha_\lambda$ (λ is finite or countable ordinal), such that $\alpha_\lambda = \alpha$ and every α_i is an axiom or a formula from T, or it is derived from the preceding formulas by an inference rule, with the exception that the rule R2 can be applied to the theorems only. The sequence $\alpha_0, \alpha_1, \ldots, \alpha$ is a *proof* of α from T.

We write $\vdash$ instead of $\vdash_{Ax(\mathrm{LPP}_1^{\mathrm{conf}})}$ when it is clear from context.

Note that the length of a proof is any countable *successor* ordinal. Let us illustrate this with a simple example in which we apply the inference rule R3. If $T = \{\gamma \to P_{\geq \frac{1}{2}-\frac{1}{k}}\alpha \mid k \in \mathbb{N}, k \geq 2\}$, then we can enumerate the elements of T: $\alpha_0 = \gamma \to P_{\geq 0}\alpha$, $\alpha_1 = \gamma \to P_{\geq \frac{1}{2}-\frac{1}{3}}\alpha$, $\ldots$, $\alpha_n = \gamma \to P_{\geq \frac{1}{2}-\frac{1}{n+2}}\alpha, \ldots$

If we denote by α_ω (where ω denotes the smallest infinite ordinal) the formula $\gamma \to P_{\geq \frac{1}{2}}\alpha$, then the infinite sequence

$$\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots, \alpha_\omega$$

is a proof of $\gamma \to P_{\geq \frac{1}{2}} \alpha$ from T (obtained by the application of R3). We can observe that the length of that proof is the successor ordinal $\omega + 1$.

Note that if we would remove α_ω from the sequence, we would obtain the sequence $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ of the length ω, which is a limit ordinal. That sequence does not have the last element and it is not considered a proof by Definition 4.1.

DEFINITION 4.2 (Consistency).
A set of formulas T is *inconsistent* if $T \vdash \bot$, otherwise it is *consistent*.

T is a *maximal consistent set* (mcs) of formulas if it is consistent and every proper superset of T is inconsistent.

At the end of this section, we show that the axiom system $Ax(\mathrm{LPP}_1^{\mathrm{conf}})$ is sound.

THEOREM 4.3 (Soundness).
The axiomatization $Ax(\mathrm{LPP}_1^{\mathrm{conf}})$ is sound with respect to the class of structures $\mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$.

PROOF. We can show that every instance of any axiom scheme holds in every structure and that the inference rules preserve the validity. Let us consider the axioms A7 and A9 and the rule R5. For A7, assume that $M \in \mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$ and w is a world of the model M, such that $M, w \models c_{\geq r}(\alpha, \beta)$. Then, $\mu(w)([\beta]) > 0$, so $M, w \models P_{>0}\beta$. Now, let us consider A9. Suppose that $M, w \models (P_{\geq t}\alpha \wedge P_{\geq s}\beta) \wedge c_{\geq r}(\alpha, \beta)$. Then, $\mu(w)([\alpha]) \geq t$, $\mu(w)([\beta]) \geq s$, $\mu(w)([\beta]) \geq 0$ and $\mu(w)([\alpha]|[\beta]) - \mu(w)([\alpha]) \geq r$, i.e. $\mu(w)([\alpha \wedge \beta]) \geq \mu(w)([\beta])(r + \mu(w)([\alpha]))$. This means that $\mu(w)([\alpha \wedge \beta]) \geq s(r+t)$. Therefore, $M, w \models P_{\geq s(r+t)}(\alpha \wedge \beta)$.

Now, let us consider R5. In order to show that it preserves validity, we will prove that whenever premises of the rule hold in a world, then the conclusion also holds in the same world. Assume that $M, w \models \{\gamma \to P_{>0}\beta\} \cup \{\gamma \to ((P_{\leq t}\alpha \wedge P_{\leq s}\beta) \to P_{\leq s(r+t)}(\alpha \wedge \beta)) \mid t, s \in [0,1]_Q\}$. If $M, w \not\models \gamma$, we have $M, w \models \gamma \to c_{\leq r}(\alpha, \beta)$. Now, suppose that $M, w \models \gamma$. Then, $M, w \models P_{>0}\beta$, i.e $\mu(w)([\beta]) > 0$, and $M, w \models (P_{\leq t}\alpha \wedge P_{\leq s}\beta) \to P_{\leq s(r+t)}(\alpha \wedge \beta))$ for all $t, s \in [0,1]_Q$. If the numbers $t, s \in [0,1]_Q$ are such that $t \geq \mu(w)([\alpha])$ and $s \geq \mu(w)([\beta])$, then $M, w \models P_{\leq t}\alpha \wedge P_{\leq s}\beta$, so $M, w \models P_{\leq s(r+t)}(\alpha \wedge \beta)$, i.e. $\mu(w)([\alpha \wedge \beta]) \leq s(r+t)$. Using the fact that rational numbers are dense in reals, we conclude $\mu(w)([\alpha \wedge \beta]) \leq \mu(w)([\beta])(r + \mu(w)([\alpha]))$ i.e. $\mu(w)([\alpha]|[\beta]) - \mu(w)([\alpha]) \leq r$. Assuming that $\mu(w)([\beta]) > 0$, we have that $M, w \models c_{\leq r}(\alpha, \beta)$. Thus, $M, w \models \gamma \to c_{\leq r}(\alpha, \beta)$. $\square$

5 Axiomatization issues

We start with the observation that the logic $\mathrm{LPP}_1^{\mathrm{conf}}$ is not compact. Indeed, let T be the set

$$\{c_{>0}(p,q)\} \cup \{c_{<\frac{1}{n}}(p,q) \mid n \in \mathbf{N}\},$$

where p and q are propositional letters. For every finite subset T' of T, we have the largest $k \in \mathbf{N}$ such that $c_{<\frac{1}{k}}(p,q) \in T'$. It is easy to see that there is an $\mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$-measurable structure $M_{T'} = (W', Prob', v')$ and $w' \in W'$ such that $\mu(w')([q]) > 0$, $\mu(w')([p]|[q]) - \mu(w')([p]) = \frac{1}{k+1}$ and $M_{T'}, w' \models T'$. However, there is no $\mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$-measurable structure $M = (W, Prob, v)$ and $w \in W$ such that $M, w \models T$, since for every $m > 0$ if $\mu(w)([q]) > 0$ and $\mu(w)([p]|[q]) - \mu(w)([p]) = m$, there is a $k \in \mathbf{N}$ such that $\frac{1}{k} < m$ and $M, w \not\models c_{<\frac{1}{k}}(p,q)$. If $\mu(w)([q]) > 0$ and $\mu(w)([p]|[q]) - \mu(w)[p]) = 0$, then $M, w \not\models c_{>0}(p,q)$. If $\mu(w)([q]) = 0$, then T is not satisfiable. Thus, every finitary subset T' of T is $\mathrm{LPP}^{\mathrm{conf}}_{1,\mathrm{Meas}}$-satisfiable, but the set T itself is not. Therefore, compactness theorem does not holds for $\mathrm{LPP}^{\mathrm{conf}}_{1}$.

Since the logic is not compact, there is no finitary axiomatization which is strongly complete. In the following section, we will employ infirnitary rules to obtain strong completeness.

An obvious alternative to an infinitary axiomatization is to develop a finitary system which would be weakly complete ('a formula is a theorem iff it is valid'). However, in order to formalize the degrees of confirmation, such a logic would need to allow reasoning about linear combinations of conditional probabilities. That task turned out to be very hard to accomplish. Fagin*et al.* [14] faced problems when they tried to represent conditional probabilities via a logical language with polynomial weight formulas that allow products of terms, e.g.

$$w(p_1 \wedge p_2) \cdot (w(p_1) + w(p_2)) \geq w(p_1) \cdot w(p_2).$$

Note that the above formula represents the sentence 'the conditional probability of p_2 given p_1 plus the conditional probability of p_1 given p_2 is at least 1'. The authors observed that even for obtaining the weak completeness additional expressiveness is needed. Thus, they introduced a first-order language such that variables can appear in formulas. For example, $(\forall x)(\exists y)[(3+x)w(\varphi)w(\psi)+2w(\varphi \vee \psi) \geq z]$ is a well-formed formula of that language. The corresponding axiom system contains the standard first-order axiomatization and axioms for real closed fields.

As an alternative, the researchers from the field of probability logic use the infinitary approaches [6] and fuzzy approaches [25]. Marchioni and Godo [25]consider the probability of a conditional event of the form 'α given β' as the truth-value of the fuzzy proposition $P(\alpha|\beta)$ which is read as '$P(\alpha|\beta)$ is probable.'

In the case of first-order probability logics, the situation is even worse, since the set of valid formulas of the considered logics is not recursively enumerable [1, 20]. As a consequence, no finitary axiomatization, which would be even weakly complete, is possible.

6 Completeness of $Ax(\mathrm{LPP}^{\mathrm{conf}}_{1})$

In this section, we show that the axiomatization $Ax(\mathrm{LPP}^{\mathrm{conf}}_{1})$ is strongly complete for the logic $\mathrm{LPP}^{\mathrm{conf}}_{1}$, i.e. we prove that every consistent set of formulas has a model. Completeness is proved in several steps, along the lines of Henkin construction. First, we prove that the deduction theorem holds for $Ax(\mathrm{LPP}^{\mathrm{conf}}_{1})$, using the implicative form of the infinitary rules. Then, we use the deduction theorem to show that we can extend an arbitrary consistent set of formulas T to an mcs (Lindenbaum's theorem). The standard technique needs to be adapted in presence of infinitary inference rules. Third, we use mcs to construct the canonical model. Finally, we show that the canonical model is indeed a model of T.

6.1 Lindenbaum's theorem

We start by showing that the Deduction theorem holds.

THEOREM 6.1 (Deduction theorem).
Let α and β be formulas and T a set of formulas. Then,

$$T \cup \{\alpha\} \vdash \beta \text{ iff } T \vdash \alpha \rightarrow \beta.$$

PROOF. Here, we will consider the nontrivial direction—from left to right, i.e. $T \cup \{\alpha\} \vdash \beta$ implies $T \vdash \alpha \rightarrow \beta$. So, let us assume that $T \cup \{\alpha\} \vdash \beta$. We proceed by the length of the inference. Here, we only focus on the case when β is obtained either by R2 or by R4, while the cases of applications of other infinitary rules can be handled in a similar way. Let us consider the case where $\beta = P_{\geq 1}\beta'$ is obtained from $T \cup \{\alpha\}$ by the rule R2. In that case, we have

$$T \cup \{\alpha\} \vdash \beta'$$
$$T \cup \{\alpha\} \vdash P_{\geq 1}\beta'.$$

Since R2 can be applied to theorems only, we know that β' is a theorem. Therefore, $P_{\geq 1}\beta'$ is a theorem as well, so we have

$$\vdash P_{\geq 1}\beta'$$
$$\vdash P_{\geq 1}\beta' \to (\alpha \to P_{\geq 1}\beta')$$
$$\vdash \alpha \to P_{\geq 1}\beta'$$
$$T \vdash \alpha \to P_{\geq 1}\beta'.$$

Now, we consider the case when β is derived by R4. Suppose that β is the formula $\gamma \to c_{\geq r}(\alpha', \beta')$, obtained from the set of premises $\{\gamma \to P_{>0}\beta'\} \cup \{\gamma \to ((P_{\geq t}\alpha' \wedge P_{\geq s}\beta') \to P_{\geq s(r+t)}(\alpha' \wedge \beta')) \mid t, s \in [0,1]_Q\}$. By the induction hypothesis,

$$T \vdash \alpha \to (\gamma \to P_{>0}\beta'), \text{ and}$$
$$T \vdash \alpha \to (\gamma \to ((P_{\geq t}\alpha' \wedge P_{\geq s}\beta') \to P_{\geq s(r+t)}(\alpha' \wedge \beta'))), \text{ for every } t, s \in [0,1]_Q.$$

Then, by propositional reasoning, we have

$$T \vdash (\alpha \wedge \gamma) \to P_{>0}\beta', \text{ and}$$
$$T \vdash (\alpha \wedge \gamma) \to ((P_{\geq t}\alpha' \wedge P_{\geq s}\beta') \to P_{\geq s(r+t)}(\alpha' \wedge \beta')) \text{ for every } t, s \in [0,1]_Q.$$

Applying R4, we obtain

$$T \vdash (\alpha \wedge \gamma) \to c_{\geq r}(\alpha', \beta').$$

Using A1 and R1, we obtain

$$T \vdash \alpha \to (\gamma \to c_{\geq r}(\alpha', \beta'))$$
$$T \vdash \alpha \to \beta.$$

□

Next, we prove some statements which are crucial for the proof of the completeness theorem.

THEOREM 6.2 (Lindenbaum's theorem).
Every consistent set of formulas can be extended to an mcs.

PROOF. Let T be an arbitrary consistent set of formulas. Assume that $\{\gamma_i \mid i = 0, 1, 2, \dots\}$ is an enumeration of all formulas from $For_{\mathrm{LPP}_1^{\mathrm{conf}}}$. We construct the set T^* recursively, in the following way.

1. $T_0 = T$.
2. If the formula γ_i is consistent with T_i, then $T_{i+1} = T_i \cup \{\gamma_i\}$.
3. If the formula γ_i is not consistent with T_i, then there are four cases:
   (a) if $\gamma_i = \gamma' \to P_{\geq r}\alpha$, then

$$T_{i+1} = T_i \cup \{\gamma' \to P_{<r-\frac{1}{k}}\alpha\},$$

   where k is a positive integer such that $r - \frac{1}{k} \geq 0$ and T_{i+1} is consistent;
   (b) if $\gamma_i = \gamma' \to c_{\geq r}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\gamma_i'\}$ where

$$\gamma_i' = \begin{cases} \gamma' \to P_{=0}\beta, & T_i \cup \{\gamma' \to P_{=0}\beta\} \nvdash \bot \\ \gamma' \to (P_{\geq t}\alpha \wedge P_{\geq s}\beta \wedge P_{<s(r+t)}(\alpha, \beta)), & T_i \cup \{\gamma' \to P_{=0}\beta\} \vdash \bot \end{cases}$$

and t and s are two rational numbers from the unit interval such that T_{i+1} is consistent;

(c) if $\gamma_i = \gamma' \to c_{\leq r}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\gamma_i'\}$ where

$$\gamma_i' = \begin{cases} \gamma' \to P_{=0}\beta, & T_i \cup \{\gamma' \to P_{=0}\beta\} \nvdash \bot \\ \gamma' \to (P_{\leq t}\alpha \wedge P_{\leq s}\beta \wedge P_{>s(r+t)}(\alpha, \beta)), & T_i \cup \{\gamma' \to P_{=0}\beta\} \vdash \bot \end{cases}$$

and t and s are two rational numbers from the unit interval such that T_{i+1} is consistent;

(d) otherwise, $T_{i+1} = T_i$.

4. $T^* = \bigcup_{n=0}^{\infty} T_n$.

First, using Theorem 6.1, one can prove that the set T^* is correctly defined, i.e. there exist k, t and s from Steps 3(a)–3(c) of the construction. Here, we will consider Step 3(c), while the other two steps can be shown in a similar way.

Let us assume that $T \cup \{\gamma' \to c_{\leq r}(\alpha, \beta)\}$ is inconsistent. Then, the set $T \cup \{c_{\leq r}(\alpha, \beta)\}$ is inconsistent as well. From Theorem 6.1, we obtain $T \vdash \neg c_{\leq r}(\alpha, \beta)$. Now, suppose that the set $T \cup \{\gamma' \to P_{=0}\beta\}$ is inconsistent and that the set $T \cup \{\gamma' \to (P_{\leq t}\alpha \wedge P_{\leq s}\beta \wedge P_{>s(r+t)}(\alpha, \beta))\}$ is inconsistent for every t and s. By Theorem 6.1, we obtain that $T \vdash P_{>0}\beta$ and $T \vdash \neg(P_{\leq t}\alpha \wedge P_{\leq s}\beta \wedge P_{>s(r+t)}(\alpha, \beta))$, for every t and s. Consequently,

$$T \vdash \top \to P_{>0}\beta$$

and

$$T \vdash \top \to ((P_{\leq t}\alpha \wedge P_{\leq s}\beta) \to P_{\leq s(r+t)}(\alpha, \beta)),$$

for all t and s, so from R5, we derive

$$T \vdash \top \to c_{\leq r}(\alpha, \beta).$$

Note that this contradicts our assumption that $T \cup \{c_{\leq r}(\alpha, \beta)\}$ is an inconsistent set. Thus, there are rational numbers t and s such that the set

$$T \cup \{\gamma' \to (P_{\leq t}\alpha \wedge P_{\leq s}\beta \wedge P_{>s(r+t)}(\alpha, \beta))\}$$

is consistent.

Next, we prove that T^* is an mcs. Note that every T_i is consistent by the construction. This still does not imply consistency of $T^* = \bigcup_{n=0}^{\infty} T_n$ because of the presence of the infinitary rules. In order to prove the consistency of T^*, we first show that T^* is deductively closed. If the formula γ is an instance of some axiom, then $\gamma \in T^*$ by the construction of T^*. Next, we prove that T^* is closed under the inference rules. Here, we show that T^* is closed under the rule $R5$; the other cases are similar.

First, we show that for every $\gamma' \in For_{\mathrm{LPP}_1^{\mathrm{conf}}}$ either $\gamma' \in T^*$ or $\neg\gamma' \in T^*$ holds. Let i and j be the nonnegative integers such that $\gamma_i = \gamma'$ and $\gamma_j = \neg\gamma'$. Then, either γ' or $\neg\gamma'$ is consistent with $T_{max\{i,j\}}$. If $T_{max\{i,j\}}$ is not consistent with γ' and $\neg\gamma'$, then by Theorem 6.1, $T_{max\{i,j\}}$ will bi-inconsistent. Then, either $\gamma' \in T_{i+1}$ or $\neg\gamma' \in T_{j+1}$, so either $\gamma' \in T^*$ or $\neg\gamma' \in T^*$.

Let us show that T^* is closed under the inference rule R5. Assume that

$$\gamma' \to P_{>0}\beta, \gamma' \to ((P_{\leq t}\alpha \wedge P_{\leq s}\beta) \to P_{\leq s(r+t)}(\alpha, \beta)) \in T^*$$

for all $r, s \in [0,1]_Q$. We need to show that $\gamma' \to c_{\leq r}(\alpha, \beta) \in T^*$. Assume that $\gamma' \to c_{\leq r}(\alpha, \beta) \notin T^*$. Then, by maximality of T^*, $\neg(\gamma' \to c_{\leq r}(\alpha, \beta)) \in T^*$. Thus, $\gamma' \in T^*$, so there is i such that $\gamma' \in T_i$. Let j be a nonnegative integer such that $\gamma_j = \gamma' \to c_{\leq r}(\alpha, \beta)$. By Step 3(c) of the construction of T^*, $\gamma' \to P_{=0}\beta \in T_{j+1}$, or there are $t', s' \in [0,1]_Q$ such that $\gamma' \to (P_{\leq t'}\alpha \wedge P_{\leq s'}\beta \wedge P_{>s'(r+t')}(\alpha, \beta)) \in$

T_{j+1}. Suppose that $\gamma' \to P_{=0}\beta \in T_{j+1}$, and let k be the nonnegative integer such that $\gamma_k = \gamma' \to P_{>0}\beta$. Then,

$$T_{\max\{i,k+1\}} \vdash P_{>0}\beta.$$

Note that we also have $T_{\max\{i,j+1\}} \vdash P_{=0}\beta$. Consequently, $T_{\max\{i,j+1,k+1\}} \vdash \perp$, a contradiction.

Now, suppose that $\gamma' \to (P_{\leq t'}\alpha \wedge P_{\leq s'}\beta \wedge P_{>s'(r+t')}(\alpha,\beta)) \in T_{j+1}$, where $t', s' \in [0,1]_Q$. Let k' be the nonnegative integer such that $\gamma_{k'} = \gamma' \to ((P_{\leq t'}\alpha \wedge P_{\leq s'}\beta) \to P_{\leq s'(r+t')}(\alpha,\beta))$. Then, $T_{\max\{i,k'+1\}} \vdash (P_{\leq t'}\alpha \wedge P_{\leq s'}\beta) \to P_{\leq s'(r+t')}(\alpha,\beta)$. On the other hand,

$$T_{\max\{i,j+1\}} \vdash P_{\leq t'}\alpha \wedge P_{\leq s'}\beta \wedge P_{>s'(r+t')}(\alpha,\beta).$$

Thus, $T_{\max\{i,j+1,k'+1\}} \vdash \perp$, a contradiction. Consequently, the set T^* is deductively closed.

From the fact that T^* is deductively closed, we can prove that T^* is consistent. Indeed, if T^* is inconsistent, there is a formula $\gamma' \in For_{\mathrm{LPP}_1^{\mathrm{conf}}}$ such that $T^* \vdash \gamma' \wedge \neg\gamma'$. But then, there is a nonnegative integer i such that $\gamma' \wedge \neg\gamma' \in T_i$, a contradiction. □

6.2 *Canonical model*

Now, we are ready to prove our main result: the axiomatization $Ax(\mathrm{LPP}_1^{\mathrm{conf}})$ is strongly complete for the class of $\mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$-structures. We use mcs to build a special measurable structure, so-called *canonical model*. For a given consistent set T, we show that there is a world in the canonical model in which all the formulas of its maximal consistent extension T^* hold. Recall that the existence of such extension is guaranteed by Theorem 6.2.

DEFINITION 6.3 (Canonical model).
The canonical model M_C is the tuple $(W, Prob, v)$ where

- W is the set of all mcs;
- for every world w and every propositional letter $p \in \mathcal{P}$, $v(w,p) = true$ iff $p \in w$;
- $Prob(w) = (W(w), H(w), \mu(w))$ such that
  - $W(w) = W$,
  - $H(w) = \{\{w' \in W \mid \alpha \in w'\} \mid \alpha \in For_{\mathrm{LPP}_1^{\mathrm{conf}}}\}$,
  - $\mu(w) : H(w) \to [0,1]$ such that $\mu(w)(\{w' \in W \mid \alpha \in w'\}) = \sup\{r \in [0,1]_Q \mid P_{\geq r}\alpha \in w\}$.

We use the following notation to refer to the elements of $H(w)$ from the canonical model:

$$[\![\alpha]\!] = \{w' \in W \mid \alpha \in w'\}.$$

Next, we want to show that $M_C \in \mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$. We start by proving that M_C is an $\mathrm{LPP}_1^{\mathrm{conf}}$-structure.

LEMMA 6.4
The canonical model M_C is a $\mathrm{LPP}_1^{\mathrm{conf}}$-structure.

PROOF. To prove the statement we will show that

(1) $H(w)$ is an algebra of subsets of W. It easy to see that $W = [\![\alpha \vee \neg\alpha]\!]$, so from the definition of $H(w)$ we have that $W \in H(w)$. Also, if $[\![\alpha]\!], [\![\beta]\!] \in H(w)$ then $[\![\neg\alpha]\!] \in H(w)$ and $[\![\alpha \vee \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!] \in H(w)$. Thus, $H(w)$ is algebra of subsets of W.

Next, we show that $\mu(w)$ is correctly defined.

(2) If $[\![\alpha]\!] = [\![\beta]\!]$, then $\sup\{r \in [0,1]_Q \mid P_{\geq r}\alpha \in w\} = \sup\{r \in [0,1]_Q \mid P_{\geq r}\beta \in w\}$ (in other words, $\mu(w)([\![\alpha]\!]) = \mu(w)([\![\beta]\!])$)

We proved that if $[\![\alpha]\!] \subseteq [\![\beta]\!]$, then $\mu(w)([\![\alpha]\!]) \leq \mu(w)([\![\beta]\!])$. From $[\![\alpha]\!] \subseteq [\![\beta]\!]$, we get $\vdash \neg(\alpha \wedge \neg\beta)$ and $\vdash P_{\geq 1}(\alpha \to \beta)$. Now, we must show that if $P_{\geq r}\alpha \in w$, then $P_{\geq r}\beta \in w$, i.e. $\mu(w)([\![\alpha]\!]) \leq \mu(w)([\![\beta]\!])$. It is sufficient to show that $\vdash P_{\geq 1}(\alpha \to \beta) \to (P_{\geq r}\alpha \to P_{\geq r}\beta)$.

We know that $\vdash \neg\alpha \vee \neg\bot$, so by the inference rule R2 it is $\vdash P_{\geq 1}(\neg\alpha \vee \neg\bot)$. From the axioms A2 and A5, we have

$\vdash P_{\geq 0}\neg\bot$,

$\vdash (P_{\geq r}\alpha \wedge P_{\geq 0}\neg\bot \wedge P_{\geq 1}(\neg\alpha \vee \neg\bot)) \to P_{\geq r}(\alpha \vee \bot)$.

By the rule R1, we get

$\vdash P_{\geq r}\alpha \to P_{\geq r}(\alpha \vee \bot)$.

Instances of the axioms A6 and A1 are

$\vdash (P_{\leq 1-r}(\neg\alpha \wedge \neg\bot) \wedge P_{<r}\neg\neg\alpha) \to P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)$,

$\vdash (\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha$.

By the inference rule R2, we have

$\vdash P_{\geq 1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)$, i.e. $\vdash \neg P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)$.

Therefore,

$$\vdash (P_{\leq 1-r}(\neg\alpha \wedge \neg\bot) \wedge P_{<r}\neg\neg\alpha) \to (P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha) \wedge \neg P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)),$$

i.e.

$$\vdash (P_{\leq 1-r}(\neg\alpha \wedge \neg\bot) \wedge P_{<r}\neg\neg\alpha) \to \bot.$$

From the axiom A1, we actually have $\vdash P_{\leq 1-r}(\neg\alpha \wedge \neg\bot) \to \neg P_{<r}\neg\neg\alpha$, i.e. $\vdash P_{\geq r}(\alpha \vee \bot) \to P_{\geq r}\neg\neg\alpha$. So, we get $\vdash P_{\geq r}\alpha \to P_{\geq r}\neg\neg\alpha$. Now, from axiom A6, we have

$$\vdash (P_{\leq 1-r}\neg\alpha \wedge P_{<r}\beta) \to P_{<1}(\neg\alpha \vee \beta),$$

i.e. $\vdash P_{>1-r}\neg\alpha \vee P_{\geq r}\beta \vee P_{<1}(\neg\alpha \vee \beta)$. By propositional reasoning, we have $\vdash P_{<r}\alpha \vee P_{\geq r}\beta \vee \neg P_{\geq 1}(\alpha \to \beta)$. Using the definition of probability operator $P_{<r}$ and axiom A1, we obtain

$$\vdash P_{\geq 1}(\alpha \to \beta) \to (P_{\geq r}\alpha \to P_{\geq r}\beta).$$

Therefore, $\mu(w)([\![\alpha]\!]) \leq \mu(w)([\![\beta]\!])$.

Next, we show that $\mu(w)$ is a finitely additive probability measure.

(3) $\mu(w)([\![\alpha]\!]) \geq 0$. Since $P_{\geq 0}\alpha$ is an axiom, it belongs T^* and $\mu(w)([\![\alpha]\!]) \geq 0$.

(4) $\mu(w)([\![\alpha]\!] \cup [\![\beta]\!]) = \mu(w)([\![\alpha]\!]) + \mu(w)([\![\beta]\!])$ for all disjoint $[\![\alpha]\!]$ and $[\![\beta]\!]$. First, we show that $\mu(w)([\![\alpha]\!]) = 1 - \mu(w)([\![\neg\alpha]\!])$. Let $r = \mu([\![\alpha]\!]) = \sup\{s \in [0,1]_Q \mid P_{\geq s}\alpha \in w\}$. Suppose that $r = 1$ so $P_{\geq 1}\alpha \in w$ (R3). Thus, $\neg P_{>0}\neg\alpha \in w$. If for some $r > 0$, $P_{\geq r}\neg\alpha \in w$, it must be $P_{>0}\neg\alpha \in w$ and it is a contradiction. It follows that $\mu(w)([\![\neg\alpha]\!]) = 0$. Suppose now that $r < 1$. Then, for every rational number $r' \in (r,1]_Q$, $\neg P_{\geq r'}\alpha = P_{<r'}\alpha \in w$. By Axiom A4, we get $P_{\leq r'}\alpha \in w$ and $P_{\geq 1-r'}(\neg\alpha) \in w$. Also, if there is a rational number $r" \in [0,r)_Q$ such that $P_{\geq 1-r"}(\neg\alpha) \in w$ then $\neg P_{>r"}\alpha \in w$, a contradiction. Hence, $\sup\{r \in [0,1]_Q \mid P_{\geq r}(\neg\alpha) \in w\} = 1 - \sup\{r \in [0,1]_Q \mid P_{\geq r}\alpha \in w\}$, i.e. $\mu(w)([\![\alpha]\!]) = 1 - \mu(w)([\![\neg\alpha]\!])$. Now, let $[\![\alpha]\!] \cap [\![\beta]\!] = \emptyset$, $\mu(w)([\![\alpha]\!]) = r$ and $\mu(w)([\![\beta]\!]) = s$. Since $[\![\beta]\!] \subseteq [\![\neg\alpha]\!]$, by the above steps, we have $r + s \leq r + (1-r) = 1$. Suppose that $r > 0$ and $s > 0$, then for every $r' \in [0,r]_Q$ and every $s' \in [0,s]_Q$, we have $P_{\geq r'}\alpha$ and $P_{\geq s'}\beta$ are in w. It follows by A5 that $P_{\geq r'+s'}(\alpha \vee \beta) \in w$. Hence, $r + s \leq t_0 = \sup\{t \in [0,1]_Q \mid P_{\geq t}(\alpha \vee \beta) \in w\}$.

If $r+s=1$, then the statement trivially holds. Suppose that $r+s<1$. If $r+s<t_0$, then for every $t' \in (r+s,t_0)_Q$, we have $P_{\geq t'}(\alpha\vee\beta)\in w$. We can choose rational numbers $r" > r$ and $s" > s$ such that $\neg P_{\geq r"}\alpha, P_{<r"}\alpha \in w, \neg P_{\geq s"}\beta, P_{<s"}\beta \in w$ and $r"+s" = t' \leq 1$. Using A4, we have $P_{\leq r"}\alpha \in w$. By A6, we get $P_{<r"+s"}(\alpha\vee\beta)\in w, \neg P_{\geq r"+s"}(\alpha\vee\beta)\in w$ and $\neg P_{\geq t'}(\alpha\vee\beta)\in w$, a contradiction. Hence, $r+s=t_0$ and $\mu(w)([\![\alpha]\!]\cup[\![\beta]\!]) = \mu(w)([\![\alpha]\!])+\mu(w)([\![\beta]\!])$. Finally, suppose that $r=0$ or $s=0$, then we can do the same as above with the only exception that $r'=0$ or $s'=0$. □

LEMMA 6.5 (Truth lemma).
Let M_C be the canonical model and $\gamma \in For_{\mathrm{LPP}_1^{\mathrm{conf}}}$. Then, for every world w from M_C,

$$\gamma \in w \text{ iff } M_C, w \models \gamma.$$

PROOF. We use induction on the complexity of the formula γ. If γ is a propositional letter, the statement follows from the construction of M_C. The cases when γ is a conjunction or a negation are straightforward.

Let γ be $P_{\geq r}\alpha$. If $P_{\geq r}\alpha \in w$, then $\sup\{s\in[0,1]_Q \mid P_{\geq s}\alpha\in w\} = \mu(w)([\![\alpha]\!]) \geq r$ and $M_C, w \models P_{\geq r}\alpha$. Suppose now that $M_C, w \models P_{\geq r}\alpha$. It follows that $\sup\{s\in[0,1]_Q \mid P_{\geq s}\alpha\in w\} \geq r$, i.e. $\mu(w)([\![\alpha]\!]) \geq r$. If $\mu(w)([\alpha]) > r$, then by the definition of the canonical model, $P_{\geq r}\alpha \in w$. If it is $\mu(w)([\![\alpha]\!]) = r$, then by R3 and the fact that w is deductively closed, we have that $P_{\geq r}\alpha\in w$.

Let γ be of the form $c_{\geq r}(\alpha,\beta)$.

($\Rightarrow$) Assume that $c_{\geq r}(\alpha,\beta) \in w$. Let $\{t_n \mid n\in\mathbf{N}\}$ and $\{s_n \mid n\in\mathbf{N}\}$ be two strictly increasing sequences of numbers from $[0,1]_Q$, such that $\lim_{n\to\infty} t_n = \mu(w)([\![\alpha]\!])$ and $\lim_{n\to\infty} s_n = \mu(w)([\![\beta]\!])$. Let n be any number from $\mathbf{N}$. Then, $w \vdash P_{\geq t_n}\alpha \wedge P_{\geq s_n}\beta$. Using the assumption $c_{\geq r}(\alpha,\beta)\in w$, the axioms A7 and A9 and propositional reasoning, we obtain $w\vdash P_{>0}\beta$ and $w\vdash P_{\geq s_n(r+t_n)}(\alpha\wedge\beta)$. Finally, by Definition 6.3, we have $\mu(w)([\![\beta]\!]) > 0$ and $\mu(w)([\![\alpha\wedge\beta]\!]) \geq lim_{n\to\infty}s_n(r+t_n) = \mu(w)([\![\beta]\!])(r+\mu(w)([\![\alpha]\!]))$, i.e.

$$\mu(w)([\![\beta]\!]) > 0$$

and

$$\mu(w)([\![\alpha]\!] | [\![\beta]\!]) - \mu(w)([\![\alpha]\!]) \geq r.$$

($\Leftarrow$) Now, assume that $\mu(w)([\![\beta]\!]) > 0$ and $\mu(w)([\![\alpha]\!] | [\![\beta]\!]) - \mu(w)([\![\alpha]\!]) \geq r$, i.e. $\mu(w)([\![\alpha\wedge\beta]\!]) \geq \mu(w)([\![\beta]\!])(r+\mu(w)([\![\alpha]\!]))$. We will show that

$$w \vdash P_{>0}\beta$$

and

$$w \vdash (P_{\geq t}\alpha \wedge P_{\geq s}\beta) \rightarrow P_{\geq s(r+t)}(\alpha\wedge\beta) \text{ for all } t,s\in[0,1]_Q.$$

Suppose that $w \nvdash P_{>0}\beta$. By maximality of $w\vdash P_{=0}\beta$, i.e. $\mu(w)([\![\beta]\!]) = 0$, a contradiction. So we have that $w\vdash P_{>0}\beta$.

If $t > \mu(w)([\![\alpha]\!])$ or $s > \mu(w)([\![\beta]\!])$, then $w\nvdash P_{\geq t}\alpha \wedge P_{\geq s}\beta$. By maximality of w, $w\vdash \neg(P_{\geq t}\alpha\wedge P_{\geq s}\beta)$ and consequently $w\vdash (P_{\geq t}\alpha\wedge P_{\geq s}\beta)\rightarrow P_{\geq s(r+t)}(\alpha\wedge\beta)$. If $t\leq\mu(w)([\![\alpha]\!])$ and $s\leq \mu(w)([\![\beta]\!])$, then $s(r+t)\leq\mu(w)([\![\alpha\wedge\beta]\!])$ by assumption, so $w\vdash P_{\geq s(r+t)}(\alpha\wedge\beta)$ by Definition 6.3. Now, the result follows from the fact that w is deductively closed.

The case when γ is $c_{\leq r}(\alpha,\beta)$ can be proved in a similar way. □

Hence, we have shown that for every formula $\alpha \in For_{\mathrm{LPP}_1^{\mathrm{conf}}}$ and every world w from M_C the equality $[\alpha]_{M_C,w} = [\![\alpha]\!]$ holds. Consequently, every $[\alpha]_{M_C,w}$ belongs to algebra $H(w)$ of the canonical model. Combining that fact with Lemma 6.4, we obtain the following corollary.

COROLLARY 6.6
The canonical model M_C is an $\mathrm{LPP}_1^{\mathrm{conf}}$-measurable structure which is a model for every consistent set T.

Now, we formulate the completeness theorem for our logic.

THEOREM 6.7 (Strong completeness of $\mathrm{LPP}_1^{\mathrm{conf}}$).
A set of formulas T is consistent iff T is $\mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$-satisfiable.

PROOF. Note that the direction form right to left follows from Theorem 4.3. For the other direction, suppose that T is a consistent set of formulas. By Theorem 6.2, there is a maximal consistent superset T^* of T. From the previous corollary, we have that $M_C \in \mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$, so we only need to show that M_C is a model of T^*. By Lemma 6.5, if T is consistent set we know that T^* is a world in M_C, so we obtain $M_C, T^* \models T$. $\square$

Finally, let us recall that the usual formulation of strong completeness is

$$T \vdash \alpha \text{ iff } T \models \alpha.$$

It is well known that this standard formulation is equivalent to the formulation of Theorem 6.7.

7 Decidability of $\mathrm{LPP}_1^{\mathrm{conf}}$

In this section, we prove that the logic $\mathrm{LPP}_1^{\mathrm{conf}}$ is decidable. In the proof, we use the method of filtration and reduction to finite systems of inequalities. First, we show that an $\mathrm{LPP}_1^{\mathrm{conf}}$-formula is satisfiable iff it is satisfiable in a model with a finite number of worlds.

THEOREM 7.1
If an $\mathrm{LPP}_1^{\mathrm{conf}}$-formula α is satisfiable in a model $M \in \mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$, then it is satisfied in a model $M^* \in \mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$ with a finite number of worlds.

PROOF. Let w be a world from M such that $M, w \models \alpha$. Let $Subf(\alpha)$ be the set of all subformulas of α and $k = |Subf(\alpha)|$. By $\sim$, we denote the equivalence relation over $W \times W$, where $w \sim w'$ iff for every $\beta \in Subf(\alpha)$, $w \models \beta$ iff $w' \models \beta$. The quotient set $W_{/\sim}$ is finite and $|W_{/\sim}| \leq 2^{|Subf(\alpha)|}$. Now, for every class C_i, we choose an element and denote it w_i. We consider the model $M^* = (W^*, Prob^*, v^*)$, where

- $W^* = \{w_i \mid C_i \in W_{/\sim}\}$;
- $Prob^*(w_i) = (W^*(w_i), H^*(w_i), \mu^*(w_i))$ such that
  - $W^*(w_i) = \{w_j \in W^* \mid (\exists u \in C_{w_j}) u \in W(w_i)\}$,
  - $H^*(w_i)$ is the power set of $W^*(w_i)$,
  - $\mu^*(w_i)(\{w_j\}) = \mu(w_i)(C_{w_j})$ and for any $D \in H^*(w_i)$, $\mu^*(w_i)(D) = \sum_{w_j \in D} \mu^*(w_i)(\{w_j\})$;
- $v^*(w_i, p) = v(w_i, p)$.

It can be shown that $M^* \in \mathrm{LPP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$. Now, we want to prove that for any $\beta \in Subf(\alpha)$, $M, w \models \beta$ iff $M^*, w_i \models \beta$ where w_i represents C_w in M^*. The proof is by induction on the complexity of the formulas. Let us briefly consider the case when $\beta = c_{\geq r}(\alpha', \beta')$, while the proofs for the other cases are similar.

$$
\begin{aligned}
&M, w \models \beta \\
&\text{iff } M, w_i \models \beta \\
&\text{iff } \mu(w_i)([\beta']) > 0 \text{ and } \mu(w_i)([\alpha' \wedge \beta']) \geq \mu(w_i)([\beta'])(r + \mu(w_i)([\alpha])) \\
&\text{iff } \sum_{C_u : M, u \models \beta'} \mu(w_i)(C_u) > 0 \text{ and} \\
&\sum_{C_u : M, u \models \alpha' \wedge \beta'} \mu(w_i)(C_u) \geq \Big(\sum_{C_u : M, u \models \beta'} \mu(w_i)(C_u) \Big) \Big(r + \sum_{C_u : M, u \models \alpha'} \mu(w_i)(C_u) \Big) \\
&\text{iff } \sum_{C_u : M^*, u \models \beta'} \mu^*(w_i)(\{u\}) > 0 \text{ and} \\
&\sum_{C_u : M^*, u \models \alpha' \wedge \beta'} \mu^*(w_i)(\{u\}) \geq \Big(\sum_{C_u : M^*, u \models \beta'} \mu^*(w_i)(\{u\}) \Big) \Big(r + \sum_{C_u : M^*, u \models \alpha'} \mu^*(w_i)(\{u\}) \Big) \\
&\text{iff } \mu^*(w_i)([\beta']) > 0 \text{ and} \\
&\qquad \mu^*(w_i)([\alpha' \wedge \beta']) \geq \mu^*(w_i)([\beta'])(r + \mu^*(w_i)([\alpha])) \\
&\text{iff } M^*, w_i \models \beta.
\end{aligned}
$$

□

Note that there are infinitely many finite models from $\mathrm{LPP}^{\mathrm{conf}}_{1,\mathrm{Meas}}$ with at most $2^{|Subf(\alpha)|}$ worlds because there are infinitely many possibilities for real-valued probabilities. Thus, the previous theorem does not directly imply decidability. In order to show decidability, we will translate the problem of satisfiability of a formula to the problem of satisfiability of finite sets of equations and inequalities.

THEOREM 7.2
Satisfiability problem for $\mathrm{LPP}^{\mathrm{conf}}_{1}$ is decidable.

PROOF. Let us describe the procedure which checks satisfiability of a formula α. We use the notions introduced in the proof of Theorem 7.1. First, we transform the formula α to a disjunction of the formulas of the form $\bigwedge_{k=1}^{|Subf(\alpha)|} \beta_k$, where

$$\beta_k \in Subf(\alpha) \cup \{\neg\beta \mid \beta \in Subf(\alpha)\},$$

and each subformula of α appears exactly once (with or without the leading negation and assuming that $\neg\neg\beta$ is the same as β). In each world $w \in W^*$, exactly one formula of the form $\bigwedge_{k=1}^{|Subf(\alpha)|} \beta_k$ holds. Denote that formula by α_w. In order to shorten the notation, we write

$$\beta \in \alpha_w$$

to denote that β is a conjunct in α_w. For every

$$l \leq 2^{Subf(\alpha)},$$

we will consider l formulas of the above form such that the following two conditions hold:

- the chosen formulas are not necessarily different, but each α_w does not contain both β and $\neg\beta$ in the top conjunction;
- at least one α_w then must contain the formula α in the top conjunction.

For every world w_i, $i < l$, we consider specific equations and inequalities that we describe below. We chose the variables of the form y_{w_i,w_j} which represent the values $\mu(w_i)(\{w_j\})$.

The inequality (1) below assures that all the probability measures are nonnegative, and the equality (2) states that the probability of the set of all possible worlds has to be equal to 1. Furthermore, each of (3)–(8) refers to a specific conjunct in α_w. It is easy to see that (3), (5) and (6) correspond to the second, the third and the fourth conditions of the satisfiability relation from Definition 3.5, respectively. Similarly, (4), (7) and (8) deal with negative literals and correspond to the combination of the fifth condition with the second, the third and the fourth conditions (respectively) from Definition 3.5.

Now, we state the equations and inequalities (recall that we write $\beta \in \alpha_w$ to denote that β is a conjunct in α_w):

(1) $y_{w_i,w_j} \geq 0$, for every world w_j;

(2) $\displaystyle\sum_{w_j \in W(w_i)} y_{w_i,w_j} = 1$;

(3) $\displaystyle\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j} \geq r$, for every $P_{\geq r}\beta \in \alpha_{w_i}$;

(4) $\displaystyle\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j} < r$, for every $\neg P_{\geq r}\beta \in \alpha_{w_i}$;

(5) $\displaystyle\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j} > 0$ and

$$\sum_{w_j:\gamma\wedge\beta\in\alpha_{w_j}} y_{w_i,w_j} \geq \Big(\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j}\Big)\Big(r + \sum_{w_j:\gamma\in\alpha_{w_j}} y_{w_i,w_j}\Big),$$

for every $c_{\geq r}(\gamma,\beta) \in \alpha_{w_i}$;

(6) $\displaystyle\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j} > 0$ and

$$\sum_{w_j:\gamma\wedge\beta\in\alpha_{w_j}} y_{w_i,w_j} \leq \Big(\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j}\Big)\Big(r + \sum_{w_j:\gamma\in\alpha_{w_j}} \mu(w_i)(\{w_j\})\Big),$$

for every $c_{\leq r}(\gamma,\beta) \in \alpha_{w_i}$;

(7) $\displaystyle\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j} = 0$ or

$$\sum_{w_j:\gamma\wedge\beta\in\alpha_{w_j}} y_{w_i,w_j} < \Big(\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j}\Big)\Big(r + \sum_{w_j:\gamma\in\alpha_{w_j}} y_{w_i,w_j}\Big),$$

for every $\neg c_{\geq r}(\gamma,\beta) \in \alpha_{w_i}$;

(8) $\displaystyle\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j} = 0$ or

$$\sum_{w_j:\gamma\wedge\beta\in\alpha_{w_j}} y_{w_i,w_j} > \Big(\sum_{w_j:\beta\in\alpha_{w_j}} y_{w_i,w_j}\Big)\Big(r + \sum_{w_j:\gamma\in\alpha_{w_j}} y_{w_i,w_j}\Big),$$

for every $\neg c_{\leq r}(\gamma,\beta) \in \alpha_{w_i}$.

The equations and inequalities (1)–(8) form not one, but a number of finite systems of linear equations and inequalities. Note that adding either (7) or (8) to any system *Sys* of equations and inequalities results with a disjunction of two different extensions of *Sys*. For the purpose of this proof, the fact that we always have finitely many systems is sufficient and it is enough if one of the systems is solvable. Note that all those systems are in the language of real closed fields, and it is well known that the theory of real closed fields is decidable. Since we have finitely many possibilities for the choice of l, and (for each l) finitely many possibilities to chose l formulas α_w, our logic $\mathrm{LPP}_1^{\mathrm{conf}}$ is decidable as well. □

8 The first-order logic $\mathrm{LFOP}_1^{\mathrm{conf}}$

In this section, we present the logic $\mathrm{LFOP}_1^{\mathrm{conf}}$, the first-order extension of $\mathrm{LPP}_1^{\mathrm{conf}}$. Due to similarities in syntax and semantics, we avoid repetition of some technical details that were already presented in detail in the propositional case.

The language of the logic $\mathrm{LFOP}_1^{\mathrm{conf}}$ consists of

- a denumerable set of variables $Var = \{x, y, z, ...\}$;
- universal quantifier $\forall$, and classical propositional connectives;
- for every integer $k \geq 0$, denumerably many function symbols $F_0^k, F_1^k, \dots$ of arity k;
- for every integer $k \geq 1$, denumerably many relation symbols $P_0^k, P_1^k, \dots$ of arity k;
- a list of unary probability operators $P_{\geq r}$, for every $r \in [0,1]_Q$; and
- a list of binary probability operators $c_{\geq r}$,$c_{\leq r}$, for every $r \in [-1,1]_Q$.

The function symbols of arity 0 are called constant symbols. Terms and formulas are defined as usual, as well as the notion of a term that is free for a variable. Sentences are formulas without free variables.

EXAMPLE 8.1
Consider the following sentence:

> 'The chance that all your colleagues know your secret would increase (for r) if at least one of them is aware of it.'

If the predicate symbol S denotes 'knows the secret', then the sentence can be formalized as follows:

$$c_{\geq r}((\forall x)S(x), (\exists x)S(x)).$$

An $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure is a tuple $M = (W, D, I, Prob)$ where

- W is a non-empty set of worlds;
- D is non-empty domain for every $w \in W$;
- I assigns an interpretation $I(w)$ to every $w \in W$ such that for all i and k,
  - $I(w)(F_i^k)$ is a function from D^k to D;
  - for every $w' \in W$, $I(w)(F_i^k) = I(w')(F_i^k)$;
  - $I(w)(P_i^k)$ is a relation over D^k to
- $Prob(w) = (W(w), H(w), \mu(w))$ is a triple where
  - $W(w)$ is non-empty subset of W,
  - $H(w)$ is an algebra of subsets of $W(w)$,
  - $\mu(w) : H(w) \longrightarrow [0,1]$ is a finitely additive measure.

Note that we made two assumptions which are somewhat standard for first-order modal logics. The first one is that the domain is fixed in a model (in other words, the domain is the same in all the worlds of a considered model). Intuitively, this means that there is no uncertainty regarding which objects exist. The second assumption is that the terms are *rigid*, i.e. for every model their meanings are the same in all the worlds of a considered model.

Let $M = (W, D, I, Prob)$ be an $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure. A *variable valuation* v assigns some element of the corresponding domain to every variable x, i.e. $v(w)(x) \in D$. If $w \in W$, $d \in D$ and v is a valuation, then $v[d/x]$ is a valuation same as v except that $v[d/x](w)(x) = d$.

The *value* of a term t, denoted by $I(w)(t)_v$, is

- if t is a variable x, then $I(w)(x)_v = v(w)(x)$; and
- if $t = F_i^m(t_1, ..., t_m)$, then $I(w)(t)_v = I(w)(F_i^m)(I(w)(t_1)_v, ..., I(w)(t_m)_v)$.

Next, we introduce the concept of a truth-value.

DEFINITION 8.2
The truth-value of a formula α in a world $w \in W$ of a $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure $M = (W, D, I, Prob)$, under a valuation v (denoted by $I(w)(\alpha)_v$) is as follows.

- If $\alpha = P_i^m(t_1, ..., t_m)$, then $I(w)(\alpha)_v = true$; if $(I(w)(t_1)_v, ..., I(w)(t_m)_v) \in I(w)(P_i^m)$, otherwise $I(w)(\alpha)_v = false$.
- If $\alpha = P_{\geq r}\beta$, then $I(w)(\alpha)_v = true$; if $\mu(w)(\{u \in W(w) \mid I(u)(\beta)_v = true\}) \geq r$, otherwise $I(w)(\alpha)_v = false$.
- If $\alpha = c_{\geq r}(\beta, \gamma)$, then $I(w)(\alpha)_v = true$; if $\mu(w)(\{u \in W(w) \mid I(u)(\gamma)_v = true\}) > 0$ and $\mu(w)(\{u \in W(w) \mid I(u)(\beta)_v = true\} | \{u \in W(w) \mid I(u)(\gamma)_v = true\}) - \mu(w)(\{u \in W(w) \mid I(u)(\beta)_v = true\}) \geq r$, otherwise $I(w)(\alpha)_v = false$.
- If $\alpha = c_{\leq r}(\beta, \gamma)$, then $I(w)(\alpha)_v = true$; if $\mu(w)(\{u \in W(w) \mid I(u)(\gamma)_v = true\}) > 0$ and $\mu(w)(\{u \in W(w) \mid I(u)(\beta)_v = true\} | \{u \in W(w) \mid I(u)(\gamma)_v = true\}) - \mu(w)(\{u \in W(w) \mid I(u)(\beta)_v = true\}) \leq r$, otherwise $I(w)(\alpha)_v = false$.
- If $\alpha = \neg\beta$, then $I(w)(\alpha)_v = true$; if $I(w)(\beta)_v = false$, otherwise $I(w)(\alpha)_v = false$.
- If $\alpha = \beta \wedge \gamma$, then $I(w)(\alpha)_v = true$; if $I(w)(\beta)_v = true$ and $I(w)(\gamma)_v = true$, otherwise $I(w)(\alpha)_v = false$.
- If $\alpha = (\forall x)\beta$, then $I(w)(\alpha)_v = true$; if for every $d \in D$, $I(w)(\beta)_{v[d/x]} = true$, otherwise $I(w)(\alpha)_v = false$.

A formula holds in a world w from an $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure $M = (W, D, I, Prob)$ (denoted by $(M, w) \models \alpha$) if for every valuation v, $I(w)(\alpha)_v = true$. If $d \in D$, we will use $(M, w) \models \alpha(d)$ to denote that $I(w)(\alpha(x))_{v[d/x]} = true$, for every valuation v. A formula is valid in an $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure $M = (W, D, I, Prob)$ (denoted by $M \models \alpha$), if it is satisfied in every world w from W. A formula α is valid if for every $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure M, $M \models \alpha$. A sentence α is satisfiable if there is a world w in an $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure M such that $(M, w) \models \alpha$. A set T of sentences is satisfiable if there is a world w in an $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure M such that $(M, w) \models \alpha$ for every $\alpha \in T$.

Like in the propositional case, we will consider only $\mathrm{LFOP}_1^{\mathrm{conf}}$-measurable structures: an $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure M is measurable if for every sentence α, every valuation v and every world w from M, the set

$$[\alpha]_{M,w}^v = \{u \in W(w) \mid I(u)(\alpha)_v = true\}$$

belongs to $H(w)$. If α is a sentence, we omit the superscript v in $[\alpha]^v_{M,w}$. We denote the set of all measurable $\mathrm{LFOP}_1^{\mathrm{conf}}$-structure with $\mathrm{LFOP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$. The definition of a model is the same as in the propositional case.

Our axiomatic system for the logic $\mathrm{LFOP}_1^{\mathrm{conf}}$ contains all the axioms and inference rules from Section 4 and, in addition, the following axiom schemes:

(A11) $(\forall x)(\alpha \to \beta) \to (\alpha \to (\forall x)\beta)$ where x is not free in α;
(A12) $(\forall x)(\alpha(x) \to \alpha(t/x)$, where $\alpha(t/x)$ is obtained by substituting all free occurrences of x in $\alpha(x)$ by the term t which is free for x in $\alpha(x)$;

and the inference rule

(R6) from α infer $(\forall x)\alpha$.

Recall that we use fixed domain $\mathrm{LFOP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$-measurable models with rigid terms, which is similar to the objectual interpretation for first order modal logics [17]. If we reject the assumption that the terms are rigid, then the standard first order axiom A12 is not sound.

The notions of theorems and deducibility are defined as in Section 4.

THEOREM 8.3 (Soundness).
The axiomatization $Ax(\mathrm{LFOP}_1^{\mathrm{conf}})$ is sound with respect to the class of $\mathrm{LFOP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$-structures.

PROOF. Here, we will consider only the axiom A11. Let $M = (W, D, I, Prob)$ be an $\mathrm{LFOP}_{1,\mathrm{Meas}}^{\mathrm{conf}}$ model. Suppose that for some $w \in W$, $M, w \not\models (\forall x)(\alpha \to \beta) \to (\alpha \to (\forall x)\beta)$. It means that for some valuation v, $I(w)((\forall x)(\alpha \to \beta))_v = true$, $I(w)(\alpha)_v = true$ and $I(w)((\forall x)\beta)_v = false$. So, for some valuation v, for every $d \in D$, $I(w)(\alpha \to \beta)_{v[d/x]} = true$, $I(w)(\alpha)_v = true$ and for some $d' \in D$, $I(w)(\beta)_{v[d'/x]} = false$. For d', we also have that $I(w)(\alpha)_{v[d'/x]} = true$, which implies that $I(w)(\alpha \to \beta)_{v[d'/x]} = false$, a contradiction. □

In the completeness proof, we follow the ideas from the propositional case. We omit the proof of deduction theorem, which is a straightforward adaptation of the proof of Theorem 6.1.

In the construction of the canonical model, we will use a special kind of mcs called *saturated* sets.

DEFINITION 8.4
An mcs T is saturated if it satisfies

if $\neg(\forall x)\alpha \in T$, then for some term t, $\neg\alpha(t) \in T$.

Now, we show that we can extend any consistent set of sentences to a saturated set. For that, we need to extend the language of $\mathrm{LFOP}_1^{\mathrm{conf}}$ with a countably infinite set of novel constant symbols C (i.e. the elements of C do not belong to the considered first-order language).

THEOREM 8.5 (Lindenbaum's theorem).
Every consistent set of sentences can be extended to a saturated set of sentences of the language extended with a countably infinite set of novel constant symbols C.

PROOF. Consider a consistent set T, and let $\alpha_0, \alpha_1, \ldots$ be an enumeration of all sentences from $For_{\mathrm{LFOP}_1^{\mathrm{conf}}}$. Let T^* denote the set of sentences obtained by Steps (1)–(4) from the proof of Theorem 6.2 with one additional requirement in Step (3):

- if the set T_{i+1} is obtained by adding a formula of the form $\neg(\forall x)\beta(x)$ to the set T_i, then for some $c \in C$, $\neg\beta(c)$ is also added to T_{i+1}, so that T_{i+1} is consistent.

Let us consider the only new step. Suppose that for some $i > 0$, a formula of the form $\neg(\forall x)\beta(x)$ is consistently added to T_i. If there is a constant symbol $c \in C$ such that $\neg\beta(c) \in T_i$, then $T_i \cup \{\neg(\forall x)\beta(x), \neg\beta(c)\}$ is consistent. Suppose that there is no such c. Since $T_i \cup \{\neg(\forall x)\beta(x)\}$ is obtained by adding only finitely many formulas to T, and T does not contain constants from C, there is at least one constant $c \in C$ which does not appear in $T_i \cup \{\neg(\forall x)\beta(x)\}$. If $T_i \cup \{\neg(\forall x)\beta(x), \neg\beta(c)\} \vdash \bot$, then by deduction theorem, we obtain that $T_i \cup \{\neg(\forall x)\beta(x)\} \vdash \beta(c)$, and since c does not appear in $T_i \cup \{\neg(\forall x)\beta(x)\}$, we obtain $T_i \cup \{\neg(\forall x)\beta(x)\} \vdash (\forall x)\beta(x)$. Thus, $T_i \vdash (\forall x)\beta(x)$. It follows that the set T_i is not consistent, a contradiction.

Finally, in order to prove that T^* is consistent, we need to show that T^* is deductively closed (i.e. $T^* \vdash \alpha$ implies $\alpha \in T^*$), while the construction guarantees that T^* is both maximal and saturated. The only case that does not appear in the proof of Theorem 6.2 is when $T^* \vdash (\forall x)\beta(x)$ is obtained from $T^* \vdash \beta(x)$ by the inference rule R6. Since $\beta(x)$ has one free variable, and T_i and T^* are sets of sentences, $\beta(x)$ does not belong to T^*. By classical first-order reasoning, we have that $T^* \vdash \beta(c)$ holds for every constant $c \in C$, and from the induction hypothesis $\beta(c) \in T^*$. If $(\forall x)\beta(x) \notin T^*$, the construction of the set T^* guarantees that there are some $i > 0$ and $c \in C$ such that $\beta(c), \neg\beta(c) \in T_i$ for some $c \in C$, a contradiction. □

A canonical model $M_C = (W, D, I, Prob)$ is a tuple such that

- W is the set of all saturated sets of formulas;
- D is the set of all variable-free terms;
- for every $w \in W$, $I(w)$ is an interpretation such that
  - for every function symbol F_i^m, $I(w)(F_i^m) : D^m \to D$ such that for all variable-free terms $t_1, ..., t_m$, $I(w)(F_i^m) : \langle t_1, ..., t_m\rangle \to (F_i^m(t_1, ..., t_m))$; and
  - for every relation symbol P_i^m, $I(w)(P_i^m) = \{\langle t_1, ..., t_m\rangle \mid P_i^m(t_1, ..., t_m) \in w\}$, for all variable-free terms $t_1, ..., t_m$;
- for every $w \in W$, $Prob(w) = (W(w), H(w), \mu(w)$ such that
  - $W(w) = W$;
  - $H(w)$ is the class of sets $[\![\alpha]\!] = \{w' \in W \mid \alpha \in w'\}$, for every sentence α; and
  - for every set $[\![\alpha]\!] \in H(w)$, $\mu(w)([\![\alpha]\!]) = \sup\{s \in [0,1]_Q \mid P_{\geq s}\alpha \in w\}$.

Similarly as in the propositional case, it can be proved that the canonical model is indeed a measurable structure and that the following result holds.

THEOREM 8.6 (Strong completeness of $\text{LFOP}_1^{\text{conf}}$).
A set of sentences T is consistent iff T is $\text{LFOP}_{1,\text{Meas}}^{\text{conf}}$-satisfiable.

9 The Logic $\text{LPP}_1^{\text{Fr(n),conf}}$

The goal of this section is to present some restrictions of the previously developed logics $\text{LPP}_1^{\text{conf}}$ and $\text{LFOP}_1^{\text{conf}}$ by assuming that probability measures in their semantics have specific finite ranges and to show that finitary axiomatizations can be obtained under those restrictions. For each $n \in \mathbb{N}$, we consider the probability measures whose range is the set

$$Range(n) = \{0, \frac{1}{n}, \ldots, \frac{(n-1)}{n}, 1\},$$

so there are actually denumerably many different logics that we denote by $\mathrm{LPP}_1^{\mathrm{Fr(n),conf}}$ and $\mathrm{LFOP}_1^{\mathrm{Fr(n),conf}}$. The process of obtaining the semantics and axiomatization of $\mathrm{LPP}_1^{\mathrm{Fr(n),conf}}$ from the ones of $\mathrm{LPP}_1^{\mathrm{conf}}$ is exactly the same as the process of moving from $\mathrm{LFOP}_1^{\mathrm{conf}}$ to $\mathrm{LFOP}_1^{\mathrm{Fr(n),conf}}$. Regarding the axiomatization, the process consists of replacing the three infinitary rules, R3-R5, with three novel axioms. Because of the similarities between the propositional and the first-order case, in the rest of this section, we focus on the propositional case and present the logic $\mathrm{LPP}_1^{\mathrm{Fr(n),conf}}$.

The syntax of $\mathrm{LPP}_1^{\mathrm{Fr(n),conf}}$ is the same as the syntax of $\mathrm{LPP}_1^{\mathrm{conf}}$. In the semantics of $\mathrm{LPP}_1^{\mathrm{Fr(n),conf}}$, we consider the subclass of $\mathrm{LPP}_1^{\mathrm{conf}}$-structures where probability measures have the range *Range(n)*, i.e. for every world of an $\mathrm{LPP}_1^{\mathrm{conf}}$-structure

$$\mu(w) : H(w) \to Range(n).$$

We define the class of $LPP_{1,Meas}^{Fr(n),conf}$-models in the same way as in Section 3.2.

We consider three additional axiom schemes for the logic

(F1) $\bigwedge_{k=0}^{n-1} P_{>\frac{k}{n}}\alpha \to P_{\geq\frac{k+1}{n}}\alpha$

(F2) $((P_{>0}\beta) \wedge \bigwedge_{k,l=0}^{n}((P_{\geq\frac{k}{n}}\alpha \wedge P_{\geq\frac{l}{n}}\beta) \to P_{\geq\frac{l}{n}(r+\frac{k}{n})}(\alpha\wedge\beta))) \to c_{\geq r}(\alpha,\beta)$

(F3) $((P_{>0}\beta) \wedge \bigwedge_{k,l=0}^{n}((P_{\leq\frac{k}{n}}\alpha \wedge P_{\leq\frac{l}{n}}\beta) \to P_{\leq\frac{l}{n}(r+\frac{k}{n})}(\alpha\wedge\beta))) \to c_{\leq r}(\alpha,\beta)$.

We assume that formulas respect Definition 3.1. For example, we consider only those instances of the axioms F2 and F3 satisfying $0 \leq \frac{l}{n}(r+\frac{k}{n}) \leq 1$.

Note that the axiomatization $Ax(\mathrm{LPP}_1^{\mathrm{Fr(n),conf}})$ is obviously sound for the class of structures $LPP_{1,Meas}^{Fr(n),conf}$. This follows directly from Theorem 4.3. Next, we show that the new axioms are sound as well.

THEOREM 9.1 (Soundness).
Axioms F1–F3 are sound with respect to the class of structures $LPP_{1,Meas}^{Fr(n),conf}$.

PROOF. Let us consider the axiom F2. Let $M \in LPP_{1,Meas}^{Fr(n),conf}$, and let w be a world in a model M such that $M,w \models (P_{>0}\beta) \wedge \bigwedge_{k,l=0}^{n}((P_{\geq\frac{k}{n}}\alpha \wedge P_{\geq\frac{l}{n}}\beta) \to P_{\geq\frac{l}{n}(r+\frac{k}{n})}(\alpha\wedge\beta))$. Then, we have that $M,w \models P_{>0}\beta$ and $M,w \models (P_{\geq\frac{k}{n}}\alpha \wedge P_{\geq\frac{l}{n}}\beta) \to P_{\geq\frac{l}{n}(r+\frac{k}{n})}(\alpha\wedge\beta)$ for all $k,l=0,\dots,n$. So we have that

$$\mu(w)([\beta]) > 0. \tag{1}$$

Let $t=\mu(w)([\alpha])$ and $s=\mu(w)([\beta])$. Since $\mu(w) : H(w) \to Range(n)$, there are k and l such that $t=\frac{k}{n}$ and $s=\frac{l}{n}$. Hence, $M,w \models (P_{\geq\frac{k}{n}}\alpha \wedge P_{\geq\frac{l}{n}}\beta)$, by soundness of R1, we obtain $M,w \models P_{\geq\frac{l}{n}(r+\frac{k}{n})}(\alpha\wedge\beta)$. Then, we also have

$$\mu(w)([\alpha\wedge\beta]) \geq \frac{l}{n}(r+\frac{k}{n}) = \mu(w)([\beta])(r+\mu(w)([\alpha])), \tag{2}$$

so from (1) and (2), we get $M,w \models c_{\geq r}(\alpha,\beta)$.

The case when we consider F3 is similar to the one presented above. The case of the axiom F1 can be found in [29]. □

Now, we introduce the finitary axiomatic system $Ax(\mathrm{LPP}_1^{\mathrm{Fr(n),conf}})$. It contains

- the axiom schemes A1-A10 and F1-F3 and
- the inference rules R1 and R2.

Next, we prove that this axiomatization is strongly complete with respect to the class of models $LPP_{1,Meas}^{Fr(n),conf}$.

THEOREM 9.2 (Strong completeness).
A set of formulas T is consistent with respect to the axiomatization $Ax(\mathrm{LPP}_1^{\mathrm{Fr(n),conf}})$ if and only if T is $LPP_{1,Meas}^{Fr(n),conf}$-satisfiable.

PROOF. Completeness is proved in two steps. In the first step, we show that the axiom system obtained by adding the inference rules R3–R5 to $Ax(\mathrm{LPP}_1^{\mathrm{Fr(n),conf}})$ is complete. In the second part, we show that we can remove the rules R3–R5 from the axiomatization as they are derivable from the rest of the axiomatization. This results with the complete axiomatic system $Ax(\mathrm{LPP}_1^{\mathrm{Fr(n),conf}})$.

The first part is almost identical to the proof of completeness of $\mathrm{LPP}_1^{\mathrm{conf}}$ logic. The difference is in the construction of the canonical model. Since *Range* is a finite set, the measures in the canonical model are defined by

$$\mu(w)([\![\alpha]\!]) = \max\{r \mid r \in Range, P_{\geq r}\alpha \in w\}.$$

Using the axiom $F1$, one can show that the constructed canonical model indeed belongs to the class $LPP_{1,Meas}^{Fr(n),conf}$.

Now, we want to show that we can exclude the infinitary rules from the axiomatization. First, it is shown in [29] that R3 can be excluded in presence of the axiom F1.

Next, we show that the axiom F2 can replace the rule R4. The idea is to show that R4 is derivable from F2 and the propositional reasoning (A1 and R1). In other words, for given α, β and γ and fixed $r \in [0,1]_Q$, we assume the set of premises

$$P = \{\gamma \to P_{>0}\beta\} \cup \{\gamma \to ((P_{\geq t}\alpha \wedge P_{\geq s}\beta) \to P_{\geq s(r+t)}(\alpha \wedge \beta)) | t, s \in [0,1]_Q\}$$

of the rule R4, and we infer its conclusion

$$\gamma \to c_{\geq r}(\alpha, \beta)$$

using F2. Consider the finite set $S = \{\gamma \to P_{>0}\beta\} \cup \{\gamma \to ((P_{\geq \frac{k}{n}}\alpha \wedge P_{\geq \frac{l}{n}}\beta) \to P_{\geq \frac{l}{n}(r+\frac{k}{n})}(\alpha \wedge \beta)) \mid k, l \in [0,1]_{\mathbb{N}}, \frac{l}{n}(r+\frac{k}{n}) \in [0,1]\}$. Obviously $S \subseteq P$.

Since $(\phi \to \psi) \to [(\gamma \to \phi) \to (\gamma \to \psi)]$ and $[\gamma \to (\phi \wedge \psi)] \leftrightarrow [(\gamma \to \phi) \wedge (\gamma \to \psi)]$ are propositional tautologies, we obtain

$$\bigwedge S \to [\gamma \to ((P_{>0}\beta) \wedge \bigwedge_{k,l=0}^{n}((P_{\geq \frac{k}{n}}\alpha \wedge P_{\geq \frac{l}{n}}\beta) \to P_{\geq \frac{l}{n}(r+\frac{k}{n})}(\alpha \wedge \beta))],$$

by propositional reasoning. From this formula, using F2 (and A1 and R1, similarly as above), we obtain

$$\bigwedge S \to (\gamma \to c_{\geq r}(\alpha, \beta)).$$

Thus, since $S \subseteq P$, we proved $\gamma \to c_{\geq r}(\alpha, \beta)$ using F2, A1 and R1.

In a similar way, one can show that F3 can replace the rule R5. □

As a consequence, since the above finitary system is strongly complete, we have that the logic $\mathrm{LPP}_1^{\mathrm{Fr(n),conf}}$ is compact.

10 Conclusion

In this paper, we presented the propositional probabilistic logic $\mathrm{LPP}_1^{\mathrm{conf}}$ for reasoning about degrees of confirmation and its first-order extension $\mathrm{LFOP}_1^{\mathrm{conf}}$. The languages of $\mathrm{LPP}_1^{\mathrm{conf}}$ and $\mathrm{LFOP}_1^{\mathrm{conf}}$ extend the languages of LPP_1 and LFOP_1 (respectively) from [28] with the binary operators that model the measure of confirmation. We proposed axiomatizations for $\mathrm{LPP}_1^{\mathrm{conf}}$ and $\mathrm{LFOP}_1^{\mathrm{conf}}$ and prove strong completeness. Since the logics are not compact, the axiomatizations contain infinitary rules of inference. Then, we simplified the semantics and we achieved compactness using probabilistic functions with finite ranges. For those simplified logics, we provide finitary axiomatizations.

We also proved that the problem of checking whether a probabilistic formula of the logic $\mathrm{LPP}_1^{\mathrm{conf}}$ is satisfiable is decidable. We combined the method of filtration [23] and a reduction to a system of polynomial inequalities.

There exist several complete logical formalisms for qualitative and quantitative reasoning about evidence [21, 31, 35]. However, to the best of our knowledge, the only logic in which Bayesian confirmation notions were formalized is [7], where the classificatory concept of confirmation is modelled through the binary operators $\uparrow$ ('confirms') and $\downarrow$ ('disconfirms'). Since our logic $\mathrm{LPP}_1^{\mathrm{conf}}$ has a richer language in which we can express degrees of confirmation, those operators can be modelled in $\mathrm{LPP}_1^{\mathrm{conf}}$: $\alpha \uparrow \beta$ as $c_{>0}(\alpha, \beta)$ and $\alpha \downarrow \beta$ as $c_{<0}(\alpha, \beta)$.

It is worth mentioning that our confirmation operators can be modelled in the logical language with polynomial weight formulas from [14] that we described in Section 5. However, for obtaining the weak completeness of that logic, additional expressiveness was needed and the language was extended to a first-order language such that variables can appear in formulas.

Finally, let us propose an avenue for further research. Recall that in this paper, we modelled the difference measure. We chose this measure simply because it is the most standard measure of confirmation. However, we can adapt the technique developed here to capture the other popular measures from the literature (see, e.g. [33]). For example, Carnap's relevance measure

$$\mu(A \wedge B) - \mu(A)\mu(B)$$

can be axiomatized by replacing A7–A10 and R4 and R5 with the following axiom schemes and inference rules:

(A7') $(P_{\geq t}\alpha \wedge P_{\geq s}\beta \wedge c_{\geq r}(\alpha, \beta)) \to P_{\geq r+st}(\alpha \wedge \beta)$;
(A8') $(P_{\leq t}\alpha \wedge P_{\leq s}\beta \wedge c_{\leq r}(\alpha, \beta)) \to P_{\leq r+st}(\alpha \wedge \beta)$;
(R4') from the set of premises

$$\{\gamma \to ((P_{\geq t}\alpha \wedge P_{\geq s}\beta) \to P_{\geq r+st}(\alpha \wedge \beta)) \mid t, s \in [0,1]_Q\}$$

infer $\gamma \to c_{\geq r}(\alpha, \beta)$;
(R5') from the set of premises

$$\{\gamma \to ((P_{\leq t}\alpha \wedge P_{\leq s}\beta) \to P_{\leq r+st}(\alpha \wedge \beta)) \mid t, s \in [0,1]_Q\}$$

infer $\gamma \to c_{\leq r}(\alpha, \beta)$.

For axiomatizing Carnap's relevance measure, we need only eight axiom schemes. Note that we can also apply the similar technique for axiomatizing log–ratio measure

$$c(\alpha,\beta) = \log\big[\frac{P(\alpha|\beta)}{P(\alpha)}\big],$$

but the decidability results are not clear. In that case, we cannot translate a formula to an existential sentence in the first-order language of fields, as we did in Section 7, so we cannot apply the procedure from [14].

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References

[1] M. Abadi and J. Y. Halpern. Decidability and expressiveness for first-order logics of probability. *Information and Computation*, **112**, 1–36, 1994.

[2] N. Alechina. Logic with probabilistic operators. In *Proceedings of the ACCOLADE '94*, pp. 121–138, 1995. https://dare.uva.nl/search?identifier=7c46827b-9212-4a56-aa52-b2b034898e6a.

[3] R. Carnap. *Logical Foundations of Probability*, 2nd edn. University of Chicago Press, 1962.

[4] S. Dautovic, D. Doder and Z. Ognjanovic. Reasoning about degrees of confirmation. In *Third International Conference on Logic and Argumentation, CLAR 2020*, M. Dastani, H. Dong and L. van der Torre, eds, Hangzhou, China, 6–9 April 2020, Vol. 12061 of Lecture Notes in Computer Science, pp. 80–95. Springer, 2020.

[5] J. P. Delgrande, B. Renne and J. Sack. The logic of qualitative probability. *Artificial Intelligence*, **275**, 457–486, 2019.

[6] D. Doder, B. Marinković, P. Maksimović and A. Perović. A logic with conditional probability operators. *Publications de l'Institut Mathmatique*, **87**, 85–96, 2010.

[7] D. Doder and Z. Ognjanović. Probabilistic logics with independence and confirmation. *Studia Logica*, **105**, 943–969, 2017.

[8] D. Doder. A logic with big-stepped probabilities that can model nonmonotonic reasoning of system p. *Publications de l'Institut Mathmatique*, **90**, 13–22, 2011.

[9] D. Doder and Z. Ognjanovic. A probabilistic logic for reasoning about uncertain temporal information. In *Proceedings of the Thirty-First Conference on Uncertainty in Artificial Intelligence, UAI 2015*, M. Meila and T. Heskes, eds, 12–16 July 2015, Amsterdam, The Netherlands, pp. 248–257. AUAI Press, 2015.

[10] D. Doder, Z. Ognjanović and Z. Marković. An axiomatization of a first-order branching time temporal logic. *Journal of Universal Computer Science*, **16**, 1439–1451, 2010.

[11] D. Doder, N. Savic and Z. Ognjanovic. Multi-agent logics for reasoning about higher-order upper and lower probabilities. *Journal of Logic, Language and Information*, **29**, 77–107, 2020.
[12] E. Eells and B. Fitelson. Measuring confirmation and evidence. *Journal of Philosophy*, **97**, 663–672, 2000.
[13] R. Fagin and J. Y. Halpern. Reasoning about knowledge and probability. *Journal of the ACM*, **41**, 340–367, 1994.
[14] R. Fagin, J. Y. Halpern and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation*, **87**, 78–128, 1990.
[15] B. Fitelson. The plurality of Bayesian measures of confirmation and the problem of measure sensitivity. *Philosophy of Science*, **66**, 378, 1999.
[16] A. Frisch and P. Haddawy. Anytime deduction for probabilistic logic. *Artificial Intelligence*, **69**, 93–122, 1994.
[17] D. M. Gabbay and I. M. Hodkinson. An axiomitization of the temporal logic with until and since over the real numbers. *Journal of Logic and Computation*, **1**, 229–259, 1990.
[18] H. Gaifman. A theory of higher order probabilities. In *Proceedings of the Theoretical Aspects of Reasoning about Knowledge*, J. Y. Halpern ed., pp. 275–292. Morgan Kaufmann, San Mateo, California, 1986.
[19] D. Guelev. A propositional dynamic logic with qualitative probabilities. *Journal of Philosophical Logic*, **28**, 575–605, 1999.
[20] J. Y. Halpern. An analysis of first-order logics of probability. *Artificial Intelligence*, **46**, 311–350, 1990.
[21] J. Y. Halpern and R. Pucella. A logic for reasoning about evidence. *Journal of Artificial Intelligence Research*, **26**, 1–34, 2006.
[22] A. Heifetz and P. Mongin. Probability logic for type spaces. *Games and Economic Behavior*, **35**, 31–53, 2001.
[23] G. E. Hughes and M. J. Cresswell. *A Companion to Modal Logic*. Methuen, London; New York, 1984.
[24] H. J. Keisler. Probability quantifiers. In *Model Theoretic Logic*, J. Barwise and S. Feferman, eds, pp. 509–556. Springer, Berlin, 1985.
[25] E. Marchioni and L. Godo. A logic for reasoning about coherent conditional probability: A modal fuzzy logic approach. In *9th European Conference on Logics in Artificial Intelligence, JELIA 2004*, J. J. Alferes and J. A. Leite, eds, Lisbon, Portugal, 27–30 September 2004, Vol. 3229 of Lecture Notes in Computer Science, pp. 213–225. Springer, 2004.
[26] B. Marinkovic, Z. Ognjanovic, D. Doder and A. Perovic. A propositional linear time logic with time flow isomorphic to ω^2. *Journal of Applied Logic*, **12**, 208–229, 2014.
[27] N. Nilsson. Probabilistic logic. *Artificial Intelligence*, **28**, 71–87, 1986.
[28] Z. Ognjanović and M. Rašković. Some first-order probability logics. *Theoretical Computer Science*, **247**, 191–212, 2000.
[29] Z. Ognjanović, M. Rašković and Z. Marković. *Probability Logics: Probability-Based Formalization of Uncertain Reasoning*. Springer, Cham, Switzerland, 2016.
[30] Z. Ognjanovic, D. Doder and Z. Markovic. A branching time logic with two types of probability operators. In *5th International Conference on Scalable Uncertainty Management, SUM 2011*, S. Benferhat and J. Grant, eds, Dayton, OH, USA, 10–13 October 2011, Vol. 6929 of Lecture Notes in Computer Science, pp. 219–232. Springer, 2011.
[31] Z. Ognjanovic, Z. Markovic, M. Raskovic, D. Doder and A. Perovic. A propositional probabilistic logic with discrete linear time for reasoning about evidence. *Annals of Mathematics and Artificial Intelligence*, **65**, 217–243, 2012.

[32] N. Savic, D. Doder and Z. Ognjanovic. Logics with lower and upper probability operators. *International Journal of Approximate Reasoning*, **88**, 148–168, 2017.

[33] K. Tentori, V. Crupi, N. Bonini and D. Osherson. Comparison of confirmation measures. *Cognition*, **103**, 107–119, 2007.

[34] S. Tomovic, Z. Ognjanovic and D. Doder. Probabilistic common knowledge among infinite number of agents. In *13th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, ECSQARU 2015*, S. Destercke and T. Denoeux, eds, Compiègne, France, 15–17 July 2015, Vol. 9161 of Lecture Notes in Computer Science, pp. 496–505. Springer, 2015.

[35] J. van Benthem and E. Pacuit. Dynamic logics of evidence-based beliefs. *Studia Logica*, **99**, 61–92, 2011.

[36] W. van der Hoek. Some considerations on the logic PDF. *Journal of Applied Non-Classical Logics*, **7**, 287–307, 1997.

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