



Uniform Lyndon Interpolation for Basic Non-normal Modal Logics

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Abstract. In this paper, a proof-theoretic method to prove uniform Lyndon interpolation for non-normal modal logics is introduced and applied to show that the logics E, M, MC, EN, MN have that property. In particular, these logics have uniform interpolation. Although for some of them the latter is known, the fact that they have uniform Lyndon interpolation is new. Also, the proof-theoretic proofs of these facts are new, as well as the constructive way to explicitly compute the interpolants that they provide. It is also shown that the non-normal modal logics EC and ECN do not have Craig interpolation, and whence no uniform (Lyndon) interpolation.

Keywords: Non-normal modal logics · Uniform interpolation · Uniform Lyndon interpolation · Craig interpolation

1 Introduction

Uniform interpolation (UIP), a strengthening of interpolation in which the interpolant only depends on the premise or the conclusion of an implication, is an intriguing logical property. One of the reasons is that it is hard to predict which logic does have the property and which does not. Well-behaved logics like K and KD have it, but then, other well-known modal logics, such as K4, do not. Early results on the subject were by Shavrukov [17], who proved UIP for the modal logic GL, and by Ghilardi [4] and Visser [19], who independently proved the same for K, followed later by Bílková, who showed that KT has the property as well [2]. Surprisingly, K4 and S4 do not have UIP, although they do have interpolation [2, 4]. Pitts provided the first proof-theoretic proof of UIP, for intuitionistic propositional logic, IPC, the smallest intermediate logic [13]. Results from [5, 9] imply that there are exactly seven intermediate logics with interpolation and that they are exactly the intermediate logics with UIP. Pitts' result is especially important to us, as also in our paper the approach is proof-theoretic.

The study of UIP in the context of non-normal modal logics has a more recent history. The area is less explored than its normal counterpart, but for

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several well-known non-normal logics UIP has been established, for example, for the monotone logic M [14], a result later extended in [12, 15] to other non-normal modal and conditional logics, such as E and basic conditional logic CK.

Non-normal modal logics are modal logics in which the K -axiom, i.e. the axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, does not hold but a weaker version that is given by the following E -rule does:

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$$

Thus the minimal non-normal modal logic, E, is propositional logic plus the E -rule above. Over the last decades non-normal modal logics have emerged in various fields, such as game theory and epistemic and deontic logic [3]. Two well-known non-normal modal logics that are investigated in this paper are natural weakenings of the principle $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$ that implies K over E. Namely, the two principles:

$$(M) \quad \Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi) \quad (C) \quad (\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi).$$

Because the K -axiom holds in the traditional relational semantics for modal logic, non-normal modal logics require different semantics, of which the most well-known is neighborhood semantics. As we do not need semantics in this paper, we refer the interested reader to the textbook [11].

Our interest in the property of UIP for non-normal modal logics lies in the fact that it can be used as a tool in what we would like to call *universal proof theory*, the area where one is concerned with the general behavior of proof systems, investigating problems such as the existence problem (when does a theory have a certain type of proof system?) and the equivalence problem (when are two proof systems equivalent?). The value of UIP for the existence problem has been addressed in a series of recent papers in which a method is developed to prove UIP that applies to many intermediate, (intuitionistic) modal, and substructural (modal) logics [1, 6, 7]. The proof-theoretic method makes use of sequent calculi, and shows that general conditions on the calculi imply UIP for the corresponding logic. Thus implying that any logic without UIP cannot have a sequent calculus satisfying these conditions. The generality of the conditions, such as closure under weakening, makes this into a powerful tool, especially for those classes of logics in which UIP is rare, such as intermediate logics. Note that in principle other regular properties than UIP could be used in this method, as long as the property is sufficiently rare to be of use.

In this paper we do not focus on the connection with the existence problem as just described, but rather aim to show the flexibility and utility of our method to prove UIP by showing that it can be extended to (yet) another class of logics, namely the class of non-normal modal logics, that it is constructive and can be easily adapted to prove not only UIP but even uniform Lyndon interpolation. Uniform Lyndon interpolation (ULIP) is a strengthening of UIP in which the interpolant respects the polarity of propositional variables (a definition follows in the next section). It first occurred in [8], where it was shown that several normal

modal logics, including K and KD, have that property. In this paper we show that the non-normal modal logics E, M, MC, EN, MN have uniform Lyndon interpolation and the interpolant can be constructed explicitly from the proof. In the last part of this paper we show that the non-normal modal logics EC and ECN do not have interpolation, and whence no uniform (Lyndon) interpolation either. This surprising fact makes EC and ECN potential candidates for our approach to the existence problem discussed above, but that we have to leave for another paper.

Our proof-theoretic method to prove UIP makes use of sequent calculi for non-normal modal logics that are equivalent or equal to calculi introduced in [10]. In this paper it is also shown that E, M, MC, EN, MN have Craig interpolation. The proof that these logics have UIP is not a mere extension of the proof that they have interpolation but requires a very different approach. That the logics E and M have UIP has already been established in [12, 14], but that they have uniform Lyndon interpolation is, as far as we know, a new insight. Interestingly, for logics with LIP, that fact does not always follow easily from the proof that they have IP, as is for example the case for GL [16]. But for our method this indeed is the case: the proof of UIP easily implies ULIP. Thus the hard work lies in proving the former, in a way that turns out to imply the latter. Therefore we consider the proof-theoretic method to prove uniform interpolation for non-normal modal logics the main contribution of this paper, as until now such proofs have always been semantical in nature. In [15] the search for proof-theoretic techniques to prove uniform interpolation in the setting of non-normal modal logics is explicitly mentioned in the conclusion of that paper.

2 Preliminaries

Set $\mathcal{L} = \{\wedge, \vee, \rightarrow, \perp, \Box\}$ as the language of modal logics. We use \top and $\neg A$ as abbreviations for $\perp \rightarrow \perp$ and $A \rightarrow \perp$, respectively, and write $\varphi \in \mathcal{L}$ to indicate that φ is a formula in the language \mathcal{L} . The *weight of a formula* is defined inductively by: $w(\perp) = w(p) = 0$, for any atomic p and $w(A \odot B) = w(A) + w(B) + 1$, for any $\odot \in \{\wedge, \vee, \rightarrow\}$, and $w(\Box A) = w(A) + 1$.

Definition 1. *The sets of positive and negative variables of a formula $\varphi \in \mathcal{L}$, denoted by $V^+(\varphi)$ and $V^-(\varphi)$, respectively, are defined recursively by:*

- $V^+(p) = \{p\}$, $V^-(p) = V^+(\top) = V^-(\top) = V^+(\perp) = V^-(\perp) = \emptyset$, for atom p ,
- $V^+(\varphi \odot \psi) = V^+(\varphi) \cup V^+(\psi)$ and $V^-(\varphi \odot \psi) = V^-(\varphi) \cup V^-(\psi)$, for $\odot \in \{\wedge, \vee\}$,
- $V^+(\varphi \rightarrow \psi) = V^-(\varphi) \cup V^+(\psi)$ and $V^-(\varphi \rightarrow \psi) = V^+(\varphi) \cup V^-(\psi)$,
- $V^+(\Box \varphi) = V^+(\varphi)$ and $V^-(\Box \varphi) = V^-(\varphi)$.

Define $V(\varphi)$ as $V^+(\varphi) \cup V^-(\varphi)$. For an atomic formula p , a formula φ is called p^+ -free (p^- -free), if $p \notin V^+(\varphi)$ ($p \notin V^-(\varphi)$). It is called p -free if $p \notin V(\varphi)$. Note that a formula is p -free iff p occurs nowhere in it.

For the sake of brevity, when we want to refer to both $V^+(\varphi)$ and $V^-(\varphi)$, we use the notation $V^\dagger(\varphi)$ with the condition “for any $\dagger \in \{+, -\}$ ”. If we want to refer to one of $V^+(\varphi)$ and $V^-(\varphi)$ and its dual, we write $V^\circ(\varphi)$ for the one we intend and $V^\diamond(\varphi)$ ¹ for the other one. For instance, if we state that for any atomic formula p , any $\circ \in \{+, -\}$ and any p° -free formula φ , there is a p° -free formula ψ such that $\varphi \vee \psi \in L$, we are actually stating that if φ is p^+ -free, there is a p^- -free ψ such that $\varphi \vee \psi \in L$ and if φ is p^- -free, there is a p^+ -free ψ such that $\varphi \vee \psi \in L$.

Definition 2. A logic L is a set of formulas in \mathcal{L} extending the set of classical tautologies, CPC, and closed under substitution and modus ponens $\varphi, \varphi \rightarrow \psi \vdash \psi$.

Definition 3. A logic L has Lyndon interpolation property (LIP) if for any formulas $\varphi, \psi \in \mathcal{L}$ such that $L \vdash \varphi \rightarrow \psi$, there is a formula $\theta \in \mathcal{L}$ such that $V^\dagger(\theta) \subseteq V^\dagger(\varphi) \cap V^\dagger(\psi)$, for any $\dagger \in \{+, -\}$ and $L \vdash \varphi \rightarrow \theta$ and $L \vdash \theta \rightarrow \psi$. A logic has Craig interpolation (CIP) if it has the above properties, omitting all the superscripts $\dagger \in \{+, -\}$.

Definition 4. A logic L has uniform Lyndon interpolation property (ULIP) if for any formula $\varphi \in \mathcal{L}$, atom p , and $\circ \in \{+, -\}$, there are p° -free formulas, $\forall^\circ p\varphi$ and $\exists^\circ p\varphi$, such that $V^\dagger(\exists^\circ p\varphi) \subseteq V^\dagger(\varphi)$ and $V^\dagger(\forall^\circ p\varphi) \subseteq V^\dagger(\varphi)$, for any $\dagger \in \{+, -\}$ and

- (i) $L \vdash \forall^\circ p\varphi \rightarrow \varphi$,
- (ii) for any p° -free formula ψ if $L \vdash \psi \rightarrow \varphi$ then $L \vdash \psi \rightarrow \forall^\circ p\varphi$,
- (iii) $L \vdash \varphi \rightarrow \exists^\circ p\varphi$, and
- (iv) for any p° -free formula ψ if $L \vdash \varphi \rightarrow \psi$ then $L \vdash \exists^\circ p\varphi \rightarrow \psi$.

A logic has uniform interpolation property (UIP) if it has all the above properties, omitting the superscripts $\circ, \dagger \in \{+, -\}$, everywhere.

Remark 1. As the formulas $\forall^\circ p\varphi$ and $\exists^\circ p\varphi$ are provably unique, using the functional notation of writing $\forall^\circ p\varphi$ and $\exists^\circ p\varphi$ as the functions with the arguments $\circ \in \{+, -\}$, p and φ is allowed.

Theorem 1. If a logic L has ULIP, then it has both LIP and UIP.

Proof. For UIP, set $\forall p\varphi = \forall^+ p\forall^- p\varphi$ and $\exists p\varphi = \exists^+ p\exists^- p\varphi$. We only prove the claim for $\forall p\varphi$, as the case for $\exists p\varphi$ is similar. First, it is clear that $V^\dagger(\forall p\varphi) \subseteq V^\dagger(\varphi)$, for any $\dagger \in \{+, -\}$. Hence, we have $V(\forall p\varphi) \subseteq V(\varphi)$. Moreover, $\forall p\varphi$ is p -free. Because $\forall^- p\varphi$ is p^- -free by definition and as $V^-(\forall p\varphi) \subseteq V^-(\forall^- p\varphi)$, the formula $\forall^+ p\forall^- p\varphi$ is also p^- -free. As $\forall^+ p\forall^- p\varphi$ is p^+ -free by definition, we have $p \notin V(\forall p\varphi) = V^+(\forall p\varphi) \cup V^-(\forall p\varphi)$. For condition (i) in Definition 4, as $L \vdash \forall^+ p\forall^- p\varphi \rightarrow \forall^- p\varphi$ and $L \vdash \forall^- p\varphi \rightarrow \varphi$, we have $L \vdash \forall p\varphi \rightarrow \varphi$. For condition (ii), if $L \vdash \psi \rightarrow \varphi$, for a p -free ψ , then ψ is also p^- -free and hence $L \vdash \psi \rightarrow \forall^- p\varphi$. As ψ is also p^+ -free, we have $L \vdash \psi \rightarrow \forall^+ p\forall^- p\varphi$.

¹ The label \diamond has nothing to do with the modal operator $\diamond = \neg\Box\neg$.

For LIP, assume $L \vdash \varphi \rightarrow \psi$. For any $\dagger \in \{+, -\}$, set $P^\dagger = V^\dagger(\varphi) - [V^\dagger(\varphi) \cap V^\dagger(\psi)]$. Define $\theta = \exists^+ P^+ \exists^- P^- \varphi$, where by $\exists^\dagger \{p_1, \dots, p_n\}^\dagger$ we mean $\exists p_1^\dagger \dots \exists p_n^\dagger$. Since θ is p^\dagger -free for any $p \in P^\dagger$ and any $\dagger \in \{+, -\}$, we have $V^\dagger(\theta) \subseteq V^\dagger(\varphi) - P^\dagger \subseteq V^\dagger(\varphi) \cap V^\dagger(\psi)$. For the provability condition, it is clear that $L \vdash \varphi \rightarrow \theta$ and as ψ is p^\dagger -free for any $p \in P^\dagger$, we have $L \vdash \theta \rightarrow \psi$.

2.1 Sequent Calculi

We use capital Greek letters and the bar notation in $\bar{\varphi}$ and \bar{C} to denote multisets. A *sequent* is an expression in the form $\Gamma \Rightarrow \Delta$, where Γ (the *antecedent*) and Δ (the *succedent*) are multisets of formulas. It is interpreted as $\bigwedge \Gamma \rightarrow \bigvee \Delta$. For sequents $S = (\Gamma \Rightarrow \Delta)$ and $T = (\Pi \Rightarrow \Lambda)$ we denote the sequent $\Gamma, \Pi \Rightarrow \Delta, \Lambda$ by $S \cdot T$, and the multisets Γ and Δ by S^a and S^s , respectively. Define $V^+(S) = V^-(S^a) \cup V^+(S^s)$ and $V^-(S) = V^+(S^a) \cup V^-(S^s)$ and the *weight of a sequent* as the sum of the weights of the formulas occurring in that sequent. A sequent S is *lower than* a sequent T , if the weight of S is less than the weight of T .

$$\begin{array}{c}
 \frac{}{\Gamma, p \Rightarrow p, \Delta} \\
 \\
 \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} L\wedge \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} R\wedge \\
 \\
 \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} L\vee \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta} R\vee \\
 \\
 \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta} L\rightarrow \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \rightarrow \psi, \Delta} R\rightarrow
 \end{array}$$

Fig. 1. The sequent calculus **G3cp**. In the axiom, p must be an atomic formula.

In this paper we are interested in modal extensions of the well-known sequent calculus **G3cp** from [18] (Fig. 1) for classical logic CPC and its extension by the following two weakening rules, denoted by **G3W**:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} Lw \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta} Rw$$

In each rule in **G3W**, the multiset Γ (res. Δ) is called the left (res. right) *context*, the formulas outside $\Gamma \cup \Delta$ are called the *active formulas* of the rule and the only formula in the conclusion outside $\Gamma \cup \Delta$ is called the *main formula*. If S is the conclusion of an instance of a rule R , we say that R is *backwards applicable* to S . The modal rules by which we extend **G3cp** or **G3W** are given in Fig. 2. For any such rule (X) except (EC), (N) and (NW), if we add it to **G3W** we denote the resulting system by **GX**, and if we add (N) to that system we get **GXN**. Note that **GMCN** is the usual system for the logic K. If we add (EC) to **G3cp**, we get **GEC** and if we also add the rule (NW), we get **GECN**. Note that **GEC** and **GECN** have no explicit weakening rules.

The systems **GEC** and **GECN** are introduced in [10]. The others are equivalent to the systems introduced in [10]. The only difference is that in our representation, the weakening rules are explicitly present, while the extra context in the conclusion of the modal rules are omitted. We will present the systems as such for convenience in our later proofs. As the systems **GE**, **GM**, **GMC**, **GEN** and **GMN** are equivalent to the systems introduced in [10], they all admit the cut rule and the contraction rules. Moreover, the logics of these systems, i.e., the sets of formulas φ for which the systems prove $(\Rightarrow \varphi)$ are the well-known basic non-normal modal logics E, M, MC, EN and MN, respectively. The logics of the systems **GEC** and **GECN** are the logics EC and ECN, respectively [10].

$$\frac{\varphi \Rightarrow \psi \quad \psi \Rightarrow \varphi}{\Box \varphi \Rightarrow \Box \psi} E \quad \frac{\varphi_1, \dots, \varphi_n \Rightarrow \psi \quad \psi \Rightarrow \varphi_1 \quad \dots \quad \psi \Rightarrow \varphi_n}{\Sigma, \Box \varphi_1, \dots, \Box \varphi_n \Rightarrow \Box \psi, \Lambda} EC \ (n \geq 1)$$

$$\frac{\varphi \Rightarrow \psi}{\Box \varphi \Rightarrow \Box \psi} M \quad \frac{\Rightarrow \psi}{\Rightarrow \Box \psi} N \quad \frac{\Rightarrow \psi}{\Sigma \Rightarrow \Box \psi, \Lambda} NW \quad \frac{\varphi_1, \dots, \varphi_n \Rightarrow \psi}{\Box \varphi_1, \dots, \Box \varphi_n \Rightarrow \Box \psi} MC$$

Fig. 2. The modal rules

Here are some remarks about the rules introduced above. First, for any rule the weight of each premise is less than the weight of its conclusion. Specifically, the weight of $\Gamma, \Sigma \Rightarrow \Delta, \Lambda$ is less than the weight of $\Box \Gamma, \Sigma \Rightarrow \Box \Delta, \Lambda$, as long as $\Gamma \cup \Delta$ is non-empty. Second, in any rule in **G3W**, if we add a multiset, both to the antecedent (succedent) of the premises and to the antecedent (succedent) of the conclusion, the result remains an instance of the rule. We call this property the *context extension property*. Conversely, if a multiset is a sub-multiset of the left (right) context of the rule, then if we eliminate this multiset both from the premises and the conclusion, the result remains an instance of the rule. We call this property the *context restriction property*. Third, for any rule in **G3W** and any $\circ \in \{+, -\}$, if the main formula φ is in the antecedent, then for any active formula α in the antecedent of a premise and any active formula β in the succedent of a premise, we have $V^\circ(\alpha) \cup V^\circ(\beta) \subseteq V^\circ(\varphi)$, and if φ is in the succedent, we have $V^\circ(\alpha) \cup V^\circ(\beta) \subseteq V^\circ(\varphi)$ (note the use of \circ and \diamond). We call this property, the *variable preserving property*. As a consequence of this property for the rule $\frac{S_1 \dots S_n}{S}$ in **G3W**, we have $\bigcup_{i=1}^n V^\circ(S_i) \subseteq V^\circ(S)$.

3 Uniform Lyndon Interpolation

In this section, we prove ULIP for the logics E, M, MC, EN, and MN. To this end, we need to first extend the notion to the sequent calculi of these logics. Since all these logics are classical, we only define the universal quantifier, as the existential quantifier is constructed by the universal quantifier and negation.

Definition 5. Let G be one of the sequent calculi introduced in Preliminaries. G has uniform Lyndon interpolation property (ULIP) if for any sequent S , any atom p and any $\circ \in \{+, -\}$, there exists a formula $\forall^\circ pS$ such that:

- (var) $\forall^\circ pS$ is p° -free and $V^\dagger(\forall^\circ pS) \subseteq V^\dagger(S)$, for any $\dagger \in \{+, -\}$,
- (i) $S \cdot (\forall^\circ pS \Rightarrow)$ is derivable in G ,
- (ii) for any sequent $\Gamma \Rightarrow \Delta$ such that $p \notin V^\circ(\Gamma \Rightarrow \Delta)$, if $S \cdot (\Gamma \Rightarrow \Delta)$ is derivable in G then $(\Gamma \Rightarrow \forall^\circ pS, \Delta)$ is derivable in G .

$\forall^\circ pS$ is called a uniform \forall°_p -interpolant of S in G . For any set of rules \mathcal{R} of G , a formula $\forall^\circ_{\mathcal{R}} pS$ is called a uniform \forall°_p -interpolant of S with respect to \mathcal{R} , if it satisfies the conditions (var) and (i), when $\forall^\circ pS$ is replaced by $\forall^\circ_{\mathcal{R}} pS$, and:

- (ii') for any sequent $\Gamma \Rightarrow \Delta$ such that $p \notin V^\circ(\Gamma \Rightarrow \Delta)$, if there is a derivation of $S \cdot (\Gamma \Rightarrow \Delta)$ in G whose last inference rule is an instance of a rule in \mathcal{R} , then $(\Gamma \Rightarrow \forall^\circ_{\mathcal{R}} pS, \Delta)$ is derivable in G .

Remark 2. As the formula $\forall^\circ pS$ is provably unique, using the functional notation of writing $\forall^\circ pS$ as a function with the arguments $\circ \in \{+, -\}$, p and S is allowed. The same does not hold for $\forall^\circ_{\mathcal{R}} pS$. However, as there is no risk of confusion and we will be specific about the construction of the formula $\forall^\circ_{\mathcal{R}} pS$, we will also use the functional notation in this case.

The following theorem connects ULIP for sequent calculi to the original version.

Theorem 2. Let G be one of the sequent calculi introduced in Preliminaries and L be its logic. Then, G has ULIP iff L has ULIP.

Proof. If G has ULIP, set $\forall^\circ pA = \forall^\circ p(\Rightarrow A)$ and $\exists^\circ pA = \neg \forall^\circ p \neg A$. Conversely, if L has ULIP, set $\forall^\circ p(\Gamma \Rightarrow \Delta) = \forall^\circ p(\bigwedge \Gamma \rightarrow \bigvee \Delta)$.

Our strategy to prove ULIP for the logics E, M, MC, EN, and MN is to prove ULIP for their sequent calculi. From now on, up to Subsect. 3.1, we assume that G is one of **GE**, **GM**, **GMC**, **GEN**, and **GMN**. As stated previously, backward applications of the rules decreases the weight of the sequent. Using this property and recursion on the weight of the sequents, for any given sequent $S = (\Gamma \Rightarrow \Delta)$, any atom p and any $\circ \in \{+, -\}$, we first define a p° -free formula $\forall^\circ pS$ and then by induction on the weight of S , we prove that $\forall^\circ pS$ meets the conditions in Definition 5. Towards that end, both in the definition of $\forall^\circ pS$ and in the proof of its properties, we must address all the rules of the system G , one by one. To make the presentation uniform, modular, and more clear, we divide the rules of G into two families: the rules of **G3W** and the modal rules specific for G . The rules in the first class has one of the following forms:

$$\frac{\{ \Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta \}_i}{\Gamma, \varphi \Rightarrow \Delta} \qquad \frac{\{ \Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta \}_i}{\Gamma \Rightarrow \varphi, \Delta}$$

where Γ and Δ are free for all multiset substitutions, and $\bar{\varphi}_i$'s and $\bar{\psi}_i$'s are multisets of formulas (possibly empty). The rules have the variable preserving condition, i.e., given $\circ \in \{+, -\}$, for the left rule $\bigcup_i \bigcup_{\theta \in \bar{\varphi}_i} V^\circ(\theta) \cup \bigcup_i \bigcup_{\theta \in \bar{\psi}_i} V^\circ(\theta) \subseteq V^\circ(\varphi)$, and for the right one $\bigcup_i \bigcup_{\theta \in \bar{\varphi}_i} V^\circ(\theta) \cup \bigcup_i \bigcup_{\theta \in \bar{\psi}_i} V^\circ(\theta) \subseteq V^\circ(\varphi)$. Rather than addressing each rule in **G3W**, we simply address these two forms.

Lemma 1. *For any sequent S , atom p and $\circ \in \{+, -\}$, a uniform \forall_p° -interpolant of S with respect to the set of all axioms of G exists.*

Proof. Let us define a formula $\forall_{ax}^\circ pS$: if S is provable, define it as \top , otherwise, define it as the disjunction of all p° -free formulas in S^s and the negation of all p° -free formulas in S^a . We show that $\forall_{ax}^\circ pS$ is the uniform \forall_p° -interpolant of S with respect to the set of axioms of G . It is easy to see that $\forall_{ax}^\circ pS$ is p° -free, $V^\dagger(\forall_{ax}^\circ pS) \subseteq V^\dagger(S)$, for $\dagger \in \{+, -\}$ and $S \cdot (\forall_{ax}^\circ pS \Rightarrow)$ is provable in G . To prove the condition (i') in Definition 5, if S is provable, then as $\forall_{ax}^\circ pS = \top$, we have $\bar{C} \Rightarrow \forall_{ax}^\circ pS, \bar{D}$. If S is not provable, then let $S \cdot (\bar{C} \Rightarrow \bar{D})$ be an axiom. There are two cases to consider. First, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ is in the form $\Gamma, q \Rightarrow q, \Delta$, where q is an atomic formula. Then, if $q \notin \bar{C}$ and $q \notin \bar{D}$, we have $q \in \Gamma \cap \Delta$ and hence the sequent S is provable which contradicts our assumption. Therefore, either $q \in \bar{C}$ or $q \in \bar{D}$. If $q \in \bar{C} \cap \bar{D}$, then $\bar{C} \Rightarrow \forall_{ax}^\circ pS, \bar{D}$ is provable. Hence, we assume either $q \in \bar{C}$ and $q \notin \bar{D}$ or $q \notin \bar{C}$ and $q \in \bar{D}$. In the first case, if $q \in \bar{C}$, it is p° -free and since it occurs in Δ , it is a disjunct in $\forall_{ax}^\circ pS$. Hence, $\bar{C} \Rightarrow \forall_{ax}^\circ pS, \bar{D}$ is provable. In the second case, if $q \in \bar{D}$, it is p° -free and as $q \in \Gamma$, its negation occurs in $\forall_{ax}^\circ pS$. Therefore $\bar{C} \Rightarrow \forall_{ax}^\circ pS, \bar{D}$ is provable. If $S \cdot (\bar{C} \Rightarrow \bar{D})$ is in the form $\Gamma, \perp \Rightarrow \Delta$, then $\perp \in \bar{C}$, because otherwise, $\perp \in \Gamma$ and hence S will be provable. Now, since $\perp \in \bar{C}$, we have $\bar{C} \Rightarrow \forall_{ax}^\circ pS, \bar{D}$.

Definition 6. *Let $\mathcal{U}_p^\circ(S)$ be the statement that “all sequents lower than S have uniform \forall_p° -interpolants”. A calculus G has MUIP if for any sequent S , atom p , and $\circ \in \{+, -\}$, there exists a formula $\forall_m^\circ pS$ such that if $\mathcal{U}_p^\circ(S)$, then $\forall_m^\circ pS$ is a uniform \forall_p° -interpolant for S with respect to the set of modal rules of G .*

Theorem 3. *If a sequent calculus G has MUIP, then it has ULIP.*

Proof. Define a formula $\forall^\circ pS$ by recursion on the weight of S : if S is provable define it as \top , otherwise, define it as:

$$\bigvee_R \left(\bigwedge_i \forall^\circ pS_i \right) \vee (\forall_{ax}^\circ pS) \vee (\forall_m^\circ pS)$$

where the first disjunction is over all rules R in **G3W** backward applicable to S , where S is the consequence and S_i 's are the premises. $\forall_{ax}^\circ pS$ is a uniform \forall_p° -interpolant of S with respect to the set of axioms of G that Lemma 1 provides. $\forall_m^\circ pS$ is the formula that MUIP provides. To prove that $\forall^\circ pS$ is a \forall_p° -interpolant for S , we use induction on the weight of S to prove:

- (var) $\forall^\circ pS$ is p° -free and $V^\dagger(\forall^\circ pS) \subseteq V^\dagger(S)$, for any $\dagger \in \{+, -\}$,
- (i) $S \cdot (\forall^\circ pS \Rightarrow)$ is provable in G ,

(ii) for any p° -free sequent $\bar{C} \Rightarrow \bar{D}$, if $S \cdot (\bar{C} \Rightarrow \bar{D})$ is derivable in G then $\bar{C} \Rightarrow \forall^\circ pS, \bar{D}$ is derivable in G .

By induction hypothesis, (var), (i), and (ii) hold for all sequents T lower than S . Now, (var) also holds for $\forall^\circ pS$, because both $\forall_{ax}^\circ pS$ and $\forall_m^\circ pS$ satisfy (var) and all rules in **G3W** have the variable preserving property.

To prove (i), it is enough to show that the following are provable in G :

$$S \cdot \left(\bigwedge_i \forall^\circ pS_i \Rightarrow \right) \quad (1), \quad S \cdot (\forall_{ax}^\circ pS \Rightarrow) \quad (2), \quad S \cdot (\forall_m^\circ pS \Rightarrow) \quad (3).$$

Sequent (3) is provable by induction hypothesis and the assumption that G has MUIP. Sequent (2) is proved in Lemma 1. For the sequent (1), assume that the rule R of **G3W** is backward applicable to S , i.e., the premises of R are S_i 's and its conclusion S . As S_i 's are lower than S , by induction hypothesis we have $S_i \cdot (\forall^\circ pS_i \Rightarrow)$. Therefore, by weakening, we have $S_i \cdot (\{\forall^\circ pS_i\}_i \Rightarrow)$. Since any rule in **G3W** has the context extension property, we can add $\{\forall^\circ pS_i\}_i$ to the antecedent of both premises and conclusion and by the rule itself, we have $S \cdot (\{\forall^\circ pS_i\}_i \Rightarrow)$ and hence $S \cdot (\bigwedge_i \forall^\circ pS_i \Rightarrow)$.

For (ii), we use induction on the length of the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$. Let $S \cdot (\bar{C} \Rightarrow \bar{D})$ be derivable in G . If it is an axiom, we have $\bar{C} \Rightarrow \bar{D}, \forall_{ax}^\circ pS$ by Lemma 1, and hence $\bar{C} \Rightarrow \bar{D}, \forall^\circ pS$. If the last rule is a rule in **G3W** of the form:

$$\frac{\{\Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta\}_i}{\Gamma, \varphi \Rightarrow \Delta},$$

then there are two cases to consider, i.e., either $\varphi \in \bar{C}$ or $\varphi \in S^a$. If $\varphi \in \bar{C}$, then set $\bar{C}' = \bar{C} - \{\varphi\}$. Since $\varphi \in \bar{C}$, it is p° -free by the assumption and φ_i 's are all p° -free and ψ_i 's are all p° -free by the variable preserving property. By induction hypothesis, as $(\bar{C}', \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \bar{D})$ is p° -free and $S \cdot (\bar{C}', \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \bar{D})$ has a shorter proof, we have $\bar{C}', \bar{\varphi}_i \Rightarrow \forall^\circ pS, \bar{\psi}_i, \bar{D}$. By using the rule itself, we have

$$\frac{\{\bar{C}', \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \forall^\circ pS, \bar{D}\}_i}{\bar{C}', \varphi \Rightarrow \forall^\circ pS, \bar{D}}$$

which implies $\bar{C} \Rightarrow \forall^\circ pS, \bar{D}$.

If $\varphi \notin \bar{C}$, then both \bar{C} and \bar{D} do not contain any active formula of the rule and hence the last rule is in form

$$\frac{\{\bar{C}, \Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \bar{D}, \Delta\}_i}{\bar{C}, \Gamma, \varphi \Rightarrow \bar{D}, \Delta}.$$

By context restriction property, if we erase \bar{C} and \bar{D} both on the premises and the consequence of the last rule, the rule remains valid and it changes to:

$$\frac{\{\Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta\}_i}{\Gamma, \varphi \Rightarrow \Delta}.$$

Therefore, the rule is backward applicable to $S = (\Gamma, \varphi \Rightarrow \Delta)$. Set $S_i = (\Gamma, \bar{\varphi}_i \Rightarrow \bar{\psi}_i, \Delta)$. As the weight of S_i 's are less than the weight of S and $S_i \cdot (\bar{C} \Rightarrow \bar{D})$

are provable, by induction hypothesis, we have $\bar{C} \Rightarrow \forall^\circ pS_i, \bar{D}$. Hence, $\bar{C} \Rightarrow \bigwedge_i \forall^\circ pS_i, \bar{D}$ and as $\bigwedge_i \forall^\circ pS_i$ is a disjunct in $\forall^\circ pS$, we have $\bar{C} \Rightarrow \forall^\circ pS, \bar{D}$. The case where the last rule is in **GW3** with its main formula in the antecedent is similar. For the modal rules, by induction hypothesis $\mathcal{U}_p^\circ(S)$ and the assumption that G has MUIP, we get that $\forall_m^\circ pS$ is a uniform \forall_p° -interpolant for S with respect to the set of modal rules of G . By (ii') in Definition 5, this gives $\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D}$ and hence $\bar{C} \Rightarrow \forall^\circ pS, \bar{D}$.

In the upcoming subsections, for the following choices of the system G , we show that it has MUIP. Therefore, by Theorem 3 and Theorem 2, we will have:

Theorem 4. *Logics E, M, MC, EN and MN have ULIP, hence UIP and LIP.*

3.1 Modal Logics M and MN

Let G be either **GM** or **GMN**. We will show that G has MUIP. To define $\forall_m^\circ pS$, if $\neg \mathcal{U}_p^\circ(S)$, define $\forall_m^\circ pS$ as \perp . If $\mathcal{U}_p^\circ(S)$, (i.e., for any sequent T lower than S a uniform \forall_p° -interpolant, denoted by $\forall^\circ pT$, exists), define $\forall_m^\circ pS$ in the following way: if S is provable, define it as \top , otherwise, if it is of the form $(\Box\varphi \Rightarrow)$, define $\forall_m^\circ pS = \neg\Box\neg\forall^\circ pS'$, where $S' = (\varphi \Rightarrow)$, if S is of the form $(\Rightarrow \Box\psi)$, define $\forall_m^\circ pS = \Box\forall^\circ pS''$, where $S'' = (\Rightarrow \psi)$, and otherwise, define $\forall_m^\circ pS = \perp$. Note that $\forall_m^\circ pS$ is well-defined as we have $\mathcal{U}_p^\circ(S)$ and S' and S'' are lower than S .

To show that G has MUIP, we assume $\mathcal{U}_p^\circ(S)$ to prove the three conditions (var), (i) and (ii') in Definition 5 for $\forall_m^\circ pS$. First, note that using $\mathcal{U}_p^\circ(S)$ on $(\varphi \Rightarrow)$ and $(\Rightarrow \psi)$ that are lower than $(\Box\varphi \Rightarrow)$ and $(\Rightarrow \Box\psi)$, respectively, the variable conditions are implied from (var) for S' and S'' , respectively.

For (i), if S is provable, there is nothing to prove. Otherwise, if $S = (\Box\varphi \Rightarrow)$ then $\forall_m^\circ pS = \neg\Box\neg\forall^\circ pS'$. As S' is lower than S , we have $(\varphi, \forall^\circ pS' \Rightarrow)$ by $\mathcal{U}_p^\circ(S)$, which implies $(\varphi \Rightarrow \neg\forall^\circ pS')$. Using the rule (M), we get $(\Box\varphi \Rightarrow \Box\neg\forall^\circ pS')$, which is equivalent to $(\Box\varphi, \neg\Box\neg\forall^\circ pS' \Rightarrow)$. Hence, $S \cdot (\forall_m^\circ pS \Rightarrow)$ is provable.

If S is not provable and $S = (\Rightarrow \Box\psi)$, we have $\forall_m^\circ pS = \Box\forall^\circ pS''$. Using $\mathcal{U}_p^\circ(S)$ on S'' and the fact that S'' is lower than S , we have $(\forall^\circ pS'' \Rightarrow \psi)$ and by the rule (M), we can show that $S \cdot (\Box\forall^\circ pS'' \Rightarrow)$ is provable in G . If S is not provable and has none of the mentioned forms, as $\forall_m^\circ pS = \perp$, there is nothing to prove.

For (ii'), let $S \cdot (\bar{C} \Rightarrow \bar{D})$ be derivable in G for a p° -free sequent $\bar{C} \Rightarrow \bar{D}$ and the last rule is a modal rule. We want to show that $\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D}$ is derivable in G . If the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is (M), the sequent must have the form $(\Box\varphi \Rightarrow \Box\psi)$ and the rule must be in form:

$$\frac{\varphi \Rightarrow \psi}{\Box\varphi \Rightarrow \Box\psi} M$$

If S is provable, as $\forall_m^\circ pS = \top$, we clearly have $\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D}$. Assume S is not provable and hence $\bar{C} \cup \bar{D}$ cannot be empty. Therefore, there are three cases to consider, either \bar{C} is $\Box\varphi$ or \bar{D} is $\Box\psi$ or both. First, if $\bar{C} = \Box\varphi$ and $\bar{D} = \emptyset$, then, $S = (\Rightarrow \Box\psi)$ and φ is p° -free. Set $S'' = (\Rightarrow \psi)$. Then $\forall_m^\circ pS = \Box\forall^\circ pS''$. As S''

is lower than S , by $\mathcal{U}_p^\circ(S)$ we have $(\varphi \Rightarrow \forall^\circ pS'')$. Using the modal rule (M) , we have $(\Box\varphi \Rightarrow \Box\forall^\circ pS'')$ and hence $(\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D})$.

In the second case, assume $\bar{C} = \emptyset$ and $\bar{D} = \Box\psi$. Hence, $S = (\Box\varphi \Rightarrow)$ and ψ is p° -free. Set $S' = (\varphi \Rightarrow)$. Hence, $\forall_m^\circ pS = \neg\Box\neg\forall^\circ pS'$. Since $(\varphi \Rightarrow \psi)$ is provable in G and S' is lower than S , by \mathcal{U}_p° we have $(\Rightarrow \forall^\circ pS', \psi)$, or equivalently $(\neg\forall^\circ pS' \Rightarrow \psi)$. Using the rule (M) , we get $(\Box\neg\forall^\circ pS' \Rightarrow \Box\psi)$ or equivalently $(\Rightarrow \neg\Box\neg\forall^\circ pS', \Box\psi)$. Therefore, we have $(\Rightarrow \forall_m^\circ pS, \Box\psi)$ or $(\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D})$.

In the third case, if $\bar{C} = \Box\varphi$ and $\bar{D} = \Box\psi$, then S is the empty sequent and $\bar{C} \Rightarrow \bar{D}$ is provable. Hence, $\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D}$ is also provable.

For the case $G = \mathbf{GMN}$, if $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Rightarrow \Box\psi)$ is proved by the rule (N) , it must have the following form:

$$\frac{\Rightarrow \psi}{\Rightarrow \Box\psi} N$$

Then $\bar{C} = \emptyset$ and there are two cases to consider. The first case is when $S = (\Rightarrow \Box\psi)$ and $\bar{D} = \emptyset$. Then, it means that S is provable which contradicts our assumption. The second case is when $S = (\Rightarrow)$ and $\bar{D} = \Box\psi$. Hence, $\bar{C} \Rightarrow \bar{D}$ is provable and we have the provability of $\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D}$ in G .

3.2 Modal Logic MC

Similar to the argument of the previous subsection, to define $\forall_m^\circ pS$, if $\neg\mathcal{U}_p^\circ(S)$, define $\forall_m^\circ pS$ as \perp . If $\mathcal{U}_p^\circ(S)$, (i.e., for any sequent T lower than S the uniform \forall_p° -interpolant, denoted by $\forall^\circ pT$, exists), define $\forall_m^\circ pS$ as the following: if S is provable, define $\forall_m^\circ pS = \top$. Otherwise, if S is of the form $(\Box\varphi_1, \dots, \Box\varphi_i \Rightarrow)$, for some $i \geq 1$, define $\forall_m^\circ pS = \neg\Box\neg\forall^\circ pS'$, where $S' = (\varphi_1, \dots, \varphi_i \Rightarrow)$. If S is of the form $(\Rightarrow \Box\psi)$, define $\forall_m^\circ pS = \Box\forall^\circ pS''$, where $S'' = (\Rightarrow \psi)$. If S is of the form $(\Box\varphi_1, \dots, \Box\varphi_i \Rightarrow \Box\psi)$, for some $i \geq 1$, define $\forall_m^\circ pS = \Box\forall^\circ pS''$, where $S'' = (\varphi_1, \dots, \varphi_i \Rightarrow \psi)$. Otherwise, define $\forall_m^\circ pS = \perp$. Note that $\forall_m^\circ pS$ is well-defined as we assumed $\mathcal{U}_p^\circ(S)$ and in each case S' or S'' are lower than S .

To show that \mathbf{GMC} has MUIP, we assume $\mathcal{U}_p^\circ(S)$ to prove the three conditions (var) , (i) and (ii') in Definition 5 for $\forall_m^\circ pS$. The condition (var) is an immediate consequence of $\mathcal{U}_p^\circ(S)$ and the fact that S' or S'' are lower than S . For (i) , if S is provable, there is nothing to prove. If S is of the form $(\Box\varphi_1, \dots, \Box\varphi_i \Rightarrow)$ and $\forall_m^\circ pS = \neg\Box\neg\forall^\circ pS'$, where $S' = (\varphi_1, \dots, \varphi_i \Rightarrow)$, as S' is lower than S , by $\mathcal{U}_p^\circ(S)$ we have $(\varphi_1, \dots, \varphi_i, \forall^\circ pS' \Rightarrow)$ or equivalently $(\varphi_1, \dots, \varphi_i \Rightarrow \neg\forall^\circ pS')$. Using the rule (MC) , we get $(\Box\varphi_1, \dots, \Box\varphi_i \Rightarrow \Box\neg\forall^\circ pS')$, which is equivalent to $(\Box\varphi_1, \dots, \Box\varphi_i, \neg\Box\neg\forall^\circ pS' \Rightarrow)$ and hence $S \cdot (\forall_m^\circ pS \Rightarrow)$.

If S is of the form $(\Rightarrow \Box\psi)$ and $S'' = (\Rightarrow \psi)$, or S is of the form $(\Box\varphi_1, \dots, \Box\varphi_i \Rightarrow \Box\psi)$, for some $i \geq 1$ and S'' is of the form $(\varphi_1, \dots, \varphi_i \Rightarrow \psi)$, we have $\forall_m^\circ pS = \Box\forall^\circ pS''$. In both cases, using $\mathcal{U}_p^\circ(S)$ on S'' , we have either $\forall^\circ pS'' \Rightarrow \psi$ or $\varphi_1, \dots, \varphi_i, \forall^\circ pS'' \Rightarrow \psi$, respectively. In both cases, using the rule (MC) , we can show that $S \cdot (\Box\forall^\circ pS'' \Rightarrow)$ is provable and hence $S \cdot (\forall_m^\circ pS \Rightarrow)$.

For (ii') , let $S \cdot (\bar{C} \Rightarrow \bar{D})$ be derivable in \mathbf{GMC} and the last rule is the modal rule (MC) , for a p° -free sequent $\bar{C} \Rightarrow \bar{D}$. We want to show that $\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D}$

is derivable in **GMC**. If S is provable, as $\forall_m pS = \top$, we have $\bar{C} \Rightarrow \forall_m pS, \bar{D}$. Therefore, we assume that S is not provable. As the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is (MC) , the sequent must have the form $(\Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Box\psi)$ and the rule is:

$$\frac{\varphi_1, \dots, \varphi_n \Rightarrow \psi}{\Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Box\psi} MC$$

Then, there are two cases to consider, either $\bar{D} = \Box\psi$ or $\bar{D} = \emptyset$. First, assume S is of the form $(\Box\varphi_1, \dots, \Box\varphi_i \Rightarrow)$, for $i \leq n$, then $\bar{C} = \Box\varphi_{i+1}, \dots, \Box\varphi_n$ and $\bar{D} = \Box\psi$ and hence $\varphi_{i+1}, \dots, \varphi_n \Rightarrow \psi$ is p° -free. Set $S' = (\varphi_1, \dots, \varphi_i \Rightarrow)$. By the form of S , we have $\forall_m pS = \neg\Box\neg\forall^\circ pS'$. As S' is lower than S , by $\mathcal{U}_p^\circ(S)$, we have $(\varphi_{i+1}, \dots, \varphi_n \Rightarrow \forall^\circ pS', \psi)$. Hence, by moving $\forall^\circ pS'$ to the left, applying the rule (MC) and moving back, we have $(\Box\varphi_{i+1}, \dots, \Box\varphi_n \Rightarrow \neg\Box\neg\forall^\circ pS', \Box\psi)$ or equivalently $(\bar{C} \Rightarrow \forall_m pS, \bar{D})$.

If S is of the form $\Box\varphi_1, \dots, \Box\varphi_i \Rightarrow \Box\psi$, for some $i \leq n$, we must have $\bar{C} = \Box\varphi_{i+1}, \dots, \Box\varphi_n$ and $\bar{D} = \emptyset$. Hence, $\varphi_{i+1}, \dots, \varphi_n$ are p° -free. Note that $i < n$, because if $i = n$, then S will be provable that contradicts our assumption. Set $S'' = (\varphi_1, \dots, \varphi_i \Rightarrow \psi)$. As S'' is lower than S , by $\mathcal{U}_p^\circ(S)$ we have $\varphi_{i+1}, \dots, \varphi_n \Rightarrow \forall^\circ pS''$. By the fact that $i < n$, we can apply the rule (MC) to prove $\Box\varphi_{i+1}, \dots, \Box\varphi_n \Rightarrow \Box\forall^\circ pS''$ and hence $(\bar{C} \Rightarrow \forall_m pS, \bar{D})$.

3.3 Modal Logics E and EN

Let G be **GE** or **GEN**. Similar to the argument of the previous subsection, if $\neg\mathcal{U}_p^\circ(S)$, define $\forall_m pS$ as \perp . If $\mathcal{U}_p^\circ(S)$, then: if S is provable in G , define $\forall_m pS = \top$. Otherwise, if $S = (\Box\varphi \Rightarrow)$ and both $(\neg\forall^\circ pS' \Rightarrow \varphi)$ and $(\varphi \Rightarrow \neg\forall^\circ pS')$ are provable in G , define $\forall_m pS = \neg\Box\neg\forall^\circ pS'$ for $S' = (\varphi \Rightarrow)$. If S has the form $(\Rightarrow \Box\psi)$ and both $(\forall^\circ pS'' \Rightarrow \psi)$ and $(\psi \Rightarrow \forall^\circ pS'')$ are provable in G , define $\forall_m pS = \Box\forall^\circ pS''$ for $S'' = (\Rightarrow \psi)$. Otherwise, define $\forall_m pS = \perp$. Note that $\forall^\circ pS$ is well-defined as S' and S'' are lower than S and we assumed $\mathcal{U}_p^\circ(S)$.

To show that G has MUIP we assume $\mathcal{U}_p^\circ(S)$ to prove (var) , (i) and (ii') in Definition 5 for $\forall_m pS$. Condition (var) is a consequence of $\mathcal{U}_p^\circ(S)$ and that S' or S'' are lower than S . For (i) , if S is provable, there is nothing to prove. If $S = (\Box\varphi \Rightarrow)$ and $S' = (\varphi \Rightarrow)$ and both $(\neg\forall^\circ pS' \Rightarrow \varphi)$ and $(\varphi \Rightarrow \neg\forall^\circ pS')$ are provable in G , then using the rule (E) , we have $(\Box\varphi \Rightarrow \Box\neg\forall^\circ pS')$ which implies $(\Box\varphi, \neg\Box\neg\forall^\circ pS' \Rightarrow)$ and hence $S \cdot (\forall_m pS \Rightarrow)$ is provable in G .

If $S = (\Rightarrow \Box\psi)$ and $S'' = (\Rightarrow \psi)$ and both $(\forall^\circ pS'' \Rightarrow \psi)$ and $(\psi \Rightarrow \forall^\circ pS'')$ are provable in G , then using the rule (E) , we have $(\Box\forall^\circ pS'' \Rightarrow \Box\psi)$ and hence $S \cdot (\forall_m pS \Rightarrow)$ is provable in G . If $\forall_m pS = \perp$, there is nothing to prove.

For (ii') , if S is provable, then $\forall_m pS = \top$ and hence $\bar{C} \Rightarrow \forall_m pS, \bar{D}$. Therefore, assume that S is not provable. If the last rule used in the proof of $S \cdot (\bar{C} \Rightarrow \bar{D})$ is the rule (E) , the sequent $S \cdot (\bar{C} \Rightarrow \bar{D})$ is of the form $\Box\varphi \Rightarrow \Box\psi$. There are four cases to consider based on if \bar{C} or \bar{D} are empty or not. First, if $\bar{C} = \bar{D} = \emptyset$, then S is provable which contradicts our assumption. If S is the empty sequent (\Rightarrow) , then $\bar{C} \Rightarrow \bar{D}$ is provable and hence $\bar{C} \Rightarrow \forall_m pS, \bar{D}$ is provable.

If $S = (\Box\varphi \Rightarrow)$, then $\bar{C} = \emptyset$ and $\bar{D} = \Box\psi$ and hence ψ is p° -free. Set $S' = (\varphi \Rightarrow)$ and as the last rule is (E) , both $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$ are provable. By $\mathcal{U}_p^\circ(S)$ and the fact that S' is lower than S , we have $(\varphi, \forall^\circ pS' \Rightarrow)$ or equivalently, $(\varphi \Rightarrow \neg\forall^\circ pS')$. Again by $\mathcal{U}_p^\circ(S)$ for S' , the provability of $S' \cdot (\Rightarrow \bar{D}) = (\varphi \Rightarrow \psi)$ and the fact that $(\Rightarrow \psi)$ is p° -free, we have $(\Rightarrow \forall^\circ pS', \psi)$ or equivalently, $(\neg\forall^\circ pS' \Rightarrow \psi)$. Since $(\varphi \Rightarrow \psi)$ and $(\psi \Rightarrow \varphi)$ are provable, by cut we can prove the equivalence between φ, ψ and $\neg\forall^\circ pS'$. Using this fact, we have:

$$\frac{\psi \Rightarrow \neg\forall^\circ pS' \quad \neg\forall^\circ pS' \Rightarrow \psi}{\Box\neg\forall^\circ pS' \Rightarrow \Box\psi} E$$

Hence, $(\Rightarrow \neg\Box\neg\forall^\circ pS', \Box\psi)$. Then, as $S = (\Box\varphi \Rightarrow)$ and both $(\neg\forall^\circ pS' \Rightarrow \varphi)$ and $(\varphi \Rightarrow \neg\forall^\circ pS')$ are provable in G , by definition we have $\forall_m^\circ pS = \neg\Box\neg\forall^\circ pS'$ and hence $(\Rightarrow \neg\Box\neg\forall^\circ pS', \Box\psi) = (\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D})$ is provable in G . The last case where $S = (\Rightarrow \Box\psi)$ and $\bar{C} = \Box\varphi$ and $\bar{D} = \emptyset$ is similar.

For the case $G = \mathbf{GEN}$, if $S \cdot (\bar{C} \Rightarrow \bar{D}) = (\Rightarrow \Box\psi)$ is proved by the rule (N) , it must have the form $\frac{\Rightarrow \psi}{\Rightarrow \Box\psi} N$. Then $\bar{C} = \emptyset$ and there are two cases. First, $S = (\Rightarrow \Box\psi)$ and $\bar{D} = \emptyset$, which means that S is provable which contradicts our assumption. Second, if $S = (\Rightarrow)$ and $\bar{D} = \Box\psi$, and hence $\bar{C} \Rightarrow \bar{D}$ is provable, we have the provability of $\bar{C} \Rightarrow \forall_m^\circ pS, \bar{D}$ in G .

4 Modal Logics EC and ECN

In this section we prove that the logics EC and ECN do not enjoy the Craig interpolation property. To this end, we set $\varphi = \Box(\neg q \wedge r)$ and $\psi = \Box(p \wedge q) \rightarrow \Box\perp$, where p, q , and r are three distinct atomic formulas and show that if L is either EC or ECN, the formula $\varphi \rightarrow \psi$ is provable in L , while there is no formula θ such that $V(\theta) \subseteq \{q\}$ and both formulas $\varphi \rightarrow \theta$ and $\theta \rightarrow \psi$ are provable in L .

To show that $\varphi \rightarrow \psi$ is in EC and hence ECN, we use the following proof tree in **GE**C:

$$\frac{p \wedge q, \neg q \wedge r \Rightarrow \perp \quad \perp \Rightarrow p \wedge q \quad \perp \Rightarrow \neg q \wedge r}{\Box(p \wedge q), \Box(\neg q \wedge r) \Rightarrow \Box\perp} EC$$

Now, for the sake of contradiction, assume that the interpolant θ for $\varphi \rightarrow \psi$ exists. Let G be either **GE**C or **GE**CN. Hence, both $\Box(\neg q \wedge r) \Rightarrow \theta$ and $\Box(p \wedge q), \theta \Rightarrow \Box\perp$ are provable in G . We first analyse the general form of θ .

First, note that by a simple induction on the structure of the formulas in the language \mathcal{L} , it is possible to show that any formula A is **G3cp**-equivalent to a CNF-style formula $\bigwedge_{i \in I} \bigvee_{j \in J_i} L_{ij}$, where I and J_i 's are (possibly empty) finite sets, $V(L_{ij}) \subseteq V(A)$, and each L_{ij} is either an atomic formula, the negation of an atomic formula, $\Box C$ or $\neg\Box C$, for a formula C . In particular, the formula θ is **G3cp**-equivalent to a CNF-style formula in the form $\bigwedge_{i \in I} \bigvee_{j \in J_i} L_{ij}$. W.l.o.g., assume that for any $i \in I$, it is impossible to have both an atomic formula and its negation in $\{L_{ij}\}_{j \in J_i}$, and that none of sequents $(\Rightarrow L_{ij})$ or $(L_{ij} \Rightarrow)$ are provable in G .

Back to the main argument, as $\varphi \Rightarrow \theta$ is provable in G , we have $\varphi \Rightarrow \bigwedge_{i \in I} \bigvee_{j \in J_i} L_{ij}$ which means that for every $i \in I$, we have $\varphi \Rightarrow \bigvee_{j \in J_i} L_{ij}$. Based on the form of each L_{ij} , we can transform the sequent to a provable sequent of the form $\varphi, P, \Box \Gamma \Rightarrow Q, \Box \Delta$, where P and Q are multisets of atomic formulas and Γ and Δ are multisets of formulas. We claim that for any $i \in I$, the corresponding Γ is non-empty. Suppose $\Gamma = \emptyset$. Then, we have $\varphi, P \Rightarrow Q, \Box \Delta$. This sequent must have been the conclusion of the rule (EC) , because for $G = \mathbf{GEC}$, the other possible case is being an axiom which implies either $\perp \in P$ or the existence of an atomic s in $P \cap Q$. Both contradict the structure of $\bigvee_{j \in J_i} L_{ij}$. For $G = \mathbf{GECN}$, the same holds. Moreover, if the last rule is (NW) , then for an element $\delta \in \Delta$, the sequent $(\Rightarrow \delta)$ and hence $(\Rightarrow \Box \delta)$ must be provable in G which contradicts the structure of L_{ij} 's again. Therefore, $T = (\varphi, P \Rightarrow Q, \Box \Delta)$ is the consequence of (EC) and hence, it has the form $(\Sigma, \Box \alpha_1, \dots, \Box \alpha_n \Rightarrow \Box \beta, \Lambda)$ and the last rule is:

$$\frac{\alpha_1, \dots, \alpha_n \Rightarrow \beta \quad \beta \Rightarrow \alpha_1 \quad \dots \quad \beta \Rightarrow \alpha_n}{\Sigma, \Box \alpha_1, \dots, \Box \alpha_n \Rightarrow \Box \beta, \Lambda} EC$$

Now there are two cases, either $\varphi \in \Sigma$ or $\varphi \notin \Sigma$. In the first case, as the formulas outside Σ are either atomic or boxed, we must have no boxed formula outside Σ . This is impossible, as the form of the rule (EC) dictates that we must have at least one boxed formula in the antecedent of the conclusion. Hence, $\varphi \notin \Sigma$.

As all formulas in T^a (except φ) are atomic, we must have only one boxed formula in T^a , which is φ . Therefore, in the premises of the rule, we have $\neg q \wedge r \Rightarrow \beta$ and $\beta \Rightarrow \neg q \wedge r$. Since $V(\beta) \subseteq V(\theta) \subseteq \{q\}$, then β is r -free. If we once substitute \perp for r and then $\neg q$ for r , as β remains intact, we will have $\beta \Leftrightarrow \perp$ and $\beta \Leftrightarrow \neg q$, which implies the contradictory $\perp \Leftrightarrow \neg q$. Hence, Γ cannot be empty.

So far, we have proved that Γ is non-empty, for any $i \in I$. Let D_i be a formula in Γ and note that $\neg \Box D_i$ occurs as one of the L_{ij} 's. Now, as $\Box(p \wedge q), \theta \Rightarrow \Box \perp$ or equivalently $\Box(p \wedge q), \bigwedge_{i \in I} \bigvee_{j \in J_i} L_{ij} \Rightarrow \Box \perp$ is provable in G , we have $\Box(p \wedge q), \{\neg \Box D_i\}_{i \in I} \Rightarrow \Box \perp$ is provable in G . Define $\mathcal{D} = \{D_i\}_{i \in I}$. Thus $S = (\Box(p \wedge q) \Rightarrow \Box \mathcal{D}, \Box \perp)$ is provable. As all the formulas are boxed, this must have been the conclusion of the rule (EC) . The reason is that G has no weakening rules, and for $G = \mathbf{GEC}$, the only modal rule is (EC) and for $G = \mathbf{GECN}$, the last rule cannot be the rule (NW) as it implies that for one $D \in \mathcal{D}$ the sequent $(\Rightarrow D)$ is provable in G which means that $(\Rightarrow \Box D)$ and hence $(\neg \Box D \Rightarrow)$ is provable. The last contradicts with the structure of L_{ij} 's. This implies that the last inference is of the form:

$$\frac{\alpha_1, \dots, \alpha_n \Rightarrow \beta \quad \beta \Rightarrow \alpha_1 \quad \dots \quad \beta \Rightarrow \alpha_n}{\Sigma, \Box \alpha_1, \dots, \Box \alpha_n \Rightarrow \Box \beta, \Lambda} EC$$

Similar as before, there are two cases, either $\beta = \perp$ or $\beta \in \mathcal{D}$. If $\beta = \perp$, in the premises we must have $p \wedge q \Leftrightarrow \perp$ which is impossible. If $\beta \in \mathcal{D}$, it means that in the premises we had $p \wedge q \Leftrightarrow \beta$. Note that as $\beta \in \mathcal{D}$ we have $V(\beta) \subseteq V(\theta) \subseteq \{q\}$. Hence β is p -free. Substituting once \perp and then q for p , leave β intact and hence we get $\perp \Leftrightarrow \beta$ and $q \Leftrightarrow \beta$ which implies $q \Leftrightarrow \perp$, which is impossible.

Theorem 5. *Logics EC and ECN do not have CIP, hence not UIP or ULIP either.*

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