Roel Nicolai
Department of Mathematics
University of Utrecht
r.nicolai@uu.nl

History

The map projection of portolan charts

The sudden appearance of portolan charts, realistic nautical charts of the Mediterranean and Black Sea, in the last quarter of the thirteenth century, is considered to be one of the most significant events in the history of cartography. Using analysis techniques available in geodesy, Roel Nicolai showed in [9, pp. 133–200] that these charts are mosaics of regional charts that are considerably more accurate than had been assumed earlier. The good agreement of these regional charts with the Mercator map projection is even more remarkable. Map projections were unknown in the Middle Ages and the Mercator projection was developed some three centuries after the appearance of the oldest extant portolan chart. Therefore, virtually unanimous agreement exists among historians of cartography that its map projection must be coincidental. In this article, however, Nicolai shows, using probability calculus, that it is very unlikely that the map projection emerged as an unintentional by-product of the charts' construction.

Portolan charts are highly realistic medieval nautical charts of the Mediterranean and the Black Sea. They appear suddenly in the last quarter of the thirteenth century AD in the coastal zone around Genoa, Italy. Their extraordinary realism contrasts sharply with the qualitative nature of *mappae mundi*, archetypical medieval maps, based on a mental model of the world inspired by Christian religious ideas and ideas inherited from classical antiquity. Portolan charts represent an unprecedented step forward in cartography and set the standard for mapping and charting in the Age of Discovery and beyond. Portolan charts are the first maps attempted to be drawn to scale after Claudius Ptolemy's from the second century AD.

Portolan charts appear to be entirely unrelated to any other known map type from medieval times and classical antiquity. The charts appear in the world of maritime trade, apparently made by mariners for mariners, without involvement of the clerical intellectual elite of the day. Their most remarkable characteristics are their accuracy and the surprisingly good agreement of their coastlines with those of a modern map on the Mercator map projection.

The charts are hand-drawn on vellum, a fine quality of parchment. Their dimensions were commonly dictated by the size of the animal skin used, typically about 100 cm by 75 cm. That defines their scale as approximately $1:5\,500\,000$; 1 cm on the chart thus corresponds to about 55 km in the real world. The earliest portolan charts depict with remarkable accuracy the Mediterranean, the Black Sea and often the Atlantic coasts between the latitudes of the Canary Islands and the English Channel. Although the North Sea and the Baltic Sea are also drawn on many charts, these areas lack the maturity and detail of the core areas. Portolan charts are clearly nautical aids and as such they constituted a new cartographic genre; no earlier sea charts are known. The names of ports and landmarks were written at right angles with the coastline on the land side, important names in red ink and the remainder in black. Their cartographic style became the hallmark of all nautical charts until well into the eighteenth century.

A striking characteristic of these charts is the criss-cross pattern of straight lines drawn apparently at random across the entire chart. On closer inspection they form a regular pattern, known as a *wind rose*, created by interconnecting sixteen evenly spaced points on a (hidden) circle covering the larger part of the chart. The chart in Figure 1 has two wind roses, with their centres near Barcelona and Antalya respectively. The two wind roses are tangent in the middle of the chart, near Taranto. The wind rose lines were colour-coded to facilitate the selection of the correct compass bearing when navigating or planning a voyage. The availability of the wind rose

Figure 1 Portolan chart by Angelino Dulcert (1339).

provides an absolute orientation for the charts and reveals that the entire coastline image is rotated anticlockwise by about 9°. This angle remains more or less constant until about 1600 AD, when portolan charts oriented to true north begin to appear.

Most surviving charts were decorated with colourful city vignettes and pennants and were probably intended for prestige and display by their wealthy owners. However, charts for onboard use also existed, presumably as navigation aids, but most of such charts probably would have lacked these decorative elements and would have had a limited lifetime in the damp and salty offshore environment [10, p. 439].

Strange characteristics

Practically unanimous consensus exists that portolan charts are based on measurements, rather than on a purely mental model of the world. Portolan charts have a number of curious characteristics. They appear suddenly, 'out of the blue', almost fully developed; no cartographic products are known that might be seen as precursors or prototypes. Consequently, there is no 'bread crumb trail' in the historical record that might shed light on how these charts were constructed and how they acquired their high accuracy. Equally strange is that hardly any development appears to have taken place *after* their first appearance: their key characteristics do not change; their core area was copied with few changes from chart to chart for the next four centuries. They did not become gradually more accurate; on the contrary, over time their coastlines become more stylized at the expense of coastal detail. Their typical shortcomings and defects were not resolved over time either; they exhibit regional scale and orientation differences from the first chart onward, and while some change can be detected in these properties, no steady improvement is visible. Other shortcomings concern persistent errors in the details of the coastlines. All that is odd, because if medieval cartographers were capable of making such accurate charts, why did not the same skills permit them to resolve these shortcomings? The strangest property of these charts, apart from their accuracy, which will be discussed below, is the close resemblance of their Mediterranean, Black Sea and Atlantic coastlines with those of a modern map on the Mercator projection. The Mercator projection was invented by Gerard Kremer in the middle of the sixteenth century, while the oldest extant portolan chart, the so-called Carte Pisane, is dated to the last quarter of the thirteenth century. Moreover, the accuracy of portolan charts is much higher than the accuracy of contemporary, earlier and many later maps and charts. Not until the eighteenth century were new maps of comparable accuracy produced.

Existing views about the map projection

The view that the map projection of portolan charts is an unintentional by-product of the mapmaking process is firmly established in the community of researchers that studies these charts [3, pp. 385–386; 4, pp. 6–7, 186–187; 9, pp. 35, 327–335]. When referring to this issue at all, authors usually state matter-of-factly that portolan charts cannot be based on a map projection. This community consists predominantly of medievalists and historians of cartography with incidental representatives from other disciplines. Practically all have a background in the humanities rather than the exact sciences. Consequently, the hypothesis of the coincidental nature of the map projection has never been subjected to detailed numerical investigation and testing, but is taken for granted in this community.

Because the charts appeared in the maritime-commercial milieu, the commonly accepted notion is that medieval mariners systematically measured and recorded course bearing and distance sailed during their trading voyages. The data collected in that way is assumed to have provided the basis for chart construction. Many authors see confirmation of this hypothesis in the anticlockwise rotation angle of the map image of about $9°$ that all charts exhibit, which roughly agrees with the 'average' magnetic declination in the Mediterranean in the thirteenth century. Magnetic declination is the angle between true north and magnetic north; it varies by location and over time. Those authors that are specific enough to propose a construction method usually postulate central collation of all data somewhere along the Ligurian coast of Italy. Genoa and Pisa are prime candidates; it is from this area that the oldest extant portolan charts originate and both cities were dominant in maritime trade. Additionally, some unspecified schema of accuracy improvement is assumed, usually expressed in vague terms such as 'progressively better estimates became available over time', but some authors explicitly mention a process of averaging.

The historian of cartography David Woodward was one of the few who were fairly specific on how this process of mapmaking from a geometric framework of distances and bearings might have taken place. He did not mention bearings, despite compass bearings between points being crucial for the construction of such a geometric framework.

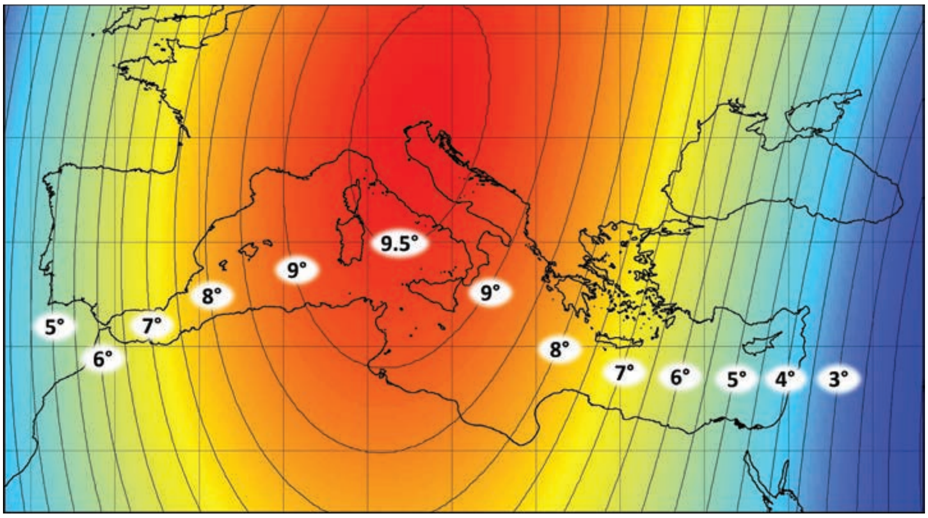

Figure 2 Magnetic declination from the paleomagnetic model CALS7k.2 for the year 1250. The contour interval is $0.5°$ and the positive values designate an easterly magnetic declination [9, p. 225].

> "The cumulative experience of several centuries of coastal and other shipping in each of these basins could have led to the independent recording of traditionally known distances. The average distances derived from both coastal traverses and cross-basin routes could then have been used in the construction of a series of separate charts of the individual basins. [...] If these routes were plotted to form networks in each of the basins, each network might have assumed the form of a self-correcting closed traverse of each basin. The rigidity of this structure would, however, have depended on the availability of cross-basin distances, acting as braces to the framework. It is thus postulated that some system of empirical or stepwise graphic method of correcting these frameworks was used to achieve a 'least-squares' result" [3, p. 188].

Woodward was well aware that the method of least squares was not available in the Middle Ages; his intention is to suggest that all data conflicts would have been resolved by distributing them over the entire network of coastal points using a less formal graphic method. Most researchers approach this explanation intuitively and accept it as true. No one has checked whether it is feasible or realistic.

The distances, corrected for nominal chart scale, and the bearings between points are assumed to have been transferred to the map by plane geometry, that is, ignoring the effects of earth curvature as well as magnetic declination, which was an unknown phenomenon at the time. More sophisticated methods cannot be assumed to have been available in the Middle Ages. Additionally, historians of cartography assume that some form of graphic adjustment was carried out by the cartographer in order to resolve the contradictions in the data due to the inevitable random errors in the measurements. The errors due to ignoring the curvature of the earth's surface would have been subsumed in that graphical adjustment and are generally downplayed as 'negligible' or 'relatively minor'. After this putative construction of the framework of the chart, the details of the coastline are presumed to have been drawn in.

The Mercator projection

In the fifteenth and sixteenth centuries mariners used nautical charts in ocean navigation which approximated the so-called *equidistant cylindrical projection* centred on the equator. Such a chart is also called a *plane chart*.

Although mariners criticized their nautical charts and attributed a host of navigation errors to them, the plane chart remained popular, mainly because it appeared to allow navigation problems to be solved by plane (Euclidean) geometry, that is, as if the earth were flat, although they were well aware of the earth's spherical shape. There were two fundamental problems with the plane chart. Firstly, a straight line on the chart does not correspond with the track of a ship on a constant course.

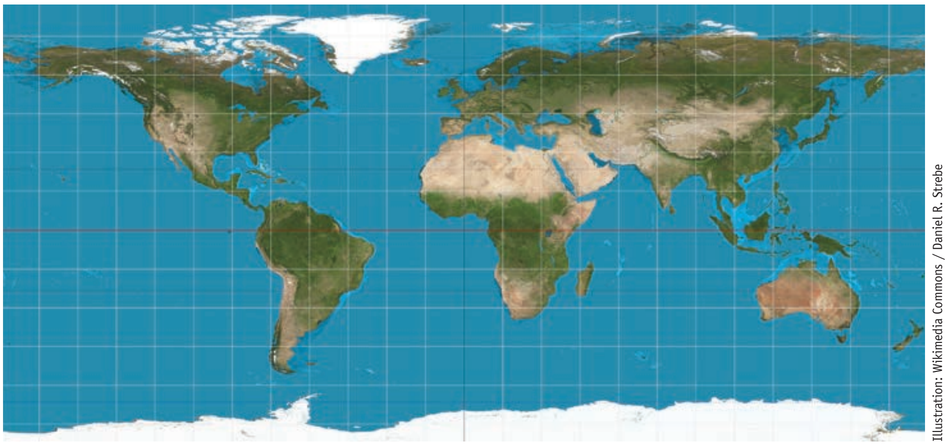

Figure 3 World map on the equidistant cylindrical projection centred on the equator. The graticule interval is $15°$.

Secondly and more importantly, an east-west line well away from the equator is shown too long on a plane chart, because on the spherical earth the meridians converge, but on the plane chart they do not.

Both problems were eventually solved by Gerard Kremer (Mercator), who, in 1569, published a world map on the carefully designed map projection that bears his name. Kremer designed the projection from the principle that any constant-course trajectory, that is, a loxodrome, should project as a straight line on the map, making the same angle with the meridians on the map as well as on the sphere. This can only be achieved at the expense of the scale of the map or chart, which consequently increases with latitude from unity at the equator to infinity at the poles. In Figure 4 it may be seen that Greenland and Africa have about the same area on the Mercator chart, while in reality Africa is fourteen times as large as Greenland. The varying latitude scale was counterintuitive to mariners, who had been educated with the idea that parallels are equidistant and expected to see this equidistance property reflected on their charts. The plane chart honours this, but the Mercator chart does not. This was one of the reasons why the Mercator projection was not an overnight success.

Gerard Kremer constructed his projection graphically and showed some early awareness of what is now known as differential geometry. Expressing latitude and longitude in radians, the corresponding line segments on the surface of the earth of two infinitesimal angular increments $d\varphi$ and $d\lambda$ are: $R \cdot d\varphi$ and $R \cdot \cos(\varphi) \cdot d\lambda$, when the radius of the spherical earth is R.

Adopting a similar coordinate system for the Mercator chart, spanned by the two orthogonal (X and Y) axes, where the positive X-axis points east and the positive Y-axis north, Mercator's solution was to define:

$$dX = R \cdot d\lambda,$$
$$dY = \frac{R \cdot d\varphi}{\cos(\varphi)}.$$

Integration of these differential quantities yields (with integration constants set to zero):

$$X = R \cdot \lambda,$$
$$Y = R \cdot \ln\left(\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right).$$

Mercator constructed the graticule of his world map graphically, accumulating increments of $R \cdot \Delta\varphi / \cos(\varphi)$ to obtain the spacing of the parallels [6,5], with $\Delta\varphi = 1°$. Although one might argue that, in principle, this would not have been beyond the capability of medieval natural philosophers, in practice it was. The development of the Mercator projection was specifically aimed at solving the problems experienced in ocean navigation and is linked to the discovery of the loxodrome by Pedro Nunes in 1537 [11]. The questions leading Gerard Mercator to his projection were never asked in the thirteenth century; at least there is no indication they were. A medieval Mercator projection would have been a (complex) solution for a non-existing problem.

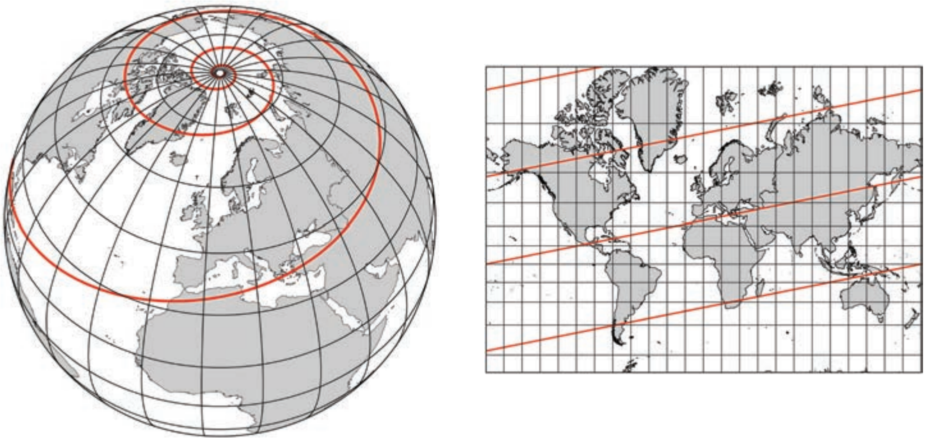

Figure 4 Loxodrome in an oblique perspective projection (left) and the Mercator projection (right) [9, p. 467, 470].

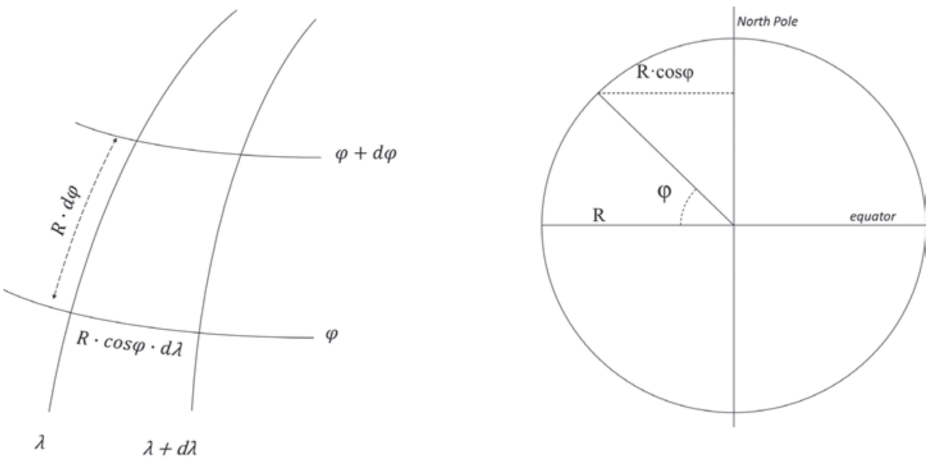

Figure 5 The geometry of the sphere.

The accuracy of portolan charts

The question of whether the Mercator map projection can emerge as an unintentional by-product of unrecognized magnetic declination and ‘plane charting’, that is, charting that ignores the effects of earth curvature, does not have a simple yes-or-no answer. While presently the curvature of the earth and magnetic declination in the thirteenth century are ‘known’ quantities, be it that magnetic declination has to be estimated using a paleomagnetic model (see Figure 2), the question of whether or not the effects of ignoring these factors would have had a measurable impact on portolan charts depends on the accuracy of those charts. If the random errors in portraying the coastlines, putatively caused by the limited accuracy of medieval navigation, are much larger than the effects of ignoring earth curvature and magnetic declination, the latter effects may not be measurable in the charts at all. This begs the question of what the accuracy of portolan charts is, which requires the concept of map accuracy to be defined first. The accuracy of a measurement is usually defined as the closeness of that measurement to its true value. One might therefore be tempted to expand this principle to a map: measure the coordinates of a set of recognisable points on the portolan chart and compare those with the ‘true’ values of those points in the ‘real world’. In similar analyses such points are commonly called *identical points*. The coordinates of the identical points on the chart would be expressed in a cartesian coordinate system, while the modern positions of these points on the spherical earth would be expressed in terms of latitude and longitude. Cartesian coordinates cannot be compared directly with curvilinear coordinates; a direct comparison is therefore impossible. The relationship between the coordinates in both systems must therefore be established first and that takes us back to the map projection. However, the map projection is not sufficient to describe the relationship between the two sets of coordinates. Because the cartesian coordinate system of the portolan chart is arbitrary, at the very least a shift, a rotation and a scale difference between the two coordinate systems will have to be solved for. Because portolan charts are quite old, allowance should be made for deformation of the vellum on which the charts were drawn. This leads to the assumption of an affine transformation, with separate rotations and scale factors for both coordinate axes.

Each map projection introduces its own specific distortions to the geometry of points on the earth’s surface; those introduced by the Mercator projection were discussed in the previous section. Earlier research has established that the map projection that approximates the coastline portrayal on portolan charts best is the Mercator projection [7, p. 146, 149].

The complete model that describes the relationship between cartesian X, Y coordinates of the portolan chart and latitude, longitude of the spherical earth (in matrix form) is as follows.

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} T_X \\ T_Y \end{pmatrix} + \begin{pmatrix} k_X & 0 \\ 0 & k_Y \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_X & \sin\theta_Y \\ -\sin\theta_X & \cos\theta_Y \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \bar{\phi} \end{pmatrix}$$

where:

X, Y: coordinates of the identical point in the cartesian coordinate system of the portolan chart;

T_x, T_y: origin shift (translation) of the internal (X,Y) coordinate system of the portolan chart;

$\bar{\varphi}$: isometric latitude of the identical point:

$$\varphi = \ln\left[\tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right)\right].$$

φ, λ: latitude and longitude of the identical point (in radians) in the reference dataset;

k_x, k_y: scale factors of the X-axis and the Y-axis. Together these parameters determine the scale of the portolan chart relative to the internal (X,Y) coordinate system. The radius R of the spherical earth is absorbed by the scale factors.

θ_x, θ_y: rotation angles about the portolan chart’s X and Y-axis respectively.

Figure 6 Composite of rectified regional charts of the Dulcert portolan chart (1339) [9, p. 198].

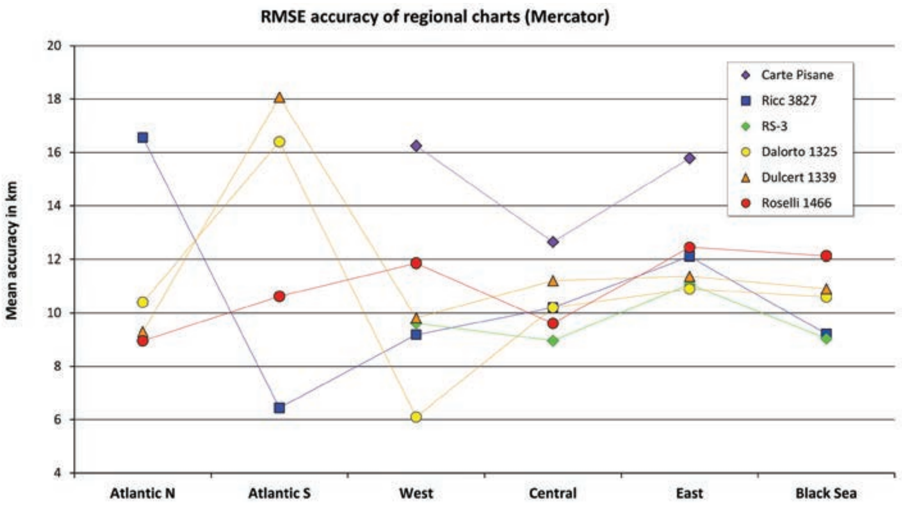

Figure 7 Mean point accuracy of regional charts.

The matrix equation above is non-linear; evaluation in 'linearized' form about approximate values is required in order to apply Least-Squares Estimation and iterate to a final solution.

Portolan charts have long been known to have regional scale variations. Best known from literature are the scale of the Atlantic coasts, which is too small, and the scale of the Black Sea, which is too large. David Woodward, the historian of cartography cited above, hypothesizes that the subbasins of the Mediterranean were charted first. This is how I approached the analysis of portolan charts, using statistical testing of the identical point residuals to establish homogeneous clusters of points. That revealed the existence of five or six regional charts, shown in Figure 6.

These regions are quite consistent for the six early portolan charts that I analysed, but their boundaries do not coincide neatly with the boundaries of subbasins. Some regional charts exhibit overlaps, with the common section of coastline modified by the cartographer to achieve a smooth join [9, p. 199]. Each region has its own distinct scale, rotation and shift parameters, which justifies the conclusion that portolan charts are indeed mosaics of regional charts. After the least-squares fit of the portolan chart to the modern dataset, the fit of the identical points will not be perfect. Small discrepancies remain, known as *residual errors* or *residuals*. The Mean Squared Error (MSE) of all residuals, or its square root, the RMSE, is a measure of how well the portolan chart agrees with a modern dataset, in other words its accuracy. The large number of identical points used, 836 for the Dulcert portolan chart, allows reliable estimates for the accuracy of the regional charts to be computed. The RMSE values of the latitude residuals and the longitude residuals were calculated separately. The larger of the two represents the accuracy of the regional chart [9, p. 129–132].

The accuracy of all regional charts, computed for six portolan charts, is shown in Figure 7. The mean of all 37 accuracy values, weighted by the number of contributing points, is 11.3 km, which is exceptionally good. With an average scale of the charts of $1:5\,500\,000$, this amounts to about 2 mm on the chart. Latitude and longitude have approximately the same accuracy.

Some geodetic quality-control concepts

Modern maps and charts are based on geodetic measurements that determine the positions (coordinates) of a set of control points covering the area to be mapped. Until about the middle of the twentieth century the only way to achieve that was by triangulation, that is, by measuring the angles of triangles spanned by any three intervisible control points. The earliest known map, or rather series of maps, to have been constructed in accordance with this principle is the so-called Cassini map of France from the first half of the eighteenth century. Broadly speaking, this is still how maps are made today.

Any measurement of a physical or geometric quantity in the real world is subject to random, or *stochastic*, variation. In a geodetic network those variations will cause discrepancies in the measured geometry of the points, for example, the sum of the measured angles of any triangle will rarely be exactly $180°$. A measurable geometric quantity, such as an angle or a distance, may thus be seen as a random or stochastic variable. These variables are assumed to have a normal or Gaussian error distribution. The misclosures in a measured geodetic control network are commonly resolved through Least-Squares Estimation, which allows computation of the vector of improved (least-squares) estimates $\hat{m}$ of the measurements vector m by minimizing the weighted sum of the squares of the vector of residuals v:

$$\hat{m} = m + v.$$

The sum of the weighted squared residuals E is found through the matrix expression: $E = v^T \cdot W \cdot v$ where the weight matrix W is the inverse of the covariance matrix Q of the measurements: $W = Q^{-1}$ and v^T is the transposed vector of residuals v [1, p. 14]. Numerical considerations usually dictate reduction of the covariance matrix Q by a constant scaling factor σ_0^2 (a scalar): $Q = \sigma_0^2 \cdot G$. In the case of the accuracy analysis of a portolan chart, discussed in the previous paragraph, the measurement variables are the cartesian X and Y coordinates of the identical points on the chart. The best fit to the modern Mercator chart by Least-Squares Estimation was conducted with $G = I$ (identity matrix). The resulting MSE of each regional chart is then an estimate of σ_0^2.

When the individual measurements are all Gaussian-distributed random variables, the sum of the weighted squared least-squares residuals will have the chi-squared distribution with b degrees of freedom, where b is the number of 'redundant' or 'surplus' measurements, that is, the number of measurements over and above the minimum measurements to calculate the coordinates of all points of the geodetic network. More commonly, however, the variable $S = E/(b \cdot \sigma_0^2)$ is used for quality control. It is virtually impossible to measure any large geodetic network without gross errors occurring in some of the measurements [1, p. 14]. The variable S is Fisher-distributed with degrees of freedom b and ∞. It is used for a statistical test to reveal the possible existence of gross errors in the measurements. In the example of Figure 8, $b = 20$. The test with *significance level* $\alpha = 1\%$ would lead to rejection of the null hypothesis (= no gross errors in

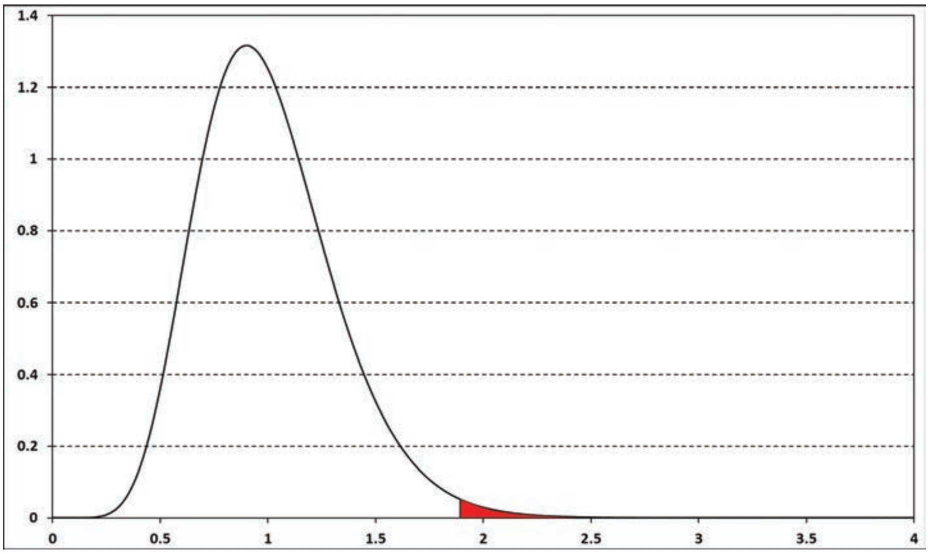

Figure 8 Fisher probability density function with degrees of freedom b and ∞ and 1% right-hand rejection zone.

any of the measurements) when S is greater than the right-hand critical value associated with the significance level. In Figure 8 that would be 1.87 for a significance level of 1%. The complement, the number range from zero to 1.87 is called the 99% *confidence interval*. This means that an extreme value of S, that is, $S > 1.87$, has a probability of occurring of less than 1%.

The F-test was introduced by Ronald A. Fisher in the 1920s and offers a means of identifying gross errors in the data. Whether or not the effect of one or more gross measurement errors in a geodetic network are discoverable with this test will depend on the magnitude of the errors and the geometric design of the network. More sensitive tests have therefore been developed for the analysis of complex geodetic networks, but those refinements will not concern us here. See [2, p. 14].

Around 1935 Jerzy Neyman and Egon Pearson placed hypothesis testing on more secure footing by introducing the concepts of Type I and Type II errors [12, pp. 282–285]. In modern medical tests a Type I error is referred to as a *false positive* result, while a Type II error is called *false negative*. A value of S smaller than its critical value does not guarantee the absence of errors in the data. The probability of a Type I error in Figures 8 and 9 is the red shaded area, that is, the probability of the null hypothesis (= no gross errors in the data) being incorrectly rejected. When the test variable S is corrupted by one or more gross errors, it is still a random variable, but with a *non-central Fisher* distribution, represented in Figure 9 by the right-hand curve. The probability of a Type II error is the grey shaded area in Figure 9 and equates to the probability that the alternative hypothesis (= there is at least one gross error in the data) is incorrectly rejected and an actual gross error in the data is not noticed. The probability of *not* making a Type II error is known as the *power of the test*. In geodetic networks the gross error(s) in the measurements lead(s) to a bias or gross error in the test variable S. Because the magnitudes of those errors are rarely known, classical Neyman–Pearson theory recommends fixing the power of the test to a suitably large number such as 80% or 90% and then working back to the magnitude of bias that may actually be found with that probability. Neyman and Pearson's work led to the sobering realization that the actual error-detection capability is considerably worse than Fisher's work seemed to suggest. For geodetic applications this was refined to considerable detail by Willem Baarda at Delft University of Technology in the 1960s [8].

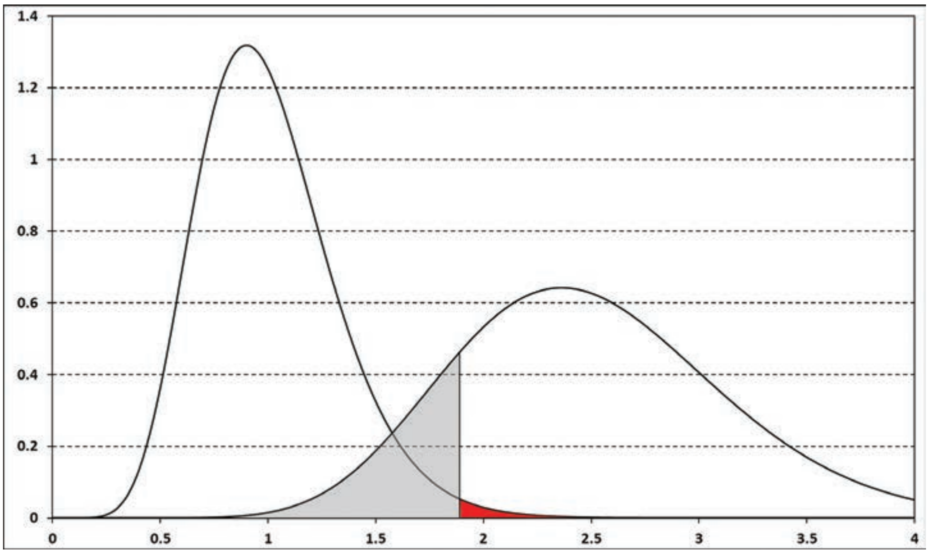

Figure 9 Central F-distribution, associated with the null hypothesis, on the left, and non-central F-distribution, associated with the alternative hypothesis, on the right.

Is the map projection coincidental?

In the Middle Ages, triangulation as a technique had not yet been invented and no suitable geodetic instruments were available anyway. Instead, a marine network of distances and compass bearings between coastal points is usually assumed as the basis for portolan chart construction.

The principle that I used to establish whether or not the map projection of portolan charts is coincidental consists of inverting Neyman and Pearson's idea of fixing the power of the test and then working out the magnitude of the gross error in the test variable. That gross error can be calculated by simulating the hypothetical medieval charting process with 'measurements' calculated exactly for the sphere, that is, spherical (loxodromic) distances and magnetic bearings between the points of a conjectural medieval maritime network. I split up the entire network into subnetworks covering the western and eastern Mediterranean and the Black Sea. Magnetic declination was estimated from the paleomagnetic model CALS7k.2 model for the year 1250 (Figure 2). The conjectural design of the western Mediterranean network, taking into account trade routes, prevailing wind direction in summer and the sailing properties of medieval ships, is shown in Figure 10, in which each line represents a bearing and distance pair.

A hypothetical medieval cartographer would have constructed his chart from this set of bearing and distance data in any of very large number of ways. It is impossible to test all, but it is possible to constrain the options to a limited number of realistic scenarios. I evaluated what I consider the most plausible scenario, assuming that our hypothetical cartographer would start in Genoa or Pisa, the area considered to be the 'birth place' of portolan charts, and that he would begin by charting the coastline using the data of the relatively short coastal legs. He would have come across his first 'loop misclosure', finding two positions for Marsala, Sicily 89 km apart. Such data conflicts due to plane charting are exclusively in east-west direction [9, pp. 471–482]. We do not know how the cartographer might have dealt with such a conflict, but my assumption is that he would have adjusted a section of the North African coastline from, for example, Algiers to Cap Bon by compressing it to eliminate the data conflict. The section before adjustment is shown in grey in Figure 10. As mentioned above, adjustment of a section of coastline was a cartographic technique that was well-understood by portolan chart makers. Portolan chart making in the medieval Mediterranean was conducted by small commercial enterprises, often within a family, and it would have been in the cartographer's interest to avoid making unnecessary, small changes to the entire chart. I used a threshold value of 15 km for data conflicts. Assuming that Corsica and Sardinia would have been charted from the Italian coast and Majorca from the Spanish, most contradictions are relatively small, with the exception of the long east-west courses to Majorca. In Figure 10 the red lines indicate misclosures greater than 15 km than cannot be reduced further by shifting locations of points without increasing the misclosures in other network points too much. It has to be borne in mind that the medieval cartographer would not have been able to separate the errors due to plane charting and the stochastic errors in the navigation measurements with which he is presumed to have been working.

This iterative process I executed yielded a set of simulated 'medieval' positions for all network points that would have formed the framework for further charting. The iterative computation of the networks for the eastern Mediterranean and the Black Sea networks yielded similar datasets. The key question is: to what extent does the shape of these networks agree with the correct modern geometry on a Mercator chart?

To answer this question, a least-squares fit of this 'medieval' set of coordinates against the corresponding set of modern Mercator coordinates was executed, applying the same affine model as used for the real portolan charts. This removed any differences in rotation, non-orthogonality of coordinate axes and (differential) scale and yielded a sum of squared residuals, but now this contains exclusively the compounded gross errors or bias, caused by ignoring earth curvature and magnetic declination. Let that bias be designated by B. Had the appropriate Mercator corrections and magnetic declination been applied prior to the network calculation, the calculation would have yielded the correct (unbiased) Mercator coordinates of all points and $B = 0$.

How large is that bias B in the test parameter S for the three networks? The consensus among historians of cartography translates to $B \approx 0$, because agreement with the Mercator projection is believed to be automatic. However, calculation of this bias yields the following figures:

$$B_{\text{Western Mediterranean}} = 76.3\ \text{km}^2,$$
$$B_{\text{Eastern Mediterranean}} = 98.2\ \text{km}^2,$$
$$B_{\text{Black Sea}} = 41.1\ \text{km}^2.$$

It was established earlier that the mean accuracy of the regional charts of a portolan chart is 11.3 km. The square of this figure, 129 km^2, is an estimate of the variance factor σ_0^2 discussed in the previous section. Using this figure to normalize the values for the biases listed above yields:

$$B'_{\text{Western Mediterranean}} = 0.59,$$
$$B'_{\text{Eastern Mediterranean}} = 0.76,$$
$$B'_{\text{Black Sea}} = 0.32.$$

The mathematical expectation of S under the null hypothesis (that is, Mercator corrections and magnetic declination corrections applied) equals unity. Under the alternative hypothesis it equals $1 + B'$:

$$E\{S \mid H_0\} = 1,$$
$$E\{S \mid H_A\} = 1 + B'.$$

Applying the statistical testing principle and using a right-hand significance level of 1% yields the following P-values (probabilities) of a Type II error occurring:

Western Mediterranean:	$P = 0.0018,$
Eastern Mediterranean:	$P = 3 \cdot 10^{-7},$
Black Sea:	$P = 0.31,$

for the western Mediterranean see Figure 11. For the eastern and western Mediterranean the F-test leads to rejection of the alter-

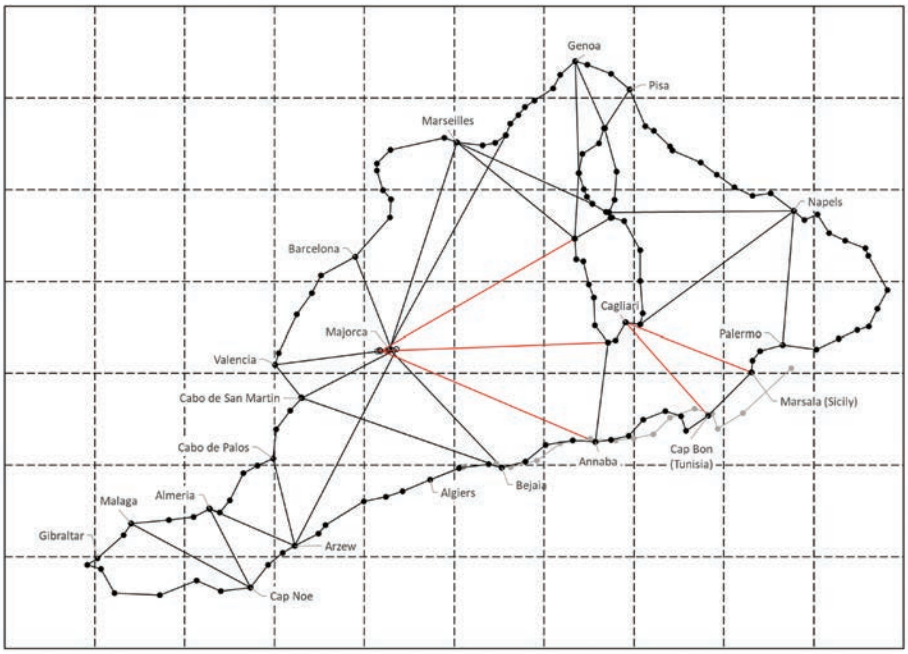

Figure 10 Conjectural geodetic network in the western Mediterranean, hypothetically underlying the construction of a portolan chart and drawn by plane charting and ignoring magnetic declination. The grid interval is 200 km.

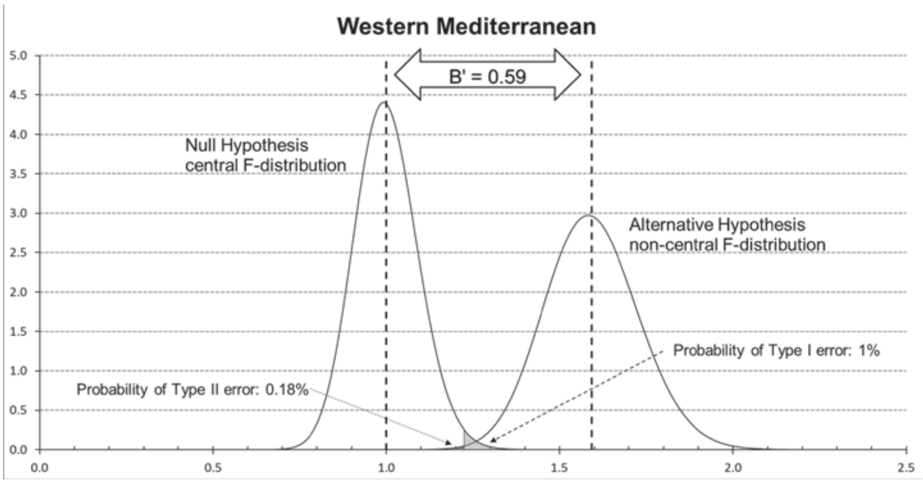

Figure 11 Probability density functions of the (normalized) MSE under the null hypothesis (= Mercator projection is not accidental) and under the alternative hypothesis (= Mercator projection is coincidental) for the western Mediterranean.

native hypothesis which postulates the coincidental emergence of the Mercator projection. Only for the Black Sea, which is much smaller and mappable with a smaller set of control points, a realistic possibility of 31% can be said to exist. The probability that the Mercator projection would result coincidentally in all three regions can be considered negligible. The corresponding P-value would be obtained by multiplying the three values shown above.

The caveat must be made that only a single charting solution has been tested. An alternative method of medieval charting than described above (see Figure 10) might reduce the bias values slightly. However, the discrepancy between the spherical geometry of the earth and the Euclidean geometry of the map in each region remains the same and a different schemes of plane charting would redistribute the inevitable data conflicts in different, but similar ways. That would result in some variations in the bias values shown, but it is unlikely that these values would become so small that a coincidental Mercator projection would become a realistic possibility, given how small the total P-value is for the scenario evaluated, but, plausible though this may be, it is impossible to prove rigorously. Previously, I had calculated a least-squares solution for the plane charting of the three networks, following Woodward's suggestion described earlier in this article [9, pp. 508–516]. Surprisingly, the bias (B) values for this solution were considerably larger: $B = 165\,\text{km}^2$, $B = 329\,\text{km}^2$ and $B = 58\,\text{km}^2$.

A coincidental emergence of the Mercator projection would have to result from a fortuitous combination of random measurement errors and arbitrary plane charting corrections, such that the resulting residuals would mimic, to a sufficient extent, the magnetic declination in every compass bearing as well as the latitude-dependent Mercator distance magnification in every distance.

Concluding remarks

The emergence of a Mercator or Mercator-like projection on medieval portolan charts as a coincidental by-product of a simple charting method that ignored earth curvature and magnetic declination is highly unlikely.

As a professional geodesist I am unable to suggest an alternative realistic mechanism that would accidentally generate the Mercator projection. I will therefore conclude that the map projection of portolan charts is most probably an intentional feature. That creates a significant challenge for historians who now will have to explain how this is possible in the light of the evidence expounded in this paper. However, no researcher worth their salt ought to shirk from such a challenge; challenges are what drives science forward.

The calculations in this article focussed on the question of the map projection of portolan charts and therefore made the implicit assumption that medieval navigation was accurate enough to yield a chart of the accuracy of the Mediterranean portolan charts. That aspect could not be addressed in this article and neither could relevant historical aspects; they are covered in my book.

References

1 W. Baarda, *Statistical Concepts in Geodesy*, Publications on Geodesy New Series, Vol. 2, No. 4, Netherlands Geodetic Commission, Delft, 1967.

2 W. Baarda, *A Testing Procedure for Use in Geodetic Networks*, Publications on Geodesy New Series, Vol. 2, No. 5, Netherlands Geodetic Commission, Delft, 1968.

3 T. Campbell, Portolan charts from the late thirteenth century to 1500, in J.B. Harley and David Woodward, eds., *The History of Cartography, Vol. 1 – Cartography in Prehistoric, Ancient and Medieval Europe and the Mediterranean*, University of Chicago Press, 1987.

4 J.A. Gaspar, *From the Portolan Chart of the Mediterranean to the Latitude Chart of the Atlantic. Cartometric Analysis and Modelling*, PhD Thesis, Universidade Nova de Lisboa, 2010.

5 J.A. Gaspar and H. Leitão, Squaring the Circle: how Mercator constructed his projection in 1569, *Imago Mundi* 66(1) (2014), 1–24.

6 W. Krücken, Ist das 'Rätsel der (Konstruktion der) Mercator-Karte von 1569' [Hermann Wagner] gelöst? Unpublished, www.wilhelm-kruecken.de.

7 S.A. Loomer, *A Cartometric Analysis of Portolan Charts: A Search for Methodology,* PhD Thesis, University of Wisconsin-Madison, 1987.

8 M. Molenaar, *De Delftse School. De ontwikkeling van een kwaliteitstheorie voor geodetische metingen, 1930–1980,* Geodetisch-Historische Monografieën Nr.4, Stichting De Hollandse Cirkel, 2020.

9 R. Nicolai, *The Enigma of the Origin of Portolan Charts. A Geodetic Analysis of the Hypothesis of a Medieval Origin*, Brill, Leiden, 2016.

10 R.J. Pujades I Bataller, *Les cartes portolanes: la representatició medieval d'una mar solcada*, trans. Richard Rees, Lunwerg Editores, Barcelona, 2007.

11 W.G.L. Randles, Pedro Nunes and the discovery of the loxodromic curve (1537), *Journal of Navigation* 50 (1997).

12 D. Spiegelhalter, *The Art of Statistics. Learning from Data*, Penguin Random House, 2020.