

Variational analysis of multiscale problems with differential constraints

Material models involving
incompressibility

Dominik Engl

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Cover: The overall cover design is inspired by incompressibility and the Poisson effect: the contraction of a material in the directions orthogonal to the direction of stretching. The three illustrations on the back cover exemplify the three major topics of this thesis. On the top left, the thin rod stands for the dimension reduction of incompressible strings and rods; here, the three mutually orthogonal vectors form a Frénét-frame that is used to describe the mid-fiber deformation and twisting effects of the cross section. On the right, the variational analysis of fiber-reinforced materials is represented by an impression of an admissible macroscopic deformation of a cuboid; the blue lines portray infinitesimally thin, long rigid fibers. The illustration on the bottom left stands for the analysis of polycrystalline plasticity, showing an example of a polycrystal in reference configuration; the arrows depict the orientation of the slip system, and the different colors indicate the boundary and interior grains of the polycrystal.

Variational analysis of multiscale problems
with differential constraints

Material models involving incompressibility

Variationele analyse van multischaalproblemen
onder differentiaalvoorwaarden

Modellen voor niet-samendrukbare materialen

(met een samenvatting in het Nederlands)

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“It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.”
- Carl Friedrich Gauß

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Chapter 1

Introduction

Identifying optimal solutions and characterizing their features is the bedrock of the calculus of variations. The appeal of this mathematical branch is that minimization problems are generally easy to formulate but much more complicated to solve; indeed, the reader will likely agree that we often seek solutions of least effort to issues in our daily lives. The importance of the calculus of variations is underlined by its vast variety of applications ranging from finding geodesics in differential geometry to optimal control theory, the analysis of electrostatics, equilibrium states in quantum mechanics, and most prominently, particularly in this thesis, materials science, see e.g. [70, 134, 182, 197]. In fact, the classical mechanics literature usually covers variational approaches to some degree, and most textbooks on the calculus of variations target aspects of elasticity and plasticity theory, among others. After all, we find plenty of variational principles emerging from physics such as Fermat’s principle of least time in optics, or Hamilton’s principle of least action (although “least” is a historical misnomer).

The first era of the modern calculus of variations was born in 1696 with Johann Bernoulli’s famous brachistochrone problem (from Ancient Greek, “shortest time”). He raised the question of which path an object that is attached to a wire and is driven frictionless only by gravity has to take to travel between two given points in a vertical plane in the shortest possible time, see Figure 1.1. His challenge, which was published in the journal *Acta Eruditorum* [32], has gained considerable attention in the calculus community with notable contributions by Leibniz, L’Hospital, Newton, and Johann’s brother Jacob Bernoulli.

In the eighteenth century, the framework of the classical methods (or nowadays so-called indirect methods) in the calculus of variations was developed by Euler and Lagrange. Euler was also the first to tackle variational problems constrained by differential equations, a topic paramount to this thesis. These classical techniques involve the generalization of the first direc-

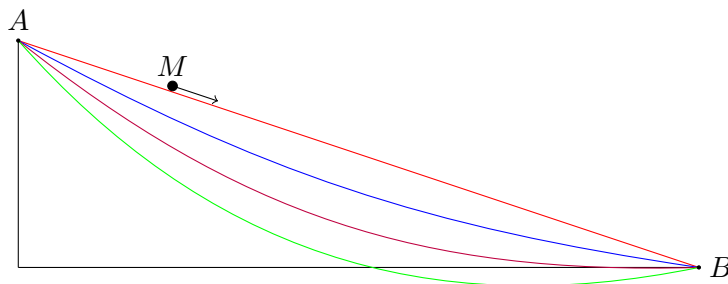


Figure 1.1: Illustration of the brachistochrone problem, in which an object M moves along a wire between two points A and B .

tional derivative of a functional, which is called the first variation, and the identification of its extremal points as stationary points with respect to the first variation of the functional. The latter commonly amounts to solving the resulting ordinary or partial differential equations called the Euler-Lagrange equations. With these contributions, determining necessary properties of minimizers was the primary focus during the eighteenth century, at the end of which Legendre was one of the first to attack the issue of sufficiency with the help of the second variation [142]; His work was later continued by Weierstraß [109, Chapter 5.3].

David Hilbert started the second era of the modern calculus of variations with his esteemed lecture in 1900 to the International Congress of Mathematicians, and the development of the so-called direct method (see Section 1.2), which he used to prove Dirichlet's principle [154, Chapter IV]. In his lecture, he presented his now well-known 23 problems, three of which directly concerned the calculus of variations and inspired many mathematicians, like Hadamard, Lebesgue, Noether, and Tonelli, to contribute to the field. For more details on the history of the calculus of variations, we refer the reader to, e.g., Giaquinta & Hildebrandt [103, 104], Goldstine [109], or Monna [154].

In contrast to the classical techniques, the direct method exploits minimality properties of the functional itself instead of determining characterizing attributes of the minimizers. From the work of Hilbert, a major branch of the calculus of variations emerged that is devoted to determining the existence of minimizers via lower semicontinuity and coercivity of the functional, the key ingredients to the direct method. We describe in Section 1.2 the general theory as well as its applications to integral functionals, which are significant for application in the context of isoperimetric problems (see, e.g., [197, Chapter 4]), classical mechanics [52, 53, 54], or imaging sciences (e.g., [45]), to mention but a few.

In the setting of integral functionals, the domain is often chosen to be a (subset of a) Sobolev space equipped with the weak topology, in which case we are lead to investigate aspects of weak coercivity and weak lower semicontinuity. While weak coercivity can simply be achieved by imposing suitable polynomial growth of the corresponding integrand, finding characterizing conditions for weak lower semicontinuity is much more delicate and is closely related to generalized notions of convexity. For standard problems involving suitable polynomial behavior, it turns out that quasiconvexity of the integrand, which was first introduced by Morrey in 1952 [160], is such a necessary and sufficient condition, see Theorem 1.15. However, this classical result does not cover extended-valued functionals emerging from constrained variational problems. In fact, the characterization of weak lower semicontinuity of unbounded functionals is still an open problem [15]. This issue complicates minimization problems involving non-convex differential constraints arising, in particular, in the context of hyperelasticity in the form of incompressibility, non-interpenetration, (local) rigidity, and orientation preservation. Such assumptions are, however, crucial in designing advanced high-tech materials, which is why it is paramount to obtain a better understanding of constrained variational problems.

In the absence of lower semicontinuity, the existence of minimizers is no longer secured. Instead, the calculus of variations is then concerned with the analysis of low-energy sequences, which often exhibit fine oscillations on smaller length scales. The most prominent example in elastostatics is the formation of microstructures in shape-memory alloys; in the simplest case, these structures are essentially described by functions whose gradients are highly oscillating between two values, although the deformation gradient may be smooth on a macroscopic level, see for example [21, 84, 162], see also Remark 1.23. To efficiently capture and describe these effects, a closely related minimization problem from which one can deduce asymptotic properties of low-energy sequences is studied. Precisely, a common approach is to determine the lower semicontinuous envelope (or relaxation) of the given functional. Under suitable coercivity assumptions, the associated relaxed variational problem then admits a solution and low-energy

sequences of the unrelaxed functional converge (up to the selection of subsequences) to minimizers of the relaxation, see Theorem 1.12. In the case of integral functionals with standard polynomial growth of the integrand, Dacorogna proved in [69] (see also [70]) the structure-preserving result that the weak lower semicontinuous envelope is determined by an integral functional, whose integrand coincides with the quasiconvex envelope of the original integrand, see also Theorem 1.22.

Another approach to effectively describe the asymptotic behavior of low-energy sequences involves the theory of Young measures, see for example [13] or [175] and the references therein. In this context, we shall also make the reader aware of the work by Kinderlehrer & Pedregal [121, 122] in which the authors derive a significant connection between quasiconvexity and gradient Young measures. In this thesis, we merely touch upon this topic in Chapter 5 as a tool.

The fundamental basis of the relation between calculus of variation and classical mechanics consists of solid grasp of material deformation behavior. To this end, one needs to understand the materials' reactions to external forces on different length scales, as well as their interaction, through suitable averaging procedures and limit processes. This is commonly referred to as multiscale modeling and, in practice, strongly narrows the availability of numerical simulations [150]. These obstacles can be overcome with the help of powerful novel mathematical concepts in the calculus of variations and asymptotic analysis such as the abstract framework of Γ -convergence introduced by Franzoni & De Giorgi in 1975 [78, 79], see also Section 1.3. This notion of variational convergence is particularly well-suited for the analysis of nonlinear problems and extraordinarily adaptable. It enables the treatment of a wide variety of problems such as dimension reduction, homogenization, discrete-to-continuum limits and relaxation, see for example [42] or [71] and the references therein. We provide a separate brief introduction to dimension reduction and homogenization in Sections 1.4 and 1.5. With this potent variational concept at hand, finding reliable macroscopic descriptions of materials with fine microstructures or inhomogeneities becomes much more accessible.

New challenges in the calculus of variations lie in the analysis of variational problems subject to non-standard constraints. Especially constraints involving non-convex (inhomogeneous) partial differential expressions are generally known to be technically demanding. In particular, many physically relevant material properties such as incompressibility, non-interpenetration of matter, orientation preservation, or rigidity prevent the application of standard theories.

The goal of this thesis is to investigate constrained variational problems attributed to important material classes involving local volume preservation. A detailed overview of the main results of this thesis is given in Section 1.6. In short, a complete hierarchy of $3d$ - $1d$ dimension reduction results for locally volume preserving elastic strings and rods is derived in Chapters 2 and 3. We provide a macroscopic understanding of high-contrast materials reinforced by rigid fibers via a compactness result of an associated homogenization framework in Chapter 4. We conclude this thesis with an analysis of the set of attainable macroscopic deformations in single-slip polycrystal plasticity in Chapter 5; these states are determined by affine boundary values of Lipschitz solutions to an inhomogeneous partial differential inclusion involving local volume preservation as well as length-preservation in specific directions.

1.1 Introduction to hyperelasticity

We start to introduce the mathematical framework for variational approaches to the Lagrangian description of elasticity theory in $n \in \{1, 2, 3\}$ dimensions. We call the reference configuration $\Omega \subset \mathbb{R}^n$ the region that an undeformed elastic body occupies at a given time. In this thesis, we are only interested in stationary, or time-independent problems. The material body is then (instantly) transformed via a deformation $u : \Omega \rightarrow \mathbb{R}^n$; the image $u(\Omega)$ is called the deformed

configuration, see Figure 1.2. To avoid confusion we point out that in the engineering community the letter u refers to the displacement, which describes the deviation from the identity, instead of the deformation itself, see, e.g., [52, 113]. In the context of continuum mechanics, it is commonly assumed that deformations are continuously differentiable and have a positive determinant everywhere. However, these conditions are often mathematically too restrictive and are therefore weakened or omitted. The key feature of hyperelastic materials is that their

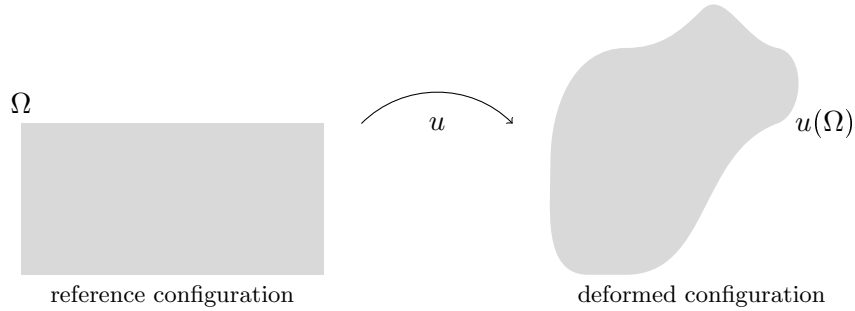


Figure 1.2: Illustration of a material body Ω being deformed by a deformation u .

so-called first Piola-Kirchhoff stress tensor $P : \Omega \times \text{GL}_+(n) \rightarrow \mathbb{R}^{n \times n}$ can be expressed as the derivative of a function $W : \Omega \times \mathbb{R}^{n \times n} \rightarrow [0, \infty]$, which is finite on $\text{GL}_+(n) := \{F \in \mathbb{R}^{n \times n} : \det F > 0\}$, with respect to its second argument, i.e.

$$P(x, F) = \partial_F W(x, F) \quad \text{for all } F \in \text{GL}_+(n) \text{ and a.e. } x \in \Omega. \quad (1.1)$$

In this setting, the internal elastic energy associated to a deformation can be described via the stored energy density (or strain energy density) W that captures the material properties. Precisely, if $\Omega \subset \mathbb{R}^n$ is the reference configuration of a given hyperelastic body and $u : \Omega \rightarrow \mathbb{R}^n$ is a deformation, then

$$\int_{\Omega} W(x, \nabla u(x)) \, dx$$

determines the elastic energy associated to u .

It is often assumed that the strain energy remains constant under any change of observers, which is incorporated into the density via

$$W(x, RF) = W(x, F) \quad \text{for all } R \in \text{SO}(n), F \in \mathbb{R}^{n \times n} \text{ and a.e. } x \in \Omega. \quad (1.2)$$

Since strain energy densities are usually normed in the sense that $W(x, \text{Id}) = 0$ for a.e. $x \in \Omega$, we find that, under the assumption of frame-indifference (1.2), they vanish on the set of rotations. A finite valued prototypical choice of W for homogeneous solids is then given by

$$W(F) = \text{dist}^p(F, \text{SO}(n)) \quad \text{with} \quad \text{dist}(F, \text{SO}(n)) = \inf_{R \in \text{SO}(n)} |F - R|, \quad (1.3)$$

for $p \geq 1$. However, such finite-valued densities do not reflect all relevant material properties. In particular, (1.3) allows an unrealistic compression to a single point as well as the change of orientation of deformations or self-interpenetration. Mathematically, these phenomena are eliminated by introducing the constraint requiring positive determinants and a singularity at $\det F = 0$, i.e.,

$$W(x, F) = \infty \text{ if } \det F \leq 0 \quad \text{and} \quad W(x, F) \rightarrow \infty \text{ as } \det F \rightarrow 0 \quad \text{for a.e. } x \in \Omega. \quad (1.4)$$

Combining the frame-indifference (1.2) with (1.4) then ensures that there exists another density $\widetilde{W} : \Omega \times \text{GL}_+(n) \cap \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow [0, \infty]$ such that

$$W(x, F) = \widetilde{W}(x, F^T F)$$

for all $F \in \text{GL}_+(n)$, due to the standard polar decomposition $F = R\sqrt{F^T F}$ for some suitable $R \in \text{SO}(n)$. If F is taken to be the gradient of a deformation, then the quantity $F^T F$ is called the right Cauchy-Green strain tensor and determines the square of local change of length applied by the deformation. In some applications, the deformation behavior of solids does not depend on the direction of external influences like mechanical loading. Such materials are called isotropic and their stored energy densities satisfy the relation

$$W(x, FR) = W(x, F) \quad \text{for all } R \in \text{SO}(n), F \in \mathbb{R}^{n \times n} \text{ and a.e. } x \in \Omega. \quad (1.5)$$

Combining (1.2) with (1.4) and (1.5), we obtain that the strain energy essentially depends only on the eigenvalues of $F^T F$ or the singular values of F . In engineering terms, the energy density depends only on the principal invariants I_1, I_2, I_3 of $F^T F$,

$$\begin{aligned} I_1 &:= \text{Tr}(F^T F) = |F|^2, \\ I_2 &:= \frac{1}{2}((\text{Tr}(F^T F))^2 - \text{Tr}(F^T F)^2) = |\text{adj } F|^2, \\ I_3 &:= \det F, \end{aligned} \quad (1.6)$$

where $\text{adj } F$ is the classical adjoint, or adjugate, of F . This fact is of particular relevance for the non-standard theory to prove the existence of minimizers of the total elastic energy, see Remark 1.18 below. Finally, we note that, similarly to (1.4), another common assumption is that infinite stretching also costs an infinite amount of energy, i.e.,

$$W(x, F) \rightarrow \infty \text{ if } |F| \rightarrow \infty \quad \text{for a.e. } x \in \Omega.$$

Before we continue, we shall present some standard examples of stored energy densities describing homogeneous materials. Inhomogeneities are commonly modeled by incorporating the dependence on the space variable directly into the occurring material constants. Recalling (1.1), we note that strain energy densities are specified only up to a translation.

Example 1.1. a) The stored energy density for Saint-Venant Kirchhoff materials is defined by

$$W(F) = \frac{\lambda}{2} (\text{Tr} \frac{1}{2}(F^T F - \text{Id}))^2 - \mu |\frac{1}{2}(F^T F - \text{Id})|^2, \quad F \in \mathbb{R}^{n \times n}, \quad (1.7)$$

where $\lambda, \mu > 0$ are the so-called Lamé coefficients; the latter is also called the shear modulus. The associated first Piola-Kirchhoff stress tensor is then

$$P(F) = \frac{\lambda}{2} \text{Tr}(F^T F - \text{Id}) - \mu(F^T F - \text{Id}), \quad F \in \mathbb{R}^{n \times n}.$$

If $F \in \mathbb{R}^{n \times n}$ is chosen to be a deformation gradient, then the quantity resulting from $\frac{1}{2}(F^T F - \text{Id})$ is called the Green-Lagrange (or Green-Saint Venant) strain tensor and can be used as a metric for the deviation of the gradient from a rotation. The factor $\frac{1}{2}$ is included to make the transition to linearized elasticity consistent.

In applications, it is often more convenient to write (1.7) in terms of the singular values $v_1(F) \leq \dots \leq v_n(F)$ of F and different elastic constants. Precisely,

$$W(F) = \frac{E}{8(1+\nu)} \sum_{i=1}^n (v_i(F)^2 - 1)^2 + \frac{E\nu}{8(1+\nu)(1-2\nu)} \left(\sum_{i=1}^n v_i(F)^2 - 2 \right)^2, \quad F \in \mathbb{R}^{2 \times 2} \quad (1.8)$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \text{and} \quad \nu = \frac{\lambda}{2(\lambda + \mu)} \quad (1.9)$$

are respectively called the Young modulus and the Poisson ratio.

b) Compressible Neo-Hookean materials [183] are described by simple energy densities of the form

$$W(F) = \begin{cases} a|F|^2 + \Gamma(\det F) & \text{if } \det F > 0, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{n \times n} \quad (1.10)$$

with $a > 0$ and $\Gamma : (0, \infty) \rightarrow [0, \infty)$ is given by $\Gamma(d) = c_1 d^2 - c_2 \log d$ for suitable constants $c_1, c_2 > 0$. In toy models, the orientation preservation and the term Γ governing the determinant are left out.

c) The energy densities describing compressible Mooney-Rivlin materials [155, 183] are similar to (1.10) but include an additional term measuring the $(n-1) \times (n-1)$ minors of $F \in \mathbb{R}^{n \times n}$. To be precise,

$$W(F) = \begin{cases} a|F|^2 + b|\text{adj } F|^2 + \Gamma(\det F) & \text{if } \det F > 0, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{n \times n}$$

with $a, b > 0$ and Γ as in b).

d) Compressible Ogden materials [169, 170] represent an even broader class of materials and are described by

$$W(F) = \begin{cases} \sum_{i=1}^M a_i(x) \text{Tr } C^{\frac{\gamma_i}{2}} + \sum_{j=1}^N b_j \text{Tr}(\text{adj } C)^{\frac{\delta_j}{2}} + \Gamma(\det F) & \text{if } \det F > 0, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{n \times n} \quad (1.11)$$

where $C = F^T F$ describes the right Cauchy-Green strain tensor, and $a_i, b_i > 0$, $\gamma_i, \delta_j \geq 1$ for all $i \in \{1, \dots, M\}$ and $j \in \{1, \dots, N\}$ for $M, N \in \mathbb{N}$. In this setting, the function $\Gamma : (0, \infty) \rightarrow (0, \infty)$ is assumed to be convex and to explode with infinite compression, i.e., $\lim_{d \rightarrow 0} \Gamma(d) = \infty$.

e) Local volume preservation is modeled by subjecting the deformation gradients to the non-convex constraint $\det \nabla u = 1$. Hence, incompressible variants of the materials in a) - d) are then described (up to a translation) by a constrained energy density of the form

$$W(F) = \begin{cases} W_0(F) & \text{if } \det F = 1, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{n \times n} \quad (1.12)$$

where $W_0 : \mathbb{R}^{n \times n} \rightarrow [0, \infty]$ is chosen to be one of the functions in (1.7) - (1.11). Another model for rubber-like materials in three dimensions was introduced by Yeoh in [202, 203] and is given by

$$W(F) = \begin{cases} \sum_{i=1}^3 a_i (\text{Tr}(F^T F - \text{Id}))^i & \text{if } \det F = 1, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{3 \times 3}$$

We highlight that (1.4) is often neglected in many variational models for the sake of mathematical simplicity. In this thesis, (1.4) is generally disregarded, if not mentioned otherwise.

In particular, the standard theory on the existence of minimizers of integral functionals, which we discuss in Section 1.2, is built on finite-valued energy functionals with suitable growth and convexity conditions. The next proposition shows that many toy models working with convexity of W are automatically excluded by (1.4).

Proposition 1.2 (Non-existence of a convex energy density [134, Proposition 2.3.4]). *If $n \geq 2$, then there exist no function $W : \Omega \times \mathbb{R}^{n \times n} \rightarrow [0, \infty]$ such that $W(x, \cdot)$ is convex and finite on $\text{GL}_+(n)$ for almost every $x \in \Omega$, and satisfies (1.4).*

External forces play a crucial role in the deformation behavior of elastic materials and we model their effect with the help of the total elastic energy (or system energy) given by

$$\mathcal{E}(u) = \int_{\Omega} W(\nabla u(x)) \, dx - \int_{\Omega} f(x) \cdot u(x) \, dx, \quad (1.13)$$

where $f : \Omega \times \mathbb{R}^n$ is the force density of a so-called dead load like gravity.

The variational approach to elasticity theory involves the minimization of such energy functionals since determining its (local) minimizers is critical in understanding the material's deformation behavior. The first ansatz is to seek solutions that are continuously differentiable with a positive Jacobian, but the minimization over this class of functions is rather delicate and does not account for microstructures such as laminates. Instead, the system energy (1.13) is minimized in a (subset of a) reflexive Sobolev space $W^{1,p}(\Omega; \mathbb{R}^n)$ for some $p > 1$ throughout this thesis. Although highly significant for applications, we do not concern ourselves with questions of regularity here. If fracture behavior of a material is of interest, then (1.13) can instead be analyzed on the space of functions with bounded variation $BV(\Omega; \mathbb{R}^n)$; for an introduction to these spaces, see for example [6, 10].

The importance of suitably minimizing (1.13) is underlined by the next proposition which relates minimizers of (1.13) to solutions to the equilibrium equations of the first Piola-Kirchhoff tensor, see for example [134, Theorem 1.2.6] and [182, Theorem 3.1]. The reader will note that, considering (1.4), the growth assumption on the gradient of the elastic energy density is a mathematical simplification. Of course, including the constraint of positive determinants makes the minimization much more involved since minimizers of (1.13) need not satisfy the associated Euler-Lagrange equations, see [23, 24]. This constrained setting is generally more complex and and requires a more delicate approach, see [15, Section 2.4].

Proposition 1.3 (Euler-Lagrange equations of hyperelastic materials). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain of hyperelastic material with strain energy density $W : \Omega \times \mathbb{R}^{n \times n} \rightarrow [0, \infty)$, that is,*

$$P(x, F) = \partial_F W(x, F) \quad x \in \Omega, F \in \mathbb{R}^{n \times n},$$

where $P : \Omega \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ is the first Piola-Kirchhoff stress tensor. Further let $g \in W^{1,p}(\Omega; \mathbb{R}^n)$, $f \in L^q(\Omega; \mathbb{R}^n)$ with $\frac{1}{q} + \frac{1}{p} = 1$, and suppose that W is Carathéodory, twice continuously differentiable in its second argument, and that P satisfies

$$|P(x, F)| \leq C(1 + |F|^p)$$

for a constant $C > 0$, all $F \in \mathbb{R}^{n \times n}$ and a.e. $x \in \Omega$.

Then, every solution $\bar{u} \in g + W_0^{1,p}(\Omega; \mathbb{R}^n)$ to the minimization problem

$$\inf \left\{ \int_{\Omega} W(x, \nabla u(x)) \, dx - \int_{\Omega} f(x) \cdot u(x) \, dx : u \in g + W_0^{1,p}(\Omega; \mathbb{R}^n) \right\} \quad (1.14)$$

satisfies the Euler-Lagrange equations

$$\begin{aligned} -\operatorname{div} P(x, \nabla \bar{u}(x)) &= f(x) && \text{for a.e. } x \in \Omega, \\ \bar{u}(x) &= g(x) && \text{for a.e. } x \in \partial\Omega. \end{aligned} \quad (1.15)$$

in the distributional sense.

The Dirichlet boundary condition on all of $\partial\Omega$ can be weakened to an equality only on a subset $\Gamma \subset \partial\Omega$ instead. The equations (1.15) in their classical sense are referred to as the equilibrium equations of the first Piola-Kirchhoff stress tensor. In their weak (or distributional) formulation, they are called the principle of virtual work in the reference configuration, see e.g. [134, Section 1.2.4]. If, for example, $W : \Omega \times \mathbb{R}^{n \times n}$ is twice continuously differentiable and (1.14) has a solution $\bar{u} \in W^{2,p}(\Omega; \mathbb{R}^3)$, then \bar{u} satisfies (1.15) almost everywhere in Ω due to the Fundamental Lemma of the calculus of variations.

1.2 The direct method in the calculus of variations

As we have seen earlier, minimizing the system energy of a hyperelastic material is the key to understanding its deformation behavior. In the calculus of variations, there are two schools of thought on how to obtain minimizers of given functionals: the classical (or indirect) approaches, which deal with characteristics of the minimizers, and the direct method, for which properties of the functional itself are of significance. The indirect methods involve seeking critical (or stationary) points of the functional via its first variation, which is a generalization of the directional derivative, and then determining positive definiteness of the second variation in a neighborhood of such points. With the direct method, on the other hand, we are only tasked with the question whether a (global) minimizer exists. For a comprehensive introduction, see e.g., [70, 96, 182].

In the finite dimensional case, the indirect methods (in this case simply known as curve sketching) are well-known by every mathematician and finding local and global minimizers leads to algebraic equations. However, many real-world problems, like the energetic approach to elasticity theory, require approaches in an infinite-dimensional framework such as an energy functional defined on a suitable function space. In such settings, critical points are determined by ordinary or partial differential equations, the so-called Euler-Lagrange equations. Obviously, these problems are inherently more challenging and solutions may not even exist. In fact, the following famous example by Weierstraß [201], in which he criticized Dirichlet's principle on the assumption of existence of minimizers of integral functional, shows that minimizers do not necessarily exist.

Example 1.4 (Weierstraß 1870). The functional

$$\mathcal{I} : X = \{u \in C^1([-1, 1]) : u(-1) = -1, u(1) = 1\} \rightarrow \mathbb{R}, \quad u \mapsto \int_{-1}^1 (xu'(x))^2 dx, \quad (1.16)$$

has no minimizer in X . One can prove this by first establishing that $\inf_{u \in X} \mathcal{I}(u) = 0$; since \mathcal{I} is non-negative, it is sufficient to show that $\inf_{u \in X} \mathcal{I}(u) \leq 0$. To this end, consider for every $\varepsilon > 0$ the function $u_\varepsilon \in X$ given by

$$u_\varepsilon(x) = \frac{\arctan(\frac{x}{\varepsilon})}{\arctan(\frac{1}{\varepsilon})}, \quad x \in [-1, 1], \quad (1.17)$$

see also Figure 1.3, and compute that

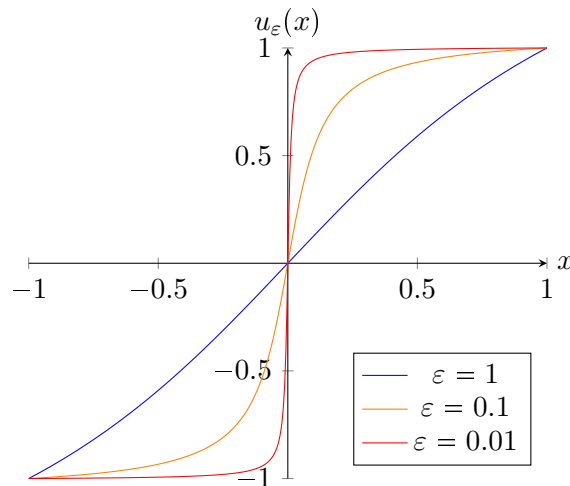


Figure 1.3: Illustration of the functions u_ε as in (1.17) for three values of ε .

$$\inf_{u \in X} \mathcal{I}(u) \leq \mathcal{I}(u_\varepsilon) < \int_{-1}^1 (x^2 + \varepsilon^2) (u'_\varepsilon(x))^2 dx = \left(\frac{1}{\arctan(\frac{1}{\varepsilon})} \right)^2 \int_{-1}^1 \frac{\varepsilon^2}{x^2 + \varepsilon^2} dx = \frac{2\varepsilon}{\arctan(\frac{1}{\varepsilon})}.$$

Taking the limit $\varepsilon \rightarrow 0$ then yields that

$$\inf_{u \in X} \mathcal{I}(u) \leq \lim_{\varepsilon \rightarrow 0} \frac{2\varepsilon}{\arctan(\frac{1}{\varepsilon})} = 0,$$

which shows that $\inf_{u \in X} \mathcal{I}(u) = 0$.

However, \mathcal{I} can not attain this value since any minimizer $u \in X$ has to satisfy $u' = 0$ everywhere on $[-1, 1]$, as well as the two boundary conditions $u(-1) = -1$ and $u(1) = 1$, which is a contradiction.

With the help of the the direct method in the calculus of variations, we are able to determine the existence of minimizers based on the properties of the sublevel sets of the functional. First, we introduce the general theory and we then turn towards the application to integral functionals relevant in the context of materials science. We begin this section in the setting that X is a topological Hausdorff space and $\mathcal{I} : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is a given functional; we are interested in finding solutions to the variational problem

$$\inf \{ \mathcal{I}(u) : u \in X \}. \quad (1.18)$$

The direct method in the calculus of variations gives two sufficient conditions for the solvability of (1.18): lower semicontinuity and coercivity. In the following, we shall define and discuss these concepts.

Definition 1.5 (Lower semicontinuity). *We say that \mathcal{I} is lower semicontinuous if*

$$\liminf_{k \rightarrow \infty} \mathcal{I}(u_k) \geq \mathcal{I}(u)$$

for every $u \in X$ and every sequence $(u_k)_k \subset X$ such that $u_k \rightarrow u$ in X .

Definition 1.6 (Coercivity). *The function \mathcal{I} is said to be coercive if every sequence $(u_k)_k \subset X$ with $\sup_k \mathcal{I}(u_k) < \infty$ has a converging subsequence with limit in X .*

It is worth pointing out that several different notions of lower semicontinuity and coercivity can be found in the literature.

Remark 1.7. a) Many authors call \mathcal{I} lower semicontinuous if its sublevel sets $\{u \in X : \mathcal{I}(u) \leq t\}$ for all $t \in \mathbb{R}$ are closed, cf. [96, Definition 3.1]. On the other hand, \mathcal{I} is called sequentially lower semicontinuous if its sublevel sets are sequentially closed, which is equivalent to Definition 1.5, see [96, Proposition 3.4]. Naturally, every lower semicontinuous function is also sequentially lower semicontinuous; the two notions coincide if, e.g., X is first-countable.

b) Depending on the general framework, the literature provides vastly different definitions of the concept of coercivity, see for example [71, Definition 1.12], [96, Definition 3.14], or [105, Chapter 1, Definition 1], although they have the similar spiritual origins. If \mathcal{I} satisfies the condition in Definition 1.6, then [71, Definition 1.12] calls \mathcal{I} sequentially coercive.

c) In the context of this thesis, we focus purely on the sequential versions of lower semicontinuity and coercivity, which is why we omit the word “sequential” in the context of these two notions for the remainder of this work.

d) In the spirit of c), if X is a normed space endowed with the weak topology, we call \mathcal{I} weakly lower semicontinuous if \mathcal{I} is lower semicontinuous in the sense of Definition 1.5 where the convergence in X means weak convergence. Similarly, we say that \mathcal{I} is weakly coercive if sequences of bounded energy have weakly convergent subsequences with limits in X .

We are now in a position to formulate the direct method in the calculus of variations to which we shall also present a proof for the purpose of illustration. We refer the reader to, e.g., [71, Theorem 1.15].

Theorem 1.8 (Direct method). *If \mathcal{I} is coercive and lower semicontinuous, then there exists a solution to the minimization problem (1.18).*

Proof. Assume that \mathcal{I} is not identical to ∞ , otherwise every point in X is a minimizer. Let $(u_k)_k \subset X$ be a minimizing sequence, i.e.,

$$\lim_{k \rightarrow \infty} \mathcal{I}(u_k) = \inf_{u \in X} \mathcal{I}(u) < \infty.$$

Then, we deduce that there exists a constant $C \in \mathbb{R}$ such that $\mathcal{I}(u_k) \leq C$ for all k sufficiently large. Since \mathcal{I} is coercive, there exist a subsequence of $(u_k)_k$ (not relabeled) and $\bar{u} \in X$ such that $u_k \rightarrow \bar{u}$ in X . Finally, we conclude from the lower semicontinuity of \mathcal{I} that

$$\inf_{u \in X} \mathcal{I}(u) = \lim_{k \rightarrow \infty} \mathcal{I}(u_k) = \liminf_{k \rightarrow \infty} \mathcal{I}(u_k) \geq \mathcal{I}(\bar{u}) \geq \inf_{u \in X} \mathcal{I}(u),$$

which proves that \bar{u} is a minimizer of \mathcal{I} . □

It is worth pointing out that there also exists another version of Theorem 1.8 in which the weaker topological versions of coercivity and lower semicontinuity are assumed, cf. Remark 1.7; the proof is then based on Cantor’s intersection theorem instead of the analysis of minimizing sequences, see [105, Proposition 1].

Looking back at Example 1.4, we see that the reason why the direct method in the calculus of variations fails for (1.16) is the lack of suitable coercivity. In fact, it is easy to prove that every minimizing sequence of (1.16) converges to the sign-function locally uniformly in $[-1, 1] \setminus \{0\}$.

Next, we briefly discuss the aspect of incorporating constraints into minimization problems.

Remark 1.9 (Constrained variational problems). In the following, let $\mathcal{A} \subset X$ describe the set of constraints which the functional \mathcal{I} shall be subject to. We are then lead to the

constrained minimization problem

$$\inf\{\mathcal{I}(u) : u \in \mathcal{A}\}. \quad (1.19)$$

The solvability of (1.19) depends on the topological properties of both \mathcal{A} and \mathcal{I} (or the restricted functional $\mathcal{I}|_{\mathcal{A}}$).

a) Suppose that \mathcal{A} is sequentially closed and describes the set of constraints. It is then straightforward to show that, if \mathcal{I} is lower semicontinuous and coercive in X , then $\mathcal{I}|_{\mathcal{A}}$ is also lower semicontinuous and coercive in \mathcal{A} , respectively. In view of Theorem 1.8, we see that (1.19) has a solution if \mathcal{I} is lower semicontinuous and coercive in the larger space X .

b) In the context of normed spaces X endowed with the weak topology, it is clear that (1.19) can be solved if \mathcal{A} is convex and closed with respect to the norm on X . Indeed, any such set \mathcal{A} is weakly (sequentially) closed, i.e., if a sequence $(u_k)_k \subset \mathcal{A}$ converges weakly to some $u \in X$, then $u \in \mathcal{A}$.

c) Constraints are often directly incorporated into the functional itself. Precisely, one introduces a new extended-valued function $\mathcal{J} : X \rightarrow \overline{\mathbb{R}}$ which coincides with \mathcal{I} on \mathcal{A} and is set to infinity everywhere else, that is,

$$\mathcal{J}(u) = \begin{cases} \mathcal{I}(u) & \text{if } u \in \mathcal{A}, \\ \infty & \text{otherwise.} \end{cases} \quad (1.20)$$

Then, we see that the level sets of \mathcal{J} satisfy

$$\{u \in X : \mathcal{J}(u) \leq t\} = \{u \in X : \mathcal{I}(u) \leq t\} \cap \mathcal{A}$$

for every $t \in \mathbb{R}$ so that minimizing sequences for the constrained variational problem are necessarily contained in \mathcal{A} .

d) In the spirit of c), the minimization problem (1.18) with $\mathcal{I} : X \rightarrow \overline{\mathbb{R}}$ extended-valued, can be considered to be constrained by the effective domain $\mathcal{A} = \{u \in X : \mathcal{I}(u) < \infty\}$ of \mathcal{I} .

In light of Theorem 1.8, the existence of minimizers may fail due to a lack of coercivity or lower semicontinuity. The theory of relaxation is concerned with the description of minimizing sequences $(u_k)_k \subset X$ of functionals \mathcal{I} that are not lower semicontinuous. One approach is to find a related functional whose minimizers exist and are limit points of $(u_k)_k$.

Definition 1.10 (Lower semicontinuous envelope). We define the lower semicontinuous envelope $\text{lsc } \mathcal{I} : X \rightarrow \overline{\mathbb{R}}$ of \mathcal{I} as

$$\text{lsc } \mathcal{I}(u) = \sup\{\Phi(u) : \Phi : X \rightarrow \overline{\mathbb{R}} \text{ is lower semicontinuous and } \Phi \leq \mathcal{I}\}, \quad (1.21)$$

where lower semicontinuity is understood as in Definition 1.5.

Remark 1.11. a) Along the lines of Remark 1.7 a), the literature makes a distinction between lower semicontinuous and sequential lower semicontinuous envelopes, see [96, Definition 3.8]. Since we focus solely on the latter in this thesis, we shall continue to omit the word “sequential”.

b) In spirit of 1.7 c), if X is a normed space endowed with the weak topology, then we define the weak lower semicontinuous envelope of \mathcal{I} as in Definition 1.10 where the strong convergence in X is replaced by weak convergence. We write $\text{wlsc } \mathcal{I}$ instead of $\text{lsc } \mathcal{I}$ to highlight the weak convergence in X .

c) The (weak) lower semicontinuous envelope is (weakly) lower semicontinuous by design, cf. [96, Proposition 3.5]. In fact, $\text{lsc } \mathcal{I}$ (or $\text{wlsc } \mathcal{I}$) is the largest (weakly) lower semicontinuous function which lies below \mathcal{I} .

We conclude this general setting of the direct method with the main result in the theory of relaxation. The next theorem is a direct consequence of, for example, [71, Theorem 3.5 & 3.8], where the topological version of lower semicontinuity is used, cf. Remark 1.7 a).

Theorem 1.12 (Relaxation). *It holds that*

$$\inf\{\text{lsc } \mathcal{I}(u) : u \in X\} = \inf\{\mathcal{I}(u) : u \in \mathcal{A}\},$$

and if \mathcal{I} is coercive, then $\text{lsc } \mathcal{I}$ is also coercive; in particular, $\text{lsc } \mathcal{I}$ has a minimizer in X . Moreover, any minimizing sequence $(u_k)_k \subset X$ of \mathcal{I} has a subsequence which converges to a minimizer of $\text{lsc } \mathcal{I}$.

For the remainder of this section, we focus on variational problems in Sobolev spaces, which are particularly useful in the context of nonlinear hyperelasticity. Hence, we consider for $p \in (1, \infty)$ and $m, n \in \mathbb{N}$ the energy functional

$$\mathcal{I} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow \mathbb{R}_\infty, \quad u \mapsto \int_\Omega W(x, \nabla u(x)) \, dx - \int_\Omega f(x) \cdot u(x) \, dx, \quad (1.22)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain representing the reference configuration of a material body. Here, $f \in L^q(\Omega; \mathbb{R}^3)$ is the force density with $\frac{1}{q} + \frac{1}{p} = 1$, and $W : \Omega \times \mathbb{R}^{m \times n} \rightarrow [0, \infty]$ is the non-negative stored energy density, which is assumed to be Carathéodory (or a Carathéodory function), i.e.,

$$\begin{aligned} x &\mapsto W(x, F) \text{ is Lebesgue-measurable for every } F \in \mathbb{R}^{m \times n}, \\ F &\mapsto W(x, F) \text{ is continuous for almost every } x \in \Omega. \end{aligned}$$

We say that W is homogeneous if W is constant in the space variable x , otherwise W is called inhomogeneous.

We shall now discuss coercivity and lower semicontinuity for functionals of the form (1.22). Since bounded sets in $W^{1,p}(\Omega; \mathbb{R}^n)$ are weakly sequentially precompact for $p \in (1, \infty)$, it is common to equip $W^{1,p}(\Omega; \mathbb{R}^m)$ with the weak topology. The direct method in the calculus of variations now leads us to study the weak coercivity and weak lower semicontinuity of \mathcal{I} (recall Remark (1.7) d)) Naturally, these characteristics are inherited by the properties of the corresponding densities W and f . However, it is often sufficient to analyze only W , for the additional force term

$$W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow \mathbb{R}, \quad u \mapsto \int_\Omega f(x) \cdot u(x) \, dx$$

simply acts as a weakly sequentially continuous perturbation of the internal energy

$$W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow \mathbb{R}_\infty, \quad u \mapsto \int_\Omega W(x, \nabla u(x)) \, dx. \quad (1.23)$$

Compared to the general theory for integral functionals with a single density, this additive decoupling of the deformation from its gradient makes the analysis of weak coercivity and lower semicontinuity much more straightforward.

While establishing coercivity of (1.23) is just a matter of assuming boundary values (or vanishing mean value) and suitable polynomial estimates of W from below, finding (necessary and) sufficient conditions on W such that (1.23) is weakly lower semicontinuous is more involved and requires a generalized notion of convexity.

Definition 1.13 (Quasiconvexity [70, Definition 5.1 ii]). A locally bounded and Borel-measurable function $h : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is called quasiconvex if

$$h(F) \leq \int_D h(F + \nabla \varphi(x)) \, dx, \quad (1.24)$$

for every open bounded set $D \subset \mathbb{R}^n$, every $F \in \mathbb{R}^{m \times n}$, and every $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^m)$.

Before we discuss necessity and sufficiency of quasiconvexity of $W(x, \cdot)$ for \mathcal{I} to be weakly lower semicontinuous, a remark on some characteristics and generalizations of quasiconvexity is in order.

Remark 1.14. a) Quasiconvexity was first introduced by Morrey in [160, Definition 2.2] and has since seen a few generalizations. The property in Definition 1.13 is sometimes referred to as $W^{1,\infty}$ -quasiconvexity. For $p \in [1, \infty)$, one calls a locally bounded Borel function $h : \mathbb{R}^{m \times n} \rightarrow [0, \infty)$ $W^{1,p}$ -quasiconvex, if (1.24) is satisfied for all $\varphi \in W_0^{1,p}(D; \mathbb{R}^m)$, cf. [26, Definition 2.1].

If h has p -growth, that is,

$$|h(F)| \leq C(|F|^p + 1)$$

for a constant $C > 0$ and all $F \in \mathbb{R}^{m \times n}$, then h is $W^{1,\infty}$ -quasiconvex if and only if h is $W^{1,p}$ -quasiconvex. This is a direct consequence of the density of Lipschitz functions in the space of $W^{1,p}$ -Sobolev functions, see e.g. [182, Lemma 5.2 ii)].

b) It is also worth pointing out that $W^{1,p}$ -quasiconvexity for $p \in [1, \infty]$ can also be defined for extended-valued functions, see [26, Definition 2.1]. However, this property has merely been proved to be a necessary condition for weak lower semicontinuity of the associated integral functional, which is why we omit the extended-valued case in Definition 1.13, cf. [70, Remark 5.2 (v)] and the references therein.

c) If (1.24) is satisfied for one open bounded set D , then it holds also for any other such set, cf. [70, Remark 5.2 (iv)].

d) Every convex function is quasiconvex as a consequence of Jensen's inequality, see e.g., [182, Proposition 5.1]. In the scalar case, that is, $m = 1$ or $n = 1$, quasiconvexity coincides with the classical notion of convexity as direct result of the fact that quasiconvex functions are rank-one convex, see Definition 1.17 ii) and Theorem 1.19 below.

e) Following [182, page 107], we conclude this remark with a geometric interpretation of quasiconvexity which underlines its plausibility the context of elasticity theory.

Let $D = B(0, 1) \subset \mathbb{R}^3$ represent the reference configuration of a homogeneous elastic body with a quasiconvex energy density $W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$. Consider the affine deformation $u_1(x) = x_0 + Ax$ for $x_0 \in \mathbb{R}^3$ and $A \in \mathbb{R}^{3 \times 3}$ and set $u_2(x) = u_1(x) + \varphi(x)$ for some $\varphi \in W_0^{1,\infty}(B(0, 1); \mathbb{R}^3)$. The function u_2 then describes a deformation with the same boundary values as u_1 but with an additional internal distortion. In light of the quasiconvexity of W , we then estimate that

$$\int_{B(0,1)} W(\nabla u_1(x)) \, dx = \int_{B(0,1)} W(A) \, dx \leq \int_{B(0,1)} W(A + \nabla \varphi) \, dx = \int_{B(0,1)} W(\nabla u_2(x)) \, dx,$$

which means that the purely affine deformation u_1 is energetically preferable.

Under suitable growth conditions on the energy density W , we can then formulate a characterization of weak lower semicontinuity of \mathcal{I} . The following result is a direct consequence of [70, Theorem 3.15, Lemma 3.18], [182, Theorem 5.16], and the specific additive structure of \mathcal{I} as in (1.22).

Theorem 1.15 (Characterization of weak lower semicontinuity). *Let $f \in L^q(\Omega; \mathbb{R}^m)$ and suppose that the Carathéodory function $W : \Omega \times \mathbb{R}^{m \times n} \rightarrow [0, \infty)$ satisfies*

$$W(x, F) \leq C(|F|^p + 1)$$

with a constant $C > 0$, for almost every $x \in \Omega$ and every $F \in \mathbb{R}^{m \times n}$. Then, the associated energy functional \mathcal{I} as in (1.22) is weakly lower semicontinuous if and only if $W(x, \cdot)$ is quasiconvex for almost every $x \in \Omega$.

Hence, we see that, under suitable growth conditions, weak lower semicontinuity of \mathcal{I} is equivalent to the quasiconvexity of the density W . In view of the direct method in the calculus of variations, cf. Theorem 1.8, we then summarize the following result on the existence of minimizers for variational problems in Sobolev spaces with given boundary values.

Theorem 1.16 (Existence of minimizers). *Let $f \in L^q(\Omega; \mathbb{R}^m)$ and assume that $W : \Omega \times \mathbb{R}^{m \times n} \rightarrow [0, \infty)$ is Carathéodory and satisfies the growth condition*

$$c|F|^p - C \leq W(x, F) \leq C(|F|^p + 1) \quad (1.25)$$

for some constants $c, C > 0$, almost every $x \in \Omega$ and every $F \in \mathbb{R}^{m \times n}$; suppose further that $W(x, \cdot)$ is quasiconvex for almost every $x \in \Omega$.

Then, there exists a solution to the minimization problem

$$\inf \left\{ \int_{\Omega} W(x, \nabla u(x)) \, dx - \int_{\Omega} f(x) \cdot u(x) \, dx : u \in g + W_0^{1,p}(\Omega; \mathbb{R}^m) \right\}$$

for every boundary value $g \in W^{1,p}(\Omega; \mathbb{R}^m)$.

Before we turn our attention towards a relaxation result for integral functionals, we provide the reader with some additional context for generalized notions of convexity. We point out that quasiconvexity of a function is non-local property, cf. [132, Theorem 1]. It is therefore often challenging to determine whether a given function is quasiconvex. Clearly, convex functions are quasiconvex in light of Jensen's inequality. However, classical convexity is often too restrictive for applications, see Proposition 1.2, which is why we are led to study closely related notions of convexity, see e.g. [70, Definition 5.1].

Definition 1.17 (Other notions of convexity). *Let $h : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}_{\infty}$.*

i) We say that h is polyconvex if there exists a convex function $g : \mathbb{R}^{\tau(m,n)} \rightarrow \mathbb{R}_{\infty}$ with

$$h(F) = g(M(F)) \quad (1.26)$$

for all $F \in \mathbb{R}^{m \times n}$ where $\tau(m, n) = \sum_{k=1}^{\min(m,n)} \binom{m}{k} \binom{n}{k}$ and

$$M : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{\tau(m,n)}, \quad F \mapsto (F, \text{adj}_2(F), \dots, \text{adj}_{\min(m,n)}(F)); \quad (1.27)$$

here, $\text{adj}_k(F)$ describes the matrix of all $k \times k$ minors of F for $k \in \{2, \dots, \min(m, n)\}$.

ii) The function h is called rank-one convex if

$$h(\lambda F + (1 - \lambda)G) \leq \lambda h(F) + (1 - \lambda)h(G)$$

for every $\lambda \in [0, 1]$ and all $F, G \in \mathbb{R}^{m \times n}$ with $\text{rank}(F - G) \leq 1$.

Now, we briefly comment on the importance of polyconvexity

Remark 1.18. a) In the special cases $m = n = 3$ and $m = n = 2$, (1.26) and (1.27) reduce to

$$h(F) = \begin{cases} g(F, \det F) & \text{if } m = n = 2, \\ g(F, \operatorname{adj} F, \det F) & \text{if } m = n = 3, \end{cases} \quad (1.28)$$

where $\operatorname{adj} F$ denotes the classical adjoint (or adjugate) of F . Moreover, if $m = 1$ or $n = 1$, it is obvious that polyconvexity coincides with the classical notion of convexity. It is also worth mentioning that the choice of g is not unique, see [70, Remark 5.2 vii)].

b) Polyconvexity is particularly relevant in the context of nonlinear elasticity under the assumption (1.4), which heavily penalizes the elastic energy if the material is compressed to a point, or if the deformation changes orientation. Obviously, this condition is incompatible with standard p -growth assumptions as in (1.25). In this case, it was established in [26, Corollary 3.2] that quasiconvexity of $W(x, \cdot)$ is a necessary but no longer sufficient condition for \mathcal{I} to be weakly lower semicontinuous. Moreover, the characterization of weak lower semicontinuity for unbounded functionals is still an open question, cf. [15].

Although classical convexity of the integrand is sufficient in the extended-valued case, this assumption is too strong and narrows its range of applications drastically, see Proposition 1.2. In light of this complication, John Ball introduced in [14, Definition 4.2] the notion of polyconvexity for extended-valued maps, which (under suitable coercivity assumptions) turns out to be sufficient for weak lower semicontinuity of the associated integral functional, cf. [14, Theorem 7.3].

As we can see in Example 1.1 b) - d), many stored energy densities W used in the context of hyperelasticity are polyconvex. With (1.28) in mind, it is also noteworthy that if W satisfies (1.2) - (1.5), then it can be written as a function of the three principal invariants (1.6).

c) If $h \in C^2(\mathbb{R}^{m \times n})$, then rank-one convexity is equivalent to the ellipticity condition

$$\sum_{i,j=1}^m \sum_{k,l=1}^n \frac{\partial^2 h(F)}{\partial F_{i,k} \partial F_{j,l}} \xi_i \xi_j \zeta_k \zeta_l \geq 0$$

for every $F \in \mathbb{R}^{m \times n}$, $\xi \in \mathbb{R}^m$, and $\zeta \in \mathbb{R}^n$. This highlights the fact that rank-one convexity is, in contrast to quasiconvexity, a local property in the sense that it can be described by a finite number of its derivatives.

The relaxation of constrained functionals turns out to be much more difficult

In the next theorem, for which we refer to [70, Theorem 5.3], we see the relation between the various different concepts of convexity in Definitions 1.13 and 1.17.

Theorem 1.19 (Comparison of different notions of convexity). *The following chains of implications are true:*

i) Let $h : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$, then it holds that

$$h \text{ convex} \Rightarrow h \text{ polyconvex} \Rightarrow h \text{ quasiconvex} \Rightarrow h \text{ rank-one convex}.$$

ii) If $h : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}_\infty$ is extended-valued, then

$$h \text{ convex} \Rightarrow h \text{ polyconvex} \Rightarrow h \text{ rank-one convex}.$$

iii) If $m = 1$ or $n = 1$, then all the notions of convexity in i) and ii) are equivalent.

Although extended-valued quasiconvexity functionals are generally not rank-one convex, Conti shows in [59] that this implication is indeed valid if the functional is finite on $\text{Sl}(2) : \{Fn \times n : \det F = 1\}$.

In general, all the reversed implications in Theorem 1.19 are false: a classical example of a polyconvex function that is not convex is $h(F) = \det F$ for $F \in \mathbb{R}^{2 \times 2}$; there are no elementary examples of quasiconvex functions that are not polyconvex, but the reader will find a few examples in [70, Section 5.3]; for the dimensions $m \geq 3$ and $n \geq 2$ Šverák provided in [199] the first rank-one convex function that is not quasiconvex. Whether this statement is also valid in the case $m = n = 2$ is still an open question.

In [148] it is demonstrated that rank rank-one convexity implies polyconvexity in two dimensions if the functional is objective, isotropic and isochoric. The authors continued their work in the planar case [101] and showed that the additional constraint of incompressibility is sufficient for rank-one convex functions to be polyconvex.

We conclude this section with a brief discussion on relaxation of integral functionals as in (1.22) with respect to the weak topology. From Theorem 1.15, we deduce that \mathcal{I} is weakly lower semicontinuous if and only if W is quasiconvex in the second variable, assuming that W has suitable growth. A lack of quasiconvexity of W may thus result in the absence of minimizers of \mathcal{I} and minimizing sequences may not converge.

Definition 1.20 (Generalized convex envelopes). For $h : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}_\infty$, we define quasiconvex envelope of h as

$$h^{\text{qc}} : \mathbb{R}^{m \times n} \rightarrow \overline{\mathbb{R}}, \quad F \mapsto \sup\{\phi(F) : \phi : \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \text{ is quasiconvex and } \phi \leq h\}. \quad (1.29)$$

The convex, polyconvex, and rank-one convex envelopes h^c , h^{pc} , and h^{rc} are defined analogously.

With the help of Theorem 1.19, we deduce for $h : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}_\infty$ that

$$h^c \leq h^{\text{pc}} \leq h^{\text{qc}} \leq h^{\text{rc}} \leq h.$$

Note that the generalized convex envelopes may attain the values $\pm\infty$.

Example 1.21 (Saint Venant-Kirchhoff materials). Let us consider the stored energy density $W : \mathbb{R}^{2 \times 2} \rightarrow [0, \infty)$ given by (1.8) for $n = 2$ with Young's modulus E and Poisson ratio ν as in (1.9). In this case, W satisfies (1.25) for $p = 4$.

In three dimensions $n = 3$, Raoult proved in [179] that the Saint Venant-Kirchhoff density is not polyconvex and in [180, Theorem 1.1, Corollary 1.2] the author showed that it is not even quasi-convex or rank-one convex (similar arguments can also be applied to the case $n = 2$). Its quasiconvex envelope in three dimensions was then first established in [138]. In the following, we present an explicit formula in the case $n = 2$ for which we refer to [70, Theorem 6.29]. If we define $\hat{W} : \mathbb{R}^{2 \times 2} \rightarrow [0, \infty)$ as

$$\hat{W}(F) = \begin{cases} W(F) & \text{if } F \notin D_1 \cap D_2 \\ \frac{E}{8(1+\nu^2)}(v_2(F)^2 - 1)^2 & \text{if } F \in D_2 \\ 0 & \text{if } F \in D_1 \end{cases}$$

for $F \in \mathbb{R}^{2 \times 2}$, where

$$\begin{aligned} D_1 &= \{F \in \mathbb{R}^{2 \times 2} : (1 - \nu)v_1(F)^2 + \nu v_2(F)^2 < 1 \text{ and } v_2(F) < 1\} \\ &= \{F \in \mathbb{R}^{2 \times 2} : v_1(F) \leq v_2(F) < 1\}, \\ D_2 &= \{F \in \mathbb{R}^{2 \times 2} : (1 - \nu)v_1(F)^2 + \nu v_2(F)^2 < 1 \text{ and } v_2(F) \geq 1\}, \end{aligned}$$

then all generalized convex envelopes coincide with \hat{W} , i.e.,

$$W^c = W^{pc} = W^{qc} = W^{rc} = \hat{W}.$$

We now draw the connection between the weak lower semicontinuous envelope of \mathcal{I} and the quasiconvex envelope of W in its second argument. The proof can be based on the arguments presented in [96, Proposition 3.16], [182, Theorem 7.6], and [70, Theorem 9.1, Remark 9.4 v)].

Theorem 1.22 (Relaxation for integral functionals). *Let $f \in L^q(\Omega; \mathbb{R}^m)$, $g \in W^{1,p}(\Omega; \mathbb{R}^m)$, and let $W : \Omega \times \mathbb{R}^{m \times n} \rightarrow [0, \infty)$ be a Carathéodory function satisfying (1.25). Assume further that there exists an increasing and continuous function $w : [0, \infty) \rightarrow [0, \infty)$ such that $w(0) = 0$ and*

$$|W(x, F) - W(y, F)| \leq w(|x - y|)(1 + |F|^p)$$

for almost every $x, y \in \Omega$ and every $F \in \mathbb{R}^{m \times n}$.

Then, the weak lower semicontinuous envelope $\text{wlsc } \mathcal{I}$ of \mathcal{I} , cf. Remark 1.11 b), is given by

$$\text{wlsc } \mathcal{I}(u) = \int_{\Omega} W^{qc}(x, \nabla u(x)) \, dx - \int_{\Omega} f(x) \cdot u(x) \, dx, \quad u \in g + W_0^{1,p}(\Omega; \mathbb{R}^m),$$

where $W^{qc}(x, \cdot)$ is the finite-valued quasiconvex envelope of $W(x, \cdot)$ for $x \in \Omega$, see (1.29).

The same result holds true if the space $g + W_0^{1,p}(\Omega; \mathbb{R}^m)$ is replaced by $W^{1,p}(\Omega; \mathbb{R}^m) \cap L_0^p(\Omega; \mathbb{R}^m)$.

Finding the relaxation of extended-valued functionals with non-standard constraints is generally more challenging. In the last few years, there has been significant progress in finding the lower semicontinuous envelope of energy functionals with determinant constraints such as physical growth conditions (1.4) or incompressibility (1.12). In particular, Conti & Dolzmann [63] were the first to obtain a relaxation result under the assumption that the quasiconvex envelope of the integrand is polyconvex; Cicalese & Fusco [57] later derived a generalized version for inhomogeneous densities also depending on lower-order terms. In this spirit, we shall also mention the article [102] where the authors determine an explicit formula for the quasiconvex envelope of an incompressible isotropic energy density in two dimensions, and the paper [151] which deals with necessary and sufficient conditions for polyconvexity of constrained isotropic densities.

We conclude this section with a brief discussion on the significance of the quasiconvex envelope for the formation of microstructures in materials science following the ideas presented in [182, Sections 1.8 and 8.3].

Remark 1.23 (Formation of microstructures). It has been experimentally observed that the macroscopically attainable deformations, which correspond to minimizers of the strain energy, also minimize the corresponding integrand W (approximately) pointwise. We shall describe the effect of relaxation and the occurrence of microstructures with the help of the idealized energy density

$$\mathcal{I}_K : W^{1,2}(B(0, 1); \mathbb{R}^3) \cap L_0^2(B(0, 1); \mathbb{R}^3) \rightarrow [0, \infty), \quad u \mapsto \int_{B(0, 1)} \text{dist}^2(\nabla u, K) \, dx, \quad (1.30)$$

where $B(0, 1) \subset \mathbb{R}^3$ is the unit ball in three dimensions, and K is a given compact set; we also write $W_K(F) = \text{dist}^2(F, K)$ for $F \in \mathbb{R}^{3 \times 3}$. For example, in the context of shape-memory alloys, we have $K = \text{SO}(3)$ in the high-temperature austenite phase or $K = \bigcup_{i=1}^N \text{SO}(3)U_i$ with

$U_1, \dots, U_N \in \text{GL}_+(3)$ for $N \in \mathbb{N}$ in the low-temperature martensite phase. From the structure of (1.30), we then expect the observed macroscopic deformations to have gradients in the set K .

Due to Theorem 1.22, the relaxation of \mathcal{I}_K is an integral functional whose integrand coincides with W_K^{qc} , the quasiconvex envelope of W_K . This relaxed energy density then vanishes exactly on

$$K^{\text{qc}} := \{F \in \mathbb{R}^{3 \times 3} : h(F) \leq \sup_{G \in K} h(G) \text{ for all quasiconvex } h : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}\},$$

the so-called quasiconvex hull of K , see [204, Definition 2.3 - Proposition 2.5]. We shall point out that, determining an explicit expression of K^{qc} is generally a complicated task.

From the theory of gradient Young measures, one obtains for every $F \in K^{\text{qc}}$ a sequence $(\varphi_i)_i \subset W_0^{1,\infty}(B(0,1); \mathbb{R}^3)$ with uniformly bounded gradients such that

$$\int_{B(0,1)} \text{dist}^2(F + \nabla \varphi_i(x), K) \, dx \rightarrow 0,$$

see for example [162, Theorem 4.10] or [182, Section 8.3]. Hence, the collection of all observed macroscopic strains can, in fact, be larger than the expected set K . In particular, the material is able to attain the linear boundary condition $u = Fx$ on $\partial\Omega$ with $F \in K^{\text{qc}}$ for only a small energy cost, although the internal structure may be highly intricate. If W_K is quasiconvex, then we infer from the discussion in Remark 1.14 e) that there is no formation of microstructures since the purely affine deformation is energetically favorable.

1.3 Convergence of variational problems: Γ -convergence

While the previous section covers the standard theory for a single minimization problem, we focus now on the analysis of sequences of variational problems. In particular, we are interested in the asymptotic behavior of (almost) minimizers and the respective infima.

To this end, we present in this section the notion of Γ -convergence for sequences of functionals, which was first introduced by Franzoni & De Giorgi in [78, 79]. Proving a Γ -convergence result consists of identifying matching lower and upper bounds on the limit functional, alongside an asymptotic coercivity (or compactness) result. For this section, we refer primarily to the standard work by Braides [42], and the more advanced and in-depth book [71] by Dal Maso. The former serves as a gentle introduction to Γ -convergence for functionals defined on metric spaces while the latter also contains, among many other topics, an exhaustive analysis in the general topological setting, which is discussed briefly at the end of this section.

We will see that every Γ -limit is lower semicontinuous, and coercive, if the sequence of functionals is coercive in an asymptotical or uniform sense with respect to the sequence index. The direct method in the calculus of variations (see Theorem 1.8) then ensures that the every such Γ -limit admits at least one minimizer; moreover every sequence of (almost) minimizers converges to a minimizer of the corresponding Γ -limit, see Proposition 1.31 for more details. Therefore, Γ -convergence turns out to be highly valuable tool for the description of the asymptotic behavior of parameter-dependent variational problems rather than simply being a concept of convergence for sequences of functionals. Moreover, we will see that Γ -convergence is also invariant under continuous perturbations, which renders it particularly useful for applications in materials science where the system energy can be described by the sum of an internal energy and a (weakly) continuous force term.

In this section, we also introduce K -convergence for a sequence of sets, which is related to Γ -convergence via epigraphs and indicator functions in the sense of convex analysis. This concept

of convergence can also be used to determine (bounds on) the effective domain of Γ -limits of constrained functionals in the sense of (1.20)

If not specified otherwise, we will assume throughout this section that X is a metric space equipped with a metric d ; moreover, for $k \in \mathbb{N}$ let $\mathcal{I}_k, \mathcal{I} : X \rightarrow \mathbb{R}_\infty$ be functionals on X , and let $\mathcal{A}_k, \mathcal{A} \subset X$ be subsets of X . We begin this section with the definition of Γ -convergence [42, Definition 1.5].

Definition 1.24 (Γ -convergence). *The sequence $(\mathcal{I}_k)_k$ Γ -converges to \mathcal{I} in X (or with respect to the metric d) if the following two conditions are fulfilled:*

i) *Every sequence $(u_k)_k \subset X$ with $u_k \rightarrow u$ for $u \in X$ satisfies*

$$\liminf_{k \rightarrow \infty} \mathcal{I}_k(u_k) \geq \mathcal{I}(u);$$

ii) *For every $u \in X$ there exists a sequence $(u_k)_k \subset X$ such that $u_k \rightarrow u$ and*

$$\limsup_{k \rightarrow \infty} \mathcal{I}_k(u_k) \leq \mathcal{I}(u).$$

If i) and ii) are satisfied, then we also write $\mathcal{I} = \Gamma\text{-}\lim_{k \rightarrow \infty} \mathcal{I}_k$.

Note that for every u and $(u_k)_k$ as in Definition 1.24 ii), it also holds that

$$\limsup_{k \rightarrow 0} \mathcal{I}_k(u_k) \leq \mathcal{I}(u) \leq \liminf_{k \rightarrow 0} \mathcal{I}_k(u_k),$$

and thus

$$\lim_{k \rightarrow 0} \mathcal{I}_k(u_k) = \mathcal{I}(u).$$

Therefore, such a sequence $(u_k)_k$ is also referred to as a so-called recovery sequence.

Naturally, not every sequence of functionals has a Γ -limit. However, we can define related quantities that always exist and yield lower and upper bounds on a possible Γ -limit, see e.g. [42, Definition 1.24]

Remark 1.25 (Lower and upper Γ -limits). For $u \in X$, we define the lower and upper Γ -limit of $(\mathcal{I}_k)_k$ as

$$\begin{aligned} \Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{I}_k(u) &= \inf \left\{ \liminf_{k \rightarrow \infty} \mathcal{I}_k(u_k) : u_k \rightarrow u \text{ in } X \right\}, \\ \Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{I}_k(u) &= \inf \left\{ \limsup_{k \rightarrow \infty} \mathcal{I}_k(u_k) : u_k \rightarrow u \text{ in } X \right\}. \end{aligned} \tag{1.31}$$

As pointed out in [42, Remark 1.26], the infima in the previous definition are, in fact, attained. It is then evident that $(\mathcal{I}_k)_k$ Γ -converges to \mathcal{I} if and only if

$$\mathcal{I} = \Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{I}_k = \Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{I}_k.$$

Next, we shall introduce K-convergence [71, Remark 8.2], which is a notion of convergence for sequences of sets, in the spirit of Remark 1.25. We will see in Propositions 1.27 and 1.28, the relation between Γ -convergence and K-convergence.

Definition 1.26 (K-convergence). *The the lower and upper K-limits $K\text{-}\liminf_{k \rightarrow \infty} \mathcal{A}_k \subset X$ and $K\text{-}\limsup_{k \rightarrow \infty} \mathcal{A}_k \subset X$ are defined as follows:*

- i) The set $\text{K-lim inf}_{k \rightarrow \infty} \mathcal{A}_k$ consists of all points $u \in X$ such that there exist $N \in \mathbb{N}$ and a sequence $(u_k)_k \subset X$ with $u_k \rightarrow u$ in X and $u_k \in \mathcal{A}_k$ for every $k > N$;
- ii) The set $\text{K-lim sup}_{k \rightarrow \infty} \mathcal{A}_k$ consists of all $u \in X$ such that there exists a subsequence $(\mathcal{A}_{k_j})_k$ of $(\mathcal{A}_k)_k$ and a sequence $(u_j)_j$ with $u_j \rightarrow u$ in X and $u_j \in \mathcal{A}_{k_j}$ for every $j \in \mathbb{N}$.

We say that $(\mathcal{A}_k)_k$ *K-converges* in X , or *converges in X in the sense of Kuratowski*, to the set \mathcal{A} if

$$\text{K-lim sup}_{k \rightarrow \infty} \mathcal{A}_k \subset \mathcal{A} \subset \text{K-lim inf}_{k \rightarrow \infty} \mathcal{A}_k;$$

in this case, we write $\mathcal{A} = \text{K-lim}_{k \rightarrow \infty} \mathcal{A}_k$.

K-convergence plays an important role in determining an upper bound on the effective domain of Γ -limits. Precisely, if $\mathcal{A}_k = \{u \in X : \mathcal{I}_k(u) < \infty\}$ describes the set of admissible states of \mathcal{I}_k and $(\mathcal{I}_k)_k$ is Γ -convergent, then

$$\{u \in X : (\Gamma\text{-lim}_{k \rightarrow \infty} \mathcal{I}_k)(u) < \infty\} \subset \text{K-lim inf}_{k \rightarrow \infty} \mathcal{A}_k.$$

Under suitable assumptions on the structure and properties (like growth and coercivity) of the functionals \mathcal{I}_k , we even find that

$$\{u \in X : (\Gamma\text{-lim}_{k \rightarrow \infty} \mathcal{I}_k)(u) < \infty\} = \text{K-lim}_{k \rightarrow \infty} \mathcal{A}_k,$$

see, for example, the analysis of fiber-reinforced materials in the framework of homogenization in Chapter 4.

In the sequel, we discuss the relation between K-convergence and Γ -convergence. First, we will see that convergence in the sense of Kuratowski is equivalent to Γ -convergence of the corresponding sequence of indicator functions

$$\chi_E : X \rightarrow \mathbb{R}_\infty, \quad u \mapsto \begin{cases} 0 & \text{if } u \in E, \\ \infty & \text{otherwise,} \end{cases} \quad \text{for } E \subset X, \quad (1.32)$$

see [71, Proposition 4.15].

Proposition 1.27 (Γ -convergence of indicator functions). *The sequence $(\mathcal{A}_k)_k$ converges to \mathcal{A} in the sense of Kuratowski if and only if $(\chi_{\mathcal{A}_k})_k$ Γ -converges to $\chi_{\mathcal{A}}$ in X , where $\chi_{(\cdot)}$ is defined as in (1.32). Moreover, if*

$$\mathcal{A}' = \text{K-lim inf}_{k \rightarrow \infty} \mathcal{A}_k \quad \text{and} \quad \mathcal{A}'' = \text{K-lim sup}_{k \rightarrow \infty} \mathcal{A}_k,$$

then

$$\chi_{\mathcal{A}'} = \Gamma\text{-lim sup}_{k \rightarrow \infty} \chi_{\mathcal{A}_k} \quad \text{and} \quad \chi_{\mathcal{A}''} = \Gamma\text{-lim inf}_{k \rightarrow \infty} \chi_{\mathcal{A}_k}.$$

Second, we define the epigraph of a function $\mathcal{I} : X \rightarrow \overline{\mathbb{R}}$ as

$$\text{epi}(\mathcal{I}) = \{(u, t) \in X \times \mathbb{R} : \mathcal{I}(u) \leq t\};$$

it is worth pointing out that $\text{epi}(\chi_{\mathcal{A}}) = \mathcal{A} \times [0, \infty)$ for any $\mathcal{A} \subset X$. The next proposition, which can be found in [71, Theorem 4.16], shows that Γ -convergence is equivalent to K-convergence of the corresponding sequence of epigraphs in $X \times \mathbb{R}$, which is why Γ -convergence is sometimes referred to as epi-convergence.

Proposition 1.28 (K-convergence of epigraphs). *The sequence $(\mathcal{I}_k)_k$ Γ -converges to \mathcal{I} in X if and only if $(\text{epi}(\mathcal{I}_k))_k$ K-converges to $\text{epi}(\mathcal{I})$ in $X \times \mathbb{R}$. Furthermore, if*

$$\mathcal{I}' = \Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{I}_k \quad \text{and} \quad \mathcal{I}'' = \Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{I}_k,$$

then

$$\text{epi}(\mathcal{I}') = \text{K-lim sup}_{k \rightarrow \infty} \text{epi}(\mathcal{I}_k) \quad \text{and} \quad \text{epi}(\mathcal{I}'') = \text{K-lim inf}_{k \rightarrow \infty} \text{epi}(\mathcal{I}_k).$$

One of the most important properties of Γ -convergence is the stability under continuous perturbations. This is particularly relevant for applications in materials science, where total energy functionals are described by the sum of an internal energy and a continuous force term, see (1.22). For the following statement, we refer the reader to [42, Remark 1.7].

Proposition 1.29 (Stability under continuous perturbation). *Let $\mathcal{J} : X \rightarrow \overline{\mathbb{R}}$ be continuous and assume that $(\mathcal{I}_k)_k$ Γ -converges to \mathcal{I} in X as $k \rightarrow \infty$. Then, the sequence $(\mathcal{I}_k + \mathcal{J})_k$ Γ -converges to $\mathcal{I} + \mathcal{J}$ in X as $k \rightarrow \infty$.*

In view of the direct method in the calculus of variations, it is also crucial to highlight that every Γ -limit is lower semicontinuous, see e.g. [42, Proposition 1.28].

Proposition 1.30 (Lower semicontinuity). *The lower and upper Γ -limits $\Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{I}_k$ and $\Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{I}_k$ are lower semicontinuous. In particular, if $(\mathcal{I}_k)_k$ Γ -converges, then its Γ -limit is lower semicontinuous.*

Moreover, if $\mathcal{I}_k = \mathcal{I}_0 : X \rightarrow \overline{\mathbb{R}}$ for every $k \in \mathbb{N}$, then $\Gamma\text{-}\lim_{k \rightarrow \infty} \mathcal{I}_k = \text{lsc } \mathcal{I}_0$, cf. (1.21).

As a consequence of the last statement in Proposition 1.30, there is no topology on the space of functionals from X to $\overline{\mathbb{R}}$ describing Γ -convergence. However, on the space of lower semicontinuous functionals from X to $\overline{\mathbb{R}}$, such a topology does exist; for more details we refer to [71, Chapter 10]. It is worth mentioning that the sequence $(\mathcal{I}_k)_k$ Γ -converges to \mathcal{I} in X if and only if $(\text{lsc } \mathcal{I}_k)_k$ Γ -converges to \mathcal{I} in X [42, Proposition 1.32]. In particular, this means that Γ -convergence always includes an additional relaxation process.

Also note that lower and upper K-limits are closed in X , due to Proposition 1.28; in particular, the constant sequence $\mathcal{A}_k = \mathcal{A}_0 \subset X$ K-converges to the closure $\overline{\mathcal{A}_0}$.

The next proposition establishes that Γ -convergence is essentially a notion of convergence of variational problems. Given suitable coercivity of the sequence $(\mathcal{I}_k)_k$, it is then possible to show that every Γ -limit has a minimizer and limit points of minimizing sequences are minimizers of the Γ -limit. Precisely, we obtain from [42, Theorem 1.21] the following existence result.

Proposition 1.31 (Convergence of minimizers). *Assume that $(\mathcal{I}_k)_k$ Γ -converges to \mathcal{I} and that there exists a compact set $K \subset X$ such that $\inf_{u \in X} \mathcal{I}_k(u) = \inf_{u \in K} \mathcal{I}_k(u)$ for all $k \in \mathbb{N}$ sufficiently large. Then, \mathcal{I} has a global minimizer and*

$$\min_{u \in X} \mathcal{I}(u) = \lim_{k \rightarrow \infty} \inf_{u \in X} \mathcal{I}_k(u).$$

Furthermore, if $(u_k)_k$ is a sequence of almost-minimizers, i.e.,

$$\lim_{k \rightarrow \infty} \mathcal{I}(u_k) = \lim_{k \rightarrow \infty} \inf_{u \in X} \mathcal{I}_k(u),$$

then $(u_k)_k$ has a subsequence that converges to a minimizer of \mathcal{I} .

Remark 1.32 (Asymptotic coercivity). Usually, Γ -convergence results are coupled with an additional compactness result for sequences of bounded energy of the following form: For any $(u_k)_k \subset X$ with $\sup_k \mathcal{I}_k(u_k) < \infty$ there exists a converging subsequence; this condition can be viewed as an asymptotic generalization of the notion of coercivity as in Definition 1.6.

In Proposition 1.31, the assumption on the existence of a compact set $K \subset X$ such that $\inf_{u \in X} \mathcal{I}_k(u) = \inf_{u \in K} \mathcal{I}_k(u)$ can be replaced by the weaker condition of asymptotic coercivity as above. In this case, Proposition 1.31 can be viewed as a generalization of Theorem 1.12 for parameter-dependent problems.

Now, we generalize the definition of Γ -convergence and K-convergence to topological spaces X . In this generalized setting, it is possible to introduce two different notions of Γ -convergence: a topological and a sequential one.

Remark 1.33 (Generalization to topological spaces). In this remark, we assume that X is a topological space.

a) (Sequential definition) We say that $(\mathcal{I}_k)_k$ Γ -converges sequentially with respect to the topology in X (or Γ -converges with respect to the convergence of sequences in X) if the two conditions i) and ii) in Definition 1.24 are satisfied, cf. [100, Definition 2.1]. Similarly, we can define a sequential notion of Kuratowski-convergence for set-valued sequences.

b) (Topological definition) The sequence $(\mathcal{I}_k)_k$ Γ -converges to \mathcal{I} in X if

$$\mathcal{I}(u) = \Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{I}_k(u) = \Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{I}_k(u)$$

for every $u \in X$, where

$$\begin{aligned} \Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{I}_k(u) &= \sup_{U \in \mathcal{N}(u)} \liminf_{k \rightarrow \infty} \inf_{v \in U} \mathcal{I}_k(v), \\ \Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{I}_k(u) &= \sup_{U \in \mathcal{N}(u)} \limsup_{k \rightarrow \infty} \inf_{v \in U} \mathcal{I}_k(v), \end{aligned} \tag{1.33}$$

see [71, Definition 4.1]. Here, $\mathcal{N}(u)$ denotes the set of all open neighborhoods of u . It is worth mentioning that $\mathcal{N}(u)$ can be replaced by a base for the neighborhood system of $u \in X$, which highlights the local character of Γ -convergence, cf. [71, Remark 4.3]. Precisely, if another sequence $(\mathcal{J}_k)_k$ of functionals $\mathcal{J}_k : X \rightarrow \overline{\mathbb{R}}$ coincides with $(\mathcal{I}_k)_k$ on an open subset $U \subset X$, then

$$\Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{I}_k = \Gamma\text{-}\liminf_{k \rightarrow \infty} \mathcal{J}_k \quad \text{and} \quad \Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{I}_k = \Gamma\text{-}\limsup_{k \rightarrow \infty} \mathcal{J}_k$$

on all of U .

The topological definition of Kuratowski convergence is similar to Γ -convergence, see [71, Definition 4.10, Remark 4.11]. Precisely, for the sequence of sets $(\mathcal{A}_\varepsilon)_\varepsilon$ we define its upper K-limit $\text{K-lim sup}_{k \rightarrow \infty} \mathcal{A}_k$ as the set of all points $u \in X$ such that for all $U \in \mathcal{N}(u)$ there exists $N \in \mathbb{N}$ such that $U \cap \mathcal{A}_k \neq \emptyset$ for all $k > N$. Analogously, the set $\text{K-lim inf}_{k \rightarrow \infty} \mathcal{A}_k$ consists of all $u \in X$ such that for all $U \in \mathcal{N}(u)$ and all $N \in \mathbb{N}$ there exists $k > N$ such that $U \cap \mathcal{A}_k \neq \emptyset$.

Assuming that X satisfies the first countability axiom, the identities in (1.33) can be rewritten as (1.31), see [71, Proposition 8.1] for a detailed proof. In particular, if X is first countable, then the topological and sequential notions of Γ -convergence coincide.

We close this section with a discussion on the topological and sequential notions of Γ - and K-convergence with respect to the weak topology in Banach spaces. The arguments we refer to are essentially based on the local metrizability of the weak topology, see [71, Proposition 8.7].

Remark 1.34 (The weak topology). In the following, let X be a reflexive, separable Banach space endowed with its weak topology and suppose that X is compactly embedded into another Banach space Y . By d we denote the metric on X induced by the norm of Y , i.e., $d(u, v) = \|u - v\|_Y$ for $u, v \in X$.

A standard example, that is useful in many applications, is $X = W^{1,p}(\Omega; \mathbb{R}^m)$ and $Y = L^p(\Omega; \mathbb{R}^m)$ (or their intersection with $L_0^p(\Omega; \mathbb{R}^m)$ for models where the functionals \mathcal{I}_k merely depend on the gradient instead of the full potential) for a Lipschitz domain $\Omega \subset \mathbb{R}^n$ and $p \in (1, \infty)$.

According to [71, Proposition 8.16], if the sequence of functionals $(\mathcal{I}_k)_k$ is asymptotically coercive in the sense of Remark 1.32, then the topological and sequential definitions of Γ -convergence with respect to the weak topology in X coincide. It is particularly remarkable that the associated Γ -limit (if it exists) then has a minimizer.

Let \mathcal{I}' , \mathcal{I}'' be the lower and upper Γ -limit of $(\mathcal{I}_k)_k$ with respect to the weak topology on X as in (1.33) and let \mathcal{I}'_d , \mathcal{I}''_d be the corresponding limits with respect to the metric d on X . In light of [71, Remark 8.9, Proposition 8.10], it holds that $\mathcal{I}' = \mathcal{I}'_d$ and $\mathcal{I}'' = \mathcal{I}''_d$. The upper and lower limits \mathcal{I}' and \mathcal{I}'' can be rewritten as (1.31) where the norm convergence is replaced by the weak convergence in X .

Moreover, \mathcal{I}' and \mathcal{I}'' are weakly lower semicontinuous with respect to the weak topology in X , in the sense of Remark 1.7 d). In particular, if $(\mathcal{I}_k)_k$ is the constant sequence, then $\mathcal{I}' = \mathcal{I}'' = \text{wlscl}$. This implies, in particular, that lower semicontinuity with respect to d is equivalent to weak lower semicontinuity in X , see also [137, Lemma 5].

We close this remark, with a brief comment on Kuratowski-convergence. Suppose that the sequence of sets $(\mathcal{A}_k)_k$ is equi-bounded, i.e.,

$$\sup_{k \in \mathbb{N}} \sup_{u \in \mathcal{A}_k} \|u\|_X < \infty.$$

Then, the lower and upper K-limits of $(\mathcal{A}_k)_k$ with respect to the weak topology are characterized by i) and ii) of Definition 1.26, where the norm convergence is replaced by weak convergence in X ; in other words, the topological and sequential notions of K-convergence are the same.

1.4 Applications of Γ -convergence: Dimension reduction

Thin structures - material objects, which are thin in one or more dimensions - have a distinctly different deformation behavior than bulk materials. To describe this phenomenon, classical approaches based on asymptotic expansions, cf. [53, 195], have been very successful in the small-strain setting of linear elasticity. However, to deal with large deformations and the resulting geometric nonlinearity, a fitting mathematical framework is needed. Since system energy minimization is crucial in determining the material response to external forces, the theory of Γ -convergence, which is essentially a notion of convergence for variational problems (see Proposition 1.31), is the ideal tool to obtain a qualitative understanding of the deformation behavior of thin structures. For a broader perspective on the theory of dimension reduction, see also Chapters 2 and 3.

To introduce the concept of dimension reduction we present first a simple $3d$ - $2d$ reduction result based on [137] and [42, Chapter 14]. Since Chapters 2 and 3 deal with a complete hierarchy of incompressible string and rod models, we also discuss related results exhibiting higher rigidity as well as the corresponding $3d$ - $1d$ cases in Remarks 1.37 and 1.38.

Let $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ describe a membrane with thickness $\varepsilon > 0$ and cross section $\omega \subset \mathbb{R}^2$, where ω is a bounded Lipschitz domain, see also Figure 1.4.

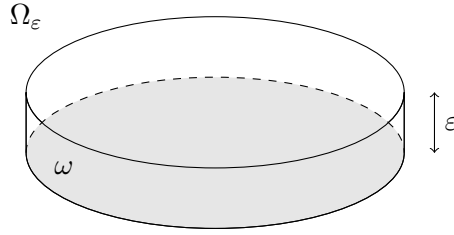


Figure 1.4: Illustration of the thin structure $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ with circular cross section $\omega \subset \mathbb{R}^2$.

The corresponding strain energy shall be given by

$$\mathcal{J}_\varepsilon : W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3) \rightarrow [0, \infty), \quad v \mapsto \int_{\Omega_\varepsilon} W(\nabla v(y)) \, dy$$

with a homogeneous energy density $W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ which satisfies standard p -growth and coercivity (1.25) for $p > 1$.

Now, the immediate question is whether the sequence $(\mathcal{J}_\varepsilon)_\varepsilon$ of internal energies of the thin three dimensional material converges in a meaningful way to a limit energy representing the corresponding lower dimensional material object. To avoid trivial limits, the standard strategy is to normalize the energy \mathcal{J}_ε by unit volume, which leads us to study the sequence $(\frac{1}{\varepsilon^2} \mathcal{J}_\varepsilon)_\varepsilon$ instead. The next step is then to introduce a change of variables $v(y) = u(x)$ with $y = (x_1, x_2, \varepsilon x_3)$ for $x \in \Omega := \Omega_1$, which allows us to redefine the normalized energy on a fixed space. Precisely, we now have $\frac{1}{\varepsilon^2} \mathcal{J}_\varepsilon(v) = \mathcal{I}_\varepsilon(u)$ with

$$\mathcal{I}_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty), \quad u \mapsto \int_{\Omega} W(\nabla^\varepsilon u(x)) \, dx \quad (1.34)$$

where $\nabla^\varepsilon u$ describes the so-called rescaled deformation gradient

$$\nabla^\varepsilon u = (\partial_1 u | \partial_2 u | \frac{1}{\varepsilon} u_\varepsilon).$$

We are now in a position to prove a Γ -convergence result for the sequence of internal energies $(\mathcal{I}_\varepsilon)_\varepsilon$, which is based on [137] and [42, Chapter 14].

Theorem 1.35 (Membrane model). *The sequence $(\mathcal{I}_\varepsilon)_\varepsilon$ as in (1.34) Γ -converges sequentially with respect to the weak topology in $W^{1,p}(\Omega; \mathbb{R}^3)$ to the functional*

$$\mathcal{I} : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty], \quad u \mapsto \begin{cases} \int_{\omega} \overline{W}^{\text{qc}}(\partial_1 u | \partial_2 u) \, dx_1 \, dx_2 & \text{if } u \in W^{1,p}(\omega; \mathbb{R}^3) \\ \infty & \text{otherwise,} \end{cases} \quad (1.35)$$

where $\overline{W} : \mathbb{R}^{3 \times 2} \rightarrow [0, \infty)$ is the two-dimensional density given by

$$\overline{W}(G) := \min_{\xi \in \mathbb{R}^3} W(G | \xi)$$

and $(\cdot)^{\text{qc}}$ is the quasiconvex envelope as in Definition 1.20.

Furthermore, every sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ with $\sup_\varepsilon \mathcal{I}_\varepsilon(u_\varepsilon) < \infty$ and vanishing mean value has a weakly converging subsequence with limit in $W^{1,p}(\omega; \mathbb{R}^3)$.

Remark 1.36. a) First, we observe that the limit energy $\mathcal{I}(u)$ is finite if and only if the deformation u is independent of the x_3 variable. Properties of this kind are the main feature

of every dimension reduction result, for they illustrate the idea that the elastic energy of a thin structure can essentially be described by merely considering deformations of the associated lower-dimensional material object.

b) In the spirit of a), we observe that the energy density W of the fully three-dimensional object changes in the limit accordingly. In fact, we see that W undergoes the transformation into the reduced density \overline{W} via minimization in the last column of the rescaled deformation gradients.

c) If we replace the space $W^{1,p}(\Omega; \mathbb{R}^3)$ by $W^{1,p}(\Omega; \mathbb{R}^3) \cap L_0^p(\Omega; \mathbb{R}^3)$, then the sequence $(\mathcal{I}_\varepsilon)_\varepsilon$ also Γ -converges with respect to the weak topology in said space, see also Remark 1.34. This is due to Poincaré's inequality for functions with vanishing mean value.

d) As we outlined in Proposition 1.30 and Remark 1.34, the Γ -limit \mathcal{I} as in (1.35) is weakly lower semicontinuous. Recalling Theorem 1.22 on the relaxation of integral functionals, the appearance of the quasiconvex envelope is therefore a natural consequence.

e) If W is an energy density describing a Saint Venant-Kirchhoff material as in (1.8) for $n = 3$, then W satisfies (1.25) for $p = 4$, see also (1.7). In this case, the limit density $\overline{W}^{\text{qc}} : \mathbb{R}^{3 \times 2} \rightarrow [0, \infty)$ was first calculated by Le Dret & Raoult in [137, Proposition 16] and it is given by

$$\begin{aligned} \overline{W}^{\text{qc}}(G) = & \frac{E}{8} [v_2(G) - 1]_+^2 + \frac{E}{8(1 - \nu^2)} [v_1(G)^2 + \nu v_2(G)^2 - (1 + \nu)]_+^2 + \\ & \frac{E}{8(1 - \nu^2)(1 - 2\nu)} [\nu(v_1(G)^2 + v_2(G)^2) - (1 + \nu)]_+^2, \end{aligned}$$

where $[x]_+^2 = x^2$ for $x \geq 0$ and $[x]_+^2 = 0$ of $x < 0$, and $v_1(G) \leq v_2(G)$ are the right singular values of $G \in \mathbb{R}^{3 \times 2}$, i.e., the ordered eigenvalues of $\sqrt{G^T G}$. In the remark following [137, Proposition 16], the reader will find a more detailed discussion on the properties of \overline{W}^{qc} . In particular, the authors point out on which subsets of $\mathbb{R}^{3 \times 2}$ the quasiconvex envelope \overline{W}^{qc} vanishes or coincides with \overline{W} (similar to Example 1.21).

Proof. We sketch a simplified proof focusing on the key arguments.

Step 1: Compactness. Let $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ be a sequence of bounded energy and assume that $(u_\varepsilon)_\varepsilon$ has vanishing mean value (which we can always assume since translations do not yield any change of energy), i.e.,

$$\mathcal{I}_\varepsilon(u_\varepsilon) < K \quad \text{and} \quad \int_\Omega u_\varepsilon(x) \, dx = 0 \quad (1.36)$$

for a constant $K > 0$ and every ε . Combining (1.36) with the p -growth (1.25) and Poincaré's inequality, we deduce that $(u_\varepsilon)_\varepsilon$ is bounded in $W^{1,p}(\Omega; \mathbb{R}^3)$. Due to standard Sobolev embeddings, we can thus find $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ and a (non-relabeled) subsequence such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$. Furthermore, the limit u satisfies $\partial_3 u = 0$, since $\partial_3 u_\varepsilon \rightarrow 0$ strongly in $L^p(\Omega; \mathbb{R}^3)$. This is why we may identify u with an element in $W^{1,p}(\omega; \mathbb{R}^3)$.

Step 2: Lower bound. Now, let $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$ and assume without loss of generality that $\liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u_\varepsilon) < \infty$, otherwise there is nothing to prove. From Step 1, we obtain that the limit u only depends on the cross-section variables. Since W has standard p -growth and coercivity (1.25), it is straightforward to prove that the quasiconvex limit density \overline{W}^{qc} satisfies

$$c|G|^p - C \leq \overline{W}^{\text{qc}}(G) \leq C(|G|^p + 1)$$

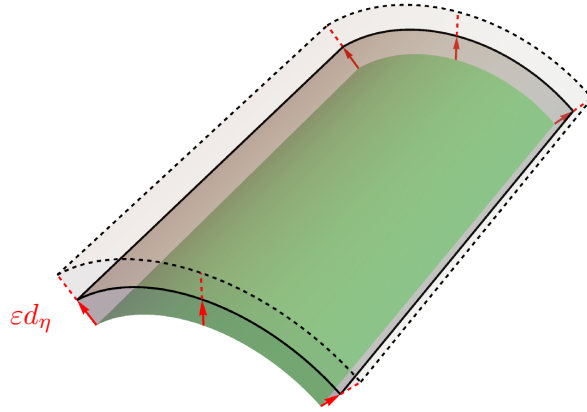


Figure 1.5: An example of deformations as in (1.37). The green surface is the image $u(\omega)$ of the cross section ω under the lower-dimensional transformation u , and the red arrows represent the direction along which $u(\omega)$ is fattened.

for all $G \in \mathbb{R}^{3 \times 2}$. In view of Theorem 1.15, we thus obtain that

$$W^{1,p}(\omega; \mathbb{R}^3) \rightarrow \mathbb{R}, \quad u \mapsto \int_{\Omega} \overline{W}^{\text{qc}}(\nabla u(x)) \, dx$$

is weakly lower-semicontinuous. The lower bound is then established as follows:

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon}(u_{\varepsilon}) &= \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} W(\partial_1 u_{\varepsilon} | \partial_2 u_{\varepsilon} | \frac{1}{\varepsilon} \partial_3 u_{\varepsilon}) \, dx \\ &\geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \overline{W}(\partial_1 u_{\varepsilon} | \partial_2 u_{\varepsilon}) \, dx \geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \overline{W}^{\text{qc}}(\partial_1 u_{\varepsilon} | \partial_2 u_{\varepsilon}) \geq \mathcal{I}(u). \end{aligned}$$

Step 3: Upper bound. For this part, we assume for the sake of simplicity that $p = 2$ and W is convex, otherwise the recovery sequences constructed in the sequel need more refinement since we need to invoke an additional relaxation step. We find for any $u \in W^{1,2}(\omega; \mathbb{R}^3)$ a function $d \in L^2(\omega; \mathbb{R}^3)$ such that

$$\overline{W}(\partial_1 u | \partial_2 u) = W(\partial_1 u | \partial_2 u | d) \text{ a.e. in } \omega,$$

for which we choose a sequence $(d_{\eta})_{\eta} \subset C^{\infty}(\overline{\omega}; \mathbb{R}^3)$ such that $d_{\eta} \rightarrow d$ strongly in $L^2(\omega; \mathbb{R}^3)$. We then define for each $\eta > 0$ a sequence of deformations on all of Ω by thickening the one given by u along the direction d , i.e.,

$$u_{\eta,\varepsilon}(x) = u(x_1, x_2) + \varepsilon x_3 d_{\eta}(x_1, x_2), \quad x = (x_1, x_2, x_3) \in \Omega = \omega \times (0, L), \quad (1.37)$$

see also Figure 1.5.

Observe that $u_{\eta,\varepsilon} \rightarrow u$ strongly in $W^{1,2}(\Omega; \mathbb{R}^3)$ as $\varepsilon \rightarrow 0$ for every η and that $u_{\eta,\varepsilon}$ is bounded in $W^{1,2}(\Omega; \mathbb{R}^3)$ uniformly in η . We then calculate that

$$\begin{aligned} \mathcal{I}_{\varepsilon}(u_{\eta,\varepsilon}) &= \int_{\Omega} \overline{W}(\partial_1 u | \partial_2 u) + W(\nabla^{\varepsilon} u_{\eta,\varepsilon}) - W(\partial_1 u | \partial_2 u | d) \, dx \\ &= \mathcal{I}(u) + \int_{\Omega} W(\nabla^{\varepsilon} u_{\eta,\varepsilon}) - W(\partial_1 u | \partial_2 u | d) \, dx, \end{aligned}$$

where the last difference can be estimated via the local Lipschitz continuity of W and Hölder's inequality. We then obtain that

$$\limsup_{\eta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u_{\eta,\varepsilon}) = \mathcal{I}(u).$$

Finally, we choose a diagonal subsequence $(u_\varepsilon)_\varepsilon = (u_{\eta(\varepsilon),\varepsilon})_\varepsilon$ such that

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u_\varepsilon) = \mathcal{I}(u) \quad \text{and} \quad u_\varepsilon \rightarrow u \text{ strongly in } W^{1,2}(\Omega; \mathbb{R}^3).$$

□

Remark 1.37 (External forces and scaling regimes). In reality, we often need to analyze the interplay between the internal elastic energy and the acting forces, which leads us to consider energy functionals of the form

$$W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3) \ni v \mapsto \int_{\Omega_\varepsilon} W(\nabla v) \, dx - \int_{\Omega_\varepsilon} f_\varepsilon \cdot v \, dx, \quad (1.38)$$

see also (1.13) or (1.22). For the sake of simplicity, it is commonly assumed that $f_\varepsilon \in L^q(\omega; \mathbb{R}^3)$, with $\frac{1}{q} + \frac{1}{p} = 1$, is independent of x_3 and that

$$f_\varepsilon = \varepsilon^\alpha f \text{ for } \alpha \geq 0 \text{ and } f \in L^q(\omega; \mathbb{R}^3).$$

It is known that the resulting normalized system energy

$$\mathcal{J}_\varepsilon^{(\alpha)} : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow \mathbb{R}, \quad u \mapsto \int_{\Omega} W(\nabla^\varepsilon u) \, dx - \frac{1}{\varepsilon^\alpha} \int_{\Omega} f \cdot u \, dx$$

has (almost-)minimizers $u_\varepsilon \in W^{1,p}(\Omega; \mathbb{R}^3)$ such that $\mathcal{J}_\varepsilon^{(\alpha)}(u_\varepsilon) = \mathcal{O}(\varepsilon^{\beta(\alpha)})$ with

$$\beta(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in [0, 2), \\ 2\alpha - 2 & \text{if } \alpha \in [2, \infty), \end{cases} \quad (1.39)$$

where \mathcal{O} is the Landau symbol, cf. [99]. Hence, after scaling $\mathcal{J}_\varepsilon^{(\alpha)}$ by the factor $\varepsilon^{\beta(\alpha)}$ and neglecting the term describing external forces, it remains to determine the asymptotic behavior of the elastic strain energies

$$\mathcal{I}_\varepsilon^{(\alpha)} : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty), \quad u \mapsto \varepsilon^{-\beta(\alpha)} \int_{\Omega} W(\nabla^\varepsilon u) \, dx.$$

Depending on the scaling parameter α , and suitable assumptions on the energy density W , we observe a different asymptotic behavior of the corresponding energy functionals $\mathcal{I}_\varepsilon^{(\alpha)}$, which correspond to thin structures of different stiffness.

For instance, the case $\alpha = 0$, which we treated in Theorem 1.35, essentially describes elastic membranes and was first studied by Le Dret & Raoult in [137]. Note that, in this case, the term describing the external body forces is merely a weakly continuous perturbation and may therefore be neglected for the asymptotic analysis, see Proposition 1.29. In this limit model, we observe that the elastic vanishingly thin structure exhibits high flexibility for there are no constraints on the set of admissible limit deformations, aside from the independence of x_3 .

The scaling regimes $\alpha \geq 2$, for which $p = 2$ is assumed, are more difficult as they require a powerful tool to describe limits of sequences with bounded energy. In 2002, Friesecke & James & Müller [98] published their famous geometric rigidity result, which constitutes a nonlinear

generalization of Korn's inequality. In their work, the authors used their new rigidity estimate to prove the Γ -convergence result for $\alpha = 2$, in which the limit energy is finite only on the set of isometries, and describes Kirchhoff's plate theory.

Their pioneering work has since become the basis of many asymptotic analyses results. This includes, the authors' later contributions to the cases $\alpha > 2$ [99], as well as analogous Γ -convergence results for 3d-1d reduction [156, 157, 184, 185], to name but a few.

We close this section with a brief comment on 3d-1d reductions, in which the considered thin structure is small in two directions.

Remark 1.38 (3d-1d reductions). For $\varepsilon > 0$, let $\Omega_\varepsilon = (0, L) \times \varepsilon\omega$ be the reference configuration of a thin tube with length $L > 0$ and shrinking cross section $\varepsilon\omega \subset \mathbb{R}^2$, where $\omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, cf. Figure 1.6.

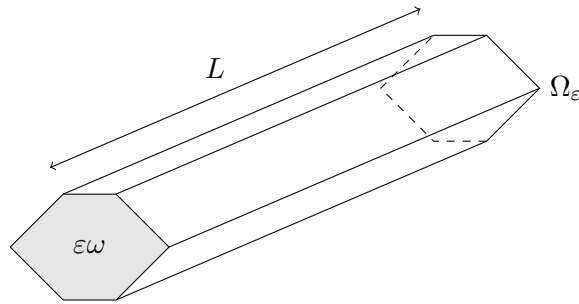


Figure 1.6: Illustration of the thin structure $\Omega_\varepsilon = (0, L) \times \varepsilon\omega$ with hexagonal $\omega \subset \mathbb{R}^2$.

The elastic system energy is then given as in (1.38) with

$$f_\varepsilon = \varepsilon^\alpha \text{ for } \alpha \geq 0 \text{ and } f \in L^q(0, L; \mathbb{R}^3).$$

Analogously to Remark 1.37, we determine the asymptotic variational behavior of the rescaled sequence

$$\mathcal{I}_\varepsilon^{(\alpha)} : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty), \quad u \mapsto \int_\Omega W(\nabla^\varepsilon u) \, dx - \varepsilon^{\alpha-\beta(\alpha)} \int_\Omega f \cdot u \, dx$$

with β as in (1.39), and the rescaled gradient for 3d-1d problems $\nabla^\varepsilon u = (\partial_1 u|_{\frac{1}{\varepsilon}} \partial_2 u|_{\frac{1}{\varepsilon}} \partial_3 u)$.

The scaling regime $\alpha = 0$, which leads to a limit model describing highly flexible elastic strings, was studied by Acerbi & Buttazzo & Percivale in [1]. The asymptotic variational analysis for the case $\alpha = 2$ is based on the geometric rigidity result and the rigorous derivation of Kirchhoff's plate theory [98]. Naturally, the limit model is much more restrictive than in the regime $\alpha = 0$, in that the deformation behavior is determined by a Frénet-Frame which describes the strain of the mid fiber and rotation effects of the cross section, see [156]. The regimes $\alpha > 2$, for which we refer to [157, 184, 185] yield limit energies that display even more rigidity in the sense that every sequence of bounded energy converges to a fixed rotation similar to the analogous 3d-2d reduction.

1.5 Applications of Γ -convergence: Homogenization

The theory of homogenization concerns problems with highly oscillating behavior. In the framework of nonlinear elasticity, this includes energy functionals describing material bodies with a large amount of heterogeneities on a microscopic scale, see Figure 1.7. The goal is to obtain an

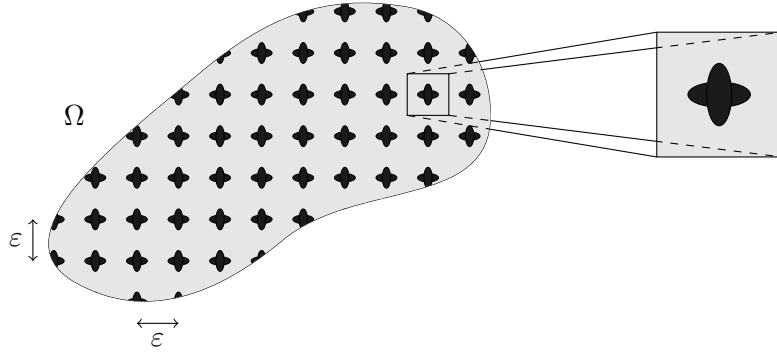


Figure 1.7: Illustration of a domain Ω containing ε -periodically distributed inhomogeneities in two dimensions.

effective macroscopic description of the inhomogeneous material via averaged quantities. Since the deformation behavior of materials can be described by minimizing the associated system energy, we are naturally interested in the Γ -limit of such energies, which reduces the complex heterogeneous multiscale material problem into a simpler one. The hence obtained limit models provide structural insight into mechanical material behavior and make numerical implementations feasible.

The first pioneering homogenization results in this variational framework are due to Marcellini [147] for convex and to Braides [41] and Müller [161] for non-convex integral functionals, both with standard polynomial growth assumptions. In the following, we shall present the standard variational homogenization theory, see e.g., [42, Chapter 3] or [43], and illustrate the result with a specific choice of the energy density.

Let $\Omega \subset \mathbb{R}^n$ be the reference configuration of an elastic body in $n \in \{1, 2, 3\}$ dimensions. We are particularly interested in periodically distributed inhomogeneities in the reference configuration $\Omega \subset \mathbb{R}^3$, where Ω is a bounded Lipschitz domain. The internal energy density $W : \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow [0, \infty)$ is therefore assumed to be 1-periodic in the space variable, i.e.,

$$W(x + k, F) = W(x, F) \text{ for all } k \in \mathbb{Z}^n \text{ and all } F \in \mathbb{R}^{n \times n}.$$

Note that, it is then sufficient to define $W(\cdot, F)$ on the unit cell $[0, 1]^n$ for every $F \in \mathbb{R}^n$. To model an increasing amount of microscopic inhomogeneities, we consider the strain energy given by

$$\mathcal{I}_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^n) \rightarrow [0, \infty), \quad u \mapsto \int_\Omega W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx, \quad (1.40)$$

where we also assume that W is Carathéodory and satisfies standard p -growth and coercivity for $p > 1$, cf. (1.25).

Theorem 1.39 (Homogenization). *The sequence $(\mathcal{I}_\varepsilon)_\varepsilon$ as in (1.40) Γ -converges sequentially with respect to the weak topology in $W^{1,p}(\Omega; \mathbb{R}^n)$ to the functional*

$$\mathcal{I} : W^{1,p}(\Omega; \mathbb{R}^n) \rightarrow [0, \infty), \quad u \mapsto \int_\Omega W_{\text{hom}}(\nabla u) dx,$$

where $W_{\text{hom}} : \mathbb{R}^{n \times n} \rightarrow [0, \infty)$ is the homogenized density given by the asymptotic homogenization formula

$$W_{\text{hom}}(F) = \liminf_{\varepsilon \rightarrow 0} \left\{ \int_{(0,1)^n} W\left(\frac{x}{\varepsilon}, \nabla \varphi(x) + F\right) dx : \varphi \in W_0^{1,p}(\Omega; \mathbb{R}^n) \right\} \quad (1.41)$$

Furthermore, every sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^n)$ with $\sup_\varepsilon \mathcal{I}_\varepsilon(u_\varepsilon) < \infty$ and vanishing mean value has a weakly converging subsequence.

Remark 1.40 (Interpretation of the result). a) The limit density W_{hom} describes a homogeneous material since it no longer depends on the spatial variable.

b) If we define \mathcal{I}_ε and \mathcal{I} on the space $W^{1,p}(\Omega; \mathbb{R}^n) \cap L_0^p(\Omega; \mathbb{R}^n)$, then the sequence $(\mathcal{I}_\varepsilon)_\varepsilon$ also Γ -converges to \mathcal{I} with respect to the weak topology, see Remark 1.34. This is a direct consequence of Poincaré's inequality; the proof can be easily adapted to include the vanishing mean value.

c) For specific choices of the energy density W , we can find an explicit expression of the homogenized density W_{hom} as in (1.41). In particular, for $n = 1$ and

$$W(x, F) = a(x)|F|^2 \quad \text{with} \quad a(x) = \begin{cases} \alpha_1 & \text{if } x \in [0, \frac{1}{2}), \\ \alpha_2 & \text{if } x \in [\frac{1}{2}, 1), \end{cases}, \quad x \in \Omega, F \in \mathbb{R}^{1 \times 1} \cong \mathbb{R} \quad (1.42)$$

for $\alpha_2 > \alpha_1 > 0$, which is a toy model based on the Neo-Hookean ansatz (1.10), we then obtain that

$$W_{\text{hom}}(F) = \alpha^* |F|^2, \quad \text{where} \quad \alpha^* = \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2} = \left(\int_0^1 \frac{1}{a(t)} dt \right)^{-1}$$

is called the harmonic mean of α_1 and α_2 , see [42, Section 12.2.2] or [43, Exercise 13.1]. It may be surprising that the homogenized strain energy involves the harmonic mean instead of the arithmetic one. In fact, this is a direct consequence of the choice of topology on $W^{1,p}(\Omega; \mathbb{R}^n)$, see also d) of this remark.

d) Replacing the weak topology on $W^{1,p}(\Omega; \mathbb{R}^n)$ with the standard norm-topology yields a different homogenization result. In fact, choosing W as in (1.42) leads to a Γ -limit of integral form with homogenized energy density

$$W_{\text{hom}}(F) = \bar{\alpha} |F|^2, \quad \text{where} \quad \bar{\alpha} = \frac{1}{2}(\alpha_1 + \alpha_2)$$

is the arithmetic mean of α_1 and α_2 . However, no subsequence of $(\mathcal{I}_\varepsilon)_\varepsilon$ satisfies the asymptotic coercivity condition in Remark 1.32 with respect to the standard norm-topology in $W^{1,p}(\Omega; \mathbb{R}^n)$. Since this condition is one of the main ingredients in proving the existence of minimizers of Γ -limits, it is preferable to choose the weak topology on $W^{1,p}(\Omega; \mathbb{R}^n)$ instead.

Proof. For the sake of simplicity, we prove Theorem 1.39 only for the specific case (1.42), normalizing the domain $\Omega = (0, 1)$. We also provide a proof for the statement in Remark 1.40 d) since the arguments are very similar to the setting with the weak topology.

A crucial observation for this proof is that the function $a_\varepsilon := a(\frac{\cdot}{\varepsilon})$ satisfies

$$\frac{1}{a_\varepsilon} \xrightarrow{*} \int_0^1 \frac{1}{a(t)} dt = \frac{1}{\alpha^*} \quad \text{in } L^\infty(0, 1). \quad (1.43)$$

as a consequence of the Riemann Lebesgue Lemma on the weak convergence of oscillating sequences.

Step 1: Compactness. The asymptotic coercivity is much more straightforward than in the case of dimension reduction problems. In fact, if $(u_\varepsilon)_\varepsilon \subset W^{1,2}(0, 1)$ is a sequence of bounded energy and vanishing mean value, then we obtain from (1.25) and Poincaré's inequality that there exists a subsequence of $(u_\varepsilon)_\varepsilon$ converging weakly to some $u \in W^{1,2}(0, 1)$.

Step 2: Lower bound. Now, let $(u_\varepsilon)_\varepsilon \subset W^{1,2}(0,1)$ with weak limit $u \in W^{1,2}(0,1)$. Then, we derive from (1.43) that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon \rightarrow 0} \int_0^1 a_\varepsilon(x) |u'_\varepsilon|^2 dx \\ &= \liminf_{\varepsilon \rightarrow 0} \int_0^1 a_\varepsilon(x) \left(u'_\varepsilon(x) - \frac{\alpha^*}{a_\varepsilon(x)} u'(x) \right)^2 dx + 2 \int_0^1 \alpha^* u'_\varepsilon(x) u'(x) dx - \int_0^1 \frac{(\alpha^*)^2}{a_\varepsilon(x)} |u'_\varepsilon(x)|^2 dx \\ &\geq \liminf_{\varepsilon \rightarrow 0} 2 \int_0^1 \alpha^* u'_\varepsilon(x) u'(x) dx - \int_0^1 \frac{(\alpha^*)^2}{a_\varepsilon(x)} |u'_\varepsilon(x)|^2 dx \\ &= 2 \int_0^1 \alpha^* |u'(x)|^2 dx - \int_0^1 \frac{(\alpha^*)^2}{\alpha^*} |u'_\varepsilon(x)|^2 dx = \int_0^1 \alpha^* |u'(x)|^2 dx. \end{aligned}$$

This concludes the proof of the liminf-inequality.

We observe that the corresponding lower bound in the setting of Remark 1.40 d) is much easier. In this case, we exploit that

$$a_\varepsilon \xrightarrow{*} \int_0^1 a(t) dt = \bar{\alpha} \quad \text{in } L^\infty(0,1) \quad (1.44)$$

and that $(u_\varepsilon)_\varepsilon$ converges strongly in $W^{1,2}(0,1)$.

Step 3: Upper bound. For $u \in W^{1,2}(0,1)$, we define a recovery sequence $(u_\varepsilon)_\varepsilon \subset W^{1,2}(0,1)$ by

$$u_\varepsilon(x) = \int_0^x \frac{\alpha^*}{a_\varepsilon(t)} u'(t) dt, \quad x \in (0,1).$$

It is obvious that $u'_\varepsilon = \frac{\alpha^*}{a_\varepsilon} u'$ and we may thus conclude from (1.43) that $u_\varepsilon \rightharpoonup u$ in $W^{1,2}(0,1)$. For the associated elastic energy we estimate

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(u_\varepsilon) &= \limsup_{\varepsilon \rightarrow 0} \int_0^1 a_\varepsilon(x) \left(\frac{\alpha^*}{a_\varepsilon(x)} \right)^2 |u'(x)|^2 dx \\ &= \limsup_{\varepsilon \rightarrow 0} \int_0^1 \frac{(\alpha^*)^2}{a_\varepsilon(x)} |u'(x)|^2 dx = \int_0^1 \alpha^* |u'(x)|^2 dx. \end{aligned}$$

In the setting of Remark 1.40 d), where the weak topology in $W^{1,2}(0,1)$ is replaced by the strong one, we simply choose the constant recovery sequence and exploit (1.44). \square

1.6 Outline of the thesis

We first focus on the variational asymptotic analysis of elastic thin structures, cf. Section 1.4, with non-standard energy densities. Precisely, Chapters 2 and 3 are concerned with the derivation of a complete hierarchy of one-dimensional string and rod theories, starting out from three-dimensional models in nonlinear elasticity subject to local volume-preservation. These chapters are identical to the published articles

- D. Engl and C. Kreisbeck. Asymptotic variational analysis of incompressible elastic strings. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, pages 1–28, 2020
- D. Engl and C. Kreisbeck. Theories for incompressible rods: A rigorous derivation via Γ -convergence. *Asymptotic Analysis*, 124:1–28, 2021

Similar to what we discussed in Section 1.4, in particular Remark 1.38, we consider the system energy of fully three-dimensional elastic bodies of cylindrical shape $\Omega = (0, L) \times \varepsilon\omega$ with a shrinking cross section; here $\omega \subset \mathbb{R}^2$ is a simply connected bounded Lipschitz domain and $L > 0$, see also Figure 1.6. The fundamental difference to the standard theory presented before is the additional constraint of incompressibility, which we incorporate directly into the functionals as in (1.20). Since the system energies of hyperelastic materials are integral functionals we can equivalently introduce the constraint into the stored energy densities. Precisely, we set

$$W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty], \quad F \mapsto \begin{cases} W_0(F) & \text{if } \det F = 1, \\ \infty & \text{otherwise,} \end{cases} \quad (1.45)$$

where $W_0 : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ is a finite-valued energy density used in the analysis of compressible materials. After the usual normalization by unit volume and a rescaling argument, cf. Remark 1.38, we are led to study the constrained strain energies

$$\mathcal{I}_\varepsilon^{(\alpha)} : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty], \quad u \mapsto \begin{cases} \varepsilon^{-\beta(\alpha)} \int_\Omega W_0(\nabla^\varepsilon u(x)) \, dx & \text{if } \det \nabla^\varepsilon u = 1, \\ \infty & \text{otherwise} \end{cases} \quad (1.46)$$

with $p > 1$, and $\beta(\alpha)$ as in (1.39) for $\alpha \geq 0$; here $\nabla^\varepsilon u = (\partial_1 u|_\varepsilon^\perp \partial_2 u|_\varepsilon^\perp \partial_3 u)$.

Although the construction of recovery sequences follows a similar approach in all scaling regimes, we split the asymptotic analysis of (1.46) into two parts for thematic consistency within overarching proof strategies of the Γ -convergence results. Chapter 2 is devoted to the scaling regimes $\alpha < 2$, which lead to (degenerate) models for incompressible strings; in Chapter 3 we then focus on the remaining regimes $\alpha \geq 2$, where we obtain energy functionals describing incompressible rods.

In case $\alpha \in [0, 2)$, local volume preservation affects either the limit densities through a constrained minimization in the last two columns of the (rescaled) gradients, or the effective domain of the Γ -limit in the regime $\alpha \in (0, 2)$. It is particularly striking that, compressions and stretches of the string are admissible deformations in all scaling regimes. In the more physically relevant case $\alpha = 0$, we obtain the following result:

Theorem 1.41. *If W_0 satisfies standard p -growth and coercivity (1.25), then the sequence $(\mathcal{I}_\varepsilon^{(0)})_\varepsilon$ as in (1.46) Γ -converges sequentially with respect to the weak topology in $W^{1,p}(\Omega; \mathbb{R}^3)$ to $\mathcal{I}^{(0)} : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty]$ defined by*

$$\mathcal{I}^{(0)}(u) = \begin{cases} \int_0^L \overline{W}^c(u'(x_1)) \, dx_1 & \text{if } u \in W^{1,p}(0, L; \mathbb{R}^3), \\ \infty & \text{otherwise,} \end{cases}$$

where $\overline{W} : \mathbb{R}^3 \rightarrow [0, \infty]$ is given by

$$\overline{W}(\xi) = \inf_{A \in \mathbb{R}^{3 \times 2}} W(\xi|A) = \begin{cases} \min_{A \in \mathbb{R}^{3 \times 2}, \det(\xi|A)=1} W_0(\xi|A) & \text{if } \xi \neq 0, \\ \infty & \text{otherwise,} \end{cases} \quad \xi \in \mathbb{R}^3$$

and $(\cdot)^c$ is the convex envelope as in (1.24).

Moreover, every sequence of bounded energy in $W^{1,p}(\Omega; \mathbb{R}^3)$ with vanishing mean value has a weakly convergent subsequence with limit in $W^{1,p}(0, L; \mathbb{R}^3)$.

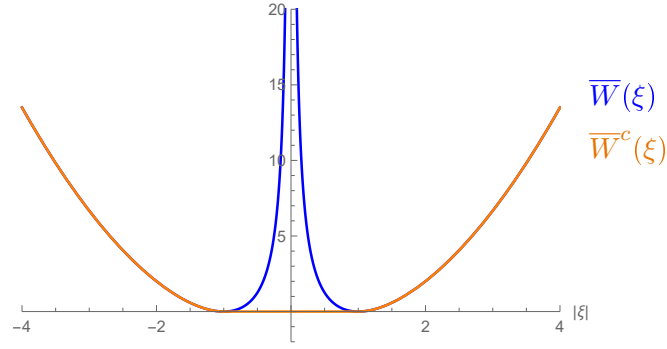


Figure 1.8: The minimized energy \bar{W} and its convex envelope \bar{W}^c for the Neo-Hookean model (1.47) with $\mu = 1$

We observe that the local volume preservation only affects the finite values attained by the Γ -limit $\mathcal{I}^{(0)}$. In fact, the structure of the limit functional is essentially the same as in the compressible case of $3d-1d$ or $3d-2d$ reductions as in Section 1.4. In contrast to the standard theory, the minimized energy \bar{W} explodes near the origin as a consequence of the constraint of incompressibility (1.45); in particular, \bar{W} does not have standard p -growth, which requires a more careful approach for the construction of recovery sequences. However, this singularity at the origin vanishes after applying the convex envelope (which coincides with the quasiconvex envelope in $\mathbb{R}^{3 \times 1} \cong \mathbb{R}^3$); the resulting function then does satisfy standard p -growth and coercivity, see Figure 1.8 for illustration. We illustrate our findings by providing explicit expressions of the limit densities in incompressible Neo-Hookean and Yeoh models (recall Example 1.1 *b*), *e*). For example, we derive for

$$W(F) = \begin{cases} \mu \operatorname{Tr}(F^T F - \operatorname{Id}) = \mu(|F|^2 - 3) & \text{if } \det F = 1, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{3 \times 3} \quad (1.47)$$

with $\mu > 0$, that

$$\bar{W}^c(\xi) = \begin{cases} \mu(|\xi|^2 + 2|\xi|^{-1} - 3) & \text{if } |\xi| \geq 1, \\ 0 & \text{if } |\xi| \leq 1, \end{cases} \quad \xi \in \mathbb{R}^3,$$

see also Figure 1.8.

One of the main difficulties in the proof of the Γ -limit is to establish recovery sequences that accommodate the nonlinear, nonconvex differential constraint imposed by the incompressibility. To this end, we combine suitable modifications of the classical constructions in the unconstrained case, in which we need to carefully circumvent gradients close to the origin due to the singularity of \bar{W} , with the help of an inner perturbation argument tailored for $3d-1d$ dimension reduction problems. The main idea of the latter is to modify recovery sequences from the associated compressible case, which will satisfy the constraint of incompressibility only approximately, in such a way that the resulting sequence satisfies the differential inclusion $\det \nabla^\varepsilon u_\varepsilon = 1$ exactly almost everywhere.

In case $\alpha \geq 2$, we assume that $p = 2$ and that W is a frame-indifferent and sufficiently regular strain energy density with a single well at $\operatorname{SO}(3)$. In these scaling regimes, we obtain that the set of admissible deformations in the limit model is exactly the same as in the corresponding compressible case as a consequence of the inherent higher rigidity of these models in the compressible cases. For example, if $\alpha = 2$ then the Γ -limit $\mathcal{I}^{(2)}$ of $(\mathcal{I}_\varepsilon^{(2)})_\varepsilon$ is finite only on

$$\mathcal{A}^{(2)} := \{(u, D) \in H^2(0, L; \mathbb{R}^3) \times H^1(0, L; \mathbb{R}^{3 \times 2}) : (u'|D) \in \operatorname{SO}(3) \text{ a.e. in } (0, L)\}. \quad (1.48)$$

Here, the function u represents the deformation of the mid-fiber and D consists of the two directors describing rotation effects of the cross section, see also Figure 1.9. Exactly as in the

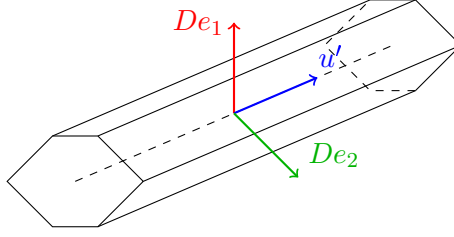


Figure 1.9: Illustration of a thin rod and the Frénet Frame $(u'|D)$ as in (1.48).

corresponding compressible case, we find that the rescaled gradients of sequences of bounded energy converge strongly in $L^2(\Omega; \mathbb{R}^{3 \times 3})$ to some $(u'|D)$ with $(u, D) \in \mathcal{A}^{(2)}$. In light of this strong convergence, the constraint of incompressibility would carry over to the limit but the triple $(u'|D)$ is already a Frénet-frame and thus satisfies $\det(u'|D) = 1$ almost everywhere anyway.

However, the local volume preservation does affect the strain energy densities of the resulting Γ -limits in that they are determined by minimization problems involving a trace constraint that arises from a second-order linearization of the determinant condition on the (rescaled) deformation gradient. In the case $\alpha = 2$, the limit density is given by

$$\mathbb{R}_{\text{skew}}^{3 \times 3} \ni F \mapsto \min_{\beta \in H^1(\omega; \mathbb{R}^3), \text{Tr}(F(x_2 e_2 + x_3 e_3)|\tilde{\nabla}\beta)=0} \int_{\omega} Q(F(x_2 e_2 + x_3 e_3)|\tilde{\nabla}\beta) \, d\tilde{x}, \quad (1.49)$$

where $\tilde{x} = (x_2, x_3)$ and $\tilde{\nabla} = (\partial_2 | \partial_3)$, and $Q(F) = \nabla^2 W_0(\text{Id})[F, F]$ for $F \in \mathbb{R}^{3 \times 3}$ is the quadratic form emerging from the second derivative of W_0 at the identity.

We identify the Euler-Lagrange equations of the limit energies in the scaling regime $\alpha = 2$, and observe that the corresponding critical points are thus also affected by the local volume preservation. In particular, under simplifying assumptions, stationary points of the Γ -limit are determined by a set of ordinary differential equations involving the Young modulus for incompressible materials, which emerges from the compressible case in the limit of diverging first Lamé coefficient λ .

The proofs of the lower bounds rely on suitable constraint regularization in the sense that we approximate the incompressible energy density W by penalized strain energy densities $W_k : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ for $k \in \mathbb{N}$, defined by

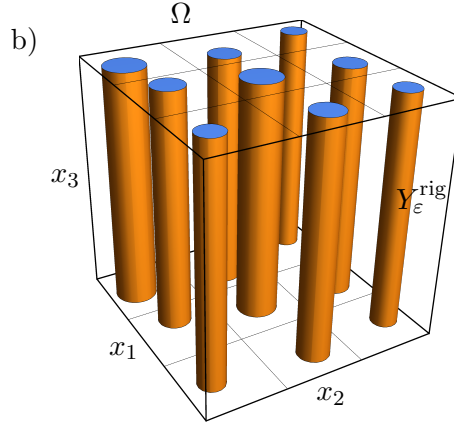
$$W_k(F) = W_0(F) + \frac{k}{2}(\det F - 1)^2, \quad F \in \mathbb{R}^{3 \times 3}.$$

We point out that the second derivative of W_k at the identity satisfies

$$Q_k(F) = \nabla^2 W_k(\text{Id})[F, F] = Q(F) + k \text{Tr}(F), \quad F \in \mathbb{R}^{3 \times 3},$$

which points towards the vanishing trace constraint mentioned above, see (1.49). The upper bounds require a careful, explicit construction of locally volume-preserving recovery sequences. After decoupling the cross-section variables with the help of divergence-free extensions, we apply an inner perturbation argument that we introduced in the scaling regimes $\alpha < 2$ to enforce the desired non-convex determinant constraint.

In Chapter 4, we turn our attention towards the asymptotic variational analysis of a non-standard model in the context of homogenization theory. More precisely, we determine the effective macroscopic deformation behavior of high-contrast fiber-reinforced elastic materials. This chapter is available as the the preprint

Figure 1.10: Illustration of a collection of fibers embedded into a cuboid Ω .

- D. Engl, C. Kreisbeck, and A. Ritorto. Asymptotic analysis of deformation behavior in high-contrast fiber-reinforced materials: rigidity and anisotropy. *Preprint, arXiv:2105.03971*, 2021

As for the basic geometric setup, we assume that the three-dimensional reference configuration is given by $\Omega = \omega \times (0, L)$ for a bounded Lipschitz domain $\omega \subset \mathbb{R}^2$ and height $L > 0$; we suppose further that each scaled and translated unit cell $\varepsilon(k + [0, 1)^2)$ with $\varepsilon > 0$ and $k \in \mathbb{Z}^2$ shall contain the cross section ω_ε^k of a rigid fiber of height L . The collection of all these fibers, which need not be periodically distributed, is then denoted by $Y_\varepsilon^{\text{rig}} := \bigcup_{k \in \mathbb{Z}^2} \omega_\varepsilon^k$, see Figure 1.10 for a simple illustration of the geometric framework. The starting point of our analysis is an internal elastic energy functional of the form

$$\mathcal{I}_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty], \quad u \mapsto \int_\Omega W(x, \nabla u) \, dx \quad (1.50)$$

for $p > 2$ with an inhomogeneous constrained energy density

$$W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty], \quad F \mapsto \begin{cases} W_{\text{soft}}(F) & \text{if } x \notin Y_\varepsilon^{\text{rig}}, \\ 0 & \text{if } x \in Y_\varepsilon^{\text{rig}} \text{ and } F \in \text{SO}(3), \\ \infty & \text{if } x \in Y_\varepsilon^{\text{rig}} \text{ and } F \notin \text{SO}(3), \end{cases} \quad (1.51)$$

where $W_{\text{soft}} : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ is supposed to have p -growth and coercivity as in (1.25), as well as other standard properties like frame-indifference and a single well at $\text{SO}(3)$. Note that the constrained homogeneous density $W|_{Y_\varepsilon^{\text{rig}}}$ on the fibers can be viewed as having infinitely large elastic constants.

The primary difference to standard models as in Section 1.5 lies in the inhomogeneous non-convex differential constraint given by W and the (possibly) non-periodic distribution of the inhomogeneities. Hence, the set of admissible deformations for the functionals \mathcal{I}_ε as in (1.50) is given by

$$\mathcal{A}_\varepsilon = \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : \nabla u \in \text{SO}(3) \text{ a.e. in } \Omega \cap Y_\varepsilon^{\text{rig}}\}, \quad (1.52)$$

which significantly impacts the set on which the associated sequential Γ -limit of $(\mathcal{I}_\varepsilon)_\varepsilon$ with respect to the weak topology (if existent) is finite. From a material science point of view, (1.52) imposes the restriction that every deformation is (locally) rigid on the fibers within the elastic material. In particular, the deformation is incompressible on the fibers, and preserves all angles and the length in every direction.

Our contribution to this model lies in the full characterization of the effective domain of a possible Γ -limit of $(\mathcal{I}_\varepsilon)_\varepsilon$ with respect to weak convergence. In other words, we derive a sequential Kuratowski-convergence result with respect to the weak topology in $W^{1,p}(\Omega; \mathbb{R}^3)$, cf. Remark 1.33 a), for the sequence of sets $(\mathcal{A}_\varepsilon)_\varepsilon$. We show that, if the cross section ω has a sufficiently regular boundary, then the associated limit set consists of deformations that exhibit a highly restrictive anisotropic deformation behavior in the sense that the strain in the direction of the fibers has unit length and merely depends on the cross-section variables.

Theorem 1.42 (Characterization of limit deformations). *Assume that $\omega \subset \mathbb{R}^2$ is bi-Lipschitz homeomorphic to the unit ball and $p > 2$. Then, the sequence $(\mathcal{A}_\varepsilon)_\varepsilon$ with \mathcal{A}_ε as in (1.52) K -converges sequentially with respect to the weak topology in $W^{1,p}(\Omega; \mathbb{R}^3)$ to*

$$\begin{aligned} \mathcal{A} &= \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : \partial_3 u \in W^{1,p}(\omega; \mathcal{S}^2)\} \\ &= \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u(x) = x_3 \Sigma(x') + d(x') \text{ for a.e. } x = (x', x_3) \in \Omega \\ &\quad \text{with } \Sigma \in W^{1,p}(\omega; \mathcal{S}^2), d \in W^{1,p}(\omega; \mathbb{R}^3)\} \\ &= \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u(x) = R(x')x + b(x') \text{ for a.e. } x = (x', x_3) \in \Omega \\ &\quad \text{with } R \in W^{1,p}(\omega; \text{SO}(3)), b \in W^{1,p}(\omega; \mathbb{R}^3)\}. \end{aligned}$$

However, the construction of approximating sequences is more delicate here due to the higher flexibility and connectedness of the soft material component. We overcome these technical challenges by a careful approximation of the identity that is constant on the rigid components, combined with a lifting in fiber bundles for Sobolev functions. The proof of the upper bound for \mathcal{A} emerges as a natural generalization and suitable modification of a state-of-the-art asymptotic rigidity analysis for high-contrast layered composites [50, 51]. We can view this step as the compactness (or asymptotic coercivity) result for the energy functionals $(\mathcal{I}_\varepsilon)_\varepsilon$ as in (1.50), which is significantly more intricate than the standard theory presented in Section 1.5 due to the non-convex differential constraint of rigidity on the fibers. Constructing approximating sequences, which constitutes the lower bound of the sequential K -convergence result, is more intricate here than in the setting of layered composites due to the higher flexibility resulting from the connectedness of the soft material component. We solve these issues by a meticulous approximation of the identity that is constant on the cross section of the rigid fibers, along with a lifting result for Sobolev functions with values in the base of a fiber bundle.

We illustrate our results with a few examples of attainable macroscopic deformations. We draw the comparison to high-contrast layered materials as in [50, 51] and briefly comment on incompressibility of elements in \mathcal{A} .

We then conclude Chapter 4, with an analysis of a modification of \mathcal{I}_ε as in (1.50) where we introduce an additional second-order regularization in the cross-section variables. The class of such strain energies play a vital role in the analysis of so-called non-simple materials. We prove that the resulting limit model is trivial in the sense that the set of admissible limit deformations consists purely of rigid body motions, which have no contribution to the elastic energy, cf. (1.51).

We close this thesis in Chapter 5 with an analysis of observed macroscopic strains in single-slip polycrystal plasticity, for which we refer to the preprint

- D. Engl and C. Kreisbeck. On the interplay of anisotropy and geometry for polycrystals in single-slip crystal plasticity. *Preprint, arXiv:2109.01022*, 2021

Our model emerges from an energetic variational framework of finite single-crystal plasticity (cf. [48, 152, 149, 172]), where we adapt a two-dimensional geometrically nonlinear model in which the deformation gradient $\nabla u = F$ has a multiplicative decomposition $F = F_{\text{el}} F_{\text{pl}}$ into an

elastic part F_{el} and a plastic one F_{pl} . We invoke a setting with one active slip system under the assumption of elastically rigid behavior [58, 68], in which F_{el} is contained in the set of rotations $\text{SO}(2)$ almost everywhere and does not contribute to the elastoplastic energy, and F_{pl} is assumed to be a simple shear $\text{Id} + \gamma s \otimes m$ along a slip direction $s \in \mathcal{S}^1$ and slip-plane normal $m \in \mathcal{S}^1$, where γ measures the plastic slip. Admissible deformations are therefore subject to the differential constraint

$$\nabla u = F \in \mathcal{M}_s := \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fs| = 1\};$$

In particular, every deformation needs to be incompressible, a constraint significant throughout this thesis.

In our variational polycrystalline model based on [19, 58, 68], determining the macroscopically attainable strains of a polycrystal with reference configuration $\Omega \subset \mathbb{R}^2$ amounts to identifying affine boundary data $F \in \mathbb{R}^{2 \times 2}$ such that there exists a Lipschitz solution $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ to the non-convex inhomogeneous partial differential inclusion

$$\begin{cases} \nabla u(x) \in \mathcal{M}_{R_*(x)e_1} & \text{for a.e. } x \in \Omega, \\ u(x) = Fx & \text{for } x \in \partial\Omega. \end{cases} \quad (P_{\mathcal{M}})$$

Here, the piecewise constant function $R_* : \Omega \rightarrow \text{SO}(2)$ describes the texture of the polycrystal, i.e., the decomposition into single-crystals and their associated slip orientations, see also Figure 1.11. The goal of Chapter 5 is the characterization of the set

$$\mathcal{F}_{\mathcal{M}}(\Omega, R_*) = \{F \in \mathbb{R}^{2 \times 2} : \text{there exists } u \in W^{1,\infty}(\Omega; \mathbb{R}^2) \text{ that solves } (P_{\mathcal{M}})\}, \quad (1.53)$$

which represents the domain of the macroscopic elastoplastic energy density (see 5.1.1 for a detailed energetic approach to this model). To this end, we determine inner and outer bounds on $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ by analyzing various issues related to the solvability of $(P_{\mathcal{M}})$ and its relaxed version

$$\begin{cases} \nabla u(x) \in \mathcal{N}_{R_*(x)e_1} & \text{for } x \in \Omega, \\ u(x) = Gx & \text{for } x \in \partial\Omega \end{cases} \quad (P_{\mathcal{N}})$$

with

$$\mathcal{N}_s := \mathcal{M}_s^{\text{qc}} = \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fs| \leq 1\},$$

where $(\cdot)^{\text{qc}}$, is the (finite) quasiconvex convex hull [70, Definition 7.25]. Under suitable assumptions on the shapes and orientations of the grains, our obtained bounds coincide, leading to a full characterization of the set of attainable macroscopic strains of the polycrystal.

First, we present and discuss a geometry-independent inner bound by identifying deformations of globally constant strain of the relaxed problem $(P_{\mathcal{N}})$. From these affine solutions, Lipschitz solutions to the unrelaxed problem $(P_{\mathcal{M}})$ are generated with the help of convex integration and relaxation results along the lines of [164] and [68]. We show that this inner bound, which consists of the finite intersection $\bigcap_{x \in \Omega} \mathcal{N}_{R_*(x)e_1}$, depends on at most three slip orientations occurring in the polycrystal, no matter the amount of grains.

Outer bounds on $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ are obtained through a generalized Hadamard compatibility condition [17] between the macroscopic and the microscopic strains at the boundary grains. To the best of our knowledge, the paper [17] has not yet been published, which is why we provide the proof of a special case that is tailored to the needs of this work.

We discuss examples of polycrystals with sufficient symmetry and selected bicrystals and show that the inner bound resulting from constant-strain solutions to $(P_{\mathcal{N}})$ is optimal and coincides with the outer bound emerging from rank-one compatibility at the boundary grains.

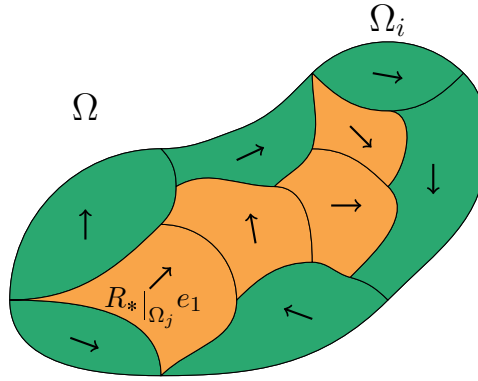


Figure 1.11: Illustration of a polycrystal with reference configuration Ω and texture R_* . The green and orange domains are the so-called boundary and interior grains; the arrows indicate the orientation of the slip direction R_*e_1 .

Finally, we produce an explicit construction to show that this inner bound is not optimal in general. Precisely, we design a polycrystal together with suitable finitely piecewise affine solution to the relaxed differential inclusion $(P_{\mathcal{N}})$ to generate a Lipschitz solution to $(P_{\mathcal{M}})$ via convex integration and relaxation results.

Chapter 2

Asymptotic variational analysis of incompressible elastic strings

This chapter corresponds to the published paper [88].

2.1 Introduction

Modern mathematical approaches to applications in materials science result in variational problems with non-standard constraints for which the classical methods of the calculus of variations do not apply. Constraints involving non-convexity, differential expressions and/or nonlocal effects are known to be particularly challenging.

In the context of the analysis of thin objects, interesting effects may occur due to the interaction between restrictive material properties and the lower-dimensional structure of the objects. We mention here a few selected examples: thin (heterogenous) films and strings subject to linear first-order partial differential equations, which are general enough to cover applications in nonlinear elasticity and micromagnetism at the same time, are studied in [129, 130, 131], cf. also [106, 128]; pointwise constraints on the stress fields appear naturally in models of perfectly plastic plates [72, 77]; for work on lower-dimensional material models that involve issues related to non-interpenetration of matter and (global) invertibility, we refer for instance to [135, 171, 186, 205]; physical growth conditions, which guarantee orientation preservation of deformation maps, have been taken into account in models of thin nematic elastomers [3] and von Kármán type rods and plates [76, 159].

This chapter is concerned with 3d-1d dimension reduction problems in nonlinear elasticity with incompressibility - a determinant constraint on the deformation gradient, which ensures local volume preservation, and is ideal to model e.g. rubber-like materials [168]. To be more specific, we provide an ansatz-free derivation of reduced models for incompressible thin tubes by means of Γ -convergence techniques (see [42, 71] for a comprehensive introduction). We take the limit of vanishing cross section, considering external loading of the order of magnitude that gives rise to string type models.

The analogous problem in the 3d-2d context, meaning for incompressible membranes, was solved independently by Trabelsi [195] and Conti & Dolzmann [61] based on different approaches. To overcome the difficulty of accommodating the nonlinear differential constraint when constructing recovery sequences, [61] involves the construction of suitable inner variations. This idea has been applied in the analysis of incompressible Kirchhoff and von Kármán plates [62, 144], and lends inspiration to this chapter, where we adapt it for 3d-1d reductions.

The first results in the literature to use Γ -convergence techniques to deduce reduced models for thin objects go back to the 1990s, with the seminal works by Acerbi, Buttazzo & Percivale

[1] on strings and Le Dret & Raoult [137] on membranes. Notice that in both papers, the authors start from unconstrained energy functionals whose energy densities satisfy standard growth. Before that, common techniques for gaining quantitative insight into thin structures relied mostly on asymptotic expansion methods, and were applied in the setting of linearized elasticity, see e.g. [53, 196]. For recent work on the (geometric) linearization of models involving energy densities with local volume-preservation constraints, we refer to [118, 145].

Over the last two decades, the fundamental results in [1, 137] have been generalized in multiple directions. This includes for instance the study of membrane theory with Cosserat vectors [39, 40], curved strings [184], inhomogeneous thin films [44], thin structures made of periodically heterogeneous material [11, 12, 136], or junctions between membranes and strings [95].

2.1.1 Problem formulation

For small $\varepsilon > 0$, let $\Omega_\varepsilon := (0, L) \times \varepsilon\omega$ with $L > 0$ and a bounded Lipschitz domain $\omega \subset \mathbb{R}^2$ represent the reference configuration of a thin unilaterally extended body. Up to translation, we may always assume that the origin lies in ω . In this chapter, we assume that $p \in (1, \infty)$.

The starting point of our analysis is a three-dimensional model in hyperelasticity with an energy functional (per unit cross section) of the form

$$\mathcal{E}_\varepsilon(v) = \frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} W(\nabla v) \, dy - \frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} f_\varepsilon \cdot v \, dy, \quad v \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3);$$

here, $f_\varepsilon \in L^{p'}(\Omega_\varepsilon; \mathbb{R}^3)$ with $p' := \frac{p}{p-1} \in (1, \infty)$ the dual exponent of p , are external forces and W is a constrained stored elastic energy density enforcing incompressibility, precisely,

$$W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty], \quad F \mapsto \begin{cases} W_0(F) & \text{if } \det F = 1, \\ \infty & \text{otherwise,} \end{cases} \quad (2.1)$$

with $W_0 : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ a continuous function that is bounded from below and has suitable growth behavior. We give more details on the exact assumptions on W_0 in Section 2.2.2, see (H1)-(H3). In this model, the observed deformations of the thin object in response to external forces correspond to minimizers (or, if the latter do not exist, low energy states) of \mathcal{E}_ε .

To derive reduced one-dimensional models that capture the asymptotic behavior of these minimizers, it is technically convenient to work with functionals defined on ε -independent spaces, which can be achieved by a classical rescaling argument in the cross section. Indeed, let $u(x) := v(y)$ for $v \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$ with the parameter transformation $y = (x_1, \varepsilon x_2, \varepsilon x_3)$ for $x \in \Omega := \Omega_1$, and suppose for simplicity that f_ε is independent of the cross-section variables. Then, $\mathcal{I}_\varepsilon(u) = \mathcal{E}_\varepsilon(v)$, where

$$\mathcal{I}_\varepsilon(u) := \int_{\Omega} W(\nabla^\varepsilon u) \, dx - \int_{\Omega} f_\varepsilon \cdot u \, dx, \quad u \in W^{1,p}(\Omega; \mathbb{R}^3),$$

with $\nabla^\varepsilon u = (\partial_1 u | \frac{1}{\varepsilon} \partial_2 u | \frac{1}{\varepsilon} \partial_3 u)$ the rescaled gradient of u .

In analogy to well-known facts in the context of compressible materials (see e.g. [99]), here as well, the scaling behavior of \mathcal{I}_ε depends strongly on the external forces f_ε . Whenever f_ε is of order ε^α for some $\alpha \geq 0$, then $\inf_{u \in W^{1,p}(\Omega; \mathbb{R}^3)} \mathcal{I}_\varepsilon(u)$ behaves like ε^β with $\beta = \alpha$ if $\alpha \leq 2$ and $\beta = 2\alpha - 2$ if $\alpha \geq 2$. Depending on these scalings, one has to expect qualitatively different limit models, falling into the categories of string theory ($\alpha = 0$), rod theories ($\alpha = 2$ and $\alpha = 3$) or other intermediate theories.

Since this chapter deals with the regimes $\alpha < 2$ (the cases $\alpha \geq 2$ are addressed in Chapter 3), it is natural to consider in the following the rescaled functionals $\mathcal{I}_\varepsilon^\alpha : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty]$

for $\alpha \in [0, 2)$ defined by

$$\mathcal{I}_\varepsilon^\alpha(u) = \frac{1}{\varepsilon^\alpha} \int_\Omega W(\nabla^\varepsilon u) \, dx, \quad u \in W^{1,p}(\Omega; \mathbb{R}^3); \quad (2.2)$$

notice that one may, without loss of generality, omit here the term describing work due to the external forces, for it is merely a continuous perturbation of the total (rescaled) elastic energy.

2.1.2 Statement of the main results.

The new contribution of this chapter is a complete characterization of the Γ -limits of sequences $(\mathcal{I}_\varepsilon^\alpha)_\varepsilon$ as in (2.2) for $\varepsilon \rightarrow 0$.

To be more precise, we prove that under suitable assumptions, $(\mathcal{I}_\varepsilon^\alpha)_\varepsilon$ Γ -converges with respect to the weak topology in $W^{1,p}(\Omega; \mathbb{R}^3)$ to $\mathcal{I}^\alpha : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty]$ given for $\alpha = 0$ by

$$\mathcal{I}^0(u) = \begin{cases} |\omega| \int_0^L \overline{W}^c(u'(x_1)) \, dx_1 & \text{if } u \in W^{1,p}(0, L; \mathbb{R}^3), \\ \infty & \text{otherwise.} \end{cases} \quad (2.3)$$

and for $\alpha \in (0, 2)$ by

$$\mathcal{I}^\alpha(u) = \begin{cases} 0 & \text{if } u \in W^{1,p}(0, L; \mathbb{R}^3) \text{ with } \overline{W}^c(u'(x_1)) = 0 \text{ for a.e. } x_1 \in (0, L), \\ \infty & \text{otherwise,} \end{cases} \quad (2.4)$$

respectively, cf. Theorem 2.7 and 2.8 for all details.

The reduced energy density \overline{W} results from minimizing out the cross-section variables from W , that is,

$$\overline{W}(\xi) := \min_{A \in \mathbb{R}^{3 \times 2}} W((\xi|A)) = \min_{A \in \mathbb{R}^{3 \times 2}, \det(\xi|A)=1} W_0((\xi|A)), \quad \xi \in \mathbb{R}^3 \setminus \{0\},$$

cf. (2.7), while the convexification \overline{W}^c of \overline{W} reflects a relaxation process.

The representation formulas (2.3) and (2.4) indicate that the two regimes $\alpha = 0$ and $\alpha \in (0, 2)$ give rise to qualitatively different reduced one-dimensional models. Whereas the latter admits only restricted deformations of the thin object, which can however be obtained with zero energy, the former allows us for any deformation of the string at finite energetic cost.

Despite their differences, both cases share a feature that may seem surprising at first. In fact, the incompressibility constraint imposed on the three-dimensional elasticity models does not carry over to the reduced ones, in the sense that admissible deformations are not length preserving in general, but can undergo compression and/or stretching, cf. Remark 2.2 for a discussion of the zero-level sets of \overline{W}^c . For a similar observation in the context of incompressible membranes, see [61].

2.1.3 Approach and techniques

The proofs for the cases $\alpha = 0$ and $\alpha \in (0, 2)$ can be found in Section 2.3 and Section 2.4, respectively. Overall, our idea is to combine tools from [61] on 3d-2d dimension reduction for incompressible membranes and the references [1, 184], where the authors derive one-dimensional models for strings without volumetric constraints.

In both regimes, compactness and the liminf-inequalities are straightforward to show, as they follow immediately from the corresponding results for the unconstrained problems. However, the construction of recovery sequences is more delicate.

The difficulty is to accommodate the incompressibility condition, while approximating the desired limit deformation in an energetically optimal way. To achieve this, we take the recovery sequences from the compressible case - i.e., the ones from [1] if $\alpha = 0$, and from [184] for $\alpha \in (0, 2)$ - as a basis, and modify them with the help of an inner perturbation argument tailored for 3d-1d dimension reduction. The latter, which is stated in Lemma 2.5, is a key ingredient of the proof.

In order to apply Lemma 2.5, though, one needs sequences that are sufficiently regular and whose rescaled deformation gradients have determinant close to 1 up to a small, quantified error. Especially in the string regime $\alpha = 0$, this requires some technical effort. Indeed, with the help of Bézier curves, we establish a mollification argument for piecewise affine functions of one variable, which, amongst other useful properties, yields uniform bounds on the derivatives, see Lemma 2.6. Moreover, we construct tailored moving frames along the resulting smooth curves in order to guarantee that fattening them to tubes results in deformed configurations that are almost (with controlled errors) locally volume-preserving.

2.2 Preliminaries

To start with, we introduce notations and collect a few technical tools.

2.2.1 Notation

The following notational conventions are used throughout the chapter: Let $a \cdot b$ be the standard inner product of two vectors $a, b \in \mathbb{R}^3$, and e_1, e_2, e_3 the standard unit vectors in \mathbb{R}^3 . On the space $\mathbb{R}^{m \times n}$ of real-valued $m \times n$ matrices, we denote the Frobenius-norm by $|\cdot|$ and write $|F|_p := \left(\sum_{i=1}^m \sum_{j=1}^n |e_i \cdot F e_j|^p \right)^{\frac{1}{p}}$ with $F \in \mathbb{R}^{m \times n}$ for any $p > 1$; naturally, there is a constant $c(p) > 0$ such that

$$\frac{1}{\sqrt{c(p)}} |\cdot|_p \leq |\cdot| \leq \sqrt{c(p)} |\cdot|_p. \quad (2.5)$$

Moreover, the closure of a given set $U \subset \mathbb{R}^n$ is denoted by \overline{U} , whereas U^c stands for the convex hull of U . Accordingly, the convex envelope of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is f^c . For the zero level set of f , we use $L_0(f)$.

The partial derivative of $v : U \rightarrow \mathbb{R}^m$, where $U \subset \mathbb{R}^n$ is open, with respect to the i -th variable is denoted by $\partial_i v$, and gradients by ∇v . If v depends only on one real variable, meaning if $n = 1$, we simplify $\partial_1 v$ to v' . The rescaled gradient of $v : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is given by

$$\nabla^\varepsilon v := \left(\partial_1 v \Big|_{\frac{1}{\varepsilon}} \partial_2 v \Big|_{\frac{1}{\varepsilon}} \partial_3 v \right). \quad (2.6)$$

We employ standard notation for Lebesgue and Sobolev spaces as well as for spaces of continuous and k -times continuously differentiable functions; in particular, $L^p(U; \mathbb{R}^m)$, $W^{1,p}(U; \mathbb{R}^m)$ and $C^k(\overline{U}; \mathbb{R}^m)$ with $k \in \mathbb{N}_0$ for an open set $U \subset \mathbb{R}^n$. We shall equip the latter with the norm

$$\|u\|_{C^k(\overline{U}; \mathbb{R}^m)} := \sum_{i=0}^k \max_{x \in \overline{U}} |\nabla^i u(x)|.$$

To shorten notation, let $L^p(a, b; \mathbb{R}^m) := L^p((a, b); \mathbb{R}^m)$ and $W^{1,p}(a, b; \mathbb{R}^m) := W^{1,p}((a, b); \mathbb{R}^m)$ if $U = (a, b) \subset \mathbb{R}$.

Furthermore, $A_{\text{pw}}(I; \mathbb{R}^m)$ with an open interval $I \subset \mathbb{R}$ is defined as the space of continuous piecewise affine functions with values in \mathbb{R}^m .

If $\Omega = (0, L) \times \omega$ as in the introduction, without explicit mention, we identify any $u : (0, L) \rightarrow \mathbb{R}^3$ with its trivial extension to a function on Ω ; in particular, given sufficient regularity, $\partial_1 u$ and u' are used interchangeably.

Finally, $\mathcal{O}(\cdot)$ is the well-known Landau symbol.

2.2.2 Hypotheses and properties of W_0 and \overline{W}

Consider the following regularity and growth assumptions for the energy density $W_0 : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$:

(H1) W_0 is continuous on $\mathbb{R}^{3 \times 3}$;

(H2) there are constants $C_2, c_2 > 0$ such that

$$c_2|F|^p - C_2 \leq W_0(F) \leq C_2(|F|^p + 1) \quad \text{for all } F \in \mathbb{R}^{3 \times 3};$$

(H3) there are constants $C_3, c_3 > 0$ such that

$$c_3 \operatorname{dist}^p(F, \operatorname{SO}(3)) \leq W_0(F) \leq C_3 \operatorname{dist}^p(F, \operatorname{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3}.$$

Clearly, (H3) implies (H2). Notice that hypotheses (H1) and (H2), which appear in our proofs for the regimes $\alpha \in [0, \frac{1}{2}]$ (cf. Theorems 2.7 and 2.8), are standard assumptions for energy densities - possibly with multi-well character - used in the literature to describe thin elastic objects like strings and membranes (see e.g. [1, 137] for the compressible and [61] for the incompressible setting). For technical reasons, we replace (H2) in the intermediate regimes $\alpha \geq \frac{1}{2}$ (see Theorem 2.8) with the stronger condition (H3). The latter corresponds to a single-well type energy and is common in the theory of rods and plates [98, 156, 184, 62].

Recalling the definition of W in (2.1), we define $\overline{W} : \mathbb{R}^3 \rightarrow [0, \infty]$ by minimizing out the cross-section variables, that is,

$$\overline{W}(\xi) = \inf_{A \in \mathbb{R}^{3 \times 2}} W((\xi|A)) = \begin{cases} \inf_{A \in \mathbb{R}^{3 \times 2}, \det(\xi|A)=1} W_0((\xi|A)) & \text{if } \xi \neq 0, \\ \infty & \text{otherwise,} \end{cases} \quad (2.7)$$

for $\xi \in \mathbb{R}^3$. Notice that the hypotheses (H1) and (H2) guarantee that the infima in (2.7) are attained.

As we detail next, the convexification \overline{W}^c of \overline{W} inherits growth properties from W_0 .

Lemma 2.1. *Let W_0 satisfy (H1) and (H2). There are constants $C_4, c_4 > 0$ such that*

$$c_4|\xi|^p - C_4 \leq \overline{W}^c(\xi) \leq C_4(|\xi|^p + 1) \quad \text{for all } \xi \in \mathbb{R}^3. \quad (2.8)$$

In particular, $\overline{W}^c : \mathbb{R}^3 \rightarrow [0, \infty)$ is continuous as a finite-valued convex function.

Proof. Indeed, (H2), along with the observation that

$$\min_{A \in \mathbb{R}^{3 \times 2}, \det(\xi|A)=1} |(\xi|A)|^p \leq c(p) \min_{x, y \in \mathbb{R}^3, (x \times y) \cdot \xi = 1} |\xi|^p + |x|^p + |y|^p = c(p) (|\xi|^p + 2|\xi|^{-\frac{p}{2}}) \quad (2.9)$$

for any $\xi \in \mathbb{R}^3 \setminus \{0\}$, cf. (2.5), allows us to infer that

$$c_2|\xi|^p - C_2 \leq \overline{W}(\xi) \leq c(p)C_2(|\xi|^p + 2|\xi|^{-\frac{p}{2}}) + C_2 \quad (2.10)$$

for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, which after convexification gives rise to (2.8). \square

The following lemma collects a few basic properties of the zero level sets of \overline{W} and \overline{W}^c .

Remark 2.2. Suppose that W_0 satisfies (H1) and (H2).

a) The growth assumption (H3) implies immediately that $L_0(W) = L_0(W_0) = \text{SO}(3)$.

b) If W_0 is frame-indifferent, i.e., $W_0(RF) = W_0(F)$ for all $F \in \mathbb{R}^{3 \times 3}$ and any $R \in \text{SO}(3)$, then $\overline{W}(\xi)$ with $\xi \in \mathbb{R}^3$ depends de facto only on $|\xi|$, i.e., $W(\xi) = f(|\xi|)$ for $\xi \in \mathbb{R}^3$ with some $f : [0, \infty) \rightarrow \mathbb{R}$. Indeed, following the strategy in [1, Theorem 4.1], we compute for any $R \in \text{SO}(3)$ and $\xi \neq 0$,

$$\begin{aligned} \overline{W}(R\xi) &= \min\{W_0((R\xi|A)) : A \in \mathbb{R}^{3 \times 2}, \det(R\xi|A) = 1\} \\ &= \min\{W_0(R(\xi|A)) : A \in \mathbb{R}^{3 \times 2}, \det(R(\xi|A)) = 1\} \\ &= \min\{W_0((\xi|A)) : A \in \mathbb{R}^{3 \times 2}, \det(\xi|A) = 1\} = \overline{W}(\xi). \end{aligned}$$

In this case, $\overline{W}^c = f^c(|\cdot|)$ and $L_0(\overline{W}^c) = L_0(\overline{W})^c = \{\xi \in \mathbb{R}^3 : |\xi| \leq \max_{t \geq 0, t \in L_0(f)} t\}$.

c) A frame-indifferent single-well energy density W_0 vanishing at the identity has $\text{SO}(3)$ as zero level set of W_0 . Hence, $L_0(\overline{W}) = \{\xi \in \mathbb{R}^3 : |\xi| = 1\}$ in light of b), and after convexification,

$$L_0(\overline{W}^c) = L_0(\overline{W})^c = \{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}.$$

For illustration, we discuss two examples of energy densities with an incompressibility constraint \overline{W} from the mechanics literature. We provide explicit formulas for the minimized and convexified densities \overline{W}^c , which fully determine the models resulting from our 3d-1d dimension reduction analysis.

Example 2.3. a) Consider the stored energy density of an incompressible homogeneous neo-Hookean solid, i.e.,

$$W(F) = \begin{cases} \mu \text{Tr}(F^T F - \text{Id}) & \text{if } \det F = 1, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{3 \times 3},$$

where $\mu > 0$ is a given shear modulus. With $W_0(F) = \mu \text{Tr}(F^T F - \text{Id}) = \mu(|F|^2 - 3)$ for $F \in \mathbb{R}^{3 \times 3}$, which satisfies the assumptions (H1) and (H2) for $p = 2$, this model fits into the framework of this chapter for the scaling regimes $\alpha \in [0, \frac{1}{2})$. An optimization argument as in (2.9) yields that $\overline{W}(\xi) = \mu(|\xi|^2 + 2|\xi|^{-1} - 3)$ for $\xi \in \mathbb{R}$, and thus, by convexification,

$$\overline{W}^c(\xi) = \begin{cases} \mu(|\xi|^2 + 2|\xi|^{-1} - 3) & \text{if } |\xi| > 1, \\ 0 & \text{if } |\xi| \leq 1, \end{cases} \quad \xi \in \mathbb{R}^3.$$

Inserting this expression into the limit functionals \mathcal{I}^0 as in (2.3) and \mathcal{I}^α in (2.4) shows, in particular, that the one-dimensional string models allow for compression (that is, $|u'| \leq 1$ for $u \in W^{1,2}(0, L; \mathbb{R}^3)$) at no energetic cost.

b) A widely used material class where incompressibility constitutes a mechanical key feature are rubber-like ones. Following the phenomenological description of carbon-black-filled rubber introduced by Yeoh in [202, 203] leads us to study the energy density

$$W(F) = \begin{cases} \sum_{k=1}^3 a_k (\text{Tr}(F^T F - \text{Id}))^k & \text{if } \det F = 1, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{3 \times 3},$$

with given material constants $a_1, a_2, a_3 \geq 0$. In analogy to a), the growth conditions (H1) and (H2) hold with $p = 6$, and we conclude in light of the monotonicity of $t \mapsto t^k$ on $[0, \infty)$ for $k \in \mathbb{N}$ that

$$\overline{W}^c(\xi) = \begin{cases} \sum_{k=1}^3 a_k (|\xi|^2 + 2|\xi|^{-1} - 3)^k & \text{if } |\xi| \geq 1, \\ 0 & \text{if } |\xi| < 1, \end{cases} \quad \xi \in \mathbb{R}^3.$$

Here as well, the resulting lower-dimensional limit energies do not penalize compression.

To conclude this subsection, we briefly comment on the connection between models with and without incompressibility.

Remark 2.4. Formally, three-dimensional models for incompressible elastic materials arise through a limit passage from compressible ones with a suitable penalization term that diverges whenever the Jacobian determinant of the deformation field deviates from one.

Let us exemplify this idea in the context of this chapter - on the basis of a simple scenario. Suppose that $\alpha = 0$, $p \geq 3$ and consider a continuous $\psi : \mathbb{R} \rightarrow [0, \infty)$ satisfying $\psi(0) = 0$ as well as standard q -growth and q -coercivity for some $q \in [1, \frac{p}{3}]$. For $k \in \mathbb{N}$, we define

$$\mathcal{I}_{\varepsilon,k}^0(u) = \int_{\Omega} W_k(\nabla^\varepsilon u) \, dx, \quad u \in W^{1,p}(\Omega; \mathbb{R}^3),$$

where

$$W_k(F) := W_0(F) + k\psi(\det F - 1)$$

for $F \in \mathbb{R}^{3 \times 3}$ with $W_0 : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ satisfying (H1) and (H2). Notice that each W_k satisfies all the assumptions (in particular, standard p -growth and p -coercivity) necessary for the classical analysis of compressible elastic strings in [1]. On the other hand, it is straight-forward to verify that the sequence $(\mathcal{I}_{\varepsilon,k}^0)_k$ Γ -converges with respect to the weak topology in $W^{1,p}(\Omega; \mathbb{R}^3)$ to $\mathcal{I}_\varepsilon^0$ defined as in (2.2) for any $\varepsilon > 0$.

2.2.3 Technical tools

The following auxiliary result on inner perturbations is a key ingredient for the construction of recovery sequences, both in the regimes $\alpha = 0$ and $\alpha \in (0, 2)$, since it allows us to modify sequences subject to the incompressibility constraint in an approximate sense into ones that satisfy it exactly.

Analogous techniques applicable to 3d-2d dimension reduction problems were first introduced in [61] and later exploited in [62, 144]. We adapt the method to the 3d-1d context, where perturbing one of the cross-section variables (instead of both) is enough to realize the desired determinant condition.

Lemma 2.5 (Inner perturbation). *Let $\gamma > 0$ and $J \subset J' \subset \mathbb{R}$ be bounded closed intervals such that $0 \in J$ and J is compactly contained in the interior of J' . Further, let $Q_L := [0, L] \times J \times J \subset \mathbb{R}^3$ and analogously, $Q'_L := [0, L] \times J' \times J'$, and define $\Pi_3 : Q_L \rightarrow J$ by $x \mapsto x_3$ for $x \in Q_L$.*

If $(v_\varepsilon)_\varepsilon \subset C^2(Q'_L; \mathbb{R}^3)$ is such that

$$\|\det \nabla^\varepsilon v_\varepsilon - 1\|_{C^1(Q'_L)} = \mathcal{O}(\varepsilon^\gamma), \quad (2.11)$$

then there exists a sequence $(\Phi_\varepsilon)_\varepsilon \subset C^1(Q_L; Q'_L)$ with functions of the form

$$\Phi_\varepsilon(x) = (x_1, x_2, \varphi_\varepsilon(x)), \quad x \in Q_L,$$

where $\varphi_\varepsilon \in C^1(Q_L; J')$ for $\varepsilon > 0$ is such that

$$\|\varphi_\varepsilon - \Pi_3\|_{C^1(Q_L; \mathbb{R}^3)} = \mathcal{O}(\varepsilon^\gamma), \quad (2.12)$$

and the perturbed sequence $(u_\varepsilon)_\varepsilon \subset C^1(Q_L; \mathbb{R}^3)$ defined by $u_\varepsilon := v_\varepsilon \circ \Phi_\varepsilon$ satisfies

$$\det \nabla^\varepsilon u_\varepsilon = 1 \quad \text{everywhere in } Q_L \text{ for } \varepsilon > 0. \quad (2.13)$$

Proof. We subdivide the construction of a sequence $(\varphi_\varepsilon)_\varepsilon \in C^1(Q_L; J')$ satisfying the conditions (2.12) and (2.13) into two steps. The arguments are strongly inspired by the ideas and techniques of [61, 62].

Note that according to (2.11), there is a constant $l > 0$ such that

$$\|\det \nabla^\varepsilon v_\varepsilon\|_{C^0(Q'_L)} \geq l \quad (2.14)$$

for all $\varepsilon > 0$ sufficiently small.

Step 1: Implementation of the determinant constraint. Recalling the definition of the rescaled gradients in (2.6), we deduce from the chain rule that $u_\varepsilon = v_\varepsilon \circ \Phi_\varepsilon$ satisfies

$$\begin{aligned} \det \nabla^\varepsilon u_\varepsilon &= \det(\nabla v_\varepsilon \circ \Phi_\varepsilon) \det \nabla^\varepsilon \Phi_\varepsilon = \frac{1}{\varepsilon^2} \det(\nabla v_\varepsilon \circ \Phi_\varepsilon) \det \nabla \Phi_\varepsilon \\ &= \det(\nabla^\varepsilon v_\varepsilon \circ \Phi_\varepsilon) \det \nabla \Phi_\varepsilon = \det(\nabla^\varepsilon v_\varepsilon \circ \Phi_\varepsilon) \partial_3 \varphi_\varepsilon \end{aligned}$$

on Q_L . Hence, condition (2.13) is fulfilled if φ_ε solves the following initial value problem: For each $x_1 \in [0, L]$ and $x_2 \in J$,

$$\begin{cases} \partial_3 \varphi_\varepsilon(x_1, x_2, x_3) &= \frac{1}{\det \nabla^\varepsilon v_\varepsilon(x_1, x_2, \varphi_\varepsilon(x_1, x_2, x_3))} \quad \text{for } x_3 \in J, \\ \varphi_\varepsilon(x_1, x_2, 0) &= 0; \end{cases} \quad (2.15)$$

notice that in view of (2.14), the denominator on the right-hand of the differential equation is in particular non-zero; also, the choice of initial conditions is indeed admissible, considering that $0 \in J$.

The existence of a unique solution $\varphi_\varepsilon \in C^1(Q_L; J')$ to the initial value problem in (2.15) with continuously differentiable dependence on the parameters x_1 and x_2 follows from standard ODE theory. More precisely, the argument is based on Banach's fixed point theorem, see e.g. [200, III, §13, Satz II and IV]; note that in our case, the contraction may be defined on $C^0(Q_L; J')$, since (2.11) implies that for any $\phi \in C^0(Q_L; J')$ and $x \in Q_L$,

$$\int_0^{x_3} \frac{1}{\det \nabla^\varepsilon v_\varepsilon(x_1, x_2, \phi(x_1, x_2, s))} ds \in J',$$

provided ε is small enough.

Step 2: Estimates for φ_ε . To verify (2.12) for the previously constructed φ_ε , we are going to show that

$$\|\partial_3 \varphi_\varepsilon - 1\|_{C^0(Q_L)} = \mathcal{O}(\varepsilon^\gamma), \quad (2.16)$$

$$\|\varphi_\varepsilon - \Pi_3\|_{C^0(Q_L)} = \mathcal{O}(\varepsilon^\gamma), \quad (2.17)$$

$$\|\partial_1 \varphi_\varepsilon\|_{C^0(Q_L)} = \mathcal{O}(\varepsilon^\gamma) \quad \text{and} \quad \|\partial_2 \varphi_\varepsilon\|_{C^0(Q_L)} = \mathcal{O}(\varepsilon^\gamma). \quad (2.18)$$

Indeed, (2.16) follows from

$$|\partial_3 \varphi_\varepsilon - 1| = \frac{|\det(\nabla^\varepsilon v_\varepsilon \circ \Phi_\varepsilon) - 1|}{|\det(\nabla^\varepsilon v_\varepsilon \circ \Phi_\varepsilon)|} \leq \frac{1}{l} |\det(\nabla^\varepsilon v_\varepsilon \circ \Phi_\varepsilon) - 1| \leq \frac{1}{l} \|\det \nabla^\varepsilon v_\varepsilon - 1\|_{C^0(Q_L)}$$

in combination with (2.11), and it suffices for (2.17) to observe that

$$|\varphi_\varepsilon(x) - x_3| \leq \left| \int_0^{x_3} \partial_3 \varphi_\varepsilon(x_1, x_2, s) - 1 ds \right| \leq |x_3| \|\partial_3 \varphi_\varepsilon - 1\|_{C^0(Q_L)} \quad (2.19)$$

for any $x \in Q_L$.

For the proof of (2.18), it is convenient to rewrite (2.15) equivalently in terms of the integral equation

$$\int_0^{\varphi_\varepsilon(x)} \det \nabla^\varepsilon v_\varepsilon(x_1, x_2, s) \, ds = x_3 \quad \text{for } x \in Q_L. \quad (2.20)$$

By the Leibniz integral rule, differentiating (2.20) with respect to x_1 leads to

$$\int_0^{\varphi_\varepsilon(x)} \partial_1 [\det \nabla^\varepsilon v_\varepsilon(x_1, x_2, s)] \, ds + \partial_1 \varphi_\varepsilon(x) \det \nabla^\varepsilon v_\varepsilon(x_1, x_2, \varphi_\varepsilon(x)) = 0$$

for $x \in Q_L$, and hence, along with (2.14),

$$|\partial_1 \varphi_\varepsilon(x)| \leq \frac{1}{l} \left| \int_0^{\varphi_\varepsilon(x)} \partial_1 \det \nabla^\varepsilon v_\varepsilon(x_1, x_2, s) \, ds \right| \leq \frac{|\varphi_\varepsilon(x)|}{l} \|\nabla(\det \nabla^\varepsilon v_\varepsilon)\|_{C^0(Q_L)}.$$

In view of (2.19) and (2.11), this gives the first part of (2.18). The second part involving $\partial_2 \varphi_\varepsilon$ follows in the same way. \square

The next lemma provides a technical tool that, intuitively speaking, allows us to round off the corners of a piecewise affine curve in such a way that the resulting mollification is a regular curve and still piecewise affine on most of its domain. This can be achieved with the help of Bézier curves, see e.g. [93]. Although there is a substantial literature on the subject, we have not been able to track down the specific statement needed for the construction of a recovery sequence in Theorem 2.7. We present a self-contained proof in Section 2.5.

Lemma 2.6 (Mollification via Bézier curves). *Let $k \in \mathbb{N}$, $q \geq 1$ and $u \in A_{\text{pw}}(0, L; \mathbb{R}^3)$. Further, let Γ be the finite set of points in $(0, L)$ where u is not differentiable, and suppose that $u' \neq 0$ a.e. in $(0, L)$.*

Then there exists a sequence $(u_i)_i \subset C^k([0, L]; \mathbb{R}^3)$ with the following three properties:

- (i) (Uniform bounds) $c \leq |u'_i| \leq C$ in $[0, L]$ for all $i \in \mathbb{N}$ with constants $c, C > 0$ depending only on u ;
- (ii) (Affinity) $u_i = u$ on $[0, L] \setminus \Gamma_i$ with $\Gamma_i = \{t \in [0, L] : \text{dist}(t, \Gamma) \leq \frac{1}{i}\}$ for $i \in \mathbb{N}$ large enough;
- (iii) (Convergence) $u_i \rightarrow u$ in $W^{1,q}(0, L; \mathbb{R}^3)$ as $i \rightarrow \infty$.

2.3 The regime $\alpha = 0$

The first main result derives an effective one-dimensional model for incompressible elastic strings via a Γ -convergence analysis of the energies $\mathcal{I}_\varepsilon^\alpha$ with $\alpha = 0$ in the limit of vanishing ε . Its two main characteristics reflect an optimization over all deformations of the cross section at finite thickness and a relaxation procedure minimizing the energy over possible microstructures.

Theorem 2.7. *For $\varepsilon > 0$, let $\mathcal{I}_\varepsilon^0$ as in (2.2), assuming that W_0 satisfies (H1) and (H2). Then, $(\mathcal{I}_\varepsilon^0)_\varepsilon$ Γ -converges sequentially regarding the weak topology in $W^{1,p}(\Omega; \mathbb{R}^3)$ to*

$$\mathcal{I}^0 : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty], \quad u \mapsto \begin{cases} |\omega| \int_0^L \overline{W}^c(u') \, dx_1 & \text{if } u \in W^{1,p}(0, L; \mathbb{R}^3), \\ \infty & \text{otherwise,} \end{cases}$$

with $\overline{W} : \mathbb{R}^3 \rightarrow [0, \infty)$ defined as in (2.7).

Furthermore, every sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ with $\int_\Omega u_\varepsilon \, dx = 0$ and $\sup_{\varepsilon > 0} \mathcal{I}_\varepsilon^0(u_\varepsilon) < \infty$ is relatively weakly compact.

Proof. The overall idea of the proof follows along the lines of [61], but the arguments need to be suitably modified for this setting of 3d-1d reduction. This involves a tailored mollification and frame construction for piecewise affine regular curves, as well as elements from [1], see also [184]. The key ingredient for realizing the volumetric constraint in the construction of the recovery sequence is Lemma 2.5.

Part I: Lower bound and compactness. Let $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ be a sequence of functions with zero mean value and uniformly bounded energy regarding $(\mathcal{I}_\varepsilon^0)_\varepsilon$. Due to $W \geq W_0$ and the fact that, by assumption, W_0 satisfies the necessary properties to apply the compactness of the corresponding compressible problem (see e.g. [1, Theorem 2.1]), we conclude the existence of a subsequence of $(u_\varepsilon)_\varepsilon$ converging weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$ to some $u \in W^{1,p}(0, L; \mathbb{R}^3)$. Since \overline{W}^c is convex, continuous and bounded from below, the functional

$$L^p(\Omega; \mathbb{R}^3) \ni v \mapsto \int_{\Omega} \overline{W}^c(v) \, dx$$

is L^p -weakly lower semi-continuous (see e.g. [70, Theorem 1.3]), and hence,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^0(u_\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} W(\nabla^\varepsilon u_\varepsilon) \, dx \geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} \overline{W}^c(\partial_1 u_\varepsilon) \, dx \\ &\geq \int_{\Omega} \overline{W}^c(\partial_1 u) \, dx = |\omega| \int_0^L \overline{W}^c(u') \, dx_1, \end{aligned}$$

which is the desired liminf-inequality.

Part II: Upper bound. Let $u \in W^{1,p}(0, L; \mathbb{R}^3)$. For easier reading, we subdivide the argument into several steps.

Step 1: Affine approximation. Considering that $A_{\text{pw}}(0, L; \mathbb{R}^3)$ is dense in $W^{1,p}(0, L; \mathbb{R}^3)$, one can find a sequence $(\tilde{v}_j)_j \subset A_{\text{pw}}(0, L; \mathbb{R}^3)$ such that

$$\tilde{v}_j \rightarrow u \quad \text{in } W^{1,p}(0, L; \mathbb{R}^3). \quad (2.21)$$

Then, in light of the continuity of \overline{W}^c and its p -growth (2.8), we can pass to the limit via the Vitali-Lebesgue convergence theorem to obtain

$$\lim_{j \rightarrow \infty} \int_0^L \overline{W}^c(\tilde{v}_j') \, dx_1 = \int_0^L \overline{W}^c(u') \, dx_1. \quad (2.22)$$

Step 2: Relaxation. Next, we will construct a sequence $(v_j)_j \subset A_{\text{pw}}(0, L; \mathbb{R}^3)$ such that $v_j \rightharpoonup u$ in $W^{1,p}(0, L; \mathbb{R}^3)$ and

$$\lim_{j \rightarrow \infty} \int_0^L \overline{W}(v_j') \, dx_1 \leq \int_0^L \overline{W}^c(u') \, dx_1; \quad (2.23)$$

in particular, this means that $v_j' \neq 0$ a.e. in $(0, L)$ for $j \in \mathbb{N}$.

For $j \in \mathbb{N}$, let \tilde{v}_j be the function from Step 1, and denote the finitely many disjoint open subintervals of $(0, L)$ on which \tilde{v}_j' is constant by $\tilde{I}_j^{(n)}$ with $n = 1, \dots, \tilde{N}_j$; notice that

$$|(0, L) \setminus \bigcup_{n=1}^{\tilde{N}_j} \tilde{I}_j^{(n)}| = 0.$$

The idea is to modify \tilde{v}_j suitably on each $\tilde{I}_j^{(n)}$. To be precise, by classical arguments in convex analysis and relaxation theory (see e.g. [70, Theorem 2.35]), one can find functions $\phi_j^{(n)} \in W_0^{1,p}(\tilde{I}_j^{(n)}; \mathbb{R}^3) \cap A_{\text{pw}}(0, L; \mathbb{R}^3)$, such that

$$\int_{\tilde{I}_j^{(n)}} \overline{W}(\tilde{v}_j' + (\phi_j^{(n)})') \, dx_1 \leq \overline{W}^c(\tilde{v}_j') + \frac{1}{j}$$

and

$$\int_{\tilde{I}_j^{(n)}} |\phi_j^{(n)}|^p \, dx_1 \leq \frac{1}{j^p} |\tilde{I}_j^{(n)}|; \quad (2.24)$$

the add-on (2.24) follows via a simple refinement argument, just replace $\phi_j^{(n)}$ with a piecewise affine function that consists of multiple scaled copies of the latter.

Define $v_j := \tilde{v}_j + \sum_{n=1}^{\tilde{N}_j} \phi_j^{(n)} \mathbb{1}_{\tilde{I}_j^{(n)}}$, where $\mathbb{1}_I$ for some $I \subset \mathbb{R}$ denotes the associated indicator function with values 0 and 1. Then, $\|v_j - \tilde{v}_j\|_{L^p(0, L; \mathbb{R}^3)} \leq \frac{\sqrt[p]{L}}{j}$ and

$$\begin{aligned} \int_0^L \overline{W}(v_j') \, dx_1 &= \sum_{n=1}^{\tilde{N}_j} \int_{\tilde{I}_j^{(n)}} \overline{W}(\tilde{v}_j' + (\phi_j^{(n)})') \, dx_1 \\ &\leq \sum_{n=1}^{\tilde{N}_j} (\overline{W}^c(\tilde{v}_j') + \frac{1}{j}) |\tilde{I}_j^{(n)}| = \int_0^L \overline{W}^c(\tilde{v}_j') \, dx_1 + \frac{L}{j}. \end{aligned}$$

Hence, letting $j \rightarrow \infty$ implies (2.23) in view of (2.22), as well as $v_j \rightarrow u$ in $L^p(0, L; \mathbb{R}^3)$ due to (2.21). The observation that $(v_j)_j$ is uniformly bounded in $W^{1,p}(0, L; \mathbb{R}^3)$ as a consequence of the lower bound in (2.10) allows us to conclude the desired weak convergence $v_j \rightharpoonup u$ in $W^{1,p}(0, L; \mathbb{R}^3)$.

Step 3: Mollification of the piecewise affine approximations. With the help of Lemma 2.6 applied to each v_j from Step 2 with $q = p$ and $k = 3$, and a diagonalization argument, we obtain a sequence of functions $(u_j)_j \subset C^3([0, L]; \mathbb{R}^3)$ with these properties:

- (i) for every $j \in \mathbb{N}$ there are $0 < l_j \leq L_j$ such that $l_j \leq |u_j'| \leq L_j$ in $[0, L]$; without loss of generality, we may assume that $l_j \leq 1$;
- (ii) for every $j \in \mathbb{N}$ there are finitely many disjoint open intervals $I_j^{(n)} \subset [0, L]$ with $n = 1, \dots, N_j$ such that the restriction of u_j to $I_j^{(n)}$ is affine and coincides with $v_j|_{I_j^{(n)}}$;
- (iii) $\lim_{j \rightarrow \infty} |\Gamma_j| (L_j^p + l_j^{-\frac{p}{2}} + 1) = 0$, where $\Gamma_j := [0, L] \setminus \bigcup_{n=1}^{N_j} I_j^{(n)}$ for $j \in \mathbb{N}$;
- (iv) $u_j - v_j \rightarrow 0$ in $W^{1,p}(0, L; \mathbb{R}^3)$, and hence by Step 2,

$$u_j \rightharpoonup u \quad \text{in } W^{1,p}(0, L; \mathbb{R}^3). \quad (2.25)$$

Step 4: Taylored frame. For any curve u_j with $j \in \mathbb{N}$ as in the previous step, let $n_j \in C^2([0, L]; \mathbb{R}^3)$ be a normal unit vector field along u_j , meaning $u_j' \cdot n_j = 0$ and $|n_j| = 1$ everywhere in $[0, L]$; we may assume without restriction that n_j is constant whenever u_j' is. Moreover, define

$$b_j := \frac{u_j' \times n_j}{|u_j' \times n_j|^2} \in C^2([0, L]; \mathbb{R}^3); \quad (2.26)$$

indeed, the denominator in (2.26) is non-zero, because n_j is orthogonal to u'_j and u_j a regular curve by Step 3 (i). By definition, the triple (u'_j, n_j, b_j) forms an orthogonal moving frame along the trajectory given by u_j . Our aim in this step is to modify this moving frame into a version that is well-suited for the construction of an approximating sequence for u along which the energies converge as well, cf. Step 5.

To this end, recall that u_j is affine on each $I_j^{(n)}$ with $n = 1, \dots, N_j$ according to Step 3 (ii), that is,

$$u'_j|_{I_j^{(n)}} = \xi_j^{(n)} \quad \text{with } \xi_j^{(n)} \in \mathbb{R}^3. \quad (2.27)$$

Let $A_j^{(n)} \in \mathbb{R}^{3 \times 2}$ be such that

$$W((\xi_j^{(n)} | A_j^{(n)})) = \min_{A \in \mathbb{R}^{3 \times 2}} W((\xi_j^{(n)} | A)) = \overline{W}(\xi_j^{(n)}); \quad (2.28)$$

in particular, $\det(\xi_j^{(n)} | A_j^{(n)}) = 1$, cf. (2.7).

Moreover, for fixed $\delta, \eta > 0$ sufficiently small, we consider compactly contained nested open subintervals $I_{j,\delta,\eta}^{(n)} \subset I_{j,\delta}^{(n)} \subset I_j^{(n)}$ with

$$|I_j^{(n)} \setminus I_{j,\delta}^{(n)}| \leq \delta \quad \text{and} \quad |I_{j,\delta,\eta}^{(n)} \setminus I_{j,\delta}^{(n)}| \leq \eta. \quad (2.29)$$

Based on these definitions, we find $\bar{n}_{j,\delta} \in C^2([0, L]; \mathbb{R}^3)$ with the properties that

$$\bar{n}_{j,\delta}|_{\Gamma_j} = n_j \quad \text{and} \quad \bar{n}_{j,\delta}|_{I_{j,\delta}^{(n)}} = A_j^{(n)} e_1 \quad \text{for } n = 1, \dots, N_j,$$

as well as

$$|\bar{n}_{j,\delta}| < R_j \quad \text{and} \quad |u'_j \times \bar{n}_{j,\delta}| > r_j \quad \text{in } [0, L], \quad (2.30)$$

where $R_j := 2 \max\{1, \max_{j=1, \dots, N_j} |A_j^{(n)} e_1|\}$ and

$$r_j := \frac{1}{2} \min\{l_j, \min_{n=1, \dots, N_j} |\xi_j^{(n)} \times A_j^{(n)} e_1|\} > 0.$$

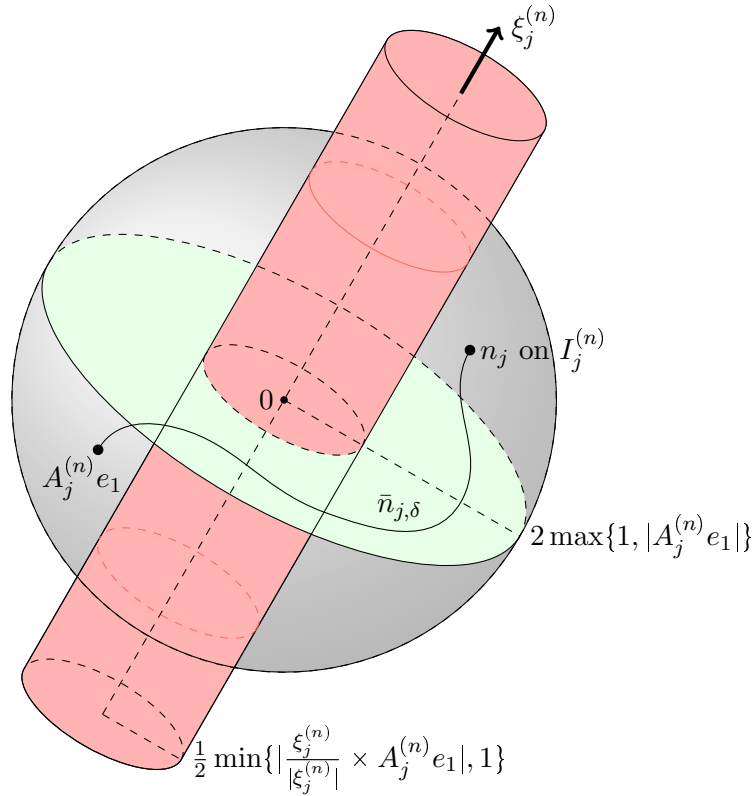
Geometrically speaking, we choose the free curve segments of $\bar{n}_{j,\delta}$ on $I_j^{(n)} \setminus I_{j,\delta}^{(n)}$ in such a way that $\bar{n}_{j,\delta}$ lies within the ball around the origin of radius $2 \max\{1, |A_j^{(n)} e_1|\}$, but in the complement of the cylinder centered in the origin with axis pointing in the direction of $\xi_j^{(n)}$ and circular cross section of radius $\frac{1}{2} \min\{|\frac{\xi_j^{(n)}}{|\xi_j^{(n)}|} \times A_j^{(n)} e_1|, 1\}$. Such a choice is possible, because the selected two radii guarantee that both the value of n_j on $I_j^{(n)}$ and $A_j^{(n)} e_1$ are contained in the specified path-connected region, see Figure 2.1 for illustration.

As a customized replacement for b_j from (2.26), we introduce $\bar{b}_{j,\delta,\eta} \in C^2([0, L]; \mathbb{R}^3)$ given by

$$\bar{b}_{j,\delta,\eta} = \bar{b}_{j,\delta} + \sum_{n=1}^{N_j} \psi_{j,\delta,\eta}^{(n)} (A_j^{(n)} e_2 - \hat{b}_j^{(n)})$$

where $\psi_{j,\delta,\eta}^{(n)} : [0, L] \rightarrow [0, 1]$ are smooth cut-off functions with compact support in $I_{j,\delta}^{(n)}$ satisfying $\psi_{j,\delta,\eta}^{(n)} = 1$ on $I_{j,\delta,\eta}^{(n)}$,

$$\bar{b}_{j,\delta} := \frac{u'_j \times \bar{n}_{j,\delta}}{|u'_j \times \bar{n}_{j,\delta}|^2} \quad \text{and} \quad \hat{b}_j^{(n)} := \frac{\xi_j^{(n)} \times A_j^{(n)} e_1}{|\xi_j^{(n)} \times A_j^{(n)} e_1|^2};$$

Figure 2.1: Sketch of the construction of $\bar{n}_{j,\delta}$ on $I_j^{(n)} \setminus I_{j,\delta}^{(n)}$.

we remark that the last two quantities are well-defined due to (2.30) and the fact that

$$\det(\xi_j^{(n)} | A_j^{(n)}) = (\xi_j^{(n)} \times A_j^{(n)} e_1) \cdot A_j^{(n)} e_2 \neq 0.$$

Next, we collect a few useful properties of the newly constructed moving frames $(u'_j, \bar{n}_{j,\delta}, \bar{b}_{j,\delta,\eta})$. Setting

$$F_{j,\delta,\eta} := (\bar{n}_{j,\delta} | \bar{b}_{j,\delta,\eta}) \in C^2([0, L]; \mathbb{R}^{3 \times 2}),$$

we observe that

$$\begin{aligned} \det(u'_j | F_{j,\delta,\eta}) &= \det(u'_j | \bar{n}_{j,\delta} | \bar{b}_{j,\delta}) + \sum_{n=1}^{N_j} \psi_{j,\delta,\eta}^{(n)} [\det(\xi_j^{(n)} | A_j^{(n)}) - \det(\xi_j^{(n)} | A_j^{(n)} e_1 | \hat{b}_j^{(n)})] \\ &= \det(u'_j | \bar{n}_{j,\delta} | \bar{b}_{j,\delta}) = 1, \end{aligned} \quad (2.31)$$

and, in view of (2.5),

$$|F_{j,\delta,\eta}|^p = c(p)(|\bar{n}_{j,\delta}|^p + |\bar{b}_{j,\delta,\eta}|^p) \leq c(p)C(L_j^p + l_j^{-\frac{p}{2}} + 1) \quad (2.32)$$

with a constant $C > 0$ independent of j, δ and η ; see again Step 3, where the constants L_j and l_j have been introduced. Indeed, to see (2.32), we infer from (2.30) together with the estimate

$$|\xi_j^{(n)} \times A_j^{(n)} e_1| |A_j^{(n)} e_2| \geq |\det(\xi_j | A_j^{(n)} e_1 | A_j^{(n)} e_2)| = 1 \quad (2.33)$$

that

$$|\bar{n}_{j,\delta}| \leq R_j \leq 2(1 + \max_{n=1,\dots,N_j} |A_j^{(n)} e_1|), \quad (2.34)$$

and

$$\begin{aligned} |\bar{b}_{j,\delta,\eta}| &\leq |u'_j \times \bar{n}_{j,\delta}|^{-1} + \max_{n=1,\dots,N_j} (|A_j^{(n)} e_2| + |\xi_j^{(n)} \times A_j^{(n)} e_1|^{-1}) \leq r_j^{-1} + 2 \max_{n=1,\dots,N_j} |A_j^{(n)} e_2| \\ &\leq 2(l_j^{-1} + 2 \max_{n=1,\dots,N_j} |A_j^{(n)} e_2|); \end{aligned}$$

in the last inequality, we have used in particular that

$$r_j \geq \frac{1}{2} \min\{l_j, \min_{n=1,\dots,N_j} |A_j^{(n)} e_2|^{-1}\} = \frac{1}{2} \min\{l_j, (\max_{n=1,\dots,N_j} |A_j^{(n)} e_2|)^{-1}\}$$

due to (2.33).

Due to (2.28) and the growth properties of \bar{W} and W_0 from (2.10) and (H2), there exists a constant $C > 0$ such that

$$|A_j^{(n)}|^p \leq C(|\xi_j^{(n)}|^p + |\xi_j^{(n)}|^{-\frac{p}{2}} + 1) \quad (2.35)$$

for all $n = 1, \dots, N_j$ and $j \in \mathbb{N}$. Hence, in view of (2.27) and Step 3(i), combining (2.35) with (2.33) and (2.34) eventually implies (2.32).

Step 5: Recovery sequence. Let $J \subset J' \subset \mathbb{R}$ as in Lemma 2.5 and $\bar{\omega} \subset J \times J$. We start by considering for fixed $j \in \mathbb{N}$ and $\delta, \eta > 0$ the auxiliary sequence $(v_{j,\delta,\eta,\varepsilon})_\varepsilon \subset C^2(Q'_L; \mathbb{R}^3)$ given by

$$v_{j,\delta,\eta,\varepsilon}(x) := u_j(x_1) + \varepsilon x_2 \bar{n}_{j,\delta}(x_1) + \varepsilon x_3 \bar{b}_{j,\delta,\eta}(x_1) \quad \text{for } x \in Q'_L, \quad (2.36)$$

cf. e.g. [1, Proposition 3.3]. Clearly,

$$\|v_{j,\delta,\eta,\varepsilon} - u_j\|_{C^1(\bar{\Omega}; \mathbb{R}^3)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (2.37)$$

Using (2.31) and the uniform boundedness of $\bar{n}_{j,\delta}$, $\bar{b}_{j,\delta,\eta}$ and their derivatives uniformly in $[0, L]$, we obtain for the following terms involving the rescaled gradients of $v_{j,\delta,\eta,\varepsilon}$ that

$$\|\det \nabla^\varepsilon v_{j,\delta,\eta,\varepsilon} - 1\|_{C^1(\bar{\Omega})} = \mathcal{O}(\varepsilon), \quad (2.38)$$

and

$$\|W_0(\nabla^\varepsilon v_{j,\delta,\eta,\varepsilon}) - W_0((u'_j|F_{j,\delta,\eta}))\|_{C^0(\bar{\Omega})} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (2.39)$$

As a consequence of the choices in Step 4, we find that $(u'_j|F_{j,\delta,\eta}) = (\xi_j^{(n)}|A_j^{(n)})$ on $I_{j,\delta,\eta}^{(n)}$ for all $n = 1, \dots, N_j$ and $\varepsilon > 0$. Hence, along with (2.5), (2.7), (2.31), (2.28) and (H2),

$$\begin{aligned} \int_{\Omega} W_0((u'_j|F_{j,\delta,\eta})) \, dx &\leq |\omega| \sum_{n=1}^{N_j} \bar{W}(\xi_j^{(n)}) |I_{j,\delta,\eta}^{(n)}| + c(p) C_2 |\omega| \int_{[0,L] \setminus \bigcup_{n=1}^{N_j} I_{j,\delta,\eta}^{(n)}} |u'_j|^p + |F_{j,\delta,\eta}|^p + 1 \, dx_1 \\ &\leq |\omega| \int_0^L \bar{W}(v'_j) \, dx_1 + C |\omega| (|\Gamma_j| + N_j \delta \eta) (L_j^p + l_j^{-\frac{p}{2}} + 1) \end{aligned} \quad (2.40)$$

with $C > 0$ dependent on p , but independent of j, δ, η and ε ; in the last estimate, we have exploited (2.27) in combination with Step 3(ii), as well as (2.32) and (2.29).

What prevents a suitably diagonalized version of $(v_{j,\delta,\eta,\varepsilon})_\varepsilon$ from being a valid recovery sequence is its failure to satisfy the incompressibility constraint. This issue can be overcome by modifying the sequence according to Lemma 2.5, which is applicable due to (2.38). Precisely, one obtains $(u_{j,\delta,\eta,\varepsilon})_\varepsilon \subset C^1(\bar{\Omega}; \mathbb{R}^3)$ such that $\det \nabla^\varepsilon u_{j,\delta,\eta,\varepsilon} = 1$ for every $\varepsilon > 0$ and

$$\|u_{j,\delta,\eta,\varepsilon} - v_{j,\delta,\eta,\varepsilon}\|_{C^1(\bar{\Omega}; \mathbb{R}^3)} = \mathcal{O}(\varepsilon^2); \quad (2.41)$$

the latter follows from (2.12) in combination with the special structure of $v_{j,\delta,\eta,\varepsilon}$ in (2.36).

Hence, $u_{j,\delta,\eta,\varepsilon} \rightarrow u_j$ uniformly on $\bar{\Omega}$ and

$$\mathcal{I}_\varepsilon^0(u_{j,\delta,\eta,\varepsilon}) - \int_{\Omega} W_0(\nabla^\varepsilon v_{j,\delta,\eta,\varepsilon}) \, dx = \int_{\Omega} W_0(\nabla^\varepsilon u_{j,\delta,\eta,\varepsilon}) \, dx - \int_{\Omega} W_0(\nabla^\varepsilon v_{j,\delta,\eta,\varepsilon}) \, dx \rightarrow 0 \quad (2.42)$$

as $\varepsilon \rightarrow 0$.

Joining (2.39) and (2.40) with (2.42), under consideration of (2.23), (2.29) and Step 3 (iii), gives

$$\limsup_{j \rightarrow \infty} \limsup_{\delta \rightarrow 0} \limsup_{\eta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^0(u_{j,\delta,\eta,\varepsilon}) \leq |\omega| \int_0^L \overline{W}^c(u') \, dx_1 = \mathcal{I}^0(u).$$

Together with (2.25), (2.37) and (2.41), we can finally extract a diagonal sequence $(u_\varepsilon)_\varepsilon$ in the sense of Attouch [9, Lemma 1.15, Corollary 1.16] such that

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^0(u_\varepsilon) \leq \mathcal{I}^0(u)$$

and $u_\varepsilon \rightarrow u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$, which concludes the proof. \square

2.4 The regime $0 < \alpha < 2$

In the intermediate scaling regime $\alpha \in (0, 2)$, all admissible deformations for the one-dimensional limit model can be realized with zero energy, as our next theorem shows.

Theorem 2.8. *For $\varepsilon > 0$ and $p > \max\{1, \alpha\}$, let $\mathcal{I}_\varepsilon^\alpha$ with $0 < \alpha < 2$ as in (2.2) such that W_0 satisfies (H1), as well as (H2) if $\alpha < \frac{1}{2}$, and (H3) if $\alpha \geq \frac{1}{2}$.*

Then, $(\mathcal{I}_\varepsilon^\alpha)_\varepsilon$ Γ -converges sequentially with respect to the weak topology in $W^{1,p}(\Omega; \mathbb{R}^3)$ to

$$\mathcal{I}^\alpha : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, \infty], \quad u \mapsto \begin{cases} 0 & \text{if } u \in W^{1,p}(0, L; \mathbb{R}^3) \text{ with } u' \in L_0(\overline{W})^c \text{ a.e. in } (0, L), \\ \infty & \text{otherwise,} \end{cases}$$

where $L_0(\overline{W})$ is the zero level set of \overline{W} as in (2.7).

Moreover, any $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ with $\int_{\Omega} u_\varepsilon \, dx = 0$ for all $\varepsilon > 0$ and $\sup_{\varepsilon > 0} \mathcal{I}_\varepsilon^\alpha(u_\varepsilon) < \infty$ is relatively weakly compact.

Remark 2.9. a) Trivially, if the zero level set of \overline{W} is empty, which is the case when $L_0(W_0) \cap \text{Sl}(3) = \emptyset$, the limit functional \mathcal{I}^α takes the value ∞ everywhere.

b) Recall Remark 2.2 c), which shows that if W_0 is frame-indifferent and has a single-well energy at $\text{SO}(3)$, then,

$$\overline{W}^c(\xi) = 0 \text{ if and only if } |\xi| \leq 1.$$

The interpretation of Theorem 2.8 in this case is that no energy is required to compress the one-dimensional limit object. Stretching, on the other hand, has infinite energetic cost and is

therefore forbidden. It is interesting to observe that a comparison with [184, Theorem 4.5], where no incompressibility constraint is imposed, yields no difference for the resulting string models.

c) By a slight adaptation (in fact, a simplification) of the proof below, the statement of Theorem 2.8 remains true if W is replaced with a continuous density W_0 that satisfies the growth assumption (H2) if $\alpha < 1$ and (H3) if $\alpha \geq 1$. This observation allows to weaken the hypotheses on the energy densities in [184, Theorem 4.5], where 3d-1d dimension reduction is performed in the unconstrained case, for $\alpha < 1$; in particular, the result becomes applicable for energies of multi-well type.

Proof of Theorem 2.8. Under consideration of Lemma 2.5, the proof of the upper bound comes down to a modification and generalization of the construction in [184, Theorem 4.5]. The compactness and lower bound follow as an immediate consequence of the respective results for the case $\alpha = 0$.

Part I: Lower bound and compactness. Let $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ be a sequence of functions with vanishing mean value. If $(u_\varepsilon)_\varepsilon$ has uniformly bounded energy, then there is a constant $C > 0$ such that

$$\mathcal{I}_\varepsilon^0(u_\varepsilon) \leq C\varepsilon^\alpha$$

for all $\varepsilon > 0$. By Theorem 2.7, a subsequence of $(u_\varepsilon)_\varepsilon$ (not relabeled) converges weakly to some one-dimensional function $u \in W^{1,p}(0, L; \mathbb{R}^3)$ in $W^{1,p}(\Omega; \mathbb{R}^3)$, and

$$0 = \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^0(u_\varepsilon) \geq |\omega| \int_0^L \overline{W}^c(u') \, dx_1.$$

Thus, $u' \in L_0(\overline{W}^c) = L_0(\overline{W})^c$ in $(0, L)$, which concludes the first part of the proof.

Part II: Upper bound. Let $u \in W^{1,p}(0, L; \mathbb{R}^3)$ and assume that $u' \in L_0(\overline{W})^c$ a.e. in $(0, L)$, otherwise there is nothing to prove. We proceed in two steps.

Step 1: Basic construction for piecewise affine functions. We address first the special case when $u \in A_{\text{pw}}(0, L; \mathbb{R}^3)$ and $u' \in L_0(\overline{W})$ a.e. in $(0, L)$.

Let

$$0 = t^{(0)} < t^{(1)} < \dots < t^{(N-1)} < t^{(N)} = L$$

be a partition of the interval such that the restrictions of u' to the intervals $(t^{(n-1)}, t^{(n)})$ are constant, with values $\xi^{(n)} \in L_0(\overline{W})$, respectively. For any $n = 1, \dots, N$, we select $A^{(n)} \in \mathbb{R}^{3 \times 2}$ to be a solution to the minimization problem in (2.7) defining $\overline{W}(\xi^{(n)})$, so that

$$\det(\xi^{(n)} | A^{(n)}) = 1 \quad \text{and} \quad W((\xi^{(n)} | A^{(n)})) = \overline{W}(\xi^{(n)}) = 0. \quad (2.43)$$

Next, we set $\mathcal{M} := \text{SO}(3)$ if $\alpha \geq \frac{1}{2}$ and $\mathcal{M} := \text{Sl}(3)$ if $\alpha < \frac{1}{2}$, and exploit the fact that \mathcal{M} is a path-connected smooth manifold to obtain

$$P^{(n)} \in C^\infty([0, 1]; \mathcal{M})$$

such that $P^{(n)}(0) = (\xi^{(n)} | A^{(n)})$ and $P^{(n)}(1) = (\xi^{(n+1)} | A^{(n+1)})$ for $n = 1, \dots, N-1$. It is convenient to reparametrize $P^{(n)}$ in the following way:

Fix $0 < \beta < \frac{1}{2}$ (to be specified later) and let $\psi \in C^\infty([0, 1]; [0, 1])$ be a transition function that vanishes in a neighbourhood of 0, takes the value 1 close to 1 and satisfies $|\psi'| \leq 2$. For $\varepsilon > 0$ sufficiently small, we set

$$P_\varepsilon(t) := (P^{(n)} \circ \psi)\left(\frac{t - t^{(n)}}{\varepsilon^\beta}\right) \quad \text{for } t \in [t^{(n)}, t^{(n)} + \varepsilon^\beta]$$

for $n = 1, \dots, N-1$. Regarding the scaling behavior of P_ε and its derivatives one finds that

$$\|P_\varepsilon\|_{C^0(\Gamma_\varepsilon; \mathbb{R}^{3 \times 3})} = \mathcal{O}(1), \quad \|P'_\varepsilon\|_{C^0(\Gamma_\varepsilon; \mathbb{R}^{3 \times 3})} = \mathcal{O}(\varepsilon^{-\beta}) \quad \text{and} \quad \|P''_\varepsilon\|_{C^0(\Gamma_\varepsilon; \mathbb{R}^{3 \times 3})} = \mathcal{O}(\varepsilon^{-2\beta}), \quad (2.44)$$

where $\Gamma_\varepsilon := \bigcup_{n=1}^{N-1} [t^{(n)}, t^{(n)} + \varepsilon^\beta]$.

Now, let $J \subset J' \subset \mathbb{R}$ be closed and bounded intervals as in Lemma 2.5 and $\bar{\omega} \subset J \times J$. With inspiration from [184, Theorem 4.5], we define an auxiliary sequence $(v_\varepsilon)_\varepsilon$ of functions on the cuboid $Q'_L := [0, L] \times J' \times J'$; precisely, for $\varepsilon > 0$ and $x \in Q'_L$,

$$v_\varepsilon(x) = \begin{cases} (\xi^{(1)}|A^{(1)})x_\varepsilon + b_\varepsilon^{(1)} & \text{if } x_1 \in [t^{(0)}, t^{(1)}], \\ \int_{t^{(n)}}^{x_1} P_\varepsilon(t)e_1 dt & \text{if } x_1 \in [t^{(n)}, t^{(n)} + \varepsilon^\beta] \text{ with } n = 1, \dots, N-1, \\ \quad + P_\varepsilon(x_1)(x_\varepsilon - x_1e_1) + d_\varepsilon^{(n)} & \\ (\xi^{(n)}|A^{(n)})x_\varepsilon + b_\varepsilon^{(n)} & \text{if } x_1 \in [t^{(n)} + \varepsilon^\beta, t^{(n+1)}] \text{ with } n = 1, \dots, N-1; \end{cases}$$

here, $x_\varepsilon := (x_1, \varepsilon x_2, \varepsilon x_3)$, and the translation vectors $b_\varepsilon^{(n)}, d_\varepsilon^{(n)} \in \mathbb{R}^3$ are chosen in such a way that v_ε is continuous. It is immediate to see that

$$v_\varepsilon \rightarrow u \quad \text{uniformly in } \bar{\Omega} \text{ as } \varepsilon \rightarrow 0. \quad (2.45)$$

Let us collect some further useful properties of the functions v_ε . In fact, v_ε is not only continuous, but by construction even smooth, so in particular, $v_\varepsilon \in C^2(Q'_L; \mathbb{R}^3)$, and a calculation of the rescaled gradients gives

$$\nabla^\varepsilon v_\varepsilon(x) = \begin{cases} (\xi^{(1)}|A^{(1)}) & \text{if } x_1 \in [t^{(0)}, t^{(1)}] \\ P_\varepsilon(x_1) & \text{if } x_1 \in [t^{(n)}, t^{(n)} + \varepsilon^\beta] \text{ with } n = 1, \dots, N-1, \\ \quad + \varepsilon P'_\varepsilon(x_1)(x_2e_2 + x_3e_3) \otimes e_1 & \\ (\xi^{(n)}|A^{(n)}) & \text{if } x_1 \in [t^{(n)} + \varepsilon^\beta, t^{(n+1)}] \text{ with } n = 1, \dots, N-1. \end{cases}$$

Since $\beta < \frac{1}{2}$, the sequence $(\nabla^\varepsilon v_\varepsilon)_\varepsilon$ is bounded in $C^0(Q'_L; \mathbb{R}^{3 \times 3})$. Moreover, the function v_ε satisfies the incompressibility condition exactly except on sets of small measure, where $\det \nabla^\varepsilon v_\varepsilon$ is close to 1. To quantify this statement, we compute

$$\det \nabla^\varepsilon v_\varepsilon(x) = 1 + \varepsilon \det(P'_\varepsilon(x_1)(x_2e_2 + x_3e_3)|P_\varepsilon(x_1)e_2|P_\varepsilon(x_1)e_3)$$

for $x \in Q_\varepsilon := \Gamma_\varepsilon \times J' \times J'$, and observe that

$$\det \nabla^\varepsilon v_\varepsilon = 1 \quad \text{on } Q'_L \setminus Q_\varepsilon. \quad (2.46)$$

Thus, it follows in view of (2.44) that

$$\|\det \nabla^\varepsilon v_\varepsilon - 1\|_{C^0(Q'_L)} = \mathcal{O}(\varepsilon^{1-\beta}) \quad \text{and} \quad \|\nabla(\det \nabla^\varepsilon v_\varepsilon)\|_{C^0(Q'_L; \mathbb{R}^3)} = \mathcal{O}(\varepsilon^{1-2\beta}). \quad (2.47)$$

Now, with (2.47) at hand, we are in the position to apply Proposition 2.5 to the sequence $(v_\varepsilon)_\varepsilon$ with $\gamma = 1 - 2\beta$ to obtain a modified sequence $(u_\varepsilon)_\varepsilon \subset C^1(\bar{\Omega}; \mathbb{R}^3)$ that satisfies $\det \nabla^\varepsilon u_\varepsilon = 1$ everywhere in Ω , namely

$$u_\varepsilon(x) := v_\varepsilon(x_1, x_2, \varphi_\varepsilon(x)), \quad x \in \Omega,$$

with $\varphi_\varepsilon \in C^1(Q_L; J')$ such that (2.12) holds. Notice that the inner perturbation defining u_ε corresponds to the identity map on $Q_L \setminus Q_\varepsilon$, since, due to (2.46), the ordinary differential equation in (2.15) reduces to $\partial_3 \varphi_\varepsilon = 1$ on this set; thus, along with (2.43),

$$u_\varepsilon = v_\varepsilon \quad \text{and} \quad \nabla^\varepsilon u_\varepsilon = \nabla^\varepsilon v_\varepsilon \in L_0(W) \subset L_0(W_0) \quad \text{on } \Omega \setminus Q_\varepsilon. \quad (2.48)$$

Furthermore, as a consequence of (2.12),

$$\|u_\varepsilon - v_\varepsilon\|_{C^0(\bar{\Omega}; \mathbb{R}^3)} = \mathcal{O}(\varepsilon^{2-2\beta}) \quad \text{and} \quad \|\nabla^\varepsilon u_\varepsilon - \nabla^\varepsilon v_\varepsilon\|_{C^0(\bar{\Omega}; \mathbb{R}^{3 \times 3})} = \mathcal{O}(\varepsilon^{1-2\beta}), \quad (2.49)$$

and therefore, also $(\nabla^\varepsilon u_\varepsilon)_\varepsilon$ is bounded in $C^0(\bar{\Omega}; \mathbb{R}^{3 \times 3})$. Along with (2.45), it follows that

$$u_\varepsilon \rightharpoonup u \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^3).$$

We are now in the position to conclude the proof Step 1 by showing that,

$$\lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^\alpha(u_\varepsilon) = 0 = \mathcal{I}^\alpha(u). \quad (2.50)$$

The two cases $\alpha < \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, call for a different reasoning, which we will detail next.

Step 1a: The case $\alpha \geq \frac{1}{2}$. Let $\beta < \min\{\frac{1}{2}, \frac{p-\alpha}{2p-1}\}$. Then, joining (H3), (2.48), (2.44) and (2.49) with the observations that $|Q_\varepsilon| = \mathcal{O}(\varepsilon^\beta)$, $\det \nabla^\varepsilon u_\varepsilon = 1$, and $P_\varepsilon \in \text{SO}(3)$ pointwise, gives rise to the following estimate,

$$\begin{aligned} \mathcal{I}_\varepsilon^\alpha(u_\varepsilon) &= \frac{1}{\varepsilon^\alpha} \int_\Omega W_0(\nabla^\varepsilon u_\varepsilon) \, dx \leq \frac{C_3}{\varepsilon^\alpha} \int_\Omega \text{dist}^p(\nabla^\varepsilon u_\varepsilon, \text{SO}(3)) \, dx \\ &\leq \frac{2^{p-1}C_3}{\varepsilon^\alpha} \left(\int_{Q_\varepsilon} \text{dist}^p(\nabla^\varepsilon v_\varepsilon, \text{SO}(3)) \, dx + \int_{Q_\varepsilon \cap \Omega} |\nabla^\varepsilon v_\varepsilon - \nabla^\varepsilon u_\varepsilon|^p \, dx \right) \\ &\leq \frac{2^{p-1}C_3}{\varepsilon^\alpha} \left(\varepsilon^p |J|^4 |Q_\varepsilon| \|P'_\varepsilon\|_{C^0(\Gamma_\varepsilon; \mathbb{R}^{3 \times 3})}^p + |Q_\varepsilon| \|\nabla^\varepsilon v_\varepsilon - \nabla^\varepsilon u_\varepsilon\|_{C^0(\bar{\Omega}; \mathbb{R}^{3 \times 3})}^p \right) \\ &= \mathcal{O}(\varepsilon^{p-\alpha-\beta(p-1)}) + \mathcal{O}(\varepsilon^{p-\alpha-\beta(2p-1)}) = \mathcal{O}(\varepsilon^{p-\alpha-\beta(2p-1)}). \end{aligned}$$

The choice of β yields (2.50).

Step 1b: The case $\alpha < \frac{1}{2}$. Let $\alpha < \beta < \frac{1}{2}$. We invoke (2.48) and (H2), as well as $\det \nabla^\varepsilon u_\varepsilon = 1$ in Ω , $|Q_\varepsilon| = \mathcal{O}(\varepsilon^\beta)$, and the uniform boundedness of $\nabla^\varepsilon u_\varepsilon$ to infer that

$$\begin{aligned} \mathcal{I}_\varepsilon^\alpha(u_\varepsilon) &= \frac{1}{\varepsilon^\alpha} \int_\Omega W_0(\nabla^\varepsilon u_\varepsilon) \, dx \leq \frac{1}{\varepsilon^\alpha} \int_{Q_\varepsilon \cap \Omega} W_0(\nabla^\varepsilon u_\varepsilon) \, dx \\ &\leq \frac{C_2}{\varepsilon^\alpha} \int_{Q_\varepsilon \cap \Omega} |\nabla^\varepsilon u_\varepsilon|^p + 1 \, dx \leq C_2 |Q_\varepsilon| \varepsilon^{-\alpha} (\|\nabla^\varepsilon u_\varepsilon\|_{C^0(\bar{\Omega}; \mathbb{R}^{3 \times 3})}^p + 1) = \mathcal{O}(\varepsilon^{\beta-\alpha}). \end{aligned}$$

Step 2: Relaxation and approximation. To address the general case, let $u \in W^{1,p}(0, L; \mathbb{R}^3)$ such that $u' \in L_0(\bar{W})^c$ a.e. in $(0, L)$. By standard tools from convex and asymptotic analysis (cf. e.g. Caratheodory's theorem and the Riemann-Lebesgue lemma), there is a sequence $(u_j)_j \subset A_{\text{pw}}(0, L; \mathbb{R}^3)$ such that $u'_j \in L_0(\bar{W})$ a.e. in $(0, L)$ and

$$u_j \rightharpoonup u \quad \text{in } W^{1,p}(0, L; \mathbb{R}^3).$$

Now, Step 1 applied for each fixed $j \in \mathbb{N}$ provides sequences $(u_{j,\varepsilon})_\varepsilon \subset C^1(\bar{\Omega}; \mathbb{R}^3)$ with the properties that $u_{j,\varepsilon} \rightharpoonup u_j$ in $W^{1,p}(\Omega; \mathbb{R}^3)$ and $\lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^\alpha(u_{j,\varepsilon}) = 0$. Extracting a diagonal sequence $(u_\varepsilon)_\varepsilon$ with the help of a generalized version of Attouch's diagonalization lemma (see e.g. [94, proof of Proposition 1.11 (p. 449)]) finally gives the sought after recovery sequence for u . \square

2.5 Auxiliaries

Proof of Lemma 2.6. The idea of the proof is to mollify u using Bézier curves with sufficiently many control points to ensure the desired C^k -regularity. This way, the mollified curve has derivatives that lie in the convex hull of two neighbouring slopes of u . However, if the latter happen to be anti-parallel, then, by design, the derivative of the mollified curve vanishes at some point. To circumvent this issue, we perturb u via a suitable loop construction.

Since $u \in A_{\text{pw}}(0, L; \mathbb{R}^3)$ is piecewise affine and $u' \neq 0$ almost everywhere, there is a partition $0 =: t^{(0)} < t^{(1)} < \dots < t^{(N-1)} < t^{(N)} := L$ of the interval $[0, L]$ and vectors $\xi^{(n)} \in \mathbb{R}^3 \setminus \{0\}$ such that

$$u' = \xi^{(n)} \quad \text{on } (t^{(n-1)}, t^{(n)}) \quad \text{for } n = 1, \dots, N. \quad (2.51)$$

Step 1: The case without reversions. First, we will prove the statement under the assumption that neither two $\xi^{(n)}$ and $\xi^{(n+1)}$ from (2.51) are anti-parallel. Without loss of generality, it suffices to detail the case $N = 2$, where u' takes only the two values $\xi^{(1)}$ and $\xi^{(2)}$. For general N , one can simply repeat the same construction.

Step 1a: Definition of suitable Bézier curves. For $\eta > 0$ sufficiently small, we choose $2k + 1$ control points around $u(t^{(1)})$ by

$$u(t_{\eta, m}) \quad \text{with } t_{\eta, m} := t^{(1)} - (k - m)\frac{\eta}{k} \quad \text{for } m = 0, \dots, 2k. \quad (2.52)$$

Then,

$$u(t_{\eta, m+1}) - u(t_{\eta, m}) = \begin{cases} \frac{\eta}{k} \xi^{(1)} & \text{if } m \in \{0, \dots, k-1\}, \\ \frac{\eta}{k} \xi^{(2)} & \text{if } m \in \{k, \dots, 2k-1\}. \end{cases} \quad (2.53)$$

Based on the control points in (2.52), we consider the Bézier curve $B_\eta : \mathbb{R} \rightarrow \mathbb{R}^3$ by

$$B_\eta(t) = \sum_{m=0}^{2k} b_{m, 2k}(t) u(t_{\eta, m}), \quad t \in \mathbb{R}, \quad (2.54)$$

where $b_{r,s} : \mathbb{R} \rightarrow \mathbb{R}$ are the Bernstein polynomials, cf. Lemma 2.10.

After suitable reparametrization, (2.54) provides a mollification of u via

$$u_\eta(t) = \begin{cases} B_\eta\left(\frac{t - t_{\eta, 0}}{2\eta}\right), & \text{for } t \in [t^{(1)} - \eta, t^{(1)} + \eta] = [t_{\eta, 0}, t_{\eta, 2k}], \\ u(t), & \text{otherwise,} \end{cases} \quad t \in [0, L], \quad (2.55)$$

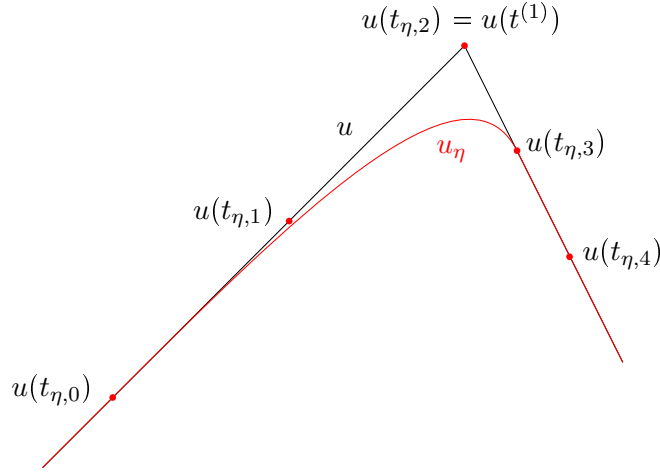
see Figure 2.2.

Step 1b: Regularity of u_η . Next, we verify that u_η as constructed in Step 1a is indeed k -times continuously differentiable on $[0, L]$. Indeed, it is enough to check that

$$B'_\eta(0) = 2\eta \xi^{(1)} \quad \text{and} \quad B'_\eta(1) = 2\eta \xi^{(2)}, \quad (2.56)$$

and that for any $j \in \mathbb{N}$ with $2 \leq j \leq k$,

$$\frac{d^j}{dt^j} B_\eta(0) = \frac{d^j}{dt^j} B_\eta(1) = 0. \quad (2.57)$$

Figure 2.2: Illustration of u_η for the case $k = 2$.

As for (2.56), we obtain with the help of Lemma 2.10 a), c) and (2.53) that for all $t \in \mathbb{R}$,

$$\begin{aligned}
 B'_\eta(t) &= 2k \sum_{m=0}^{2k} (b_{m-1,2k-1}(t) - b_{m,2k-1}(t)) u(t_{\eta,m}) \\
 &= 2k \sum_{m=0}^{2k-1} b_{m,2k-1}(t) (u(t_{\eta,m+1}) - u(t_{\eta,m})) \\
 &= 2\eta \left(\sum_{m=0}^{k-1} b_{m,2k-1}(t) \xi^{(1)} + \sum_{m=k}^{2k-1} b_{m,2k-1}(t) \xi^{(2)} \right) \\
 &= 2\eta (\lambda(t) \xi^{(1)} + (1 - \lambda(t)) \xi^{(2)}),
 \end{aligned} \tag{2.58}$$

where $\lambda(t) := \sum_{m=0}^{k-1} b_{m,2k-1}(t) \in [0, 1]$ for $t \in \mathbb{R}$. Due to Lemma 2.10 b), $\lambda(0) = 1$ and $\lambda(1) = 0$, which yields (2.56).

Similar calculations, invoking again the properties of Bernstein polynomials, in particular Lemma 2.10 d), give (2.57).

Step 1c: Uniform bounds and convergence of $(u_\eta)_\eta$. As a consequence of (2.58), the first derivative of u_η stays within the line segment connecting $\xi^{(1)}$ and $\xi^{(2)}$. In other words, it is a convex combination of these two vectors; formally,

$$u'_\eta \in [\xi^{(1)}, \xi^{(2)}] := \{\xi \in \mathbb{R}^3 : \xi = \lambda \xi^{(1)} + (1 - \lambda) \xi^{(2)} \text{ with } \lambda \in [0, 1]\}.$$

Since $\xi^{(1)}$ and $\xi^{(2)}$ are not antiparallel, it follows that $0 \notin [\xi^{(1)}, \xi^{(2)}]$. Hence, in view of the compactness of the line segment $[\xi^{(1)}, \xi^{(2)}]$, one can find constants $c, C > 0$ independent of η such that

$$c \leq |u'_\eta(t)| \leq C$$

for all $t \in [0, L]$. Moreover, along with (2.55),

$$\int_0^L |u' - u'_\eta|^q dt \leq 2\eta (C + |\xi^{(1)}| + |\xi^{(2)}|)^q,$$

and therefore,

$$u_\eta \rightarrow u \text{ in } W^{1,q}(0, L; \mathbb{R}^3) \text{ as } \eta \rightarrow 0$$

by Poincaré's inequality. After passing to a suitable discrete sequence, this completes the proof of statement under the assumption that the curve u is free of reversion.

Step 2: The general case with reversions. The idea is to reduce the argument to the situation of Step 1 via a loop construction and to conclude with a diagonalization argument.

In the following, let I stand for the index set consisting of all $n \in \{1, \dots, N-1\}$ such that $\xi^{(n)}$ and $\xi^{(n+1)}$ are anti-parallel, that is,

$$\xi^{(n+1)} = -\nu_n \xi^{(n)}$$

for some $\nu_n > 0$.

Step 2a: Loop construction. Without loss of generality, I is a singleton, say $I = \{1\}$; otherwise the argument below can be performed analogously for all (finitely many) elements in I . Besides, as in Step 1, we take $N = 2$ to keep notations simple.

For $\delta > 0$ sufficiently small, define $u_\delta \in A_{\text{pw}}(0, L; \mathbb{R}^3)$ via linear interpolation such that

$$u_\delta = u \quad \text{on } [0, t^{(1)} - \delta] \cup [t^{(1)} + \delta\sigma, L] \quad \text{and} \quad u_\delta(t^{(1)}) = u(t^{(1)}) + \delta\xi_\perp^{(1)}, \quad (2.59)$$

where $\xi_\perp^{(1)}$ is a non-zero vector orthogonal to $\xi^{(1)}$, and $\sigma = 1$ if $\nu_1 \neq 1$ and $\sigma = \frac{1}{2}$ if $\nu_1 = 1$, see Figure 2.3.

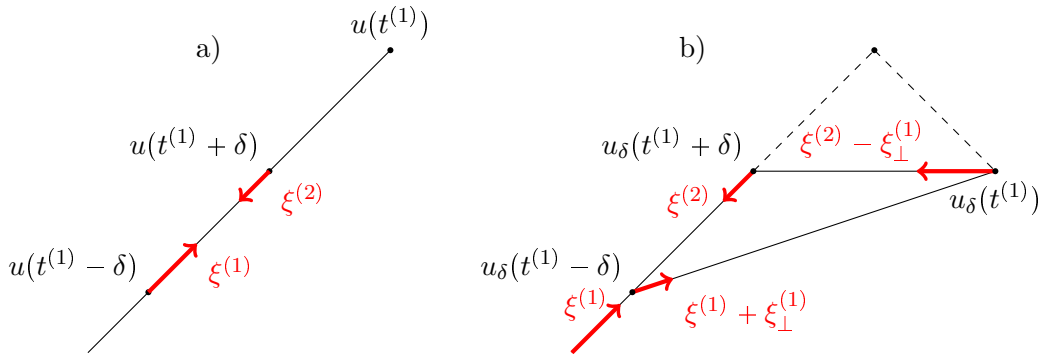


Figure 2.3: Illustration of a) a curve u that reverses its path at the point $t^{(1)}$; b) the modified curve u_δ resulting from the loop construction in the case $\nu_1 \neq 1$.

By design,

$$u'_\delta \in \{\xi^{(1)}, \xi^{(2)}, \xi^{(1)} + \xi_\perp^{(1)}, \xi^{(2)} - \frac{1}{\sigma}\xi_\perp^{(1)}\} \quad \text{a.e. in } [0, L],$$

so that in particular,

$$c \leq \|u'_\delta\|_{C^0([0, L]; \mathbb{R}^3)} \leq C, \quad (2.60)$$

with constants $c, C > 0$ depending only on u , and thus, independent of δ . Therefore, since u_δ differs from u only on a set of measure $\delta(1 + \sigma)$, we conclude together with Poincaré's inequality that

$$u_\delta \rightarrow u \quad \text{in } W^{1,q}(0, L; \mathbb{R}^3) \text{ as } \delta \rightarrow 0. \quad (2.61)$$

Step 2b: Diagonalization. By applying the results of Step 1 to each u_δ and accounting for (2.60) and (2.59), we obtain sequences $(u_{\delta,i})_i \subset C^k([0, L]; \mathbb{R}^3)$ such that

- (i) $_{\delta}$ $c \leq \|u'_{\delta,i}\|_{C^0([0,L];\mathbb{R}^3)} \leq C$ for all $i \in \mathbb{N}$ and $\delta > 0$ with constants $c, C > 0$ depending on u ;
- (ii) $_{\delta}$ $u_{\delta,i} = u$ on $[0, L] \setminus \Gamma_{2i}^{\delta}$, where $\Gamma_i^{\delta} := \{t \in [0, L] : \text{dist}(t, \Gamma^{\delta}) \leq \frac{1}{i}\}$ for i sufficiently large and Γ^{δ} denotes the set of points in $(0, L)$ where u_{δ} is not differentiable;
- (iii) $_{\delta}$ $u_{\delta,i} \rightarrow u_{\delta}$ in $W^{1,q}(0, L; \mathbb{R}^3)$ as $i \rightarrow \infty$;

In consideration of (2.61), (ii) $_{\delta}$ and (ii) $_{\delta}$, we can pick a diagonal sequence $(u_i)_i \subset C^k([0, L]; \mathbb{R}^3)$ with $u_i := u_{\delta(i),i}$ such that $\Gamma_{2i}^{\delta(i)} \subset \Gamma_i$ for all $i \in \mathbb{N}$, and $u_i \rightarrow u$ in $W^{1,q}(0, L; \mathbb{R}^3)$ as $i \rightarrow \infty$ by Attouch's lemma, which proves the statement. \square

The next lemma gathers some basic facts about Bernstein polynomials, which were an important ingredient in the definition of Bézier curves in the previous proof. For more details, we refer the reader e.g. to [83, 93].

Lemma 2.10. *For $r \in \mathbb{Z}$ and $s \in \mathbb{N}$, let $b_{r,s} : \mathbb{R} \rightarrow \mathbb{R}$ be the corresponding Bernstein polynomial, i.e.,*

$$b_{r,s}(t) = \begin{cases} \binom{s}{r} (1-t)^{s-r} t^r & \text{if } 0 \leq r \leq s, \\ 0 & \text{otherwise,} \end{cases} \quad t \in \mathbb{R}.$$

Then the following properties hold:

a) (binomial theorem)

$$\sum_{m=0}^s b_{m,s} = 1;$$

b) (values at 0 and 1)

$$b_{r,s}(0) = \begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{if } r \neq 0, \end{cases} \quad \text{and} \quad b_{r,s}(1) = \begin{cases} 1 & \text{if } r = s, \\ 0 & \text{if } r \neq s; \end{cases}$$

c) (first derivative)

$$b'_{r,s} = p(b_{r-1,s-1} - b_{r,s-1});$$

d) (higher-order derivatives)

$$\frac{d^j}{dt^j} b_{r,s} = \frac{s!}{(s-j)!} \sum_{m=\max\{0, r-s+j\}}^{\min\{j, r\}} (-1)^{m+j} \binom{j}{m} b_{r-m, s-j}$$

for any natural number $j \leq s$.

Chapter 3

Theories for incompressible rods: A rigorous derivation via Γ -convergence

This chapter has been published in [90].

3.1 Introduction

The study of the deformation behavior of thin structures in response to external forces dates back centuries, with pioneering contributions on the bending of elastic rods by Euler and Bernoulli, and the formulation of a plate theory by Kirchhoff. And still, nowadays, the topic has not lost any of its relevance, as numerous modern applications in technology and recent developments in the life sciences demonstrate. Thinking, for instance, of carbon nanowires and printed electronics in computer devices, of fiber-reinforced and layered composites in materials science, or of cell membranes and DNA strands in biology, current research directions require a profound understanding of elastic bodies with a small extension in one or two spatial dimensions.

Whereas classical approaches based on asymptotic expansions had been remarkably fruitful in the small-strain setting of linear elasticity, see e.g. [53, 196], accounting for large deformations calls for a mathematical framework that is well-suited to deal with geometric nonlinearity. In a variational context, dimension reduction via Γ -convergence [42, 71] allows to establish a rigorous connection between fully three-dimensional models in hyperelasticity and lower-dimensional theories for thin structures.

The first results in this spirit are due to Acerbi, Buttazzo & Percivale [1], who proved a 3d-1d reduction for elastic strings, and to Le Dret & Raoult [137, 139], who deduced a model for two-dimensional elastic membranes from the Γ -limit of elastic energy functionals for vanishing thickness. A few years later, an ansatz-free derivation of Kirchhoff's plate theory was obtained independently in [98, 173]. These seminal findings, in particular, the quantitative geometric rigidity estimate by Friesecke, James & Müller [98], actuated substantial efforts towards a systematic analysis of different types of thin structures, with contributions by many authors. We highlight here a few selected examples. Considering that the scaling of the acting external forces has a decisive influence on the resulting lower-dimensional models, a complete hierarchy of plate models was derived in [99]; details about the asymptotic analysis one-dimensional objects, precisely strings and rods, in various scaling regimes can be found in [156, 157, 185]. Recently, various special features of thin structures have been investigated, including small-scale heterogeneities in plates and rods, which require a combination of dimension reduction and homoge-

nization techniques [116, 166, 167], global invertibility aiming to avoid self-interpenetration of matter [27, 115, 171], or thin objects made of materials with pre-existing strain [36, 55, 56, 125]. While the previously mentioned works rely on Γ -convergence, and hence (assuming suitable compactness) imply that (almost) minimizers converge, we refer e.g. to [47, 76, 158, 159, 163] for statements on the convergence of equilibria.

An important class of thin structures are those made of incompressible materials, which are commonly used to describe rubber-like substances [2, 82, 168], and thus, occur e.g. in blood vessels, tires, seat belts, etc. From an analytical point of view, there is recent work on membranes [61], plates in the Kirchhoff [62] and von Kármán regime [143, 144], hyperelastic shells [5], and strings (see Chapter 2. The challenge in the mathematical analysis of these models lies in the non-convex constraint imposed on the elastic energy functionals to guarantee local volume-preservation; precisely, the Jacobian determinant of admissible deformation fields has to be constant and equal to one, cf. [63].

Our intention with this thesis is to close a gap in the literature by deriving a hierarchy of theories for incompressible rods, including, in particular, the Kirchhoff- and von Kármán-type cases. To this end, we characterize the asymptotic behavior in the limit of vanishing cross-section of suitably rescaled elastic energy functionals subject to a local volume-preservation constraint.

The chapter is organized as follows: In the remainder of the introduction, we give the precise problem formulation, announce the main results, give insight into our methodological approach, and specify relevant notation. As preliminaries, we discuss in Section 3.2 several properties of the limit densities arising through the dimension reduction procedures, and Section 3.3 collects the most important technical tools for proving the upper bounds. The core of this work, Section 3.4, contains the main Γ -convergence result for elastic rods in the Kirchhoff regime under the assumption of incompressibility. We identify the reduced Γ -limit for shrinking cross-section, determine the corresponding Euler-Lagrange equations, and specify our findings for the case of isotropic materials. Finally, the asymptotic analysis for the three remaining scaling regimes is presented in Section 3.5.

3.1.1 Problem formulation

Throughout the chapter, let $L > 0$ and $\omega \subset \mathbb{R}^2$ be a bounded, simply connected Lipschitz domain of unit measure, i.e., $\mathcal{L}^2(\omega) = |\omega| = \int_{\omega} d\tilde{x} = 1$, such that

$$\int_{\omega} x_2 d\tilde{x} = \int_{\omega} x_3 d\tilde{x} = \int_{\omega} x_2 x_3 d\tilde{x} = 0; \quad (3.1)$$

see Section 5.2.1, in particular, (3.11), for the use of notation.

For small $\varepsilon > 0$, we introduce $\Omega_{\varepsilon} = (0, L) \times \varepsilon\omega \subset \mathbb{R}^3$ as the reference configuration of a thin, incompressible body of length L with cross-section $\varepsilon\omega$, and define its total energy through the functional $\mathcal{E}_{\varepsilon} : H^1(\Omega_{\varepsilon}; \mathbb{R}^3) \rightarrow \mathbb{R}_{\infty} := \mathbb{R} \cup \{\infty\}$ with

$$\mathcal{E}_{\varepsilon}(v) = \int_{\Omega_{\varepsilon}} W(\nabla v) dy - \int_{\Omega_{\varepsilon}} f_{\varepsilon} \cdot v dy, \quad v \in H^1(\Omega_{\varepsilon}; \mathbb{R}^3); \quad (3.2)$$

here, $W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty]$ is the constrained elastic energy density given by

$$W(F) = \begin{cases} W_0(F) & \text{for } \det F = 1, \\ \infty & \text{otherwise,} \end{cases} \quad (3.3)$$

where $W_0 : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ satisfies the following hypotheses:

(H1) W_0 is twice continuously differentiable in a neighborhood of $\text{SO}(3)$;

(H2) $W_0(\text{Id}) = 0$ and there is $C_1 > 0$ such that $W_0(F) \geq C_1 \text{dist}^2(F, \text{SO}(3))$ for all $F \in \mathbb{R}^{3 \times 3}$;

(H3) W_0 is frame indifferent, i.e., $W_0(RF) = W_0(F)$ for all $F \in \mathbb{R}^{3 \times 3}$ and $R \in \text{SO}(3)$.

The vector field $f_\varepsilon \in L^2(\Omega_\varepsilon; \mathbb{R}^3)$ in (3.2) describes the external forces acting on the body. For the sake of simplicity, f_ε is assumed to be independent of the cross-section variables, and can therefore be interpreted as an element of $L^2(0, L; \mathbb{R}^3)$. Regarding the scaling properties of f_ε , we suppose the existence of an $\alpha \geq 0$ and a suitable $f \in L^2(0, L; \mathbb{R}^3)$ such that

$$f_\varepsilon = \varepsilon^\alpha f; \quad (3.4)$$

we suppose that the body forces average out to zero, i.e.,

$$\int_0^L f \, dx_1 = 0. \quad (3.5)$$

With the intended asymptotic analysis of the energies in (3.2) in mind, it is technically convenient to perform a change of variables that allows us to replace \mathcal{E}_ε with functionals defined on a fixed, parameter-independent space. Indeed, with $y = (x_1, \varepsilon x_2, \varepsilon x_3)$ for $x = (x_1, x_2, x_3) \in \Omega := \Omega_1$, and $u \in H^1(\Omega; \mathbb{R}^3)$ given by $u(x) = v(y)$ for $v \in H^1(\Omega_\varepsilon; \mathbb{R}^3)$, the normalization of \mathcal{E}_ε per unit volume turns into $\mathcal{J}_\varepsilon : H^1(\Omega; \mathbb{R}^3) \rightarrow \mathbb{R}_\infty$,

$$\mathcal{J}_\varepsilon(u) = \int_\Omega W(\nabla^\varepsilon u) \, dx - \int_\Omega f_\varepsilon \cdot u \, dx, \quad u \in H^1(\Omega; \mathbb{R}^3),$$

with the rescaled deformation gradient

$$\nabla^\varepsilon u = (\partial_1 u | \frac{1}{\varepsilon} \partial_2 u | \frac{1}{\varepsilon} \partial_3 u).$$

It is well-known that the scaling behavior of \mathcal{J}_ε depends on the parameter-dependence of the external forces; with α as in (3.4), one has that

$$\mathcal{J}_\varepsilon \sim \begin{cases} \alpha & \text{if } \alpha \in [0, 2), \\ 2\alpha - 2 & \text{if } \alpha \geq 2. \end{cases}$$

see [99] for more details.

The scaling regimes $\alpha \in [0, 2)$, which give rise to (degenerate) models for incompressible strings, are studied in Chapter 2. In this chapter, we focus on the cases $\alpha \geq 2$ to deduce a hierarchy of theories for incompressible rods. Hence, the relevant functionals for us to work with are the rescaled energies

$$\mathcal{J}_\varepsilon^{(\alpha)} : H^1(\Omega; \mathbb{R}^3) \rightarrow \mathbb{R}_\infty, \quad u \mapsto \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W(\nabla^\varepsilon u) \, dx - \frac{1}{\varepsilon^{\alpha-2}} \int_\Omega f \cdot u \, dx, \quad (3.6)$$

or, if we intend to consider exclusively the elastic energy contribution,

$$\mathcal{I}_\varepsilon^{(\alpha)} : H^1(\Omega; \mathbb{R}^3) \rightarrow [0, \infty], \quad u \mapsto \frac{1}{\varepsilon^{2\alpha-2}} \int_\Omega W(\nabla^\varepsilon u) \, dx. \quad (3.7)$$

The main results of this chapter are characterizations of the Γ -limits $\mathcal{I}^{(\alpha)}$ of the sequences of energy functionals $(\mathcal{I}_\varepsilon^{(\alpha)})_\varepsilon$ for all $\alpha \geq 2$ (see Theorem 3.12 and Theorem 3.18); the explicit formulas of $\mathcal{I}^{(\alpha)}$ in the four qualitatively different regimes $\alpha = 2$, $\alpha \in (2, 3)$, $\alpha = 3$, and $\alpha > 3$ can be found in (3.33), (3.62) and (3.64), respectively.

3.1.2 Approach and techniques

We adopt and tailor the methodology from [62] for incompressible plates to our situation of 3d-1d dimension reduction. In doing so, it is essential to exploit the available results in [156, 157, 185] on the asymptotic analysis of compressible rods in the various scaling regimes. In the following, we give a brief overview of the ideas behind the three steps for proving Γ -convergence, namely, compactness, lower and upper bound. For a comprehensive introduction to variational convergence and its properties, see e.g. [42, 71].

All compactness properties emerge as an immediate consequence of the literature on the respective compressible cases, considering that $W_0 \leq W$, cf. (3.3).

The key ingredient for the lower bound is a suitable approximation of the determinant constraint with the help of a suitable penalization term. To be more precise, we consider for each $k \in \mathbb{N}$ a penalized energy density $W_k : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ given by

$$W_k(F) = W_0(F) + \frac{k}{2}(\det F - 1)^2, \quad F \in \mathbb{R}^{3 \times 3}; \quad (3.8)$$

clearly, the sequence $(W_k)_k$ is increasing, and converges pointwise to W for $k \rightarrow \infty$. Since each W_k meets the requirement for densities in the limit theory of compressible rods, the Γ -limits of the functionals $(\mathcal{I}_{k,\varepsilon}^{(\alpha)})_{k,\varepsilon}$ with

$$\mathcal{I}_{k,\varepsilon}^{(\alpha)}(u) = \frac{1}{\varepsilon^{2\alpha-2}} \int_{\Omega} W_k(\nabla^\varepsilon u) \, dx, \quad u \in H^1(\Omega; \mathbb{R}^3), \quad (3.9)$$

are well-established; let us call the above-mentioned Γ -limits $\mathcal{I}_k^{(\alpha)}$. Showing that the pointwise limit of these $\mathcal{I}_k^{(\alpha)}$ for $k \rightarrow \infty$ is exactly $\mathcal{I}^{(\alpha)}$ yields the desired liminf inequality; our proof is based on the monotonicity and pointwise convergence of corresponding limit densities, which are in general defined only implicitly via infinite-dimensional minimization problems, cf. Corollary 3.7.

The upper bounds require the construction of energetically optimal approximating sequences of locally volume-preserving deformations for any admissible limit state. As a starting point, we take a sequence $(y_\varepsilon)_\varepsilon$ inspired by the recovery sequences from the unconstrained settings for the finite-valued density W_0 (under consideration of the respective scaling regime). These recovery sequences typically involve higher-order terms in ε that are determined by the solutions to the variational problems arising in the definition of the limit densities. In our incompressible setting, the corresponding minimization problem features a trace constraint - connected with the determinant constraint through linearization - from which we can deduce that

$$\det \nabla^\varepsilon y_\varepsilon - 1 \sim \varepsilon^\gamma \quad (3.10)$$

for some $\gamma > 0$ depending on the scaling regime α .

With (3.10) at hand, an inner perturbation argument in the spirit of [61], adapted for 3d-1d reductions in Chapter 2 (see also Lemma 3.10 below), allows us to replace $(y_\varepsilon)_\varepsilon$ by a sequence that satisfies the incompressibility constraint exactly. We remark that this auxiliary result is only applicable if the cross-section variables of y_ε are decoupled, for which we extend y_ε suitably to a cuboid containing Ω . Technically, this task reduces to finding a divergence-free extension (see e.g. [120]) in the cross-section direction.

Overall, our analysis shows that the following diagram commutes:

3.1.3 Notation

The following notations are used throughout the chapter. Let e_1, e_2, e_3 be the standard unit basis vectors in \mathbb{R}^3 , and let $a^\perp := (-a_2, a_1)$ for $a \in \mathbb{R}^2$. For any two vectors $a, b \in \mathbb{R}^n$, we

$$\begin{array}{ccc}
\mathcal{I}_{k,\varepsilon}^{(\alpha)} & \xrightarrow{k \rightarrow \infty} & \mathcal{I}_{\varepsilon}^{(\alpha)} \\
\downarrow \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} & & \downarrow \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \\
\mathcal{I}_k^{(\alpha)} & \xrightarrow{k \rightarrow \infty} & \mathcal{I}^{(\alpha)}
\end{array}$$

denote by $a \cdot b$ their standard inner product and by $a \otimes b \in \mathbb{R}^{n \times n}$ their tensor product, that is, componentwise, $(a \otimes b)_{ij} = a_i b_j$ for $i, j = 1, \dots, n$. The inner product on the space of matrices $\mathbb{R}^{m \times n}$ is given by $A : B = \text{Tr}(AB^T)$ for $A, B \in \mathbb{R}^{m \times n}$, where Tr is the trace operator and B^T the transpose of B . The induced norms on \mathbb{R}^n and $\mathbb{R}^{m \times n}$ are both denoted by $|\cdot|$. Moreover, $A^{\text{sym}} = \frac{1}{2}(A^T + A)$ refers to the symmetric part of $A \in \mathbb{R}^{n \times n}$, we use Id for the identity matrix in $\mathbb{R}^{n \times n}$, $\text{SO}(n) \subset \mathbb{R}^{n \times n}$ is the rotation group, and $\mathbb{R}_{\text{skew}}^{n \times n}$ stands for the space of skew-symmetric $n \times n$ matrices. If $g : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$, $b \in \mathbb{R}^3$ and $A \in \mathbb{R}^{3 \times 2}$, we simplify the expression $g((b|A))$ to $g(b|A)$.

Furthermore, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we write $x = (x_1, \tilde{x})$ with

$$\tilde{x} = (x_2, x_3) \in \mathbb{R}^2; \quad (3.11)$$

in particular, the points of any subset of $U \subset \mathbb{R}^2$ are addressed by $\tilde{x} \in U$. Likewise, we split the components of \mathbb{R}^3 -valued maps, that is $w = (w_1, \tilde{w})$ for $w : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^3$. We denote the partial derivative of a function $w : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^m$ with respect to x_i for $i = 1, 2, 3$ by $\partial_i w$. If w depends solely on the x_1 -variable, we use $\partial_1 w$ and w' interchangeably. The gradient of w is often split like

$$\nabla w = (\partial_1 w | \tilde{\nabla} w) \quad \text{with} \quad \tilde{\nabla} w := (\partial_2 w | \partial_3 w).$$

The rescaled gradient of w can then be expressed as $\nabla^\varepsilon w = (\partial_1 w | \frac{1}{\varepsilon} \tilde{\nabla} w)$. Note that whenever a function is defined on a subset of \mathbb{R}^2 , we call its two-dimensional independent variable $\tilde{x} = (x_2, x_3)$, and we write $\tilde{\nabla}$, $\tilde{\Delta}$, and $\tilde{\text{div}}$ to indicate its gradient, Laplacian and divergence.

If U is a subset of \mathbb{R}^n , then \bar{U} is its closure, and $\mathcal{L}^n(U)$, or simply $|U|$, denotes its Lebesgue measure (provided U is measurable). For any open $U \subset \mathbb{R}^n$, we adopt the standard notation for vector-valued Sobolev spaces $H^1(U; \mathbb{R}^m)$ and the space of k -times continuously differentiable functions $C^k(\bar{U}; \mathbb{R}^m)$. The space of Lebesgue-square-integrable Banach-space-valued functions is denoted by $L^2(U; \mathcal{V})$ for a Banach space \mathcal{V} . In the case where U is an interval $(a, b) \subset \mathbb{R}$, we shorten the notation $L^2(a, b; \mathcal{V}) := L^2((a, b); \mathcal{V})$ and $H^1(a, b; \mathbb{R}^m) := H^1((a, b); \mathbb{R}^m)$. For scalar-valued functions, we often drop the image space in our notation, writing e.g., $H^1(U)$ instead of $H^1(U; \mathbb{R})$. Without explicit mention, functions $(0, L) \rightarrow \mathbb{R}^m$ are identified with their constant extension onto $(0, L) \times U$ for $U \subset \mathbb{R}^2$.

Furthermore, for any open subset $U \subset \mathbb{R}^2$, let

$$L_0^2(U; \mathbb{R}^m) = \left\{ w \in L^2(U; \mathbb{R}^m) : \int_U w \, dx = 0 \right\}.$$

Moreover, we identify $L^2(0, L; H^1(U; \mathbb{R}^3))$ with a function in $L^2((0, L) \times U; \mathbb{R}^3)$, and similarly for other spaces.

We employ the standard notation $\mathcal{O}(\cdot)$ and $o(\cdot)$ for the Landau symbols. Finally, speaking of “sequences” with index $\varepsilon > 0$, means that ε can stand for any non-negative sequence $(\varepsilon_j)_j$ with $\lim_{j \rightarrow \infty} \varepsilon_j = 0$.

3.2 Properties of the limit densities

Here, we introduce and discuss relevant expressions for the formulation of the reduced limit problems, meaning the Γ -limits $\mathcal{I}^{(\alpha)}$ for $\alpha \geq 2$.

We start by defining $Q : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ as the quadratic form of linearized elasticity resulting from the second derivative of the energy density W_0 at the identity, i.e.,

$$Q(F) = \nabla^2 W_0(\text{Id})[F, F] \quad \text{for } F \in \mathbb{R}^{3 \times 3},$$

cf. (H1). Due to (H2), Taylor expansion around the identity up to second order yields

$$W_0(F) = \frac{1}{2}Q(F) + \mathcal{O}(|F - \text{Id}|^3), \quad (3.12)$$

and along with (H3), one has that

$$Q(F) = Q(F^{\text{sym}}) \geq C_Q |F^{\text{sym}}|^2, \quad (3.13)$$

for all $F \in \mathbb{R}^{3 \times 3}$ with a constant $C_Q > 0$, see e.g. [98, 156]. In the following, we denote by L the symmetric fourth order tensor such that

$$Q(F) = LF : F \quad (3.14)$$

for all $F \in \mathbb{R}^{3 \times 3}$. For any affine $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, we define $Q^\xi : H^1(\omega; \mathbb{R}^3) \rightarrow [0, \infty]$ by setting

$$Q^\xi(\beta) = \begin{cases} \int_\omega Q(\xi | \tilde{\nabla} \beta) \, d\tilde{x} & \text{if } \text{Tr}(\xi | \tilde{\nabla} \beta) = 0 \text{ a.e. in } \omega, \\ \infty & \text{otherwise,} \end{cases} \quad (3.15)$$

for $\beta \in H^1(\omega; \mathbb{R}^3)$.

Next, we address the issue of minimizing Q^ξ in $H^1(\omega; \mathbb{R}^3)$ for any affine $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. We will see that unique minimizers can be found in a certain subspace V^ξ of $H^1(\omega; \mathbb{R}^3)$ depending on ξ . Precisely, we consider the linear space $\mathcal{V}^\xi \subset H^1(\omega; \mathbb{R}^3) \cap L_0^2(\omega; \mathbb{R}^3)$ that encompasses all functions with the property

$$\int_\omega \tilde{\nabla} \beta \, d\tilde{x} = 0 \quad \text{if } \xi(0) = 0 \quad \text{and} \quad \int_\omega \tilde{x}^\perp \cdot \tilde{\beta} \, d\tilde{x} = 0 \quad \text{if } \xi(0) \neq 0;$$

recall the notation $\beta = (\beta_1, \tilde{\beta})$. With this choice of spaces, Korn's inequality holds in the following form: There exists a constant $C_K > 0$ depending only on ω such for all $\beta \in \mathcal{V}^\xi$,

$$\|(\tilde{\nabla} \tilde{\beta})^{\text{sym}}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})} \geq C_K \|\tilde{\nabla} \tilde{\beta}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}; \quad (3.16)$$

indeed, if $\xi(0) = 0$, it suffices to invoke the well-known mean-value version of Korn's inequality (see e.g. [97]). In the case $\xi(0) \neq 0$, on the other hand, one observes that \mathcal{V}^ξ contains no non-trivial infinitesimal rigid displacements (cf. also [157, Remark 4.1]), and hence, (3.16) follows from [126, Theorem 4.4].

The next results provides the existence of a unique solution to the problem of minimizing the functional Q^ξ from (3.15).

Lemma 3.1 (Minimization of Q^ξ). *Let $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be affine. Then, the functional Q^ξ has a unique minimizer with zero mean value, called β^ξ , which lies in \mathcal{V}^ξ .*

Proof. Let us start by observing that the constraint in Q^ξ is invariant under certain affine translations; precisely, if

$$\eta(\tilde{x}) = A\tilde{x} + b \quad \text{for } \tilde{x} \in \omega \text{ with } A \in \mathbb{R}^{3 \times 2} \text{ such that } A_{21} + A_{32} = 0 \text{ and } b \in \mathbb{R}^3, \quad (3.17)$$

then for any $\beta \in H^1(\omega; \mathbb{R}^3)$,

$$\text{Tr}(\xi|\tilde{\nabla}(\beta - \eta)) = \text{Tr}(\xi|\tilde{\nabla}\beta) - \text{Tr}(0|A) = \text{Tr}(\xi|\tilde{\nabla}\beta).$$

Next, we prove that

$$\inf_{\beta \in H^1(\omega; \mathbb{R}^3)} Q^\xi(\beta) = \inf_{\beta \in \mathcal{V}^\xi} Q^\xi(\beta). \quad (3.18)$$

To this end, it suffices to show that one can find for any $\beta \in H^1(\omega; \mathbb{R}^3)$ with $\text{Tr}(\xi|\tilde{\nabla}\beta) = 0$ a function η as in (3.17) such that

$$\beta - \eta \in \mathcal{V}^\xi \quad \text{and} \quad Q^\xi(\beta - \eta) \leq Q^\xi(\beta). \quad (3.19)$$

Indeed, for linear ξ , meaning $\xi(0) = 0$, we specialize the coefficients in (3.17) to $b = \int_\omega \beta \, d\tilde{x}$ and $A = \int_\omega \tilde{\nabla}\beta \, d\tilde{x}$; notice that $A_{21} + A_{32} = \int_\omega \text{div}\tilde{\beta} \, d\tilde{x} = - \int_\omega \xi_1 \, d\tilde{x} = 0$ and $\int_\omega (\beta - \eta) \, d\tilde{x} = 0$ due to (3.1). Clearly, $\int_\omega \tilde{\nabla}(\beta - \eta) \, d\tilde{x} = 0$, and along with $|\omega| = 1$, (3.1), the linearity of ξ , the symmetry of the fourth-order tensor L , and (3.13), we conclude that

$$\begin{aligned} Q^\xi(\beta - \eta) &= Q^\xi(\beta) + Q(0|A) - 2 \int_\omega L(\xi|\tilde{\nabla}\beta) : (0|A) \, d\tilde{x} \\ &= Q^\xi(\beta) + Q(0|A) - 2L(0|A) : \int_\omega (\xi|\tilde{\nabla}\beta) \, d\tilde{x} \\ &= Q^\xi(\beta) + Q(0|A) - 2L(0|A) : (0|A) = Q^\xi(\beta) - Q(0|A) \leq Q^\xi(\beta), \end{aligned}$$

recalling (3.14).

Otherwise, if $\xi(0) \neq 0$, take η as in (3.17) with $A = \nu(e_3| - e_2)$ and

$$\nu = \frac{\int_\omega \beta \cdot \tilde{x}^\perp \, d\tilde{x}}{\int_\omega |\tilde{x}|^2 \, d\tilde{x}},$$

as well as a translation vector $b = \int_\omega \beta \, d\tilde{x}$. By construction, $\beta - \eta \in \mathcal{V}^\xi$, and (3.13) in combination with the antisymmetry of $(0|A)$ implies $Q^\xi(\beta - \eta) = Q^\xi(\beta)$. This proves (3.19), and thus, also (3.18).

The existence of a minimizer of Q^ξ in \mathcal{V}^ξ is a straight-forward application of the direct method, given (3.13) in combination with (3.16) and Poincaré's inequality, as well as the quadratic and linear structure of Q and the trace-constraint, respectively. In view of (3.18), then also Q^ξ has a minimizer in $H^1(\omega; \mathbb{R}^3)$, whose uniqueness up to translations follows from the strict convexity on symmetric matrices of the integrand of Q^ξ . \square

The following two remarks provide some additional insight into the properties of the minimizers β^ξ of Q^ξ . First, we derive necessary conditions for the minimizers β^ξ of Q^ξ in the form of (weak) Euler-Lagrange equations; for related statements in the context of compressible rods, see [156, Remark 3.4] and [157, Remark 4.1]. The second aspect concerns the linear and continuous dependence of β^ξ on ξ .

Remark 3.2 (Euler-Lagrange equations). Let $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be an affine function.

As a consequence of the Lagrange-multiplier theory for constrained optimization (see e.g. [176, Theorem 3.63]), we obtain that β is a minimizer of Q^ξ if and only if

(i) the Euler-Lagrange equations

$$\int_{\omega} L(\xi|\tilde{\nabla}\beta) : (0|\tilde{\nabla}\phi) \, d\tilde{x} = -\frac{1}{2} \int_{\omega} \lambda^{\xi} \tilde{\operatorname{div}}\tilde{\phi} \, d\tilde{x} \quad (3.20)$$

hold for all test functions $\phi = (\phi_1, \tilde{\phi}) \in H^1(\omega; \mathbb{R}^3)$ with a function $\lambda^{\xi} \in L^2(\omega)$, and

(ii) β satisfies the trace condition $\operatorname{Tr}(\xi|\tilde{\nabla}\beta) = 0$, or equivalently,

$$\tilde{\operatorname{div}}\tilde{\beta} = -\xi_1. \quad (3.21)$$

Notice that the Lagrange-multiplier λ^{ξ} is unique; this follows from the surjectivity of the divergence operator $\tilde{\operatorname{div}}$ as a map from $H^1(\omega; \mathbb{R}^2) \rightarrow L^2(\omega)$, cf. [107, Chapter I, Corollary 2.4].

Remark 3.3 (Linear and continuous dependence on ξ). The considerations in Remark 3.2 imply that both the Lagrange multiplier $\lambda^{\xi} \in L^2(\omega)$ and β^{ξ} (recall the definition in Lemma 3.1) depend linearly on ξ . Furthermore, since the space of affine functions is finite-dimensional, we conclude that $\xi \mapsto \beta^{\xi}$ is a bounded, linear map from the subspace of affine functions in $L^2(\omega; \mathbb{R}^3)$ into $H^1(\omega; \mathbb{R}^3)$.

We continue with a convergence statement that identifies Q^{ξ} as the Γ -limit of a sequence of finite-valued functionals.

Lemma 3.4 (Q^{ξ} as a Γ -limit). *For $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ affine and $k \in \mathbb{N}$, let $Q_k^{\xi} : H^1(\omega; \mathbb{R}^3) \rightarrow [0, \infty)$ be given by*

$$Q_k^{\xi}(\beta) = \int_{\omega} \left(Q(\xi|\tilde{\nabla}\beta) + k \left(\operatorname{Tr}(\xi|\tilde{\nabla}\beta) \right)^2 \right) \, d\tilde{x}. \quad (3.22)$$

Then, $\Gamma\text{-}\lim_{k \rightarrow \infty} Q_k^{\xi} = Q^{\xi}$ with respect to the weak topology in $H^1(\omega; \mathbb{R}^3)$.

Moreover, every sequence $(\beta_k)_k \subset \mathcal{V}^{\xi}$ with $\sup_{k \in \mathbb{N}} Q_k^{\xi}(\beta_k) < \infty$ admits a convergent subsequence (not relabeled) such that $\beta_k \rightarrow \beta$ in $H^1(\omega; \mathbb{R}^3)$ with $\beta \in \mathcal{V}^{\xi}$ and $\operatorname{Tr}(\xi|\tilde{\nabla}\beta) = 0$.

Proof. Fix an arbitrary affine function $\xi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

Step 1: Liminf-inequality. Let $\beta_k \rightarrow \beta$ in $H^1(\omega; \mathbb{R}^3)$, and assume without loss of generality that

$$\infty > \liminf_{k \rightarrow \infty} Q_k^{\xi}(\beta_k) = \lim_{k \rightarrow \infty} Q_k^{\xi}(\beta_k).$$

Then, $\operatorname{Tr}(\xi|\tilde{\nabla}\beta_k) \rightarrow 0$ in $L^2(\omega)$, which implies in particular that $\operatorname{Tr}(\xi|\tilde{\nabla}\beta) = 0$. Since Q is convex, we infer by weak lower semicontinuity that

$$\liminf_{k \rightarrow \infty} Q_k^{\xi}(\beta_k) \geq \liminf_{k \rightarrow \infty} \int_{\omega} Q(\xi|\tilde{\nabla}\beta_k) \, d\tilde{x} \geq \int_{\omega} Q(\xi|\tilde{\nabla}\beta) \, d\tilde{x} = Q^{\xi}(\beta).$$

Step 2: Limsup-inequality. Let $\beta \in H^1(\omega; \mathbb{R}^3)$ such that $\operatorname{Tr}(\xi|\tilde{\nabla}\beta) = 0$ a.e. in ω . It is immediate to see that the constant sequence $(\beta_k)_k$ with $\beta_k = \beta$ for $k \in \mathbb{N}$ is a recovery sequence.

Step 3: Compactness. Let $(\beta_k)_k \subset \mathcal{V}^\xi$ be a sequence of uniformly bounded energy for $(Q_k^\xi)_k$. With the help of (3.13), (3.16), and Young's inequality, we estimate for every $k \in \mathbb{N}$ that

$$\begin{aligned} Q_k^\xi(\beta_k) &\geq \int_{\omega} Q(\xi|\tilde{\nabla}\beta_k) \, d\tilde{x} \geq C_Q \int_{\omega} |(\xi|\tilde{\nabla}\beta_k)^{\text{sym}}|^2 \, d\tilde{x} \\ &\geq C_Q \|(\tilde{\nabla}\beta_k)^{\text{sym}}\|_{L^2(\omega; \mathbb{R}^{2 \times 2})}^2 + \frac{C_Q}{4} \|\tilde{\nabla}(\beta_k \cdot e_1)\|_{L^2(\omega; \mathbb{R}^2)}^2 - \frac{C_Q}{2} \|\xi\|_{L^2(\omega; \mathbb{R}^3)}^2 \\ &\geq C \|\tilde{\nabla}\beta_k\|_{L^2(\omega; \mathbb{R}^{3 \times 2})}^2 - \frac{C_Q}{2} \|\xi\|_{L^2(\omega; \mathbb{R}^3)}^2 \end{aligned}$$

with constants $C = C_Q \min\{C_K^2, \frac{1}{4}\}$. Since $\beta_k \in \mathcal{V}^\xi$ has vanishing mean value, we conclude from Poincaré's inequality that $(\beta_k)_k$ is bounded in $H^1(\omega; \mathbb{R}^3)$. Hence, the statement follows by the weak compactness of $H^1(\omega; \mathbb{R}^3)$ and the weak closedness of \mathcal{V}^ξ . \square

Remark 3.5 (Minimizers of Q_k^ξ). Analogous arguments to those in Lemma 3.1 show that for every $k \in \mathbb{N}$, the unique minimizer of Q_k^ξ with vanishing mean value is an element of \mathcal{V}^ξ .

Next, we introduce some further notation that will be needed to express the energy densities of the intended limit problems. Let $Q^* : \mathbb{R}_{\text{skew}}^{3 \times 3} \times \mathbb{R} \rightarrow [0, \infty)$ be given by

$$Q^*(F, t) = \min\{Q^\xi(\beta) : \beta \in H^1(\omega; \mathbb{R}^3), \xi(\tilde{x}) = F(x_2e_2 + x_3e_3) + te_1 \text{ for } \tilde{x} \in \omega\}. \quad (3.23)$$

Note that Q^* is well-defined according to Lemma 3.1 and finite, since for any $(F, t) \in \mathbb{R}_{\text{skew}}^{3 \times 3}$, there exists $\beta \in H^1(\omega; \mathbb{R}^3)$ such that the trace constraint in (3.15) for $\xi(\tilde{x}) = F(x_2e_2 + x_3e_3) + te_1$, i.e., $\text{Tr}(F(x_2e_2 + x_3e_3) + te_1|\tilde{\nabla}\beta) = 0$, or equivalently,

$$\partial_2\beta_2 + \partial_3\beta_3 = x_2F_{12} + x_3F_{13} + t,$$

is satisfied. Moreover, owing to the linear dependence of β^ξ on ξ , as deduced at the end of Remark 3.2, and the properties of Q , Q^* is a positive-definite quadratic form.

Remark 3.6 (Additive splitting of Q^*). In analogy to the compressible case (see [157, Remark 4.4]), Q^* can be split additively into two quadratic expressions that depend only on either F or t ; precisely, it holds that

$$Q^*(F, t) = Q^*(F, 0) + \alpha t^2$$

for $(F, t) \in \mathbb{R}_{\text{skew}}^{3 \times 3} \times \mathbb{R}$, where $\alpha \in \mathbb{R}$ results from a finite-dimensional constrained quadratic optimization problem, namely,

$$\alpha = \min_{a, b \in \mathbb{R}^3, a_2 + b_3 = -1} Q(e_1|a|b).$$

As a direct consequence of Lemma 3.4 (see also Remark 3.5) and the classical properties of Γ -convergence, which include the convergence of minima (see e.g. [42, 71]), we derive a useful approximation for Q^* . The next result enters into the proof of the lower bounds, cf. Theorem 3.12 ii') and Theorem 3.18 ii').

Corollary 3.7 (Pointwise approximation of Q^*). For $k \in \mathbb{N}$ and $(F, t) \in \mathbb{R}_{\text{skew}}^{3 \times 3} \times \mathbb{R}$, let

$$Q_k^*(F, t) = \min\{Q_k^\xi(\beta) : \beta \in H^1(\omega; \mathbb{R}^3), \xi(\tilde{x}) = F(x_2e_2 + x_3e_3) + te_1 \text{ for } \tilde{x} \in \omega\}, \quad (3.24)$$

where Q_k^ξ as in (3.22). Then, $Q_k^* \rightarrow Q^*$ pointwise as $k \rightarrow \infty$.

We conclude this section with a brief discussion of the important special case (for applications), where Q^* emerges from an isotropic energy density W_0 , i.e., $W_0(FS) = W_0(F)$ for all $S \in \text{SO}(3)$ and $F \in \mathbb{R}^{3 \times 3}$. In this situation, the minimization problem characterizing Q^* can be reduced to solving a Laplace problem with suitable Neumann boundary conditions. Under the additional geometric assumption that the cross section ω is a circle, we present a fully explicit expression for Q^* .

Example 3.8 (Isotropic case). If W_0 satisfies (H1)-(H3) and is isotropic, then the associated quadratic form is

$$Q(F) = \nabla^2 W_0(\text{Id})[F, F] = 2\mu |F^{\text{sym}}|^2 + \lambda (\text{Tr } F)^2, \quad F \in \mathbb{R}^{3 \times 3}, \quad (3.25)$$

with Lamé constants $\lambda \in \mathbb{R}$ and $\mu > 0$ such that $2\mu + 3\lambda > 0$. One can show that

$$Q^*(F, t) = 3\mu \left(F_{12}^2 \int_{\omega} x_2^2 \, d\tilde{x} + F_{13}^2 \int_{\omega} x_3^2 \, d\tilde{x} + t^2 \right) + \mu\tau F_{23}^2 \quad (3.26)$$

for $(F, t) \in \mathbb{R}_{\text{skew}}^{3 \times 3} \times \mathbb{R}$. Here, τ denotes the torsional rigidity defined by

$$\tau := \int_{\omega} (|\tilde{x}|^2 - \tilde{x}^\perp \cdot \tilde{\nabla} \varphi) \, d\tilde{x},$$

and $\varphi : \omega \rightarrow \mathbb{R}$ is a solution to the Neumann problem

$$\begin{cases} \tilde{\Delta} \varphi = 0 & \text{in } \omega, \\ \tilde{\nabla} \varphi \cdot \nu = \tilde{x}^\perp \cdot \nu & \text{on } \partial\omega \end{cases} \quad (3.27)$$

where ν is the outer normal vector to $\partial\omega$.

By Corollary 3.7, (3.26) follows from a pointwise limit procedure, once explicit expressions for Q_k^* with $k \in \mathbb{N}$ are available. Indeed, one can extract from the literature on the theory of compressible rods, precisely, from [156, Remark 3.5] and [185, Remark 4.2], that

$$Q_k^*(F, t) = \frac{\mu(3\lambda + 3k + 2\mu)}{\lambda + k + \mu} \left(F_{12}^2 \int_{\omega} x_2^2 \, d\tilde{x} + F_{13}^2 \int_{\omega} x_3^2 \, d\tilde{x} + t^2 \right) + \mu\tau F_{23}^2,$$

and hence, letting $k \rightarrow \infty$ implies the stated expression for Q^* .

We point out that, in contrast to the situation without the incompressibility constraint, Q^* in (3.26) does not depend on the first Lamé coefficient λ . As a consistency check, observe that the trace-free constraint in (3.15) makes Q^ξ , and thus also Q^* , independent of λ .

Example 3.9 (Isotropic case with circular cross section). Suppose in addition to the set-up of the previous example that ω is a circle around the origin with unit measure, i.e., $\omega = \{\tilde{x} \in \mathbb{R}^2 : |\tilde{x}|^2 \leq \frac{1}{\pi}\}$. Then, the outer unit normal vector to $\partial\omega$ becomes $\sqrt{\pi}\tilde{x}$, which yields a trivial solution to (3.27), meaning, $\varphi = 0$. Due to $\int_{\omega} x_2^2 \, d\tilde{x} = \int_{\omega} x_3^2 \, d\tilde{x} = \frac{1}{4\pi}$ and $\tau = \int_{\omega} x_2^2 \, d\tilde{x} + \int_{\omega} x_3^2 \, d\tilde{x} = \frac{1}{2\pi}$, formula (3.26) simplifies to

$$\begin{aligned} Q^*(F, t) &= \frac{3\mu}{4\pi} (F_{12}^2 + F_{13}^2) + \frac{\mu}{2\pi} F_{23}^2 + 3\mu t^2 \\ &= \frac{3\mu}{4\pi} |F|^2 - \frac{\mu}{4\pi} F_{23}^2 + 3\mu t^2 \end{aligned}$$

for $(F, t) \in \mathbb{R}_{\text{skew}}^{3 \times 3} \times \mathbb{R}$.

3.3 Technical tools for the upper bounds

Inner perturbation arguments have proven to be useful cornerstones when it comes to the construction of locally volume-preserving deformations. The latter are needed to find recovery sequences in dimension reduction problems with an incompressibility constraint. In [61, Proposition 5.1], Conti & Dolzmann established a first lemma of this type for 3d-2d reductions in the context of incompressible membranes. We tailored the statement for 3d-1d reductions in Chapter 2 on incompressible strings. Since the arguments in Lemma 2.5 are symmetric in the two cross-section variables x_2 and x_3 , we can state a slightly modified version, which is formulated in a way that allows its direct application in Sections 3.4 and 3.5.

Lemma 3.10 (Inner perturbations). *Let $\gamma, \kappa > 0$ and $J \subset J' \subset \mathbb{R}$ be bounded closed intervals such that $0 \in J$ and J is compactly contained in the interior of J' . Further, let $Q_L := [0, L] \times J \times J$ and $Q'_L := [0, L] \times J' \times J'$.*

If $(y_\varepsilon)_\varepsilon \subset C^2(Q'_L; \mathbb{R}^3)$ satisfies

$$\|\partial_3 y_\varepsilon\|_{C^1(Q'_L; \mathbb{R}^3)} = \mathcal{O}(\varepsilon^\kappa) \quad \text{or} \quad \|\partial_2 y_\varepsilon\|_{C^1(Q'_L; \mathbb{R}^3)} = \mathcal{O}(\varepsilon^\kappa)$$

and

$$\|\det \nabla^\varepsilon y_\varepsilon - 1\|_{C^1(Q'_L)} = \mathcal{O}(\varepsilon^\gamma), \quad (3.28)$$

then there exists a sequence $(u_\varepsilon)_\varepsilon \subset C^1(Q_L; \mathbb{R}^3)$ with

$$\det \nabla^\varepsilon u_\varepsilon = 1 \quad \text{everywhere in } Q_L$$

for ε sufficiently small, and

$$\|u_\varepsilon - y_\varepsilon\|_{C^1(Q_L; \mathbb{R}^3)} = \mathcal{O}(\varepsilon^{\gamma+\kappa}). \quad (3.29)$$

Replacing $\mathcal{O}(\varepsilon^\gamma)$ with $o(\varepsilon^\gamma)$ in (3.28) yields (3.29) with right-hand side $o(\varepsilon^{\gamma+\kappa})$.

With the help of divergence-free extensions in the cross-section variables, we can prove the following approximation result, which is going to be another useful ingredient for the proof of the upper bounds in Sections 3.4 and 3.5.

Lemma 3.11 (Approximation under divergence constraints). *Let $\beta \in L^2(0, L; H^1(\omega; \mathbb{R}^3))$ and $\rho \in L^2((0, L) \times \omega)$ with $\rho(x_1, \cdot)$ affine for almost every $x_1 \in (0, L)$ be related via*

$$\widetilde{\operatorname{div}} \tilde{\beta} = \rho.$$

Further, let $(\rho_\delta)_\delta \subset C^2([0, L] \times \mathbb{R}^2)$ be a sequence of functions that are affine in the cross-section variables satisfying $\rho_\delta \rightarrow \rho$ in $L^2((0, L) \times \omega)$ as $\delta \rightarrow 0$.

Then, there exists a sequence $(\beta_\delta)_\delta \subset C^2([0, L] \times \mathbb{R}^2; \mathbb{R}^3)$ with

$$\widetilde{\operatorname{div}} \tilde{\beta}_\delta = \rho_\delta$$

for every δ and $\beta_\delta \rightarrow \beta$ in $L^2(0, L; H^1(\omega; \mathbb{R}^3))$ as $\delta \rightarrow 0$.

Proof. Due to the structural properties of ρ and ρ_δ , one can find $a, b, c \in L^2(0, L)$ and $a_\delta, b_\delta, c_\delta \in C^2([0, L])$ such that

$$\begin{aligned} \rho(x) &= a(x_1)x_2 + b(x_1)x_3 + c(x_1), \\ \rho_\delta(x) &= a_\delta(x_1)x_2 + b_\delta(x_1)x_3 + c_\delta(x_1). \end{aligned}$$

With the definitions

$$\begin{aligned}\Xi(x) &:= \frac{1}{2}(a(x_1)x_2^2 + c(x_1)x_2)e_2 + \frac{1}{2}(b(x_1)x_3^2 + c(x_1)x_3)e_3, \\ \Xi_\delta(x) &:= \frac{1}{2}(a_\delta(x_1)x_2^2 + c_\delta(x_1)x_2)e_2 + \frac{1}{2}(b_\delta(x_1)x_3^2 + c_\delta(x_1)x_3)e_3,\end{aligned}$$

it holds that

$$\Xi_\delta \rightarrow \Xi \quad \text{in } L^2((0, L) \times \omega) \text{ as } \delta \rightarrow 0, \quad (3.30)$$

and by straight-forward calculation,

$$\partial_2 \Xi_2 + \partial_3 \Xi_3 = \rho \quad \text{and} \quad \partial_2(\Xi_\delta \cdot e_2) + \partial_3(\Xi_\delta \cdot e_3) = \rho_\delta. \quad (3.31)$$

Hence, $\beta - \Xi \in L^2(0, L; H_{\text{div}}^1(\omega; \mathbb{R}^3))$, and after divergence-free extension in the cross-section variables according to [120, Proposition 3.1, Corollary 3.2], one can view $\beta - \Xi$ as an element of $L^2(0, L; H_{\text{div}}^1(\mathbb{R}^2; \mathbb{R}^3))$, where

$$H_{\text{div}}^1(U; \mathbb{R}^3) = \{w \in H^1(U; \mathbb{R}^3) : \partial_2 w_2 + \partial_3 w_3 = 0 \text{ a.e. in } U\}$$

for an open subset $U \subset \mathbb{R}^2$.

A standard mollification argument yields a sequence $(\hat{\beta}_\delta)_\delta \subset C^2([0, L] \times \mathbb{R}^2; \mathbb{R}^3)$ of functions that are divergence-free in the last two variables, i.e., for any δ

$$\partial_2(\hat{\beta}_\delta \cdot e_2) + \partial_3(\hat{\beta}_\delta \cdot e_3) = 0 \quad \text{in } [0, L] \times \mathbb{R}^2, \quad (3.32)$$

such that $\hat{\beta}_\delta \rightarrow \beta - \Xi$ in $L^2(0, L; H^1(\mathbb{R}^2; \mathbb{R}^3))$ as $\delta \rightarrow 0$.

Finally, in view of (3.31) and (3.32) as well as (3.30), setting $\beta_\delta = \hat{\beta}_\delta + \Xi_\delta$ provides the desired sequence. \square

3.4 The regime $\alpha = 2$

The following Γ -convergence theorem is the first main result of this chapter. It provides a reduced one-dimensional model for incompressible rods, which involves, besides the deformation of the mid-fiber, quantities related to bending and torsion effects.

Theorem 3.12 (Γ -limit for $\alpha = 2$). *Let $\mathcal{I}_\varepsilon^{(2)}$ for $\varepsilon > 0$ be the functional introduced in (3.7) with $\alpha = 2$. Moreover, let*

$$\begin{aligned}\mathcal{I}^{(2)} : H^2(0, L; \mathbb{R}^3) \times H^1(0, L; \mathbb{R}^{3 \times 2}) &\rightarrow [0, \infty], \\ (u, D) &\mapsto \begin{cases} \frac{1}{2} \int_0^L Q^*(A(x_1), 0) \, dx_1 & \text{for } (u, D) \in \mathcal{A}^{(2)}, \\ \infty & \text{otherwise,} \end{cases}\end{aligned} \quad (3.33)$$

where $A := R^T R'$ with $R := (u'|D)$, Q^* is defined in (3.23), and

$$\mathcal{A}^{(2)} := \{(u, D) \in H^2(0, L; \mathbb{R}^3) \times H^1(0, L; \mathbb{R}^{3 \times 2}) : (u'|D) \in \text{SO}(3) \text{ a.e. in } (0, L)\}.$$

i) (Compactness) *For every sequence $(u_\varepsilon)_\varepsilon \subset H^1(\Omega; \mathbb{R}^3) \cap L_0^2(\Omega; \mathbb{R}^3)$ and $\sup_{\varepsilon > 0} \mathcal{I}_\varepsilon^{(2)}(u_\varepsilon) < \infty$, there exists a subsequence (not relabeled) and $(u, D) \in \mathcal{A}^{(2)}$ such that*

$$\begin{aligned}u_\varepsilon &\rightarrow u \text{ in } H^1(\Omega; \mathbb{R}^3), \\ \frac{1}{\varepsilon} \tilde{\nabla} u_\varepsilon &\rightarrow D \text{ in } L^2(\Omega; \mathbb{R}^{3 \times 2}).\end{aligned} \quad (3.34)$$

ii) (Variational limit) *The sequence $(\mathcal{I}_\varepsilon^{(2)})_\varepsilon$ Γ -converges for $\varepsilon \rightarrow 0$ to $\mathcal{I}^{(2)}$ with respect to the convergence (3.34), that is, the following two conditions are fulfilled:*

ii') (Lower bound) Let $(u_\varepsilon)_\varepsilon \subset H^1(\Omega; \mathbb{R}^3)$ satisfy (3.34) for $(u, D) \in \mathcal{A}^{(2)}$, then

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{(2)}(u_\varepsilon) \geq \mathcal{I}^{(2)}(u, D);$$

ii'') (Upper bound) For every $(u, D) \in \mathcal{A}^{(2)}$ there exists a sequence $(u_\varepsilon)_\varepsilon \subset H^1(\Omega; \mathbb{R}^3)$ satisfying (3.34) and

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{(2)}(u_\varepsilon) \leq \mathcal{I}^{(2)}(u, D).$$

Proof. Ad i). Let $(u_\varepsilon)_\varepsilon$ be a sequence with uniformly bounded energy and vanishing mean value. It follows from $W_0 \leq W$ and hypothesis (H2) that

$$\frac{C_1}{\varepsilon^2} \int_{\Omega} \text{dist}^2(\nabla^\varepsilon u_\varepsilon, \text{SO}(3)) \leq \frac{1}{\varepsilon^2} \int_{\Omega} W_0(\nabla^\varepsilon u_\varepsilon) \, dx \leq \mathcal{I}_\varepsilon^{(2)}(u_\varepsilon) \leq C$$

for a constant $C > 0$. The statement i) is now an immediate consequence of the compactness result in [156, Theorem 2.1].

Ad ii'). Recalling the definitions of the energy densities W_k in (3.8) and the associated auxiliary functionals $\mathcal{I}_{k,\varepsilon}^{(2)}$ from (3.9), we obtain

$$\mathcal{I}_\varepsilon^{(2)}(u_\varepsilon) \geq \mathcal{I}_{k,\varepsilon}^{(2)}(u_\varepsilon)$$

for every $\varepsilon > 0$ and $k \in \mathbb{N}$. The lower bound in the compressible case [156, Theorem 3.1] yields that

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{(2)}(u_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \mathcal{I}_{k,\varepsilon}^{(2)}(u_\varepsilon) \geq \mathcal{I}_k^{(2)}(u, D), \quad (3.35)$$

where

$$\mathcal{I}_k^{(2)}(u, D) = \frac{1}{2} \int_0^L Q_k^*(A(x_1), 0) \, dx_1$$

with $A = R^T R'$, $R = (u'|D)$, and Q_k^* defined as in (3.24). In view of Corollary 3.7 and the monotonicity of Q_k^* with respect to k , meaning, $Q_k^* \leq Q_{k+1}^*$ for all $k \in \mathbb{N}$, the theorem on monotone convergence implies that $\mathcal{I}_k^{(2)}(u, D) \rightarrow \mathcal{I}^{(2)}(u, D)$ for $k \rightarrow \infty$. Thus, together with (3.35), we finally conclude that

$$\liminf_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{(2)}(u_\varepsilon) \geq \lim_{k \rightarrow \infty} \mathcal{I}_k^{(2)}(u, D) = \mathcal{I}^{(2)}(u, D).$$

Ad ii''). Let us fix $(u, D) \in \mathcal{A}^{(2)}$. We split the proof of the upper bound into three steps, starting with the construction of recovery sequences in the case when we have extra regularity for the refined limit deformations.

Step 1: Recovering smooth limit functions. Let $u \in C^3([0, L]; \mathbb{R}^3)$, $D \in C^2([0, L]; \mathbb{R}^{3 \times 2})$ such that $R = (u'|D) \in \text{SO}(3)$ everywhere in $[0, L]$. Moreover, suppose that $\beta \in C^2([0, L] \times \mathbb{R}^2; \mathbb{R}^3)$ satisfies

$$\text{Tr} \left(A(x_1)(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta(x) \right) = 0 \quad (3.36)$$

for every $x \in [0, L] \times \mathbb{R}^2$. Note that in the following, we drop the arguments x_1 and x in our notation when they are clear from the context.

Now, let $J \subset J' \subset \mathbb{R}$ be two intervals as in Lemma 3.10 with $\omega \subset J \times J$. Inspired by the recovery sequence in the situation without incompressibility, see [156, Theorem 3.1], we define for every $\varepsilon > 0$ and $x \in Q'_L := [0, L] \times J' \times J'$,

$$y_\varepsilon(x) = u(x_1) + \varepsilon R(x_1)(x_2 e_2 + x_3 e_3) + \varepsilon^2 R(x_1) \beta(x).$$

By construction, $(y_\varepsilon)_\varepsilon \subset C^2(Q'_L; \mathbb{R}^3)$ converges to (u, D) in the sense of (3.34) and has the property that

$$\|\partial_3 y_\varepsilon\|_{C^1(Q'_L; \mathbb{R}^{3 \times 3})} = \mathcal{O}(\varepsilon). \quad (3.37)$$

The rescaled gradient of y_ε is given by

$$\nabla^\varepsilon y_\varepsilon = R + \varepsilon R(A(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta) + \varepsilon^2 \partial_1(R\beta) \otimes e_1, \quad (3.38)$$

and hence,

$$\det(\nabla^\varepsilon y_\varepsilon) = \det(R^T \nabla^\varepsilon y_\varepsilon) = 1 + \varepsilon \operatorname{Tr}(A(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta) + \mathcal{O}(\varepsilon^2),$$

in view of the identity $\det(\operatorname{Id} + F) = 1 + \operatorname{Tr} F + \operatorname{Tr} \operatorname{cof} F + \det F$ for every $F \in \mathbb{R}^{3 \times 3}$. Together with the vanishing trace assumption (3.36), we conclude that

$$\|\det(\nabla^\varepsilon y_\varepsilon) - 1\|_{C^1(Q'_L)} = \mathcal{O}(\varepsilon^2). \quad (3.39)$$

In light of (3.37) and (3.39), we can now apply Proposition 3.10 with the choices $\gamma = 2$ and $\kappa = 1$ to obtain a sequence $(u_\varepsilon)_\varepsilon \subset C^1(\bar{\Omega}; \mathbb{R}^3)$ satisfying $\det \nabla^\varepsilon u_\varepsilon = 1$ everywhere in $\bar{\Omega}$ and

$$\|y_\varepsilon - u_\varepsilon\|_{C^1(\bar{\Omega}; \mathbb{R}^3)} = \mathcal{O}(\varepsilon^3); \quad (3.40)$$

due to the convergence behavior of $(y_\varepsilon)_\varepsilon$, this shows in particular that $(u_\varepsilon)_\varepsilon$ converges to (u, D) in the sense of (3.34) as well.

Moreover, the combination of (3.38) and (3.40) gives that

$$R^T \nabla^\varepsilon u_\varepsilon = \operatorname{Id} + \varepsilon(A(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta) + \mathcal{O}(\varepsilon^2),$$

and hence, by the Taylor expansion in (3.12),

$$W(\nabla^\varepsilon u_\varepsilon) = W_0(\nabla^\varepsilon u_\varepsilon) = W_0(R^T \nabla^\varepsilon u_\varepsilon) = \frac{\varepsilon^2}{2} Q(A(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta) + \mathcal{O}(\varepsilon^3);$$

also, we have used here (3.3) under consideration of $\det \nabla^\varepsilon u_\varepsilon = 1$ and the assumption of frame indifference (H3). Altogether, this shows that

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{(2)}(u_\varepsilon) \leq \frac{1}{2} \int_{\Omega} Q(A(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta) \, dx.$$

Step 2: Approximation and optimization. To approximate $(u, D) \in \mathcal{A}^{(2)}$ suitably by smooth functions, we invoke the same argument as in [156, Theorem 3.1], which provides a sequence $(R_\delta)_\delta \subset C^2([0, L]; \operatorname{SO}(3))$ such that

$$R_\delta \rightarrow R = (u' | D) \quad \text{in } H^1(0, L; \mathbb{R}^{3 \times 3}) \text{ for } \delta \rightarrow 0,$$

and thus, $(R_\delta)_\delta$ converges to R also uniformly, which implies in particular that

$$A_\delta := (R_\delta)^T R'_\delta \rightarrow R^T R' = A \text{ in } L^2(0, L; \mathbb{R}^{3 \times 3}). \quad (3.41)$$

For any δ , we consider

$$u_\delta(x_1) := \int_0^{x_1} R_\delta(t) e_1 dt - c_\delta \quad \text{and} \quad D_\delta(x_1) := (R_\delta(x_1) e_2 | R_\delta(x_1) e_3) \quad \text{for } x_1 \in (0, L),$$

where $c_\delta \in \mathbb{R}$ is chosen in such a way that u_δ has the same mean value as u . Then, $R_\delta = (u'_\delta | D_\delta)$, and

$$\begin{aligned} u_\delta &\rightarrow u \text{ in } H^2(0, L; \mathbb{R}^3), \\ D_\delta &\rightarrow D \text{ in } H^1(0, L; \mathbb{R}^{3 \times 2}) \end{aligned} \tag{3.42}$$

as $\delta \rightarrow 0$.

Next, we introduce the functions

$$\begin{aligned} \xi(x) &:= A(x_1)(x_2 e_2 + x_3 e_3), \\ \xi_\delta(x) &:= A_\delta(x_1)(x_2 e_2 + x_3 e_3), \end{aligned}$$

for $x \in (0, L) \times \omega$, and take $\beta_A(x_1, \cdot)$ for $x_1 \in (0, L)$ as the unique solution to the minimization problem defining $Q^*(A(x_1), 0)$, that is,

$$\beta_A(x) = \beta^{\xi(x_1, \cdot)}(\tilde{x}) \quad \text{for } x = (x_1, \tilde{x}) \in (0, L) \times \omega, \tag{3.43}$$

cf. (3.23) and Lemma 3.1. Notice that in light of Remarks 3.3 and 3.2, $\beta_A \in L^2(\omega; H^1(\omega; \mathbb{R}^3))$.

Considering (3.41), Corollary 3.11 applied with $\rho = -\xi \cdot e_1$, $\rho_\delta = -\xi_\delta \cdot e_1$ and β as in (3.43) gives rise to a sequence $(\beta_\delta)_\delta \in C^2([0, L] \times \mathbb{R}^2; \mathbb{R}^3)$ that satisfies the trace condition

$$\text{Tr}(\xi_\delta | \tilde{\nabla} \beta_\delta) = 0$$

on all of $[0, L] \times \mathbb{R}^2$ and

$$\beta_\delta \rightarrow \beta \text{ in } L^2(0, L; H^1(\omega; \mathbb{R}^3)). \tag{3.44}$$

Step 3: Diagonalization. For every δ , we repeat Step 1 with $u = u_\delta$, $D = D_\delta$ and $\beta = \beta_\delta$ to obtain a sequence $(u_{\delta, \varepsilon})_\varepsilon \subset C^1(\bar{\Omega}; \mathbb{R}^3)$ satisfying $\det \nabla^\varepsilon u_{\delta, \varepsilon} = 1$ for all $\varepsilon > 0$ sufficiently small, and

$$\begin{aligned} u_{\delta, \varepsilon} &\rightarrow u_\delta \text{ in } H^1(\Omega; \mathbb{R}^3), \\ \frac{1}{\varepsilon} \tilde{\nabla} u_{\delta, \varepsilon} &\rightarrow D_\delta \text{ in } L^2(\Omega; \mathbb{R}^{3 \times 2}), \end{aligned} \tag{3.45}$$

as well as

$$\limsup_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{(2)}(u_{\delta, \varepsilon}) \leq \limsup_{\delta \rightarrow 0} \frac{1}{2} \int_\Omega Q(A_\delta(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta_\delta) dx = \mathcal{I}^{(2)}(u, D). \tag{3.46}$$

For the last equality, we have exploited (3.41), (3.44), the optimality of β from (3.43), and the fact that Q is a quadratic form.

Finally, we extract a diagonal sequence $(u_\varepsilon)_\varepsilon$ from $(u_{\delta, \varepsilon})_{\delta, \varepsilon}$ in the sense of Attouch [9, Lemma 1.15, 1.16] to conclude the proof of the upper bound. Indeed, combining (3.42) with (3.45), and (3.46) gives $(u_\varepsilon)_\varepsilon$ satisfying (3.34) and

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{(2)}(u_\varepsilon) \leq \mathcal{I}^{(2)}(u, D),$$

as claimed. □

Remark 3.13 (Comparison with models involving compressibility). Even though the reduced limit energies derived from three-dimensional nonlinear elasticity with or without incompressibility constraint are different, their qualitative structure turns out to be rather similar, in the sense that they are both integral functionals with identical domains, as a comparison between Theorem 3.12 and [156, Theorem 3.1] reveals.

Intuitively speaking, this means that the presence of local volume preservation does not affect the set of admissible deformations in the resulting limit problems, which may seem surprising. In fact, the volumetric constraint in the definition of $\mathcal{I}_\varepsilon^{(2)}$ translates asymptotically for $\varepsilon \rightarrow 0$ into a trace constraint, which only enters in the formula for the finite-valued limit energy density.

Notice that the same effect can be observed in the cases $\alpha > 2$ (see Section 3.5), as well as the string regime $\alpha = 0$, which is analyzed in Chapter 2.

Remark 3.14 (Incorporating external forces). The statement of Theorem 3.12 still holds if we replace the sequence of elastic energy functionals $(\mathcal{I}_\varepsilon^{(2)})_\varepsilon$ with the system energies $(\mathcal{J}_\varepsilon^{(2)})_\varepsilon$ as in (3.6); the corresponding Γ -limit (with respect to the convergence (3.34)) is given by

$$\mathcal{J}^{(2)}(u, D) = \mathcal{I}^{(2)}(u, D) - \int_0^L f \cdot u \, dx_1 \quad (3.47)$$

for $(u, D) \in H^2(0, L; \mathbb{R}^3) \times H^1(0, L; \mathbb{R}^{3 \times 2})$, given that the external force term constitutes a continuous perturbation.

Furthermore, we observe that introducing

$$h : (0, L) \rightarrow \mathbb{R}^3, \quad t \mapsto \int_0^t f(x_1) \, dx_1, \quad (3.48)$$

as the primitive of f allows us, in view of (3.5), to rewrite (3.47) as

$$\mathcal{J}^{(2)}(u, D) = \mathcal{I}^{(2)}(u, D) + \int_0^L h \cdot u' \, dx_1 \quad (3.49)$$

for $(u, D) \in \mathcal{A}^{(2)}$. Hence, just like $\mathcal{I}^{(2)}$, the functional $\mathcal{J}^{(2)}$ is invariant under translation.

In the second part of this section, we complement the asymptotic analysis of the sequence $(\mathcal{J}_\varepsilon^{(2)})_\varepsilon$ by calculating the Euler-Lagrange equations of the limit functional $\mathcal{J}^{(2)}$, which characterize its stationary points. First, let us briefly introduce the necessary notation of the stress and its moments of first order. For $(u, D) \in \mathcal{A}^{(2)}$, let

$$A := R^T R' \in L^2(\omega; \mathbb{R}_{\text{skew}}^{3 \times 3}), \quad (3.50)$$

recalling that $R = (u'|D)$. Furthermore, let $\beta_A \in L^2(0, L; H^1(\omega; \mathbb{R}^3))$ be such that $\beta_A(x_1, \cdot)$ is a solution to the variational problem defining $Q^*(A(x_1), 0)$ for $x_1 \in (0, L)$, cf. (3.23) and (3.43). The stress $E \in L^2((0, L) \times \omega; \mathbb{R}^{3 \times 3})$ associated with (u, D) is then given as

$$E = L(A(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta_A) \quad (3.51)$$

with L as in (3.14), and $\tilde{E}, \hat{E} \in L^2(0, L; \mathbb{R}^{3 \times 3})$ denote the first-order moments of E , i.e.,

$$\tilde{E} = \int_\omega x_2 E \, d\tilde{x} \quad \text{and} \quad \hat{E} = \int_\omega x_3 E \, d\tilde{x}. \quad (3.52)$$

Proposition 3.15 (Euler-Lagrange equations). *Let $\mathcal{J}^{(2)}$ be as in (3.49) with (3.48) and (3.5). Then, $(u, D) \in \mathcal{A}^{(2)}$ is a stationary point of $\mathcal{J}^{(2)}$ if and only if*

$$\begin{aligned}\check{E}'_{11} - \check{E}'_{22} &= A_{13}(\hat{E}_{21} - \check{E}_{31}) - A_{23}(\hat{E}_{11} - \hat{E}_{33}) - h \cdot De_1, \\ \hat{E}'_{11} - \hat{E}'_{33} &= -A_{12}(\hat{E}_{21} - \check{E}_{31}) + A_{23}(\check{E}_{11} - \check{E}_{22}) - h \cdot De_2, \\ \hat{E}'_{21} - \check{E}'_{31} &= A_{12}(\hat{E}_{11} - \hat{E}_{33}) - A_{13}(\check{E}_{11} - \check{E}_{22}),\end{aligned}$$

and

$$\begin{aligned}\check{E}_{11}(0) - \check{E}_{22}(0) &= \check{E}_{11}(L) - \check{E}_{22}(L) = 0, \\ \hat{E}_{11}(0) - \hat{E}_{33}(0) &= \hat{E}_{11}(L) - \hat{E}_{33}(L) = 0, \\ \hat{E}_{21}(0) - \check{E}_{31}(0) &= \hat{E}_{21}(L) - \check{E}_{31}(L) = 0,\end{aligned}$$

where A, \check{E}, \hat{E} are defined as in (3.50) and (3.52), respectively.

Proof. The calculation of the first variation of $\mathcal{J}^{(2)}$, which we will identify with a functional on $H^1(0, L; \text{SO}(3))$ in the following, can be done similarly to [158, Lemma 2.3]. Precisely, for any $(u, D) \in \mathcal{A}^{(2)}$ and $B \in H^1(0, L; \mathbb{R}_{\text{skew}}^{3 \times 3})$, we consider a curve

$$\gamma : (-1, 1) \rightarrow H^1(0, L; \text{SO}(3)) \quad \text{with } \gamma(0) = R = (u'|D) \text{ and } \partial_s \gamma(0) = RB;$$

notice that the tangent space of $H^1(0, L; \text{SO}(3))$ at R can be identified with $RH^1(0, L; \mathbb{R}_{\text{skew}}^{3 \times 3})$. Evaluating $\mathcal{J}^{(2)}$ along this curve gives

$$\mathcal{J}^{(2)}(\gamma(s)) = \frac{1}{2} \int_0^L Q^*(\gamma(s)^T \gamma(s)', 0) \, dx_1 + \int_0^L h \cdot \gamma(s) e_1 \, dx_1$$

for $s \in (-1, 1)$. In view of

$$\begin{aligned}\frac{d}{ds} \Big|_{s=0} \gamma(s)^T \gamma(s)' &= -B(R^T R') + (R^T R')B + B' \\ &= AB - BA + B' =: H \in L^2(0, L; \mathbb{R}_{\text{skew}}^{3 \times 3}),\end{aligned} \tag{3.53}$$

we find that

$$\begin{aligned}\frac{d}{ds} \Big|_{s=0} \mathcal{J}^{(2)}(\gamma(t)) &= \int_{\Omega} E : (H(x_2 e_2 + x_3 e_3) | 0 | 0) \, dx \\ &\quad + \int_{\Omega} E : (0 | \tilde{\nabla} \beta_H) \, dx + \int_0^L h \cdot R B e_1 \, dx_1,\end{aligned} \tag{3.54}$$

where $\beta_H \in L^2(0, L; H^1(\omega; \mathbb{R}^3))$ is such that $\beta_H(x_1, \cdot)$ solves the minimization problem in (3.23) with the argument $(H(x_1), 0)$, cf. Remark 3.3.

In view of (3.52), the first integral in (3.54) can be rewritten as

$$\int_0^L \check{E} e_1 \cdot H e_2 + \hat{E} e_1 \cdot H e_3 \, dx_1. \tag{3.55}$$

The treatment of the second term in (3.54) exploits the Euler-Lagrange equations (3.20) and the trace condition (3.21) applied to every affine function $\xi(\tilde{x}) = H(x_1)(x_2 e_2 + x_3 e_3)$ with $x_1 \in (0, L)$. If we write $\lambda_H(x_1, \cdot)$ for the corresponding Lagrange multipliers, λ_H can be viewed as an element

in $L^2((0, L) \times \omega)$ due to the linear dependence on the affine input pointed out in Remark 3.2. Let us introduce $\check{\lambda}_H, \hat{\lambda}_H \in L^2(0, L)$ as the first moments of λ_H , that is,

$$\check{\lambda}_H(x_1) = \int_{\omega} \lambda_H x_2 \, d\tilde{x} \quad \text{and} \quad \hat{\lambda}_H(x_1) = \int_{\omega} \lambda_H x_3 \, d\tilde{x}.$$

Then,

$$\begin{aligned} \int_{\Omega} E : (0 | \tilde{\nabla} \beta_H) \, dx &= -\frac{1}{2} \int_{\Omega} \lambda_H \tilde{\text{div}} \tilde{\beta}_H \, dx = \frac{1}{2} \int_{\Omega} \lambda_H (H_{12} x_2 + H_{13} x_3) \, dx \\ &= \frac{1}{2} \int_0^L \check{\lambda}_H e_1 \cdot H e_2 + \hat{\lambda}_H e_1 \cdot H e_3 \, dx_1. \end{aligned} \quad (3.56)$$

On the other hand, the choice of test fields $\phi = (0, \frac{1}{2}x_2^2, 0)$ and $\phi = (0, 0, \frac{1}{2}x_3^2)$ in (3.20) yields that

$$\check{\lambda}_H = -2\check{E}_{22} \quad \text{and} \quad \hat{\lambda}_H = -2\hat{E}_{33}. \quad (3.57)$$

Therefore, by joining (3.54) with (3.55) and (3.56), we find that

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \mathcal{J}^{(2)}(\gamma(t)) &= \int_0^L (\check{E} e_1 - \check{E}_{22} e_1) \cdot H e_2 \\ &\quad + (\hat{E} e_1 - \hat{E}_{33} e_1) \cdot H e_3 + h \cdot R B e_1 \, dx_1. \end{aligned} \quad (3.58)$$

To conclude, it suffices now to specialize B to three classes of test fields, recalling that H depends on B through (3.53): For $\psi, \theta, \sigma \in H^1(0, L)$, we plug

$$\begin{aligned} B &= \psi e_1 \otimes e_2 - \psi e_2 \otimes e_1, \\ B &= \theta e_1 \otimes e_3 - \theta e_3 \otimes e_1, \\ B &= \sigma e_2 \otimes e_3 - \sigma e_3 \otimes e_2, \end{aligned}$$

into (3.58), which gives rise to the system

$$\begin{aligned} \int_0^L (\check{E}_{11} - \check{E}_{22}) \psi' + A_{13}(\hat{E}_{21} - \check{E}_{31}) \psi - A_{23}(\hat{E}_{11} - \hat{E}_{33}) \psi - h \cdot R e_2 \psi \, dx_1 &= 0, \\ \int_0^L (\hat{E}_{11} - \hat{E}_{33}) \theta' - A_{12}(\hat{E}_{21} - \check{E}_{31}) \theta + A_{23}(\check{E}_{11} - \check{E}_{22}) \theta - h \cdot R e_3 \theta \, dx_1 &= 0, \\ \int_0^L (\hat{E}_{21} - \check{E}_{31}) \sigma' + A_{12}(\hat{E}_{11} - \hat{E}_{33}) - \sigma A_{13}(\check{E}_{11} - \check{E}_{22}) \sigma \, dx_1 &= 0. \end{aligned}$$

This corresponds to the weak formulation of the stated equations and boundary conditions. \square

In the special case of isotropic incompressible rods with circular cross section, the characterizing equations for stationary points simplify considerably.

Example 3.16. We adopt the setting of Example 3.9, that is, ω is a circle around the origin with unit measure and W_0 is supposed to be isotropic, which implies that Q is of the form (3.25) with Lamé coefficients $\lambda \in \mathbb{R}$ and $\mu > 0$ giving rise to a positive bulk modulus, that is, $2\mu + 3\lambda > 0$. Then, under the assumptions of Proposition 3.15, $(u, D) \in \mathcal{A}^{(2)}$ is a stationary point of $\mathcal{J}^{(2)}$ if and only if

$$\begin{cases} A'_{12} = -\frac{4\pi}{3\mu} h \cdot D e_1, & A_{12}(0) = A_{12}(L) = 0, \\ A'_{13} = -\frac{4\pi}{3\mu} h \cdot D e_2, & A_{13}(0) = A_{13}(L) = 0, \\ A_{23} = 0. \end{cases} \quad (3.59)$$

Indeed, in this situation, L takes the form

$$LF = 2\mu F^{\text{sym}} + \lambda \text{Tr}(F) \text{Id} \quad \text{for } F \in \mathbb{R}^{3 \times 3},$$

and β_A can be determined to be

$$\beta_A(x) = -\frac{1}{4}(A_{12}(x_2^2 - x_3^2) + 2A_{13}x_2x_3)e_2 - \frac{1}{4}(A_{13}(x_3^2 - x_2^2) + 2A_{12}x_2x_3)e_3$$

for $x \in (0, L) \times \omega$; note that β_A emerges from the corresponding expression in the compressible case as the limit for diverging first Lamé coefficient. Then, the stress as defined in (3.51) becomes

$$\begin{aligned} E &= 2\mu(A(x_2e_2 + x_3e_3)|\tilde{\nabla}\beta_A)^{\text{sym}} \\ &= \mu \begin{pmatrix} 2(A_{12}x_2 + A_{13}x_3) & A_{23}x_3 & -A_{23}x_2 \\ A_{23}x_3 & -A_{12}x_2 - A_{13}x_3 & 0 \\ -A_{23}x_2 & 0 & -A_{12}x_2 - A_{13}x_3 \end{pmatrix} \end{aligned}$$

and, in view of (3.1), the first bending moments are

$$\check{E} = \frac{\mu}{4\pi} \begin{pmatrix} 2A_{12} & 0 & -A_{23} \\ 0 & -A_{12} & 0 \\ -A_{23} & 0 & -A_{12} \end{pmatrix} \quad \text{and} \quad \hat{E} = \frac{\mu}{4\pi} \begin{pmatrix} 2A_{13} & A_{23} & 0 \\ A_{23} & -A_{13} & 0 \\ 0 & 0 & -A_{13} \end{pmatrix}.$$

Finally, we insert these expressions into the equations of Proposition 3.15, which gives rise to (3.59).

We conclude the study of the regime $\alpha = 2$ with a brief comparison of the Euler-Lagrange equations for rods with and without a local volume-preservation constraint.

Remark 3.17 (Comparison with compressible rods). a) The difference between the result of Proposition 3.15 and [158, Lemma 2.3] lies in the presence of non-trivial bending terms \check{E}_{22} and \hat{E}_{33} . The latter arise as moments of the Lagrange multipliers that are necessary to accommodate the trace constraint in the minimization problem defining Q^* , cf. (3.57) and Remark 3.2.

b) In the special case of rods with circular cross-section of isotropic material, the structure of the Euler-Lagrange equations is identical, but the constant coefficients vary. To be more precise, the analogue of the factor $\frac{4\pi}{3\mu}$ in (3.59) is $\frac{4\pi(\lambda+\mu)}{\mu(3\lambda+2\mu)}$ when the assumption of incompressibility is dropped. The connection between these factors becomes apparent in the limit of diverging first Lamé coefficients. Notice that these factors are the Young modulus 1.9 for compressible and incompressible materials multiplied by 4π .

3.5 The regimes $\alpha > 2$

This section covers the asymptotic analysis in all the remaining scaling regimes. Like in the setting without incompressibility, these regimes share the common feature that the limit deformations correspond to rigid body motions. In order to extract more refined information on the reduced limit problems, it is useful to estimate the deviation of low energy sequences $(u_\varepsilon)_\varepsilon \subset H^1(\Omega; \mathbb{R}^3)$ (after suitable translation, global rotation, and scaling) from the identity. To this end, we follow [157, 185] in considering sequences $(v_\varepsilon)_\varepsilon \subset H^1(0, L; \mathbb{R}^2)$, $(w_\varepsilon)_\varepsilon \subset H^1(0, L)$ given by

$$\begin{aligned} v_\varepsilon &= \frac{1}{\varepsilon^{\alpha-2}} \int_\omega \tilde{u}_\varepsilon \, d\tilde{x}, \\ w_\varepsilon &= \frac{1}{\varepsilon^{\alpha-1}} \left(\int_\omega |\tilde{x}|^2 \, d\tilde{x} \right)^{-1} \int_\omega \tilde{u}_\varepsilon \cdot \tilde{x}^\perp \, d\tilde{x}; \end{aligned} \tag{3.60}$$

in the regime $\alpha \geq 3$, we also use $(z_\varepsilon)_\varepsilon \subset H^1(0, L)$ with

$$z_\varepsilon(x_1) = \frac{1}{\varepsilon^{\alpha-1}} \int_\omega (u_\varepsilon \cdot e_1 - x_1) \, d\tilde{x} \quad \text{for } x_1 \in (0, L), \quad (3.61)$$

which represent (appropriately scaled) versions of averaged length changes perpendicular and in-line with the midfiber, as well as torsion effects, respectively

Next, we introduce the limit energies in dependence of α . For the scaling regime $\alpha \in (2, 3)$, let $\mathcal{I}^{(\alpha)} : H^2(0, L; \mathbb{R}^2) \times H^1(0, L) \rightarrow [0, \infty)$ be given by

$$\mathcal{I}^{(\alpha)}(v, w) = \frac{1}{2} \int_0^L Q^*(B'(x_1), 0) \, dx_1, \quad (3.62)$$

where $Q^*(\cdot, 0)$ is the quadratic form in (3.23) (see also Remark 3.6) and $B \in H^1(0, L; \mathbb{R}_{\text{skew}}^{3 \times 3})$ is defined as

$$B = \begin{pmatrix} 0 & -v'_1 & -v'_2 \\ v'_1 & 0 & -w \\ v'_2 & w & 0 \end{pmatrix}. \quad (3.63)$$

In the von Kármán-type regime $\alpha = 3$ and for $\alpha > 3$, we define $\mathcal{I}^{(\alpha)} : H^2(0, L; \mathbb{R}^2) \times H^1(0, L) \times H^1(0, L) \rightarrow [0, \infty)$ via

$$\mathcal{I}^{(\alpha)}(v, w, z) = \frac{1}{2} \int_0^L Q^*(B'(x_1), s^{(\alpha)}(x_1)) \, dx_1; \quad (3.64)$$

here, the stored energy density Q^* results from the constrained variational problem defined in (3.23), B is as in (3.63) and $s^{(\alpha)} \in L^2(0, L)$ is given by

$$s^{(\alpha)} = \begin{cases} z' + \frac{1}{2}|v'|^2 & \text{for } \alpha = 3, \\ z' & \text{for } \alpha > 3. \end{cases} \quad (3.65)$$

With these definitions at hand, we can formulate the following Γ -convergence result.

Theorem 3.18 (Γ -limit for $\alpha > 2$). *Let $\mathcal{I}_\varepsilon^{(\alpha)}$ for $\varepsilon > 0$ be the functional introduced in (3.7) with $\alpha > 2$ and let $\mathcal{I}^{(\alpha)}$ as in (3.62) and (3.64), respectively.*

i) (Compactness) For every sequence $(\bar{u}_\varepsilon)_\varepsilon \subset H^1(\Omega; \mathbb{R}^3)$ with $\sup_{\varepsilon > 0} \mathcal{I}_\varepsilon^{(\alpha)}(\bar{u}_\varepsilon) < \infty$ there exist sequences of translations $(\bar{d}_\varepsilon)_\varepsilon \subset \mathbb{R}^2$, rotations $(\bar{R}_\varepsilon)_\varepsilon \in \text{SO}(3)$, and $\bar{R} \in \text{SO}(3)$ with $\bar{R}_\varepsilon \rightarrow \bar{R}$, as well as $v \in H^2(0, L; \mathbb{R}^2)$ and $w \in H^1(0, L)$ such that, with $u_\varepsilon := \bar{R}_\varepsilon \bar{u}_\varepsilon - \bar{d}_\varepsilon$, the following convergences hold up to the selection of subsequences:

$$\begin{aligned} v_\varepsilon &\rightarrow v \text{ in } H^1(0, L; \mathbb{R}^2), \\ w_\varepsilon &\rightarrow w \text{ in } H^1(0, L), \\ \frac{1}{\varepsilon^{\alpha-2}}(\nabla^\varepsilon u_\varepsilon - \text{Id}) &\rightarrow B \text{ in } L^2(\Omega; \mathbb{R}^{3 \times 3}); \end{aligned} \quad (3.66)$$

recall the definitions of v_ε , w_ε and B in (3.60) and (3.63), respectively. Additionally, if $\alpha \geq 3$, there exists $z \in H^1(0, L)$ such that $(z_\varepsilon)_\varepsilon \subset H^1(0, L)$ as in (3.61) fulfills

$$z_\varepsilon \rightarrow z \text{ in } H^1(0, L). \quad (3.67)$$

ii) (Variational limit) If $\alpha \in (2, 3)$, the sequence $(\mathcal{I}_\varepsilon^{(\alpha)})_\varepsilon$ Γ -converges to $\mathcal{I}^{(\alpha)}$ for $\varepsilon \rightarrow 0$ regarding the convergence (3.66). For $\alpha \geq 3$, $\mathcal{I}^{(\alpha)}$ is the Γ -limit of $(\mathcal{I}_\varepsilon^{(\alpha)})_\varepsilon$ with respect to the convergence (3.66) and (3.67).

Proof. Ad *i*). Since any sequence $(\bar{u}_\varepsilon)_\varepsilon$ with uniformly bounded energy satisfies

$$\frac{1}{\varepsilon^{2\alpha-2}} \int_{\Omega} W_0(\nabla^\varepsilon \bar{u}_\varepsilon) \, dx \leq \frac{1}{\varepsilon^{2\alpha-2}} \int_{\Omega} W(\nabla^\varepsilon \bar{u}_\varepsilon) \, dx = \mathcal{I}_\varepsilon^{(\alpha)}(\bar{u}_\varepsilon) \leq C$$

for a constant $C > 0$, and W_0 satisfies (H2), the statement follows directly from the literature on the compressible case. The compactness result for the von Karman-type case $\alpha = 3$ was first proven in [157, Theorem 2.2], for the remaining $\alpha > 2$, we refer to [185, Theorem 3.3], where all scaling regimes are covered in the more general context of curved rods.

Ad *ii*). As pointed out in the introductory Section 3.1.2, the Γ -limits $\mathcal{I}_k^{(\alpha)}$ of $(\mathcal{I}_{k,\varepsilon}^{(\alpha)})_\varepsilon$ as in (3.9) in the unconstrained setting provide lower bounds for the incompressible limit energy, which implies that

$$\mathcal{I}^{(\alpha)} \geq \sup_{k \in \mathbb{N}} \mathcal{I}_k^{(\alpha)}$$

with

$$\begin{cases} \mathcal{I}_k^{(\alpha)}(v, w) = \frac{1}{2} \int_0^L Q_k^*(B'(x_1), 0) \, dx_1, & \text{if } \alpha \in (2, 3), \\ \mathcal{I}_k^{(\alpha)}(v, w, z) = \frac{1}{2} \int_0^L Q_k^*(B'(x_1), s^{(\alpha)}(x_1)) \, dx_1, & \text{if } \alpha \geq 3, \end{cases}$$

cf. (3.63) and (3.65). The sought liminf-inequality follows with the help of Corollary 3.7.

For easier reading, we copy the structure of the proof of Theorem 3.12 and subdivide the arguments for the upper bound in three steps.

Step 1: Recovering smooth limit functions. Let $v \in C^3([0, L]; \mathbb{R}^2)$, $w \in C^2([0, L])$, and $B \in C^2([0, L]; \mathbb{R}_{\text{skew}}^{3 \times 3})$ as in (3.63), and if $\alpha \geq 3$, let also $z \in C^2([0, L])$. We choose $\beta \in C^2([0, L] \times \mathbb{R}^2; \mathbb{R}^3)$ such that

$$\begin{cases} \text{Tr}(B'(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta) = 0, & \text{if } \alpha \in (2, 3), \\ \text{Tr}(B'(x_2 e_2 + x_3 e_3) + s^{(\alpha)} e_1 | \tilde{\nabla} \beta) = 0 & \text{if } \alpha \geq 3. \end{cases} \quad (3.68)$$

Furthermore, let $Q_L \subset Q'_L$ be cubes as in Lemma 3.10 such that Q_L contains Ω .

The basis for our construction of locally volume-preserving approximations $(u_\varepsilon)_\varepsilon$ are the recovery sequences from the literature on the compressible cases [156, 185]. If $\alpha \in (2, 3)$, we set

$$y_\varepsilon(x) = \int_0^{x_1} R_\varepsilon(s) e_1 \, ds + \varepsilon R_\varepsilon(x_1) (x_2 e_2 + x_3 e_3) + \varepsilon^\alpha \beta(x)$$

for $x \in Q'_L$, where R_ε is the $\text{SO}(3)$ -valued matrix exponential $R_\varepsilon := \exp(\varepsilon^{\alpha-2} B)$ with B as in (3.63), cf. [185, Theorem 5.2, (5.24)]. For $\alpha \geq 3$, consider

$$y_\varepsilon(x) = x_\varepsilon + \varepsilon^{\alpha-2} (0, v(x_1)) + \varepsilon^{\alpha-1} B(x_1) (x_2 e_2 + x_3 e_3) + \varepsilon^{\alpha-1} z(x_1) e_1 + \varepsilon^\alpha \beta^{(\alpha)}(x)$$

for $x \in Q'_L$, where $x_\varepsilon = (x_1, \varepsilon x_2, \varepsilon x_3)$ and

$$\beta^{(\alpha)}(x) := \begin{cases} \beta(x) - \frac{1}{2} (x_2 \gamma(x_1) + x_3 \bar{\gamma}(x_1)), & \text{if } \alpha = 3, \\ \beta(x), & \text{if } \alpha > 3, \end{cases}$$

with $\gamma := 2wv'_2 e_1 + (w^2 + |v'_1|^2) e_2 + v'_1 v'_2 e_3$ and $\bar{\gamma} := -2wv'_1 e_1 + v'_1 v'_2 e_2 + (w^2 + |v'_2|^2) e_3$; for more details in the case $\alpha = 3$, see [157, Theorem 3.1 and (4.14), (4.15)], [185, Theorem 5.1], and for $\alpha \geq 3$, [185, Theorem 5.1].

By these constructions, $(y_\varepsilon)_\varepsilon$ satisfies the desired convergences (3.66), and if $\alpha \geq 3$ also (3.67). The specific structure of y_ε makes it immediate to see that

$$\|\partial_3 y_\varepsilon\|_{C^1(Q'_L; \mathbb{R}^3)} = \mathcal{O}(\varepsilon).$$

Moreover, (3.68) in conjunction with the computations in [157, 185] shows that

$$\|\det \nabla^\varepsilon y_\varepsilon - 1\|_{C^1(Q'_L)} = o(\varepsilon^{\alpha-1});$$

let us remark that in the case $\alpha = 3$, one even obtains that the deviation of $\det \nabla^\varepsilon y_\varepsilon$ from 1 behaves like $\mathcal{O}(\varepsilon^3)$, but indeed, $o(\varepsilon^2)$ is sufficient for our purposes.

Therefore, we can now apply Lemma 3.10 to find a sequence $(u_\varepsilon)_\varepsilon \subset C^1(\bar{\Omega}; \mathbb{R}^3)$ such that

$$\det \nabla^\varepsilon u_\varepsilon = 1 \quad \text{in } \bar{\Omega}$$

and

$$\|u_\varepsilon - y_\varepsilon\|_{C^1(\bar{\Omega}; \mathbb{R}^3)} = o(\varepsilon^\alpha).$$

This yields in particular, that $(u_\varepsilon)_\varepsilon$ converges as in (3.66), and additionally, if $\alpha \geq 3$, that $(u_\varepsilon)_\varepsilon$ satisfies (3.67). Analogously to [185, Theorem 5.1, Theorem 5.2], we obtain that

$$W(\nabla^\varepsilon u_\varepsilon) \leq \begin{cases} \frac{1}{2} \varepsilon^{2\alpha-2} Q(B'(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta) + o(\varepsilon^{2\alpha-2}), & \text{if } \alpha \in (2, 3), \\ \frac{1}{2} \varepsilon^{2\alpha-2} Q(B'(x_2 e_2 + x_3 e_3) + s^{(\alpha)} e_1 | \tilde{\nabla} \beta) + o(\varepsilon^{2\alpha-2}), & \text{if } \alpha = 3, \end{cases}$$

and thus,

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{(\alpha)}(u_\varepsilon) = \begin{cases} \frac{1}{2} \int_{\Omega} Q(B'(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta) \, dx, & \text{if } \alpha \in (2, 3), \\ \frac{1}{2} \int_{\Omega} Q(B'(x_2 e_2 + x_3 e_3) + s^{(\alpha)} e_1 | \tilde{\nabla} \beta) \, dx, & \text{if } \alpha = 3. \end{cases} \quad (3.69)$$

Step 2: Approximation and optimization. Let $v \in H^2(0, L; \mathbb{R}^2)$, $w \in H^1(0, L)$, and, if $\alpha \geq 3$, $z \in H^1(0, L)$. Then, there are $(\hat{v}_\delta)_\delta \subset C^3([0, L]; \mathbb{R}^2)$, $(\hat{w}_\delta)_\delta \subset C^2([0, L])$ such that

$$\begin{aligned} \hat{v}_\delta &\rightarrow v \text{ in } H^2(0, L; \mathbb{R}^2), \\ \hat{w}_\delta &\rightarrow w \text{ in } H^1(0, L). \end{aligned}$$

Furthermore, in the case $\alpha \geq 3$, let $(\hat{z}_\delta)_\delta \subset C^2([0, L])$ such that

$$\hat{z}_\delta \rightarrow z \text{ in } H^1(0, L).$$

We define $\beta \in L^2(0, L; H^1(\omega; \mathbb{R}^3))$ as follows: if $\alpha \in (2, 3)$ and $\alpha \geq 3$, then $\beta(x_1, \cdot)$ is the (unique) solution with vanishing mean value for the minimization problem defining $Q^*(B'(x_1), 0)$ and $Q^*(B'(x_1), s^{(\alpha)}(x_1))$, respectively, cf. also Lemma 3.1.

Now, we apply Corollary 3.11 with

$$\rho(x) = \begin{cases} B'(x_1)(x_2 e_2 + x_3 e_3) \cdot e_1, & \text{if } \alpha \in (2, 3), \\ B'(x_1)(x_2 e_2 + x_3 e_3) \cdot e_1 + s^{(\alpha)}(x_1) & \text{if } \alpha \geq 3, \end{cases}$$

and

$$\rho_\delta(x) = \begin{cases} B'_\delta(x_1)(x_2 e_2 + x_3 e_3) \cdot e_1, & \text{if } \alpha \in (2, 3), \\ B'_\delta(x_1)(x_2 e_2 + x_3 e_3) \cdot e_1 + s^{(\alpha)}_\delta(x_1) & \text{if } \alpha = 3, \end{cases}$$

where B_δ and $s_\delta^{(\alpha)}$ are given as in (3.63) and (3.65) with v, w replaced by their approximations $\hat{v}_\delta, \hat{w}_\delta$. This provides us with a sequence $(\beta_\delta)_\delta \subset C^2([0, L] \times \mathbb{R}^2; \mathbb{R}^3)$ such that

$$\begin{cases} \operatorname{Tr} (B'_\delta(x_2 e_2 + x_3 e_3) | \tilde{\nabla} \beta_\delta) = 0, & \text{if } \alpha \in (2, 3), \\ \operatorname{Tr} (B'_\delta(x_2 e_2 + x_3 e_3) + s_\delta^{(\alpha)} e_1 | \tilde{\nabla} \beta_\delta) = 0 & \text{if } \alpha \geq 3, \end{cases} \quad (3.70)$$

and $\beta_\delta \rightarrow \beta$ in $L^2(0, L; H^1(\omega; \mathbb{R}^3))$.

Step 3: Diagonalization. Exactly as in Step 3 of the proof of Theorem 3.12, we apply Step 1 for every δ with $v = \hat{v}_\delta, w = \hat{w}_\delta, z = \hat{z}_\delta$ if $\alpha \geq 3$, and the approximation of the optimal choice for β from Step 2, i.e., $\beta = \beta_\delta$. This way, we obtain a sequences $(u_{\delta, \varepsilon})_\varepsilon \subset H^1(\Omega; \mathbb{R}^3)$ that converge in the sense of (3.66), and (3.67) if $\alpha \geq 3$, and satisfy in view of (3.69), (3.23) and (3.70),

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^{(\alpha)}(u_{\delta, \varepsilon}) = \begin{cases} \frac{1}{2} \int_\Omega Q^*(B'_\delta, 0) \, dx, & \text{if } \alpha \in (2, 3), \\ \frac{1}{2} \int_\Omega Q^*(B'_\delta, s_\delta^{(\alpha)}) \, dx, & \text{if } \alpha = 3. \end{cases}$$

To finalize the proof, it suffices to extract a suitable diagonal sequence $(u_\varepsilon)_\varepsilon$ according to Atouch's lemma. \square

Chapter 4

Asymptotic analysis of deformation behavior in high-contrast fiber-reinforced materials: Rigidity and anisotropy

The arXiv preprint [\[91\]](#) is identical to this chapter.

4.1 Introduction

Designing new composite materials with advanced mechanical features is an important agenda in the engineering sciences, with relevance for many branches of industry. The properties of a composite are tightly related to the characteristics and structure of its microscopic heterogeneities, and it is well-known that the deformation behavior on a macroscopic scale may differ substantially from the way its components deform individually [\[119, 153, 198\]](#). A significant class of such materials with particular relevance for manufacturing lightweight structures are reinforced high-contrast composites; indeed, the combination of an elastically softer matrix medium with embedded stiff components of different shapes, such as fibers, gives rise to a light, yet strong material.

Motivated by these applications, an extensive body of mathematical literature on the homogenization of stiff fibered structures has emerged, providing various modeling approaches and techniques to pave the way for a reliable prediction of effective material response in reaction to external forces. It is important to notice that scaling between the fiber thickness and the elastic properties plays a crucial role for the resulting homogenized model. We highlight here a few selected works. In [\[87, 177\]](#) and [\[46, 86\]](#), the authors study variational homogenization via Γ -convergence of Saint-Venant Kirchhoff energy functionals for vanishingly small fibers with suitable adhesion conditions and diverging Lamé coefficients in the elastically linear and nonlinear setting, respectively. As a consequence of the choice of scaling relations between the elastic constants, the fiber thickness and adhesive parameters in the papers [\[86, 177\]](#), the derived limit models describe second-gradient materials. Moreover, nonlocal effects have been observed to arise in models of homogenized fiber-reinforced structures, see e.g., [\[29\]](#) or [\[30, 174, 188\]](#) in the context of linear elasticity, as well as [\[28\]](#), where additional torsion effects are taken into account.

In this chapter, we study a phenomenological model for composites reinforced by parallel long fibers, which are assumed to be fully rigid; further modeling hypotheses are that the matrix material adheres to the fibers along all interfaces and that the relative volume percentage of the

rigid components stays within fixed scale-invariant bounds. As we will see, the presence of rigid fibers gives rise to global restrictions of the material response, which sets this work apart from the aforementioned references dealing with stiff reinforcements. Our goal here is to contribute to a qualitative understanding of the nonlinear model introduced in detail below by identifying, via a rigorous limit analysis, the class of anisotropic deformations that can be attained on a macroscopic scale.

We begin the description of the model with the basic geometric set-up of the elastic body and the embedded fibers. Henceforth, let $\Omega = \omega \times (0, L) \subset \mathbb{R}^3$ be the reference configuration of a cylindrical body with height $L > 0$ and cross section $\omega \subset \mathbb{R}^2$, where ω is a bounded Lipschitz domain. Whenever indicated, we assume additionally that ω satisfies the following hypothesis:

- (H) The domain ω is bi-Lipschitz homeomorphic to the open unit disk in \mathbb{R}^2 , i.e., there exists a Lipschitz map $\phi : B(0, 1) \rightarrow \omega$ whose inverse exists and is also Lipschitz.

Examples of sets with the property (H) include in particular rectangles (see e.g. [110]) or simply connected bounded domains with smooth boundary (see e.g. [194, Chapter 5.4]).

To model the distribution of the fibers inside the body, consider a periodic lattice on \mathbb{R}^2 with unit cell $Y = [0, 1]^2$ and a small length scale parameter $\varepsilon > 0$. We suppose that each scaled and translated cell $\varepsilon(k + Y)$ with $k \in \mathbb{Z}^2$ contains the cross section of one fiber, described by a domain $\omega_\varepsilon^k \subset \mathbb{R}^2$, which is assumed to satisfy two technical conditions. First, the fiber cross-sections are required to have (relative to their size) a fixed minimal distance to the boundary of their surrounding cell, precisely,

$$\omega_\varepsilon^k \subset \varepsilon(k + [\alpha, 1 - \alpha]^2) \quad (4.1)$$

for a given $\alpha \in (0, \frac{1}{2})$. As a second hypothesis, let each ω_ε^k contain a square of side length $\varepsilon\delta$ with fixed $\delta > 0$ such that $\delta + 2\alpha < 1$, i.e., there is $a_\varepsilon^k \in \varepsilon(k + Y)$ such that

$$S_\varepsilon^k := a_\varepsilon^k + \varepsilon(-\frac{\delta}{2}, \frac{\delta}{2})^2 \subset \omega_\varepsilon^k; \quad (4.2)$$

this guarantees that the measure of the fiber cross-sections scales like ε^2 , in particular, $|\omega_\varepsilon^k| \geq \delta^2 \varepsilon^2$. The two assumptions (4.1) and (4.2) are illustrated in Figure 4.1 a).

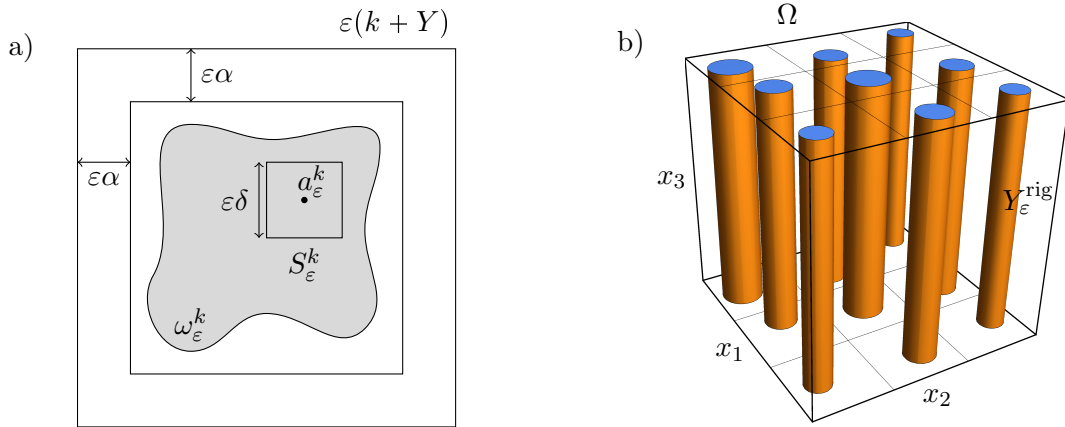


Figure 4.1: Illustration of: a) the scaled and translated unit cell $\varepsilon(k + Y)$, which contains the fiber cross-section ω_ε^k (shaded in grey) and the square S_ε^k in compliance with the minimal distance $\varepsilon\alpha$ from the boundary; b) a collection of fibers embedded into the cuboid Ω .

Considering the definition of the fiber cross-sections, we now introduce the set

$$Y_\varepsilon^{\text{rig}} = \bigcup_{k \in \mathbb{Z}^2} \omega_\varepsilon^k \times \mathbb{R}; \quad (4.3)$$

then, $Y_\varepsilon^{\text{rig}} \cap \Omega$ is the collection of all fibers in Ω , and $Y_\varepsilon^{\text{rig}} \cap \Omega$ corresponds to the matrix material, see Figure 4.1 b). Notice that in this set-up, the fiber cross-sections need not be periodically distributed. The analysis of a model where the fibers are distributed randomly without the confinement of the periodic lattice is an interesting problem, but beyond the scope of this work.

With the geometric set-up in place, we now describe the possible deformations that a body with fibers $Y_\varepsilon^{\text{rig}} \cap \Omega$ can undergo in response to external forces via Sobolev maps $u_\varepsilon : \Omega \rightarrow \mathbb{R}^3$. The rigidity of the fibers, which prevents any form of non-trivial elastic deformation, is reflected in the requirement that the deformation gradient ∇u_ε restricted to each fiber is a local orientation-preserving isometry, or equivalently, by well-known rigidity results (cf. e.g. [181]), a global rotation. Switching to the macroscopic point of view, one obtains the class of attainable effective deformations exactly as the weak limits of sequences $(u_\varepsilon)_\varepsilon$ when the scaling parameter ε tends to 0. One may think of $(u_\varepsilon)_\varepsilon$ as a sequence of uniformly bounded energy for functionals of the form

$$u \mapsto \int_{\Omega \setminus Y_\varepsilon^{\text{rig}}} W_{\text{soft}}(\nabla u) \, dx + \int_{Y_\varepsilon^{\text{rig}} \cap \Omega} W_{\text{rig}}(\nabla u) \, dx, \quad (4.4)$$

where the elastic energy density $W_{\text{soft}} : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty)$ satisfies standard properties including frame-indifference, suitable growth and coercivity assumptions, and vanishes on the identity, like, for instance, Saint Venant-Kirchhoff type densities; moreover, $W_{\text{rig}} : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty]$ is given by $W_{\text{rig}}(F) = 0$ for $F \in \text{SO}(3)$ and $W_{\text{rig}} = \infty$ otherwise, and can be interpreted as an energy density with infinitely large elastic constants.

The main result of this chapter, stated in Theorem 4.1, is a complete characterization of the weak limits of such sequences $(u_\varepsilon)_\varepsilon$. It shows that the latter exhibit a restrictive anisotropic material response in the sense that the strain in the direction of the fibers merely depends on the cross-section variables, shows higher regularity, and most importantly, has unit length. Geometrically, this means that any vertical line in the reference configuration, which may be viewed as an infinitesimally thin fiber, can only be rotated and shifted. Several examples of macroscopically observable deformations are illustrated in Section 4.5.

Theorem 4.1 (Characterization of limit deformations). *Let $p > 2$ and $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain that satisfies the hypothesis (H). With*

$$\mathcal{A}_\varepsilon := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : \nabla u \in \text{SO}(3) \text{ a.e. in } Y_\varepsilon^{\text{rig}} \cap \Omega\}$$

for $\varepsilon > 0$, the set of weak limits

$$\mathcal{A} := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : \text{there is } (u_\varepsilon)_\varepsilon \text{ with } u_\varepsilon \in \mathcal{A}_\varepsilon \text{ and } u_\varepsilon \rightharpoonup u \text{ in } W^{1,p}(\Omega; \mathbb{R}^3)\} \quad (4.5)$$

admits the three equivalent characterizations

$$\begin{aligned} \mathcal{A} &= \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : \partial_3 u \in W^{1,p}(\omega; \mathcal{S}^2)\} \\ &= \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u(x) = x_3 \Sigma(x') + d(x') \text{ for a.e. } x = (x', x_3) \in \Omega \\ &\quad \text{with } \Sigma \in W^{1,p}(\omega; \mathcal{S}^2), d \in W^{1,p}(\omega; \mathbb{R}^3)\} \\ &= \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u(x) = R(x')x + b(x') \text{ for a.e. } x = (x', x_3) \in \Omega \\ &\quad \text{with } R \in W^{1,p}(\omega; \text{SO}(3)), b \in W^{1,p}(\omega; \mathbb{R}^3)\}. \end{aligned}$$

There are different facets to the general discussion of this result and its placement in the related literature we wish to mention:

- (i) Theorem 4.1 is a natural extension of the findings in the paper [50] by Christowiak & Kreisbeck from two to three dimensions. Considering that it is not possible to distinguish between layers and fibers in 2d, also the reference [51], where the same authors study layered reinforcements, provides another natural three-dimensional extension of [50]. Whereas the codimension of the layer reinforcements in [51] is one, the codimension of the rigid components for our model with fibers is two, making the latter clearly more flexible. For more details, see Remark 4.19.
- (ii) Generally speaking, the concept of asymptotic rigidity, as introduced in [50, 51], refers to global geometric constraints that emerge in the limit of functions that are (almost) local isometries on suitably arranged, and increasingly refined, disconnected parts of their domain, cf. also [73]. Analogues in the case that a domain consists of only one connected component are the well-known classical rigidity statements by Liouville for smooth and Reshetnyak [181] for Sobolev functions. In that spirit, Theorem 4.1 represents asymptotic rigidity for fiber structures.
- (iii) We point out that our main theorem provides a basis for future efforts regarding the homogenization via Γ -convergence of variational models for elastic materials reinforced with rigid long fibers. If one aims for a statement in analogy to [51], where reinforcements in the form of rigid layers are studied, then Theorem 4.1 implies that the Γ -limit for $\varepsilon \rightarrow 0$ of energy functionals as in (4.4), which are constrained by the non-convex sets \mathcal{A}_ε , are finite exactly on the limit set \mathcal{A} . In other words, the domain of the Γ -limit can be identified as a direct consequence. Notice, however, that the approximation we obtain from Theorem 4.1 for elements in \mathcal{A} will not be a recovery sequence in general.
- (iv) Within a broader theoretical context of asymptotic analysis, one may interpret the previous theorem as a Γ -convergence result for characteristic functions (in the sense of convex analysis) associated with \mathcal{A}_ε , or equivalently, as the characterization of the Kuratowski limit of the sequence of sets $(\mathcal{A}_\varepsilon)_\varepsilon$, both with respect to weak convergence in $W^{1,p}(\Omega; \mathbb{R}^3)$; for more on these concepts, see e.g. [42, 71].

The proof of Theorem 4.1 falls naturally into two parts, the necessity and sufficiency, which we approach with different techniques. To see the necessity, it is possible to generalize and adapt the arguments in [50, 51], which again rely on a compactness result for sequences of piecewise constant rotations similar to [98, Theorem 4.1]. The required new ingredient is a suitable estimate for rotations on neighboring fibers, see Lemma 4.6. The sufficiency, on the other hand, calls for an explicit construction of approximating sequences in \mathcal{A}_ε . Since the soft matrix components have an additional degree of freedom compared to the layer case and are connected (cf. (i) above), this construction is more involved. After evoking a lifting in fiber bundles for Sobolev functions, we consider a composition with a careful approximation of the identity that is constant on the rigid components (see Lemma 4.12).

The effect of higher-order regularizations in material models has been a subject of intense study, and especially, weaker penalizations that do not involve the full Hessian of the deformations (in the second-order case), have come into the focus more recently [31, 73, 75]. In this spirit, we complement our model with an anisotropic partial regularization, precisely, a uniform bound on the second derivatives in the cross-section variables. The next theorem shows that this is already enough to deprive the macroscopic material response of any flexibility, that is, only rigid body motions can occur.

Theorem 4.2 (Rigid macroscopic behavior through partial regularization). *Let $p > 1$ and $u \in W^{1,p}(\Omega; \mathbb{R}^3)$. Suppose there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ such that $u_\varepsilon \in \mathcal{A}_\varepsilon$ for all ε and*

$$\sup_\varepsilon \max_{i,j \in \{1,2\}} \|\partial_i \partial_j u_\varepsilon\|_{L^p(\Omega; \mathbb{R}^3)}^p < \infty, \quad (4.6)$$

and $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$ as $\varepsilon \rightarrow 0$. Then, u is a rigid body motion, i.e.,

$$u(x) = Rx + b, \quad x \in \Omega,$$

with a rotation $R \in \text{SO}(3)$ and a translation vector $b \in \mathbb{R}^3$.

This work is organized as follows. After introducing the relevant notations, we collect in Section 4.2 a few technical tools for working with manifold-valued, particularly $\text{SO}(3)$ -valued, Sobolev functions, including a lifting, extension, and density result. Section 4.3 is concerned with identifying necessary structural properties of weak limits of sequences $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon \in \mathcal{A}_\varepsilon$. Moreover, we formulate in Proposition 4.10 a generalization of Theorem 4.1, where the exact differential inclusion $\nabla u_\varepsilon \in \text{SO}(3)$ a.e. in $Y_\varepsilon^{\text{rig}} \cap \Omega$ is replaced by a suitable approximate variant (cf. (4.30)); from a modeling point of view, the latter makes the transition from rigid to elastically deformable, yet, very stiff fibers. In Section 4.4, we detail the construction of suitable approximating sequences, showing the sufficiency part of Theorem 4.1. In combination with the necessity from Section 4.3 and a lifting argument, the proof of Theorem 4.1 is then completed. The intention of Section 4.5 is to illustrate the obtained analytical results from a perspective of materials engineering. To this end, we present several examples of attainable macroscopic deformations, comment on conditions for incompressibility, and make a brief comparison with the setting of layered composites. Finally, the chapter concludes with the proof of Theorem 4.2 in Section 4.6.

4.2 Preliminaries and tools

4.2.1 Notation

The standard unit vectors in \mathbb{R}^n are e_i for $i = 1, \dots, n$. For $a, b \in \mathbb{R}^3$, we denote their cross product as $a \times b$ and their scalar product as $a \cdot b$. The two-dimensional unit sphere \mathcal{S}^2 consists of all unit vectors in \mathbb{R}^3 . On the matrix space $\mathbb{R}^{m \times n}$, we work with the standard Frobenius norm given by $|A| = \sqrt{\text{Tr}(A^T A)}$ for $A \in \mathbb{R}^{m \times n}$, where Tr is the trace operator and $A^T \in \mathbb{R}^{n \times m}$ denotes the transpose of A . By $\text{SO}(n)$, we denote the special orthogonal group of matrices in $\mathbb{R}^{n \times n}$, and $\text{Id}_{\mathbb{R}^n \times n}$ stands for the identity matrix in $\mathbb{R}^{n \times n}$, while $\text{id}_{\mathbb{R}^n}$ denotes the identity map on \mathbb{R}^n . For $t \in \mathbb{R}$, $[t]$ and $\lceil t \rceil$ are the smallest integer not less and the largest integer not greater than t , respectively.

We write $B(x, r)$ for an open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$. If $A, B \subset \mathbb{R}^n$, then $A \Subset B$ means that A is compactly contained in B . Moreover, we refer to an open, connected, and non-empty set $U \subset \mathbb{R}^n$ as a domain. The Lebesgue measure is denoted by $|\cdot|$ and $\#(\cdot)$ symbolizes the counting measure.

We use the splitting $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ and write $x = (x', x_3)$ with $x' = (x_1, x_2) \in \mathbb{R}^2$. The projection onto the first two coordinates of the set $A \subset \mathbb{R}^3$ is denoted by $A' \subset \mathbb{R}^2$. Further, if $u : \mathbb{R}^3 \supset U \rightarrow \mathbb{R}^m$ is (weakly) differentiable, we split its (weak) gradient into $\nabla u = (\nabla' u | \partial_3 u)$ with $\nabla' u = (\partial_1 u | \partial_2 u)$. For $d \in \mathbb{R}^3$, let $\partial_d u = (\nabla u)d$ be the directional derivative of u in the direction d . In particular, if $d = e_k$, $k \in \{1, 2, 3\}$, we write $\partial_k u$ instead of $\partial_{e_k} u$. In case that $U \subset \mathbb{R}^2$ and $u : U \rightarrow \mathbb{R}^m$ is (weakly) differentiable, then the (weak) gradient of u is denoted by $\nabla' u$.

For $U \subset \mathbb{R}^n$ open and $1 \leq p \leq \infty$, we use the standard notation for Lebesgue and Sobolev spaces, $L^p(U; \mathbb{R}^m)$ and $W^{1,p}(U; \mathbb{R}^m)$. Replacing \mathbb{R}^m by an embedded submanifold \mathcal{M} of \mathbb{R}^m , we set

$$W^{1,p}(U; \mathcal{M}) := \{u \in W^{1,p}(U; \mathbb{R}^m) : u \in \mathcal{M} \text{ a.e. in } U\},$$

and analogously for the Lebesgue spaces. Without further mention, elements in $W^{1,p}(\omega; \mathbb{R}^3)$ are identified with functions in $W^{1,p}(\Omega; \mathbb{R}^3)$ via constant extension in x_3 -direction.

Finally, we use generic constants $C > 0$ that may differ from one line to the other. Families indexed with $\varepsilon > 0$ refer to any sequence $(\varepsilon_j)_j$ such that $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$.

4.2.2 Tools for manifold-valued Sobolev functions

In this subsection, we collect a few auxiliary results. All the statements hold in more generality, but we present them here in a way that is tailored for applying them in later sections.

First, we present a lifting result for Sobolev functions with values in \mathcal{S}^2 , which builds on the theory of fiber bundles, see e.g. [85, 189, 190] for selected references on the topic. For the reader's convenience, we give the following definition: Let E, B, F be differentiable manifolds and $\pi : E \rightarrow B$ a differentiable map. The tuple (E, B, F, π) is called a fiber bundle, if there exists an open cover $(U_\tau)_\tau$ of B and diffeomorphisms $\phi_\tau : U_\tau \times F \rightarrow \pi^{-1}(U_\tau)$ such that $\pi \circ \phi_\tau$ projects canonically onto the first coordinate. One usually refers to E as the total space, B is the base space, F the fiber, U_τ a trivial neighborhood, and ϕ_τ a trivialization.

Here, we are particularly interested in the projection

$$\pi : \text{SO}(3) \rightarrow \mathcal{S}^2, \quad R \mapsto Re_3, \quad (4.7)$$

which maps each rotation to its third column. In light of [141, Example 7.20 a), Theorem 7.15], the map π is a smooth submersion, and hence the tuple $(\text{SO}(3), \mathcal{S}^2, \text{SO}(2), \pi)$ is a fiber bundle, as a consequence of Ehresmann's lemma [85].

With this fact in mind, the following lemma is a simple modification of the lifting result for Sobolev functions defined on the open unit disk in [33, Proposition 5]; the latter, in turn, builds on similar arguments as [190, Proposition 4.10].

Lemma 4.3 (Lifting of \mathcal{S}^2 -valued Sobolev functions). *Let $\omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain that satisfies the hypothesis (H) and let $\Sigma \in W^{1,p}(\omega; \mathcal{S}^2)$ for $p > 2$. Then, there exists $R \in W^{1,p}(\omega; \text{SO}(3))$ such that $Re_3 = \Sigma$ a.e. in ω .*

Proof. Let $\phi : B(0, 1) \rightarrow \omega$ be the bi-Lipschitz map according to assumption (H). Then,

$$\tilde{\Sigma} := \Sigma \circ \phi \in W^{1,p}(B(0, 1); \mathcal{S}^2),$$

see e.g. [108, Corollary 1, page 303]. Now, we apply [33, Proposition 5] to find a function $\tilde{R} \in W^{1,p}(B(0, 1); \text{SO}(3))$ such that $\tilde{R}e_3 = \tilde{\Sigma}$. The composition $R := \tilde{R} \circ \phi^{-1} \in W^{1,p}(\omega; \text{SO}(3))$ is then the sought lift; indeed,

$$Re_3 = \pi \circ R = \pi \circ \tilde{R} \circ \phi^{-1} = \tilde{\Sigma} \circ \phi^{-1} = \Sigma \circ \phi \circ \phi^{-1} = \Sigma,$$

with π as in (4.7). □

Remark 4.4 (Non-uniqueness of lifts). This remark shows that the liftings obtained in Lemma 4.3 are generally not unique. To give an explicit example, let $\Sigma \in W^{1,p}(\omega; \mathcal{S}^2)$ for $p > 2$ and assume that there is $\eta > 0$ such that the intersection of the image $\Sigma(\omega)$ with the cylinder $B(0, \eta) \times \mathbb{R} \subset \mathbb{R}^3$ is empty. Under this assumption, $\Sigma_1^2 + \Sigma_2^2 = 1 - \Sigma_3^2 \geq \eta^2$.

Then, the functions $R, S \in W^{1,p}(\omega; \text{SO}(3))$ defined by

$$Re_3 = \Sigma, \quad Re_2 = \frac{1}{\sqrt{\Sigma_1^2 + \Sigma_2^2}}(-\Sigma_2, \Sigma_1, 0), \quad Re_1 = Re_2 \times Re_3,$$

and

$$Se_3 = \Sigma, \quad Se_2 = \frac{1}{\sqrt{\Sigma_1^2 + \Sigma_2^2}}(-\Sigma_1\Sigma_3, -\Sigma_2\Sigma_3, 1 - \Sigma_3^2), \quad Se_1 = Se_2 \times Se_3,$$

are both lifts of Σ in the sense of Lemma 4.3.

Next, we present a elementary extension result on Sobolev functions with values in $\text{SO}(3)$, based on nearest point projections. The reader will notice that the result is also true more generally for compact embedded submanifolds of \mathbb{R}^n .

Lemma 4.5 (Sobolev-extension of $\text{SO}(3)$ -valued maps). *Let $2 < p \leq \infty$ and $U \subset \mathbb{R}^2$ a bounded Lipschitz domain. If $R \in W^{1,p}(U; \text{SO}(3))$, then there exist an open set $V \subset \mathbb{R}^2$ with $U \Subset V$ and an extension $\bar{R} \in W^{1,p}(V; \text{SO}(3))$ of R .*

Proof. According to standard Sobolev theory, the function $R \in W^{1,p}(U; \text{SO}(3))$ can be extended to an element in $W^{1,p}(\mathbb{R}^2; \mathbb{R}^{3 \times 3})$ with compact support, which we denote again by R . From the tubular neighborhood theorem [141, Propositions 6.17 and 6.18], we conclude the existence of an open bounded neighborhood $\mathcal{T} \subset \mathbb{R}^{3 \times 3}$ of $\text{SO}(3)$ with $0 \notin \mathcal{T}$ and a smooth retraction map $r : \mathcal{T} \rightarrow \text{SO}(3)$, i.e.,

$$r|_{\text{SO}(3)} = \text{id}_{\mathbb{R}^{3 \times 3}}.$$

By shrinking \mathcal{T} if necessary, we may assume that r is smooth up to the boundary.

Let us consider the preimage

$$V = R^{-1}(\mathcal{T}).$$

We observe that V is open, since R is continuous by Sobolev embedding and that V is bounded as a consequence of $0 \notin \mathcal{T}$ and the fact that R vanishes outside of a bounded set. Besides, $U \subset R^{-1}(\text{SO}(3))$ is compactly contained in V .

Finally, we define $\bar{R} = r \circ R|_V : V \rightarrow \text{SO}(3)$, which, in view of

$$\bar{R}|_U = r|_{\text{SO}(3)} \circ R|_U = R|_U,$$

is indeed an extension of R . As \bar{R} is the composition of a smooth function (up to the boundary) with a $W^{1,p}$ -Sobolev map on the bounded set V , it follows that $\bar{R} \in W^{1,p}(V; \text{SO}(3))$. This proves the claim. \square

Finally, we state for the reader's convenience a special case of a well-known density result for manifold-valued Sobolev functions, see e.g. [114, Theorem 2.1].

Lemma 4.6 (Density of smooth functions). *Let $U \subset \mathbb{R}^2$ be open and bounded. The set of smooth functions $C^\infty(U; \text{SO}(3))$ is dense in $W^{1,p}(U; \text{SO}(3))$ for all $p \geq 2$.*

4.3 Proof of necessity in Theorem 4.1

Before we go into details of the asymptotic analysis of weakly convergent sequences as in the definition of \mathcal{A} (see (4.5)), let us point out a basic observation about the structure of elements $u_\varepsilon \in \mathcal{A}_\varepsilon$. As a consequence of the well-known rigidity result by Reshetnyak [181], it holds that u_ε is a rigid body motion on each connected component of $Y_\varepsilon^{\text{rig}} \cap \Omega$, or in other words, on each individual fiber inside Ω .

The next lemma establishes the connection between neighboring fibers by estimating the difference between their associated rotations. It constitutes a generalization of [50, Lemma 2.4] to three dimensions, a broader class of domains, and $p \geq 1$. For an illustration of the geometric set-up, see Figure 4.2.

Lemma 4.7. *For given $m \in \mathbb{R}$ and $L_1, L_2, L_3 > 0$, let $E = E' \times (0, L_3) \subset \mathbb{R}^3$ with*

$$E' = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < L_1, mx_1 < x_2 < mx_1 + L_2\} \quad (4.8)$$

and $d = (1 + m^2)^{-1/2}(e_1 + me_2) \in \mathbb{R}^3$. Further, let $w_i : E \rightarrow \mathbb{R}^3$ for $i = 1, 2$ be affine functions given by $w_i(x) = A_i x + b_i$ for $x \in E$ with $A_i \in \mathbb{R}^{3 \times 3}$ and $b_i \in \mathbb{R}^3$.

If $v \in W^{1,p}(E; \mathbb{R}^3)$ with $p \geq 1$ satisfies the partial boundary conditions

$$v = w_1 \text{ on } \partial E \cap \{x_1 = 0\} \quad \text{and} \quad v = w_2 \text{ on } \partial E \cap \{x_1 = L_1\} \quad (4.9)$$

in the sense of traces, then

$$\int_E |\partial_d v|^p \, dx \geq \frac{C|E|L_3^p}{(1 + |m|^p)L_1^p} |(A_2 - A_1)e_3|^p \quad (4.10)$$

with a constant $C > 0$ depending only on p .

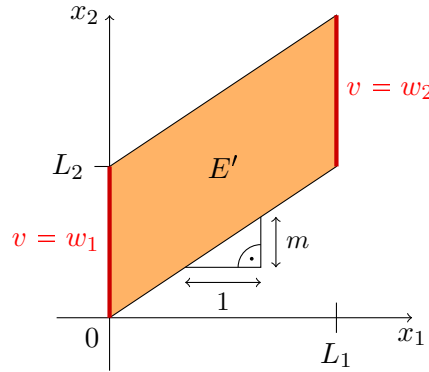


Figure 4.2: Illustration of the set E' with slope m as in (4.8) and the boundary values of v .

Proof. Since the inequality (4.10) is continuous in v with respect to the $W^{1,p}$ -norm, it suffices by a density argument to prove the statement for smooth functions v that attain the boundary values (4.9) classically.

A change of variables gives

$$\begin{aligned} \int_E |\partial_d v(x)|^p \, dx &= \frac{1}{(1 + m^2)^{p/2}} \int_Q |\partial_1 u(y)|^p \, dy \\ &= \frac{1}{(1 + m^2)^{p/2}} \int_0^{L_3} \int_0^{L_2} \int_0^{L_1} |\partial_1 u(y_1, y_2, y_3)|^p \, dy_1 \, dy_2 \, dy_3, \end{aligned} \quad (4.11)$$

where $u(y) = v(y_1, y_2 + my_1, y_3)$ for $y = (y_1, y_2, y_3) \in Q = (0, L_1) \times (0, L_2) \times (0, L_3)$. By Jensen's inequality, applied iteratively to the one-dimensional integrals over $(0, L_1)$ and $(0, L_2)$, we obtain in view of the assumption on the boundary values of v in (4.9) that

$$\begin{aligned} \int_Q |\partial_1 u(y)|^p dx &\geq L_1^{1-p} \int_0^{L_3} \int_0^{L_2} |v(L_1, y_2 + mL_1, y_3) - v(0, y_2, y_3)|^p dy_2 dy_3 \\ &\geq (L_1 L_2)^{1-p} \int_0^{L_3} \left| \int_0^{L_2} \sum_{k=2}^3 y_k (A_2 - A_1) e_k dy_2 + L_1 A_2 e_1 + mL_1 A_2 e_2 + b_2 - b_1 \right|^p dy_3 \\ &\geq \frac{L_2}{L_1^{p-1}} \min_{b \in \mathbb{R}^3} \int_0^{L_3} |y_3 (A_2 - A_1) e_3 + b|^p dy_3. \end{aligned} \quad (4.12)$$

Next, we address the optimization problem in (4.12). The observation that any minimizer is parallel to $(A_2 - A_1)e_3$, which follows from the elementary estimate

$$|a + b|^2 \geq |a|^2 + |b|^2 - 2|a||b| = \left| a - \frac{|b|}{|a|}a \right|^2 \quad \text{for any } a, b \in \mathbb{R}^3 \text{ with } a \neq 0,$$

allows us to reduce (4.12) to a one-dimensional minimization. Then,

$$\min_{b \in \mathbb{R}^3} \int_0^{L_3} |y_3 (A_2 - A_1) e_3 + b|^p dy_3 = \min_{\lambda \in \mathbb{R}} \int_0^{L_3} |y_3 + \lambda|^p |(A_2 - A_1) e_3|^p dy_3. \quad (4.13)$$

We join (4.13) with (4.12) and (4.11) to conclude that

$$\int_E |\partial_d v(x)|^p dx \geq \frac{L_3^{p+1} L_2}{2^p (p+1) (1+m^2)^{p/2} L_1^{p-1}} |(A_2 - A_1) e_3|^p.$$

This finishes the proof, considering that $|E| = L_1 L_2 L_3$ and $(1+m^2)^{p/2} \leq c(1+|m|^p)$ with a constant $c > 0$ depending on p . \square

With these preparations, we can now prove the following proposition, which implies the necessity statement of Theorem 4.1. The arguments combine ideas from the related papers [50, 98] along with the new estimates from Lemma 4.7.

Proposition 4.8. *Let $p > 1$ and suppose that $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ is a sequence with*

$$\nabla u_\varepsilon \in \text{SO}(3) \text{ a.e. in } Y_\varepsilon^{\text{rig}} \cap \Omega \quad (4.14)$$

for all ε , such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$ for $u \in W^{1,p}(\Omega; \mathbb{R}^3)$. Then,

$$\partial_3 u \in W^{1,p}(\omega; \mathbb{R}^3) \text{ with } |\partial_3 u| = 1 \text{ a.e. in } \omega,$$

or equivalently, there are $\Sigma \in W^{1,p}(\omega; \mathcal{S}^2)$ and $d \in W^{1,p}(\omega; \mathbb{R}^3)$ with

$$u(x) = x_3 \Sigma(x') + d(x'), \quad x \in \Omega.$$

Proof. By an exhaustion argument, it suffices to prove for any cylindrical open set $U = U' \times (0, L) \subset \Omega$ with $U' \Subset \omega$ that $\partial_3 u \in W^{1,p}(U'; \mathbb{R}^3)$ and $|\partial_3 u| = 1$ a.e. in U' .

We define for $\varepsilon > 0$,

$$P_\varepsilon^k = \varepsilon(k + Y) \times (0, L) \quad \text{for } k \in \mathbb{Z}^2, \quad (4.15)$$

and let the index set I_ε label the cuboids P_ε^k overlapping with U i.e., $I_\varepsilon = \{k \in \mathbb{Z}^2 : P_\varepsilon^k \cap U \neq \emptyset\}$. Choosing $\varepsilon > 0$ sufficiently small, we may assume that

$$U \subset \bigcup_{k \in I_\varepsilon} P_\varepsilon^k \subset \Omega. \quad (4.16)$$

We split the rest of the proof in four steps.

Step 1: Rigidity and approximation by piecewise affine functions. In this step, we tailor the strategy of [50, Proposition 2.1] to our setting, where the rigid components are thin in two directions instead of one. Precisely, for every $k \in I_\varepsilon$, we apply the well-known rigidity result by Reshetnyak [181] in $Y_\varepsilon^{\text{rig}} \cap P_\varepsilon^k$ (recall $Y_\varepsilon^{\text{rig}}$ from (4.3)), to find rotation matrices $R_\varepsilon^k \in \text{SO}(3)$ and translation vectors $b_\varepsilon^k \in \mathbb{R}^3$ such that

$$u_\varepsilon(x) = R_\varepsilon^k x + b_\varepsilon^k \quad \text{for } x \in Y_\varepsilon^{\text{rig}} \cap P_\varepsilon^k. \quad (4.17)$$

For every $\varepsilon > 0$, we consider the auxiliary function $w_\varepsilon = \sigma_\varepsilon + b_\varepsilon \in L^\infty(\Omega; \mathbb{R}^3)$ given by

$$\sigma_\varepsilon(x) = \sum_{k \in I_\varepsilon} (R_\varepsilon^k x) \mathbb{1}_{P_\varepsilon^k}(x), \quad \text{and} \quad b_\varepsilon(x) = \sum_{k \in I_\varepsilon} b_\varepsilon^k \mathbb{1}_{P_\varepsilon^k}(x), \quad x \in \Omega.$$

We show that

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - w_\varepsilon\|_{L^p(U; \mathbb{R}^3)} = 0. \quad (4.18)$$

Indeed, with $u_\varepsilon = w_\varepsilon$ in $Y_\varepsilon^{\text{rig}} \cap P_\varepsilon^k$ for each index $k \in I_\varepsilon$ and Poincaré's inequality applied in the cross-section variables, it follows that

$$\begin{aligned} \int_{P_\varepsilon^k} |u_\varepsilon - w_\varepsilon|^p dx &= \int_0^L \int_{(P_\varepsilon^k \setminus Y_\varepsilon^{\text{rig}})'} |u_\varepsilon - w_\varepsilon|^p dx' dx_3 \\ &\leq C\varepsilon^p \int_0^L \int_{(P_\varepsilon^k \setminus Y_\varepsilon^{\text{rig}})'} |\nabla' u_\varepsilon - \nabla' w_\varepsilon|^p dx' dx_3 \\ &\leq C\varepsilon^p \int_{P_\varepsilon^k \setminus Y_\varepsilon^{\text{rig}}} |\nabla u_\varepsilon - R_\varepsilon^k|^p dx \leq C\varepsilon^p \int_{P_\varepsilon^k} |\nabla u_\varepsilon - \nabla w_\varepsilon|^p dx \\ &\leq C\varepsilon^p (\|\nabla u_\varepsilon\|_{L^p(P_\varepsilon^k; \mathbb{R}^{3 \times 3})}^p + |P_\varepsilon^k|); \end{aligned}$$

here, we have used in particular that $u_\varepsilon - w_\varepsilon = 0$ on $\partial\omega_\varepsilon^k = \partial(Y_\varepsilon^{\text{rig}} \cap P_\varepsilon^k)'$ in the sense of traces and that the Poincaré constant scales linearly with the diameter of the domain. Summing over all $k \in I_\varepsilon$ then yields

$$\|u_\varepsilon - w_\varepsilon\|_{L^p(U; \mathbb{R}^3)}^p \leq C\varepsilon^p (\|u_\varepsilon\|_{W^{1,p}(\Omega; \mathbb{R}^3)}^p + |\Omega|)$$

in light of (4.16). Since $(u_\varepsilon)_\varepsilon$ is bounded in $W^{1,p}(\Omega; \mathbb{R}^3)$ as a weakly convergent sequence, we conclude (4.18).

As a consequence, $(w_\varepsilon)_\varepsilon$ is bounded in $L^p(U; \mathbb{R}^3)$ and, thus, as $(\sigma_\varepsilon)_\varepsilon$ is uniformly bounded in $L^\infty(U; \mathbb{R}^3)$, the sequence $(b_\varepsilon)_\varepsilon$ is bounded in $L^p(U; \mathbb{R}^3)$ as well. Therefore, (after passing to non-relabeled subsequences) there exist limit functions $\sigma \in L^\infty(U; \mathbb{R}^3)$ and $\hat{b} \in L^p(U; \mathbb{R}^3)$ such that $\sigma_\varepsilon \xrightarrow{*} \sigma$ in $L^\infty(U; \mathbb{R}^3)$ and $b_\varepsilon \rightharpoonup \hat{b}$ in $L^p(U; \mathbb{R}^3)$. Since this implies $w_\varepsilon \rightharpoonup \sigma + \hat{b}$ in $L^p(U; \mathbb{R}^3)$, we finally deduce the identity

$$u = \sigma + \hat{b}, \quad (4.19)$$

in light of the weak convergence of $(u_\varepsilon)_\varepsilon$ and (4.18); note that \hat{b} is independent of x_3 .

Step 2: Compactness of an auxiliary function. For $\varepsilon > 0$, define $\Sigma_\varepsilon \in L^\infty(\omega; \mathbb{S}^2)$ by

$$\Sigma_\varepsilon(x') = \sum_{k \in I_\varepsilon} R_\varepsilon^k e_3 \mathbb{1}_{\varepsilon(k+Y)}(x'), \quad x' \in \omega, \quad (4.20)$$

with R_ε^k as in (4.17). We will show with the help of the Fréchet-Kolmogorov compactness theorem and the estimates of Lemma 4.7 that $(\Sigma_\varepsilon)_\varepsilon$ has a strongly convergent subsequence. The proof strategy is inspired by and follows the lines of [98, Theorem 4.1]. The main difference in our approach is the estimates for the rotations on two neighboring cells, as derived in Lemma 4.7.

Let us introduce the following sets: For $\varepsilon > 0$ and $k \in \mathbb{Z}^2$, we take $E_\varepsilon^{k,\rightarrow}, E_\varepsilon^{k,\uparrow} \subset \mathbb{R}^2$ to be the open parallelograms determined by the two parallel lines

$$a_\varepsilon^k + \varepsilon \left(\left\{ \frac{\delta}{2} \right\} \times \left(-\frac{\delta}{2}, \frac{\delta}{2} \right) \right) \quad \text{and} \quad a_\varepsilon^{k+e_1} + \varepsilon \left(\left\{ -\frac{\delta}{2} \right\} \times \left(-\frac{\delta}{2}, \frac{\delta}{2} \right) \right), \quad (4.21)$$

and

$$a_\varepsilon^k + \varepsilon \left(\left(-\frac{\delta}{2}, \frac{\delta}{2} \right) \times \left\{ \frac{\delta}{2} \right\} \right) \quad \text{and} \quad a_\varepsilon^{k+e_2} + \varepsilon \left(\left(-\frac{\delta}{2}, \frac{\delta}{2} \right) \times \left\{ -\frac{\delta}{2} \right\} \right), \quad (4.22)$$

respectively; for simplicity, we restrict ourselves to the special case when the centers a_ε^k of the squares S_ε^k (cf. (4.2)) are periodically arranged and given by $a_\varepsilon^k = \varepsilon(k + a)$ with a fixed $a \in [\alpha + \frac{\delta}{2}, 1 - \alpha - \frac{\delta}{2}]^2$. This specific choice of a_ε^k means that the $E_\varepsilon^{k,\rightarrow}$ and $E_\varepsilon^{k,\uparrow}$ are in fact rectangles, as illustrated in Figure 4.3. In Remark 4.9 below, we explain how the arguments can be modified to cover the general case.

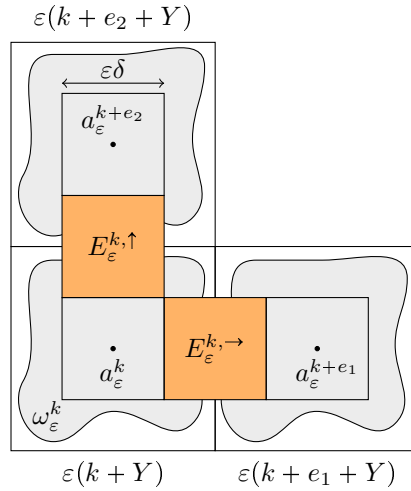


Figure 4.3: Illustration of a scaled and translated unit cell and two of its neighbors; the rectangles $E_\varepsilon^{k,\rightarrow}$ and $E_\varepsilon^{k,\uparrow}$ connect the horizontally and vertically neighboring squares $S_\varepsilon^k = a_\varepsilon^k + \varepsilon(-\frac{\delta}{2}, \frac{\delta}{2})^2$, respectively.

Moreover, let N_ε^k be the union of $\varepsilon(k + Y)$ and its eight neighboring cells, i.e.,

$$N_\varepsilon^k = \bigcup_{e \in J_k} \varepsilon(e + Y) \quad \text{with } J_k = k + \{0, \pm e_1, \pm e_2, (\pm 1, \pm 1)\} \subset \mathbb{Z}^2;$$

observe that $N_\varepsilon^k \subset \omega$ for ε sufficiently small.

Let $x' \in \varepsilon(k + Y)$ and $y' \in N_\varepsilon^k$ with $k \in I_\varepsilon$. Then,

$$|\Sigma_\varepsilon(x') - \Sigma_\varepsilon(y')|^p \leq C \left(\sum_{e \in J_k \cap (J_k - e_1)} |(R_\varepsilon^{e+e_1} - R_\varepsilon^e) e_3|^p + \sum_{e \in J_k \cap (J_k - e_2)} |(R_\varepsilon^{e+e_2} - R_\varepsilon^e) e_3|^p \right).$$

To each term on the right-hand side, we now apply Lemma 4.7, or the analogon thereof for switched roles of the variables x_1 and x_2 , with $m = 0$ and the domains $E = E_\varepsilon^{k,\rightarrow} \times (0, L)$ and $E = E_\varepsilon^{k,\uparrow} \times (0, L)$ (cf. (4.21) and (4.22)), respectively, to find that

$$\begin{aligned} & |\Sigma_\varepsilon(x') - \Sigma_\varepsilon(y')|^p \\ & \leq C\varepsilon^{p-2} \left(\sum_{e \in J_k \cap (J_k - e_1)} \|\partial_1 u_\varepsilon\|_{L^p(E_\varepsilon^{e,\rightarrow} \times (0, L); \mathbb{R}^3)}^p + \sum_{e \in J_k \cap (J_k - e_2)} \|\partial_2 u_\varepsilon\|_{L^p(E_\varepsilon^{e,\uparrow} \times (0, L); \mathbb{R}^3)}^p \right) \\ & \leq C\varepsilon^{p-2} \|\nabla' u_\varepsilon\|_{L^p(N_\varepsilon^k \times (0, L); \mathbb{R}^{3 \times 2})}^p, \end{aligned} \quad (4.23)$$

where $C > 0$ depends on L , p and δ .

Now, let $\xi \in \mathbb{R}^2$ be such that $|\xi| < \frac{1}{2} \text{dist}(U', \partial\omega)$ and set $m_\varepsilon = \lceil \frac{|\xi|}{\varepsilon} \rceil$ with $|\xi|_\infty := \max\{|\xi_1|, |\xi_2|\}$. Choosing $m_\varepsilon + 1$ points $0 = \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(m_\varepsilon)} = \xi$ such that $|\xi^{(j+1)} - \xi^{(j)}|_\infty \leq \varepsilon$ for every $j = 0, \dots, m_\varepsilon - 1$ generates a discrete path from the origin to ξ with maximal step width ε , and it follows via a telescoping sum argument and the discrete Hölder's inequality that

$$|\Sigma_\varepsilon(x') - \Sigma_\varepsilon(x' + \xi)|^p \leq m_\varepsilon^{p-1} \sum_{j=0}^{m_\varepsilon-1} |\Sigma_\varepsilon(x' + \xi^{(j)}) - \Sigma_\varepsilon(x' + \xi^{(j+1)})|^p.$$

After integration over $\varepsilon(k + Y)$ and along with (4.23), one obtains

$$\int_{\varepsilon(k+Y)} |\Sigma_\varepsilon(x' + \xi) - \Sigma_\varepsilon(x')|^p dx' \leq C m_\varepsilon^{p-1} \varepsilon^p \sum_{j=0}^{m_\varepsilon-1} \|\nabla' u_\varepsilon\|_{L^p((N_\varepsilon^k + \xi^{(j)}) \times (0, L); \mathbb{R}^{3 \times 2})}^p.$$

By summing over all $k \in I_\varepsilon$, we infer in view of $m_\varepsilon \leq 2 \frac{|\xi|}{\varepsilon} + 1$ and the boundedness of the sequence $(\nabla' u_\varepsilon)_\varepsilon \subset L^p(\Omega; \mathbb{R}^{3 \times 2})$ that

$$\begin{aligned} & \int_{U'} |\Sigma_\varepsilon(x' + \xi) - \Sigma_\varepsilon(x')|^p dx' \\ & \leq C m_\varepsilon^{p-1} \varepsilon^p \sum_{j=0}^{m_\varepsilon-1} \sum_{k \in I_\varepsilon} \|\nabla' u_\varepsilon\|_{L^p((N_\varepsilon^k + \xi^{(j)}) \times (0, L); \mathbb{R}^{3 \times 2})}^p \\ & \leq C m_\varepsilon^p \varepsilon^p \|\nabla' u_\varepsilon\|_{L^p(\Omega; \mathbb{R}^{3 \times 2})}^p \leq C (|\xi|^p + \varepsilon^p), \end{aligned} \quad (4.24)$$

for $\varepsilon > 0$ sufficiently small, with a constant $C > 0$ depending on L , p and δ .

Hence, the Fréchet-Kolmogorov theorem (see e.g. [4, Theorem 4.16]) implies the existence of $\Sigma \in L^p(U'; \mathbb{R}^3)$ and a subsequence (not relabeled) of $(\Sigma_\varepsilon)_\varepsilon$ such that

$$\Sigma_\varepsilon \rightarrow \Sigma \text{ in } L^p(U'; \mathbb{R}^3) \text{ and also pointwise a.e. in } U'. \quad (4.25)$$

Since $|\Sigma_\varepsilon| = 1$ in U' (cf. (4.20)), the limit function Σ satisfies

$$|\Sigma| = 1 \text{ a.e. in } U'.$$

Step 3: Regularity of Σ . If we divide (4.24) by $|\xi|^p$ and take the limit $\varepsilon \rightarrow 0$, it follows under consideration of (4.25) that

$$\left\| \frac{\Sigma(\cdot + \xi) - \Sigma}{|\xi|} \right\|_{L^p(U'; \mathbb{R}^3)}^p \leq C, \quad (4.26)$$

which implies $\Sigma \in W^{1,p}(U'; \mathbb{R}^3)$.

Notice that the above-mentioned exhaustion argument exploits that the constant in (4.26) is independent of U' .

Step 4: Properties of the limit deformation u . Recall that $u = \sigma + \hat{b}$ with $\partial_3 \hat{b} = 0$ according to (4.19). Then,

$$\partial_3 u = \partial_3(\sigma + \hat{b}) = \partial_3 \sigma = \Sigma, \quad (4.27)$$

where the last identity follows from the observation that $\partial_3 \sigma_\varepsilon = \Sigma_\varepsilon \rightarrow \Sigma$ by (4.25).

As an immediate consequence of (4.27),

$$\partial_3 u \in W^{1,p}(U'; \mathbb{R}^3) \text{ with } |\partial_3 u| = 1 \text{ a.e. in } U',$$

or equivalently, $u(x) = x_3 \Sigma(x') + d(x')$ with some $d \in W^{1,p}(U'; \mathbb{R}^3)$. This concludes the proof. \square

Remark 4.9 (General distribution of fiber cross-sections). In Step 2 of the previous proof, it is assumed for simplicity that the squares S_ε^k inside the fiber cross-sections ω_ε^k are periodically distributed. This remark addresses the necessary adaptations in order to cover the non-periodic case, where the sets $E_\varepsilon^{k,\rightarrow}$ and $E_\varepsilon^{k,\uparrow}$ in (4.21) and (4.22) may be general parallelograms. Indeed, in this case, we can derive the estimate

$$\int_{U'} |\Sigma_\varepsilon(x' + \xi) - \Sigma_\varepsilon(x')|^p dx' \leq C \sup_{k \in I_\varepsilon} (1 + \max\{|m_\varepsilon^{k,\rightarrow}|, |m_\varepsilon^{k,\uparrow}|\})^p (|\xi|^p + \varepsilon^p), \quad (4.28)$$

where $C > 0$ depends on L , δ and p ; here, the quantities $m_\varepsilon^{k,\rightarrow}, m_\varepsilon^{k,\uparrow} \in \mathbb{R}$ are the slopes corresponding to $E_\varepsilon^{k,\rightarrow}$ and $E_\varepsilon^{k,\uparrow}$, respectively, cf. Lemma 4.7. Under the assumptions (4.1) and (4.2), it holds that

$$\sup_{k \in I_\varepsilon} \max\{|m_\varepsilon^{k,\rightarrow}|, |m_\varepsilon^{k,\uparrow}|\} \leq \frac{1 - 2\alpha - \delta}{2\alpha}, \quad (4.29)$$

which follows from a simple geometric argument, see Figure 4.4. Thus, combining (4.29) with (4.28) yields (4.24) with a constant depending on L , p and δ , as well as on α .

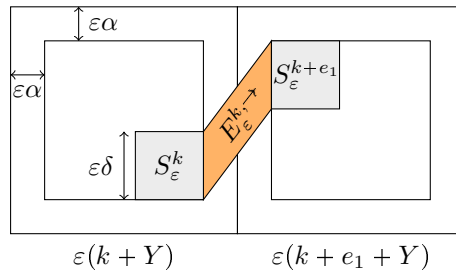


Figure 4.4: Illustration of $E_\varepsilon^{k,\rightarrow}$ in the non-periodic case. The corresponding slope is determined by the side length $\varepsilon\delta$ of S_ε^k and the parameter α as in (4.1).

The following shows the exact differential inclusion (4.14) in Proposition 4.8 can be weakened to an approximate one, namely to (4.30) as below, without changing the result. Nevertheless, we have decided to provide the reader with both proofs, as they use different techniques and Section 4.6 builds on the arguments of Proposition 4.8.

Proposition 4.10. *Let $p > 1$ and suppose that $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ is a sequence satisfying*

$$\int_{Y_\varepsilon^{\text{rig}} \cap \Omega} \text{dist}^p(\nabla u_\varepsilon, \text{SO}(3)) \, dx \leq C\varepsilon^\beta \quad (4.30)$$

for all ε , with a constant $C > 0$ and $\beta > 2p$. If $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$ for $u \in W^{1,p}(\Omega; \mathbb{R}^3)$, then there exist $\Sigma \in W^{1,p}(\omega; \mathbb{S}^2)$ and $d \in W^{1,p}(\omega; \mathbb{R}^3)$ such that

$$u(x) = x_3 \Sigma(x') + d(x'), \quad x \in \Omega. \quad (4.31)$$

Proof. We use the same basic set-up as in the proof of Proposition 4.8. Recall in particular that $U = U' \times (0, L) \subset \Omega$ with $U' \Subset \omega$, P_ε^k from (4.15) for $\varepsilon > 0$ and $k \in \mathbb{Z}^2$, and the index set I_ε ; furthermore, ε is supposed to be small enough such that (4.16) holds.

The rest of the proof is organized in two steps.

Step 1: Fiber-wise approximation by rigid body motions. We apply the quantitative geometric rigidity estimate by Friesecke, James & Müller ([98, Theorem 3.1]) to each stiff sub-fiber $S_\varepsilon^k \times (0, L)$ for $k \in I_\varepsilon$ with S_ε^k as in (4.2). This yields rotations $R_\varepsilon^k \in \text{SO}(3)$ such that

$$\|\nabla u_\varepsilon - R_\varepsilon^k\|_{L^p(S_\varepsilon^k \times (0, L); \mathbb{R}^{3 \times 3})} \leq C\varepsilon^{-2} \|\text{dist}(\nabla u_\varepsilon, \text{SO}(3))\|_{L^p(S_\varepsilon^k \times (0, L))}, \quad (4.32)$$

with a constant $C > 0$ independent of k and ε . Notice that the scaling factor ε^{-2} in (4.32) is correlated with the size of the squares S_ε^k . Indeed, this follows from a related scaling analysis for objects that are thin in one dimension as in [99, Theorem 6] and [49, Theorem 3.2.6], if we evoke the argument twice in two different directions.

Let $w_\varepsilon := \sigma_\varepsilon + b_\varepsilon \in L^p(\Omega; \mathbb{R}^3)$ with

$$\sigma_\varepsilon(x) = \sum_{k \in I_\varepsilon} R_\varepsilon^k x \mathbb{1}_{P_\varepsilon^k}(x) \quad \text{and} \quad b_\varepsilon(x) = \sum_{k \in I_\varepsilon} b_\varepsilon^k \mathbb{1}_{P_\varepsilon^k}(x), \quad x \in \Omega,$$

where $b_\varepsilon^k = \int_{S_\varepsilon^k \times (0, L)} u_\varepsilon(x) - R_\varepsilon^k x \, dx$. This choice of translations implies

$$\int_{S_\varepsilon^k \times (0, L)} u_\varepsilon - w_\varepsilon \, dx = 0,$$

and hence, enables an application of Poincaré's inequality with mean-value condition on the sub-fibers. In doing so, we obtain

$$\|u_\varepsilon - w_\varepsilon\|_{L^p(S_\varepsilon^k \times (0, L); \mathbb{R}^3)} \leq C \|\nabla u_\varepsilon - R_\varepsilon^k\|_{L^p(S_\varepsilon^k \times (0, L); \mathbb{R}^{3 \times 3})} \quad (4.33)$$

with a constant $C > 0$ that does not depend on ε and k .

Next, we utilize the previous estimates on (parts of) the stiff fibers in order to control the difference $u_\varepsilon - w_\varepsilon$ also on the soft components. To this end, we cover each P_ε^k with $k \in I_\varepsilon$ by finitely many shifted versions of $S_\varepsilon^k \times (0, L)$; in view of (4.1) and (4.2), there are translation vectors $d_{\varepsilon, n}^k \in \mathbb{R}^2$ with $n = 1, \dots, N := \lceil \delta^{-2} \rceil$ satisfying

$$P_\varepsilon^k \subset \bigcup_{n=1}^N (S_\varepsilon^k + d_{\varepsilon, n}^k) \times (0, L) \quad (4.34)$$

up to a set of zero measure. Then,

$$\begin{aligned} \int_{(S_\varepsilon^k + d_{\varepsilon, n}^k) \times (0, L)} |u_\varepsilon - w_\varepsilon|^p \, dx &\leq C \int_{S_\varepsilon^k \times (0, L)} |u_\varepsilon - w_\varepsilon|^p \, dx \\ &\quad + C \int_{S_\varepsilon^k \times (0, L)} |(u_\varepsilon - w_\varepsilon)(x) - (u_\varepsilon - w_\varepsilon)(x + d_{\varepsilon, n}^k)|^p \, dx \\ &\leq C \left(\|u_\varepsilon - w_\varepsilon\|_{L^p(S_\varepsilon^k \times (0, L); \mathbb{R}^3)}^p + |d_{\varepsilon, n}^k|^p \|\nabla' u_\varepsilon - \nabla' w_\varepsilon\|_{L^p(P_\varepsilon^k; \mathbb{R}^{3 \times 2})}^p \right), \end{aligned}$$

and consequently, in view of (4.34),

$$\int_{P_\varepsilon^k} |u_\varepsilon - w_\varepsilon|^p \, dx \leq C(\|u_\varepsilon - w_\varepsilon\|_{L^p(S_\varepsilon^k \times (0,L);\mathbb{R}^3)}^p + \varepsilon^p \|\nabla u_\varepsilon\|_{L^p(P_\varepsilon^k;\mathbb{R}^{3 \times 3})}^p + \varepsilon^p |P_\varepsilon^k|).$$

Summing this inequality over all $k \in I_\varepsilon$, we conclude together with (4.33) and (4.32) that

$$\int_U |u_\varepsilon - w_\varepsilon|^p \, dx \leq C(\varepsilon^{-2p} \|\text{dist}(\nabla u_\varepsilon, \text{SO}(3))\|_{L^p(Y_\varepsilon^{\text{rig}} \cap \Omega)}^p + \varepsilon^p \|u_\varepsilon\|_{W^{1,p}(\Omega;\mathbb{R}^3)}^p + \varepsilon^p |\Omega|),$$

and finally in light of assumption (4.30),

$$\|u_\varepsilon - w_\varepsilon\|_{L^p(U;\mathbb{R}^3)} \leq C(\varepsilon^{\frac{\beta}{p}-2} + \varepsilon), \quad (4.35)$$

where the constant $C > 0$ is independent of ε .

Since $\beta > 2p$, this shows in particular that the sequences $(w_\varepsilon)_\varepsilon$ and $(u_\varepsilon)_\varepsilon$ have an identical weak L^p -limit, namely u .

Step 2: Compactness result. We consider the functions

$$\Sigma_\varepsilon(x') = \sum_{k \in I_\varepsilon} R_\varepsilon^k e_3 \mathbb{1}_{\varepsilon(k+Y)}(x'), \quad x' \in \omega,$$

with R_ε^k as in Step 1, and observe that $\Sigma_\varepsilon = \partial_3 \sigma_\varepsilon = \partial_3 w_\varepsilon$.

Let $\xi \in \mathbb{R}^2$ with $|\xi| < \frac{1}{2} \text{dist}(U', \partial\omega)$. In analogy to the optimization argument in the proof of Lemma 4.7,

$$\begin{aligned} \|w_\varepsilon(\cdot + \xi) - w_\varepsilon\|_{L^p(U;\mathbb{R}^3)}^p &= \sum_{k \in I_\varepsilon} \int_{P_\varepsilon^k \cap U} |(R_\varepsilon^{k+\lfloor \frac{\xi}{\varepsilon} \rfloor} - R_\varepsilon^k)x + R_\varepsilon^{k+\lfloor \frac{\xi}{\varepsilon} \rfloor} \xi + b_\varepsilon^{k+\lfloor \frac{\xi}{\varepsilon} \rfloor} - b_\varepsilon^k|^p \, dx \\ &\geq C \sum_{k \in I_\varepsilon} |R_\varepsilon^{k+\lfloor \frac{\xi}{\varepsilon} \rfloor} e_3 - R_\varepsilon^k e_3|^p |P_\varepsilon^k \cap U| \geq C \|\Sigma_\varepsilon(\cdot + \xi) - \Sigma_\varepsilon\|_{L^p(U';\mathbb{R}^3)}^p, \end{aligned} \quad (4.36)$$

where $[\eta] = ([\eta_1], [\eta_2])$ for $\eta \in \mathbb{R}^2$. The left-hand side in (4.36) can be estimated from above by

$$\begin{aligned} \|w_\varepsilon(\cdot + \xi) - w_\varepsilon\|_{L^p(U;\mathbb{R}^3)} &\leq \|w_\varepsilon(\cdot + \xi) - u_\varepsilon(\cdot + \xi)\|_{L^p(U;\mathbb{R}^3)} + \|u_\varepsilon(\cdot + \xi) - u_\varepsilon\|_{L^p(U;\mathbb{R}^3)} \\ &\quad + \|w_\varepsilon - u_\varepsilon\|_{L^p(U;\mathbb{R}^3)} \\ &\leq 2\|w_\varepsilon - u_\varepsilon\|_{L^p(U;\mathbb{R}^3)} + |\xi| \|\nabla' u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^{3 \times 2})}. \end{aligned}$$

Hence, it follows along with (4.35) that

$$\|\Sigma_\varepsilon(\cdot + \xi) - \Sigma_\varepsilon\|_{L^p(U;\mathbb{R}^3)} \leq C(|\xi| + \varepsilon^{\frac{\beta}{p}-2} + \varepsilon),$$

with $C > 0$ independent of ε . The Fréchet-Kolmogorov theorem [4, Theorem 4.16] then implies once again that there exists $\Sigma \in L^p(U';\mathbb{R}^3)$ and a non-relabelled subsequence of $(\Sigma_\varepsilon)_\varepsilon$ with

$$\Sigma_\varepsilon \rightarrow \Sigma \text{ in } L^p(U';\mathbb{R}^3) \text{ and pointwise a.e. in } U',$$

and consequently, $\Sigma \in \mathcal{S}^2$ a.e. in U' .

Step 3: Conclusion. The statement follows immediately, if we repeat Steps 3 and 4 in the proof of Proposition 4.8. \square

We conclude this section with a brief comment on the validity of Proposition 4.10 for scaling exponents $\beta \leq 2p$.

Remark 4.11 (Optimal scaling exponent). Note that the statement of Proposition 4.10 is not true for $\beta \leq p$ in general. If we consider for simplicity a suitable cuboid $\Omega \subset \mathbb{R}^3$, then this is a direct consequence of the related theory of layered composites in [51, Section 2 and Corollary 3.8].

To see this, we first introduce the collection of rigid layers

$$X_\varepsilon^{\text{rig}} = \bigcup_{i \in \mathbb{Z}} \varepsilon(i + [\alpha, 1 - \alpha)) \times \mathbb{R}^2 \quad (4.37)$$

for $\varepsilon > 0$ with $\alpha \in (0, \frac{1}{2})$ as in (4.1) and observe that $Y_\varepsilon^{\text{rig}} \subset X_\varepsilon^{\text{rig}}$. Following [51, Example 2.3 and Lemma 2.1]), let $u_\varepsilon : \Omega \rightarrow \mathbb{R}^3$ be a Lipschitz function that induces uniform bending of all stiff layers in Ω ; naturally, u_ε also deforms all fibers in Ω in the same way. The elastic energy contribution of u_ε on the stiff components can be estimated by

$$\int_{Y_\varepsilon^{\text{rig}} \cap \Omega} \text{dist}^p(\nabla u_\varepsilon, \text{SO}(3)) \, dx \leq \int_{X_\varepsilon^{\text{rig}} \cap \Omega} \text{dist}^p(\nabla u_\varepsilon, \text{SO}(3)) \, dx \leq C\varepsilon^p,$$

where the constant $C > 0$ is independent of ε . Since the weak $W^{1,p}$ -limit of $(u_\varepsilon)_\varepsilon$ cannot be expressed in the form (4.31), it is confirmed that Proposition 4.10 fails for $\beta \leq p$.

Whether Proposition 4.10 is valid in the scaling regimes $\beta \in (p, 2p]$, though, remains an interesting open problem.

4.4 Proof of sufficiency in Theorem 4.1

The main ingredient for the construction of approximating sequences in the proof of Theorem 4.1 is a suitable approximation of the identity in two dimensions that is constant on the cross section of the fibers. The following lemma can be viewed as a generalization of the one-dimensional result in [50, Lemma 4.3].

Lemma 4.12 (Approximation of the identity). *Let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $\alpha \in (0, \frac{1}{2})$. Further, let $\omega_\varepsilon^k \subset \mathbb{R}^2$ with $\varepsilon > 0$ and $k \in \mathbb{Z}^2$ be open domains satisfying (4.1), i.e.,*

$$\omega_\varepsilon^k \subset \varepsilon(k + [\alpha, 1 - \alpha]^2).$$

Then, there exists a sequence $(\varphi_\varepsilon)_\varepsilon \subset W^{1,\infty}(U; \mathbb{R}^2)$ with the following properties:

- i) $\sup_{\varepsilon > 0} \|\nabla' \varphi_\varepsilon\|_{L^\infty(U; \mathbb{R}^{2 \times 2})} < \frac{1}{\alpha}$;
- ii) φ_ε is constant on $\omega_\varepsilon^k \cap U$ for every $k \in \mathbb{Z}^2$ and $\varepsilon > 0$;
- iii) $\varphi_\varepsilon(\varepsilon(k + [-\alpha, 1 - \alpha]^2)) \cap \varphi_\varepsilon(\varepsilon(j + [-\alpha, 1 - \alpha]^2)) = \emptyset$ for all $k, j \in \mathbb{Z}^2$ with $k \neq j$ and every $\varepsilon > 0$;
- iv) $\varphi_\varepsilon \rightarrow \text{id}_{\mathbb{R}^2}$ uniformly in U as $\varepsilon \rightarrow 0$.

Proof. We consider the translated unit cell $Z := Y - \alpha(e_1 + e_2) = [-\alpha, 1 - \alpha]^2$ and its partition $Z = \bigcup_{i=1}^4 Z_i$ with

$$Z_1 = [-\alpha, \alpha]^2, \quad Z_2 = [-\alpha, \alpha) \times [\alpha, 1 - \alpha), \quad Z_3 = [\alpha, 1 - \alpha)^2, \quad Z_4 = [\alpha, 1 - \alpha) \times [-\alpha, \alpha),$$

for every $\varepsilon > 0$ and every $k \in \mathbb{Z}^2$, which implies *iii*).

As for the proof of *iv*), we use the Riemann-Lebesgue lemma on the weak convergence of periodically oscillating sequences in conjunction with (4.38) to conclude that

$$\nabla' \varphi_\varepsilon \xrightarrow{*} \int_Z \nabla' \varphi \, dz' = \frac{1}{2\alpha} (|Z_1|(e_1|e_2) + |Z_2|(e_1|0) + |Z_4|(0|e_2)) = \text{Id}_{\mathbb{R}^2 \times 2} = \nabla' \text{id}_{\mathbb{R}^2} \quad (4.42)$$

in $L^\infty(U; \mathbb{R}^{2 \times 2})$ as $\varepsilon \rightarrow 0$. In light of (4.41), Poincaré's inequality yields for any $q > 2$ the existence of a constant $C = C(U, q)$ such that

$$\left\| \varphi_\varepsilon - \int_U x' \, dx' \right\|_{L^q(U; \mathbb{R}^2)} = \left\| \varphi_\varepsilon - \int_U \varphi_\varepsilon \, dx' \right\|_{L^q(U; \mathbb{R}^2)} \leq C \|\nabla' \varphi_\varepsilon\|_{L^q(U; \mathbb{R}^{2 \times 2})} \leq \frac{C}{\alpha},$$

with the last estimate due to *i*). Thus, as a uniformly bounded family in $W^{1,q}(U; \mathbb{R}^2)$ satisfying (4.42), every subsequence of $(\varphi_\varepsilon)_\varepsilon$ has a weakly converging subsequence with limit $\text{id}_{\mathbb{R}^2} + d$ for some $d \in \mathbb{R}^2$. By (4.41), the shift vector d needs to vanish. Thus, we obtain via compact Sobolev embedding that $\varphi_\varepsilon \rightarrow \text{id}_{\mathbb{R}^2}$ uniformly as $\varepsilon \rightarrow 0$, as stated. \square

First, we present the general idea of how to construct approximating sequences under the simplifying assumption of Lipschitz regularity. This approach serves as a basis for proving the analogous statement for Sobolev functions in Proposition 4.14.

Proposition 4.13 (Approximation of Lipschitz functions). *Let $R \in W^{1,\infty}(\omega; \text{SO}(3))$ and $b \in W^{1,\infty}(\omega; \mathbb{R}^3)$. If $u \in W^{1,\infty}(\Omega; \mathbb{R}^3)$ is given by*

$$u(x) = R(x')x + b(x'), \quad x \in \Omega,$$

then there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,\infty}(\Omega; \mathbb{R}^3)$ with

$$\nabla u_\varepsilon \in \text{SO}(3) \text{ a.e. in } Y_\varepsilon^{\text{rig}} \cap \Omega$$

for all ε , such that $u_\varepsilon \xrightarrow{} u$ in $W^{1,\infty}(\Omega; \mathbb{R}^3)$ as $\varepsilon \rightarrow 0$.*

Proof. We start by extending $R \in W^{1,\infty}(\omega; \text{SO}(3))$ according to Lemma 4.5, that is, we consider $R \in W^{1,\infty}(V; \text{SO}(3))$, where $V \subset \mathbb{R}^2$ is an open set with $\omega \Subset V$. Moreover, one can also find a Lipschitz extension of b in $W^{1,\infty}(V; \mathbb{R}^3)$, still called b , by standard Sobolev theory, see e.g., [92, Theorem 1, Section 3.1].

Further, let $(\varphi_\varepsilon)_\varepsilon$ be the sequence of Lipschitz functions that results from approximating the identity on \mathbb{R}^2 according to the construction in Lemma 4.12 with $U = \omega$. Since $(\varphi_\varepsilon)_\varepsilon$ converges uniformly to $\text{id}_{\mathbb{R}^2}$ on ω as $\varepsilon \rightarrow 0$, we may assume that $\varphi_\varepsilon(\omega) \subset V$ for all ε sufficiently small. For such $\varepsilon > 0$, we can therefore define

$$u_\varepsilon(x) = R(\varphi_\varepsilon(x'))x + b(\varphi_\varepsilon(x')), \quad x \in \Omega. \quad (4.43)$$

Since the composition of two Lipschitz maps is again Lipschitz, one has that $u_\varepsilon \in W^{1,\infty}(\Omega; \mathbb{R}^3)$. By Lemma 4.12 *ii*), φ_ε is constant on all the connected components of $(Y_\varepsilon^{\text{rig}})' \cap \omega = \bigcup_{k \in \mathbb{Z}^2} w_\varepsilon^k \cap \omega$, which yields

$$\nabla u_\varepsilon = R \circ \varphi_\varepsilon \in \text{SO}(3) \text{ a.e. in } Y_\varepsilon^{\text{rig}} \cap \Omega.$$

It remains to prove that $u_\varepsilon \xrightarrow{*} u$ in $W^{1,\infty}(\Omega; \mathbb{R}^3)$. Indeed, the uniform convergence of $(u_\varepsilon)_\varepsilon$ follows from the Lipschitz continuity of R and b along with the uniform convergence of $(\varphi_\varepsilon)_\varepsilon$ to

the identity map on \mathbb{R}^2 , precisely,

$$\begin{aligned} \sup_{x \in \Omega} |u_\varepsilon(x) - u(x)| &= \sup_{x \in \Omega} |(R(\varphi_\varepsilon(x')) - R(x'))x + b(\varphi_\varepsilon(x')) - b(x')| \\ &\leq C \sup_{x' \in \omega} (|R(\varphi_\varepsilon(x')) - R(x')| + |b(\varphi_\varepsilon(x')) - b(x')|) \\ &\leq C \sup_{x' \in \omega} |\varphi_\varepsilon(x') - x'| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Finally, it needs to be shown that $(\nabla u_\varepsilon)_\varepsilon$ is uniformly essentially bounded. We use the chain rule and exploit Lemma 4.12 i) to obtain the estimate

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^\infty(\Omega; \mathbb{R}^3)} &= \text{ess sup}_{x \in \Omega} |\nabla R(\varphi_\varepsilon(x')) \nabla \varphi_\varepsilon(x') x + R(\varphi_\varepsilon(x')) + \nabla b(\varphi_\varepsilon(x')) \nabla \varphi_\varepsilon(x')| \\ &\leq C \text{ess sup}_{x \in \Omega} (1 + |\nabla R(\varphi_\varepsilon(x'))| + |\nabla b(\varphi_\varepsilon(x'))|) \\ &\leq C(1 + \|R\|_{W^{1,\infty}(V; \mathbb{R}^{3 \times 3})} + \|b\|_{W^{1,\infty}(V; \mathbb{R}^3)}), \end{aligned}$$

which concludes the proof. \square

Now we can address the analogy of Proposition 4.13 in the setting of $W^{1,p}$ -functions. The proof strategy is similar but requires a refined reasoning, since the composition of Sobolev with Lipschitz functions may not be Sobolev anymore, see the work by Conti & Dolzmann [63, Appendix A].

Proposition 4.14 (Approximation of Sobolev functions). *Let $R \in W^{1,p}(\omega; \text{SO}(3))$ and $b \in W^{1,p}(\omega; \mathbb{R}^3)$ with $p \geq 2$. If $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ is given by*

$$u(x) = R(x')x + b(x'), \quad x \in \Omega, \quad (4.44)$$

then there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,p}(\Omega; \mathbb{R}^3)$ with

$$\nabla u_\varepsilon \in \text{SO}(3) \text{ a.e. in } Y_\varepsilon^{\text{rig}} \cap \Omega$$

for every ε , such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$ as $\varepsilon \rightarrow 0$.

Proof. Let $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ satisfy (4.44). The task is to imitate the construction in (4.43), while making sure that the approximating sequence actually lies in $W^{1,p}(\Omega; \mathbb{R}^3)$.

To this end, we first approximate R and b in $W^{1,p}$ by smooth functions R_η and b_η with $\eta > 0$ small. However, the natural approach of choosing $u_{\eta,\varepsilon}$ according to (4.43) for each η and concluding by a diagonalization argument, is not easily accessible. This is because the required uniform L^p -bounds on $\nabla u_{\eta,\varepsilon}$ are not trivial to obtain due to the lack of invertibility of the approximation of the identity φ_ε (cf. Lemma 4.12), which prevents a classical change of variables. To overcome this issue, we proceed similarly to [63, Lemma A.1] and refine the definition of $u_{\eta,\varepsilon}$ by introducing small translations of the independent variables.

We detail these arguments in the following three steps.

Step 1: Extension and approximation. In analogy to Proposition 4.13, we first extend R and b to $R \in W^{1,p}(V; \text{SO}(3))$, $b \in W^{1,p}(V; \mathbb{R}^3)$ with an open set $V \subset \mathbb{R}^2$ such that $\omega \Subset V$. Moreover, let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain with $\omega \Subset U \Subset V$ and $(\varphi_\varepsilon)_\varepsilon$ the approximation of the identity from Lemma 4.12. Suppose in the following that $\varepsilon > 0$ is so small that $\varphi_\varepsilon(U) + B(0, \varepsilon) \subset V$.

According to Lemma 4.6 and standard Sobolev theory, there are approximating sequences $(R_\eta)_\eta \subset C^\infty(V; \text{SO}(3))$ and $(b_\eta)_\eta \subset C^\infty(V; \mathbb{R}^3)$ such that

$$R_\eta \rightarrow R \text{ in } W^{1,p}(V; \text{SO}(3)) \quad \text{and} \quad b_\eta \rightarrow b \text{ in } W^{1,p}(V; \mathbb{R}^3), \quad (4.45)$$

respectively; notice that we may assume without loss of generality that $(R_\eta)_\eta \subset C^\infty(\bar{V}; \text{SO}(3))$ and $(b_\eta)_\eta \subset C^\infty(\bar{V}; \mathbb{R}^3)$, since otherwise, we introduce an intermediate set \tilde{V} with $U \Subset \tilde{V} \Subset V$ and choose ε even smaller to guarantee that $\varphi_\varepsilon(U) + B(0, \varepsilon) \subset \tilde{V}$.

Step 2: Construction of the approximating sequence. Similarly to (4.43), we define for $\eta > 0$, $\varepsilon > 0$ sufficiently small, and $a \in B(0, \varepsilon) \subset \mathbb{R}^2$ the Lipschitz functions

$$u_{\eta, \varepsilon}^a(x) = R_\eta(\varphi_\varepsilon(x') + a)x + b_\eta(\varphi_\varepsilon(x') + a), \quad x \in U \times (0, L). \quad (4.46)$$

Then,

$$\nabla u_{\eta, \varepsilon}^a \in \text{SO}(3) \text{ a.e. in } Y_\varepsilon^{\text{rig}} \cap U, \quad (4.47)$$

since φ_ε is constant on $\omega_\varepsilon^k \cap U$ for each $k \in \mathbb{Z}^2$ according to Lemma 4.12 ii). Further, we infer from Lemma 4.12 iv) that any sequence $(u_{\eta, \varepsilon}^{a_\varepsilon})_\varepsilon$ with $a_\varepsilon \in B(0, \varepsilon)$ converges uniformly for $\varepsilon \rightarrow 0$ to a limit function u_η given by

$$u_\eta(x) = R_\eta(x')x + b_\eta(x'), \quad x \in U \times (0, L);$$

in particular,

$$u_{\eta, \varepsilon}^{a_\varepsilon} \rightarrow u_\eta \text{ in } L^p(U \times (0, L); \mathbb{R}^3) \text{ as } \varepsilon \rightarrow 0.$$

Since also $u_\eta \rightarrow u$ in $W^{1,p}(U \times (0, L); \mathbb{R}^3)$ for $\eta \rightarrow 0$ due to (4.45), a diagonalization argument in the sense of Attouch provides a diagonal sequence $(u_\varepsilon)_\varepsilon = (u_{\eta(\varepsilon), \varepsilon}^{a_\varepsilon})_\varepsilon$ such that

$$u_\varepsilon \rightarrow u \text{ in } L^p(U \times (0, L); \mathbb{R}^3).$$

In view of (4.47), we know that $\nabla u_\varepsilon \in \text{SO}(3)$ almost everywhere in $Y_\varepsilon^{\text{rig}} \cap U$.

Step 3: Choice of suitable translations. It remains to show that the construction of sequences $(u_\varepsilon)_\varepsilon$ in Step 2 gives rise to a sequence that converges weakly in $W^{1,p}(U; \mathbb{R}^3)$. This follows immediately, if we can select translations $a_\varepsilon \in B(0, \varepsilon) \subset \mathbb{R}^2$ such that

$$\|\nabla u_{\eta, \varepsilon}^{a_\varepsilon}\|_{L^p(\Omega; \mathbb{R}^{3 \times 3})} \leq C \quad (4.48)$$

with a constant $C > 0$ independent of η and ε .

To this end, consider the index set $I_\varepsilon = \{k \in \mathbb{Z}^2 : |\varepsilon(k + Z) \cap \omega| > 0\}$, recalling $\Omega = \omega \times (0, L)$ and the notation $Z = [-\alpha, 1 - \alpha]^2$ from the proof of Lemma 4.12, and observe that

$$\omega \subset \bigcup_{k \in I_\varepsilon} \varepsilon(k + Z) \subset U \quad (4.49)$$

for ε sufficiently small. For such ε , $k \in I_\varepsilon$ and $\eta > 0$, we define the function

$$h_{\eta, \varepsilon}^k : B(0, \varepsilon) \rightarrow [0, \infty), \quad a \mapsto \|\nabla u_{\eta, \varepsilon}^a\|_{L^p(\varepsilon(k+Z) \times (0, L); \mathbb{R}^{3 \times 3})}^p,$$

with $u_{\eta, \varepsilon}^a$ as in (4.46). For the mean value of $h_{\eta, \varepsilon}^k$, we obtain from the chain and product rule, together with Lemma 4.12 i), Fubini's theorem, and a change of variables that

$$\begin{aligned} \int_{B(0, \varepsilon)} h_{\eta, \varepsilon}^k(a) \, da &= \int_{B(0, \varepsilon)} \int_0^L \int_{\varepsilon(k+Z)} |\nabla u_{\eta, \varepsilon}^a|^p \, dx' \, dx_3 \, da \\ &\leq C \int_{\varepsilon(k+Z)} \int_{B(0, \varepsilon)} |\nabla' R_\eta(\varphi_\varepsilon(x') + a)|^p + |\nabla' b_\eta(\varphi_\varepsilon(x') + a)|^p + 1 \, da \, dx' \\ &\leq C \frac{|\varepsilon(k+Z)|}{|B(0, \varepsilon)|} \int_{B(0, \varepsilon) + \varphi_\varepsilon(\varepsilon(k+Z))} |\nabla' R_\eta(a)|^p + |\nabla' b_\eta(a)|^p + 1 \, da \\ &\leq C \int_{B(0, \varepsilon) + \varphi_\varepsilon(\varepsilon(k+Z))} |\nabla' R_\eta(a)|^p + |\nabla' b_\eta(a)|^p + 1 \, da. \end{aligned} \quad (4.50)$$

where $C > 0$ is a constant that depends only on p and α .

By Lemma 4.12 iii), the sets $\varphi_\varepsilon(\varepsilon(k + Z))$ with different $k \in \mathbb{Z}^2$ are disjoint for every $\varepsilon > 0$. Therefore, every $x' \in \varphi_\varepsilon(U) + B(0, \varepsilon)$ is contained in at most 9 sets of the form $\varphi_\varepsilon(\varepsilon(k + Z)) + B(0, \varepsilon)$ with $k \in \mathbb{Z}^2$, see also Figure 4.6. Summing over all $k \in I_\varepsilon$ in (4.50) then implies in view

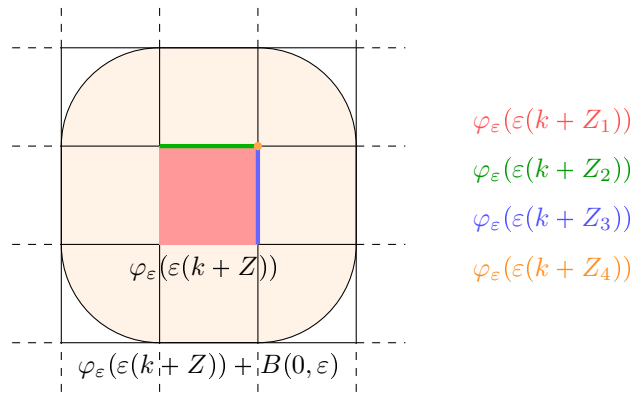


Figure 4.6: Illustration of $\varphi_\varepsilon(\varepsilon(k + Z))$, its components $\varphi_\varepsilon(\varepsilon(k + Z_k))$ for $k = 1, \dots, 4$, as well as its enlarged version $\varphi_\varepsilon(\varepsilon(k + Z)) + B(0, \varepsilon)$ for some $k \in \mathbb{Z}^2$ and $\varepsilon > 0$.

of (4.49) and the choice of V that

$$\begin{aligned} \int_{B(0, \varepsilon)} \|\nabla u_{\eta, \varepsilon}^a\|_{L^p(\Omega; \mathbb{R}^3)}^p da &\leq C \int_V |\nabla' R_\eta(a)|^p + |\nabla' b_\eta(a)|^p + 1 da \\ &\leq C(\|R_\eta\|_{W^{1,p}(V; \mathbb{R}^3 \times 3)}^p + \|b_\eta\|_{W^{1,p}(V; \mathbb{R}^3)}^p + 1), \end{aligned}$$

and we conclude, due to (4.45), that

$$\int_{B(0, \varepsilon)} \|\nabla u_{\eta, \varepsilon}^a\|_{L^p(\Omega; \mathbb{R}^3)}^p da \leq C \quad (4.51)$$

with a constant $C > 0$ independent of η and ε . Hence, there exists for each ε a subset $E_\varepsilon \subset B(0, \varepsilon)$ of positive measure such that

$$\|\nabla u_{\eta, \varepsilon}^a\|_{L^p(\Omega; \mathbb{R}^3)}^p \leq C$$

for all $a \in E_\varepsilon$ with the same constant as in (4.51), so that choosing any $a_\varepsilon \in E_\varepsilon \subset B(0, \varepsilon)$ yields the desired bound (4.48). This finishes the proof. \square

The proof of Theorem 4.1 is essentially a consequence of Propositions 4.14 and 4.8. The gap in the different representations can be closed by a lifting argument.

Proof of Theorem 4.1. It follows from Proposition 4.8 that any $u \in \mathcal{A}$ (see (4.5)) can be represented as

$$u(x) = x_3 \Sigma(x') + d(x'), \quad x \in \Omega, \quad (4.52)$$

with $\Sigma \in W^{1,p}(\omega; \mathcal{S}^2)$ and $d \in W^{1,p}(\omega; \mathbb{R}^3)$. On the other hand, Proposition 4.14 yields that any function

$$u(x) = R(x')x + b(x'), \quad x \in \Omega, \quad (4.53)$$

with $R \in W^{1,p}(\omega; \text{SO}(3))$ and $b \in W^{1,p}(\omega; \mathbb{R}^3)$ lies in \mathcal{A} . If the set of all functions of the form (4.52) and (4.53) are denoted by \mathcal{A}_Σ and \mathcal{A}_R , respectively, we have

$$\mathcal{A}_R \subset \mathcal{A} \subset \mathcal{A}_\Sigma.$$

To conclude the proof, it suffices to show that the sets \mathcal{A}_R and \mathcal{A}_Σ coincide under the assumption (H). While $\mathcal{A}_R \subset \mathcal{A}_\Sigma$ is immediately clear, the converse inclusion is more delicate as it requires the construction of an $\text{SO}(3)$ -valued Sobolev function whose third column coincides with Σ . By the lifting result in Lemma 4.3, we find for any u as in (4.52) a function $R \in W^{1,p}(\omega; \text{SO}(3))$ such that $Re_3 = \Sigma$, and hence, setting

$$b(x') = d(x') - x_1 R(x')e_1 - x_2 R(x')e_2, \quad x' \in \omega,$$

implies (4.53). \square

Even though the representation of functions in the form (4.53) is essential for our construction of an approximating sequence in Propositions 4.13 and 4.14, we point out that the choice of R is not unique, cf. Remark 4.4. More intuitive from a geometric point of view is (4.52); indeed, one can describe the image of Ω under u as the image of the cross section ω under the map d , fattened linearly in x_3 in the direction of the vector field Σ . A few illustrative examples are presented in the next section.

4.5 Examples of effective deformations

Before we focus on explicit examples of limit functions as characterized in Theorem 4.1, which describe the macroscopically attainable deformations with rigid fiber-reinforcements, let us briefly address the issue of incompressibility, or in other words, local volume preservation, of such maps.

Throughout this section, we consider $u \in \mathcal{A} \subset W^{1,p}(\Omega; \mathbb{R}^3)$ with $p > 2$ (see (4.5)) of the form

$$u(x) = x_3 \Sigma(x') + d(x'), \quad x \in \Omega, \quad (4.54)$$

with given $\Sigma \in W^{1,p}(\omega; \mathcal{S}^2)$ and $d \in W^{1,p}(\omega; \mathbb{R}^3)$, cf. also Remark 4.4.

The next lemma gives a necessary condition for the incompressibility of such deformations.

Lemma 4.15. *Let u as in (4.54) be incompressible, i.e., $\det \nabla u = 1$ a.e. in Ω . Then,*

$$\partial_1 \Sigma \parallel \partial_2 \Sigma \text{ a.e. in } \omega. \quad (4.55)$$

Proof. In fact, the condition (4.55) results from a second-order linearization in x_3 -direction of the incompressibility constraint, $\det \nabla u = 1$ a.e. in Ω . By the multi-linearity of the determinant, it follows for a.e. $x = (x', x_3) \in \Omega$ that

$$\begin{aligned} 0 &= \partial_3^2 \det \nabla u(x) = \partial_3^2 \det (\partial_1 d(x') + x_3 \partial_1 \Sigma(x') | \partial_2 d(x') + x_3 \partial_2 \Sigma(x') | \Sigma(x')) \\ &= 2 \det (\partial_1 \Sigma(x') | \partial_2 \Sigma(x') | \Sigma(x')). \end{aligned}$$

Thus, $\partial_1 \Sigma, \partial_2 \Sigma$ and Σ are linearly dependent a.e. in ω , that is, for a.e. $x' \in \omega$, there exists $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \setminus \{0\}$ such that

$$\lambda_1 \partial_1 \Sigma + \lambda_2 \partial_2 \Sigma + \lambda_3 \Sigma = 0. \quad (4.56)$$

Since $|\Sigma| = 1$ and $\partial_i \Sigma \cdot \Sigma = 0$ a.e. in ω for $i \in \{1, 2\}$, it follows from scalar multiplication of (4.56) with Σ that $\lambda_3 = 0$, which shows (4.55). \square

In the following, we provide a few illustrative examples of (compressible and incompressible) deformations of the form (4.54), where Ω is always a suitable open cuboid.

Example 4.16 (Σ is constant). If Σ is constant, the image $u(\Omega)$ corresponds to the deformed cross section $d(\omega)$ thickened in the direction of Σ by the height of Ω . A first example of an incompressible deformation of this type is

$$u(x) = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ -x_1^2 - x_2^2 \end{pmatrix}, \quad x \in \Omega, \quad (4.57)$$

where ω is transformed into (parts of) a paraboloid, see Figure 4.7a). Another classical example in this context is a simple shear in e_2 -direction, i.e.,

$$u(x) = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ \gamma x_1 + x_2 \\ 0 \end{pmatrix}, \quad x \in \Omega, \quad (4.58)$$

with shear parameter $\gamma \in \mathbb{R}$, see Figure 4.7b).

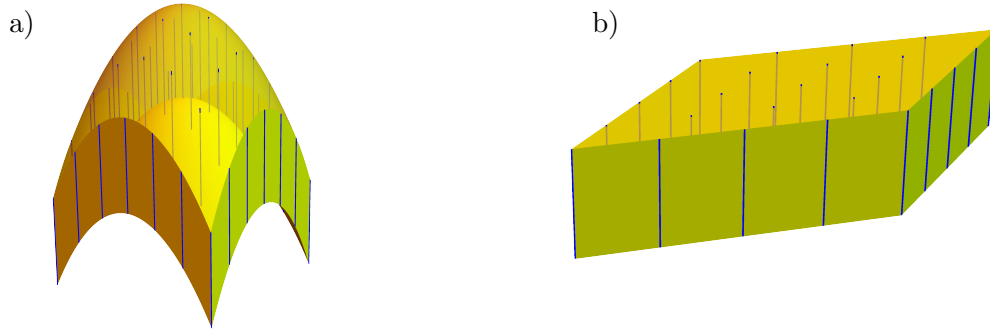


Figure 4.7: The images of $\Omega = (-1, 1)^2 \times (0, 1)$ under a) (4.57) and b) (4.58) for $\gamma = 1$.

Example 4.17 (Σ depends on only one variable). Suppose that $\partial_2 \Sigma = 0$. In this case, the direction along which the transformed cross section $d(\omega)$ is thickened depends in general non-trivially on x_1 . A locally volume-preserving deformation describing a twist in e_1 -direction is given by

$$u(x) = x_3 \begin{pmatrix} 0 \\ -\sin(\frac{x_1}{\pi}) \\ \cos(\frac{x_1}{\pi}) \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \cos(\frac{x_1}{\pi}) \\ x_2 \sin(\frac{x_1}{\pi}) \end{pmatrix}, \quad x \in \Omega, \quad (4.59)$$

see Figure 4.8a). As a second example, consider

$$u(x) = \frac{x_3}{r} \begin{pmatrix} x_1 \\ 0 \\ \sqrt{r^2 - x_1^2} \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 + e^{x_2} \\ \sqrt{r^2 - x_1^2} \end{pmatrix}, \quad x \in \Omega, \quad (4.60)$$

with $r > 0$. Notice that u as in (4.60) is not incompressible despite satisfying the necessary condition (4.55), since u involves a stretch in e_2 -direction, see Figure 4.8b).

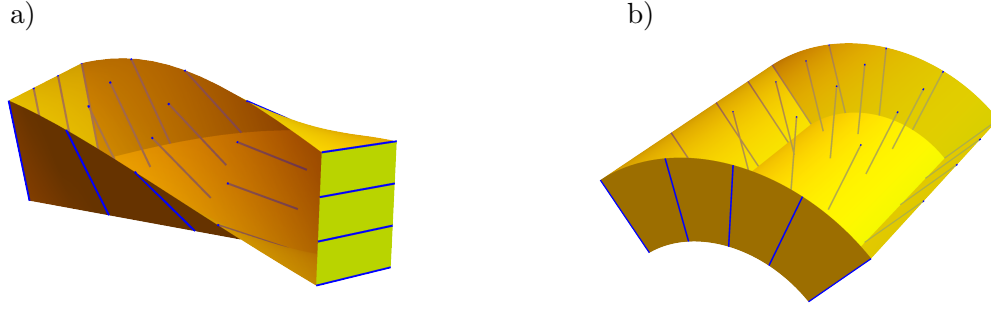


Figure 4.8: a) The image of $\Omega = (0, 4) \times (0, 1)^2$ under (4.59). b) The image of $\Omega = (-1, 1)^2 \times (0, 1)$ under (4.60) with $r = \frac{3}{2}$.

Example 4.18 (Σ depends on both cross-section variables). Consider the following modification of (4.57),

$$u(x) = \frac{x_3}{\sqrt{4(x_1^2 + x_2^2) + 1}} \begin{pmatrix} 2x_1 \\ 2x_2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ -x_1^2 - x_2^2 \end{pmatrix}, \quad x \in \Omega. \quad (4.61)$$

In this case, the vector field Σ is orthogonal to the surface of the paraboloid $x' \mapsto (x_1, x_2, -x_1^2 - x_2^2)$, see Figure 4.9a). Since $\partial_1 \Sigma$ is not parallel to $\partial_2 \Sigma$, u as in (4.61) is not locally volume-preserving according to Lemma 4.15. Another example that does not satisfy the condition (4.55) either is

$$u(x) = \frac{x_3}{\sqrt{2}\sqrt{x_1^2 + x_2^2 + 1}} \begin{pmatrix} -x_1 - x_2 \\ x_1 - x_2 \\ \sqrt{2} \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \quad x \in \Omega, \quad (4.62)$$

depicted in Figure 4.9b).

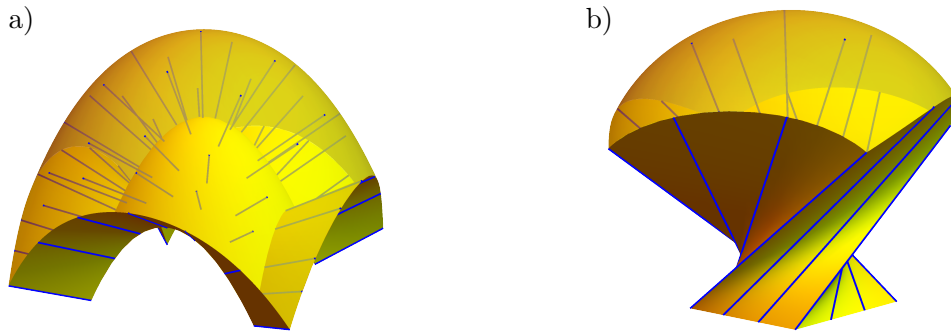


Figure 4.9: a) The image of $\Omega = (-1, 1)^2 \times (0, 1)$ under (4.61). b) The image of $\Omega = (-1, 1)^2 \times (0, 4)$ under (4.62).

Remark 4.19 (Comparison with layered composites). Let $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ be an open cuboid. As proven in [51, Theorem 1.1], the admissible effective deformations of a composite with rigid layers, or mathematically speaking, the weak $W^{1,p}$ -limits of sequences

$(u_\varepsilon)_\varepsilon$ such that

$$u_\varepsilon \in \mathcal{B}_\varepsilon := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : \nabla u \in \text{SO}(3) \text{ a.e. in } X_\varepsilon^{\text{rig}} \cap \Omega\}$$

for $\varepsilon > 0$, with $X_\varepsilon^{\text{rig}}$ as introduced in (4.37), are characterized by

$$\begin{aligned} \mathcal{B} := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u(x) = R(x_1)x + b(x_1) \text{ for a.e. } x \in \Omega \\ \text{with } R \in W^{1,p}((0, L_1); \text{SO}(3)), b \in W^{1,p}((0, L_1); \mathbb{R}^3)\}. \end{aligned}$$

Comparing with the characterization of \mathcal{A} in Theorem 4.1 shows that $\mathcal{B} \subset \mathcal{A}$. This observation backs the intuition that composites with rigid fiber reinforcements are more flexible in their deformation behavior than those with rigid layers.

It is evident that the macroscopic deformations in (4.58) and (4.59) can be attained also by layered materials, meaning that they are elements of \mathcal{B} . On the other hand, the deformations presented in Example 4.18, as well as (4.57) and (4.60) show that the inclusion $\mathcal{B} \subset \mathcal{A}$ is strict, which underlines the higher flexibility fiber-reinforced materials.

4.6 Regularization in the cross-section variables

This section is concerned with the proof of Theorem 4.2, where we additionally assume that second derivatives in the cross-section variables of weakly convergent sequences $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon \in \mathcal{A}_\varepsilon$ exist, and are L^p -bounded uniformly in ε . In this case, the weak $W^{1,p}$ -limit of $(u_\varepsilon)_\varepsilon$ corresponds to a rigid body motion. We begin our analysis with two auxiliary results.

First, the above-mentioned assumption of higher regularity allows us to improve the estimates in Lemma 4.7 with the help of a one-dimensional Poincaré estimate.

Lemma 4.20. *Let $E \subset \mathbb{R}^3$, $d \in \mathbb{R}^2$ and $w_1, w_2 : E \rightarrow \mathbb{R}^3$ be as in Lemma 4.7. If $v \in W^{1,p}(E; \mathbb{R}^3)$ with $p \geq 1$ satisfies*

$$\max_{i,j \in \{1,2\}} \|\partial_i \partial_j v\|_{L^p(E; \mathbb{R}^3)}^p < \infty$$

and if

$$v = w_1 \text{ a.e. in } E \cap ((0, \mu) \times \mathbb{R}^2) \quad \text{and} \quad v = w_2 \text{ a.e. in } E \cap ((L_1 - \mu, L_1) \times \mathbb{R}^2)$$

for some $\mu > 0$, then

$$\int_E |\partial_d^2 v|^p \, dx \geq \frac{C|E|L_3^p}{(1 + |m|p)^2 L_1^{2p}} |(A_2 - A_1)e_3|^p \quad (4.63)$$

with a constant $C > 0$ that depends only on p .

Proof. We may assume without loss of generality that $w_1 = 0$, otherwise consider $\tilde{v} = v - w_1$ in place of v . Then, by assumption, $\partial_d v = 0$ a.e. in $E \cap ((0, \mu) \times \mathbb{R}^2)$, and thus, with $u(y) = v(y_1, y_2 + my_1, y_3)$ for $y \in Q = (0, L_1) \times (0, L_2) \times (0, L_3)$,

$$\partial_1 u = 0 \quad \text{a.e. in } (0, \mu) \times (0, L_2) \times (0, L_3).$$

Via the same change of variables as in Lemma 4.7, we can therefore deduce with Poincaré's inequality, applied to $\partial_1 u$ in y_1 -direction, that

$$\begin{aligned} \int_E |\partial_d^2 v|^p \, dx &= \frac{1}{(1 + m^2)^p} \int_0^{L^3} \int_0^{L^2} \int_0^{L_1} |\partial_1(\partial_1 u)|^p \, dy_1 \, dy_2 \, dy_3 \\ &\geq \frac{C}{(1 + m^2)^p L_1^p} \int_Q |\partial_1 u|^p \, dy = \frac{C}{(1 + m^2)^{p/2} L_1^p} \int_E |\partial_d v|^p \, dx, \end{aligned}$$

where $C^{-1}L_1^p$ with $C > 0$ depending only on p is the optimal Poincaré constant. In combination with Lemma 4.7, which implies

$$\int_E |\partial_d v|^p dx \geq \frac{C|E|L_3^p}{(1+|m|^p)L_1^p} |A_2 e_3|^p,$$

we obtain the desired estimate. \square

Second, we prove that strongly L^p -convergent functions that are constant rotations on the rigid components have a constant limit in the set of rotations. In particular, when applied to gradient fields, the next result can be seen as a companion to Proposition 4.8 for strongly converging sequences in $W^{1,p}(\Omega; \mathbb{R}^3)$.

Lemma 4.21. *Let $U \subset \mathbb{R}^3$ be an open set and let $(V_\varepsilon)_\varepsilon \subset L^p(U; \mathbb{R}^{3 \times 3})$ with $p \geq 1$ satisfy*

$$V_\varepsilon \in \text{SO}(3) \text{ a.e. in } Y_\varepsilon^{\text{rig}} \cap U$$

for all ε . If $V \in L^p(U; \mathbb{R}^{3 \times 3})$ is such that $V_\varepsilon \rightarrow V$ in $L^p(U; \mathbb{R}^{3 \times 3})$ as $\varepsilon \rightarrow 0$, then

$$V \in \text{SO}(3) \text{ a.e. in } U.$$

Proof. Assume to the contrary that there is a $\gamma > 0$ and an open cube $Q \subset U$ such that

$$\text{dist}(V(x), \text{SO}(3)) > \gamma \quad \text{for a.e. } x \in Q.$$

Since

$$\gamma < \text{dist}(V(x), \text{SO}(3)) \leq |V(x) - V_\varepsilon(x)| \quad \text{for a.e. } x \in Y_\varepsilon^{\text{rig}} \cap Q,$$

it follows that, up to a set of measure zero,

$$Y_\varepsilon^{\text{rig}} \cap Q \subset \{x \in Q : |V_\varepsilon(x) - V(x)| > \gamma\}.$$

Recalling the definition of $Y_\varepsilon^{\text{rig}}$, each fiber cross-section w_ε^k with $\varepsilon > 0$ and $k \in \mathbb{Z}^2$ contains a square S_ε^k with $|S_\varepsilon^k| = \delta^2 \varepsilon^2$, where $\delta \in (0, 1 - 2\alpha)$ and $\alpha \in (0, \frac{1}{2})$ are given parameters, cf. (4.2). We define the index set

$$J_\varepsilon = \{k \in \mathbb{Z}^2 : S_\varepsilon^k \subset Q\},$$

and observe that, for sufficiently small ε , the cardinality of J_ε scales like ε^{-2} , precisely, $\#J_\varepsilon \geq c\varepsilon^{-2}$ with a geometric constant $c > 0$ depending only on Q . Consequently,

$$\begin{aligned} |\{x \in Q : |V_\varepsilon(x) - V(x)| > \gamma\}| &\geq |Y_\varepsilon^{\text{rig}} \cap Q| = \sum_{k \in \mathbb{Z}^2} |(\omega_k^\varepsilon \times \mathbb{R}) \cap Q| \\ &\geq |Q|^{1/3} \sum_{k \in J_\varepsilon} |S_\varepsilon^k| = |Q|^{1/3} \#J_\varepsilon \varepsilon^2 \delta^2 \geq c\delta^2 |Q|^{1/3} > 0 \end{aligned} \tag{4.64}$$

for all ε small enough.

On the other hand, the strong convergence of $(V_\varepsilon)_\varepsilon$ in $L^p(U; \mathbb{R}^{3 \times 3})$ implies its convergence in measure on Q and hence, in particular,

$$|\{x \in Q : |V_\varepsilon(x) - V(x)| > \gamma\}| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

This, however, is in contradiction with (4.64). \square

We are now in the position to present the proof of Theorem 4.2.

Proof of Theorem 4.2. Clearly, the assumptions of Proposition 4.8 are satisfied. We therefore know that the limit function u can be represented as

$$u(x) = x_3 \Sigma(x') + d(x'), \quad x \in \Omega, \quad (4.65)$$

with $\Sigma \in W^{1,p}(\omega; \mathcal{S}^2)$ and $d \in W^{1,p}(\omega; \mathbb{R}^3)$. In order to prove that the additional condition (4.6) forces u to be a rigid body motion, we show first that Σ is constant and then conclude with the help of Lemma 4.21 applied to a suitably constructed matrix-valued field.

Step 1: Σ is constant. To see this, we refine the proof of Proposition 4.8 by exchanging the estimates of Lemma 4.7 with the stronger ones from Lemma 4.20. This improves the key estimate (4.24) by a factor ε^p .

In more detail, let us adopt the definitions and quantities introduced in the proof of Proposition 4.8, up to one exception: the parallelograms $E_\varepsilon^{k,\rightarrow}$ and $E_\varepsilon^{k,\uparrow}$ are determined by the two parallel boundary lines

$$a_\varepsilon^k + \varepsilon \left(\left\{ \frac{\delta}{4} \right\} \times \left(-\frac{\delta}{4}, \frac{\delta}{4} \right) \right) \quad \text{and} \quad a_\varepsilon^{k+e_1} + \varepsilon \left(\left\{ -\frac{\delta}{4} \right\} \times \left(-\frac{\delta}{4}, \frac{\delta}{4} \right) \right),$$

and

$$a_\varepsilon^k + \varepsilon \left(\left(-\frac{\delta}{4}, \frac{\delta}{4} \right) \times \left\{ \frac{\delta}{4} \right\} \right) \quad \text{and} \quad a_\varepsilon^{k+e_2} + \varepsilon \left(\left(-\frac{\delta}{4}, \frac{\delta}{4} \right) \times \left\{ -\frac{\delta}{4} \right\} \right),$$

respectively, see Figure 4.10 for an illustration in the special case when the centers a_ε^k are periodically arranged. Then, in analogy to Step 2 of Proposition 4.8, with (4.63) in place

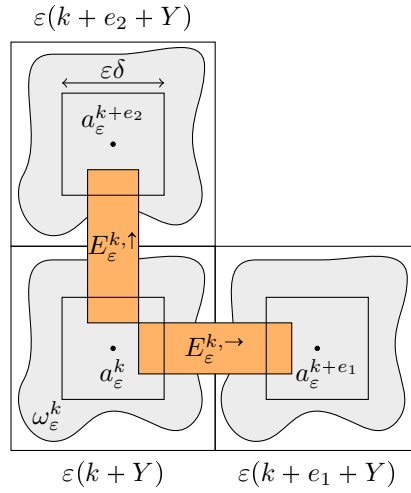


Figure 4.10: Illustration of the rectangles $E_\varepsilon^{k,\rightarrow}$ and $E_\varepsilon^{k,\uparrow}$, connecting the horizontally and vertically neighboring squares $S_\varepsilon^k = a_\varepsilon^k + \varepsilon \left(-\frac{\delta}{2}, \frac{\delta}{2} \right)^2 \subset \omega_\varepsilon^k \subset \varepsilon(k + Y)$ with a small area of overlap.

of (4.10), one obtains for Σ_ε as in (4.20) that

$$\begin{aligned} & \int_{U'} |\Sigma_\varepsilon(x' + \xi) - \Sigma_\varepsilon(x')|^p dx' \\ & \leq C\varepsilon^p (|\xi|^p + \varepsilon^p) \left(\|\partial_1^2 u_\varepsilon\|_{L^p(\Omega; \mathbb{R}^3)}^p + \|\partial_2^2 u_\varepsilon\|_{L^p(\Omega; \mathbb{R}^3)}^p \right) \leq C\varepsilon^p (|\xi|^p + \varepsilon^p) \end{aligned} \quad (4.66)$$

for any vector $\xi \in \mathbb{R}^2$ with $|\xi| < \frac{1}{2} \text{dist}(U', \partial\omega)$. In light of the convergence $\Sigma_\varepsilon \rightarrow \Sigma$ in $L^p(U'; \mathbb{R}^3)$ according to (4.25), it follows from (4.66) that

$$\int_{U'} |\Sigma(x' + \xi) - \Sigma(x')|^p dx' = 0.$$

Hence, Σ is constant in U' , and by exhaustion also in ω . This implies also that ∇u is independent of x_3 . We will prove in the next step that ∇u is constant and takes a value in $\text{SO}(3)$.

Step 2: Applying Lemma 4.21. Consider for $\varepsilon > 0$ the auxiliary matrix field

$$V_\varepsilon : U' \rightarrow \mathbb{R}^{3 \times 3}, \quad x' \mapsto \int_0^L (\nabla' u_\varepsilon(x', x_3) | \Sigma_\varepsilon(x')) dx_3;$$

by trivial extension in x_3 , one can also view V_ε as defined on U . Since $\nabla u_\varepsilon \in \text{SO}(3)$ and $\partial_3 u_\varepsilon = \Sigma_\varepsilon$ a.e. in $Y_\varepsilon^{\text{rig}} \cap U$, we infer that

$$V_\varepsilon \in \text{SO}(3) \text{ a.e. in } Y_\varepsilon^{\text{rig}} \cap U.$$

Moreover, we will see below that

$$V_\varepsilon \rightarrow \nabla u \text{ in } L^p(U; \mathbb{R}^{3 \times 3}). \quad (4.67)$$

These two observations allow us to conclude from Lemma 4.21 that $\nabla u \in \text{SO}(3)$ a.e. in U . Due to its gradient structure, however, the matrix field ∇u already has to coincide with a constant rotation on U by Reshetnyak's theorem. This property extends to Ω by exhaustion, showing that u is necessarily a rigid body motion on Ω . This gives the desired statement.

It only remains to prove (4.67). To this end, we start by observing that the first two columns $V'_\varepsilon := (V_\varepsilon e_1 | V_\varepsilon e_2)$ of V_ε satisfy

$$\int_{U'} |V'_\varepsilon(x')|^p dx' = \int_{U'} \left| \int_0^L \nabla' u_\varepsilon(x', x_3) dx_3 \right|^p dx \leq \frac{1}{L} \int_U |\nabla' u_\varepsilon(x)|^p dx = \frac{1}{L} \|\nabla' u_\varepsilon\|_{L^p(U; \mathbb{R}^{3 \times 2})}^p.$$

due to Jensen's inequality. Similarly, if we recall that u_ε is twice weakly differentiable in the cross-section variables by assumption,

$$\int_{U'} |\nabla' V'_\varepsilon(x')|^p dx' \leq 4 \max_{i,j \in \{1,2\}} \int_{U'} \left| \int_0^L \partial_i \partial_j u_\varepsilon(x', x_3) dx_3 \right|^p dx \leq \frac{4}{L} \max_{i,j \in \{1,2\}} \|\partial_i \partial_j u_\varepsilon\|_{L^p(U; \mathbb{R}^3)}^p.$$

In view of (4.6), the sequence $(V'_\varepsilon)_\varepsilon$ is therefore bounded in $W^{1,p}(U'; \mathbb{R}^{3 \times 2})$ and we can extract a non-relabeled subsequence that converges to some $V' \in W^{1,p}(U'; \mathbb{R}^{3 \times 2})$ as $\varepsilon \rightarrow 0$, both weakly in $W^{1,p}(U'; \mathbb{R}^{3 \times 2})$ and, via Sobolev embedding, strongly to $L^p(U'; \mathbb{R}^{3 \times 2})$. As $\nabla' u_\varepsilon \rightarrow \nabla' u$ in $L^p(\Omega; \mathbb{R}^{3 \times 2})$, it follows that

$$V'_\varepsilon = \int_0^L \nabla' u_\varepsilon(\cdot, x_3) dx_3 \rightarrow \int_0^L \nabla' u(\cdot, x_3) dx_3 \quad \text{in } L^p(U'; \mathbb{R}^3),$$

and thus,

$$V' = \int_0^L \nabla' u(\cdot, x_3) dx_3 = \nabla' u;$$

the last identity uses that ∇u is independent of x_3 , cf. (4.65) and Step 1. Together with (4.25) and (4.27), we finally conclude

$$V_\varepsilon = (V'_\varepsilon, \Sigma_\varepsilon) \rightarrow (\nabla' u | \Sigma) = \nabla u \text{ in } L^p(U'; \mathbb{R}^{3 \times 3}),$$

which is (4.67). □

Remark 4.22 (Local bounds on the second gradients). Notice that assumption (4.6) in the statement of Theorem 4.2 can be replaced by the weaker condition that

$$\sup_{\varepsilon} \max_{i,j \in \{1,2\}} \|\partial_i \partial_j u_{\varepsilon}\|_{L^p(K; \mathbb{R}^3)} < \infty$$

for any compact subset $K \subset \Omega$; this is immediate to verify, considering that our proof involves essentially only local arguments in the cross section.

Chapter 5

On the interplay of anisotropy and geometry for polycrystals in single-slip crystal plasticity

This chapter is available as the preprint [89] on arXiv.

5.1 Introduction

Most elastoplastic solids are polycrystalline, meaning that they consist of rotated copies of single crystals, called grains, which form patterns on a mesoscopic length scale in between the micro- and the macroscopic one. The grain structures impose restrictions on still finer substructures and highly influence the effective material response of the solid. In this chapter, we contribute to the analysis of the attainable macroscopic strains of polycrystals, focusing on a model of crystal plasticity with one active slip system and rigid elasticity. The overall goal is to obtain a deeper understanding of boundary interaction, global compatibility, and the interplay between slip mechanisms and texture in a geometrically nonlinear setting.

Our approach originates in the time-discrete variational framework introduced in [48, 149, 172] for modeling rate-independent processes arising in single-crystal finite plasticity. The first relaxation results for such problems, facilitating the description of the effective material behavior by minimizing out microstructural effects, go back to Conti & Theil [58, 68]. They determine the quasiconvexification of the elastoplastic energy density for the first time-incremental problem under the assumptions of one single slip system and elastically rigid behavior. As shown in [65], the relaxed energies of [58] result via approximation by Γ -convergence from models with elastic energy in the limit of diverging elastic constants. For more recent work on relaxation in models with two or more slip systems, we refer to [64, 66, 187]; see also [8, 7] for related studies in the context of strain-gradient plasticity.

Compared with the single-crystal case, the analysis of polycrystals holds additional challenges related to the geometry and orientation of the different grains, as well as to the compatibility of microstructures across grain boundaries; in the context of linear and nonlinear elasticity, the latter has been studied in [34, 35] and [19, 20], respectively. The description of polycrystal geometry in [19] by Ball & Carstensen in combination with the modeling of [58, 68] constitutes the basis for our framework of polycrystalline finite crystal plasticity with one active slip system under the assumption of elastically nonlinear but rigid behavior; the detailed setup is given in Section 5.1.1. Let us remark that the stress is not well-defined in this elastically rigid strain-based setting; for an analysis of a stress-based formulation of polycrystal perfect plasticity,

see [123]. Mathematically, the macroscopically attainable strains can be described via a specific inhomogeneous nonlinear differential inclusion subject to affine boundary conditions, similar to the approaches in polycrystalline shape-memory materials [35, 124], where stress-free strains are studied based on elastic energy minimization. The general theory of differential inclusions (or multi-valued differential equations) has been an active field of research over the last decades with strong methods and results, which are especially rich in the homogeneous case, see, e.g., [162, 191, 192] and the references therein. In the inhomogeneous case, where the target sets feature spatial dependence, we refer to [165] for the existence of Lipschitz solutions, and to [146] for a recent generalization of the latter in the context of Sobolev solutions. The techniques of [127] allow, among others, the construction of Sobolev functions with prescribed Jacobians under a so-called uniform tight containedness assumption. However, there is - to the best of our knowledge - currently no available abstract methodology that is able to accommodate the particular setting of this chapter - even though, considering related homogeneous inclusions does provide some partial insight.

This work addresses different aspects related to the solvability of inhomogeneous differential inclusions used to describe the macroscopic deformation behavior of elastoplastic solids. We first identify a simple geometry-independent sufficient condition by combining a new characterization of globally affine solutions to a relaxed version of the problem with well-known relaxation and convex integration results [68, 164]. On the other hand, necessary conditions are due to compatibility constraints following from a generalized Hadamard jump condition [17, 117] applied to the boundary grains. While the sufficient and necessary conditions turn out to provide a characterization for specific polycrystals, we show that they do not coincide in general though. The argument is based on an explicit construction of finitely piecewise affine maps that satisfy fixed boundary conditions and incompressibility, and is as such known to be a delicate issue. Here, we take the geometric setup of the rotated-square construction in [67, 178] as inspiration for a suitable sheared-square construction. A more detailed overview of our findings is given in Section 5.1.2.

5.1.1 Setup of the problem

In the following, we describe our two-dimensional model for single-slip polycrystal plasticity. Starting from the theory of finite plasticity for single-crystalline structures (cf. [48, 149, 152, 172]), we adopt a geometrically nonlinear model for the deformation behavior of an elastoplastic body where the deformation gradient

$$\nabla u = F = F_{\text{el}} F_{\text{pl}} \quad (5.1)$$

is split multiplicatively into an elastic part F_{el} and a plastic one F_{pl} , as proposed in [133, 140]. For recent discussions about this decomposition and possible alternative modeling approaches, see [60, 74, 80]. With (5.1) at hand, the elastoplastic energy is given in terms of the condensed energy density

$$W(F) = \min_{F = F_{\text{el}} F_{\text{pl}}} (W_{\text{el}}(F_{\text{el}}) + W_{\text{pl}}(F_{\text{pl}}) + \text{Diss}(F_{\text{pl}})), \quad (5.2)$$

where W_{el} is the elastic energy contribution, W_{pl} represents the plastic potential, and Diss encodes dissipative effects. We invoke a setting of rigid elasticity, meaning that the elastic parts F_{el} are contained in the set of rotations $\text{SO}(2)$ almost everywhere and do not contribute to the energy, i.e.,

$$W_{\text{el}}(F_{\text{el}}) = \begin{cases} 0 & \text{if } F_{\text{el}} \in \text{SO}(2), \\ \infty & \text{otherwise.} \end{cases}$$

As the plastic strain in the context of single-slip crystal plasticity is a simple shear along one active slip system, determined by a slip direction $s \in \mathcal{S}^1$ and slip-plane normal $m = s^\perp$, one has that $F_{\text{pl}} = \text{Id} + \gamma s \otimes m$, where $\gamma \in \mathbb{R}$ quantifies the plastic slip, and (5.2) becomes

$$W(F) = \begin{cases} (|Fm|^2 - 1)^{\frac{p}{2}} = |\gamma|^p & \text{if } F \in \mathcal{M}_s, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{2 \times 2}, \quad (5.3)$$

for $p \geq 1$, cf. [58, 68]. Here, the choices $p = 1$ and $p = 2$ model dissipation and linear hardening, respectively, and

$$\mathcal{M}_s := \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fs| = 1\};$$

we often use the short notation $\mathcal{M} := \mathcal{M}_{e_1}$.

In our polycrystalline setting, the active slip direction within the body is location-dependent and determined by the orientations of the individual grains. To be more precise, let us first give a definition of the term “polycrystal”, which is based on [19, Section 2] and tailored to our crystal plasticity model, see Figure 5.1 for illustration. We say that a pair (Ω, R_*) with a reference configuration $\Omega \subset \mathbb{R}^2$ and a texture $R_* : \Omega \rightarrow \text{SO}(2)$ is a (two-dimensional) polycrystal if these conditions are satisfied:

- Ω is a bounded Lipschitz and has a partition (up to a set of measure zero) into $N \in \mathbb{N}$ regular bounded Lipschitz domains $\Omega_1, \dots, \Omega_N \subset \Omega$, called the grains of (Ω, R_*) , that is,

$$\Omega = \text{int} \bigcup_{k=1}^N \overline{\Omega}_k \quad \text{and} \quad \text{int } \overline{\Omega}_k = \Omega_k \text{ for all } k \in \{1, \dots, N\},$$

where $\overline{(\cdot)}$ and $\text{int } (\cdot)$ denote the closure and interior of a set;

- $R_* : \Omega \rightarrow \text{SO}(2)$ is constant on each Ω_k for $k \in \{1, \dots, N\}$ with

$$R_*|_{\Omega_k} \neq \pm R_*|_{\Omega_l} \text{ if } \mathcal{H}^1(\partial\Omega_k \cap \partial\Omega_l) > 0 \text{ for all } k \neq l \in \{1, \dots, N\}, \quad (5.4)$$

where \mathcal{H}^1 is the one-dimensional Hausdorff-measure; otherwise, two grains can be merged into one. The image of R_* is denoted by $R_*(\Omega)$.

With the default choice of the slip direction e_1 on unrotated grains of the polycrystal (Ω, R_*) , each grain Ω_k can be viewed as a single crystal with slip direction $R_*|_{\Omega_k} e_1$. The set of attainable microscopic strains on Ω_k is then given by

$$\mathcal{M}_{R_*|_{\Omega_k} e_1} = \mathcal{M} R_*^T|_{\Omega_k}.$$

Note that polycrystals with exactly two grains differ intrinsically from single-crystals with two active slip systems subject to latent hardening, as studied in [66, 187]. This is because the slip directions are tied to the grain structure and may not be chosen arbitrarily at any point within the occupied region.

We distinguish between interior and boundary grains: the boundary of an interior grain Ω_k is contained in the boundary of others, i.e., $\partial\Omega_k \subset \bigcup_{l \neq k} \partial\Omega_l$; all remaining grains are called boundary grains. Furthermore, we introduce the points on $\partial\Omega$ where at least two boundary grains meet as the boundary dual points of (Ω, R_*) , in formulas,

$$\bigcup_{1 \leq k < l \leq N} \partial\Omega_k \cap \partial\Omega_l \cap \partial\Omega. \quad (5.5)$$

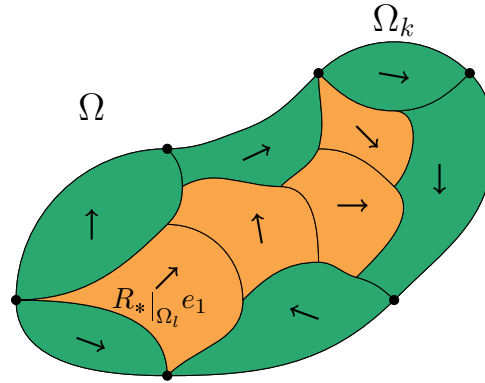


Figure 5.1: A visualization of a polycrystal (Ω, R_*) , where the green and orange domains are boundary and interior grains, respectively; the arrows indicate the orientation of the slip directions R_*e_1 and the black dots at the boundary highlight the boundary dual points.

With ν the outer unit normal of Ω , we set

$$\partial(\Omega, R_*) := \{x \in \partial\Omega : \nu(x) \text{ exists and } x \text{ is not a boundary dual point of } (\Omega, R_*)\}. \quad (5.6)$$

A variational approach to the deformation behavior of a polycrystal (Ω, R_*) requires the inhomogeneous microscopic energy density

$$W(x, F) = \begin{cases} (|FR_*(x)e_2|^2 - 1)^{\frac{p}{2}} & \text{if } F \in \mathcal{MR}_*^T(x), \\ \infty & \text{otherwise,} \end{cases} \quad x \in \Omega, \quad F \in \mathbb{R}^{2 \times 2}, \quad (5.7)$$

which results from rotated versions of the corresponding density in the single-crystalline case (5.3). Following the work by Bhattacharya & Kohn [35], we define the macroscopic energy density via averages of the microscopic one and optimization over all possible microstructures forming within the grains, that is,

$$W_{(\Omega, R_*)}(F) = \inf_{\substack{u \in W^{1,\infty}(\Omega; \mathbb{R}^2) \\ u = Fx \text{ on } \partial\Omega}} \frac{1}{|\Omega|} \int_{\Omega} W(x, \nabla u) \, dx, \quad F \in \mathbb{R}^{2 \times 2}. \quad (5.8)$$

In this chapter, our goal is to characterize (inner and outer bounds on) the domain of $W_{(\Omega, R_*)}$, which involves a deeper understanding of the inhomogeneous partial differential inclusion

$$\begin{cases} \nabla u(x) \in \mathcal{MR}_*^T(x) & \text{for a.e. } x \in \Omega, \\ u(x) = Fx & \text{for } x \in \partial\Omega, \end{cases} \quad (P_M)$$

where $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ is the unknown and $F \in \mathbb{R}^{2 \times 2}$; in fact, $W_{(\Omega, R_*)}$ is finite exactly on

$$\mathcal{F}_M(\Omega, R_*) := \{F \in \mathbb{R}^{2 \times 2} : \text{there exists a solution } u \in W^{1,\infty}(\Omega; \mathbb{R}^2) \text{ to } (P_M)\},$$

considering (5.7) and (5.8). We point out that the special linear group of degree two constitutes a trivial outer bound

$$\mathcal{F}_M(\Omega, R_*) \subset \text{Sl}(2) := \{F \in \mathbb{R}^{2 \times 2} : \det F = 1\} \quad (5.9)$$

for any polycrystal (Ω, R_*) . This follows from the observations that the solutions to (P_M) are locally volume-preserving and that the determinant is a null Lagrangian.

In the case that (Ω, R_*) is a single-crystal, i.e., the texture R_* is constant on all of Ω , the set of attainable macroscopic strains has been identified via a relaxation-type argument in [68] as $\mathcal{F}_\mathcal{M}(\Omega, R_*) = \mathcal{N}_s$ with $s = R_*e_1$ and

$$\mathcal{N}_s := \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fs| \leq 1\} = \mathcal{M}_s^{\text{pc}} = \mathcal{M}_s^{\text{qc}} = \mathcal{M}_s^{\text{rc}}, \quad (5.10)$$

where \mathcal{M}^{qc} (\mathcal{M}^{pc} , \mathcal{M}^{rc}) is the quasiconvex (polyconvex, rank-one convex) hull of \mathcal{M}_s , see (5.16) for the definitions; we set $\mathcal{N} := \mathcal{N}_{e_1}$. The proof technique of [68] exploits - besides classical results on homogeneous partial differential inclusions with Lipschitz solutions (e.g. [162, Theorem 4.10]) - the seminal work by Müller & Sverák [164], which in turn extends Gromov's theory [112, 111] of convex integration and its applications. This relaxation result in the single-crystal case motivates to study also the relaxed inclusion problem

$$\begin{cases} \nabla u(x) \in \mathcal{N}R_*^T(x) & \text{for a.e. } x \in \Omega, \\ u(x) = Fx & \text{for } x \in \partial\Omega, \end{cases} \quad (P_\mathcal{N})$$

with unknown $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ and $F \in \mathbb{R}^{2 \times 2}$, to gain insight into the structure of $\mathcal{F}_\mathcal{M}(\Omega, R_*)$.

The fact that $(P_\mathcal{M})$ and $(P_\mathcal{N})$ are non-convex inhomogeneous differential inclusion problems whose target set is unbounded makes them non-standard. In particular, the results of [127, 146, 165] are not applicable and, naturally, the classical theory of homogeneous inclusions is limited in providing useful new insight. Given that the solvability of $(P_\mathcal{M})$ and $(P_\mathcal{N})$ depends fundamentally on the interaction between the shape, size and orientation of the grains makes the analysis many-faceted.

5.1.2 Overview of the main results

Throughout this chapter, let (Ω, R_*) be a polycrystal. To analyze the set of attainable macroscopic strains $\mathcal{F}_\mathcal{M}(\Omega, R_*)$, we identify inner and outer bounds, and show that they coincide under suitable conditions on the texture.

Two geometry-independent inner bounds are tied to the assumption of constant strain, which can be traced back to the early works by Taylor [193] and Bishop & Hill [38]. Precisely, we consider the sets of globally affine solutions to $(P_\mathcal{M})$ and $(P_\mathcal{N})$, given by the finite intersections

$$\mathcal{T}_\mathcal{M}(R_*(\Omega)) := \bigcap_{x \in \Omega} \mathcal{M}R_*^T(x) \quad \text{and} \quad \mathcal{T}_\mathcal{N}(R_*(\Omega)) := \bigcap_{x \in \Omega} \mathcal{N}R_*^T(x); \quad (5.11)$$

notice that these sets are independent of the size and shape of the grains and only take the orientation of the slip systems into account. Based on the work on single-crystal plasticity by Conti & Theil [68] (see also Proposition 5.3), it holds that

$$\mathcal{T}_\mathcal{M}(R_*(\Omega)) \subset \mathcal{T}_\mathcal{N}(R_*(\Omega)) \subset \mathcal{F}_\mathcal{M}(\Omega, R_*). \quad (5.12)$$

We refer to $\mathcal{T}_\mathcal{M}(R_*(\Omega))$ and $\mathcal{T}_\mathcal{N}(R_*(\Omega))$ as the Taylor bound for the differential inclusions $(P_\mathcal{M})$ and $(P_\mathcal{N})$, in analogy to the terminology in [35, Section 2.4] and [124, Proposition 2.1] on polycrystalline shape-memory materials.

Our first main result shows that $\mathcal{T}_\mathcal{N}(R_*(\Omega))$ depends, in fact, on at most three specific orientations of the polycrystal.

Proposition 5.1 (Characterization of the Taylor bound for $(P_\mathcal{N})$). *Suppose that R_* attains the values*

$$R_{\theta_1}, \dots, R_{\theta_N} \quad \text{for } 0 = \theta_1 < \dots < \theta_N < \pi \text{ with } N \geq 2 \quad (5.13)$$

on the grains of (Ω, R_*) , and let $\theta_{N+1} = \pi$. Then,

$$\mathcal{T}_{\mathcal{N}}(R_*(\Omega)) = \mathcal{N} \cap \mathcal{N}R_{\theta_n}^T \cap \mathcal{N}R_{\theta_{n+1}}^T, \quad (5.14)$$

where $n \in \{1, \dots, N\}$ is uniquely determined by the relation $\theta_n < \frac{\pi}{2} \leq \theta_{n+1}$.

Considering that polycrystals generally consist of large number of grains with different orientations, this result simplifies the computation of this bound considerably, facilitating even an explicit analytical representation as discussed in Remark 5.8 e). In particular, we identify necessary and sufficient conditions on the slip directions such that $\mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ is trivial, i.e., identical to the set of rotations $\text{SO}(2)$, see Corollary 5.9.

A key ingredient for deriving outer bounds on the attainable macroscopic strains is the generalization of the classical Hadamard jump condition [21] formulated in Theorem 5.14. This tool puts us in the position to derive an outer bound on $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ by analyzing the rank-one compatibility between the macroscopic and microscopic strains at the boundary grains of the polycrystal. In particular, if the outer unit normal of Ω is, in a point, perpendicular to the slip orientation s_i of the associated boundary grain Ω_i , then $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ is contained in \mathcal{N}_{s_i} . The following statement is a simplified version of Proposition 5.15 below.

Proposition 5.2. *Let $\Omega_1, \dots, \Omega_M$ with $M \in \mathbb{N}$ be the boundary grains of (Ω, R_*) and let $J \subset \{1, \dots, M\}$ be the set of all indices i such that there exists $x_i \in \partial\Omega_i \cap \partial(\Omega, R_*)$ with $\nu(x_i) \cdot R_*|_{\Omega_i} e_1 = 0$. Then,*

$$\mathcal{F}_{\mathcal{M}}(\Omega, R_*) \subset \bigcap_{i \in J} \mathcal{N}R_*^T|_{\Omega_i}. \quad (5.15)$$

Observe that the outer bound in (5.15) has the same overall structure as the aforementioned inner bound $\mathcal{T}_{\mathcal{N}}(\Omega; R_*)$ and can thus be simplified and expressed in the same way, cf. Proposition 5.1. In particular, Proposition 5.2 allows us to conclude for examples of polycrystals with sufficient symmetry and selected bicrystals (see Examples 5.18 and 5.19) that the attainable macroscopic strains coincide with the globally affine solutions of $(P_{\mathcal{N}})$, i.e., $\mathcal{F}_{\mathcal{M}}(\Omega, R_*) = \mathcal{T}_{\mathcal{N}}(R_*(\Omega))$.

The Taylor bound for $(P_{\mathcal{N}})$, however, is not always optimal, meaning that the second inclusion in (5.12) is strict in general. To see this, we employ a geometric setup similar to the construction in [67, 178] and design a polycrystal together with a continuous and finitely piecewise affine solution to the relaxed problem $(P_{\mathcal{N}})$ with boundary values outside of the Taylor bound, which then gives rise to a Lipschitz solution to $(P_{\mathcal{M}})$ via Proposition 5.3.

This chapter is outlined as follows. A few preliminaries and mathematical tools for the analysis of the inclusions $(P_{\mathcal{M}})$ and $(P_{\mathcal{N}})$ and the corresponding sets \mathcal{M}_s and \mathcal{N}_s are collected in Section 5.2. The focus of Section 5.3 is the characterization and discussion of the inner Taylor bounds. Outer bounds resulting from rank-one compatibility conditions at the boundary grains are derived in Section 5.4 and illustrated by a few examples. Finally, we address the question of optimality of the Taylor bound for $(P_{\mathcal{N}})$, proving in Section 5.5 that this is not the case in general.

5.2 Preliminaries

5.2.1 Notation

Throughout this chapter, we use the following notation. The standard basis vectors in \mathbb{R}^2 are denoted by e_1, e_2 , and $\mathcal{S}^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the one-dimensional unit sphere with respect to the Euclidean norm. We write $a \cdot b$ for the standard scalar product of two vectors $a, b \in \mathbb{R}^2$,

and define their tensor product $a \otimes b$ as $(a \otimes b)_{ij} = a_i b_j$ for $i, j \in \{1, 2\}$. The space of real 2×2 matrices is equipped with the Frobenius norm $|\cdot|$. Moreover, $\text{Sl}(2)$ consists of all matrices in $\mathbb{R}^{2 \times 2}$ with determinant equal to one, and $\text{SO}(2)$ denotes the set of rotations in $\mathbb{R}^{2 \times 2}$; we write

$$R_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in \text{SO}(2)$$

for a rotation by an angle $\theta \in \mathbb{R}$, and define $a^\perp = R_{\frac{\pi}{2}} a$ for any $a \in \mathbb{R}^2$. For $x \in \mathbb{R}^2$, $\nu \in \mathcal{S}^1$, and $r > 0$, let

$$B_\nu^+(x, r) = \{y \in B(x, r) : (y - x) \cdot \nu > 0\} \quad \text{and} \quad B_\nu^-(x, r) = \{y \in B(x, r) : (y - x) \cdot \nu < 0\},$$

where $B(x, r) \subset \mathbb{R}^2$ is the open ball with center $x \in \mathbb{R}^2$ and radius $r > 0$. If not specified otherwise, the two-dimensional Lebesgue-measure of a measurable set $U \subset \mathbb{R}^2$ is denoted by $|U|$.

The product and sum of two sets $\mathcal{F}, \mathcal{G} \subset \mathbb{R}^{2 \times 2}$ are interpreted in the sense of Minkowski, i.e., $\mathcal{F}\mathcal{G} = \{FG : F \in \mathcal{F}, G \in \mathcal{G}\}$ and $\mathcal{F} + \mathcal{G} = \{F + G : F \in \mathcal{F}, G \in \mathcal{G}\}$. We work with the following definitions of (finite) generalized convex hulls: If $\mathcal{F} \subset \mathbb{R}^{2 \times 2}$, then

$$\mathcal{F}^{\text{qc}} := \{F \in \mathbb{R}^{2 \times 2} : h(F) \leq \sup_{G \in \mathcal{F}} h(G) \text{ for all } h : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R} \text{ quasiconvex}\}, \quad (5.16)$$

is the quasiconvex hull of \mathcal{F} , and \mathcal{F} is called quasiconvex if $\mathcal{F} = \mathcal{F}^{\text{qc}}$; the polyconvex and rank-one convex hulls \mathcal{F}^{pc} and \mathcal{F}^{rc} as well as polyconvexity and rank-one convexity of \mathcal{F} are introduced analogously. More details on the generalized notions of convexity for sets can be found in, e.g., [70, Chapter 7].

For a set $U \subset \mathbb{R}^2$, we define the indicator function $\mathbb{1}_U$ as $\mathbb{1}_U(x) = 1$ for $x \in U$ and $\mathbb{1}_U(x) = 0$ otherwise. Throughout, we adopt the standard notation for Lebesgue spaces, Sobolev spaces, spaces of functions of bounded variation, and spaces of continuously differentiable functions; particularly, $L^\infty(U; \mathbb{R}^2)$, $W^{1,\infty}(U; \mathbb{R}^2)$, $BV_{\text{loc}}(U; \mathbb{R}^{2 \times 2})$ and $C^1(\bar{U}; \mathbb{R}^2)$ for open $U \subset \mathbb{R}^2$.

5.2.2 Technical tools for the analysis of \mathcal{M}_s and \mathcal{N}_s

We begin with the following connection between the differential inclusion problem $(P_{\mathcal{M}})$ and its relaxed version $(P_{\mathcal{N}})$.

Proposition 5.3. *If there is a continuous, finitely piecewise affine solution to the relaxed problem $(P_{\mathcal{N}})$, then there exists a Lipschitz solution to $(P_{\mathcal{M}})$ with the same boundary values.*

This result is based on the analysis in the chapter [68] by Conti & Theil on a model in single-crystal plasticity with one active slip system. More precisely, it follows from [68, Theorem 4] applied grain-wise to each of the affine components of the solution to $(P_{\mathcal{N}})$, along with [68, Lemma 2], which again relies on the convex integration theory by Müller & Šverák [164, Theorem 1.3].

As a consequence of Proposition 5.3, one obtains an inner bound on $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ by identifying globally affine solutions to $(P_{\mathcal{N}})$, cf. (5.11) and Section 5.3.

The subsequent auxiliaries on rank-one connectedness with the sets \mathcal{N}_s are used in Section 5.4 in the context of outer bounds resulting from the rank-one compatibility at the boundary grains. Considering that $\text{Sl}(2)$ is a trivial outer bound for all polycrystals, it suffices to restrict the discussion to this class of matrices. We first introduce the following terminology.

Definition 5.4 (ν -compatibility). *Let $\nu \in \mathcal{S}^1$, $A \in \mathbb{R}^{2 \times 2}$, and $\mathcal{B} \subset \mathbb{R}^{2 \times 2}$. We call A ν -compatible with \mathcal{B} , if A is rank-one connected to $\mathcal{B} \subset \mathbb{R}^{2 \times 2}$ along linear interfaces with normal ν , i.e., if there exists $a \in \mathbb{R}^2$ such that $a \otimes \nu \in \mathcal{B} - A$, or equivalently, if $A\nu^\perp \in \mathcal{B}\nu^\perp$.*

The next lemma gives a simple condition for the rank-one connectedness between a given element in $\text{Sl}(2)$ and \mathcal{N}_s . Our calculations are based on the observation that, for any $s \in \mathcal{S}^1$, the set \mathcal{N}_s consists of matrices of the form

$$R(\beta s \otimes s + \frac{1}{\beta} s^\perp \otimes s^\perp + \gamma s \otimes s^\perp) \quad (5.17)$$

with $R \in \text{SO}(2)$, $\beta \in (0, 1]$ and $\gamma \in \mathbb{R}$. Moreover, if $\beta > 0$ then (5.17) describes all elements in the larger set $\text{Sl}(2)$.

Lemma 5.5. *Let $s, \nu \in \mathcal{S}^1$ with $s \cdot \nu \neq 0$. Further, let $F \in \text{Sl}(2)$, which is assumed to be represented as in (5.17) with $R \in \text{SO}(2)$, $\beta > 0$, and $\gamma \in \mathbb{R}$. Then the following statements are equivalent:*

- (i) F is ν -compatible with \mathcal{N}_s ;
- (ii) F is ν -compatible with \mathcal{M}_s ;
- (iii) it holds that

$$\left(\frac{s \cdot \nu^\perp}{s \cdot \nu} \beta + \gamma \right)^2 + \frac{1}{\beta^2} \geq 1. \quad (5.18)$$

Proof. We start with a useful general equivalence: Any two matrices $G, \bar{G} \in \text{Sl}(2)$ of the form (5.17) with $R = \text{Id}$, $\bar{R} \in \text{SO}(2)$, $\beta, \bar{\beta} > 0$, $\gamma, \bar{\gamma} \in \mathbb{R}$, respectively, satisfy

$$G\nu^\perp = \bar{G}\nu^\perp \quad (5.19)$$

if and only if

$$\begin{cases} \bar{\beta} \frac{s \cdot \nu^\perp}{s \cdot \nu} + \bar{\gamma} = N_{s,\nu}(\beta, \gamma) \cdot \bar{R}s, \\ \frac{1}{\bar{\beta}} = \bar{R}s^\perp \cdot N_{s,\nu}(\beta, \gamma), \end{cases} \quad (5.20)$$

$$\frac{1}{\bar{\beta}} = \bar{R}s^\perp \cdot N_{s,\nu}(\beta, \gamma), \quad (5.21)$$

with

$$N_{s,\nu}(\beta, \gamma) = \left(\frac{s \cdot \nu^\perp}{s \cdot \nu} \beta + \gamma \right) s + \frac{1}{\beta} s^\perp;$$

this follows simply via scalar multiplication of (5.19) with $\bar{R}s$ and $\bar{R}s^\perp$.

To show the implication (i) \Rightarrow (iii), let F be as in the statement and assume without restriction that $R = \text{Id}$. Suppose that there exists $\bar{F} \in \mathcal{N}_s$ of the form (5.17) with $\bar{\beta} \in (0, 1]$, $\bar{\gamma} \in \mathbb{R}$, and $\bar{R} \in \text{SO}(2)$ such that (5.20) and (5.21) hold. Estimating the right-hand side of (5.21) yields

$$1 \leq \frac{1}{\bar{\beta}^2} \leq |N_{s,\nu}(\beta, \gamma)|^2 = \left(\frac{s \cdot \nu^\perp}{s \cdot \nu} \beta + \gamma \right)^2 + \frac{1}{\beta^2},$$

which is the desired inequality (5.18).

As for (iii) \Rightarrow (ii), assume that (5.18) is fulfilled and take $\bar{\beta} = 1$. It remains to find $\bar{\gamma} \in \mathbb{R}$ and $\bar{R} \in \text{SO}(2)$ such that (5.20) and (5.21) are satisfied. For \bar{R} , it is enough to determine $\bar{R}s^\perp$, which follows from considering solutions $\xi \in \mathcal{S}^1$ to

$$1 = \xi \cdot N_{s,\nu}(\beta, \gamma);$$

these exist since $|N_{s,\nu}(\beta, \gamma)| \geq 1$ due to (5.18). Therefore, (5.21) is true and $\bar{\gamma}$ is uniquely determined by \bar{R} , $\bar{\beta}$ and (5.20). Since the implication (ii) \Rightarrow (i) is trivial, the statement is proven. \square

The next remark addresses the case of s^\perp -compatibility with the sets \mathcal{N}_s and \mathcal{M}_s .

Remark 5.6. Let $s \in \mathcal{S}^1$ and $F \in \text{Sl}(2)$.

a) Basic geometric arguments show:

$$F \text{ is } s^\perp\text{-compatible with } \mathcal{N}_s \text{ if and only if } F \in \mathcal{N}_s;$$

$$F \text{ is } s^\perp\text{-compatible with } \mathcal{M}_s \text{ if and only if } F \in \mathcal{M}_s.$$

Observe in particular, that the equivalence (i) \Leftrightarrow (ii) in Lemma 5.5 is not valid in this case.

b) Let $\mathcal{D} \subset \mathcal{S}^1$ be dense in \mathcal{S}^1 . It holds that F is s^\perp -compatible with \mathcal{N}_s if and only if F is ν -compatible with \mathcal{N}_s for all $\nu \in \mathcal{D}$. Indeed, assuming that $\pm s^\perp \notin \mathcal{D}$, that is a consequence of (i) \Leftrightarrow (iii) in Lemma 5.5 since

$$t \mapsto (t\beta + \gamma)^2 + \frac{1}{\beta^2} \geq 1 \quad \text{on a dense subset of } \mathbb{R}$$

if and only if $|Fs| = \beta \leq 1$, which is equivalent to the s^\perp -compatibility of F with \mathcal{N}_s due to a).

The last auxiliary result follows from a slight modification of the first step in the proof of [66, Theorem 1.1], according to which the generalized convex hulls of $\mathcal{M}_{e_1} \cup \mathcal{M}_{e_2}$ coincide with $\text{Sl}(2)$. Instead of the two orthogonal slip directions e_1 and e_2 , we consider here two linearly independent orientations s, s' . Alternatively, the next proposition can be seen as a consequence of the sharper result [187, Lemma 4.17], where the lamination convex hull of $\mathcal{M}_s \cup \mathcal{M}_{s'}$ is identified as $\text{Sl}(2)$, using first-order laminates. For the reader's convenience, we include the simpler proof based on the strategy in [66].

Proposition 5.7. Let $s, s' \in \mathcal{S}^1$ with $s' \neq \pm s$. Then,

$$(\mathcal{M}_s \cup \mathcal{M}_{s'})^{\text{rc}} = (\mathcal{M}_s \cup \mathcal{M}_{s'})^{\text{qc}} = (\mathcal{M}_s \cup \mathcal{M}_{s'})^{\text{pc}} = \text{Sl}(2).$$

Proof. In light of

$$(\mathcal{N}_s \cup \mathcal{N}_{s'})^{\text{rc}} = (\mathcal{M}_s^{\text{rc}} \cup \mathcal{M}_{s'}^{\text{rc}})^{\text{rc}} = (\mathcal{M}_s \cup \mathcal{M}_{s'})^{\text{rc}} \subset (\mathcal{M}_s \cup \mathcal{M}_{s'})^{\text{pc}} \subset \text{Sl}(2),$$

cf. (5.10), it suffices to prove that $\text{Sl}(2) \subset (\mathcal{N}_s \cup \mathcal{N}_{s'})^{\text{rc}}$. To this end, we show that any $F \in \text{Sl}(2)$ can be expressed as a convex combination of rank-one connected matrices $F_+, F_- \in \mathcal{N}_s \cup \mathcal{N}_{s'}$, and exploit that the lamination convex hull is contained in the rank-one convex hull [70, Theorems 7.17 and 7.28].

Suppose without loss of generality that $F \in \text{Sl}(2) \setminus (\mathcal{N}_s \cup \mathcal{N}_{s'})$ and consider the rank-one line

$$t \mapsto F_t = F(\text{Id} + t(s + s') \otimes (s - s')),$$

along which the determinant is constant by construction; also, assume that $Fs \cdot Fs' \leq 0$, otherwise switch the roles of $s + s'$ and $s - s'$ in the rank-one line. Then it holds that

$$|F(s + s')|^2 = |Fs|^2 + 2Fs \cdot Fs' + |Fs'|^2 \leq |F(s - s')|^2,$$

with equality if and only if $Fs \cdot Fs' = 0$. If the latter is satisfied, the estimate

$$\det(Fs|Fs') = \det F \det(s|s') = \det(s|s') \leq 1,$$

implies that $\min\{|Fs|, |Fs'|\} \leq 1$. However, this contradicts $F \in \text{Sl}(2) \setminus (\mathcal{N}_s \cup \mathcal{N}_{s'})$, which is why we take $|F(s + s')| < |F(s - s')|$ in the following.

The quadratic function

$$\varphi : \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto |F_t(s - s')|^2 - |F_t(s + s')|^2$$

then satisfies $\varphi(0) < 0$ and $\varphi'' > 0$ pointwise. The latter is a direct consequence of $|F_t(s + s')| = |F(s + s')|$ for every $t \in \mathbb{R}$, utilizing that $(s + s') \cdot (s - s') = 0$. Hence, there exist $t_- < 0$ and $t_+ > 0$ with $\varphi(t_{\pm}) = 0$, so that $F_- := F_{t_-}$ and $F_+ := F_{t_+}$ satisfy $|F_{\pm}(s + s')| = |F_{\pm}(s - s')|$. Exactly as before, one concludes that $\min\{|F_{\pm}s|, |F_{\pm}s'|\} \leq 1$, and thus, $F_{\pm} \in \mathcal{N}_s \cup \mathcal{N}_{s'}$, as desired. \square

The previous result is used in Section 5.4 to explain why considering boundary dual points, cf. (5.5), does not add any non-trivial contributions to outer bounds emerging from rank-one compatibility at the boundary grains.

5.3 A geometry-independent inner bound

In this section, we identify and characterize inner bounds on $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ in terms of the globally affine solutions to $(P_{\mathcal{M}})$ and $(P_{\mathcal{N}})$. Note that such solutions merely take the occurring slip directions, but not the geometry of the grains, i.e., their size and shape, into account. The expressions to be determined are the Taylor bounds $\mathcal{T}_{\mathcal{M}}(R_*(\Omega))$ and $\mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ given by the finite intersections in (5.11), which, in light of Proposition 5.3, satisfy the relation

$$\mathcal{T}_{\mathcal{M}}(R_*(\Omega)) \subset \mathcal{T}_{\mathcal{N}}(R_*(\Omega)) \subset \mathcal{F}_{\mathcal{M}}(\Omega, R_*).$$

First, we prove Proposition 5.1, showing that $\mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ depends on at most three slip orientations, no matter the number of grains of the polycrystal (Ω, R_*) ; the same statement is valid for $\mathcal{T}_{\mathcal{M}}(R_*(\Omega))$, see Remark 5.8f) below.

Proof of Proposition 5.1. We first establish an explicit expression of the intersection $\mathcal{N} \cap \mathcal{N}R_{\theta}^T$ with $\theta \in (0, \pi)$. In light of (5.17) for $s = e_1$, it holds that $F \in \mathcal{N}$ if and only if $F = S(\beta e_1 | \frac{1}{\beta} e_2 + \gamma e_1)$ for some $S \in \text{SO}(2)$, $\beta \in (0, 1]$, and $\gamma \in \mathbb{R}$. In order to characterize β and γ such that $FR_{\theta} \in \mathcal{N}$, we observe that $\det(FR_{\theta}) = \det F = 1$ and that the constraint $|FR_{\theta}e_1| \leq 1$ can be rewritten as

$$0 \geq |FR_{\theta}e_1|^2 - 1 = \gamma^2 \sin^2 \theta + 2\gamma\beta \cos \theta \sin \theta + \frac{1}{\beta^2} \sin^2 \theta + \beta^2 \cos^2 \theta - 1. \quad (5.22)$$

As the right-hand side is quadratic in γ with positive leading coefficient, it suffices to determine its zeroes to solve the inequality (5.22). In doing so, we find that

$$\mathcal{N} \cap \mathcal{N}R_{\theta}^T = \text{SO}(2)\psi(\Lambda_{\theta}) \quad (5.23)$$

with $\psi : (0, 1] \times \mathbb{R} \rightarrow \mathcal{N}$ given by $(\beta, \gamma) \mapsto (\beta e_1 | \frac{1}{\beta} e_2 + \gamma e_1)$ and

$$\Lambda_{\theta} = \{(\beta, \gamma) \in \mathbb{R}^2 : \beta \in [\sin \theta, 1], \gamma \in \Gamma(\theta, \beta)\}, \quad (5.24)$$

where

$$\Gamma(\theta, \beta) = [\gamma_-(\theta, \beta), \gamma_+(\theta, \beta)] \subset \mathbb{R} \quad \text{with} \quad \gamma_{\pm}(\theta, \beta) = -\beta \cot \theta \pm \sqrt{(\sin \theta)^{-2} - \beta^{-2}} \quad (5.25)$$

for $\theta \in (0, \pi)$ and $\beta \in [\sin \theta, 1]$. Moreover, note that $(1, 0) \in \Lambda_{\theta}$ for any $\theta \in (0, \pi)$ and that

$$\Lambda_{\theta} = \{(1, 0)\} \quad \text{if and only if} \quad \theta = \frac{\pi}{2}.$$

In light of (5.23) and the injectivity of ψ , one immediately obtains through iteration that

$$\mathcal{T}_{\mathcal{N}}(R_*(\Omega)) = \mathcal{N} \cap \mathcal{N}R_{\theta_2}^T \cap \dots \cap \mathcal{N}R_{\theta_N}^T = \text{SO}(2)\psi(\Lambda_{\theta_2} \cap \dots \cap \Lambda_{\theta_N}), \quad (5.26)$$

recalling that $\theta_1 = 0$.

In the following, we discuss the cases $n = N$, $n = 1$, and $n \notin \{1, N\}$ separately.

Step 1: The case $n = N$. The essence of this step is the observation that the sets Λ_{θ_i} for $i = 1, \dots, N$ are strictly decreasing nested sets, that is,

$$\Lambda_{\theta_2} \supsetneq \dots \supsetneq \Lambda_{\theta_N}; \quad (5.27)$$

the representation (5.26) then simplifies to $\mathcal{T}_{\mathcal{N}}(R_*(\Omega)) = \mathcal{N} \cap \mathcal{N}R_{\theta_N}^T$, which results in (5.14), given that $\mathcal{N}R_{\pi}^T = \mathcal{N}$, cf. also Remark 5.8c). Since the sine function is strictly increasing in $(0, \frac{\pi}{2})$, it suffices for (5.27) to prove that

$$\Gamma(\theta, \beta) \subsetneq \Gamma(\tilde{\theta}, \beta) \quad \text{for any } \theta, \tilde{\theta} \in (0, \frac{\pi}{2}) \text{ with } \tilde{\theta} < \theta \text{ and } \beta \in [\sin \theta, 1]. \quad (5.28)$$

To this end, we show that $\partial_{\theta}\gamma_+(\theta, \beta) < 0$ and $\partial_{\theta}\gamma_-(\theta, \beta) > 0$ for all $\theta \in (0, \frac{\pi}{2})$ and $\beta \in (\sin \theta, 1)$, cf. (5.25); the case $\beta = 1$ is treated separately below. A direct calculation yields that

$$\begin{aligned} \partial_{\theta}\gamma_{\pm}(\theta, \beta) &= \partial_{\theta} \left(\beta \cot \theta \pm \sqrt{(\sin \theta)^{-2} - \beta^{-2}} \right) \\ &= -\frac{1}{\sin^2 \theta} \left(\beta \mp \frac{\cos \theta}{\sqrt{1 - \beta^{-2} \sin^2 \theta}} \right) = \frac{1}{\sin^2 \theta \sqrt{1 - \beta^{-2} \sin^2 \theta}} \left(\sqrt{\beta^2 - \sin^2 \theta} \mp \cos \theta \right), \end{aligned}$$

where the first factor is clearly positive; the second factor is strictly increasing in β on $(\sin \theta, 1]$ with boundary values

$$\sqrt{\beta^2 - \sin^2 \theta} \mp \cos \theta = \begin{cases} \mp \cos \theta & \text{if } \beta = \sin \theta, \\ \cos \theta \mp \cos \theta & \text{if } \beta = 1, \end{cases}$$

which implies the desired monotonicity of $\gamma_+(\theta, \beta)$ and $\gamma_-(\theta, \beta)$ in θ .

For $\beta = 1$, we compute that

$$\Gamma(\theta, 1) = \begin{cases} [-2 \cot \theta, 0] & \text{if } \theta \in (0, \frac{\pi}{2}), \\ [0, -2 \cot \theta] & \text{if } \theta \in [\frac{\pi}{2}, \pi), \end{cases} \quad (5.29)$$

from which (5.28) follows immediately in that case.

Step 2: The case $n = 1$. Now, let $\frac{\pi}{2} \leq \theta_2 < \dots < \theta_N$. In this case, formula (5.26) reduces to $\mathcal{T}_{\mathcal{N}}(R_*(\Omega)) = \mathcal{N} \cap \mathcal{N}R_{\theta_2}^T$ since the chain of inclusions in (5.27) is reversed. The latter is due to the monotonicity of the sine function in $[\frac{\pi}{2}, \pi)$, as well as the symmetry of the upper and lower bounds in (5.25) around $\frac{\pi}{2}$ in the sense that

$$\gamma_{\pm}(\frac{\pi}{2} + \theta, \beta) = -\gamma_{\mp}(\frac{\pi}{2} - \theta, \beta) \quad (5.30)$$

for $\theta \in (0, \frac{\pi}{2})$ and $\beta \in [\sin(\frac{\pi}{2} - \theta), 1]$. Precisely, combining (5.30) with the arguments of Step 1 yields that

$$\Gamma(\theta, \beta) \subsetneq \Gamma(\tilde{\theta}, \beta) \quad \text{for any } \theta, \tilde{\theta} \in (\frac{\pi}{2}, \pi) \text{ with } \theta < \tilde{\theta} \text{ and } \beta \in [\sin \theta, 1].$$

Step 3: The case $n \notin \{1, N\}$. Applying the conclusion of Step 1 to $\mathcal{N} \cap \dots \cap \mathcal{N}R_{\theta_n}^T$ and the results of Step 2 to $\mathcal{N} \cap \mathcal{N}R_{\theta_{n+1}} \cap \dots \cap \mathcal{N}R_{\theta_N}$ gives

$$\mathcal{T}_{\mathcal{N}}(R_*(\Omega)) = \mathcal{N} \cap \mathcal{N}R_{\theta_2}^T \cap \dots \cap \mathcal{N}R_{\theta_n}^T \cap \mathcal{N}R_{\theta_{n+1}}^T \cap \dots \cap \mathcal{N}R_{\theta_N}^T = \mathcal{N} \cap \mathcal{N}R_{\theta_n}^T \cap \mathcal{N}R_{\theta_{n+1}}^T,$$

as stated. \square

Remark 5.8. Let (Ω, R_*) be as in Proposition 5.1 and $n \in \{1, \dots, N\}$ with $\theta_n < \frac{\pi}{2} \leq \theta_{n+1}$.

a) Observe that $\text{SO}(2) \subset \mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ for any polycrystal (Ω, R_*) . Therefore, we call the Taylor bound $\mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ trivial if $\mathcal{T}_{\mathcal{N}}(R_*(\Omega)) = \text{SO}(2)$.

b) The Taylor bound $\mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ is polyconvex as the intersection of polyconvex hulls, and it is compact whenever (Ω, R_*) is not a single-crystal. One way to see this is via the representation formula (5.23)-(5.25), where $\mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ is expressed as the image of a compact set under a continuous map.

c) If $\theta_k \leq \frac{\pi}{2}$ (or $\theta_k \geq \frac{\pi}{2}$) for all $k \in \{2, \dots, N\}$, then only two values of R_* , precisely R_{θ_1} and R_{θ_N} (or R_{θ_1} and R_{θ_2}), are sufficient for characterizing the Taylor bound, which follows directly from Steps 1 and 2 in the proof of Proposition 5.1. This observation is in agreement with (5.14) since $\mathcal{N}R_{\theta_1}^T = \mathcal{N} = \mathcal{N}R_{\pi}^T = \mathcal{N}R_{\theta_{N+1}}^T$.

d) Proposition 5.1 shows that the Taylor bound depends on at most three different slip orientations. Indeed, it involves the slip direction $s = e_1$, corresponding to $\theta_1 = 0$, and at most two others that are closest to e_2 , see Figure 5.2. For a more general setting without the restriction $\theta_1 = 0$, we refer to Remark 5.10.

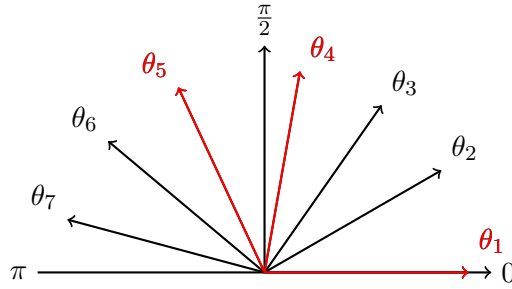


Figure 5.2: Dependence of the Taylor bound on exactly three slip directions (marked in red); here, $n = 4$ and $\mathcal{T}_{\mathcal{N}}(\{R_{\theta_1}, \dots, R_{\theta_7}\}) = \mathcal{T}_{\mathcal{N}}(\{R_{\theta_1}, R_{\theta_4}, R_{\theta_5}\})$.

e) Let $F \in \mathcal{N}$ be represented by $F = S(\beta e_1 | \frac{1}{\beta} e_2 + \gamma e_1)$ with $S \in \text{SO}(2)$, $\beta \in (0, 1]$, and $\gamma \in \mathbb{R}$. As a consequence of the previous proof, one can extract the following useful equivalence for explicit calculations: In fact, $F \in \mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ if and only if

$$(\beta, \gamma) \in \Lambda_{\theta_n} \cap \Lambda_{\theta_{n+1}}$$

or equivalently,

$$\beta \in [\max\{\sin \theta_n, \sin \theta_{n+1}\}, 1] \text{ and } \gamma \in \Gamma(\theta_n, \beta) \cap \Gamma(\theta_{n+1}, \beta),$$

see also (5.24) and (5.25). The sets Λ_θ for $\theta \in (0, \pi)$ (as well as their intersections) can be illustrated as in Figure 5.3; in particular, this figure depicts both the properties that $\Lambda_{\frac{\pi}{2}+\theta}$ emerges from $\Lambda_{\frac{\pi}{2}-\theta}$ for $\theta \in (0, \frac{\pi}{2})$ via reflection (see the green and blue areas), as well as the nested structure in (5.27) (see red and blue).

f) Regarding the Taylor bound for the unrelaxed problem $(P_{\mathcal{M}})$, we find that $\mathcal{T}_{\mathcal{M}}(R_*(\Omega))$ is compact for any non-trivial polycrystal and that

$$\mathcal{T}_{\mathcal{M}}(R_*(\Omega)) = \text{SO}(2)$$

if and only if $n \notin \{1, N\}$. It is evident from the calculations in the proof of Proposition 5.1 that

$$\mathcal{M} \cap \mathcal{M}R_\theta^T = \text{SO}(2)\psi(\{1\} \times \Gamma(1, \theta)) \quad (5.31)$$

for any $\theta \in (0, \pi)$, where $\Gamma(1, \theta)$ is given in (5.29). The identity (5.31) can be visualized by considering the “right boundary” $\Lambda_\theta \cap (\{1\} \times \mathbb{R})$ of Λ_θ in Figure 5.3.

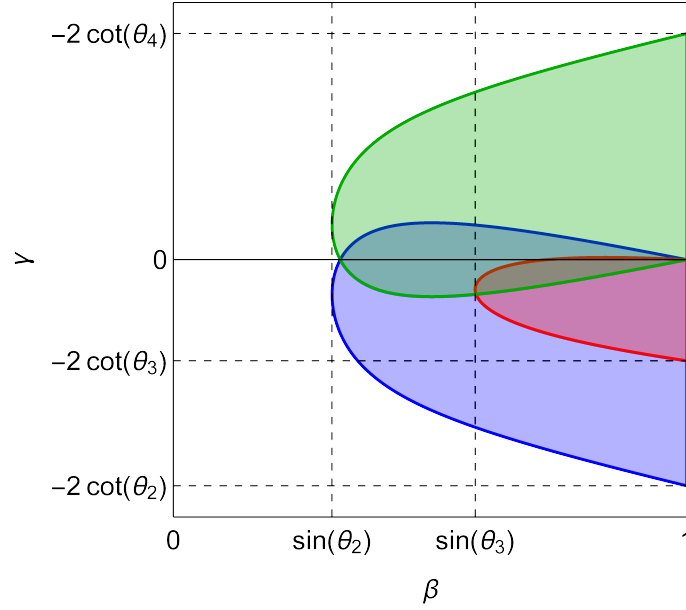


Figure 5.3: Illustration of sets Λ_θ for some choices of θ . Here, the blue, red, and green areas describe Λ_{θ_2} , Λ_{θ_3} , and Λ_{θ_4} , with $\theta_2 = \frac{\pi}{10}$, $\theta_3 = \frac{2\pi}{10}$, and $\theta_4 = \frac{9\pi}{10}$, respectively.

Based on Proposition 5.1, we now provide a necessary and sufficient condition on the orientations of a polycrystal (Ω, R_*) such that $\mathcal{T}_N(R_*(\Omega))$ is trivial.

Corollary 5.9 (Trivial Taylor bound). *Let (Ω, R_*) be as in Proposition 5.1. Then, the Taylor bound $\mathcal{T}_N(R_*(\Omega))$ is trivial, i.e., $\mathcal{T}_N(R_*(\Omega)) = \text{SO}(2)$, if and only if there exists $n \in \{1, \dots, N-1\}$ such that*

$$\frac{\pi}{2} \in [\theta_n, \theta_{n+1}] \quad \text{and} \quad \theta_{n+1} - \theta_n \leq \frac{\pi}{2}.$$

Proof. Step 1: Sufficiency. If $\theta_n = \frac{\pi}{2}$ for some $n \in \{2, \dots, N-1\}$, then any $F \in \mathcal{T}_N(R_*(\Omega))$ satisfies

$$\det F = 1, \quad |Fe_1| \leq 1, \quad |Fe_2| = |FR_{\theta_n}e_1| \leq 1.$$

Therefore, F must be a rotation. Indeed, we estimate

$$1 = \det F = (Fe_1)^\perp \cdot (Fe_2) \leq |Fe_1||Fe_2| \leq 1,$$

which implies that $|Fe_1| = |Fe_2| = 1$; in combination with $\det F = 1$, this implies $F \in \text{SO}(2)$. The same arguments can be used to treat the case $\theta_{n+1} = \frac{\pi}{2}$ for some $n \in \{1, \dots, N-1\}$.

It remains to show the claim for the case when $0 < \theta_n < \frac{\pi}{2} < \theta_{n+1} \leq \theta_n + \frac{\pi}{2} < \pi$ for $n \in \{2, \dots, N-1\}$. We apply Proposition 5.1 twice, once to (Ω, R_*) and then again to another polycrystal (Ω, \tilde{R}_*) with $\tilde{R}_*(\Omega) = \{R_{\theta_1}, R_{\theta_n}, R_{\theta_{n+1}}, R_{\theta_n + \frac{\pi}{2}}\}$ to conclude that

$$\mathcal{T}_N(R_*(\Omega)) = \mathcal{N} \cap \mathcal{N}R_{\theta_n}^T \cap \mathcal{N}R_{\theta_{n+1}}^T = \mathcal{N} \cap \mathcal{N}R_{\theta_n}^T \cap \mathcal{N}R_{\theta_{n+1}}^T \cap \mathcal{N}R_{\theta_n + \frac{\pi}{2}}^T = \mathcal{T}_N(\tilde{R}_*(\Omega)).$$

The right-hand side is trivial since $\mathcal{N}R_{\theta_n}^T \cap \mathcal{N}R_{\theta_n + \frac{\pi}{2}}^T = (\mathcal{N} \cap \mathcal{N}R_{\frac{\pi}{2}}^T)R_{\theta_n}^T = \text{SO}(2)$, which yields the statement.

Step 2: Necessity. Arguing by contraposition, assume first that either $\theta_k < \frac{\pi}{2}$ or $\theta_k > \frac{\pi}{2}$ for all $k \in \{2, \dots, N\}$. These two scenarios correspond exactly to those of Steps 1 and 2 in the proof of Proposition 5.1, where it is shown that

$$\mathcal{T}_N(R_*(\Omega)) = \text{SO}(2)\psi(\Lambda_{\theta_N}) \quad \text{or} \quad \mathcal{T}_N(R_*(\Omega)) = \text{SO}(2)\psi(\Lambda_{\theta_2}),$$

respectively, cf. (5.24). Since the sets Λ_{θ_N} and Λ_{θ_2} contain strictly more elements than $(1, 0)$ due to $\theta_2, \theta_N \neq \frac{\pi}{2}$ and since ψ is injective with $\psi(\beta, \gamma) \in \text{SO}(2)$ exactly for $(\beta, \gamma) = (1, 0)$, it follows that $\text{SO}(2) \subsetneq \mathcal{T}_N(R_*(\Omega))$.

Now it remains to address the case when there exists a $k \in \{2, \dots, N-1\}$ such that $\theta_{k+1} - \theta_k > \frac{\pi}{2}$, or equivalently,

$$0 < \theta_k < \frac{\pi}{2} < \theta_k + \frac{\pi}{2} < \theta_{k+1} < \pi. \quad (5.32)$$

To show that $\mathcal{T}_N(R_*(\Omega)) = \text{SO}(2)\psi(\Lambda_{\theta_k} \cap \Lambda_{\theta_{k+1}})$ is strictly larger than the set of rotations, we show that there exists a $\beta \in [\max\{\sin \theta_k, \sin \theta_{k+1}\}, 1)$ such that

$$\Gamma(\theta_k, \beta) \cap \Gamma(\theta_{k+1}, \beta) \neq \emptyset, \quad (5.33)$$

see (5.25) for the definition of the intervals $\Gamma(\theta, \beta)$ and the associated boundary points $\gamma_{\pm}(\theta, \beta)$. This implies that $\Lambda_{\theta_k} \cap \Lambda_{\theta_{k+1}} \supsetneq \{(1, 0)\}$, and hence, $\text{SO}(2) \subsetneq \mathcal{T}_N(R_*(\Omega))$ by the same arguments as above, as desired.

Finally, in order to verify (5.33), we observe first that

$$\gamma_+(\theta_{k+1}, \beta) \geq \gamma_-(\theta_k, \beta) \quad (5.34)$$

for any $\beta \in [\max\{\sin \theta_k, \sin \theta_{k+1}\}, 1]$; indeed, by the properties of the cotangent, the left-hand side is always non-negative and the right-hand side is non-positive, as $\theta_k \in (0, \frac{\pi}{2})$ and $\theta_{k+1} \in (\frac{\pi}{2}, \pi)$ due to (5.32). On the other hand, let

$$d(\beta) := \gamma_+(\theta_k, \beta) - \gamma_-(\theta_{k+1}, \beta) \text{ for } \beta \in [\max\{\sin \theta_k, \sin \theta_{k+1}\}, 1].$$

From (5.29), one obtains immediately that $d(1) = 0$, and $d'(1) < 0$ follows from the calculation

$$\begin{aligned} d'(1) &= \frac{d}{d\beta} \Big|_{\beta=1} (\gamma_+(\theta_k, \beta) - \gamma_-(\theta_{k+1}, \beta)) = -\cot \theta_k + \tan \theta_k + \cot \theta_{k+1} - \tan \theta_{k+1} \\ &= (\cos \theta_{k+1} \cos \theta_k + \sin \theta_{k+1} \sin \theta_k) \left(\frac{1}{\sin \theta_{k+1} \cos \theta_k} - \frac{1}{\cos \theta_{k+1} \sin \theta_k} \right) \\ &= -\frac{\sin(\theta_{k+1} - \theta_k) \cos(\theta_{k+1} - \theta_k)}{\sin \theta_{k+1} \sin \theta_k \cos \theta_{k+1} \cos \theta_k}, \end{aligned}$$

under consideration of (5.32). One can therefore find a $\beta \in [\max\{\sin \theta_k, \sin \theta_{k+1}\}, 1)$ with $d(\beta) > 0$, that is,

$$\gamma_+(\theta_k, \beta) > \gamma_-(\theta_{k+1}, \beta). \quad (5.35)$$

Consequently, the combination of (5.35) with (5.34) gives that the intersection in (5.33) is in fact trivial, which finishes the proof of the necessity. \square

Remark 5.10. We continue with two brief comments on the assumption that (Ω, R_*) is a polycrystal as in Proposition 5.1.

a) Note that considering only orientations induced by rotations with angles in $[0, \pi)$ (see (5.13)) is no real restriction, considering that plastic glide along the slip system is not uni-directed. Formally, we have that $\mathcal{N}R_\theta^T = \mathcal{N}R_{\theta \pm \pi}^T$ for any $\theta \in [0, \pi)$, and analogously, for \mathcal{M} in place of \mathcal{N} .

b) The postulate in (5.13) that the image $R_*(\Omega)$ contains the identity matrix can be made without loss of generality. If $R_*(\Omega) = \{R_{\theta_1}, \dots, R_{\theta_N}\}$ for angles $\theta_1 < \dots < \theta_N$ with $\theta_N - \theta_1 < \pi$, but not necessarily $\theta_1 = 0$, it is not hard to see that the results on the Taylor bounds in the case $\theta_1 = 0$ carry over to this more general setting. Indeed, with the new texture \tilde{R}_* on Ω given by $\tilde{R}_* = R_* R_{\theta_1}^T$, we observe that $\tilde{R}_*(\Omega) = \{R_{\theta_1 - \theta_1}, \dots, R_{\theta_N - \theta_1}\}$ and

$$\mathcal{T}_{\mathcal{N}}(\tilde{R}_*(\Omega)) = \bigcap_{x \in \Omega} \mathcal{N}\tilde{R}_*(x)^T = \bigcap_{x \in \Omega} \mathcal{N}R_{\theta_1}R_*^T(x) = \left(\bigcap_{x \in \Omega} \mathcal{N}R_*^T(x) \right) R_{\theta_1} = \mathcal{T}_{\mathcal{N}}(R_*(\Omega))R_{\theta_1};$$

and the same for \mathcal{M} instead of \mathcal{N} .

To illustrate the findings of this section, we discuss implications for random polycrystals with uniformly distributed orientations. In particular, when the number of grains diverges, then it turns out that the Taylor bound for $(P_{\mathcal{N}})$ is trivial almost surely. For an analysis of random polycrystals in the context of shape-memory alloys, see e.g., [37].

Example 5.11 (Randomized polycrystals). Let (Ω^k, R_*^k) for each $k \in \mathbb{N}$ be a polycrystal with at most $(k + 1)$ grains. In this example, the image $R_*^k(\Omega^k)$ is supposed to consist of the identity matrix Id and k rotations $R_{\theta_1}, \dots, R_{\theta_k}$ with uniformly distributed angles $\theta_1, \dots, \theta_k \in (0, \pi)$; note that we do not impose any ordering of these angles.

The basic observation for our analysis is that $\mathcal{T}_{\mathcal{N}}(R_*^k(\Omega^k)) = \text{SO}(2)$ if $(\theta_1, \dots, \theta_k) \in (0, \pi)^k$ lies in

$$\mathcal{T}_k := \{(\theta_1, \dots, \theta_k) \in (0, \pi)^k : \bigcap_{i=1}^k \mathcal{N}R_{\theta_i}^T \cap \mathcal{N} = \text{SO}(2)\}.$$

With the help of Corollary 5.9 and the elementary calculations in Lemma 5.22, we conclude that

$$\mu_k(\mathcal{T}_k) = 1 - \frac{k+1}{2^k},$$

where $\mu_k := \frac{1}{\pi^k} \lambda_k$ and λ_k is the k -dimensional Lebesgue measure. Hence, $\mu_k(\mathcal{T}_k) \rightarrow 1$ as $k \rightarrow \infty$. This means that a myriad of grains with uniformly distributed slip directions renders the Taylor bound almost surely trivial.

We conclude this subsection with a comparison of the Taylor bounds and another geometry-independent inner approximation resulting from attainable affine boundary values of a canonically associated homogeneous problem, see $(H_{\mathcal{M}})$ below. The basis of this discussion is the combination of our previous results on $\mathcal{T}_{\mathcal{M}}(R_*(\Omega))$ and $\mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ with the well-established theory of homogeneous inclusions.

Remark 5.12 (Alternative inner bound via a homogeneous inclusion). Instead of characterizing globally affine solutions to $(P_{\mathcal{M}})$ or $(P_{\mathcal{N}})$ as done to obtain the Taylor bounds, we now consider Lipschitz solutions to the homogeneous differential inclusion

$$\begin{cases} \nabla u \in \bigcap_{x \in \Omega} \mathcal{M}R_*^T(x) = \mathcal{T}_{\mathcal{M}}(R_*(\Omega)) & \text{a.e. in } \Omega, \\ u = Fx & \text{on } \partial\Omega, \end{cases} \quad (H_{\mathcal{M}})$$

where $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ is the unknown and $F \in \mathbb{R}^{2 \times 2}$. Then, the set

$$\mathcal{H}(R_*(\Omega)) := \{F \in \mathbb{R}^{2 \times 2} : \text{there exists a solution } u \in W^{1,\infty}(\Omega; \mathbb{R}^2) \text{ to } (H_{\mathcal{M}})\},$$

constitutes a geometry-independent inner bound for $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$, which, however, does not improve $\mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ if (Ω, R_*) has more than one grain. According to a classical result on differential inclusions as stated, e.g., in [162, Theorem 4.10] (applicable in view of the compactness of $\mathcal{T}_{\mathcal{M}}(R_*(\Omega))$ by Remark 5.8 f)) one has that

$$\mathcal{H}(R_*(\Omega)) \subset \mathcal{T}_{\mathcal{M}}(R_*(\Omega))^{\text{qc}} \subset \mathcal{T}_{\mathcal{M}}(R_*(\Omega))^{\text{pc}} \subset \mathcal{T}_{\mathcal{N}}(R_*(\Omega))^{\text{pc}} = \mathcal{T}_{\mathcal{N}}(R_*(\Omega)),$$

with the last identity due to Remark 5.8 b). The inclusion $\mathcal{H}(R_*(\Omega)) \subset \mathcal{T}_{\mathcal{N}}(R_*(\Omega))$ is in general even strict. For example, if $R_*(\Omega) = \{\text{Id}, R_{\frac{\pi}{6}}, R_{\frac{5\pi}{6}}\}$, then $\mathcal{T}_{\mathcal{M}}(R_*(\Omega))$ is trivial by Remark 5.8 f) so that $\mathcal{H}(R_*(\Omega)) = \text{SO}(2)$, while $\mathcal{T}_{\mathcal{N}}(R_*(\Omega)) \supsetneq \text{SO}(2)$ by Corollary 5.9.

5.4 Outer bounds resulting from boundary grains

5.4.1 Generalized Hadamard jump conditions

By the classical Hadamard jump condition, the gradients of any continuous and finitely piecewise affine function $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ need to be suitably rank-one connected. More precisely, if the sets $\{\nabla u = A\}$ and $\{\nabla u = B\}$ for matrices $A, B \in \mathbb{R}^{2 \times 2}$ are separated by a line with a normal $\nu \in \mathcal{S}^1$, then there exists some $a \in \mathbb{R}^2$ such that

$$A - B = a \otimes \nu.$$

Following-up the seminal paper by Ball & James [21], this fundamental result has seen several further generalizations over the years. For instance, in the case of Lipschitz maps, rank-one compatibility conditions between the polyconvex hulls of the sets of essential gradients (i.e., the smallest closed set containing all gradients up to a set of measure zero) on different sides of an interface are established in [17, 117]. The authors of [22, 25] show under additional regularity assumptions, namely continuous differentiability up to the interfacial boundary or locally bounded variation of the gradients, that the (approximate) gradients along the interface are rank-one connected pointwise (almost everywhere). Recently, the Hadamard jump condition was investigated in the context of moving interfaces in [81]; we also refer to this paper for a more detailed overview of the history of the problem.

The following result is a slight reformulation of [117, Corollary 4] by Iwaniec, Verchota & Vogel in the terminology of Definition 5.4, which has been suitably adapted to the needs of this chapter using the translation- and rotation-invariance of the polyconvex hull of a set $\mathcal{B} \subset \mathbb{R}^{2 \times 2}$ in the sense that $(\mathcal{B}R)^{\text{pc}} = \mathcal{B}^{\text{pc}}R$ and $(\mathcal{B} + A)^{\text{pc}} = \mathcal{B}^{\text{pc}} + A$ for any $A \in \mathbb{R}^{2 \times 2}$ and any $R \in \text{SO}(2)$.

Theorem 5.13 (Generalized Hadamard jump condition for planar interfaces). *Let $\mathcal{B} \subset \mathbb{R}^{2 \times 2}$ be closed, $A \in \mathbb{R}^{2 \times 2}$, and $\nu \in \mathcal{S}^1$. If $u \in W^{1,\infty}(B(0,1); \mathbb{R}^2)$ satisfies*

$$\begin{cases} \nabla u \in \mathcal{B} & \text{a.e. in } B_{\nu}^{+}(0,1), \\ \nabla u = A & \text{a.e. in } B_{\nu}^{-}(0,1), \end{cases}$$

then A is ν -compatible with \mathcal{B}^{pc} .

The next theorem can be seen in turn as a special case of the work by Ball & Carstensen [17], often cited in the literature, e.g., in [18, 20, 22, 25]. However, to the best of our knowledge, the reference [17] has not yet been published, which is why we include here a detailed proof for the reader's convenience. The overall strategy [16] combines a blow-up argument with Theorem 5.13.

Theorem 5.14 (Generalized Hadamard jump condition for curved interfaces). *Let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain such that $\bar{U} = \bar{U}_1 \cup \bar{U}_2$ for two disjoint Lipschitz domains $U_1, U_2 \subset U$ with interface $\Gamma := \partial U_1 \cap \partial U_2 \cap U$. Further, let $\mathcal{B} \subset \mathbb{R}^{2 \times 2}$ be a closed set, $A \in \mathbb{R}^{2 \times 2}$, and suppose that $x_0 \in \Gamma$ is a point where the outer unit normal $\nu(x_0)$ of U_1 exists. If there is a function $u \in W^{1,\infty}(U; \mathbb{R}^2)$ with*

$$\begin{cases} \nabla u \in \mathcal{B} & \text{a.e. in } U_2, \\ \nabla u = A & \text{a.e. in } U_1, \end{cases}$$

then A is $\nu(x_0)$ -compatible with \mathcal{B}^{pc} .

Proof. For $\varepsilon > 0$ sufficiently small, consider

$$v_\varepsilon : B(0, 1) \rightarrow \mathbb{R}^2, \quad v_\varepsilon(x) = \frac{1}{\varepsilon} (u(x_0 + \varepsilon x) - u(x_0)),$$

observing that $\nabla v_\varepsilon = \nabla u(x_0 + \varepsilon \cdot)$ and $v_\varepsilon(0) = 0$. Since $u \in W^{1,\infty}(U; \mathbb{R}^2)$, the sequence $(v_\varepsilon)_\varepsilon$ is bounded in $W^{1,\infty}(B(0, 1); \mathbb{R}^2)$. Consequently, there exists a compact set $K \subset \mathbb{R}^{2 \times 2}$ such that $\nabla v_\varepsilon \in K$ a.e. in $B(0, 1)$, and one can extract a subsequence of $(v_\varepsilon)_\varepsilon$ (not relabeled) such that

$$v_\varepsilon \xrightarrow{*} v \quad \text{in } W^{1,\infty}(B(0, 1); \mathbb{R}^2) \quad (5.36)$$

with $v \in W^{1,\infty}(B(0, 1); \mathbb{R}^2)$. With the transformation $\psi_\varepsilon : B(x_0, \varepsilon) \rightarrow B(0, 1)$, $x \mapsto \frac{1}{\varepsilon}(x - x_0)$, one has by assumption that $\nabla v_\varepsilon \in \mathcal{B} \cap K$ a.e. in $\psi_\varepsilon(U_1 \cap B(x_0, \varepsilon))$ and $\nabla v_\varepsilon = A$ a.e. in $\psi_\varepsilon(U_2 \cap B(x_0, \varepsilon))$. Next, we prove that

$$\begin{aligned} \text{dist}(\nabla v_\varepsilon, \mathcal{B} \cap K) &\rightarrow 0 \quad \text{in measure on } B_{\nu(x_0)}^+(0, 1), \\ |\nabla v_\varepsilon - A| &\rightarrow 0 \quad \text{in measure on } B_{\nu(x_0)}^-(0, 1) \end{aligned} \quad (5.37)$$

as $\varepsilon \rightarrow 0$. With $S_\varepsilon := (U_1 \triangle B_{\nu(x_0)}^-(x_0, \varepsilon)) \cap B(x_0, \varepsilon) = (U_2 \triangle B_{\nu(x_0)}^+(x_0, \varepsilon)) \cap B(x_0, \varepsilon)$, where \triangle denotes the symmetric difference between two sets (see Figure 5.4), the convergences in (5.37) follow immediately from

$$|\psi_\varepsilon(S_\varepsilon)| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (5.38)$$

To see the latter, recall that the interface Γ is locally the graph of a Lipschitz function and the unit outer normal $\nu(x_0)$ of U_1 exists, so that

$$\Gamma \cap B(x_0, \varepsilon) = \{x_0 - t\nu(x_0)^\perp + g(t)\nu(x_0) : t \in (-\varepsilon, \varepsilon)\} \cap B(x_0, \varepsilon)$$

with $g : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ Lipschitz continuous satisfying $g(0) = 0$ and $g'(0) = 0$. Then,

$$|\psi_\varepsilon(S_\varepsilon)| = (\det \nabla \psi_\varepsilon) |S_\varepsilon| \leq \varepsilon^{-2} \int_{-\varepsilon}^{\varepsilon} |g(t)| \, dt \leq \sup_{t \in (-\varepsilon, \varepsilon)} \left| \frac{g(t)}{t} \right| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

implying (5.38).

The remaining proof uses classical arguments from Young measure theory, see, e.g., [96, 162, 182] for general introductions. If $\{\mu_x\}_{x \in B(0, 1)}$ is the gradient Young measure generated by a (non-relabeled) subsequence of $(\nabla v_\varepsilon)_\varepsilon$, we infer along with (5.37),

$$\begin{aligned} \text{supp } \mu_x &\subset \mathcal{B} \cap K \quad \text{for a.e. } x \in B_{\nu(x_0)}^+(0, 1), \\ \text{supp } \mu_x &= \{A\} \quad \text{for a.e. } x \in B_{\nu(x_0)}^-(0, 1), \end{aligned} \quad (5.39)$$

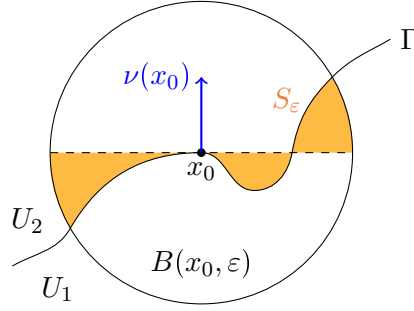


Figure 5.4: An illustration of the interface Γ in a neighborhood of x_0 , where the orange area depicts the set S_ε .

and it holds for the barycenters of μ_x that $\int_{\mathbb{R}^{2 \times 2}} M \, d\mu_x(M) = \nabla v(x)$ for a.e. $x \in B(0, 1)$ in view of (5.36). Since the quasiconvex hull $(\mathcal{B} \cap K)^{\text{qc}}$ consists of the barycenters of homogeneous gradient Young measures with support in $\mathcal{B} \cap K$, and μ_x is a homogeneous gradient Young measure itself for almost every x , it follows from (5.39) that

$$\begin{cases} \nabla v \in (\mathcal{B} \cap K)^{\text{qc}} & \text{a.e. in } B_{\nu(x_0)}^+(0, 1), \\ \nabla v = A & \text{a.e. in } B_{\nu(x_0)}^-(0, 1). \end{cases}$$

Finally, the statement follows from Theorem 5.13, according to which A is $\nu(x_0)$ -compatible with $((\mathcal{B} \cap K)^{\text{qc}})^{\text{pc}} \subset \mathcal{B}^{\text{pc}}$, noting in particular that $(\mathcal{B} \cap K)^{\text{qc}}$ is compact as the quasiconvex hull of a compact set. \square

5.4.2 Application to polycrystals

We now apply the results from Section 5.4.1 on rank-one compatibility along curved interfaces to the boundary grains of the polycrystal (Ω, R_*) . Therefore, in light of (5.9), let us introduce the set

$$\mathcal{T}_{\mathcal{N}}^\partial(\Omega, R_*) := \{F \in \text{Sl}(2) : F \text{ is } \nu(x)\text{-compatible with } \mathcal{N}R_*^T(x) \text{ for every } x \in \partial(\Omega, R_*)\}, \quad (5.40)$$

with ν the outer unit normal of Ω and $\partial(\Omega, R_*)$ as in (5.6); the texture R_* is canonically extended to $\partial\Omega$ except in the boundary dual points, cf. (5.5). Applying Theorem 5.14 locally around every $x \in \partial(\Omega, R_*)$ yields the outer bound

$$\mathcal{F}_{\mathcal{M}}(\Omega, R_*) \subset \mathcal{T}_{\mathcal{N}}^\partial(\Omega, R_*),$$

considering that $(\mathcal{M}R_*^T(x))^{\text{pc}} = (\mathcal{M}_{R_*(x)e_1})^{\text{pc}} = \mathcal{N}_{R_*(x)e_1} = \mathcal{N}R_*^T(x)$.

Notice that requiring the rank-one connectedness at boundary dual points does not improve the outer bound (5.40). Suppose that $x \in \partial\Omega$ is a boundary dual point where exactly two neighboring grains, say Ω_k and Ω_l , meet. If $\nu(x)$ exists, then Theorem 5.14 along with Proposition 5.7 and (5.4) shows that any element of $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ is $\nu(x)$ -compatible with

$$\left(\mathcal{M}R_*^T|_{\Omega_k} \cup \mathcal{M}R_*^T|_{\Omega_l} \right)^{\text{pc}} = \text{Sl}(2),$$

which is a trivial statement since $\text{Sl}(2)$ is already an outer bound for all polycrystals. In case that more than two grains meet in x , the argument is analogous.

A (possibly) larger outer bound on the set of attainable macroscopic strains of the polycrystal is $\mathcal{T}_{\mathcal{N}}^{\perp}(\Omega, R_*)$, which we define as in (5.40), but with

$$\partial_{\perp}(\Omega, R_*) = \{x \in \partial(\Omega, R_*) : \nu(x) \cdot R_*(x)e_1 = 0\}$$

in place of $\partial(\Omega, R_*)$. It is easier to characterize than $\mathcal{T}_{\mathcal{N}}^{\partial}(\Omega, R_*)$ in practice since it accounts for rank-one compatibility only at specific boundary points where an orthogonality condition between the outer unit normal and the slip orientations is satisfied. Even though the inclusion

$$\mathcal{T}_{\mathcal{N}}^{\partial}(\Omega, R_*) \subset \mathcal{T}_{\mathcal{N}}^{\perp}(\Omega, R_*)$$

is in general strict, as shown in Example 5.19 a), it is possible to provide a sufficient geometric condition on the polycrystal that ensures the equality. The next statement, a refinement of Proposition 5.2 and a direct consequence of Remark 5.6, gives more insight into the relation between these outer bounds and \mathcal{NR}_*^T .

Proposition 5.15. *Let $\Omega_1, \dots, \Omega_M$ for $M \in \mathbb{N}$ be the boundary grains of the polycrystal (Ω, R_*) and let $J = \{i \in \{1, \dots, M\} : \partial_{\perp}(\Omega, R_*) \cap \partial\Omega_i \neq \emptyset\}$.*

- a) *If $J = \{1, \dots, M\}$, then it holds that $\mathcal{T}_{\mathcal{N}}^{\partial}(\Omega, R_*) = \mathcal{T}_{\mathcal{N}}^{\perp}(\Omega, R_*)$.*
- b) *If $J \neq \emptyset$, then*

$$\mathcal{T}_{\mathcal{N}}^{\perp}(\Omega, R_*) = \bigcap_{i \in J} \mathcal{NR}_*^T|_{\Omega_i}.$$

- c) *If $J \cup J' \neq \emptyset$ with $J' = \{i \in \{1, \dots, M\} : \overline{\{\pm\nu(x) : x \in \partial(\Omega, R_*) \cap \partial\Omega_i\}} = \mathcal{S}^1\}$, then*

$$\mathcal{T}_{\mathcal{N}}^{\partial}(\Omega, R_*) \subset \bigcap_{i \in J \cup J'} \mathcal{NR}_*^T|_{\Omega_i}. \quad (5.41)$$

While J selects the boundary crystals Ω_i whose outer unit normal ν to Ω is orthogonal to the associated slip direction at some point, the index set J' identifies strongly curved boundary grains, where the image of $\pm\nu$ is dense in \mathcal{S}^1 . In Example 5.19, the combination of both these index sets enables a full characterization of $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$.

Remark 5.16 (Higher regularity of gradients). Let us comment on how requiring higher regularity for the solutions to $(P_{\mathcal{M}})$ affects the rank-one compatibility conditions at the boundary grains, and thus, the corresponding outer bounds for the effective strains. The answer depends on the geometry of the boundary grains of the polycrystal.

To be precise, let Ω be a C^1 -domain and suppose that for $F \in \mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ there is a solution $u \in C^1(\bar{\Omega}; \mathbb{R}^2)$ to $(P_{\mathcal{M}})$. It then follows from [22, Theorem 3.2] that F lies in

$$\mathcal{T}_{\mathcal{M}}^{\partial}(\Omega, R_*) := \{F \in \text{Sl}(2) : F \text{ is } \nu(x)\text{-compatible with } \mathcal{MR}_*^T(x) \text{ for every } x \in \partial(\Omega, R_*)\}.$$

If $\partial_{\perp}(\Omega, R_*) = \emptyset$, one observes that $\mathcal{T}_{\mathcal{M}}^{\partial}(\Omega, R_*) = \mathcal{T}_{\mathcal{N}}^{\partial}(\Omega, R_*)$ due to (i) \Leftrightarrow (ii) in Lemma 5.5, meaning that the added regularity does not sharpen the outer bound in this case.

On the other hand, if there exists $x \in \partial_{\perp}(\Omega, R_*)$, Remark 5.6 implies that $F \in \mathcal{MR}_*^T(x)$, instead of merely $F \in \mathcal{NR}_*^T(x)$. A comparison of Remark 5.8 f) and Corollary 5.9 therefore underlines that the polycrystal behaves more rigidly under the assumption of higher regularity.

Another natural outer bound for $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ can be derived dropping the gradient structure in the differential inclusion in $(P_{\mathcal{M}})$; for an analogous approach in the context of polycrystalline shape-memory alloys, see also [35, page 125]. Yet, it turns out that this bound is trivial due to the unboundedness of \mathcal{M} , and hence, unfit to improve $\mathcal{T}_{\mathcal{N}}^{\partial}(\Omega, R_*)$ and $\mathcal{T}_{\mathcal{N}}^{\perp}(\Omega, R_*)$.

Remark 5.17. Let us consider the inclusion problem

$$\begin{cases} U(x) \in \mathcal{M}R_*^T(x) & \text{for a.e. } x \in \Omega, \\ \int_{\Omega} U(x) \, dx = F|\Omega|, \end{cases} \quad (C_{\mathcal{M}})$$

with the unknown $U \in L^\infty(\Omega; \mathbb{R}^{2 \times 2})$ and $F \in \mathbb{R}^{2 \times 2}$, which arises from $(P_{\mathcal{M}})$ in disregard of the gradient structure. We observe that

$$\mathcal{C}_{\mathcal{M}}(\Omega, R_*) = \{F \in \mathbb{R}^{2 \times 2} : \text{there exists a solution } U \in L^\infty(\Omega; \mathbb{R}^2) \text{ to } (C_{\mathcal{M}})\}$$

constitutes an outer bound of $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$, given that $(C_{\mathcal{M}})$ clearly admits a larger class of solutions than $(P_{\mathcal{M}})$. However, under the assumption that (Ω, R_*) is not a single crystal, it holds that

$$\mathcal{C}_{\mathcal{M}}(\Omega, R_*) = \mathbb{R}^{2 \times 2}.$$

This follows from

$$\mathbb{R}^{2 \times 2} = \sum_{i=1}^N \frac{|\Omega_i|}{|\Omega|} \mathcal{M}_{R_*}^{\text{co}}|_{\Omega_i} e_1 = \sum_{i=1}^N \frac{|\Omega_i|}{|\Omega|} (\mathcal{M}R_*^T|_{\Omega_i})^{\text{co}} \subset \mathcal{C}_{\mathcal{M}}(\Omega, R_*), \quad (5.42)$$

where $(\cdot)^{\text{co}}$ denotes the convex hull and $\Omega_1, \dots, \Omega_N \subset \Omega$ with $N \geq 2$ are the grains of Ω . The first identity in (5.42) relies on the simple fact that $\mathcal{M}_s^{\text{co}} = \{F \in \mathbb{R}^{2 \times 2} : |Fs| \leq 1\}$ for any $s \in \mathcal{S}^1$, while the last inclusion is based on standard convexity arguments; essentially, it suffices to employ Carathéodory's theorem in combination with a suitable refinement of the partition given by the grains.

In the following, we present examples of polycrystals for which the set of attainable macroscopic strains $\mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ can be fully characterized with the help of the previous concepts. The analysis of polycrystalline structures with a symmetric material response and their rigidity is followed up by a brief discussion of selected bicrystals.

Example 5.18 (Polycrystals with sufficient symmetry). In analogy to [35, 124], we say that (Ω, R_*) has sufficient symmetry if there exists $R \in \text{SO}(2) \setminus \{\pm \text{Id}\}$ such that $F \in \mathcal{F}_{\mathcal{M}}(\Omega, R_*)$ if and only if $R^T F R \in \mathcal{F}_{\mathcal{M}}(\Omega, R_*)$, or equivalently, due to $R\mathcal{M} = \mathcal{M}$,

$$\mathcal{F}_{\mathcal{M}}(\Omega, R_*) = \mathcal{F}_{\mathcal{M}}(\Omega, R_*)R^T. \quad (5.43)$$

a) Let $\Omega_1, \dots, \Omega_M$ for $M \in \mathbb{N}$ be the boundary grains of (Ω, R_*) and let $J, J' \subset \{1, \dots, M\}$ be as in Proposition 5.15. If there is $i \in J \cup J'$, then (5.41) in combination with (5.43) and Lemma 5.23 yields that

$$\mathcal{F}_{\mathcal{M}}(\Omega, R_*) \subset \bigcap_{k=0}^{\infty} \mathcal{N}R_*^T|_{\Omega_i} (R^T)^k = \left(\bigcap_{k=0}^{\infty} \mathcal{N}(R^k)^T \right) R_*^T|_{\Omega_i} = \text{SO}(2).$$

Since $\text{SO}(2)$ is a trivial inner bound, the polycrystal is fully rigid in the sense that $(P_{\mathcal{M}})$ can only be solved with affine boundary values in $\text{SO}(2)$.

In view of the geometry-independence of the Taylor bound, the arguments above also show that $\mathcal{T}_{\mathcal{N}}(R_*(\Omega)) = \text{SO}(2)$ for any polycrystal (Ω, R_*) with sufficient symmetry, even if $J \cup J' = \emptyset$.

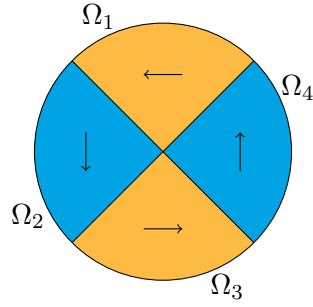


Figure 5.5: A visualization of the polycrystal (Ω, R_*) with $\Omega = B(0, 1)$ and $R_* = \text{Id} \mathbb{1}_{\Omega_1 \cup \Omega_3} + R_{\frac{\pi}{2}} \mathbb{1}_{\Omega_2 \cup \Omega_4}$.

b) A special class of polycrystals with sufficient symmetry is given as follows. Under the assumption that Ω has center of in the origin, that is, $\int_{\Omega} x \, dx = 0$, suppose that there exists a rotation $R \in \text{SO}(2) \setminus \{\pm \text{Id}\}$ such that

$$R\Omega = \Omega \quad \text{and} \quad R_*(x) = R^T R_*(Rx) \quad \text{for a.e. } x \in \Omega, \quad (5.44)$$

where $R\Omega = \{Rx : x \in \Omega\}$. A simple transformation argument shows that (5.43) is indeed satisfied. Figure 5.5 depicts an easy example of a polycrystal that fulfills (5.44) with $R = R_{\frac{\pi}{2}}$; here, $J = \{1, 2, 3, 4\}$, which implies full rigidity.

Example 5.19 (Bicrystals). Let $\Omega = B(0, 1)$ and fix slip directions $s, s' \in S^1$ with $s \neq \pm s'$.

a) Consider (Ω, R_*) with

$$R_* e_1 = \begin{cases} s & \text{in } \Omega_1 = B_{e_2}^+(0, 1), \\ s' & \text{in } \Omega_2 = B_{e_2}^-(0, 1), \end{cases}$$

see Figure 5.6 a). Recalling the definitions (5.11) and (5.40), Proposition 5.3 and Proposition 5.15 c) imply that

$$\mathcal{N}_s \cap \mathcal{N}_{s'} = \mathcal{T}_{\mathcal{N}}(R_*(\Omega)) \subset \mathcal{F}_{\mathcal{M}}(\Omega, R_*) \subset \mathcal{T}_{\mathcal{N}}^{\partial}(\Omega, R_*) = \mathcal{N}_s \cap \mathcal{N}_{s'},$$

which determines the attainable macroscopic strains of the polycrystal. If $s = e_2$, then $\partial_{\perp}(\Omega, R_*)$ consists of only a single point in $\partial\Omega_2 \cap \partial(\Omega, R_*)$ and

$$\mathcal{T}_{\mathcal{N}}^{\perp}(\Omega, R_*) = \mathcal{N}_{s'} \supsetneq \mathcal{N}_s \cap \mathcal{N}_{s'} = \mathcal{T}_{\mathcal{N}}^{\partial}(\Omega, R_*).$$



Figure 5.6: Illustration of a polycrystal as in a) with $s = e_2$ and $s' = R_{\frac{\pi}{6}} e_1$, and b) with $s = e_2$, $s' = R_{\frac{\pi}{6}} e_1$ and $\theta = \frac{\pi}{9}$.

b) As a second example, let (Ω, R_*) be given by

$$R_*e_1 = \begin{cases} s & \text{in } \Omega_1 = \{x \in B(0, 1) : x_2 > -\sin \theta\}, \\ s' & \text{in } \Omega_2 = \{x \in B(0, 1) : x_2 < -\sin \theta\}, \end{cases}$$

for $\theta \in (0, \frac{\pi}{2})$, see Figure 5.6 b). In this case, Proposition 5.15 gives rise to the characterization of $\mathcal{F}_M(\Omega, R_*)$ for $s' = R_\varphi e_1$ with $|\varphi| < \frac{\pi}{2} - \theta$, namely,

$$\mathcal{F}_M(\Omega, R_*) = \mathcal{N}_s \cap \mathcal{N}_{s'}.$$

5.5 On the non-optimality of the Taylor bound

We discuss in Example 5.19 several instances of polycrystals whose Taylor bound is optimal. Moreover, Example 5.18 demonstrates that polycrystals with sufficient symmetry are fully rigid under suitable assumptions. These results naturally raise the question of whether the Taylor bound is generally optimal. The next proposition provides a negative answer to this issue via the construction of a specific polycrystal. Our geometric setup is mainly inspired by the rotated-square approach discussed in [67, 178] in the context of stress-free martensitic inclusions in the theory of shape-memory alloys.

Proposition 5.20. *There exists a polycrystal (Ω, R_*) such that $\mathcal{T}_N(R_*(\Omega)) \subsetneq \mathcal{F}_M(\Omega, R_*)$.*

Proof. Let us start with a brief overview of our strategy for finding an explicit example of a polycrystal with non-optimal Taylor bound. To keep the arguments simple, we aim for a polycrystal (Ω, R_*) with the two orthogonal slip directions e_1 and e_2 , which guarantees a trivial Taylor bound according to Corollary 5.9, i.e., $\mathcal{T}_N(R_*(\Omega)) = \text{SO}(2)$. Necessarily, our task is then to determine an $F \in \mathcal{F}_M(\Omega, R_*) \setminus \text{SO}(2)$. As a first step, we construct a finitely piecewise affine solution to the homogeneous partial differential inclusion

$$\begin{cases} \nabla v \in \mathcal{N}_{e_1} \cup \mathcal{N}_{e_2} & \text{a.e. in } \Omega, \\ v = Fx & \text{on } \partial\Omega \end{cases} \quad (5.45)$$

for a suitable $F \notin \text{SO}(2)$. The non-empty connected components of the sets $\{\nabla v \in \mathcal{N}_{e_1}\}$ and $\{\nabla v \in \mathcal{N}_{e_2}\}$ are chosen as the polycrystalline grains $(\Omega_i)_i$, whose orientations are set accordingly to be e_1 and e_2 ; on $\{\nabla v \in \text{SO}(2)\}$, one may select either of these two orientations. This procedure provides a finitely piecewise affine solution to (P_N) . In the final step, we apply Proposition 5.3 to obtain a Lipschitz solution to (P_M) with the same affine boundary condition.

Now, let $\Omega \subset \mathbb{R}^2$ be the square with corners $(0, 0), (1, 3), (4, 2), (3, -1)$. Observe that the specific choice of Ω yields that the outer unit normal $\partial\Omega$ never attains the values e_1 or e_2 so that $\mathcal{T}_N^\perp(\Omega, R_*) = \text{Sl}(2)$. Similarly to [67, 178], we subdivide Ω into eight triangles T_1, \dots, T_8 and a square S , which represent the pieces where the solution v to the homogeneous problem (5.45) will be affine. Precisely,

$$\begin{aligned} T_1 &= \{(0, 0), (2, 0), (1, 1)\}^{\text{co}}, & T_2 &= \{(0, 0), (1, 1), (1, 3)\}^{\text{co}}, \\ T_3 &= \{(1, 1), (1, 3), (2, 2)\}^{\text{co}}, & T_4 &= \{(1, 3), (2, 2), (4, 2)\}^{\text{co}}, \\ T_5 &= \{(2, 2), (4, 2), (3, 1)\}^{\text{co}}, & T_6 &= \{(4, 2), (3, 1), (3, -1)\}^{\text{co}}, \\ T_7 &= \{(2, 0), (3, 1), (3, -1)\}^{\text{co}}, & T_8 &= \{(0, 0), (2, 0), (3, -1)\}^{\text{co}}, \\ S &= \{(1, 1), (2, 2), (3, 1), (2, 0)\}^{\text{co}}, \end{aligned} \quad (5.46)$$

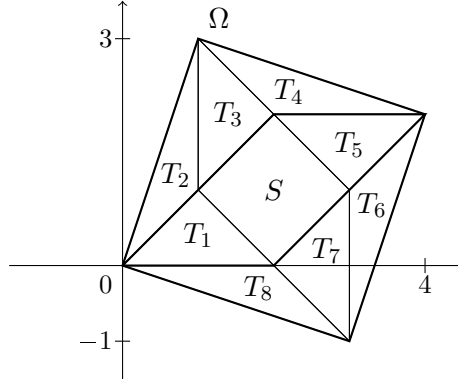


Figure 5.7: The domain Ω and its partition into the eight triangles T_1, \dots, T_8 and the square S in the center.

where $(\cdot)^\text{co}$ denotes the classical convex hull, see Figure 5.7. We stress that the area of these subsets satisfy $|T_1| = \dots = |T_8| = \frac{1}{2}|S|$ and do not represent the grains of the polycrystal.

Starting the construction of the solution v to (5.45), we set

$$\nabla v|_S = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_{e_1} \subset \mathcal{N}_{e_1}, \quad (5.47)$$

which means that S undergoes a shear in e_1 -direction with shear parameter $\gamma \in \mathbb{R}$ to be specified later.

Next, we explain how to construct the gradients in T_1 and T_5 . Inspired by the point-symmetry of the partition (5.46) of Ω , we apply affine deformations with identical strain on both these sets. To obtain a continuous deformation, the gradients of v need to be rank-one compatible along the interfaces of S and the other triangles; in particular,

$$\nabla v|_{T_1}(e_1 - e_2) = \nabla v|_S(e_1 - e_2) = \nabla v|_{T_5}(e_1 - e_2) = \begin{pmatrix} 1 - \gamma \\ -1 \end{pmatrix}, \quad (5.48)$$

which prescribes the affine deformation v on T_1 and T_5 along $e_1 - e_2$ up to translation. To fully pin down the construction, we need to designate v along another linearly independent direction. A natural choice for the latter are e_1 and $e_1 + e_2$ since they are parallel to the other edges of T_1 and T_5 . Considering that $\mathcal{N}_{e_1} \subset \text{Sl}(2)$ necessitates that the restriction of v to these two sets needs to be incompressible. Our approach to finding such a locally volume-preserving deformation works via extension of the images of the edges of S under v in a way that gives rise to

$$\nabla v|_{T_5} e_1 = \nabla v|_{T_1} e_1 = \frac{1}{2} \nabla v|_S (e_1 + e_2) = \frac{1}{2} \begin{pmatrix} 1 + \gamma \\ 1 \end{pmatrix}.$$

Hence, together with (5.48),

$$\nabla v|_{T_5} = \nabla v|_{T_1} = \frac{1}{2} \begin{pmatrix} 1 + \gamma & 3\gamma - 1 \\ 1 & 3 \end{pmatrix}. \quad (5.49)$$

If $\gamma \in [-1 - \sqrt{3}, \sqrt{3} - 1]$, then the right-hand side is contained in \mathcal{N}_{e_1} . The same strategy applied to T_3 and T_7 yields that

$$\nabla v|_{T_3} = \nabla v|_{T_7} = \frac{1}{2} \begin{pmatrix} 3 + \gamma & \gamma - 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{N}_{e_2}, \quad (5.50)$$

if $\gamma \in [1 - \sqrt{3}, 1 + \sqrt{3}]$; hence, we take $\gamma \in [-\sqrt{3} + 1, \sqrt{3} - 1]$ from now on.

Having v fixed on the triangles T_1, T_5, T_3 , and T_7 , the sought finitely piecewise affine solution to (5.45) is automatically determined on all of Ω . Indeed, the rank-one compatibility along interfaces combined with the constructions (5.49) and (5.50) require that

$$\nabla v|_{T_6} = \nabla v|_{T_2} = \frac{1}{2} \begin{pmatrix} 1 + 3\gamma & \gamma - 1 \\ 3 & 1 \end{pmatrix} \in \mathcal{N}_{e_2} \quad \text{and} \quad \nabla v|_{T_8} = \nabla v|_{T_4} = \frac{1}{2} \begin{pmatrix} 1 + \gamma & -3 + \gamma \\ 1 & 1 \end{pmatrix} \in \mathcal{N}_{e_1}. \quad (5.51)$$

We now define the polycrystal (Ω, R_*) as follows: In light of (5.47)-(5.51), let the grains of the polycrystal be

$$\Omega_1 = \text{int}(T_2 \cup T_3), \quad \Omega_2 = \text{int}(T_6 \cup T_7), \quad \Omega_3 = \text{int}(T_1 \cup T_5 \cup T_4 \cup T_8 \cup S),$$

and let the orientations be given by

$$R_* = \text{Id} \mathbb{1}_{\Omega_3} + R_{\frac{\pi}{2}} \mathbb{1}_{\Omega_1 \cup \Omega_2}. \quad (5.52)$$

Overall, the procedure above produces a finitely piecewise affine solution v to the relaxed problem (P_N) for the polycrystal (Ω, R_*) with texture R_* as in (5.52) and the boundary value $v = F_\gamma x$ on $\partial\Omega$ with

$$F_\gamma = \frac{1}{5} \begin{pmatrix} 3\gamma + 4 & 4\gamma - 3 \\ 3 & 4 \end{pmatrix} \quad \text{for } \gamma \in [-\sqrt{3} + 1, \sqrt{3} - 1],$$

see Figure 5.8 for illustration. Note that $F_\gamma \in \text{SO}(2)$ if and only if $\gamma = 0$, so that, in combination with Proposition 5.3, any $\gamma \in \mathbb{R}$ with $0 < |\gamma| < \sqrt{3} - 1$ gives rise to a solution of (P_M) with $F \notin \text{SO}(2)$. This shows that $\mathcal{T}_N(R_*(\Omega)) = \text{SO}(2) \subsetneq \mathcal{F}_M(\Omega, R_*)$ and concludes the proof. \square

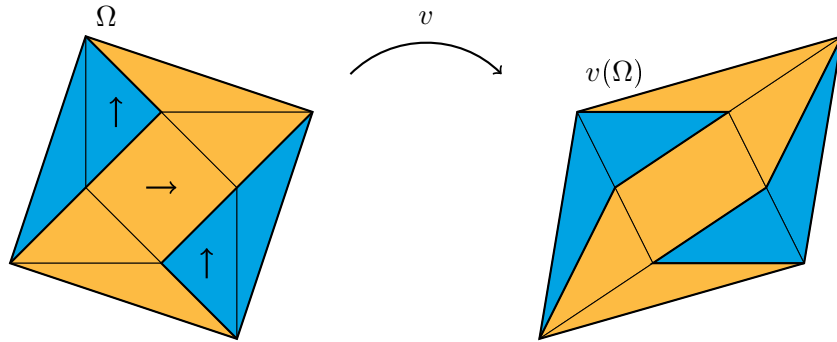


Figure 5.8: The finitely piecewise affine solution v to (P_N) with $F = F_{\frac{1}{2}}$. The orange area describes the grain with slip direction e_1 , while the blue ones represent the grains with slip direction e_2 .

Remark 5.21 (Comparison with the rotated-square construction). While our geometric framework is similar to that of [67, 178], the overall design strategy is different, given that we need to accommodate here the homogeneous differential inclusion (5.45) with a non-trivial affine boundary condition.

Unlike the rotated-square construction of [67, 178], our sheared-square construction converts any additional rotation of the center square into a rotation of the surrounding triangles T_1, \dots, T_8

by the same angle, considering that the vertices of Ω are not fixed. The boundary of Ω is thus rotated in the same way. More precisely, replacing $\nabla v|_S$ by $R\nabla v|_S$ for a rotation $R \in \text{SO}(2)$ yields the boundary value RF_γ instead of F_γ .

5.6 Auxiliaries

Lemma 5.22. *Let $k \in \mathbb{N}$ and $\mathcal{T}_k = \{(\theta_1, \dots, \theta_k) \in (0, \pi)^k : \bigcap_{i=1}^k \mathcal{N}R_{\theta_i}^T \cap \mathcal{N} = \text{SO}(2)\}$. Then it holds for the k -dimensional Lebesgue measure of \mathcal{T}_k that*

$$\lambda_k(\mathcal{T}_k) = \pi^k \left(1 - \frac{k+1}{2^k}\right). \quad (5.53)$$

Proof. For integers $1 \leq j \leq i$, let $\Sigma_{i,j}$ be the set of all injective functions $\sigma : \{1, \dots, j\} \rightarrow \{1, \dots, i\}$ and observe that $\Sigma_{i,j}$ consists of exactly $\frac{i!}{(i-j)!}$ elements; also, set $\Sigma_{i,0} = \emptyset$.

We establish (5.53) by computing the measure of the complement \mathcal{T}_k^c of \mathcal{T}_k in $(0, \pi)^k$. In view of Corollary 5.9, \mathcal{T}_k^c can be expressed as the disjoint union

$$\mathcal{T}_k^c = \bigcup_{l=0}^k T_{k,l} \cup N, \quad (5.54)$$

where $N \subset \mathbb{R}^k$ is a set of zero k -dimensional Lebesgue measure, and

$$T_{k,l} := \{(\theta_1, \dots, \theta_k) \in (0, \pi)^k : \theta_i < \frac{\pi}{2} \text{ and } \theta_j > \frac{\pi}{2} \text{ with } \theta_j - \theta_i > \frac{\pi}{2} \\ \text{for all } i \in \sigma(\{1, \dots, l\}), j \in \{1, \dots, k\} \setminus \sigma(\{1, \dots, l\}) \text{ and all } \sigma \in \Sigma_{k,l}\} \quad (5.55)$$

for $l \in \{0, \dots, k\}$. Note that $T_{k,0}$ and $T_{k,k}$ correspond to the cases $\theta_i > \frac{\pi}{2}$ and $\theta_i < \frac{\pi}{2}$ for all $i \in \{1, \dots, k\}$, respectively.

To calculate $\lambda_k(\mathcal{T}_k^c)$, it suffices to determine the measures of $T_{k,l}$ as in (5.55) and exploit (5.54). It is easy to see that

$$\lambda_k(T_{k,0}) = \lambda_k(T_{k,k}) = \left(\frac{\pi}{2}\right)^k. \quad (5.56)$$

For $l \in \{1, \dots, k-1\}$, one obtains that

$$\begin{aligned} \lambda_k(T_{k,l}) &= \binom{k}{l} \int_{(0, \frac{\pi}{2})^l} \int_{(\frac{\pi}{2} + \max\{\theta_1, \dots, \theta_l\}, \pi)^{k-l}} d(\theta_{l+1}, \dots, \theta_k) d(\theta_1, \dots, \theta_l) \\ &= \binom{k}{l} \int_{(0, \frac{\pi}{2})^l} \left(\frac{\pi}{2} - \max\{\theta_1, \dots, \theta_l\}\right)^{k-l} d(\theta_1, \dots, \theta_l) \\ &= \binom{k}{l} l! \int_0^{\frac{\pi}{2}} \int_0^{\theta_l} \dots \int_0^{\theta_2} \left(\frac{\pi}{2} - \theta_l\right)^{k-l} d\theta_1 \dots d\theta_{l-1} d\theta_l \\ &= \binom{k}{l} l \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - \theta_l\right)^{k-l} \theta_l^{l-1} d\theta_l = \binom{k}{l} l \frac{(l-1)!(k-l)!}{k!} \left(\frac{\pi}{2}\right)^k = \left(\frac{\pi}{2}\right)^k, \end{aligned} \quad (5.57)$$

where the third equality uses that $(0, \frac{\pi}{2})^l$ can (up to a null set) be split disjointly into $l!$ sets with equal measure

$$\bigcup_{\sigma \in \Sigma_{l,l}} \{(\theta_1, \dots, \theta_l) \in (0, \frac{\pi}{2})^l : \theta_{\sigma(1)} < \dots < \theta_{\sigma(l)}\};$$

moreover, the last integral in (5.57) is solved via integration by parts applied $(k-l)$ times.

The desired identity (5.53) then follows from (5.54), in view of (5.56) and (5.57). \square

Lemma 5.23. *It holds for any $R \in \text{SO}(2) \setminus \{\pm \text{Id}\}$ that*

$$\bigcap_{k=0}^{\infty} \mathcal{N}(R^k)^T = \text{SO}(2).$$

Proof. In light of Remark 5.10 a), one may assume that $R = R_\varphi$ with $\varphi \in (0, \pi)$. We set

$$\theta_j = j\varphi - \lfloor \frac{j\varphi}{\pi} \rfloor \pi \in [0, \pi)$$

for any $j \in \mathbb{N}$, where $\lfloor t \rfloor$ denotes the largest integer below $t \in \mathbb{R}$. If one can find $k, l \in \mathbb{N}$ such that

$$0 \leq \theta_k < \theta_l < \pi \quad \text{and} \quad \frac{\pi}{2} \in [\theta_k, \theta_l] \quad \text{and} \quad \theta_l - \theta_k \leq \frac{\pi}{2}, \quad (5.58)$$

then the statement follows immediately from

$$\text{SO}(2) \subset \bigcap_{k=0}^{\infty} \mathcal{N}(R^k)^T \subset \mathcal{N} \cap \mathcal{N}R_{\theta_k}^T \cap \mathcal{N}R_{\theta_l}^T = \text{SO}(2),$$

where the last identity is a consequence of Corollary 5.9.

To see (5.58), let us write $(0, \pi)$ as the disjoint union

$$(0, \pi) = \bigcup_{m=2}^{\infty} I_m \cup (0, \frac{\pi}{2}) \quad \text{with} \quad I_m := [(1 - \frac{1}{2^{m-1}})\pi, (1 - \frac{1}{2^m})\pi).$$

If $\varphi \in (0, \frac{\pi}{2})$, we take $k = \lfloor \frac{\pi}{2\varphi} \rfloor$ and $l = \lfloor \frac{\pi}{2\varphi} \rfloor + 1$, observing that $0 < \theta_k = \lfloor \frac{\pi}{2\varphi} \rfloor \varphi < \frac{\pi}{2} < (\lfloor \frac{\pi}{2\varphi} \rfloor + 1)\varphi = \theta_l < \pi$ and $\theta_l - \theta_k = \varphi < \frac{\pi}{2}$. For $\varphi \in I_m$ with $m \geq 2$, let $l = 2^{m-2}$ and $k = 2^{m-1}$. Then, $\theta_k = 2^{m-1}\varphi - (2^{m-1} - 1)\pi \in [0, \frac{\pi}{2})$ and $\theta_l = 2^{m-2}\varphi - (2^{m-2} - 1)\pi \in [\frac{\pi}{2}, \frac{3}{4}\pi)$ with

$$\theta_l - \theta_k = 2^{m-2}(\pi - \varphi) \leq \frac{2^{m-2}}{2^{m-1}}\pi = \frac{\pi}{2}.$$

□

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Summary

The development of novel high-tech materials is a critical aspect of the engineering sciences and has significance in many branches of industry. Reliable prediction of material behavior commonly requires the description of material properties on multiple length scales (e.g., the microscopic, mesoscopic, or macroscopic scale). In the past, studying energy-based multiscale models has proven remarkably successful due to their amenability to tools in the calculus of variations. However, the standard variational theories are insufficient to gain a deeper insight into advanced mechanical features, often modeled via non-convex (inhomogeneous) differential constraints. It is therefore essential to combine new techniques for constrained multiscale models with profound knowledge of the classical variational theories and materials science. In this spirit, the thesis presents new analytic results for three energy-based models whose common denominator is the constraint of incompressibility:

- i)* the derivation of one-dimensional models for three-dimensional incompressible thin structures;
- ii)* the effective deformation behavior of elastic bodies reinforced by rigid fibers;
- iii)* the material behavior of polycrystals in a model of single-slip finite crystal plasticity.

To provide the necessary background, we begin with a comprehensive overview of standard variational approaches to hyperelastic materials, which can be analyzed via elastic energy functionals. We discuss the classical theories for the existence of minimizers via the direct method and the asymptotic behavior of low-energy states via Γ -convergence. We exemplify the latter with two notably relevant applications for this thesis: dimension reduction and homogenization.

In *i)*, we focus on the dimension reduction of models for locally volume-preserving thin strings and rods. With the help of Γ -convergence, we derive lower-dimensional models that effectively describe the deformation behavior of the three-dimensional body for the entire hierarchy of scaling regimes of the external forces. We differentiate between regimes that lead to highly flexible string models and all remaining scalings, in which we derive constrained Kirchhoff- and von Kármán-type rod theories. In all physically relevant limit models, we find that incompressibility only impacts the elastic energy itself (and thus its minimizers) and not the set of admissible deformations. We overcome the differential constraint of local volume preservation by tailoring existing techniques, such as an inner perturbation argument, suitable (constraint-preserving) mollifications and approximation via penalization terms, to these new models.

We characterize in *ii)* the effective deformation behavior of hyperelastic materials reinforced by parallel, long, thin, rigid fibers. Within a suitable homogenization framework, the macroscopic responses to external forces correspond to weak Sobolev limits of functions that satisfy the challenging inhomogeneous differential constraint of rigidity on the fibers. While necessary conditions on the macroscopic deformation behavior follow similarly as in earlier works on layered composites, sufficient conditions are obtained via a careful approximation of the identity in two dimensions that accommodates the differential constraint. It turns out that the material

behavior of fibered composites is highly anisotropic in the sense that the strain in the direction of the fibers has unit length and higher regularity and only depends on the cross-section variables. We illustrate several examples of admissible deformations and set this result apart from earlier work about elastic bodies reinforced by rigid layers; in fact, our model exhibits more flexibility due to the higher co-dimension of the rigid parts and the resulting connectedness of the soft matrix. This work is the first fundamental step towards a complete homogenization result of hyperelastic bodies reinforced by fully rigid fibers.

Based on earlier work on single-crystal finite plasticity with one active slip system, we tackle in *iii*) a novel variational model for single-slip polycrystalline finite plasticity. The goal here is to describe the deformation behavior of a collection of rotated copies of single crystals, called the grains of the polycrystal. The primary challenge is the solvability of a specific inhomogeneous, non-convex differential inclusion with affine boundary values representing the admissible macroscopic strains. We determine necessary conditions by combining well-known relaxation and convex integration results with a new characterization of globally affine solutions to a relaxed differential inclusion. Sufficient conditions are derived from the generalized Hadamard jump theory and the resulting compatibility along the boundary grains. Under suitable assumptions on the geometry, that is, the orientation and shape of the grains, the necessary and sufficient conditions coincide and yield a full description of the macroscopic deformation behavior of the polycrystal.

Samenvatting

De ontwikkeling van nieuwe high-tech materialen is een cruciaal aspect van de technische wetenschappen en heeft betekenis in vele takken van de industrie. Betrouwbare voorspelling van materiaalgedrag vereist gewoonlijk de beschrijving van materiaaleigenschappen op meerdere lengteschalen (bijvoorbeeld de microscopische, mesoscopische of macroscopische schaal). In het verleden is het bestuderen van op energie gebaseerde multischaalmodellen opmerkelijk succesvol gebleken vanwege hun ontvankelijkheid voor hulpmiddelen in de variatierekening. De standaard variationele theorieën zijn echter onvoldoende om een dieper inzicht te krijgen in geavanceerde mechanische kenmerken, vaak gemodelleerd door middel van niet-convexe (inhomogene) differentiaalvoorwaarden. Het is daarom essentieel om nieuwe technieken voor multischaalmodellen met differentiaalvoorwaarden te combineren met diepgaande kennis van de klassieke variationele theorieën en de materiaalwetenschap. In deze geest presenteert het proefschrift nieuwe analytische resultaten voor drie op energie gebaseerde modellen waarvan de gemeenschappelijke deler de voorwaarde van niet-samendrukbaarheid is:

- i)* de afleiding van eendimensionale modellen voor driedimensionale niet-samendrukbare dunne structuren;
- ii)* het effectieve vervormingsgedrag van elastische lichamen versterkt met stijve vezels;
- iii)* het materiaalgedrag van polykristallen in een model van single-slip eindige kristalplasticiteit.

Om de nodige achtergrond te bieden, beginnen we met een uitgebreid overzicht van standaard variationele benaderingen van hyperelastische materialen, die kunnen worden geanalyseerd via elastische energiefunctionalen. We bespreken de klassieke theorieën voor het bestaan van minima via de directe methode en het asymptotische gedrag van lage-energietoestanden via Γ -convergentie. We illustreren dit laatste door middel van twee opmerkelijk relevante toepassingen voor dit proefschrift: dimensiereductie en homogenisatie.

In *i)* richten we ons op de dimensiereductie van modellen voor lokaal volumebehoudende dunne snaren en staven. Met behulp van Γ -convergentie leiden we lager-dimensionale modellen af die effectief het vervormingsgedrag van het driedimensionale lichaam beschrijven voor de hele hiërarchie van schalingsregimes van de externe krachten. We maken onderscheid tussen regimes die leiden tot zeer flexibele snaarmodellen en alle resterende schalingsregimes, waaruit we Kirchhoff- en von Kármán-type staaftheorieën afleiden. In alle fysiek relevante limietmodellen vinden we dat niet-samendrukbaarheid alleen de elastische energie zelf (en dus zijn minima) beïnvloedt en niet de verzameling van toelaatbare vervormingen. We overwinnen de differentiaalvoorwaarde van lokaal volumebehoud door bestaande technieken, zoals een innerlijk perturbatieargument, geschikte (voorwaarde behoudende) mollificatie en benadering via straftermen, af te stemmen op deze nieuwe modellen.

We karakteriseren in *ii)* het effectieve vervormingsgedrag van hyperelastische materialen versterkt door parallelle, lange, dunne, stijve vezels. Binnen een geschikt homogenisatiekader

komen de macroscopische reacties op externe krachten overeen met zwakke Sobolev-limieten van functies die voldoen aan de uitdagende inhomogene differentiaalvoorwaarde van stijfheid op de vezels. Terwijl noodzakelijke voorwaarden voor het macroscopische vervormingsgedrag op dezelfde manier volgen als in eerdere werken over gelaagde composieten, worden voldoende voorwaarden verkregen via een zorgvuldige benadering van de identiteit in twee dimensies die de differentiaalvoorwaarde accommodeert. Het blijkt dat het materiaalgedrag van vezelcomposieten zeer anisotroop is in die zin dat de vervorming in de richting van de vezels een eenheidslengte en een hogere regulariteit heeft en alleen afhangt van de doorsnedevariabelen. We illustreren verschillende voorbeelden van toelaatbare vervormingen en onderscheiden dit resultaat van eerder werk over elastische lichamen versterkt door stijve lagen; in feite vertoont ons model meer flexibiliteit vanwege de hogere co-dimensie van de stijve delen en de resulterende verbondenheid van het zachte materiaal. Dit werk is de eerste fundamentele stap naar een volledig homogenisatiere-sultaat van hyperelastische lichamen versterkt door volledig stijve vezels.

Gebaseerd op eerder werk aan eenkristal eindige plasticiteit met één actief slipsysteem, behandelen we in *iii*) een nieuw variationeel model voor single-slip polykristallijne eindige plasticiteit. Het doel hier is om het vervormingsgedrag te beschrijven van een verzameling geroteerde kopieën van eenkristallen, de korrels van het polykristal genoemd. De primaire uitdaging is de oplosbaarheid van een specifieke inhomogene, niet-convexe differentiaalinclusie met affine randvoorwaarden die de toelaatbare macroscopische vervormingen vertegenwoordigen. We bepalen de noodzakelijke voorwaarden door bekende resultaten van relaxatie en convexe integratie te combineren met een nieuwe karakterisering van globale affine oplossingen voor een gerelaxeerde differentiaalinclusie. Voldoende voorwaarden zijn afgeleid van de gegeneraliseerde Hadamard-sprongtheorie en de resulterende compatibiliteit langs de grenskorrels. Onder geschikte veronderstellingen over de geometrie, dat wil zeggen de oriëntatie en vorm van de korrels, vallen de noodzakelijke en voldoende voorwaarden samen en geven een volledige beschrijving van het macroscopische vervormingsgedrag van het polykristal.

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Curriculum Vitae

Dominik Engl was born on January 23, 1994 in Regensburg, Germany. In the years 2010-2012, he participated in the program “Frühstudium” in Mathematics at Universität Regensburg intended for talented high school students. After that, he started his bachelor studies in Mathematics with a minor in Physics at the same university and completed his Bachelor thesis titled *Der Auswahlssatz von Helly und seine Bedeutung für Existenzsätze von energetischen Lösungen* under the supervision of Georg Dolzmann and Carolin Kreisbeck. In 2015, he continued with a master’s in Mathematics with a minor in Actuarial Science, writing his thesis on *Dimension reduction via variational convergence in models of nonlinear elasticity* with the same supervisors.

In September 2017, he began his Ph.D. project *Variational multiscale problems with differential constraints* under Sjoerd Verduyn Lunel and Carolin Kreisbeck at Utrecht University. He attended several international conferences, workshops, and schools during these years. These visits led to a collaboration with Martin Kružík and Stefan Krömer and a research visit in early 2020 to the Institute of Information and Automation (UTIA) in Prague (funded by an NDNS+ travel grant), resulting in preliminary results of one further research project not included in this thesis. Furthermore, he co-organized the *Calculus of Variations on Schiermonnikoog* workshop in 2019. Besides his research duties, he also assumed the role of teaching assistant in several mathematics courses, where he particularly enjoyed his active involvement in the complete redesign of the course *Wiskundig modelleren*.

He started working as a Research Associate in Mathematics at Katholische Universität Eichstätt-Ingolstadt in Eichstätt in September 2021.

