

A Faster Exponential Time Algorithm for Bin Packing With a Constant Number of Bins via Additive Combinatorics

Jesper Nederlof* Jakub Pawlewicz† Céline M. F. Swennenhuis‡
Karol Węgrzycki§

Abstract

In the Bin Packing problem one is given n items with weights w_1, \dots, w_n and m bins with capacities c_1, \dots, c_m . The goal is to find a partition of the items into sets S_1, \dots, S_m such that $w(S_j) \leq c_j$ for every bin j , where $w(X)$ denotes $\sum_{i \in X} w_i$.

Björklund, Husfeldt and Koivisto (SICOMP 2009) presented an $\mathcal{O}^*(2^n)$ time algorithm for Bin Packing. In this paper, we show that for every $m \in \mathbb{N}$ there exists a constant $\sigma_m > 0$ such that an instance of Bin Packing with m bins can be solved in $\mathcal{O}(2^{(1-\sigma_m)n})$ randomized time. Before our work, such improved algorithms were not known even for m equals 4.

A key step in our approach is the following new result in Littlewood-Offord theory on the additive combinatorics of subset sums: For every $\delta > 0$ there exists an $\varepsilon > 0$ such that if $|\{X \subseteq \{1, \dots, n\} : w(X) = v\}| \geq 2^{(1-\varepsilon)n}$ for some v then $|\{w(X) : X \subseteq \{1, \dots, n\}\}| \leq 2^{\delta n}$.



*Utrecht University, The Netherlands, j.nederlof@uu.nl. Supported by the project CRACKNP that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 853234).

†Institute of Informatics, University of Warsaw, Poland, pan@mimuw.edu.pl.

‡Eindhoven University of Technology, The Netherlands, c.m.f.swennenhuis@tue.nl. Supported by the Netherlands Organization for Scientific Research under project no. 613.009.031b.

§Institute of Informatics, University of Warsaw, Poland, k.wegrzycki@mimuw.edu.pl. Supported by the grants 2016/21/N/ST6/01468 and 2018/28/T/ST6/00084 of the Polish National Science Center and project TOTAL that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 677651).

1 Introduction

A central aim in contemporary algorithm design is to minimize the worst-case complexity of an algorithm for a given (supposedly) hard computational problem in a fine-grained sense. The underlying goal is to reveal the optimal runtime witnessed by (1) an algorithm with worst-case complexity $T(n)$ on instances with parameter n , and (2) a lower bound that excludes improvements to $T(n)^{1-\varepsilon}$ time for some constant $\varepsilon > 0$. For some problems, it is an especially intriguing question whether natural runtimes of the basic algorithms solving them are optimal. One of the most important instances of such a question for an NP-complete problem is about improvements over a relatively direct¹ dynamic programming algorithm for Set Cover:

Question 1: Can Set Cover with n elements be solved in $\mathcal{O}^*((2 - \varepsilon)^n)$ time, for some $\varepsilon > 0$?

Unfortunately, Question 1 seems to have a fate similar to the Strong Exponential Time Hypothesis (that is about a similar improvement for the CNF-SAT problem): While there is an increasing interest and dependence on its validity (see e.g. [15, 39]), we seem to be far from resolving it.

Therefore, it is natural to study Question 1 for special cases of Set Cover. And indeed, improved algorithms of the type asked in Question 1 were already presented for instances with small sets [38], (more generally) large solutions [45], and for several other cases (see e.g. [27]).

However, some of the most fundamental NP-complete problems that are special cases of Set Cover such as Graph Coloring and Directed Hamiltonicity² still defy considerable research efforts to obtain the type of improved algorithms asked for in Question 1 (see e.g. [10, 22]).

Bin Packing We study one of such a fundamental NP-complete problem, the *Bin Packing problem*: Given item weights $w(1), \dots, w(n) \in \mathbb{N}$ and capacities $c_1, \dots, c_m \in \mathbb{N}$, is there a partition S_1, \dots, S_m of $[n]$ such that $w(S_j) \leq c_j$ for each $j \in [m]$? Here $w(X)$ denotes $\sum_{i \in X} w(i)$. Due to its elegant formulation and clear practical applicability, Bin Packing is a central problem in computer science. For example, it models the most basic non-trivial scheduling problem with multiple machines. While Bin Packing has been extensively studied from an approximation and online algorithms perspective [13], much less research has been devoted to *exact algorithms* for Bin Packing.

The currently fastest algorithm for Bin Packing is a consequence³ of the aforementioned algorithm for Set Cover from [9], and it runs in $\mathcal{O}^*(2^n)$ time. With Question 1 on the horizon, we ask whether this can be improved:

Question 2: Can Bin Packing with n items be solved in $\mathcal{O}((2 - \varepsilon)^n)$ time, for some $\varepsilon > 0$?

The only improvement over the $\mathcal{O}^*(2^n)$ time algorithm for Bin Packing is due to Lente et al. [40], who gave an $\mathcal{O}^*(m^{n/2})$ time algorithm. Note, that this is only an improvement for $m = 2, 3$ bins and Question 2 remained illusive for $m = 4$ already. In stark contrast, our main result is an improvement over the $\mathcal{O}^*(2^n)$ time algorithm for *every* constant number of bins:

Theorem 1.1 (Main Theorem). *For every $m \in \mathbb{N}$ there is a constant $\sigma_m > 0$ such that every Bin Packing instance with m bins can be solved in $\mathcal{O}(2^{(1-\sigma_m)n})$ time with high probability.*

While our algorithm does not resolve Question 2, we believe it makes substantial progress on it because (1) Set Cover with a constant-sized solution is as least as hard as general Set Cover, and (2) the other extreme, Set Cover with a linear number sets in the solution (and hence Bin Packing with a linear number of bins with equal capacity³), can be solved in $\mathcal{O}((2 - \varepsilon)^n)$ time (see [45]).

¹In principle, it is natural to assume the Set Cover instance has n elements and $\text{poly}(n)$ sets, but an algorithm by Björklund et al. [9] solves Set Cover instances in $\mathcal{O}^*(2^n)$ time irrespective of the number of sets.

²Krauthgamer and Trabelsi [39] rewrite a Directed Hamiltonicity instance efficiently as a Set Cover instance.

³Assuming the capacity of each bin equals c , create a Set Cover instance with all item sets of weight at most c .

1.1 Our Approach for Proving Theorem 1.1

As our starting point, we extend the methods from [8, 45] to show that instances of Bin Packing with the following restrictions admit an $\mathcal{O}^*(2^{(1-\sigma_m)n})$ time randomized algorithm for some $\sigma_m > 0$:

- (R1) the instance has anti-concentrated subset sums in the sense that $\beta(w) \leq 2^{(1-\varepsilon)n}$ for some $\varepsilon > 0$, where $\beta(w) := \max_v |\{X \subseteq \{1, \dots, n\} : w(X) = v\}|$, and
- (R2) the instance is *tight* in the sense that $\sum_{j=1}^m c_j = w([n])$.

Fix a set of bins $L \subseteq [m]$ and recall (S_1, \dots, S_m) denotes a solution. The crux of (R1) and (R2) is that they together imply that the number of candidates for $S^L := \cup_{j \in L} S_j$ is at most $2^{(1-\varepsilon)n}$ since $w(S^L) = \sum_{j \in L} c_j$. We explain in § 1.1.3 how this allows a faster algorithm via the methods of [8, 45].

However, extending this algorithm to an improved algorithm that solves *all* instances with a constant number of bins requires both new combinatorial (for relaxing (R1)) and new algorithmic (for relaxing (R2)) insights that are our main contributions. Therefore we first discuss these insights.

1.1.1 Combinatorial Ideas: Lifting Restriction (R1) via Littlewood-Offord Theory.

Our main combinatorial contribution is a new structural insight on instances that do not satisfy (R1), i.e. vectors w with $|\{X \subseteq \{1, \dots, n\} : w(X) = v\}| \geq 2^{(1-\varepsilon)n}$ for some v and $\varepsilon > 0$.

Determining the structure of such vectors w is well-known in additive combinatorics as the *Littlewood-Offord Problem*. Its rich theory has found applications ranging from pure mathematics (such as estimating the singularity of random Bernoulli matrices [51] or zeroes of random polynomials [41]), to database security [28], and to complexity theory [19, 34, 43]. See also the designated chapter in the standard textbook on additive combinatorics [50]. However, whereas most works (with notable exceptions being e.g. [29, 48]) assumed inversely *polynomially* small concentration, e.g. $\beta(w) \geq 2^n/n^{\mathcal{O}(1)}$, restriction (R1) is about inversely *exponentially* small concentration.

Recent work studied such exponentially small concentration with applications to improved exponential time algorithms for the Subset Sum problem [2, 5]. Specifically, they studied trade-off between the parameters $\beta(w)$ and $|w(2^{[n]})| := |\{w(X) : X \subseteq [n]\}|$. Two extremal cases are:

$$\begin{array}{lll} \text{If } w_a := (0, 0, \dots, 0) & \text{then} & |w_a(2^{[n]})| = 1 \text{ and } \beta(w_a) = 2^n \\ \text{If } w_b := (1, 2, \dots, 2^{n-1}) & \text{then} & |w_b(2^{[n]})| = 2^n \text{ and } \beta(w_b) = 1 \end{array}$$

One may suspect that all vectors $w \in \mathbb{Z}^n$ are a combination of these two extremes and therefore that a smooth trade-off between $\beta(w)$ and $|w(2^{[n]})|$ can be proved. This suspicion can be confirmed⁴ in the case $w \in \mathbb{F}_2^n$ where $|w(2^{[n]})|\beta(w) = 2^n$. Observe that a similar trade-off for $w \in \mathbb{Z}^n$ would allow us to lift (R1) by a simple algorithm $\mathcal{O}^*(|w(2^{[n]})|^m)$ time algorithm for Bin Packing (Lemma 3.4).

Unfortunately, this intuition is not true and the case $w \in \mathbb{Z}^n$ is far more subtle. For instance, Wiman [54] showed in his remarkable bachelor thesis that, surprisingly, vectors satisfying simultaneously both $|w(2^{[n]})| \geq 2^{(1-\varepsilon)n}$ and $\beta(w) \geq 2^{0.2563n}$ exist for any $\varepsilon > 0$. Our main combinatorial contribution is that instances with the same parameters but the roles of $\beta(w)$ and $|w(2^{[n]})|$ swapped do *not* exist:

Theorem 1.2. *Let $\varepsilon > 0$. If $\beta(w) \geq 2^{(1-\varepsilon)n}$, then $|w(2^{[n]})| \leq 2^{\delta n}$, where $\delta(\varepsilon) = \mathcal{O}_{\varepsilon \rightarrow 0} \left(\frac{\log(\log(1/\varepsilon))}{\sqrt{\log(1/\varepsilon)}} \right)$.*

⁴So $w(i)$ is a n -dimensional binary vector for every i . Then $|w(2^{[n]})| = 2^{\text{rk}_2(w)}$ and $\beta(w) = 2^{n-\text{rk}_2(w)}$ where w is interpreted as a matrix by concatenating the vectors $w(1), \dots, w(n)$ and rk_2 is the rank over \mathbb{F}_2^n .

The previous best bounds were given by Austrin et al. [2] who found a connection with *Uniquely Decodable Code Pairs (UDCPs)* from information theory (see Subsection 1.2 for details). This implies for example that if $\beta(w) \geq 2^{(1-\varepsilon)n}$, then $|w(2^{[n]})| \leq 2^{0.4228n + \sqrt{\varepsilon}}$ by a result on UDCPs from [4]. However, the reduction from [2] is symmetric with respect to swapping the roles of $\beta(w)$ and $w(2^{[n]})$, and thus by the result from [54] UDCP techniques alone are not enough to decrease the constant 0.4228 beyond 0.2563.

Therefore, we need a new ideas to reduce the constant 0.4228 to an arbitrarily small one. To do so, we first investigate the combinatorial structure of the hyperplane $H := \{x \in \mathbb{Z}^n : \langle w, x \rangle = v\}$, assuming $|H \cap \{0, 1\}^n| \geq 2^{(1-\varepsilon)n}$. Afterwards we apply an argument similar to the UDCP connection from [2]. We formally describe our approach for proving Theorem 1.2 in Section 4.

Note that Theorem 1.2 enables us to lift **(R1)**: We may assume $\beta(w) \leq 2^{(1-\varepsilon_m)n}$ where $\varepsilon_m > 0$ depends on m since otherwise the dynamic programming $\mathcal{O}^*(|w(2^{[n]})|^m)$ algorithm will be fast enough (see Lemma 3.4)

1.1.2 New Algorithmic Ideas: Lifting Restriction (R2)

As mentioned before, **(R2)** is algorithmically useful because of the following reason: We aim to detect a solution S_1, \dots, S_m to the Bin Packing instance by listing all candidates for $S^L := \bigcup_{j \in L} S_j$ for some $L \subseteq [m]$, and **(R2)** implies that $w(S^L) = \sum_{j \in L} c_j$. This allows us to narrow down the number of candidates to $2^{(1-\varepsilon)n}$ by **(R1)** (we explain in §1.1.3 why this is useful). Note this even narrows down the number of candidates for S^L if all bins have polynomially bounded *slack*, i.e., $c_j - w(S_j) \leq n^{\mathcal{O}(1)}$ since the number of possibilities of $w(S^L)$ is only $n^{\mathcal{O}(1)}$ as $m = \mathcal{O}(1)$.

But generally this strategy does not work whenever a bin has a large *slack*, that is when $c_j - w(S_j)$ is large. While reductions in several similar situations were able to turn inequalities into equalities via general rounding techniques (such as [46, 53]), we need a more sophisticated method in this paper to deal with this issue: The idea of [46] is to divide the weights by roughly $c_j - w(S_j)$ and (conservatively) round to an integer. In this case, the bin j has small slack with respect to the rounded weight function. The major complication however is that for different bins we would then need to work with differently rounded weight functions, which still does not allow us to narrow down the number of options for $w(S^L)$ and hence (via **(R1)**) S^L .

Instead we work with a rounded version w_θ of weights w where $w_\theta(i)$ is obtained from $w(i)$ by only keeping the θ most significant bits. We will show we can choose θ such that $|w_\theta(2^{[n]})| \approx 2^{\delta n}$, for some parameter δ that depends on m . We will deal with the bins in two different ways depending on whether it has large slack (i.e. is at least approximately $n2^{l-\theta}$, assuming all weights are l -bit integers) or not:

- **Large Slack Bins:** In this case, our idea is loosely inspired by rounding approximation algorithms, for e.g. for Knapsack (see e.g. [37, Section 11.8]). Observe that if some bin has large slack, we can split it in two parts. Then, we only need to keep track of the rounded weight of these parts in order to determine whether they jointly fit into the bin. Because we assumed the *upper bound* $|w_\theta(2^{[n]})| \lesssim 2^{\delta n}$ we can afford to keep track of all combinations of rounded weights as long as $\delta < 1/m$.
- **Small Slack Bins:** In this case we have a split of the bins (L, R) and all bins in L have small slack. Now we use the *lower bound* $|w_\theta(2^{[n]})| \gtrsim 2^{\delta n}$ and our additive combinatorics result guarantees $\beta(w_\theta) \leq 2^{(1-\varepsilon(\delta))n}$ for some $\varepsilon(\delta) > 0$. Now, we use the fact that all bins have small slack. Note, that there are only $n^{\mathcal{O}(1)}$ candidates for $w_\theta(S^L)$ and therefore there are at most $n^{\mathcal{O}(1)} 2^{(1-\varepsilon(\delta))n}$ candidates for S^L , which can be algorithmically exploited.

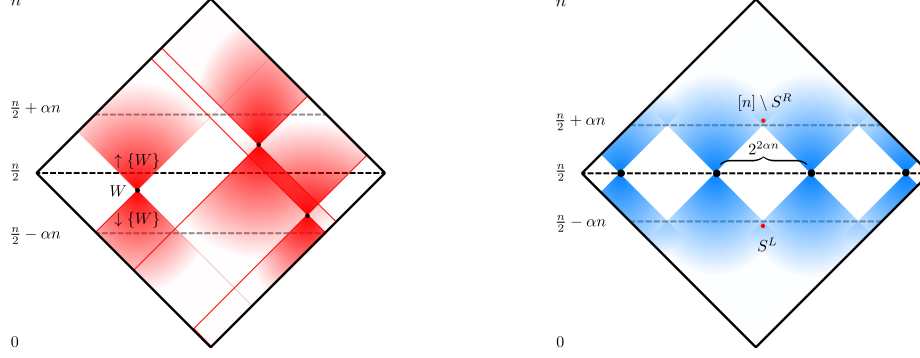


Figure 1: Schematic view of the algorithm from §1.1.3. A point in the square represents a set in $2^{[n]}$. The vertical axis corresponds to the cardinality of this set (e.g., longest horizontal line represents all sets in $\binom{[n]}{n/2}$). The left figure illustrates the analysis for the case when there exists an α -balanced solution $W \in \binom{[n]}{n/2 \pm \alpha n}$. We iterate through all W in time proportional to area the of the red region. The right figure illustrates the case of an α -unbalanced solution. A division of the solution (L, R) is witnessed by the roughly $2^{2\alpha n}$ sets W in $\binom{[n]}{n/2}$ satisfying $S^L \subseteq W \subseteq [n] \setminus S^R$.

In this informal discussion, we omitted several nontrivial technical issues. For example, in order to deal with instances with *both* a substantial number of small slack bins and large slack bins, we need to distinguish a number of cases. Due to the subtle technical issues, we need to deal with each one of them in slightly different ways. Details are postponed to Section 3.

1.1.3 Solving Instances Satisfying (R1) and (R2).

We now discuss how the methods from [8, 45] can be used to solve all instances that satisfy Restrictions **(R1)** and **(R2)** in $\mathcal{O}^*(2^{(1-\sigma_m)n})$ for some $\sigma_m > 0$. An important subroutine from [8] is an algorithm that, given a set family $\mathcal{W} \subseteq 2^{[n]}$ and set of bins L , computes for all $W \in \mathcal{W}$ whether the items in W can be divided among the bins in L . That is, it computes whether W can be a candidate for $S^L = \cup_{j \in L} S_j$. The runtime of this algorithm is $\mathcal{O}(|\downarrow \mathcal{W}|n)$, where $\downarrow \mathcal{W} := \{X \subseteq W : W \in \mathcal{W}\}$ is defined as the *down-closure* of \mathcal{W} . The analogous *up-closure* of all supersets of elements from \mathcal{W} is denoted with $\uparrow \mathcal{W}$. Let us fix a solution (S_1, \dots, S_m) . We consider two cases based on how ‘balanced’⁵ as solution is, with respect to a small parameter $\alpha \in [0, 1]$:

- **There is a $b \in [m]$ such that $\sum_{j=1}^b |S_j| \in [n/2 \pm \alpha n]$.** Observe $\bigcup_{j=1}^b S_j$ is an element of

$$\mathcal{W} := \left\{ Y \subseteq [n] : w(Y) = \sum_{j=1}^b c_j, |Y| \in \left[\frac{n}{2} \pm \alpha n \right] \right\},$$

and that $|\mathcal{W}| \leq \beta(w) \leq 2^{(1-\varepsilon)n}$ by **(R1)**. Now we can enumerate \mathcal{W} in essentially $2^{(1-\varepsilon)n}$ time. We will present an $\mathcal{O}((|\downarrow \mathcal{W}| + |\uparrow \mathcal{W}|)n)$ time algorithm that for each $W \in \mathcal{W}$ computes whether W can be divided among bins $1, \dots, b$ and $[n] \setminus W$ among bins $b+1, \dots, m$ that is based on techniques from [8]. This will detect a solution if it exists. We bound the running time using the property $|W| \in [n/2 \pm \alpha n]$. In this case we will show $|\downarrow \mathcal{W}| + |\uparrow \mathcal{W}| \leq 2^{(1-\varepsilon')n}$ and hence the algorithm is fast enough (see left Figure 1 for an illustration).

⁵The actual definition of α -balancedness (Definition 3.3) will be independent of the ordering of the bins.

- **There exists no $b \in [m]$ such that $\sum_{j=1}^b |S_j| \in [n/2 \pm \alpha n]$.** Here we can use a method from [45]: We let \mathcal{W} consist of $2^{(1-2\alpha)n}$ independently sampled subsets of $[n]$ of cardinality $n/2$. We answer *yes* if there exist $W \in \mathcal{W}$, disjoint sets $S^L = S'_1, \dots, S'_{b-1} \subseteq W$ and $S^R = S'_{b+1}, \dots, S'_m \subseteq [n] \setminus W$ such that $w(S'_j) = c_j$ for all $j \in [m] \setminus \{b\}$. This condition can also be computed in $\mathcal{O}((|\downarrow \mathcal{W}| + |\uparrow \mathcal{W}|)n)$ time by the methods of [8]. The crux is that both conditions together imply our instance is a *yes*-instance, since the remaining elements have total weight c_b by Restriction **(R2)**. Moreover, by the balancedness assumption at least $2^{2\alpha n}$ sets $W \subseteq [n]$ with the above conditions exist. Therefore the random sampling will include such a W with good probability (see right Figure 1 for an illustration).

1.2 Related Work

Littlewood Offord, UDCP's, and Exponential Time Algorithms. Two sets $A, B \subseteq \{0, 1\}^n$ form a Uniquely Decodable Code Pair (UDCP) if $|A + B| = |A| \cdot |B|$, where $A + B := \{a + b : a \in A, b \in B\}$ (and addition is in \mathbb{Z}^n). The maximal sizes of UDCP's have been very well studied in information theory. See e.g. [49, Section 3.5.1] for a (not so recent) overview. Two record upper bounds are $|A| \cdot |B| \leq 2^{1.5n}$ (from [52]) and $|A| \leq 2^{(0.4228 + \sqrt{\varepsilon})n}$ whenever $|B| \leq 2^{(1-\varepsilon)n}$ (from [4]). The study of UDCP's is relevant for this paper by the following connection shown in [3]: For any vector $w \in \mathbb{Z}^n$, there is a UDCP $A, B \subseteq \{0, 1\}^n$ such that $|A| = |w(2^{[n]})|$ and $|B| = \beta(w)$.

A study of the trade-off between the parameters $|w(2^{[n]})|$ and $\beta(w)$ was already fruitful for obtaining improved exponential time algorithms in two earlier papers in the context of the Subset Sum problem. In this problem one is given $w \in \mathbb{Z}^n$ and a target integer t and one needs to find a subset $X \subseteq [n]$ such that $w(X) = t$. First, the aforementioned paper [3] combined their connection to UDCP's with the bound from [52] to show that instances of Subset Sum satisfying $|w(2^{[n]})| \geq 2^{0.997n}$ can be solved in $\mathcal{O}(2^{0.49991n})$ time, thereby improving the best $\mathcal{O}^*(2^{n/2})$ worst case run time from [31] for these instances. Second, a slight variant of the trade-off was used in [5] to give a $\mathcal{O}(2^{0.86n})$ time algorithm that uses a random oracle and only a polynomial in the input size amount of working memory.

Exact Algorithms with Minimum Worst Case Run Time for Set Cover. Question 1 was for the first time explicitly posed in [16], who showed that a no answer to (a variant of) the question implies hardness in a fine-grained sense for the Subset Sum, Steiner Tree, and Connected Vertex Cover problems. A main motivation in [16] for posing the question was a curious reduction showing that there is no improved algorithm for counting the number of Set Cover solutions modulo 2 unless improved algorithms for CNF-Sat exist (i.e. the Strong Exponential Time Hypothesis fails). Later the assumption that no improved algorithm exists was dubbed as ‘Set Cover Conjecture’ (see e.g. [17, Conjecture 14.36]). Since then, the conjecture has been used in several works, e.g. in [1, 39].

On the positive side, (especially for this work) important algorithmic tools were developed in [9]: Fast zeta/Möbius transformations were introduced in the area of exponential time algorithms to show that Set Cover can be solved in $2^n \cdot n^{\mathcal{O}(1)}$ even when the number of sets in the input is exponential in n . One major consequence was a $2^n \cdot n^{\mathcal{O}(1)}$ time algorithm for computing whether an input graph on k vertices has a proper coloring with k colors. While for $k = 3, 4$ faster algorithms exist (see e.g. [11]), this is still the fastest algorithm for $k > 4$.

Improved algorithms for solving Set Cover instances of sets with bounded cardinality were given in [38]. Later, this was generalized to improved algorithms for Set Cover instances where the optimum is linear in the universe size [45]. Other instances that allow improved algorithm were also presented in e.g. [26].

Exact Algorithms with Minimum Worst Case Run Time for Bin Packing. In a textbook on exact exponential time algorithms it was shown that Bin Packing can be solved in time $\mathcal{O}(n \max_i w(i) 2^n)$ [23, Section 4.2.3]. A faster algorithm $\mathcal{O}^*(2^n)$ time algorithm was given in [9]. Even faster algorithms were given for $m = 2, 3$ in [40].

In [25] it was shown that Bin Packing can be solved in polynomial time if there are only a constant number of distinct items weights. In [32] Bin Packing with a constant number of bins and bounded items weights was studied. A Dynamic Programming algorithm similar to the one from 3.4 was studied: It was observed the algorithm runs in time $n^{\mathcal{O}(m)}$ if the items are polynomial in n . The authors show this run time cannot be improved to $n^{o(m/\log m)}$, unless the Exponential Time Hypothesis fails.

Heuristics for Bin Packing. The applications and combinatorial properties of Bin Packing have been studied since the 1930's [35]. To the best of our knowledge the first attempt to exactly solve Bin Packing with assistance of the modern computer was developed in the fifties by Eisemann [21], with motivation to trim losses in cutting rolls of paper. Starting from the seventies, the research on exact algorithms for Bin Packing focused on the branch-and-bound technique proposed by Eilon and Christofides [20]. These heuristics work great in practice. Nevertheless, there are no theoretical guarantees on their worst case performance.

For a modern survey and experimental evaluations of the available software see [42, 18].

Approximation Algorithms for Bin Packing. Bin Packing is one of the problems that initiated the study of approximation algorithms. The earliest one is the *First Fit* algorithm analysed by Johnson [33] that requires at most $1.7 \cdot \text{OPT} + 1$ bins. The major breakthrough was done by Karmarkar-Karp [36] who provided a polynomial time algorithm that requires at most $\text{OPT} + \mathcal{O}(\log^2(\text{OPT}))$ bins. Recently, a big leap forward was done by Rothvoß [47] who gave a polynomial time algorithm that requires only $\text{OPT} + \mathcal{O}(\log(\text{OPT}) \log \log(\text{OPT}))$ bins and Hoberg and Rothvoß [30] who improved this even further to $\text{OPT} + \mathcal{O}(\log(\text{OPT}))$ bins.

1.3 Organization

This paper is organized as follows: In Section 2 we present some preliminaries and introduce some notations. In Section 3 we present the algorithm and proof of our main theorem, assuming Theorem 1.2. The latter theorem is proved in the next two Sections 4 and 5.

2 Preliminaries

All the logarithms are base 2 unless stated otherwise. In this paper we assume that basic arithmetic operations take constant time. We use a result of Frank and Tardos [24], in a similar way to [?], to assume that $\max_i w(i) \leq 2^{n^{\mathcal{O}(1)}}$. Throughout the paper we use the \mathcal{O}^* notation to hide polynomial factors and the $\tilde{\mathcal{O}}$ notation to hide logarithmic factors. The number of bins is assumed to be constant, i.e. $m = \mathcal{O}(1)$. We say a function $f(\varepsilon) = \mathcal{O}_{\varepsilon \rightarrow 0}(g(\varepsilon))$ if there exists a positive number C and sufficiently small $\varepsilon_0 > 0$, such that $|f(\varepsilon)| \leq C \cdot g(\varepsilon)$ for all $\varepsilon < \varepsilon_0$. We use $\Omega_{\varepsilon \rightarrow 0}$ similarly to express lower bounds.

We use $[n]$ to denote the set $\{1, \dots, n\}$. If $a, b \in \mathbb{R}$ and $b \geq 0$ we let $[a \pm b]$ denote the interval $[a - b, a + b]$. If A and B are sets, we denote B^A as the set of vectors indexed by A with values from B , and we will interchangeably address these vectors as functions from A to B . If $f \in B^A$ and

$b \in B$ we denote $f^{-1}(b) := \{a \in A : f(a) = b\}$ for its inverse evaluated at b . If $x, y \in \mathbb{R}^A$ we denote $\langle x, y \rangle := \sum_{a \in A} x_a \cdot y_a$ for their inner product.

To quickly refer to properties of a solution of a Bin Packing instance we use the following notations: The function w indicated the weights of the input. It is extended to sets $X \subseteq [n]$ by defining $w(X) := \sum_{i \in X} w(i)$ and to set families $\mathcal{F} \subseteq 2^{[n]}$ by defining $\{w(X) : X \in \mathcal{F}\}$. We say a set $X \subseteq [n]$ of items can be *divided* over bins $L \subseteq [m]$ if there is a partition $X_1, \dots, X_{|L|}$ of X , such that for all $j \in \{1, \dots, |L|\}$, the set X_j can be placed in bin j , i.e., $w(X_j) \leq c_j$.

2.1 Preliminary Tools: Fast Transformations

Our algorithm will crucially relies on the following algorithmic tools and definitions tools from [8].

Definition 2.1 (Zeta and Möbius Transform). *Let $f : 2^U \rightarrow \mathbb{N}$. Then the Zeta-transform ζf and Möbius transform μf are functions from 2^U to \mathbb{N} such that for every $X \subseteq U$:*

$$(\zeta f)(X) := \sum_{Y \subseteq X} f(Y) \quad (\mu f)(X) := \sum_{Y \subseteq X} (-1)^{|U \setminus Y|} f(Y)$$

Definition 2.2. *Given $\mathcal{S} \subseteq 2^U$, the down-closure $\downarrow \mathcal{S}$ and up-closure $\uparrow \mathcal{S}$ are defined as follows:*

$$\downarrow \mathcal{S} := \{X \mid \exists S \in \mathcal{S} : X \subseteq S\} \quad \uparrow \mathcal{S} := \{X \mid \exists S \in \mathcal{S} : X \supseteq S\}.$$

Theorem 2.3 (Fast Zeta/ Möbius transform [8]). *Suppose that $f : 2^U \rightarrow \mathbb{N}$ is such that $f(X)$ can be evaluated in T time for any given $X \subseteq U$, and let $\mathcal{S} \subseteq 2^U$ be a set family. There is an algorithm that can compute for every $X \in \downarrow \mathcal{S}$ the values $(\zeta f)(X)$ and $(\mu f)(X)$. The algorithm runs in $O(|\downarrow \mathcal{S}| |U| T)$ time.*

Definition 2.4 (Cover and Dot Product). *Given $f, g : 2^U \rightarrow \mathbb{N}$, the cover product $f *_c g = h$ and the dot product $f \cdot g = h'$ are the functions $h : 2^U \rightarrow \mathbb{N}$ such that*

$$h(Z) := \sum_{X \cup Y = Z} f(X)g(Y) \quad h'(Z) := f(Z) \cdot g(Z).$$

Theorem 2.5 ([7]). $\mu((\zeta f) \cdot (\zeta g)) = f *_c g$.

Theorem 2.6. *Suppose that we have a Bin Packing instance with bin capacities c_1, \dots, c_m and item weight function w . Then for any $B \subseteq [m]$ and set $\mathcal{W} \subseteq 2^{[n]}$, computing for all $X \in \downarrow \mathcal{W}$ whether X can be divided over the bins in B can be done in time $O(|\downarrow \mathcal{W}| n)$. Similar, for any $B \subseteq [m]$ and set $\mathcal{W} \subseteq 2^{[n]}$, computing for all $X \in \uparrow \mathcal{W}$ whether $[n] \setminus X$ can be divided over the bins in B can be done in time $O(|\uparrow \mathcal{W}| n)$.*

Proof. For all $j = 1, \dots, m$ define a function $f_j : 2^{[n]} \rightarrow \{0, 1\}$ as

$$f_j(X) = \begin{cases} 1, & \text{if } w(X) \leq c_j \\ 0, & \text{otherwise.} \end{cases}$$

Assume without loss of generality that $B = \{1, \dots, d\}$. Notice that X can be divided over the bins in B if and only if $(f_1 *_c f_2 *_c \dots *_c f_d)(X) > 0$. By Theorem 2.5 we have that

$$f_1 *_c f_2 *_c \dots *_c f_d = \mu((\zeta f_1) \cdot (\zeta f_2) \cdot \dots \cdot (\zeta f_d)).$$

Then, the right hand side can be computed in $O(|\downarrow \mathcal{W}|)$ time using subsequently fast d zeta-transformation (Theorem 2.3), naïve dot product computation, and one fast Möbius-transformation (Theorem 2.3). The proof for the second part of the theorem one takes $\mathcal{W}' := \{[n] \setminus W : W \in \mathcal{W}\}$ and applies the technique above to \mathcal{W}' . Notice that indeed $\downarrow \mathcal{W}' = \uparrow \mathcal{W}$. \square

Note this can be used solve to obtain the algorithm already mention in Section 1:

Theorem 2.7 ([8]). *Bin Packing with capacities c_1, \dots, c_m can be solved in $\mathcal{O}^*(2^n)$ time.*

Proof. For $i = 1, \dots, m$ define the function $f_i : 2^{[n]} \rightarrow \{0, 1\}$ as

$$f_i(X) = \begin{cases} 1, & \text{if } w(X) \leq c_i \\ 0, & \text{otherwise.} \end{cases}$$

Note that $(f_1 *_c f_2 *_c \dots *_c f_m)([n]) > 0$ if and only if we have YES instance. By Theorem 2.5 we have that

$$f_1 *_c f_2 *_c \dots *_c f_m = \mu((\zeta f_1) \cdot (\zeta f_2) \dots (\zeta \cdot f_m)),$$

and the right hand side can be computed in $\mathcal{O}^*(2^n)$ time using subsequently fast m zeta-transformations (Theorem 2.3), naïve dot product computation, and one fast Möbius-transformation (Theorem 2.3). \square

2.2 The Entropy Function and Binomial Coefficients

We heavily use properties of the entropy function, which we will now define. Let $\mathcal{D} = (\Omega, p)$ be a discrete probability space. The *entropy* $h(\mathcal{D})$ of \mathcal{D} is defined as follows:

$$- \sum_{x \in \Omega} p(x) \log p(x). \quad (1)$$

We say $p = (p_1, \dots, p_k)$ is a probability vector if the p_i 's are non-negative and satisfy $\sum_{i=1}^k p_i = 1$. If no underlying probability space is given, we may interpret p as a probability measure over $\{1, \dots, k\}$ and thus (1) gives $h(p) = - \sum_{i=1}^k p_i \log p_i$. The *support* of p is k . If $p \in (0, 1)$, we use the shorthand notation $h(p) := h(p, 1 - p)$. If n is positive integer, we let $\binom{n}{p \cdot n}$ denote the multinomial coefficient $\binom{n}{p_1 n, p_2 n, \dots, p_k n}$. This multinomial coefficient can be approximated with $h(p)$ as follows:

Lemma 2.8 ([14], Lemma 2.2). *If p is probability vector with support at most s , then*

$$\binom{n + s - 1}{s - 1}^{-1} 2^{h(p)n} \leq \binom{n}{p \cdot n} \leq 2^{h(p)n}.$$

We will frequently use the special case $\binom{n}{p \cdot n} \leq 2^{h(p)n}$ where $p \in (0, 1)$.

The following lemma states the intuitive fact that close probability vectors have close entropy.

Lemma 2.9. *Let $p, q \in \mathbb{R}^k$ be probability vectors such that $|p_i - q_i| \leq \varepsilon$ for each $i = 1, \dots, k$. Then $|h(p) - h(q)| \leq \ln(2)k\varepsilon \log \frac{1}{\varepsilon}$.*

Proof. Recall that $h(p) = \sum_{i=1}^k p_i \log \frac{1}{p_i}$. Thus the lemma follows by applying the following inequalities to all summands of the entropy of p and q : If $x, \varepsilon, x + \varepsilon \in [0, 1]$, then we have

$$x \log \frac{1}{x} - \ln(2)\varepsilon \leq (x + \varepsilon) \log \frac{1}{x + \varepsilon} \leq x \log \frac{1}{x} + \varepsilon \log \frac{1}{\varepsilon}.$$

The second inequality is direct, and the first inequality can be derived as

$$(x + \varepsilon) \log \frac{1}{x + \varepsilon} = x \log \frac{1}{x} + x \log \frac{x}{x + \varepsilon} + \varepsilon \log \frac{1}{x + \varepsilon} \geq x \log \frac{1}{x} - x \log(1 + \frac{\varepsilon}{x}) \geq x \log \frac{1}{x} - \ln(2)\varepsilon,$$

where we use the standard fact $1 + z \leq \exp(z)$ in the last inequality. \square

3 Proof of Theorem 1.1

In this section we prove our main theorem which we first restate for convenience:

Theorem 1.1 (restated). *For every $m \in \mathbb{N}$ there is a constant $\sigma_m > 0$ such that every Bin Packing instance with m bins can be solved in $\mathcal{O}(2^{(1-\sigma_m)n})$ time with high probability.*

This section is organized as follows: In Subsection 3.1 we introduce definitions that will be used throughout this section, such as the key definition of α -balanced solutions. We then prove in Subsection 3.2 that ‘easy’ instances of Bin Packing, namely those where w generates relatively few distinct sums and those with α -unbalanced solutions (for some $\alpha > 0$), can be solved fast. We can therefore assume that there are only α -balanced solutions and that $|w_\theta(2^{[n]})| \geq 2^{\delta n}$ (for some $\delta < \frac{1}{m}$) in the rest of the section.

Subsection 3.3 introduces a few more definitions, such as the slack of a bin, which is the unused capacity of a bin in a solution. This is also where we define the ‘ θ -pruned item weights’ as the bit representation of the weights, pruned to the θ most significant bits. The parameter θ is then chosen such that $|w_\theta(2^{[n]})| \approx 2^{\delta n}$, as discussed in § 1.1.2. These definitions will be central in solving the remaining two types of instances.

In Subsection 3.4, we consider instances where at least roughly half of the items are in a bin with small slack. This is where we use the approach discussed in § 1.1.1 and apply Theorem 1.2 on the θ -pruned item weights, to conclude that $\beta(w_\theta) \leq 2^{(1-\epsilon)n}$ for some $\epsilon > 0$.

Subsection 3.5 then solves instances where at least roughly half of the items are in a bin with large slack. In the proof, we can split the large slack bins into two parts, where we use the θ -pruned item weights in each of these parts in order to determine whether they fit. Because $w_\theta(2^{[n]}) \approx 2^{\delta n}$, there are at most $2^{\delta mn}$ different tuples of weights, which we can keep track of since we assumed $\delta < \frac{1}{m}$. Furthermore, the splitting of the large slack bins into two, guarantees a constant probability to correctly guess how to split the items in small slack bins into two.

Finally, the proof of Theorem 1.1 can be found in Subsection 3.6, where we combine all these results by choosing the right parameters for δ and α based on the number of bins.

3.1 (Balanced) Solutions and witnesses

Fix an instance of Bin Packing. We begin by formally defining solutions.

Definition 3.1. *A partition S_1, \dots, S_m of $[n]$ is a solution of an instance of Bin Packing with n items and m bins, if for all $j = 1, \dots, m$ the set S_j can be put in bin j (i.e., $w(S_j) \leq c_j$).*

The following notion of a *witness* will be crucial in our approach.

Definition 3.2 ((L, R) -witnesses). *Let $L, R \subseteq [m]$. A set $W \subseteq [n]$ is an (L, R) -witness if there is a solution S_1, \dots, S_m such that $\bigcup_{j \in L} S_j \subseteq W$ and $\bigcup_{j \in R} S_j \subseteq [n] \setminus W$.*

We commonly denote $S^L := \bigcup_{j \in L} S_j$ and $S^R := \bigcup_{j \in R} S_j$. To prove that $W \subseteq [n]$ is an (L, R) witness, it is sufficient to prove that there exist $S^L \subseteq W$ and $S^R \subseteq [n] \setminus W$ such that:

- S^L can be divided over the bins in L ,
- S^R can be divided over the bins in R ,
- and $[n] \setminus (S^L \cup S^R)$ can be divided over the bins in $[m] \setminus (L \cup R)$.

Hence, finding a witness gives us a proof for existence of a solution. This will be used several times throughout this section.

Our algorithmic approach will heavily depend on whether or not the set of items can be evenly divided, which we formalize as follows:

Definition 3.3 (α -balanced solution). *Let S_1, \dots, S_m be a solution of Bin Packing. Then the solution is α -balanced if for all permutations $\pi : [m] \rightarrow [m]$ there exists an $b \in [m]$ such that $\sum_{j=1}^b |S_{\pi(j)}| \in [n/2 \pm \alpha n]$. If a solution is not α -balanced, it is called α -unbalanced.*

Hence, a solution is α -unbalanced if and only if there exists a permutation $\pi : [m] \rightarrow [m]$ and a $b \in [m]$ such that $\sum_{j=1}^{b-1} |S_{\pi(j)}| < (1/2 - \alpha)n$ and $\sum_{j=1}^b |S_{\pi(j)}| > (1/2 + \alpha)n$.

3.2 Easier instances

If the instance generates relatively few distinct sums in the sense that $|w(2^{[n]})| \leq 2^{\delta n}$ for some small δ , we can solve Bin Packing in sufficiently fast relatively easily as follows.

Lemma 3.4. *A solution of Bin Packing can be found in time $\mathcal{O}(n|w(2^{[n]})|^m)$.*

Proof. First compute $w(2^{[n]})$ in time $\mathcal{O}(|w(2^{[n]})|)$ with Lemma A.1. Subsequently, use the following Dynamic Programming algorithm: For every $i = 1, \dots, n$ and $W_1, \dots, W_m \in w(2^{[n]})$, define

$$A(i, W_1, \dots, W_m) = \begin{cases} 1, & \text{if items } 1, \dots, i \text{ can be divided over } m \text{ bins with capacities } W_1, \dots, W_m, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following recurrence relation can be easily seen to hold:

$$A(i, W_1, \dots, W_m) = \bigvee_j A(i-1, W_1, \dots, W_j - w(i), \dots, W_m).$$

Let c_1, \dots, c_m be the capacities of the bins of the bin packing instance. Then $A(n, c_1, \dots, c_m) = 1$ if and only if there is a solution. We only have to check $W_j \in w(2^{[n]})$, since those are all the possible sums that w generates and thus $A(i, W_1, \dots, W_m) = 0$ whenever $W_j \notin w(2^{[n]})$ for some j . Since we can compute each table entry in $\mathcal{O}(m)$ time, the run time follows. \square

Next we show that α -unbalanced solutions with $\alpha > 0$ can be detected quickly:

Lemma 3.5. *If a Bin Packing instance has an α -unbalanced solution, then with probability $\geq \frac{1}{2}$ it can be found in time $\mathcal{O}^*(2^{(1-f_A(\alpha))n})$ where $f_A(\alpha) = \Omega_{\alpha \rightarrow 0} \left(\frac{\alpha^2}{\log^2(\alpha)} \right)$.*

Proof. The algorithm iterates over all subsets $L, R \subseteq [m]$ such that $L \cap R = \emptyset$, $|L \cup R| = m-1$. Let $b \in [m]$ be the only element not in $L \cup R$. For each such L and R , the algorithm searches for (L, R) -witnesses of size $\frac{n}{2}$. Concretely, it samples a set \mathcal{W} of $2^{(1-2\alpha)n}$ random subsets of $[n]$ of size $\frac{n}{2}$, and it computes for every $W \in \mathcal{W}$ whether it is an (L, R) -witness as follows: First, it computes which sets from $\downarrow \mathcal{W} \cup \uparrow \mathcal{W}$ are potential candidates for S^L and S^R . This is done by computing the booleans l_X for every $X \in \downarrow \mathcal{W}$ and r_X for every $X \in \uparrow \mathcal{W}$, where

$$l_X := \begin{cases} 1 & \text{if } X \text{ can be divided over the bins in } L, \\ 0 & \text{otherwise,} \end{cases}$$

$$r_X := \begin{cases} 1 & \text{if } [n] \setminus X \text{ can be divided over the bins in } R, \\ 0 & \text{otherwise.} \end{cases}$$

This can be done in time $\mathcal{O}((|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}|)n)$ using Theorem 2.6.

Second, for each $W \in \mathcal{W}$, we search for sets $X^L \subseteq W$ and $X^R \subseteq [n] \setminus W$ of maximum weight such that they can be distributed over the bins L and R respectively. To do this, we compute l_X^* for every $X \in \downarrow\mathcal{W}$ and r_X^* for every $X \in \uparrow\mathcal{W}$, where

$$l_X^* := \max_{Y \subseteq X: l_Y=1} w(Y), \quad r_X^* := \max_{Y \supseteq X: r_Y=1} w([n] \setminus Y).$$

This can be done using Dynamic Programming with the recurrence relations

$$l_X^* = \begin{cases} w(X) & \text{if } l_X = 1, \\ \max_{i \in X} l_{X \setminus \{i\}}^* & \text{if } l_X = 0, \end{cases} \quad \text{and} \quad r_X^* = \begin{cases} w([n] \setminus X) & \text{if } r_X = 1, \\ \max_{i \notin X} r_{X \cup \{i\}}^* & \text{if } r_X = 0. \end{cases}$$

The runtime is only $\mathcal{O}((|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}|)n)$ since the values l_X^* for $X \in \downarrow\mathcal{W}$ do not depend on entries l_Y^* for $Y \notin \downarrow\mathcal{W}$, and the values r_X^* for $X \in \uparrow\mathcal{W}$ do not depend on entries r_Y^* for $Y \notin \uparrow\mathcal{W}$. Thus the algorithm only needs to evaluate $|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}|$ table entries which can be done in time $\mathcal{O}(n)$ per entry.

Third, the algorithm checks if there exists a $W \in \mathcal{W}$ such that $\sum_{i=1}^n w(i) - l_W^* - r_W^* \leq c_b$ and returns *yes* if this is the case. If for all different choices of L and R , no (L, R) -witness has been found, the algorithm returns *no*.

Correctness of Algorithm

Assume that there is an α -unbalanced solution S_1, \dots, S_m . Let $\pi : [m] \rightarrow [m]$ be a permutation of the bins such that $\sum_{j=1}^{b-1} |S_{\pi(j)}| < (1/2 - \alpha)n$ and $\sum_{j=1}^b |S_{\pi(j)}| > (1/2 + \alpha)n$ for some $b \in [m]$. Thus $|S_b| \geq 2\alpha n$. Take $L = \{\pi(1), \dots, \pi(b-1)\}$ and $R = \{\pi(b+1), \dots, \pi(m)\}$. Recall the notation $S_L = \cup_{j=1}^{b-1} S_{\pi(j)}$. Since for every $Y \in \binom{S_b}{n/2 - |S_L|}$ the set $Y \cup S_L$ is an (L, R) -witness of size $\frac{n}{2}$, there are at least $\binom{|S_b|}{n/2 - |S_L|}$ (L, R) -witnesses of cardinality $\frac{n}{2}$, which is at least $\binom{2\alpha n}{\alpha n}$ since $|S_b| \geq 2\alpha n$ and $n/2 - |S_L| \geq \alpha n$. Thus the algorithm will detect the solution with probability

$$1 - \Pr[\mathcal{W} \text{ has no witness}] = 1 - \left(1 - \frac{\binom{2\alpha n}{\alpha n}}{2^n}\right)^{|\mathcal{W}|} \geq 1 - \exp\left(-\frac{\binom{2\alpha n}{\alpha n} |\mathcal{W}|}{2^n}\right) \geq 1 - \exp\left(-\frac{n|\mathcal{W}|}{2^{(1-2\alpha)n}}\right),$$

where we use $1 + x \leq \exp(x)$ in the first inequality and $\binom{a}{a/2} \geq 2^a/a$ in the second inequality. Hence, if we take \mathcal{W} to be a set of $2^{(1-2\alpha)n}$ random subsets of $[n]$ of size $\frac{n}{2}$, with constant probability there will be an (L, R) -witness of the solution in \mathcal{W} . Notice that for any witness W it will hold that $\sum_{i=1}^n w(i) - l_W^* - r_W^* \leq c_{\pi(b)}$ and so the algorithm will return *yes* if $W \in \mathcal{W}$.

Moreover, when the algorithm finds a $W \in \mathcal{W}$ such that $\sum_{i=1}^n w(i) - l_W^* - r_W^* \leq c_{\pi(b)}$, it means there exist sets $X^L \subseteq W$ and $X^R \subseteq [n] \setminus W$ that can be divided over the bins of L and R respectively, such that $[n] \setminus (X^L \cup X^R)$ fits into bin $\pi(b)$. Therefore, W is an (L, R) -witness and we proved the existence of a solution to the Bin Packing instance.

Runtime Analysis

We are left to prove the runtime of the algorithm. Recall that the algorithm will repeat the procedure above for all $\mathcal{O}(2^m)$ combinations of L and R . The runtime per different guess of L and R is dominated by $\mathcal{O}((|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}|)n)$, hence we are left to prove that $|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}| = \mathcal{O}(2^{(1-f_A(\alpha))n})$. Let $\gamma_\alpha = \frac{\alpha}{4 \log(6/\alpha)}$. We can describe any $X \in \downarrow\mathcal{W}$ either as a set in $\binom{[n]}{(\frac{1}{2} - \gamma_\alpha)n}$ (if $|X| \leq (\frac{1}{2} - \gamma_\alpha)n$),

or as a subset of a $W \in \mathcal{W}$ together with items that W and X differ on (if $|X| \geq (\frac{1}{2} - \gamma_\alpha)n$). In the latter case, the two sets differ on at most γ_α items, since $|W| = \frac{n}{2}$. This, together with the fact that $|X|$ can only take n distinct values implies that

$$\begin{aligned} |\downarrow \mathcal{W}| &\leq n \binom{n}{(\frac{1}{2} - \gamma_\alpha)n} + |\mathcal{W}| \binom{\frac{n}{2}}{\gamma_\alpha n} \\ &\leq n 2^{h(\frac{1}{2} - \gamma_\alpha)n} + 2^{(1 - 2\alpha + \frac{1}{2}h(2\gamma_\alpha))n}. \end{aligned}$$

Notice that $|\uparrow \mathcal{W}|$ can be bounded in the same way. Now we apply Lemma B.2 with $x := \alpha$, $\gamma := \gamma_\alpha$, $b = \frac{1}{2}$ and $c = 2$. Note that indeed the condition $\gamma_\alpha \leq \frac{x}{4bc \log(\frac{12b}{x})}$ of Lemma B.2 is satisfied. Thus we obtain that $h(\frac{1}{2} - \gamma_\alpha) \geq 1 - \alpha + \frac{1}{2}h(2\gamma_\alpha) > 1 - 2\alpha + \frac{1}{2}h(2\gamma_\alpha)$. Lastly, Lemma B.1 tells us that $h(\frac{1}{2} - \gamma_\alpha) \leq 1 - \frac{2}{\ln(2)}\gamma_\alpha^2$. Hence, $|\downarrow \mathcal{W}| + |\uparrow \mathcal{W}| = \mathcal{O}(2^{(1 - f_A(\alpha))n})$ where

$$f_A(\alpha) = \frac{\alpha^2}{8 \ln(2)(\log(6/\alpha))^2} = \Omega_{\alpha \rightarrow 0} \left(\frac{\alpha^2}{\log^2(\alpha)} \right).$$

This gives us the desired runtime. □

3.3 Pruned item weights and slack

The results from the previous subsection enable us to assume that both $|w(2^{[n]})| \geq 2^{\delta n}$ for some small constant $\delta > 0$ (that we will fix later) and that there is an α -balanced solution for some $\alpha > 0$. To solve these instances of Bin Packing, we first need to define different parameters of an instance that determine our proof strategy.

Definition 3.6 (*s*-pruned item weights). *Let $l = 1 + \lceil \log(\max_i \{w(i)\}) \rceil$. For $s \in \{0, \dots, l\}$, define the s -pruned weight of an item i as*

$$w_s(i) := \lfloor w(i)/2^{l-s} \rfloor.$$

The s -pruned weight of item i comes down to pruning the l -bit representation of $w(i)$ to the s most significant bits. Indeed, $w_s(i) \leq 2^s$, and $w_0(i) = 0$ for all items i and $w_l = w$. We will need the fact that the sequence

$$1 = |w_0(2^{[n]})|, \quad |w_1(2^{[n]})|, \quad \dots, \quad |w_l(2^{[n]})| = |w(2^{[n]})|,$$

is almost non-decreasing and relatively smooth. Observe that the sequence may not be non-decreasing. For example when $w = (3, 7, 10)$ the number of bits is $l = 5$, and

$$\begin{array}{llll} w_0 = (0, 0, 0), & |w_0(2^{[n]})| = 1 & w_1 = (0, 0, 0), & |w_1(2^{[n]})| = 1 \\ w_2 = (0, 0, 1), & |w_2(2^{[n]})| = 2 & w_3 = (0, 1, 2), & |w_3(2^{[n]})| = 1 \\ w_4 = (1, 3, 5), & |w_4(2^{[n]})| = 8 & w_5 = (3, 7, 10), & |w_5(2^{[n]})| = 7. \end{array}$$

Nevertheless, we show this is only an artefact of smaller order rounding errors and that the sequence in fact is *smooth* in the following precise sense:

Lemma 3.7. *Let $w : [n] \rightarrow \mathbb{N}$ be an item weight function. Then for all $s \in [l]$:*

$$\left(\frac{1}{3n} \right) |w_s(2^{[n]})| \leq |w_{s-1}(2^{[n]})| \leq \left(\frac{3n}{2} \right) |w_s(2^{[n]})|.$$

Proof. Let $l = 1 + \lceil \log(\max_i \{w(i)\}) \rceil$. We are given $w_s(A)$ for all $A \in 2^{[n]}$. Observe, that we can bound the value of $w_{s-1}(A)$ by the following:

$$\begin{aligned} & \sum_{i \in A} w(i)/2^{l-s+1} - 1 && \leq w_{s-1}(A) && \leq \sum_{i \in A} w(i)/2^{l-s+1} \\ \Rightarrow & \frac{1}{2} \sum_{i \in A} (\lfloor w(i)/2^{l-s} \rfloor - 2) && \leq w_{s-1}(A) && \leq \frac{1}{2} \sum_{i \in A} (\lfloor w(i)/2^{l-s} \rfloor + 1) \\ \Rightarrow & \frac{1}{2} (w_s(A) - 2n) && \leq w_{s-1}(A) && \leq \frac{1}{2} (w_s(A) + n) \end{aligned}$$

Hence, for each value in $w_s(2^{[n]})$, there are at most $\frac{3n}{2}$ values in $w_{s-1}(2^{[n]})$, i.e.

$$|w_{s-1}(2^{[n]})| \leq \left(\frac{3n}{2}\right) |w_s(2^{[n]})|$$

Analogously, for a given $w_{s-1}(A)$ for any subset $A \in 2^{[n]}$. Then we can bound the value of $w_s(A)$ by the following:

$$\begin{aligned} & \sum_{i \in A} w(i)/2^{l-s} - 1 && \leq w_s(A) && \leq \sum_{i \in A} w(i)/2^{l-s} \\ \Rightarrow & 2 \sum_{i \in A} \left(\lfloor w(i)/2^{l-s+1} \rfloor - \frac{1}{2} \right) && \leq w_s(A) && \leq 2 \sum_{i \in A} (\lfloor w(i)/2^{l-s+1} \rfloor + 1) \\ \Rightarrow & 2 \left(w_{s-1}(A) - \frac{n}{2} \right) && \leq w_s(A) && \leq 2(w_{s-1}(A) + n) \end{aligned}$$

Hence, for each value in $w_{s-1}(2^{[n]})$, there are at most $3n$ values in $w_s(2^{[n]})$, i.e.

$$|w_s(2^{[n]})| \geq \left(\frac{1}{3n}\right) |w_{s-1}(2^{[n]})|$$

□

A part of our strategy is to use techniques from Lemma 3.4 to deal with some bins that are largely empty. However, the analogous Dynamic Programming table needs to be indexed by w_s for some $s \in [l]$. Therefore we will need only a finite precision (similar as in e.g. approximation schemes for Knapsack (see e.g. [37, Section 11.8])). The precision parameter we will be using is the following:

Definition 3.8 (Critical pruner). *Let $\delta \in (0, 1)$ be a fixed parameter⁶ such that $|w(2^{[n]})| \geq 2^{\delta n}$. We define the critical pruner θ as*

$$\theta := \theta(\delta) := \min \left\{ s \in \mathbb{N} : |w_s(2^{[n]})| \geq 2^{\delta n} \right\}.$$

Observe, that $|w_\theta(2^{[n]})| = \Theta(n2^{\delta n})$ by Lemma 3.7 and the fact that $w_0(2^{[n]}) = \{0\}$. Furthermore, by Corollary A.2 the critical pruner θ can be computed in $\mathcal{O}(2^{\delta n})$ time.

Definition 3.9 (Slack). *The slack of a bin j is $c_j - \sum_{i \in S_j} w(i)$. A bin has δ -large slack if it has slack at least $n \cdot 2^{l-\theta}$ and δ -small slack otherwise. We often skip δ , as δ will be fixed later in Subsection 3.6. An item is a large slack item if it is in a bin of large slack and a small slack item otherwise.*

⁶Which we will set later in Subsection 3.6

3.4 Detecting a balanced solution with many small slack items

In the next lemma we will solve Bin Packing instances with at least $(1/2 - \alpha)n$ small slack items.

Lemma 3.10. *Suppose $|w(2^{[n]})| \geq 2^{\delta n}$ and $0 < \alpha \leq 2^{-2/\delta^3}$. If a Bin Packing instance has a solution that is α -balanced and has at least $(1/2 - \alpha)n$ items with δ -small slack, then such a solution can be found in time $\mathcal{O}^*(2^{(1-f_B(\delta))n})$ for $f_B(\delta) = \Omega_{\delta \rightarrow 0}(2^{-3/\delta^3})$.*

Proof. Use Corollary A.2 to compute the critical pruner θ in time $\mathcal{O}(2^{\delta n})$. Then iterate over all combinations of sets $L, R \subseteq [m]$ that form a partition of $[m]$. For each such a partition, the algorithm searches for (L, R) -witnesses of size $\frac{n}{2} \pm \alpha n$ as follows: First, enumerate \mathcal{W} , which is defined as

$$\mathcal{W} := \left\{ W \subseteq [n] : |W| \in [\frac{n}{2} \pm \alpha n], w_\theta(W) \in \left(\sum_{j \in L} c_j / 2^{l-\theta} - (|L| + 1) \cdot n, \sum_{j \in L} c_j / 2^{l-\theta} \right) \right\}.$$

We can enumerate \mathcal{W} in time $\mathcal{O}(2^{n/2} + |\mathcal{W}|)$ with a standard Meet-in-the-Middle approach (see e.g. [6, Section 3.2] or Lemma [44, Lemma 3.8]). Next, for every $W \in \mathcal{W}$ we determine whether W is an (L, R) -witness. This is done by computing the boolean l_X for every $X \in \downarrow \mathcal{W}$ and r_X for every $X \in \uparrow \mathcal{W}$, where

$$l_X := \begin{cases} 1 & \text{if } X \text{ can be divided over the bins in } L, \\ 0 & \text{otherwise,} \end{cases}$$

$$r_X := \begin{cases} 1 & \text{if } [n] \setminus X \text{ can be divided over the bins in } R, \\ 0 & \text{otherwise.} \end{cases}$$

Using Theorem 2.6 we can do this in time $\mathcal{O}((|\downarrow \mathcal{W}| + |\uparrow \mathcal{W}|)n)$. Next, the algorithm checks for all $W \in \mathcal{W}$, whether $l_W = r_W = 1$, and if so the algorithm returns *yes*. If for no partition L, R of $[m]$ the algorithm finds a witness, the algorithm returns *no*.

Correctness of Algorithm

Assume that there is an α -balanced solution S_1, \dots, S_m . Let $\pi : [m] \rightarrow [m]$ be a permutation of the bins such that all bins with small slack have smaller index than the large slack bins, i.e. $\pi(j) \leq \pi(j')$ for all small slack bins j and large slack bins j' . Since we assumed the solution to be α -balanced, there exists a bin $b \in [m]$ such that $\sum_{j=1}^b |S_{\pi(j)}| \in [\frac{n}{2} \pm \alpha n]$. Take $L = \{\pi(1), \dots, \pi(b)\}$ and $R = \{\pi(b+1), \dots, \pi(m)\}$. Notice that $S^L = \cup_{j=1}^b S_{\pi(j)}$ is an (L, R) -witness. We will prove that in the iteration of the algorithm where correct partition L, R is chosen, it holds that $S^L \in \mathcal{W}$. Since there are at least $(1/2 - \alpha)n$ small slack items, all bins in L have small slack. Hence,

$$\sum_{j=1}^b (c_{\pi(j)} - n \cdot 2^{l-\theta}) \leq w(S^L) \leq \sum_{j=1}^b c_{\pi(j)}.$$

Then using the bound on the pruned weights of the items

$$w_\theta(S^L) \leq w(S^L) / 2^{l-\theta} \leq w_\theta(S^L) + n,$$

we can conclude that

$$\sum_{j=1}^b c_{\pi(j)} / 2^{l-\theta} - (b+1)n \leq w_\theta(S^L) \leq \sum_{j=1}^b c_{\pi(j)} / 2^{l-\theta}.$$

Combining this with the fact that $|S^L| \in [\frac{n}{2} \pm \alpha n]$, we conclude that the set S^L is present in

$$\mathcal{W} := \left\{ W \subseteq [n] : |W| \in [(\frac{1}{2} \pm \alpha)n], w_\theta(W) \in \left(\sum_{j=1}^b c_j/2^{l-\theta} - (b+1) \cdot n, \sum_{j=1}^b c_j/2^{l-\theta} \right) \right\}.$$

Notice that $l_W = r_W = 1$ if and only if W is an (L, R) -witness, since we chose L and R to partition $[m]$. Because $S^L \in \mathcal{W}$, the algorithm always returns *yes* in a yes-instance. Furthermore, when we find a W s.t., $l_W = r_W = 1$, we can conclude that there is a solution to the Bin Packing instance since all items are divided over all bins.

Runtime Analysis

It remains to analyze the runtime of the algorithm. Recall that the algorithm iterates over all $\mathcal{O}(2^m)$ combinations of L and R . Each iteration takes $\mathcal{O}((|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}|)n + 2^{n/2})$ time. First we prove that $|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}| \leq 2^{(1-f_B(\delta))n}$. Recall that θ is the critical pruner, and therefore by definition $|w_\theta(2^{[n]})| \geq 2^{\delta n}$. Theorem 4.1 states that if $\beta(w_\theta) \geq 2^{(1-\varepsilon')n}$, then $|w_\theta(2^{[n]})| \leq 2^{\delta' n}$ where $\delta' = \mathcal{O}_{\varepsilon' \rightarrow 0} \left(\frac{\log \log(1/\varepsilon')}{\sqrt{\log(1/\varepsilon')}} \right)$. By the bound $\frac{\log \log(1/\varepsilon')}{\sqrt{\log(1/\varepsilon')}} \leq \frac{1}{\sqrt[3]{\log(1/\varepsilon')}}$, this implies

$$\begin{aligned} \delta' &\leq \mathcal{O}_{\varepsilon' \rightarrow 0} \left(\frac{1}{\sqrt[3]{\log(1/\varepsilon')}} \right) \\ \Leftrightarrow \quad 2^{1/\delta'^3} &\geq \mathcal{O}_{\varepsilon' \rightarrow 0}(1/\varepsilon') \\ \Leftrightarrow \quad \varepsilon' &\leq \mathcal{O}_{\delta' \rightarrow 0} \left(2^{-1/\delta'^3} \right) \end{aligned}$$

Hence, if we denote $\varepsilon(\delta) := 2^{-1/\delta^3}$ there is some constant $\delta_0 > 0$ such that for all $\delta < \delta_0$ it holds that if $|w_\theta(2^{[n]})| \geq 2^{\delta n}$, then $\beta(w_\theta) \leq 2^{(1-\varepsilon(\delta))n}$. Note that because we only claim an asymptotic time bound in the lemma, we may assume that $\delta \leq \delta_0$. As a consequence, for fixed weight value v , there are at most $2^{(1-\varepsilon(\delta))n}$ sets $W \subseteq [n]$ that have a weight $w_\theta(W) = v$ and so $|\mathcal{W}| \leq mn \cdot 2^{(1-\varepsilon(\delta))n}$. Knowing this, we still need to bound the sizes of $\downarrow\mathcal{W}$ and $\uparrow\mathcal{W}$. We can describe all $X \in \downarrow\mathcal{W}$ either as a set in $\binom{[n]}{(\frac{1}{2}-\alpha)n}$ (if $|X| \leq (\frac{1}{2} - \alpha)n$), or as a subset of a $W \in \mathcal{W}$ together with the items that W and X differ on (if $|X| \geq (\frac{1}{2} - \alpha)n$). In the latter case, the sets differ on at most 2α items. Therefore

$$\begin{aligned} |\downarrow\mathcal{W}| &\leq \binom{n}{(\frac{1}{2}-\alpha)n} + |\mathcal{W}| \binom{n}{2\alpha n} \\ &\leq 2^{h(\frac{1}{2}-\alpha)n} + mn \cdot 2^{(1-\varepsilon(\delta)+h(2\alpha))n}. \end{aligned}$$

Notice that $|\uparrow\mathcal{W}|$ can be bounded in the same way. Now we apply Lemma B.2 with $x := \varepsilon(\delta)$, $b := 1$ and $c := 2$. We assumed $0 < \alpha \leq 2^{-2/\delta^3}$. Note there is some δ_0 such that for any δ satisfying $0 < \delta \leq \delta_0$ we have

$$\alpha \leq 2^{-2/\delta^3} \leq \frac{2^{-1/\delta^3}}{8 \log(12/2^{-1/\delta^3})} = \frac{\varepsilon(\delta)}{8 \log(12/\varepsilon(\delta))}.$$

Thus the condition of Lemma B.2 is satisfied and it states that $h(\frac{1}{2} - \alpha) \geq 1 - \varepsilon(\delta) + h(2\alpha)$. Lastly, Lemma B.1 tells us that $h(\frac{1}{2} - \alpha) \leq 1 - \frac{2}{\ln(2)}\alpha^2$. Hence, $|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}| = \mathcal{O}(2^{(1-f_B(\delta))n})$ where

$$f_B(\delta) = \frac{\varepsilon(\delta)^2}{32 \ln(2)(\log^2(6/\varepsilon(\delta)))} = \Omega_{\delta \rightarrow 0} \left(\frac{\varepsilon(\delta)^2}{\log^2 \varepsilon(\delta)} \right) = \Omega_{\delta \rightarrow 0} \left(\delta^6 \cdot 2^{-2/\delta^3} \right).$$

Since $f_B(\delta) \ll 1/2$ for $\delta \in [0, 1]$, the $\mathcal{O}(2^{n/2})$ time bound is subsumed by the $\mathcal{O}(2^{(1-f_B(\delta))n})$ term. Multiplying this term by 2^m different choices for L and R , gives us the requested runtime. \square

3.5 Detecting a balanced solution with few small slack items

We are left to prove the remaining case. In this case we will assume that the solution is α -balanced for some $0 < \alpha < \frac{1}{4m}$, $|w(2^{[n]})| \leq 2^{\delta n}$ for some $\delta > 0$ and that there are at most $(1/2 - \alpha)n$ small slack items with respect to the critical pruner θ . First we observe the following property of an α -balanced solution.

Observation 3.11. *Let S_1, \dots, S_m be an α -balanced solution for some $0 < \alpha < \frac{1}{4m}$. Assume $k, k' \in [m]$ to be two different bins with the most items. Then either:*

1. $|S_k|, |S_{k'}| \in [(1/2 \pm \alpha)n]$, or
2. $|S_j| \leq (\frac{1}{2} - \frac{1}{4m})n$ for all bins j .

Proof. If condition 2) does not hold, then we know that $|S_k| > (\frac{1}{2} - \frac{1}{4m})n$. That means that on average the other bins have $\frac{n - |S_k|}{m-1}$ items, meaning that $|S_{k'}| \geq \frac{n - |S_k|}{m-1}$. Hence,

$$\begin{aligned} |S_k| + |S_{k'}| &\geq \left(\frac{1}{2} - \frac{1}{4m} + \frac{1 - \frac{1}{2} + \frac{1}{4m}}{m-1} \right) n \\ &> \left(\frac{1}{2} - \frac{1}{4m} + \frac{\frac{1}{2}}{(m-1)} \right) n \\ &> \left(\frac{1}{2} - \frac{1}{4m} + \frac{1}{2m} \right) n \\ &= \left(\frac{1}{2} + \frac{1}{4m} \right) n \end{aligned}$$

Since the solution is α -balanced (with $\alpha \leq \frac{1}{4m}$), it means that for all permutations π , in particular those with $\pi^{-1}(k) = 1$ and $\pi^{-1}(k') = 2$, there exists an $b \in [m]$ such that $\sum_{j=1}^b |S_{\pi(j)}| \in [\frac{n}{2} \pm \alpha n]$. Because $|S_k| + |S_{k'}| > (1/2 + \alpha)n$, we know that $b = 1$ and $|S_k| \in [\frac{n}{2} \pm \alpha n]$. We can conclude the same for k' by repeating these last arguments for all permutations π with $\pi^{-1}(k) = 2$ and $\pi^{-1}(k') = 1$, and thus condition 1) must hold, and the lemma follows. \square

Lemma 3.12. *Assume $\alpha < \frac{1}{4m}$. If a solution of a Bin Packing instance with m bins is α -balanced and has at most $(1/2 - \alpha)n$ items that have δ -small slack, then with probability at least $\frac{1}{2}$ a solution can be found in time $\mathcal{O}(2^{(1-f_C(m)+\delta m)n})$ with $f_C(m) = \Omega_{m \rightarrow \infty} \left(\frac{h(\frac{1}{2m})^2}{\log^2(h(\frac{1}{2m}))} \right)$.*

Proof. For an overview of the algorithm, see Algorithm 1. Compute the critical pruner θ and the set $w_\theta(2^{[n]})$ in $\mathcal{O}(n2^{\delta n})$ time with Corollary A.2. The algorithm will search for (L, R) -witnesses for all $L, R \subseteq [m]$ such that $|R| = 1$ and $L \cap R = \emptyset$. For notation purposes, assume without loss of generality that $R = \{1\}$, $L = \{2, \dots, k\}$ and let $M = \{k+1, \dots, m\}$. Let $\mathcal{W} \subseteq \binom{[n]}{\frac{n}{2}}$ by sampled uniformly at random with size $2^{(1-g(m))n}$ where $g(m) = \frac{1}{2}h(1/(2m))$. For given L and R , we guess $a_{k+1}, \dots, a_m \in w_\theta(2^{[n]})$. Then, we compute the boolean l_X for every $X \in \downarrow \mathcal{W}$ and r_X for every $X \in \uparrow \mathcal{W}$, where

$$l_X := \begin{cases} 1 & \text{if there exists a partition } X_2, \dots, X_k, Y_{k+1}, \dots, Y_m \text{ of } X \text{ such that} \\ & \text{for all } j \in L : w(X_j) \leq c_j \text{ and for all } j' \in M : w_\theta(Y_{j'}) \leq a_{j'}, \\ 0 & \text{otherwise,} \end{cases}$$

$$r_X := \begin{cases} 1 & \text{if there exists a partition } X_1, Y_{k+1}, \dots, Y_m \text{ of } [n] \setminus X \text{ such that} \\ & w(X_1) \leq c_1 \text{ and for all } j' \in M : w_\theta(Y_{j'}) \leq c_{j'}/2^{l-\theta} - n - a_{j'}, \\ 0 & \text{otherwise.} \end{cases}$$

This can be done using Fast Zeta/Möbius transformation. For $j \in L \cup R$ define the functions $f_j : 2^{[n]} \rightarrow \{0, 1\}$ as

$$f_j(X) = \begin{cases} 1, & \text{if } w(X) \leq c_j, \\ 0, & \text{otherwise,} \end{cases}$$

and for $j' \in M$ define the functions $f_{j'}^\theta, \bar{f}_{j'}^\theta : 2^{[n]} \rightarrow \{0, 1\}$ as

$$f_{j'}^\theta(X) = \begin{cases} 1, & \text{if } w_\theta(X) \leq a_{j'}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\bar{f}_{j'}^\theta(X) = \begin{cases} 1, & \text{if } w_\theta(X) \leq c_{j'}/2^{l-\theta} - n - a_{j'}, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we observe the following:

Claim 3.13. $l_X = 1$ if and only if $(f_2 * c \cdots * c f_k * c f_{k+1}^\theta * c \cdots * c f_m^\theta)(X) > 0$.

Proof. Let us assume that $l_X = 1$. Let S'_2, \dots, S'_m be such that $f_j(S'_j) = 1$ for every j . Then S'_2, \dots, S'_m gives a non-zero contribution to $(f_2 * c \cdots * c f_k * c f_{k+1}^\theta * c \cdots * c f_m^\theta)(X)$ and hence it must be positive.

For the other way around, if $(f_2 * c \cdots * c f_k * c f_{k+1}^\theta * c \cdots * c f_m^\theta)(X) > 0$ there exist S'_2, \dots, S'_m be such that $f_j(S'_j) = 1$ for every j and $S'_2 \cup \dots \cup S'_m = X$. Observe that we can transform this into a partition S''_2, \dots, S''_m of X by choosing $S''_j \subseteq S'_j$. Because f_j does not decrease when taking subsets we know that $1 = f_j(S'_j) \leq f_j(S''_j)$, and thus $l_X = 1$. \square

Similarly, we can argue that $r_X = 1$ if and only if $(f_1 * c \bar{f}_{k+1}^\theta * c \cdots * c \bar{f}_m^\theta)([n] \setminus X) > 0$.

We can compute booleans l_x and r_X in time $\mathcal{O}((|\downarrow \mathcal{W}| + |\uparrow \mathcal{W}|)n)$ by combining Theorem 2.3 and Theorem 2.5. Finally, if we find $W \in \mathcal{W}$, such that $l_W = r_W = 1$, we can return *yes*.

Constant probability of a witness in \mathcal{W}

Recall that the set \mathcal{W} is a random subset of $\binom{[n]}{n/2}$ of size $2^{(1-g(m))n}$ with $g(m) = h(1/(2m))/2$. We first analyze the number of (L, R) -witnesses that are in $\binom{[n]}{n/2}$ and with that we will show that the probability that \mathcal{W} contains such a witness is constant. Assume that there is an α -balanced solution S_1, \dots, S_m for some $0 < \alpha < \frac{1}{4m}$. We use Observation 3.11 to conclude that either $|S_j| < (\frac{1}{2} - \frac{1}{4m})$ for all bins, or that $|S_k|, |S_{k'}| \in [\frac{n}{2} \pm \alpha n]$ for largest bins k and k' . Since we assumed that there are at most $(1/2 - \alpha)n$ small slack items, we know that in the latter case bins k and k' are therefore large slack bins. In either case, we conclude that $|S_j| \leq (\frac{1}{2} - \frac{1}{4m})n$ for all small slack bins. We will assume without loss of generality that bin 1 is the largest small slack bin and that bins $2, \dots, k$ are

Algorithm: BinPacking(w_1, \dots, w_n)

Output : Yes (whp), if an α -balanced solution with $(1/2 - \alpha)n$ small slack items exist

```

1 Compute the critical pruner  $\theta$  with Corollary A.2. // In time  $\mathcal{O}(2^{\delta n})$ 
2 Compute the set  $w_\theta(2^{[n]})$  with Lemma A.1. // In time  $\mathcal{O}(2^{\delta n})$ 
3 Choose  $\mathcal{W}$  to be a set of  $2^{(1-g(m))n}$  random subsets of  $n$  of size  $\frac{n}{2}$ .
4 for  $L, R \subseteq [m]$  such that  $|R| = 1$  and  $L \cap R = \emptyset$  do //  $m2^{m-1}$  repetitions
5   Assume without loss of generality that  $R = \{1\}$ ,  $L = \{2, \dots, k\}$ .
6   for  $a_{k+1}, \dots, a_m \in w_\theta(2^{[n]})$  do //  $|w_\theta(2^{[n]})|^{m-k}$  repetitions
7     Compute  $l_X$  for all  $X \in \downarrow \mathcal{W}$ . // In time  $\mathcal{O}((|\downarrow \mathcal{W}| + |\uparrow \mathcal{W}|)n)$ 
8     Compute  $r_X$  for all  $X \in \uparrow \mathcal{W}$ . // In time  $\mathcal{O}((|\downarrow \mathcal{W}| + |\uparrow \mathcal{W}|)n)$ 
9     if  $l_W = r_W = 1$ , for  $W \in \mathcal{W}$  then
10      | return yes.
11 return no.

```

Algorithm 1: Overview of the algorithm for Lemma 3.12

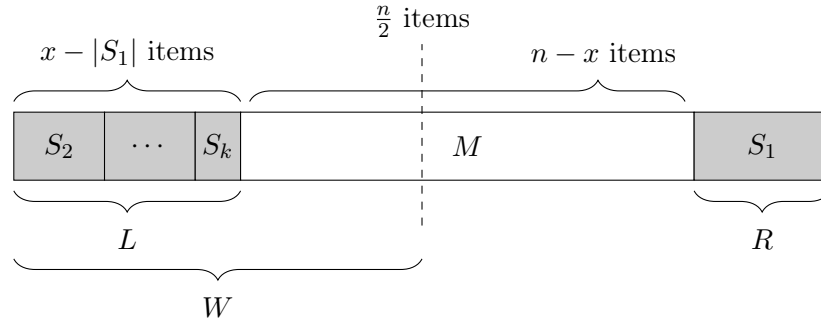


Figure 2: Overview of (L, R) -witnesses of size $\frac{n}{2}$ for explanation of equation 2. Let x be the number of small slack items. Then any such (L, R) -witness, W , must include all items of S_2, \dots, S_k and exclude any items of S_1 . The other items in W can then be any combination of large slack items, which are exactly the items in the bins of M .

the other small slack bins. Then let $L = \{2, \dots, k\}$, $R = \{1\}$ and thus $M = \{k+1, \dots, m\}$ are all large slack bins. We will lower bound the number of (L, R) -witnesses of size $\frac{n}{2}$.

Let x be the number of small slack items in the solution. Note that the number of (L, R) -witnesses of size $\frac{n}{2}$ is equal to

$$\binom{n-x}{n/2 - (x - |S_1|)}, \quad (2)$$

since the sets S_2, \dots, S_k together with any subset of $n/2 - (x - |S_1|)$ large slack items form a witness. See Figure 2 for an illustration of this. Since $\binom{n-x}{n/2 - (x - |S_1|)} \geq \binom{n-x}{n/2 - x}$, if $x \leq n/4$, there are at least $\binom{3n/4}{n/4}$ (L, R) -witnesses of size $\frac{n}{2}$.

If $x \geq \frac{n}{4}$, then notice that $\frac{x}{m} \leq |S_1| \leq (\frac{1}{2} - \frac{1}{m})n$, because S_1 is the largest small slack bin. Therefore, the number of witnesses is at least the number of ways to choose $n/2 - |S_1|$ items from M to exclude in the witness. Thus we have that the number of witnesses of size $\frac{n}{2}$ is at least

$$\binom{n-x}{n/2 - |S_1|} \geq \min \left\{ \binom{n/2}{n/2 - x/m}, \binom{n/2}{n/2 - (\frac{n}{2} - \frac{n}{m})} \right\} \geq \binom{n/2}{n/4m}.$$

So in both cases for x , we can conclude that the number of (L, R) -witnesses of size $\frac{n}{2}$ is at least $2^{g(m)n}$. Thus the algorithm will detect the solution with probability

$$1 - \Pr[W \text{ has no witness}] = 1 - \left(1 - \frac{2^{g(m)n}}{2^n}\right)^{|\mathcal{W}|} \geq 1 - \exp\left(-\frac{2^{g(m)n}|\mathcal{W}|}{2^n}\right) \geq 1 - \exp\left(-\frac{|\mathcal{W}|}{2^{(1-g(m))n}}\right),$$

where we use $1 + x \leq \exp(x)$. Hence, if we take \mathcal{W} to be a set of $2^{(1-g(m))n}$ random subsets of $[n]$ of size $\frac{n}{2}$, with constant probability there will be an (L, R) -witness of the solution in \mathcal{W} .

Correctness of Algorithm

The algorithm returns *yes* if and only if $l_W = r_W = 1$ for some $W \in \mathcal{W}$. So if it returns *yes*, there exists a partition $X_2, \dots, X_k, Y_{k+1}, \dots, Y_m$ of W and a partition $X_1, Y'_{k+1}, \dots, Y'_m$ of $[n] \setminus W$ by definition. Together they partition all items. Notice that by definition we know that X_j can be put into bin j for all $j \in L \cup R$. Hence we are left to prove that for all $j \in M$: $X_j = Y_j \cup Y'_j$ can be put into bin j . Notice that since $f_j^\theta(Y_j) = \bar{f}_j^\theta(Y'_j) = 1$ we have that

$$\begin{aligned} \sum_{i \in X_j} w_\theta(i) &\leq a_j + c_j/2^{l-\theta} - n - a_j && \implies \\ \sum_{i \in X_j} \lfloor w(i)/2^{l-\theta} \rfloor &\leq c_j/2^{l-\theta} - n && \implies \\ \sum_{i \in X_j} (w(i) - 2^{l-\theta}) &\leq c_j - n2^{l-\theta} && \implies \\ \sum_{i \in X_j} w(i) &\leq c_j, \end{aligned}$$

and so, indeed the items of X_j fit into bin j and we have a yes-instance. For the implication in the other direction, we prove that if there exists a solution, the algorithm finds it with constant probability. We already showed that with constant probability there is an (L, R) -witness $W \in \mathcal{W}$. Next, we will prove that there exist $a_{k+1}, \dots, a_m \in w_\theta(2^{[n]})$ such that $l_W = r_W = 1$ for all witnesses W . Let $S_2, \dots, S_k, S_{k+1} \cap W, \dots, S_m \cap W$ be the partition of W from the definition of l_W , and let $S_1, \dots, S_{k+1} \setminus W, \dots, S_m \setminus W$ be the partition of $[n] \setminus W$ from the definition of r_W .

Note that, for all $j \in L \cup R$ it holds that $w(S_j) \leq c_j$ because S_1, \dots, S_m is a solution. So we are left to prove that for all $j \in M$ there exists an $a_j \in w_\theta(2^{[n]})$ such that

$$w_\theta(S_j \cap W) \leq a_j \quad \text{and} \quad w_\theta(S_j \setminus W) \leq c_j/2^{l-\theta} - n - a_j.$$

Recall that we assumed that the bins of M are large slack bins. Hence we know that for $j \in M$:

$$\begin{aligned} \sum_{i \in S_j} w(i) &\leq c_j - n2^{l-\theta} && \implies \\ \sum_{i \in S_j} \lfloor w(i)/2^{l-\theta} \rfloor &\leq c_j/2^{l-\theta} - n && \implies \\ \sum_{i \in S_j} w_\theta(i) &\leq c_j/2^{l-\theta} - n - a_j + a_j. \end{aligned}$$

So, take $a_j = w_\theta(S_j \cap W) \in w_\theta(2^{[n]})$ and indeed the correctness of the algorithm follows.

Runtime Analysis

The algorithm will go through the procedure of computing the booleans l_X and r_X for all different sets $L, R \subseteq [m]$ such that $|R| = 1$ and for all different values of $a_{k+1}, \dots, a_m \in w_\theta(2^{[n]})$. This gives a total of at most $m \cdot 2^{m-1} \cdot |w_\theta(2^{[n]})|^m$ repetitions. By Lemma 3.7, we have $|w_\theta(2^{[n]})| \leq 3n|w_{\theta-1}(2^{[n]})|$. Because θ is the critical pruner, and since $w_0(2^{[n]}) = \{0\}$, we know that $|w_{\theta-1}(2^{[n]})| \leq 2^{\delta n}$. Hence, the number of repetitions is at most $\mathcal{O}((2n)^m \cdot 2^{\delta mn})$.

Now we analyze the time complexity per choice of (L, R) and a_{k+1}, \dots, a_m . Recall that we chose $\mathcal{W} \subseteq \binom{[n]}{n/2}$ as a random set of size $2^{(1-g(m))n}$. Computing all the booleans l_X and r_X can be done in $\mathcal{O}((|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}|)n)$ time. Let $\gamma_m = \frac{g(m)}{4\log(12/g(m))}$. Notice that we can describe all $X \in \downarrow\mathcal{W}$ either as a set in $\binom{[n]}{(\frac{1}{2}-\gamma_m)n}$ (if $|X| \leq (\frac{1}{2}-\gamma_m)n$), or as a subset of a set $W \in \mathcal{W}$ together with the items that W and X differ on (if $|X| \geq (\frac{1}{2}-\gamma_m)n$). In the latter case, the sets differ at most on $\gamma_m n$ items. Therefore:

$$|\downarrow\mathcal{W}| \leq \binom{n}{(\frac{1}{2}-\gamma_m)n} + |\mathcal{W}| \binom{n}{\gamma_m n} \leq 2^{h(\frac{1}{2}-\gamma_m)n} + 2^{(1-g(m)+h(\gamma_m))n}.$$

Notice that $|\uparrow\mathcal{W}|$ can be bounded in the same way. We apply Lemma B.2 with $b := c := 1$, $\gamma := \gamma_m$ and $x := g(m)$. Since $\gamma_m \leq g(m)/(4\log_2 \frac{12}{g(m)})$ it implies that $h(\frac{1}{2}-\gamma_m) \geq 1-g(m)+h(\gamma_m)$. Lastly, Lemma B.1 tells us that $h(\frac{1}{2}-\gamma_m) \leq 1 - \frac{2}{\ln(2)}(\gamma_m)^2$. Hence, $|\downarrow\mathcal{W}| + |\uparrow\mathcal{W}| = \mathcal{O}(2^{(1-f_C(m))n})$ where

$$f_C(m) = \frac{g(m)^2}{8\ln(2)(\log(12/g(m)))^2} = \frac{h(\frac{1}{2m})^2}{32\ln(2)\log(24/h(\frac{1}{2m}))^2} = \Omega_{m \rightarrow \infty} \left(\frac{h(\frac{1}{2m})^2}{\log^2(h(\frac{1}{2m}))} \right).$$

Combining this with the number of repetitions we get a run time of $\mathcal{O}(2^m \cdot 2^{(1-f_C(m)+\delta m)n})$. This gives us the requested runtime. \square

3.6 Proof of Theorem 1.1

We are now ready to prove Theorem 1.1 by combining all work of the previous sections and setting the parameters α and δ :

Proof. We will now combine all previous lemmas. An overview of the algorithm can be found in Figure 3. To facilitate the asymptotic analysis, note we can assume the number of bins m is at least m_0 for some constant m_0 . If this is not the case we can add $m_0 - m$ artificial bins with unique small capacities and a matching items. Since m_0 is constant this does not influence the asymptotic run time of the algorithm. Define $f_C(m)$ as in Lemma 3.12 as:

$$f_C(m) = \frac{h(\frac{1}{2m})^2}{32\ln(2)\log(24/h(\frac{1}{2m}))^2} = \Omega_{m \rightarrow \infty} \left(\frac{h(\frac{1}{2m})^2}{\log^2(h(\frac{1}{2m}))} \right).$$

Then, set $\delta := f_C(m)/(2m)$ and $\alpha := 2^{-2/\delta^3}$. Then $f_C(m) > 0$, $\delta > 0$ and $\alpha > 0$.

1. If $|w(2^{[n]})| \leq 2^{\delta n}$, the algorithm from Lemma 3.4 solves the instance in time $\mathcal{O}^*(2^{\frac{f_C(m)}{2}n})$.
2. If the instance has an α -unbalanced solution, the algorithm from Lemma 3.5 can detect with constant probability in time

$$\mathcal{O}^* \left(2^{\left(1 - \Omega\left(\delta^6 2^{-4/\delta^3}\right)\right)n} \right).$$

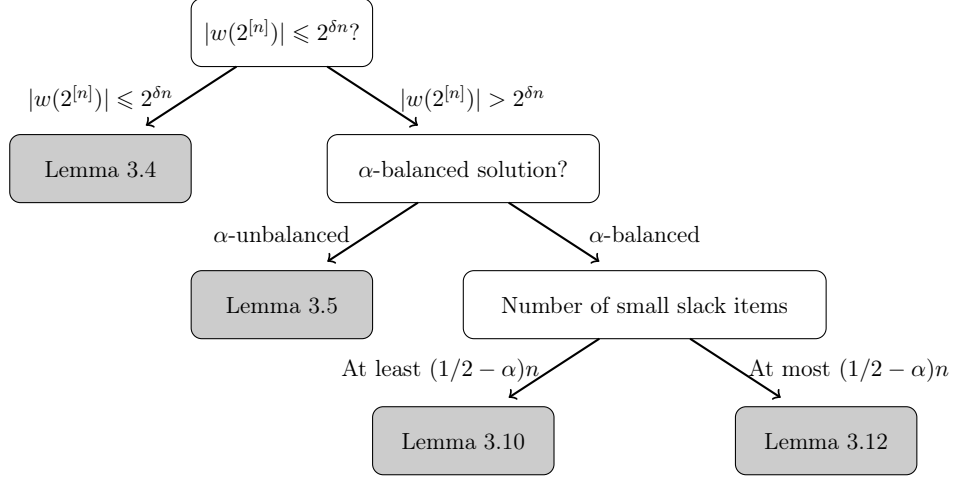


Figure 3: Overview of use of Lemma's proving Theorem 1.1.

3. If the instance has an α -balanced solution, $|w(2^{[n]})| \leq 2^{\delta n}$, and a solution with at least $(1/2 - \alpha)n$ small slack items, the upper bound $\alpha \leq 2^{-2/\delta^3}$ ensures that the solution can be detected by the algorithm from Lemma 3.10 in time

$$\mathcal{O}^* \left(2^{\left(1 - \Omega(2^{-3/\delta^3})\right)n} \right).$$

4. Otherwise, if the instance has an α -balanced solution, $|w(2^{[n]})| \leq 2^{\delta n}$, and a solution with at most $(1/2 - \alpha)n$ small slack items, the algorithm from Lemma 3.12 detect the solution with probability at least $1/2$ in time

$$\mathcal{O}^* \left(2^{\left(1 - \frac{f_C(m)}{2}\right)n} \right).$$

Thus, we obtain a probabilistic algorithm for Bin Packing that runs in time $\mathcal{O}^*(2^{(1-\sigma_m)n})$, where σ_m is a strictly positive number. Notice that any polynomial factor hidden in the \mathcal{O}^* notation can be subsumed by $\mathcal{O}(2^{(1-\sigma_m)n})$. □

4 The Littlewood Offord Theorem

In this section, we will prove our Additive Combinatorics result which we first restate for convenience:

Theorem 1.2 (restated). *Let $\varepsilon > 0$. If $\beta(w) \geq 2^{(1-\varepsilon)n}$, then $|w(2^{[n]})| \leq 2^{\delta n}$, where*

$$\delta(\varepsilon) = \mathcal{O}_{\varepsilon \rightarrow 0} \left(\frac{\log(\log(1/\varepsilon))}{\sqrt{\log(1/\varepsilon)}} \right).$$

For our proof, it will be convenient to use a reformulation of Theorem 1.2 to a version with two set families that attain the parameters and that uses vector notation (so w is a vector and $w(X)$ is the inner product $\langle w, x \rangle$ of w with the characteristic vector x of set X):

Theorem 4.1 (Theorem 1.2 reformulated). *Let $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$ be a vector with integer weights, and let $A, B \subseteq \{0, 1\}^n$ be such that $|a^{-1}(1)| = \alpha n$ for each $a \in A$ and*

- $\langle w, b \rangle = \tau$ for every $b \in B$, and
- if $a, a' \in A$ and $\langle w, a \rangle = \langle w, a' \rangle$, then $a = a'$.

If $|B| \geq 2^{(1-\varepsilon)n}$, then $|A| \leq 2^{\delta(\varepsilon)n}$, where

$$\delta(\varepsilon) = \mathcal{O}_{\varepsilon \rightarrow 0} \left(\frac{\log(\log(1/\varepsilon))}{\sqrt{\log(1/\varepsilon)}} \right).$$

We first show that this implies Theorem 4.1:

Proof of Theorem 1.2 from Theorem 4.1. Suppose w_1, \dots, w_n and τ are such that $|\{X \subseteq [n] : w(X) = \tau\}| \geq 2^{(1-\varepsilon)n}$. Then $B := \{b \in \{0, 1\}^n : \langle w, b \rangle = \tau\}$ satisfies the conditions of Theorem 4.1 and $|B| \geq 2^{(1-\varepsilon)n}$. For every $i \in w(2^{[n]})$ arbitrary choose a vector $a(i) \in \{0, 1\}^n$ such that $\langle w, a(i) \rangle = i$. Define $A' = \{a(i) : i \in w(2^{[n]})\}$. Since $|a^{-1}(1)|$ can only take n different values, there exist an α such that $|a^{-1}(1)| = \alpha n$ for at least an $1/n$ fraction of the elements of A' . This gives a set A that satisfies the condition of Theorem 4.1, and thus

$$|w(2^{[n]})| = |A'| \leq |A| \cdot n \leq 2^{\delta(\varepsilon)n + o(n)},$$

and we can use the $\mathcal{O}(\cdot)$ notation in the term $\delta(\varepsilon)$ to hide the $2^{o(n)}$ factors. □

□

The rest of this section is dedicated to the proof of Theorem 4.1. We use the following standard definitions from Additive Combinatorics: For sets X, Y we define $X + Y$ as the sumset $\{x + y : x \in X, y \in Y\}$. For an integer k , we define $k \cdot X$ as the k -fold sum $\underbrace{X + X + \dots + X}_{k \text{ times}}$.

The starting point of the proof of Theorem 4.1 is the following simple lemma that proves that $|A||k \cdot B| = |A + k \cdot B|$. It is heavily inspired by the UDCP connection from [3, Proposition 4.2].

Lemma 4.2. *If $a, a' \in A$ and $b, b' \in k \cdot B$ are such that $a + b = a' + b'$, then $(a, b) = (a', b')$.*

Proof. Note that

$$\langle w, a \rangle + \langle w, b \rangle = \langle w, a + b \rangle = \langle w, a' + b' \rangle = \langle w, a' \rangle + \langle w, b' \rangle.$$

By definition of B , we know that $\langle w, b \rangle = \langle w, b' \rangle = k \cdot \tau$, hence $\langle w, a \rangle = \langle w, a' \rangle$. Therefore by definition of set A it has to be that $a' = a$. This implies that $b = b'$, since $a + b = a' + b'$. □

Thus $|A|$ is equal to $|A + k \cdot B|/|k \cdot B|$, and we may restrict our attention to upper bounding the latter quantity for any integer $k \in \mathbb{N}$. Since this is in general not easy, we instead define a set $P \subseteq A \times k \cdot B$ of pairs such that for each $(a, b) \in P$ the distribution of the values in the vector $a + b$ is close to what one would expect for random vectors. This is useful since the control on pairs $(a, b) \in P$ gives us control on the vectors $a + b$ which allow us to upper bound P . Moreover, we also provide a lower bound that shows that P is not much smaller than $|A| \cdot |k \cdot B|$. Combining the two bounds results in the upper bound for A . We will make this more formal in the next subsections, but first we give a warm-up result that sets up the notation for the main proof.

4.1 A Warm-up with $B = \{0, 1\}^n$

Let us first investigate what happens in a case when B is equal to the whole Boolean hypercube $\{0, 1\}^n$. While $|A|$ can be easily be upper bounded by direct methods, it is instructive to see what our approach will be in this special case. In this setting we can think about vectors from B as sampled uniformly at random. Fix a parameter $0 < \alpha < 1$, let $a \in \{0, 1\}^n$ be a fixed, adversarially chosen vector with $|a^{-1}(1)| = \alpha n$, and let $b_1, \dots, b_k \in \{0, 1\}^n$ be independently sampled random vectors. Let $b = b_1 + b_2 + \dots + b_k$ and $c = a + b$. Observe that for every $i \in \{0, \dots, k\}$ and $i' \in \{0, \dots, k+1\}$

$$\mathbb{E} \left[\frac{|b^{-1}(i)|}{n} \right] = \binom{k}{i} 2^{-k}, \text{ and } \mathbb{E} \left[\frac{|c^{-1}(i')|}{n} \right] = \left((1 - \alpha) \binom{k}{i'} + \alpha \binom{k}{i' - 1} \right) 2^{-k}.$$

For further reference, we now define the found distributions explicitly:

Definition 4.3 (Altered Binomial Distribution). *For every $k \in \mathbb{N}$, we let $\text{Bin}(k)$ denote the binomial distribution $(\{0, \dots, k\}, p)$ where $p(i) = \binom{k}{i} 2^{-k}$.*

For an additional parameter $\alpha \in (0, 1)$, we define the altered binomial distribution $\text{Bin}(k, \alpha)$ as $(\{0, \dots, k+1\}, p')$ where $p'(i) = (1 - \alpha) \binom{k}{i} 2^{-k} + \alpha \binom{k}{i-1} 2^{-k}$.

Note, that $\text{Bin}(k+1) := \text{Bin}(k, 1/2)$ by Pascal's Formula. Now, we present the intuition for the random case. We have that:

$$n \cdot h(\text{Bin}(k, \alpha)) = h(c) = h(a, b) = h(a) + h(b) = h(a) + n \cdot h(\text{Bin}(k)),$$

where the second equality follows by Lemma 4.2 and the third inequality follows because a and b are independent. Thus $h(a) = n(h(\text{Bin}(k, \alpha)) - h(\text{Bin}(k)))$, and the proof in the random case can be concluded by using Lemma 4.8 in which we show that for any constant $\alpha \in (0, 1)$ it holds that $h(\text{Bin}(k, \alpha)) - h(\text{Bin}(k)) = \mathcal{O}_{k \rightarrow \infty}((\log k)/\sqrt{k})$, and the standard fact that the support of any uniform random variable of entropy h is at most 2^h .

To extend this to the setting where $B \subset \{0, 1\}^n$ we need to obtain vectors b_1, \dots, b_k that are sufficiently random. This condition will translate to $\varepsilon \leq 1/2^{\mathcal{O}(k)}$, that will enforce $k := \mathcal{O}(\log(1/\varepsilon))$. The following chain of (in-)equalities summarizes the strategy of our proof.

$$|A| = \frac{|A + k \cdot B|}{|k \cdot B|} \leq \frac{|P|}{|k \cdot B|} \cdot 2^{f(\varepsilon, k)n} \leq 2^{n(h(\text{Bin}(k+1)) - h(\text{Bin}(k)))} \cdot 2^{f(\varepsilon, k)n} \leq 2^{\delta(\varepsilon)n}$$

Lemma 4.2
Section 4.2
Section 4.3
Lemma 4.8

Now, we will make the above idea more precise.

4.2 Balanced Pairs

The following definition quantifies the ‘sufficiently random’ terms from the previous subsection by measuring how far the distribution of the values of a vector are from a given (expected) distribution.

Definition 4.4 (Balanced vectors). *Let $\mathcal{D} = (\Omega, p)$ be a discrete probability space. Fix $\gamma \in (0, 1)$. Let U be the finite universe set and let $X \subseteq U$. A mapping (or a vector) $v \in \Omega^U$ is γ - \mathcal{D} balanced for X if for all $\omega \in \Omega$ it holds that*

$$\frac{|v^{-1}(\omega) \cap X|}{|X|} \in [p(\omega) \pm \gamma].$$

As a shorthand we say a mapping (or a vector) $v \in \Omega^U$ is γ - \mathcal{D} balanced if it is γ - \mathcal{D} balanced for U . We denote the set of all γ - \mathcal{D} balanced vectors $v \in \Omega^U$ with $(\mathcal{D} \pm \gamma)^U$.

As an illustration of Definition 4.4, suppose

$$U = \{1, \dots, 6\}, \quad X = \{1, \dots, 4\}, \quad \Omega = \{0, 1\}, \quad p(0) = p(1) = \frac{1}{2}, \quad \mathcal{D} = (\Omega, p).$$

Then $(0, 1, 1, 1, 1, 1)$ is $\frac{1}{4}$ - \mathcal{D} balanced for X but not $\frac{1}{4}$ - \mathcal{D} balanced. The vector $(0, 0, 0, 0, 1, 1)$ is not $\frac{1}{4}$ - \mathcal{D} balanced for X but it is $\frac{1}{6}$ - \mathcal{D} balanced.

We will use Definition 4.4 with \mathcal{D} being the distribution we would get in the random case as outlined in Subsection 4.1 (hence \mathcal{D} will usually be $\text{Bin}(k)$ or $\text{Bin}(k, \alpha)$). Now, we prove a general upper bound on the number of γ - \mathcal{D} balanced vectors.

Lemma 4.5. *Let $\mathcal{D} = (\Omega, p)$ be a discrete probability space. The number of γ - \mathcal{D} balanced vectors is at most*

$$2^{(h(\mathcal{D}) + f(\Omega, \gamma))|U|},$$

where $f(\Omega, \gamma) := \mathcal{O}(|\Omega| \cdot \gamma \log(1/\gamma))$.

Proof. The number of γ - \mathcal{D} balanced vectors is at most $\sum_q \binom{|U|}{q \cdot |U|}$, where the sum is over all probability distributions q such that $q \cdot |U|$ is a vector with integer coordinates and $q(\omega) \in [p(\omega) \pm \gamma]$ for every $\omega \in \Omega$. Since the number of possibilities for such a q is at most $|U|^{|\Omega|}$ and $\binom{|U|}{q \cdot |U|} \leq 2^{h(q)|U|}$ by Lemma 2.8, we obtain

$$\sum_q \binom{|U|}{q \cdot |U|} \leq |U|^{|\Omega|} 2^{h(q)|U|} \leq |U|^{|\Omega|} 2^{(h(\mathcal{D}) + \ln(2)|\Omega| \gamma \log \frac{1}{\gamma})|U|} \leq 2^{h(\mathcal{D})|U|} \cdot 2^{\mathcal{O}(|U||\Omega| \gamma \log(\frac{1}{\gamma}))},$$

where the second inequality follows from Lemma 2.9 and the third comes from $|U|^{|\Omega|} \leq 2^{|U||\Omega|}$. \square

For example, Lemma 4.5 bounds the number of γ - $\text{Bin}(k)$ balanced vectors by $2^{nh(\text{Bin}(k)) + nf(\gamma, k)}$ for some positive function $f(\gamma, k) \rightarrow 0$ when $\gamma \rightarrow 0$.

With Definition 4.4 in hand, we are ready to define the set of pairs mentioned in the start of this section:

$$P := \{(a, b) \in A \times k \cdot B : b \in B \text{ is } \varepsilon^{0.01}\text{-Bin}(k) \text{ balanced for } a\}.$$

We will devote Section 5 to the proof of the following somewhat technical lemma:

Lemma 4.6. *Let $k < 0.01 \cdot \log(1/\varepsilon)$. Then, for every $a \in \{0, 1\}^n$ with $|a^{-1}(1)| > \varepsilon^{0.01}n$, there exists $E_a \subseteq k \cdot B$, such that $|E_a| \geq 2^{(h(\text{Bin}(k)) - \varepsilon^{0.1})n}$ and $(a, b) \in P$ for every $b \in E_a$.*

Note, that we can assume that $\alpha > \varepsilon^{0.01}$ because otherwise $|A| \leq \binom{n}{\varepsilon^{0.01}n} \leq 2^{\varepsilon^{0.01} \log(4/\varepsilon)n} \leq 2^{\delta n}$ and Theorem 4.1 follows automatically.

Thus, we may apply Lemma 4.6 for each $a \in A$ and obtain that

$$|P| \geq |A| \cdot 2^{(h(\text{Bin}(k)) - \varepsilon^{0.1})n}. \quad (3)$$

On the other hand, the balancedness property can be used to give an upper bound on P via Lemma 4.2. To do so, the following will be useful:

Lemma 4.7. *If $(a, b) \in P$, then $a + b$ is $(2\varepsilon^{0.01})$ - $\text{Bin}(k, \alpha)$ balanced.*

Proof. From the definition of P , vector b is $\varepsilon^{0.01}$ -Bin(k) balanced for a . So, for every $i \in \{0, \dots, k\}$:

$$|a^{-1}(1) \cap b^{-1}(i)| \in \left[\binom{k}{i} \frac{\alpha n}{2^k} \pm \varepsilon^{0.01} n \right]$$

And similarly,

$$|a^{-1}(0) \cap b^{-1}(i)| \in \left[\binom{k}{i} \frac{(1-\alpha)n}{2^k} \pm \varepsilon^{0.01} n \right]$$

It follows that for every $i \in \{0, \dots, k+1\}$ it holds that:

$$|(a+b)^{-1}(i)| \in \left[\binom{k}{i} \frac{(1-\alpha)n}{2^k} + \binom{k}{i-1} \frac{\alpha n}{2^k} \pm 2\varepsilon^{0.01} n \right].$$

□

Next we define function $\eta(a, b) := a + b$. Observe that η is an injective function on $A \times k \cdot B$ by Lemma 4.2, and since $P \subseteq A \times k \cdot B$ we have $|\eta(P)| = |P|$. By Lemma 4.7, every vector in $\eta(P)$ is $(2\varepsilon^{0.01})$ -Bin(k, α) balanced, and thus Lemma 4.5 implies

$$|P| \leq 2^{n \cdot h(\text{Bin}(k, \alpha))} \cdot 2^{\mathcal{O}(nk\varepsilon \log(1/\varepsilon))}. \quad (4)$$

4.3 Proof of Theorem 4.1

By combining (3) and (4) we obtain the following bound:

$$|A| \leq 2^{n(h(\text{Bin}(k, \alpha)) - h(\text{Bin}(k)))} \cdot 2^{\mathcal{O}((\varepsilon^{0.01} + k\varepsilon \log \frac{1}{\varepsilon})n)}. \quad (5)$$

By Lemma B.4 we in fact have that $h(\text{Bin}(k, \alpha)) \leq h(\text{Bin}(k+1))$, and thus it remains to bound the difference in entropy of two consecutive binomial distributions as follows:

Lemma 4.8. *For large enough k , we have that $h(\text{Bin}(k)) - h(\text{Bin}(k-1)) \leq \frac{\log k}{\sqrt{k}}$.*

Before we present the proof of Lemma 4.8, let us see how to use it. We choose $k := \Theta(\log(1/\varepsilon))$. Thus Lemma 4.8 implies that

$$|A| \leq 2^{n(\log k / \sqrt{k} + \varepsilon^{0.01} \log(1/\varepsilon))} = 2^{\mathcal{O}(n \cdot \delta(\varepsilon))},$$

where $\delta(\varepsilon) = \mathcal{O}_{\varepsilon \rightarrow 0} \left(\frac{\log(\log(1/\varepsilon))}{\sqrt{\log(1/\varepsilon)}} \right)$, because $\varepsilon^{0.01} \log(1/\varepsilon) \ll \delta(\varepsilon)$ for small enough ε . This finishes the proof of Theorem 4.1.

Proof of Lemma 4.8. For every $i, k \in \mathbb{N}$ such that $i \leq k$ let us define an auxiliary function:

$$f(k, i) := \frac{\binom{k}{i}}{2^k} \log \left(\frac{2^k}{\binom{k}{i}} \right).$$

Thus we have $h(\text{Bin}(k)) = \sum_{i=0}^k f(k, i)$. To relate $h(\text{Bin}(k))$ with $h(\text{Bin}(k-1))$, the following will be useful:

Claim 4.9.

$$f(k, i) \leq \begin{cases} f(k-1, i), & \text{if } i < \lfloor k/2 \rfloor, \\ f(k-1, i-1), & \text{if } i \geq k/2. \end{cases} \quad (6)$$

Proof. Define $g(x) = x \cdot \log(1/x)$. Since its derivative is $g'(x) = -\frac{\ln(x)+1}{\ln(2)}$ we have that $g(x) \leq g(x')$ whenever $x \leq x' \leq 1/e$.

Note $f(k, i) = g(\binom{k}{i}/2^k)$, and since $\binom{k}{i}/2^k \leq 1/\sqrt{k}$ by the standard bound $\binom{k}{i} \leq 2^k/\sqrt{k}$, we have $\binom{k}{i}/2^k \leq 1/e$ for $k \geq 9$ large enough. Thus to prove the claim it remains to show that

$$\binom{k}{i} 2^{-k} \leq \begin{cases} \binom{k-1}{i} 2^{-(k-1)}, & \text{if } i < \lfloor k/2 \rfloor, \\ \binom{k-1}{i-1} 2^{-(k-1)}, & \text{if } i \geq k/2. \end{cases}$$

To see this first suppose $i < \lfloor k/2 \rfloor$. Then we have that

$$\binom{k}{i} 2^{-k} = \binom{k-1}{i} \frac{k}{k-i-1} 2^{-k} \leq \binom{k-1}{i} 2^{-(k-1)}.$$

Second, if $i \geq k/2$, then we have that

$$\binom{k}{i} 2^{-k} = \binom{k-1}{i-1} \frac{k-1}{i-1} 2^{-k} \leq \binom{k-1}{i-1} 2^{-(k-1)}.$$

□

Now we can use Claim 4.9 to give the required upper bound:

$$\begin{aligned} h(\text{Bin}(k)) &= \sum_{i=0}^k f(k, i) \\ &= \left(\sum_{i=0}^{\lfloor k/2 \rfloor - 1} f(k, i) \right) + \left(\sum_{i=\lfloor k/2 \rfloor + 1}^k f(k, i) \right) + f(k, \lfloor k/2 \rfloor) \\ &\leq \left(\sum_{i=0}^{\lfloor k/2 \rfloor - 1} f(k-1, i) \right) + \left(\sum_{i=\lfloor k/2 \rfloor + 1}^k f(k-1, i-1) \right) + f(k, \lfloor k/2 \rfloor) \\ &= h(\text{Bin}(k-1)) + f(k, \lfloor k/2 \rfloor) \\ &\leq h(\text{Bin}(k-1)) + \log(k)/\sqrt{k}, \end{aligned}$$

where we use Claim 4.9 in the first inequality, and $\binom{k}{\lfloor k/2 \rfloor} \leq 2^k/\sqrt{k}$ in the second inequality. Hence $h(\text{Bin}(k)) - h(\text{Bin}(k-1)) \leq \log(k)/\sqrt{k}$. □

5 Properties of $k \cdot B$: Proof of Lemma 4.6

In this section we prove the Lemma 4.6 that we used in Section 4 to prove Theorem 1.2.

Lemma 4.6 (restated). *Let $k < 0.01 \cdot \log(1/\varepsilon)$. Then, for every $a \in \{0, 1\}^n$ with $|a^{-1}(1)| > \varepsilon^{0.01}n$, there exists $E_a \subseteq k \cdot B$, such that $|E_a| \geq 2^{(h(\text{Bin}(k)) - \varepsilon^{0.1})n}$ and $(a, b) \in P$ for every $b \in E_a$.*

Recall that

$$P := \{(a, b) \in A \times k \cdot B : b \in B \text{ is } \varepsilon^{0.01}\text{-Bin}(k) \text{ balanced for } a\}.$$

Intuitively, we prove that for any fixed set $B \subseteq \{0, 1\}^n$ there exists a large set $E_a \subseteq k \cdot B$ with the following property: for every $b \in E_a$ we can perturb $\varepsilon^{0.01}n$ entries in b , such that it is indistinguishable from a vector randomly sampled from the binomial distribution, even if we focus on a concrete subset of coordinates $a^{-1}(1) \subseteq [n]$.

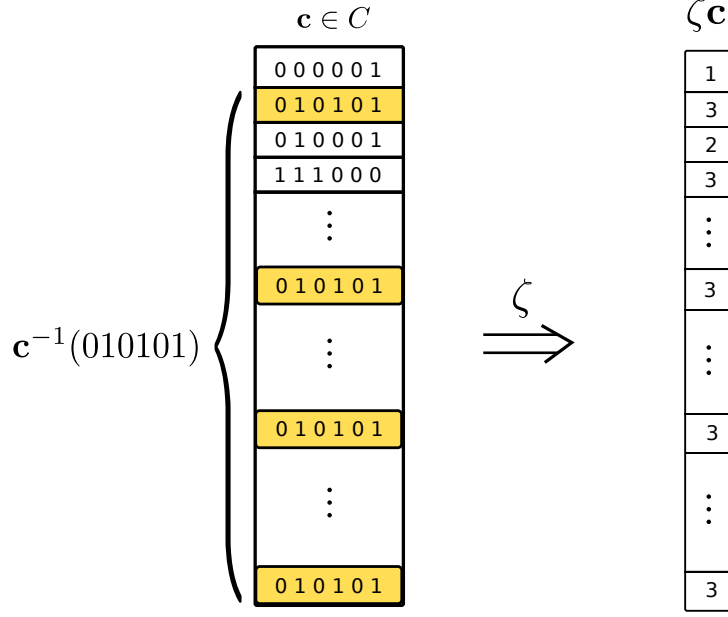


Figure 4: The ζ operation takes a $x \times y$ matrix $\mathbf{c} \in (\mathbb{Z}^x)^y$ as input and outputs a vector $\zeta(\mathbf{c}) \in \mathbb{Z}^x$ by adding all columns.

First, observe that we can interpret a tuple $(b_1, \dots, b_k) \in B^k$ as the $n \times k$ matrix with the i 'th column equal to b_i . We interchangeably address such a tuple as an $n \times k$ matrix and as an element of $(\{0, 1\}^n)^k$. To emphasize the type of such variables, we denote such matrices with bold face. For example, (b_1, \dots, b_k) is denoted with $\mathbf{b} \in (\{0, 1\}^n)^k$.

Using the notation \mathbf{b}^T to denote the transpose of a matrix, we let $C := \{\mathbf{b}^T : \mathbf{b} \in B^k\} \subseteq (\{0, 1\}^k)^n$ denote the set of matrices B^k interpreted in the transposed way.

In Section 5.1 we show how to select a subset $D \subseteq C$ of matrices in C , in such a way that for all $\mathbf{b} \in D \subseteq (\{0, 1\}^k)^n$, any column $z \in \{0, 1\}^k$ occurs $\frac{n}{2^k} \pm f(\varepsilon)n$ times in \mathbf{b} , that is $|\mathbf{b}^{-1}(z)| \in [\frac{n}{2^k} \pm f(\varepsilon)n]$.

Next, in Section 5.2 we define the operation $\zeta((a_1, \dots, a_x)) := \sum_{i=1}^x a_i$, that sums the columns of a matrix $\mathbf{a} \in (\mathbb{Z}^y)^x$ to a single column $\zeta(\mathbf{a}) \in \mathbb{Z}^y$ (see Figure 4). We consider the set $E := \{\zeta(\mathbf{b}^T) : \mathbf{b} \in D\} \subseteq \{0, \dots, k\}^n$, and argue that each vector in E is γ -Bin(k) balanced for some small $\gamma > 0$.

Finally, in Section 5.3 we take care of $a \in \{0, 1\}^n$ and select the set $E_a \subseteq E$ to be all vectors in E that are $\varepsilon^{0.01}$ -Bin(k) balanced for $a^{-1}(1)$.

Uniform distribution. We define the *uniform distribution* to be $\text{Uni}(\Omega) = (\Omega, p)$ if $p(\omega) = \frac{1}{|\Omega|}$ for each $\omega \in \Omega$. We will focus on the special cases when $\Omega = \{0, 1\}$ and $\Omega = \{0, 1\}^k$. Thus, $v \in (\text{Uni}(\{0, 1\}) \pm \gamma)^{[n]}$ means that $v^{-1}(i) \in [n/2 \pm \gamma n]$ for all $i \in \{0, 1\}$. Similarly, $\mathbf{v} \in (\text{Uni}(\{0, 1\}^k) \pm \gamma)^{[n]}$ means that $\mathbf{v}^{-1}(z) \in [n/2^k \pm \gamma n]$ for all $z \in \{0, 1\}^k$. If Ω is clear from the context, $v \in \Omega^U$ and $X \subseteq U$, we also say that a vector is γ -uniform for X to refer to the statement that it is γ -Uni(Ω) balanced for X .

Inequalities. Through the section we assume that $\varepsilon \leq 1/2^{4k}$, $\gamma := 4\sqrt{\varepsilon}$ and $k > 100$ is an integer. This means that the following inequalities hold:

$$\varepsilon^{1/2} \leq 2^k \gamma \leq \varepsilon^{1/4}, \quad (7)$$

$$100 \cdot k 2^k \gamma \log(1/(2^k \gamma)) \leq \varepsilon^{1/5}. \quad (8)$$

5.1 Constructing a set D of uniform k -tuples

We first prove the following result that will be helpful to obtain the aforementioned set C .

Lemma 5.1 (Most vectors in B are uniform). *Let $U_1 \uplus \dots \uplus U_\ell = [n]$ be a partition such that $|U_i| \geq \mu n$ for all $i \in [\ell]$. Let $\lambda \in (0, 1/2]$. For every $B \subseteq \{0, 1\}^n$ with $|B| \geq 2^{(1-\mu\lambda^2)n-o(n)}$ it holds that:*

$$|\{b \in B : b \text{ is } \lambda\text{-uniform in } U_i \text{ for every } i\}| \geq |B|/2.$$

Proof. For a fixed i we argue that number of vectors that are not λ -uniform for U_i is bounded by $2^{(1-\lambda^2\mu)n+o(n)}$. This will finish the proof, since we can sum this bound over all partitions.

Let $s = |U_i|/n$ and note that $s \geq \mu$. Observe that the number of vectors $v \in \{0, 1\}^n$ such that $|v^{-1}(1) \cap U_i| \notin [sn/2 \pm \lambda sn]$ is at most:

$$\sum_{\lambda' \notin [-\lambda, \lambda]} \binom{sn}{sn/2 - \lambda' sn} 2^{n-sn}$$

because a vector v that is not λ -uniform on U_i can be arbitrary in $[n] \setminus U_i$. We upper bound this with binary entropy function

$$\sum_{0 \leq \lambda' \leq \lambda} 2^{sn \cdot h(1/2 - \lambda') + o(n)} \cdot 2^{n-sn}.$$

The expression is maximized when $\lambda' = \lambda$ because the $h(p)$ entropy function is increasing in $[0, 0.5]$. Hence we can upper bound the expression with

$$n 2^{(1+s(h(1/2-\lambda)-1))n+o(n)}.$$

Now, we use a bound $h(1/2 - x) \leq 1 - x^2$ when $0 \leq x \leq 1/2$ (recall that $\lambda \in [0, 1/2]$) and obtain that the number of vectors that are not λ -uniform for U_i is at most

$$2^{(1-s\lambda^2)n+o(n)} \leq 2^{(1-\mu\lambda^2)n+o(n)}$$

Thus, by summing over all U_i , the number of vectors that are not λ -uniform for some U_i is at most $2^{(1-\mu\lambda^2)n+o(n)}$, and the number of vectors in B that are λ -uniform for all U_i is at least

$$|B| - 2^{(1-\mu\lambda^2)n+o(n)} \geq |B|/2,$$

and the claim follows. \square

Set a balance parameter $\gamma := 4\sqrt{\varepsilon}$, and define

$$D := C \cap (\text{Uni}(\{0, 1\}^k) \pm \gamma)^{[n]}.$$

Lemma 5.2 (Most k -tuples are uniform). *Let $k \in \mathbb{N}$ be such that $\varepsilon < 1/4^{k+2}$. Then it holds that*

$$|D| \geq \left(\frac{|B|}{2}\right)^k.$$

Proof. We denote $C_j \subseteq (\{0, 1\}^j)^{[n]}$ to select all matrices obtain by removing the first j columns of matrices C , namely

$$C_j := \{\mathbf{b}^T : \mathbf{b} \in B^j\}.$$

Thus, $C = C_k$. For $j \in \{1, \dots, k\}$, let

$$D_j := C_j \cap (\text{Uni}(\{0, 1\}^j) \pm \gamma)^{[n]}.$$

We prove that $|D_j| \geq (\frac{|B|}{2})^j$ by induction on k . First we prove the base case $j = 1$ of the induction, so $|D_1| \geq |B|/2$. This follows by applying Lemma 5.1 with $\lambda = \sqrt{\varepsilon}$ and partition $U_1 = [n]$, since it implies that

$$|B|/2 \leq \left| B \cap (\text{Uni}(\{0, 1\}) \pm \sqrt{\varepsilon})^{U_1} \right| \leq \left| C_1 \cap (\text{Uni}(\{0, 1\}) \pm \gamma)^{U_1} \right| = |D_1|.$$

The induction step with $j > 1$ is a direct consequence of the following claim, which therefore is sufficient to finish the proof.

Claim 5.3. *Let $\mathbf{b} \in D_{j-1}$. Then there are at least $|B|/2$ vectors $b_j \in B$, such that $\mathbf{b}_+ \in (\text{Uni}(\{0, 1\}^j) \pm \gamma)^{[n]}$, where \mathbf{b}_+ is obtained from \mathbf{b} by appending b_j as the j 'th row to it.*

Proof. Define a partition $\{U_z\}_{z \in \{0, 1\}^{j-1}}$ of $[n]$ by $U_z = \mathbf{b}^{-1}(z)$. Because $\mathbf{b} \in (\text{Uni}(\{0, 1\}^{(j-1)}) \pm \gamma)^{[n]}$ we know that:

$$\mu := \min_{z \in \{0, 1\}^{j-1}} |U_z|/n \geq \frac{1}{2^{j-1}} - \gamma.$$

Note, that $\mu > 1/2^j$ because we assumed that $\varepsilon < 1/4^{k+2}$ (hence $\gamma < 1/2^j$). Now, we use Lemma 5.1 with partition $\{U_z\}_{z \in \{0, 1\}^{j-1}}$ and $\lambda := 2^{j-3} \cdot \gamma$. First let us assert that the condition $|B| \geq 2^{(1-\mu\lambda^2)n+o(n)}$ holds. Recall that we assumed $|B| \geq 2^{(1-\varepsilon)n+o(n)}$ and $\mu\lambda^2 \geq \frac{1}{2^j}(2^{j-3} \cdot 4\sqrt{\varepsilon})^2 \geq 2^{j-2}\varepsilon \geq \varepsilon$ (for $j \geq 2$). Hence $|B| \geq 2^{(1-\varepsilon)n} \geq 2^{(1-\mu\lambda^2)n+f(n)}$ for some function $f \in o(n)$.

Lemma 5.1 states that there are at least $|B|/2$ vectors $b_j \in B$ such that for each $z \in \{0, 1\}^{j-1}$

$$|U_z \cap b_j^{-1}(1)| \in \left[\frac{|U_z|}{2} \pm \lambda|U_z| \right]. \quad (9)$$

We know that $|U_z| \in [\frac{n}{2^{j-1}} \pm \gamma n]$ (because $\mathbf{b} \in (\text{Uni}(\{0, 1\}^{j-1}) \pm \gamma)^{[n]}$). Thus in fact (9) can be rewritten to

$$|U_z \cap b_j^{-1}(1)| \in \left[\frac{n}{2^j} \pm (\lambda|U_z| + (\gamma/2)n) \right].$$

We bound $\lambda|U_z|$ by

$$\begin{aligned} \lambda|U_z| &= 2^{j-3}\gamma|U_z| \leq 2^{j-3}\gamma \left(\frac{n}{2^{j-1}} + \gamma n \right) = \gamma n \left(\frac{1}{4} + 2^{j-3}\gamma \right) = \gamma n \left(\frac{1}{4} + 2^{j-3}4\sqrt{\varepsilon} \right) \\ &< \gamma n \left(\frac{1}{4} + 2^{j-3}4\sqrt{1/4^{k+2}} \right) = \gamma n \left(\frac{1}{4} + 2^{j-1}/2^{k+2} \right) < \gamma n \left(\frac{1}{4} + \frac{1}{4} \right) = (\gamma/2)n, \end{aligned}$$

where we use the assumption $\varepsilon \leq \frac{1}{4^{k+2}}$ in the second line of the inequality. Thus, for every $z \in \{0, 1\}^{j-1}$ we have

$$|U_z \cap b_j^{-1}(1)| \in \left[\frac{n}{2^j} \pm 2(\gamma/2)n \right].$$

Now, observe that for all $z \in \{0, 1\}^{j-1}$ it holds that:

$$U_z \cap b_j^{-1}(1) = \mathbf{b}^{-1}(z) \cap b_j^{-1}(1) = \mathbf{b}_+^{-1}(z'),$$

where $z' \in \{0, 1\}^j$ is the vector obtained from z by adding a j -th entry with value 1. Thus vector z' fulfills the condition for \mathbf{b}_+ to be in $(\text{Uni}(\{0, 1\}^j) \pm \gamma)^{[n]}$. Similarly we can prove this condition by concatenating a 0 to the vector z . Hence, for every $\mathbf{b} \in D_{j-1}$ there are at least $|B|/2$ vectors $b_j \in B$, such that $\mathbf{b}_+ \in (\text{Uni}(\{0, 1\}^j) \pm \gamma)^{[n]}$. \square

Thus this claim proves our induction hypothesis and hence the lemma. \square

5.2 Summing tuples from D gives many distinct sums

As mentioned in the beginning of this section we define the operation $\zeta((a_1, \dots, a_x)) := \sum_{i=1}^x a_i$, that sums the columns of a matrix $\mathbf{a} \in (\mathbb{Z}^y)^x$ to a single column $\zeta(\mathbf{a}) \in \mathbb{Z}^y$ (see Figure 4).

We define E to be all sums of tuples from D :

$$E := \{\zeta(\mathbf{b}^T) : \mathbf{b} \in D\} \subseteq \{0, \dots, k\}^{[n]}$$

In fact, by the assumption on D we have the following control on the distributions of the values in the vectors in E :

Lemma 5.4. *If $v \in E$, then for $j = 0, \dots, k$ it holds that $|v^{-1}(j)| \in \left[\binom{k}{j} \frac{n}{2^k} \pm \binom{k}{j} \gamma n\right]$, i.e. every vector in E is a $(2^k \gamma)$ -Bin(k) balanced vector.*

Proof. Consider an arbitrarily vector $v \in E$ and fix $j \in \{0, \dots, k\}$. From the definition of E , there exists a vector $\mathbf{b} \in D = C \cap (\text{Uni}(\{0, 1\}^k) \pm \gamma)^{[n]}$ such that $\zeta(\mathbf{b}^T) = v$. Hence for every $z \in \{0, 1\}^k$:

$$|\mathbf{b}^{-1}(z)| \in \left[\frac{n}{2^k} \pm \gamma n\right].$$

Hence, if we sum over all vectors $z \in \{0, 1\}^k$ such that $|z^{-1}(1)| = j$ we have:

$$|v^{-1}(j)| \leq \sum_{\substack{z \in \{0, 1\}^k \\ |z^{-1}(1)| = j}} \frac{n}{2^k} + \gamma n = \binom{k}{j} \frac{n}{2^k} + \binom{k}{j} \gamma n,$$

and analogously $|v^{-1}(j)| \geq \binom{k}{j} \frac{n}{2^k} - \binom{k}{j} \gamma n$. Thus indeed v is a $(2^k \gamma)$ -Bin(k) balanced vector, as desired. \square

We now show that E is sufficiently large:

Lemma 5.5. *It holds that $|E| \geq 2^{(h(\text{Bin}(k)) - \varepsilon^{0.2})n}$.*

Proof. For a vector $v \in E$ we define

$$D_v := \{\mathbf{b} \in D : \zeta(\mathbf{b}^T) = v\}.$$

By grouping all elements of D on their image with respect to ζ :

$$|D| = \sum_{v \in E} |D_v| \leq |E| \max_{v \in E} |D_v|,$$

which can be rewritten into the following lower bound on $|E|$:

$$|E| \geq |D| / \max_{v \in E} |D_v| \geq (|B|/2)^k / \max_{v \in E} |D_v| \geq 2^{k(n-\varepsilon n+1)} / \max_{v \in E} |D_v|. \quad (10)$$

Thus in the remainder of the proof we can focus on showing that for any vector $v \in E$, $|D_v| \leq 2^{n(k-h(\text{Bin}(k))+\varepsilon^{0.2})}$; the Lemma would then follow from substituting the bound in (10).

Let $\mathbf{b} \in D_v$. This means that for every $j = 0, \dots, k$:

$$\bigcup_{\substack{z \in \{0,1\}^k \\ |z^{-1}(1)|=j}} \mathbf{b}^{-1}(z) = v^{-1}(j).$$

Thus the number of possibilities for \mathbf{b} is

$$\prod_{j=0}^k \binom{k}{j}^{|v^{-1}(j)|}.$$

We multiply this quantity with $\binom{[n]}{|v^{-1}(0)|, \dots, |v^{-1}(k)|}} = \binom{[n]}{\phi \cdot n}$ where $\phi := (|v^{-1}(0)|/n, \dots, |v^{-1}(k)|/n)$ and obtain

$$\binom{[n]}{\phi \cdot n} \prod_{j=0}^k \binom{k}{j}^{|v^{-1}(j)|} \leq 2^{kn},$$

where the inequality follows since the left hand side counts partitions of $[n]$ into $\sum_{i=0}^k \binom{k}{i} = 2^k$ parts. Thus by Lemma 2.8 we have $|D_v| \leq 2^{n(k-h(\phi))}$. Because v is γ -Bin(k) balanced (since it is in E), we have

$$h(\phi) \geq h(\text{Bin}(k)) - \ln(2)k2^k\gamma \log \frac{1}{2^{k\gamma}} \geq h(\text{Bin}(k)) - \varepsilon^{0.2},$$

where the first inequality is by Lemma 2.9, and the second inequality uses that $\gamma = 4\sqrt{\varepsilon}$ (see Inequality 8). \square

5.3 Selecting a set $E_a \subseteq E$ for every $a \in A$

Lemma 5.6. *Let $\mathcal{D} = (\Omega, p)$ be a discrete probability space, and let $X \subseteq [n]$ with $|X| = \alpha n$. The number of vectors $v \in (\mathcal{D} + 0)^{[n]}$ that are not ρ - \mathcal{D} balanced for X and $[n] \setminus X$ is at most $2^{n(h(\mathcal{D}) - \alpha^2 \min(\rho^2, \log \alpha))}$.*

Proof. We define a relation $R \subseteq (\mathcal{D} + 0)^{[n]} \times \binom{[n]}{\alpha n}$ as follows:

$$(v, X) \in R \Leftrightarrow v \text{ is not } \rho\text{-}\mathcal{D} \text{ balanced for } X.$$

Additionally, let

$$\begin{aligned} R_v &= R \cap \left(\{v\} \times \binom{[n]}{\alpha n} \right), & \text{for } v \in (\mathcal{D} + 0)^{[n]}, \\ R_X &= R \cap \left((\mathcal{D} + 0)^{[n]} \times \{X\} \right), & \text{for } X \in \binom{[n]}{\alpha n}. \end{aligned}$$

Note that $|R_X|$ is the value we want to bound. Note that the mapping $(v, X) \mapsto (v \circ \pi, \pi(X))$ for any permutation $\pi : [n] \leftrightarrow [n]$ of the index set $[n]$ is an automorphism of R (i.e., $(v, X) \in R$ if and only if $(\pi(v), \pi(X)) \in R$). Therefore, we have

$$|R| = |(\mathcal{D} + 0)^{[n]}| \cdot |R_v| = |R_X| \binom{n}{\alpha n} \quad (11)$$

for a fixed v and X . By (11) we can focus on bounding $|R_v|$ instead of $|R_X|$. To do so, note that if $(v, X) \in R$ for $X \in \binom{[n]}{\alpha n}$, there must exist $\omega \in \Omega$ such that $|X \cap v^{-1}(\omega)| \notin [p(\omega)\alpha n \pm \rho\alpha n]$. We can construct any such X by first selecting a subset of $v^{-1}(\omega)$ (which has cardinality $p(\omega)n$), and then choosing the remaining elements. Hence:

$$|R_v| \leq \sum_{\omega \in \Omega} \sum_{x \notin [-\rho, \rho]} \binom{p(\omega)n}{(p(\omega) + x)\alpha n} \binom{(1 - p(\omega))n}{(1 - p(\omega) - x)\alpha n}.$$

Next, we use Lemma B.3. In our case with $\beta := p(\omega)$, $\alpha := \alpha$ and $\rho := -\alpha x$ it implies:

$$|R_v| \leq \sum_{j \in \{0, \dots, k\}} \sum_{x \notin [-\rho, \rho]} \binom{n}{\alpha n} 2^{-F_n}$$

where

$$F = \begin{cases} (\alpha x)^2, & \text{if } |\alpha x| < \alpha(1 - \alpha)p(\omega), \\ \alpha^2 \log(1/(2\alpha)), & \text{otherwise.} \end{cases}$$

Since $|x| \geq \rho$ we have $F \geq \alpha^2 \min\{\rho^2, \log(1/(2\alpha))\}$, thus

$$|R_v| \leq \binom{n}{\alpha n} 2^{-\alpha^2 \min\{\rho^2, \log(1/(2\alpha))\}n},$$

which plugged into (11) gives the desired inequality. \square

Proof of Lemma 4.6. Recall that we assume that $\alpha > \varepsilon^{0.01}$. By Lemma 5.5 E is large, and by Lemma 5.4 each vector in E is a $(2^k\gamma)$ -Bin(k) balanced vector. By the pigeon hole principle there must be a distribution $\mathcal{D} = (\{0, 1, \dots, k\}, p)$ where $p = (p_0, \dots, p_k)$ such that $|p_0 - \binom{k}{j} 2^{-k}| \leq 2^k\gamma$ for each j and E has a subset E' of at least $|E|/n^k$ vectors that are in $\mathcal{D}^{[n]}$. Hence:

$$|E'| \geq |E|/n^k \geq 2^{(h(\mathcal{D}) - \varepsilon^{0.2}n) - o(n)}$$

Now for each $a \in A$, define E_a to be all vectors in E' that are $\varepsilon^{0.05}$ - \mathcal{D} balanced for $a^{-1}(1)$. Observe that this means that vectors in E_a are $\varepsilon^{0.01}$ -Bin(k) balanced (because $\varepsilon^{0.05} + 2^k\gamma \ll \varepsilon^{0.01}$).

Applying Lemma 5.6 with E' and $a^{-1}(1)$, we get that there are at most $2^{h(\mathcal{D}) - \alpha^2 \min(\varepsilon^{0.1}, \log(1/(2\alpha)))} \leq 2^{(h(\mathcal{D}) - \varepsilon^{0.12}n)}$ vectors in E' that are not $\varepsilon^{0.05}$ - \mathcal{D} balanced. Hence:

$$|E' \setminus E_a| \leq 2^{(h(\mathcal{D}) - \varepsilon^{0.12}n)} \leq |E'|/2$$

Now the lemma follows because

$$|E_a| \geq |E'|/2 \geq 2^{(h(\mathcal{D}) - \varepsilon^{0.2}n) - o(n)} \geq 2^{(h(\text{Bin}(k) - \ln(2)2^k\gamma \log(1/(2^k\gamma)) - \varepsilon^{0.2})n)} \geq 2^{(h(\text{Bin}(k) - \varepsilon^{0.1})n)}$$

where the last inequality follows from Lemma 2.9 and Inequality 8 (since ε is small enough). \square

6 Conclusion and Open Problems

In this paper, we present a randomized $\mathcal{O}(2^{(1-\sigma_m)n})$ time algorithm for the Bin Packing problem, where $\sigma_m > 0$ and m denotes the number of bins. This is an improvement over the state-of-the-art algorithm of Björklund et al. [9] that runs in $\mathcal{O}^*(2^n)$ time for small m . Nevertheless, it still remains to give an algorithm for Bin Packing that works in $\mathcal{O}^*((2-\varepsilon)^n)$ time for arbitrarily large number of bins for some fixed constant $\varepsilon > 0$. We believe our algorithm made significant progress on this question. One open end for further research is how the number of bins influence the complexity of an instance. By the methods of [45], instances of Bin Packing with a linear number of bins (with equal capacity) can also be solved in time $\mathcal{O}(2^{(1-\varepsilon)n})$ based on a witness sampling technique similar to what we used in some of our cases. It is thus natural to wonder whether (an extension) of the methods presented in this paper are enough to give improved algorithms for all number of bins.

Our improvement is tiny and we provide only inversely exponentially small asymptotic lower bounds on σ_m . The main bottleneck in our analysis is the Additive Combinatorics result. We conjecture that the bound on $\delta(\varepsilon)$ in Theorem 1.2 can be significantly improved. This would automatically yield a better bound on the running time of our algorithm.

We believe our Additive Combinatorics result is natural and may have applications beyond the scope of this paper. As mentioned in the introduction, Littlewood-Offord theory has a wide variety of applications, and it is natural to expect that the setting that we address may be of interest in any of these settings.

In the introduction we mentioned Question 1 as one motivation for studying improved exact exponential time algorithms for the Bin Packing problem. While it is not clear whether made direct progress on this question, we do believe that some of our ideas such as the approach to narrow down the number of witnesses may inspire future work on improved algorithms for Set Cover. For example, our algorithm gives improved run times for all Set Cover instance with family $\mathcal{F} \subseteq 2^U$ such that $\mathcal{F} = H \cap \{0, 1\}^n$ where H is some hyperplane in \mathbb{R}^n via standard methods. Our methods may also inspire progress on improved exponential time algorithms for special cases of Set Cover such as the Graph Coloring problem mentioned in the introduction. To the best of our knowledge there is still no $\mathcal{O}((2-\varepsilon)^n)$ time algorithm for some $\varepsilon > 0$ to determine whether a graph admits a proper 6-coloring (see e.g. [11]).

Acknowledgement

The research leading to the results presented in this paper was partially carried out during the Parameterized Algorithms Retreat of the University of Warsaw, PARUW 2020, held in Krynica-Zdrój in February 2020. This workshop was supported by a project that has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement No 714704 (PI: Marcin Pilipczuk).

References

- [1] A. Abboud. Fine-grained reductions and quantum speedups for dynamic programming. In C. Baier, I. Chatzigiannakis, P. Flocchini, and S. Leonardi, editors, *46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, July 9-12, 2019, Patras, Greece*, volume 132 of *LIPICs*, pages 8:1–8:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

- [2] P. Austrin, P. Kaski, M. Koivisto, and J. Nederlof. Subset sum in the absence of concentration. In E. W. Mayr and N. Ollinger, editors, *32nd International Symposium on Theoretical Aspects of Computer Science, STACS 2015, March 4-7, 2015, Garching, Germany*, volume 30 of *LIPIcs*, pages 48–61. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015.
- [3] P. Austrin, P. Kaski, M. Koivisto, and J. Nederlof. Dense subset sum may be the hardest. In N. Ollinger and H. Vollmer, editors, *33rd Symposium on Theoretical Aspects of Computer Science, STACS 2016, February 17-20, 2016, Orléans, France*, volume 47 of *LIPIcs*, pages 13:1–13:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.
- [4] P. Austrin, P. Kaski, M. Koivisto, and J. Nederlof. Sharper upper bounds for unbalanced uniquely decodable code pairs. *IEEE Trans. Inf. Theory*, 64(2):1368–1373, 2018.
- [5] N. Bansal, S. Garg, J. Nederlof, and N. Vyas. Faster space-efficient algorithms for subset sum, k-sum, and related problems. *SIAM J. Comput.*, 47(5):1755–1777, 2018.
- [6] A. Becker, J. Coron, and A. Joux. Improved generic algorithms for hard knapsacks. In K. G. Paterson, editor, *Advances in Cryptology - EUROCRYPT 2011 - 30th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Tallinn, Estonia, May 15-19, 2011. Proceedings*, volume 6632 of *Lecture Notes in Computer Science*, pages 364–385. Springer, 2011.
- [7] A. Björklund, T. Husfeldt, P. Kaski, and M. Koivisto. Fourier meets möbius: fast subset convolution. In D. S. Johnson and U. Feige, editors, *Proceedings of the 39th Annual ACM Symposium on Theory of Computing, San Diego, California, USA, June 11-13, 2007*, pages 67–74. ACM, 2007.
- [8] A. Björklund, T. Husfeldt, P. Kaski, and M. Koivisto. Counting paths and packings in halves. In A. Fiat and P. Sanders, editors, *Algorithms - ESA 2009, 17th Annual European Symposium, Copenhagen, Denmark, September 7-9, 2009. Proceedings*, volume 5757 of *Lecture Notes in Computer Science*, pages 578–586. Springer, 2009.
- [9] A. Björklund, T. Husfeldt, and M. Koivisto. Set partitioning via inclusion-exclusion. *SIAM J. Comput.*, 39(2):546–563, 2009.
- [10] A. Björklund, P. Kaski, and I. Koutis. Directed hamiltonicity and out-branchings via generalized laplacians. In I. Chatzigiannakis, P. Indyk, F. Kuhn, and A. Muscholl, editors, *44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland*, volume 80 of *LIPIcs*, pages 91:1–91:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.
- [11] J. M. Byskov. Enumerating maximal independent sets with applications to graph colouring. *Oper. Res. Lett.*, 32(6):547–556, 2004.
- [12] C. Calabro. *The exponential complexity of satisfiability problems*. PhD thesis, UC San Diego, 2009.
- [13] E. G. Coffman Jr., J. Csirik, G. Galambos, S. Martello, and D. Vigo. *Bin Packing Approximation Algorithms: Survey and Classification*, pages 455–531. Springer New York, New York, NY, 2013.
- [14] I. Csiszár and P. C. Shields. *Information theory and statistics: A tutorial*. Now Publishers Inc, 2004.

- [15] M. Cygan, H. Dell, D. Lokshtanov, D. Marx, J. Nederlof, Y. Okamoto, R. Paturi, S. Saurabh, and M. Wahlström. On problems as hard as CNF-SAT. In *Proceedings of the 27th Conference on Computational Complexity, CCC 2012, Porto, Portugal, June 26-29, 2012*, pages 74–84. IEEE Computer Society, 2012.
- [16] M. Cygan, H. Dell, D. Lokshtanov, D. Marx, J. Nederlof, Y. Okamoto, R. Paturi, S. Saurabh, and M. Wahlström. On problems as hard as CNF-SAT. *ACM Trans. Algorithms*, 12(3):41:1–41:24, 2016.
- [17] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.
- [18] M. Delorme, M. Iori, and S. Martello. Bin packing and cutting stock problems: Mathematical models and exact algorithms. *European Journal of Operational Research*, 255(1):1–20, 2016.
- [19] I. Diakonikolas and R. A. Servedio. Improved approximation of linear threshold functions. *Comput. Complex.*, 22(3):623–677, 2013.
- [20] S. Eilon and N. Christofides. The loading problem. *Management Science*, 17(5):259–268, 1971.
- [21] K. Eisemann. The trim problem. *Management Science*, 3(3):279–284, 1957.
- [22] F. V. Fomin and P. Kaski. Exact exponential algorithms. *Commun. ACM*, 56(3):80–88, 2013.
- [23] F. V. Fomin and D. Kratsch. *Exact Exponential Algorithms*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2010.
- [24] A. Frank and É. Tardos. An application of simultaneous diophantine approximation in combinatorial optimization. *Combinatorica*, 7(1):49–65, 1987.
- [25] M. X. Goemans and T. Rothvoß. Polynomiality for bin packing with a constant number of item types. In C. Chekuri, editor, *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014*, pages 830–839. SIAM, 2014.
- [26] A. Golovnev, A. S. Kulikov, and I. Mihajlin. Families with infants: A general approach to solve hard partition problems. In J. Esparza, P. Fraigniaud, T. Husfeldt, and E. Koutsoupias, editors, *Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part I*, volume 8572 of *Lecture Notes in Computer Science*, pages 551–562. Springer, 2014.
- [27] A. Golovnev, A. S. Kulikov, and I. Mihajlin. Families with infants: Speeding up algorithms for np-hard problems using FFT. *ACM Trans. Algorithms*, 12(3):35:1–35:17, 2016.
- [28] J. R. Griggs. Database security and the distribution of subset sums in \mathbb{R}^m . In *Graph Theory and Combinatorial Biology*, 1998.
- [29] G. Halász. Estimates for the concentration function of combinatorial number theory and probability. *Periodica Mathematica Hungarica*, 8(3-4):197–211, 1977.
- [30] R. Hoberg and T. Rothvoss. A logarithmic additive integrality gap for bin packing. In P. N. Klein, editor, *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19*, pages 2616–2625. SIAM, 2017.

- [31] E. Horowitz and S. Sahni. Computing partitions with applications to the knapsack problem. *J. ACM*, 21(2):277–292, 1974.
- [32] K. Jansen, S. Kratsch, D. Marx, and I. Schlotter. Bin packing with fixed number of bins revisited. *J. Comput. Syst. Sci.*, 79(1):39–49, 2013.
- [33] D. S. Johnson. *Near-optimal bin packing algorithms*. PhD thesis, Massachusetts Institute of Technology, 1973.
- [34] D. M. Kane and R. Williams. Super-linear gate and super-quadratic wire lower bounds for depth-two and depth-three threshold circuits. In D. Wichs and Y. Mansour, editors, *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2016, Cambridge, MA, USA, June 18-21, 2016*, pages 633–643. ACM, 2016.
- [35] L. V. Kantorovich. Mathematical methods of organizing and planning production. *Management science, English Translation of a 1939 paper written in Russian*, 6(4):366–422, 1960.
- [36] N. Karmarkar and R. M. Karp. An efficient approximation scheme for the one-dimensional bin-packing problem. In *23rd Annual Symposium on Foundations of Computer Science (sfcs 1982)*, pages 312–320. IEEE, 1982.
- [37] J. M. Kleinberg and É. Tardos. *Algorithm design*. Addison-Wesley, 2006.
- [38] M. Koivisto. Partitioning into sets of bounded cardinality. In J. Chen and F. V. Fomin, editors, *Parameterized and Exact Computation, 4th International Workshop, IWPEC 2009, Copenhagen, Denmark, September 10-11, 2009, Revised Selected Papers*, volume 5917 of *Lecture Notes in Computer Science*, pages 258–263. Springer, 2009.
- [39] R. Krauthgamer and O. Trabelsi. The Set Cover Conjecture and Subgraph Isomorphism with a Tree Pattern. In R. Niedermeier and C. Paul, editors, *36th International Symposium on Theoretical Aspects of Computer Science (STACS 2019)*, volume 126 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 45:1–45:15, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
- [40] C. Lenté, M. Liedloff, A. Soukhal, and V. T’Kindt. On an extension of the sort & search method with application to scheduling theory. *Theor. Comput. Sci.*, 511:13–22, 2013.
- [41] J. E. Littlewood and A. C. Offord. On the number of real roots of a random algebraic equation. *Journal of the London Mathematical Society*, s1-13(4):288–295, 1938.
- [42] S. Martello and P. Toth. *Knapsack Problems: Algorithms and Computer Implementations*. Wiley Series in Discrete Mathematics and Optimization. Wiley, 1990.
- [43] R. Meka, O. Nguyen, and V. Vu. Anti-concentration for polynomials of independent random variables. *Theory Comput.*, 12(1):1–17, 2016.
- [44] M. Mucha, J. Nederlof, J. Pawlewicz, and K. Wegrzycki. Equal-subset-sum faster than the meet-in-the-middle. In M. A. Bender, O. Svensson, and G. Herman, editors, *27th Annual European Symposium on Algorithms, ESA 2019, September 9-11, 2019, Munich/Garching, Germany*, volume 144 of *LIPIcs*, pages 73:1–73:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

- [45] J. Nederlof. Finding large set covers faster via the representation method. In P. Sankowski and C. D. Zaroliagis, editors, *24th Annual European Symposium on Algorithms, ESA 2016, August 22-24, 2016, Aarhus, Denmark*, volume 57 of *LIPIcs*, pages 69:1–69:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016.
- [46] J. Nederlof, E. J. van Leeuwen, and R. van der Zwaan. Reducing a target interval to a few exact queries. In B. Rovan, V. Sassone, and P. Widmayer, editors, *Mathematical Foundations of Computer Science 2012 - 37th International Symposium, MFCS 2012, Bratislava, Slovakia, August 27-31, 2012. Proceedings*, volume 7464 of *Lecture Notes in Computer Science*, pages 718–727. Springer, 2012.
- [47] T. Rothvoß. Approximating bin packing within $O(\log OPT \cdot \log \log OPT)$ bins. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 20–29, 2013.
- [48] M. Rudelson and R. Vershynin. The littlewood–offord problem and invertibility of random matrices. *Advances in Mathematics*, 218(2):600 – 633, 2008.
- [49] C. Schlegel and A. Grant. *Coordinated multiuser communications*. Springer, 2006.
- [50] T. Tao and V. H. Vu. *Additive combinatorics*, volume 105 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 2007.
- [51] K. Tikhomirov. Singularity of random bernoulli matrices. *Annals of Mathematics*, 191(2):593–634, 2020.
- [52] H. C. A. van Tilborg. An upper bound for codes in a two-access binary erasure channel (corresp.). *IEEE Trans. Inf. Theory*, 24(1):112–116, 1978.
- [53] V. V. Williams and R. R. Williams. Subcubic equivalences between path, matrix, and triangle problems. *J. ACM*, 65(5):27:1–27:38, 2018.
- [54] M. Wiman. Improved constructions of unbalanced uniquely decodable code pairs, 2017. Bachelor Thesis KTH.

A Computing the number of distinct sums and critical pruner

Lemma A.1. *Let $w : [n] \rightarrow \mathbb{N}$ be an item weight function. Then the set $w(2^{[n]})$ can be computed in time $O(|w(2^{[n]}|)$.*

Proof. Define for all $i \in \{0, \dots, n\}$ the set W_i as:

$$W_i = \{w(X) : X \subseteq [i]\}$$

Notice that $W_n = w(2^{[n]})$. We iterate over i to compute these sets, setting $W_0 = \emptyset$. Then to compute the next sets use that:

$$W_i = W_{i-1} \cup \{x + w(i) : x \in W_{i-1}\}$$

The total computation time can be upper bounded by $\sum_{i=1}^n 2|S_i| = \mathcal{O}(|W_n|)$. □

Corollary A.2. *Let $|w(2^{[n]}| \geq 2^{\delta n}$. Then the critical pruner θ can be computed in time $O(2^{\delta n})$.*

Proof. Recall the following definitions. Let $l = 1 + \lceil \log(\max_i \{w(i)\}) \rceil$. For $s \in \{0, \dots, l\}$, the s -pruned weight of item i is $w_s(i) := \lfloor w(i)/2^{l-s} \rfloor$. The critical pruner, θ , is $\theta = \min\{s \in \mathbb{N} : |w_s(2^{[n]})| \geq 2^{\delta n}\}$. Notice that we can assume $l = n^{O(1)}$ by [24].

The algorithm finds θ by computing $|w_s(2^{[n]})|$ using Lemma A.1 for each $s = 1, 2, \dots$ until $|w_s(2^{[n]})| \geq 2^{\delta n}$. Take this s as θ . Because $w_0(2^{[n]}) = \{0\}$ and Lemma 3.7 tells us that $(\frac{1}{3n}) |w_s(2^{[n]})| \leq |w_{s-1}(2^{[n]})|$ for any s , we know that $|w_\theta(2^{[n]})| = \mathcal{O}(2^{\delta n})$. Therefore, the algorithm will take $\mathcal{O}(2^{\delta n})$ times per iteration and will repeat at most l times, which gives the requested runtime. \square

B Inequalities with binomials and entropy

Let us start with the useful facts about binary entropy function.

$$h(x) := -x \log(x) - (1-x) \log(1-x).$$

The first derivative of binary entropy is:

$$h'(x) := \log(1-x) - \log(x)$$

The second derivative:

$$h''(x) := -\frac{1}{(\ln 2)x(1-x)}$$

and we will also need third derivative

$$h'''(x) := \frac{1-2x}{(\ln 2)x^2(1-x)^2}$$

Observe, that for $x \in [0, 0.5]$ we have that $h'(x) \geq 0$, $h''(x) \leq 0$ and $h'''(x) \geq 0$. From 4th derivative we will only need that $h^{(4)}(x) \leq 0$ when $x \in [0, 0.5]$.

Hence from Taylor expansion for $x \in [0, 0.5]$ it holds that:

$$h(x + \varepsilon) := h(x) + h'(x)\varepsilon + \frac{h''(x)}{2}\varepsilon^2 + \frac{h'''(x)}{6}\varepsilon^3 + \mathcal{O}(\varepsilon^4).$$

If we assume, that $x \in [0, 0.5]$ and $\varepsilon \leq \frac{3h''(x)}{2h'''(x)}$ then:

$$h(x + \varepsilon) \leq h(x) + h'(x)\varepsilon + \frac{h''(x)}{4}\varepsilon^2. \quad (12)$$

because $h^{(4)}(x) \leq 0$ when $x \in [0, 0.5]$.

Lemma B.1 (Theorem 2.2 from [12]).

$$\begin{aligned} \forall x \in [0, 1] : \quad & 1 - 4 \left(x - \frac{1}{2}\right)^2 \leq h(x) \leq 1 - \frac{2}{\ln(2)} \left(x - \frac{1}{2}\right)^2, \\ \forall x \in [0, 1] : \quad & \frac{x}{2 \log(\frac{6}{x})} \leq h^{-1}(x) \leq \frac{x}{\log \frac{1}{x}}, \end{aligned}$$

where the inverse entropy function $h^{-1} : [0, 1] \rightarrow [0, 1]$ is the inverse of h restricted to the interval $[0, 0.5]$.

Lemma B.2. Let $c \geq 1$, and $b \geq \frac{1}{2}$. Then for any x satisfying $0 < x < 1$:

$$h(1/2 - \gamma) \geq 1 - x + b \cdot h(c\gamma), \quad \text{if} \quad \gamma \leq \frac{x}{4bc \log(12b/x)}$$

Proof. Note that by the various assumptions of the lemma $\gamma \leq x/(2 \log^2(6/x))$, and thus

$$4\gamma^2 \leq \frac{4x^2}{2 \log^2(6/x)} \leq \frac{4}{2 \log^2 6} x/2 \leq x/2. \quad (13)$$

Now $h(\frac{1}{2} - \gamma)$ can be lower bounded by

$$\begin{aligned} &\geq 1 - 4\gamma^2, && \text{by Lemma B.1} \\ &\geq 1 - x/2, && \text{by (13)} \\ &\geq 1 - x + b \cdot h(c\gamma). \end{aligned}$$

as desired. Here, the last inequality follows because $c\gamma \leq \frac{x/2b}{2 \log(6/(x/2b))} \leq h^{-1}(\frac{x}{2b})$ by Lemma B.1, and thus $b \cdot h(c\gamma) \leq \frac{x}{2}$. Therefore we have that $b \cdot h(c\gamma) \leq \frac{x}{2}$ since h is a monotonic in $[0, 1/2]$. \square

Lemma B.3. For every $\beta, \alpha, \rho \in [0, 0.5]$ it holds that :

$$\binom{\beta n}{\alpha \beta n - \rho n} \binom{(1-\beta)n}{\alpha(1-\beta)n + \rho n} \leq \binom{n}{\alpha n} \cdot 2^{-f(\rho, \alpha)n}.$$

where

$$f(\rho, \alpha, \beta) := \begin{cases} \rho^2 & \text{if } |\rho| < \alpha(1-\alpha) \min\{\beta, (1-\beta)\} \\ -\alpha^2 \log 2\alpha & \text{otherwise} \end{cases}$$

Proof. First, observe that when $|\rho| > \alpha(1-\alpha) \min\{\beta, (1-\beta)\}$ we our expression is upper bounded by:

$$\binom{\beta n}{\alpha \beta n(1-\alpha)} \binom{(1-\beta)n}{\alpha(1-\alpha)(1-\beta)n} \leq \binom{n}{\alpha(1-\alpha)n}.$$

This however is bounded by $2^{h(\alpha(1-\alpha))n}$. Observe that $h(\alpha - \alpha^2) \leq h(\alpha) - h'(\alpha)\alpha^2 = h(\alpha) - \alpha^2(\log(\alpha) - \log(1-\alpha)) \leq h(\alpha) - \alpha^2 \log(2\alpha)$. Hence when $|\rho|$ is large we upper bound our expression with:

$$\binom{n}{\alpha n} 2^{\alpha^2 \log(2\alpha)n}.$$

Now, we consider the case of small $|\rho|$. We upper bound the expression with binary entropy.

$$\binom{\beta n}{\alpha \beta n - \rho n} \binom{(1-\beta)n}{\alpha(1-\beta)n + \rho n} = 2^{n(\beta h(\alpha - \frac{\rho}{\beta}) + (1-\beta)h(\alpha + \frac{\rho}{1-\beta}))}$$

Let us consider an exponent:

$$\beta h\left(\alpha - \frac{\rho}{\beta}\right) + (1-\beta)h\left(\alpha + \frac{\rho}{1-\beta}\right)$$

We use Inequality 12 with $x = \alpha$ and $\varepsilon := -\frac{\rho}{\beta}$ for $h(\alpha - \rho/\beta)$ and with $\varepsilon := \frac{\rho}{1-\beta}$ for $h(\alpha + \rho/(1-\beta))$.

Observe that at the beginning we assumed that $|\rho| \leq \alpha(1-\alpha) \min\{\beta, (1-\beta)\}$ hence $|\frac{\rho}{\beta}|$ and $|\frac{\rho}{1-\beta}|$ are upper bounded by $|\frac{3h''(\alpha)}{2h'''(\alpha)}|$. So, by Inequality 12:

$$\beta h\left(\alpha - \frac{\rho}{\beta}\right) + (1-\beta)h\left(\alpha + \frac{\rho}{1-\beta}\right) \leq h(\alpha) + \frac{h''(\alpha)}{4} \frac{\rho^2}{\beta(1-\beta)}.$$

Observe that first order factors cancel out. Hence

$$\binom{\beta n}{\alpha\beta n - \rho n} \binom{(1-\beta)n}{\alpha(1-\beta)n + \rho n} \leq \binom{n}{\alpha n} 2^{\frac{h''(\alpha)\rho^2}{4\beta(1-\beta)}}$$

Finally, observe that $h''(\alpha) < -1$ for all $\alpha \in [0, 0.5]$ and $\frac{1}{\beta(1-\beta)} \leq 4$ for all $\beta \in [0, 0.5]$ hence:

$$\binom{\beta n}{\alpha\beta n - \rho n} \binom{(1-\beta)n}{\alpha(1-\beta)n + \rho n} \leq \binom{n}{\alpha n} 2^{-\rho^2 n}$$

□

Lemma B.4. For all $k \in \mathbb{N}$ and $\alpha \in [0, 1]$ we have:

$$h(\text{Bin}(k, \alpha)) \leq h(\text{Bin}(k+1)).$$

Proof. Let us fix $k \in \mathbb{N}$. Recall that $\text{Bin}(k+1) := (\{0, \dots, k+1\}, p(i))$ and $\text{Bin}(k, \alpha) := (\{0, \dots, k+1\}, p_\alpha(i))$ where $p(i) := \binom{k+1}{i} \frac{1}{2^{k+1}}$ and $p_\alpha(i) := \binom{k}{i} \frac{(1-\alpha)}{2^k} + \binom{k}{i-1} \frac{\alpha}{2^k}$. Hence, we need to prove that for all $\alpha \in [0, 1]$:

$$h(p_\alpha(0), \dots, p_\alpha(k+1)) \leq h(p(0), \dots, p(k+1)).$$

Let us denote $\phi(\alpha) := h(p_\alpha(0), \dots, p_\alpha(k+1))$. First, observe that $\phi(0.5) = h(p(0), \dots, p(k+1))$ because $\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}$. Therefore we need to prove that for all $\alpha \in [0, 1]$ it holds that:

$$\phi(\alpha) \leq \phi(0.5).$$

Recall that binary entropy of multinomial is $h(a_0, \dots, a_{k+1}) := -a_0 \log(a_0) - \dots - a_{k+1} \log(a_{k+1})$ and $(x \ln(x))' = \ln(x) + 1$. Observe that function $\phi(\alpha)$ is well defined for $\alpha = 0$ and $\alpha = 1$ as limits. Moreover $\phi(\alpha) \geq 0$ for all $\alpha \in [0, 1]$.

Now, we compute the first derivative.

$$\phi'(\alpha) = -\frac{1}{2^k \ln 2} \sum_i \left[\binom{k}{i-1} - \binom{k}{i} \right] (1 + \ln(p_\alpha(i))).$$

Because $\sum_i \binom{k}{i-1} = \sum_i \binom{k}{i}$ the first derivative simplifies to:

$$\phi'(\alpha) = -\frac{1}{2^k \ln 2} \sum_i \left[\binom{k}{i-1} - \binom{k}{i} \right] \ln(p_\alpha(i))$$

Now the second derivative is

$$\phi''(\alpha) = -\frac{1}{4^k \ln 2} \sum_i \left[\binom{k}{i-1} - \binom{k}{i} \right]^2 \cdot \frac{1}{p_\alpha(i)} \leq 0,$$

thus $\phi(\alpha)$ is concave for all $\alpha \in [0, 1]$. So in order to show that the $\phi(\alpha)$ function has exactly one maximum in $\alpha = 1/2$ it is sufficient to show that $\phi'(0.5) = 0$.

Let us rearrange the sum:

$$\begin{aligned}\phi'(0.5) &= \sum_i \left[\binom{k}{i-1} - \binom{k}{i} \right] \ln(p_{0.5}(i)) = \sum_i \binom{k}{i} \ln(p_{0.5}(i+1)) - \sum_i \binom{k}{i} \ln(p_{0.5}(i)) \\ &= \sum_i \binom{k}{i} \ln \frac{p_{0.5}(i+1)}{p_{0.5}(i)}.\end{aligned}$$

Because

$$p_{0.5}(i) = \frac{1}{2^{k+1}} \binom{k+1}{i}$$

we can simplify the fraction:

$$\frac{p_{0.5}(i+1)}{p_{0.5}(i)} = \frac{\binom{k+1}{i+1}}{\binom{k+1}{i}} = \frac{k+1-i}{i+1},$$

thus

$$\begin{aligned}\phi'(0.5) &= \sum_i \binom{k}{i} \ln \frac{k+1-i}{i+1} = \sum_i \binom{k}{i} \ln(k+1-i) - \sum_i \binom{k}{i} \ln(i+1) \\ &= \sum_i \binom{k}{i} \ln(k+1-i) - \sum_i \binom{k}{k-i} \ln(k-i+1) = 0,\end{aligned}$$

which finishes the proof. □