

Geometric Embeddability of Complexes is $\exists\mathbb{R}$ -complete

MIKKEL ABRAHAMSEN

miab@di.ku.dk
University of Copenhagen

LINDA KLEIST

kleist@ibr.cs.tu-bs.de
TU Braunschweig

TILLMANN MILTZOW

t.miltzow@uu.nl
Utrecht University

November 2021

Abstract

We show that the decision problem of determining whether a given (abstract simplicial) k -complex has a geometric embedding in \mathbb{R}^d is complete for the Existential Theory of the Reals for all $d \geq 3$ and $k \in \{d-1, d\}$. This implies that the problem is polynomial time equivalent to determining whether a polynomial equation system has a real solution. Moreover, this implies NP-hardness and constitutes the first hardness result for the algorithmic problem of geometric embedding (abstract simplicial) complexes.

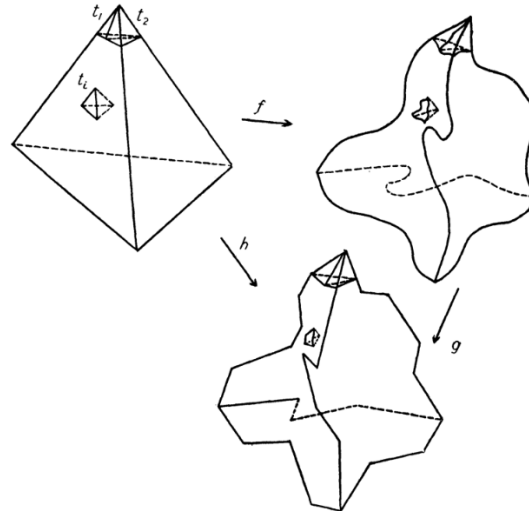


Figure 1: Illustration of different embeddings of a complex; figure taken from Bing [8, Annals of Mathematics 1959].

1 Introduction

Since the dawn of the last century, much attention has been devoted to studying embeddings of complexes [8, 21, 28, 29, 30, 39, 52, 64]. Typical types of embeddings include geometric (also referred to as linear), piecewise linear (PL), and topological embeddings, see also Figure 1. For formal definitions, we refer to Section 1.2; here we give an illustrative example. Embeddings of a 1-complex in the plane correspond to drawings of a graph in the plane. In a topological embedding, each edge is represented by a Jordan arc, in a PL embedding it is a concatenation of a finite number of segments, and in a geometric embedding each edge is represented by a segment. Unlike in higher dimensions, for the embeddability of complexes in the plane all three notions coincide.

We are interested in the problem of deciding whether a given k -complex has a linear/piecewise linear/topological embedding in \mathbb{R}^d . Several necessary and sufficient conditions are easy to identify and have been known for many decades. For instance, a k -simplex requires $k + 1$ points in general position in \mathbb{R}^d and, thus, $k \leq d$ is an obvious necessary condition. Moreover, it is straight-forward to verify that every set of n points in \mathbb{R}^3 in general position allows for a geometric embedding of any 1-complex on n vertices, i.e., the points are the vertices of a straight-line drawing of a (complete) graph. Indeed, this fact generalizes to higher dimensions: every k -complex embeds (even linearly) in \mathbb{R}^{2k+1} [39]. Van Kampen and Flores [25, 58, 64] showed that this bound is tight by providing k -complexes that do not topologically embed into \mathbb{R}^{2k} . For some time, it was believed that the existence of a topological embedding also implies the existence of a geometric embedding, e.g., Grünbaum conjectured that if a k -complex topologically embeds in \mathbb{R}^{2k} , then it also geometrically embeds in \mathbb{R}^{2k} [28]. However, this was later disproven. In particular, for every $k, d \geq 2$ with $k + 1 \leq d \leq 2k$, there exist k -complexes that have a PL embedding in \mathbb{R}^d , but no geometric embedding in \mathbb{R}^d [9, 10, 11]. In contrast, PL and topological embeddability coincides in many cases, e.g., if $d \leq 3$ [8, 45] or $d - k \geq 3$ [12]. There are many further necessary and sufficient conditions known for geometric embeddings [6, 43, 44, 58, 61, 62] and PL or/and topological embeddings [20, 26, 46, 52, 60, 63].

In recent years, the **algorithmic complexity** of deciding whether or not a given complex is embeddable gained attention. In the absence of a complete characterization, an efficient algorithm is the best tool to decide embeddability. For instance, deciding whether a 1-complex embeds in the plane corresponds to testing graph planarity and is thus polynomial time decidable [31]. Conversely, the proven non-existence of efficient algorithms may offer a rigorous proof that a complete characterization is impossible. To give a concrete example, let $\text{EMBED}_{k \rightarrow d}$ denote the algorithmic problem of determining whether a given k -complex has a PL embedding in \mathbb{R}^d . Because $\text{EMBED}_{4 \rightarrow 5}$ is known to be undecidable [24, 56], we have a proof that there does not exist an efficient algorithm for $\text{EMBED}_{4 \rightarrow 5}$ – even without any complexity assumptions (such as $\text{NP} \neq \text{P}$ or similar).

More recently, there have been several breakthroughs concerning the **PL embeddability**. For an overview of the state of the art, consider Table 1. In dimensions $d \geq 4$, the decision problem $\text{EMBED}_{k \rightarrow d}$ is polynomial-time decidable for $k < 2/3 \cdot (d - 1)$ [13, 15, 16, 33] and NP-hard for all remaining cases [37], i.e., for all $k \geq 2/3 \cdot (d - 1)$. For $d \geq 5$ and $k \in \{d - 1, d\}$, $\text{EMBED}_{k \rightarrow d}$ is even known to be undecidable; for all other NP-hard cases and $d \geq 4$ decidability is unknown. For the case $d = 3$, Matoušek, Sedgwick, Tancer, and Wagner have shown that $\text{EMBED}_{2 \rightarrow 3}$ and $\text{EMBED}_{3 \rightarrow 3}$ are decidable [36] and de Mesmay, Rieck, Sedgwick, and Tancer have proved NP-hardness [40].

Building upon [37], Skopenkov and Tancer [59] proved NP-hardness for a relaxed notion called *almost (PL/topological) embeddability* where it is only required that disjoint sets must be mapped

Table 1: Overview of the complexity of $\text{EMBED}_{k \rightarrow d}$.

$d \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	P	P	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	✗	P	D	?	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
3	✗	✗	D	?	?	P	✓	✓	✓	✓	✓	✓	✓	✓
4	✗	✗	✗	?	U	?	?	P	✓	✓	✓	✓	✓	✓
5	✗	✗	✗	✗	U	U	?	?	P	P	✓	✓	✓	✓
6	✗	✗	✗	✗	✗	U	U	?	?	?	P	P	✓	✓

- ✓ always yes
- ✗ always no
- P polynomial-time
- D decidable
- U undecidable
- NP-hard

to disjoint objects, i.e., this notion allows that two edges incident to a common vertex cross in an interior point. More precisely, they showed that recognizing almost embeddability of k -complexes in \mathbb{R}^d is NP-hard for all $d, k \geq 2$ such that $d \pmod{3} = 1$ and $2/3 \cdot (d - 1) \leq k \leq d$.

The analogous questions for **geometric embeddings** are wide open. Let $\text{GEM}_{k \rightarrow d}$ denote the algorithmic problem of determining whether a given k -complex has a geometric embedding in \mathbb{R}^d . In contrast to PL embeddability, however, it is easy to see that $\text{GEM}_{k \rightarrow d}$ is decidable for all k, d , since every instance can be expressed as a sentence in the first order theory of the reals, which is decidable. In analogy to the PL embeddings, NP-hardness has been conjectured by Skopenkov.

Conjecture ([57], Conjecture 3.2.2). $\text{GEM}_{k \rightarrow d}$ is NP-hard for all k, d with

$$\max\{3, k\} \leq d \leq 3/2 \cdot k + 1.$$

Note that these parameters correspond to the NP-hard cases of $\text{EMBED}_{k \rightarrow d}$, see also Table 1. Cardinal [18, Section 4] mentions $\text{GEM}_{2 \rightarrow 3}$ as an interesting open problem. The closely related question of polyhedral complexes, posed in the Handbook of Discrete and Computational Geometry, reads as follows: When is a given finite poset isomorphic to the face poset of some polyhedral complex in a given space \mathbb{R}^d ? [51, Problem 20.1.1]. Note that simplicial complexes are special cases of polyhedral complexes, because each simplex is a basic polyhedron. The recognition of polyhedral complexes (with triangles and quadrangles) in \mathbb{R}^3 has been claimed to be $\exists\mathbb{R}$ -complete [18, Theorem 5]. Focussing on convex polytopes, Richter-Gebert proved that recognizing convex polytopes in \mathbb{R}^4 is $\exists\mathbb{R}$ -complete [48, 49].

Our Results. In this work, we present the first results concerning Skopenkov’s conjecture for any non-trivial entry with $d \geq 3$. More precisely, we establish the exact computational complexity of $\text{GEM}_{k \rightarrow d}$ for all values $d \geq 3$ and $k \in \{d - 1, d\}$, hereby confirming the conjecture for these cases. This includes a complete understanding of the most intriguing entries with $d = 3$. Note that this also answers the computational aspects of the question from the Handbook of Discrete and Computational Geometry. Table 2 summarizes the current knowledge on the computational complexity of $\text{GEM}_{k \rightarrow d}$.

Theorem 1. *For every $d \geq 3$ and each $k \in \{d - 1, d\}$, the decision problem $\text{GEM}_{k \rightarrow d}$ is $\exists\mathbb{R}$ -complete. Moreover, the statement remains true even if a PL embedding is given.*

Table 2: Overview of the computational complexity of $\text{GEM}_{k \rightarrow d}$.

$d \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	P	P	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	✗	P	∃ℝc	?	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
3	✗	✗	∃ℝc	∃ℝc	?	?	✓	✓	✓	✓	✓	✓	✓	✓
4	✗	✗	✗	∃ℝc	∃ℝc	?	?	?	✓	✓	✓	✓	✓	✓
5	✗	✗	✗	✗	∃ℝc	∃ℝc	?	?	?	?	✓	✓	✓	✓
6	✗	✗	✗	✗	✗	∃ℝc	∃ℝc	?	?	?	?	?	✓	✓

✓ always yes
✗ always no
P polynomial-time
∃ℝc ∃ℝ-complete

Our proof implies that distinguishing between k -complexes with PL and geometric embeddings in \mathbb{R}^d is complete for $\exists\mathbb{R}$. Because $\text{NP} \subseteq \exists\mathbb{R}$, our result confirms the conjecture by Skopenkov for the corresponding values of k and d . Moreover, if $\text{NP} \neq \exists\mathbb{R}$, the problem $\text{GEM}_{k \rightarrow d}$ cannot be tackled with well developed tools for NP-complete problems such as SAT and ILP solvers. For more details, we refer to Section 1.1.

A geometric embedding of a complex can also be viewed as a *simplicial representation* of a hypergraph, i.e., a representation of a hypergraph in which every hyperedge is represented by a simplex. Of particular interest is the case of uniform hypergraphs where all hyperedges have the same number of elements. Thus, in the language of hypergraphs, our result reads as follows.

Corollary 2. *For all $d \geq 3$ and every $k \in \{d-1, d\}$, deciding whether a $(k+1)$ -uniform hypergraph has a simplicial representation in \mathbb{R}^d is $\exists\mathbb{R}$ -complete.*

Outline and techniques. Our proof of Theorem 1 consists of three steps: Establishing $\exists\mathbb{R}$ -membership, showing $\exists\mathbb{R}$ -hardness in \mathbb{R}^3 , i.e., of $\text{GEM}_{2 \rightarrow 3}$ and $\text{GEM}_{3 \rightarrow 3}$, and reducing $\text{GEM}_{k \rightarrow d}$ to $\text{GEM}_{k+1 \rightarrow d+1}$. The core of the proof lies in establishing hardness of $\text{GEM}_{2 \rightarrow 3}$.

The main idea to prove hardness of $\text{GEM}_{2 \rightarrow 3}$ is to reduce from the problem STRETCHABILITY. In STRETCHABILITY, we are given an arrangement of pseudolines (curves) in the plane and we are asked to decide whether there exists a set of straight lines that has the same combinatorial pattern as the pseudoline arrangement, see Figure 2(a) for an illustration and Section 1.2 for a formal definition. Given a pseudoline arrangement L , we construct a 2-complex C which has a geometric embedding in \mathbb{R}^3 if and only if L is stretchable. On a high level, our construction of C goes along the following lines: We add a helper triangle that contains all intersections of the pseudolines, see Figure 2(b). We place each pseudoline in \mathbb{R}^3 and replace it by a *special edge* of the complex C . We surround the special edges by tunnels, see Figure 2(c) and (d). For each crossing in L , we glue the corresponding tunnel sections together, see Figure 2(e). At last, we insert an apex u high above that is connected to all visible tunnel parts, see Figure 2(f) and we insert additional objects in order to ensure that the neighborhood of u is an essentially 3-connected graph, Figure 2(i).

It is relatively straightforward to verify that if L is stretchable, then the complex C embeds geometrically into \mathbb{R}^3 . The other direction requires more care and work: We show that a geometric embedding of C induces a line arrangement with the same combinatorics as L . The idea of the proof is to consider a small sphere around the apex u and to project its neighborhood and the special edges

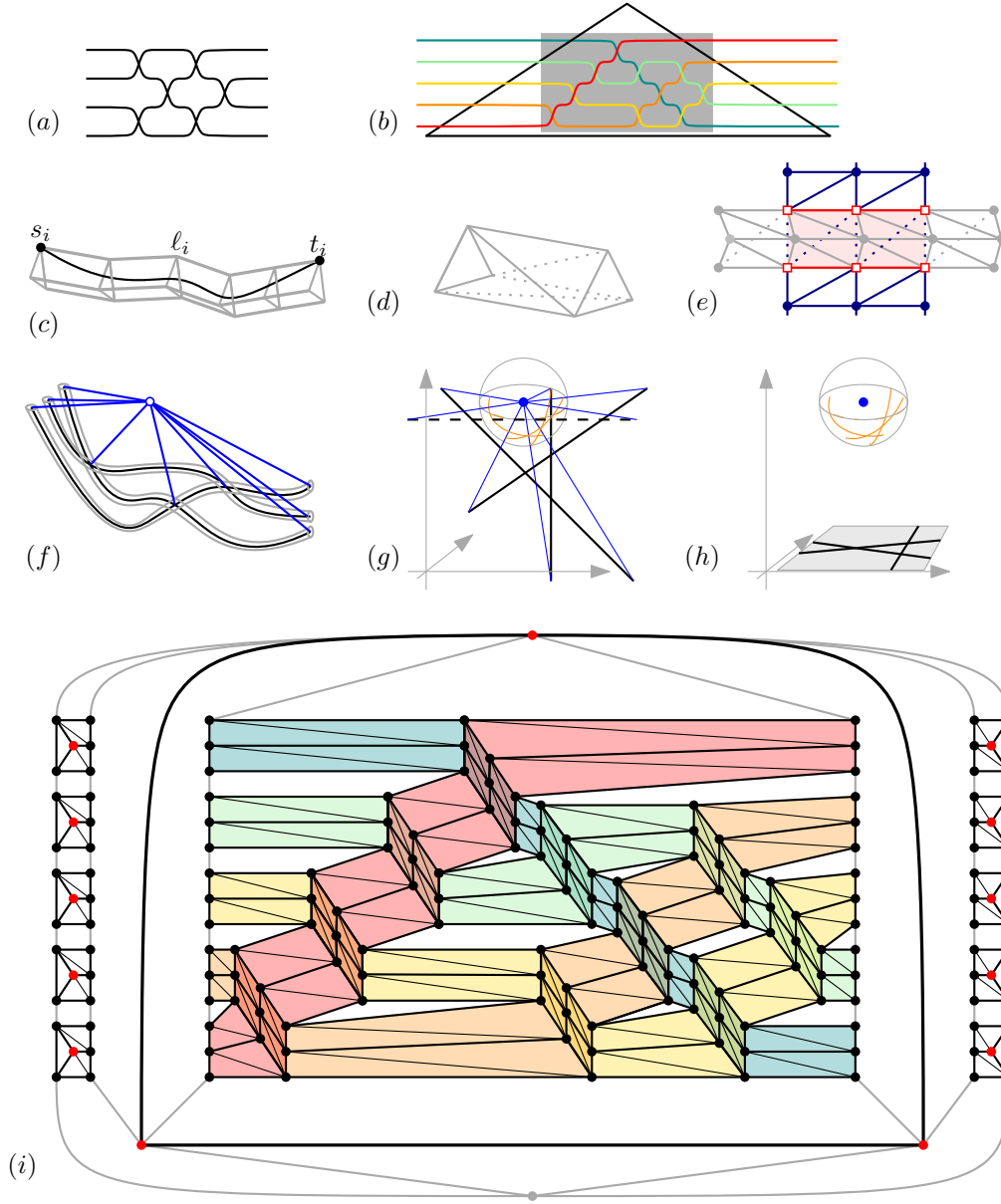


Figure 2: (a) We start with a pseudoline arrangement L . (b) We add a triangle containing all intersections of L . (c) Each pseudoline is represented by a special edge that is surrounded by a tunnel. (d) Each tunnel consists of tunnel sections. (e) For the crossings of the special edges, we identify parts of the tunnels. (f) We add an apex u and insert triangles to the visible parts of the construction; we enhance the neighborhood of the apex to an essentially 3-connected graph depicted in (i). Together, (e), (f) and (i) are crucial to enforce the correct combinatorics. (g) In the correctness proof, we use a small sphere around the apex and the projection of each special edge onto the sphere. (h) We will argue that the combinatorics on the sphere are correct and then project the special edges onto a plane. This will give a stretched arrangement. (i) The neighborhood graph of the apex u .

onto the sphere, see Figure 2(g). Because the neighborhood graph of u is essentially 3-connected by construction, all its crossing-free drawings on the sphere are equivalent. This is an crucial property to show that each special edge lies in the projection of its tunnel roof (when restriction the attention to an interesting part within the helper triangle). We remark, that our proof does not show this explicitly. Instead, we establish some even stronger properties. As a consequence, the projection of the tunnels have the intended combinatorics and thus also the special edges. At last, we project the arcs from the sphere onto a plane, see Figure 2(h). In this way, we obtain a line arrangement with the same combinatorics as L .

In order to show hardness of $\text{GEM}_{3 \rightarrow 3}$, we use a similar construction, in which we “fatten” each triangle to a tetrahedron, by adding extra vertices.

We finally present a dimension reduction, i.e., we reduce $\text{GEM}_{k \rightarrow d}$ to $\text{GEM}_{k+1 \rightarrow d+1}$. Given a k -complex C , we create a $(k+1)$ -complex C^+ that contains C and has two additional vertices a and b . Moreover, for each subset e of C , C^+ has the additional subsets $e \cup \{a\}$ and $e \cup \{b\}$. We prove that C geometrically embeds in \mathbb{R}^d if and only if C^+ geometrically embeds in \mathbb{R}^{d+1} .

In this way, we show that distinguishing PL embeddable and geometrically embeddable complexes is $\exists\mathbb{R}$ -complete.

1.1 Existential Theory of the Reals

The class of the existential theory of the reals $\exists\mathbb{R}$ (pronounced as ‘exists R’, ‘ER’, or ‘ETR’) is a complexity class which has gained a lot of interest in recent years, specifically in the computational geometry community. To define this class, we first consider the algorithmic problem *Existential Theory of the Reals* (ETR). An instance of this problem consists of a sentence of the form

$$\exists x_1, \dots, x_n \in \mathbb{R} : \Phi(x_1, \dots, x_n),$$

where Φ is a well-formed quantifier-free formula over the alphabet $\{0, 1, +, \cdot, \geq, >, \wedge, \vee, \neg\}$, and the goal is to check whether this sentence is true. As an example of an ETR-instance, consider $\exists x, y \in \mathbb{R} : \Phi(x, y) = (x \cdot y^2 + x \geq 0) \wedge \neg(y < 2x)$, for which the goal is to determine whether there exist real numbers x and y satisfying the formula $\Phi(x, y)$.

The *complexity class* $\exists\mathbb{R}$ is the family of all problems that admit a polynomial-time many-one reduction to ETR. It is known that

$$\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}.$$

The first inclusion follows from the definition of $\exists\mathbb{R}$. Showing the second inclusion was first established by Canny in his seminal paper [17]. The complexity class $\exists\mathbb{R}$ gains its significance because a number of well-studied problems from different areas of theoretical computer science have been shown to be complete for this class.

Famous examples from discrete geometry are the recognition of geometric structures, such as unit disk graphs [38], segment intersection graphs [35], STRETCHABILITY [42, 54], and order type realizability [35]. Other $\exists\mathbb{R}$ -complete problems are related to graph drawing [34], Nash-Equilibria [7, 27], geometric packing [5], the art gallery problem [3], non-negative matrix factorization [53], polytopes [22, 49], geometric linkage constructions [1], training neural networks [4], visibility graphs [19], continuous constraint satisfaction problems [41], and convex covers [2]. The fascination for the complexity class stems not merely from the number of $\exists\mathbb{R}$ -complete problems but from the large scope

of seemingly unrelated $\exists\mathbb{R}$ -complete problems. We refer the reader to the lecture notes by Matoušek [35] and surveys by Schaefer [50] and Cardinal [18] for more information on the complexity class $\exists\mathbb{R}$.

1.2 Definitions

Simplex. A k -simplex σ is a k -dimensional polytope which is the convex hull of its $k+1$ vertices V , which are not contained in the same $(k-1)$ -dimensional hyperplane. Hence, a 0-simplex corresponds to a point, a 1-simplex to a segment, and a 2-simplex to a triangle etc. The convex hull of any nonempty proper subset of V is called a *face* of σ . A *simplicial complex* K is a set of simplices satisfying the following two conditions: (i) Every face of a simplex from K is also in K . (ii) For any two simplices $\sigma_1, \sigma_2 \in K$ with a non-empty intersection, the intersection $\sigma_1 \cap \sigma_2$ is a face of both simplices σ_1 and σ_2 . The purely combinatorial counterpart to a simplicial complex is an abstract simplicial complex, which we refer to simply as a *complex*.

Complex. A *complex* $C = (V, E)$ is a finite set V together with a collection of subsets $E \subseteq 2^V$ which is closed under taking subsets, i.e., $e \in E$ and $e' \subseteq e$ imply that $e' \in E$. A k -*complex* is a complex where the largest subset contains exactly $k+1$ elements. We call a complex *pure* if all (inclusion-wise) maximal elements in E have the same cardinality.

For any vertex $v \in V$ in a k -complex $C = (V, E)$, the neighbourhood of v gives rise to a lower dimensional complex $C_v := (V', E')$, where $E' := \{e \setminus \{v\} \mid v \in e \in E\}$ and $V' := N(v) = \bigcup_{e \in E'} e$ are the *neighbors* of v . Complexes are in close relation to Hypergraphs.

Hypergraphs. Hypergraphs generalize graphs by allowing edges to contain any number of vertices. Formally, a *hypergraph* H is a pair $H = (V, E)$ where V is a set of vertices, and E is a set of non-empty subsets of V called *hyperedges* (or edges). A k -uniform hypergraph is a hypergraph such that all its hyperedges contain exactly k elements. Note that the maximal sets of a pure k -complex yield a $(k+1)$ -uniform hypergraph and vice versa. Hence, $(k+1)$ -uniform hypergraphs and pure k -complexes are in a straight-forward one-to-one correspondence. A *simplicial representation* of a $(k+1)$ -uniform hypergraph is a geometric embedding of the corresponding complex.

Geometric embeddings. A *geometric embedding* of a complex $C = (V, E)$ in \mathbb{R}^d is a function $\varphi: V \rightarrow \mathbb{R}^d$ fulfilling the following two properties: (i) for every $e \in E$, $\overline{\varphi}(e) := \text{conv}(\{\varphi(v) : v \in e\})$ is a simplex of dimension $|e| - 1$ and (ii) for every pair $e, e' \in E$, it holds that

$$\overline{\varphi}(e) \cap \overline{\varphi}(e') = \overline{\varphi}(e \cap e').$$

Note that if φ is a geometric embedding, then $\{\overline{\varphi}(e) : e \in E\}$ is a simplicial complex. The problem $\text{GEM}_{k \rightarrow d}$ asks whether a given k -complex has a geometric embedding in \mathbb{R}^d .

Topological and PL embeddings. Consider a complex $C = (V, E)$. In contrast to geometric embeddings, for PL or topological embeddings it is not sufficient to describe the mapping of the vertices V . Choose d' so large that C admits a geometric embedding $\varphi' : V \rightarrow \mathbb{R}^{d'}$, and define $S = \bigcup_{e \in E} \overline{\varphi'}(e)$. We then say that an injective and continuous function $\varphi : S \rightarrow \mathbb{R}^d$ is a *topological embedding* of C in \mathbb{R}^d . If furthermore for each $e \in E$, the image $\varphi(\overline{\varphi'}(e))$ is a finite union of connected subsets of $(|e| - 1)$ -dimensional hyperplanes, then φ is a *piecewise linear (PL) embedding*. The problem $\text{EMBED}_{k \rightarrow d}$ asks whether a given k -complex has a PL embedding in \mathbb{R}^d .

Graph Drawings. A graph is a 1-complex. A graph is *planar* if there exists a crossing-free drawing in the plane, i.e., a (topological) embedding in \mathbb{R}^2 . As mentioned above, a graph has a topological embedding in \mathbb{R}^2 if and only if it has a geometric embedding in \mathbb{R}^2 . A *plane graph* is a planar graph together with a specified crossing-free drawing. By means of stereographic projection, any graph that has a crossing-free drawing in the plane also has a crossing-free drawing on the sphere and vice versa. Two drawings of a graph (in the plane or on the sphere) are *equivalent* if they can be transformed into one another by a homeomorphism (of the plane or the sphere). In particular, two equivalent drawings have the same set of faces. Consider a plane graph G and let D' be the specified drawing of G . When talking about an (arbitrary) drawing D of G , we always mean that D is equivalent to D' .

Stretchability. A *pseudoline arrangement* is a family of curves that apart from ‘straightness’ share similar properties with a line arrangement. More formally, a (*Euclidean*) *pseudoline arrangement* is an arrangement of x -monotone curves in the Euclidean plane such that any two meet in exactly one point. In fact, each pseudoline arrangement can be encoded by a *wiring diagram*; see also Figure 4(a). A pseudoline arrangement is *stretchable* if it is combinatorially equivalent to an arrangement of straight-lines, i.e., if the arrangements can be transformed into one another by a homeomorphism of the plane. STRETCHABILITY denotes the algorithmic problem of deciding whether a given pseudoline arrangement is stretchable. In a seminal paper, Shor [54] proved that STRETCHABILITY is complete for the existential theory of the reals; for a stream-line exposition of this result see the expository paper by Matoušek [35].

1.3 Pitfalls

While the general proof ideas are fairly straightforward, our arguments in Section 2 may at first glance appear a bit tedious. In the following, we highlight one of the appearing challenges. It is easy to see that each special edge lies inside its tunnel in any geometric embedding. It follows that the projection of the special edge lies also inside the projection of the tunnel on the sphere centered at the apex. Furthermore, we know that the roof of the tunnels are seen by the apex. One may be tempted to (directly) conclude that the projection of the special edge is thus also contained in the projection of the roof; the underlying thought being that the projection of the tunnel bottom lies below the tunnel roof in the geometric representation and thus the projection of the tunnel bottom is contained in the projection of the tunnel roof. Yet, the latter is not true in general, as can be seen in Figure 3. In the figure, the tunnel bottom is not covered by the roof. We (implicitly) show that the projection of the special edge lies inside the projection of the roof by establishing some even stronger topological and geometric properties.

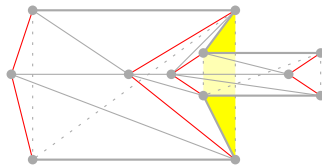


Figure 3: From the perspective of u , the tunnel bottom is not always hidden below the tunnel roof. From the three tunnel sections displayed, the bottom (yellow) of the middle section is partially visible from the apex.

2 The Proof

In this section, we prove Theorem 1. Our proof consists of the following three parts.

- a) Establishing $\exists\mathbb{R}$ -membership (Section 2.1: Lemma 3).
- b) Showing $\exists\mathbb{R}$ -hardness in \mathbb{R}^3 , i.e., of $\text{GEM}_{2 \rightarrow 3}$ and $\text{GEM}_{3 \rightarrow 3}$ (Section 2.2: Theorem 4 and Lemma 5).
- c) Reducing $\text{GEM}_{k \rightarrow d}$ to $\text{GEM}_{k+1 \rightarrow d+1}$ (Section 2.3: Lemma 6).

Together Lemmas 3, 5 and 6 and Theorem 4 prove Theorem 1.

2.1 Membership

In this subsection, we show $\exists\mathbb{R}$ -membership of $\text{GEM}_{k \rightarrow d}$. Note that this is essentially folklore [14]. We present a proof for the sake of completeness.

Lemma 3. *For all $k, d \in \mathbb{N}$, the decision problem $\text{GEM}_{k \rightarrow d}$ is contained in $\exists\mathbb{R}$.*

Proof. In order to show membership in $\exists\mathbb{R}$, we use the following characterization by Erickson, Hoog and Miltzow [23]: A problem P lies in $\exists\mathbb{R}$ if and only if there exists a real verification algorithm A for P that runs in polynomial time on the real RAM. In particular, for every yes-instance I of P there exists a polynomial sized witness w such that $A(I, w)$ returns yes, and for every no-instance I of P and any witness w , $A(I, w)$ returns no. In contrast to the definition of the complexity class NP, we also allow witnesses that consist of real numbers. Consequently, we execute A on the real RAM as well.

It remains to present a real verification algorithm for $\text{GEM}_{k \rightarrow d}$. While the witness describes the coordinates of the vertices, the algorithm checks for intersections between any two simplices. Note that each simplex is a convex set and the intersection of convex sets is a convex set as well. For any simplex S with n vertices, we can efficiently determine n linear inequalities and at most one linear equality that together describe S . Then checking for intersections can be reduced to a linear program, which is polynomial time solvable. This finishes the description of the real verification algorithm. \square

2.2 Hardness in three dimensions

This section is dedicated to proving Theorem 1 for $d = 3$ and $k \in \{2, 3\}$. The crucial part lies in the case $k = 2$. For the benefit of the reader, we include a glossary in Table 3.

Theorem 4. *The decision problem $\text{GEM}_{2 \rightarrow 3}$ is $\exists\mathbb{R}$ -hard.*

Proof. We reduce from the $\exists\mathbb{R}$ -hard problem STRETCHABILITY, as described in Section 1.2. Let L be an arrangement of n pseudolines in the plane. Every pseudoline arrangement has a representation as a wiring diagram in which each pseudoline is given by a monotone curve consisting of $2n - 1$ sections. For an illustration consider Figure 4(a); each section could be represented by a segment, however for a visual appealing display, the bend points are rounded. We add a pseudoline ℓ_0 that intersects all pseudolines in the beginning, see Figure 4(a), and call the resulting pseudoline arrangement L^* . Note that L^* is stretchable if and only if L is stretchable. For later reference, we endow a natural orientation upon each pseudoline from left to right. In the following, we construct a 2-complex $C = (V, E)$ that allows for a geometric realization if and only if L^* (and thus L) is stretchable. In order to define C , we add a helper triangle Δ to our arrangement that intersects the pseudolines of L^+ as illustrated in Figure 4(b).

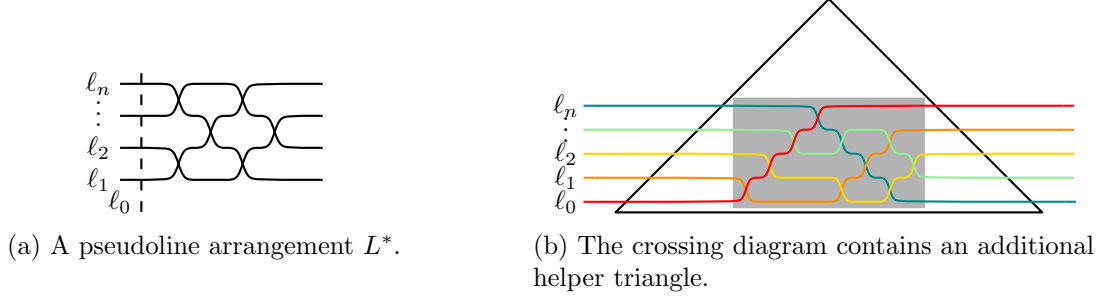


Figure 4: Adding an extra pseudoline ℓ_0 and the helper triangle Δ to the construction.

Construction of the 2-complex. In order to define C , we use the intended geometric embedding. We will refer to the subsets in C as vertices, edges, and triangles depending on whether they contain one, two or three elements. The construction has five steps.

In the first step, we place the pseudolines and the helper triangle Δ in 3-space. Each pseudoline ℓ_i lies in the plane $z = i$ such that an observer high above (at infinity) sees the wiring diagram. Similarly, we place the segments of the helper triangle Δ in 3-space such that it lies in the plane $z = n + 1$. Note that no two pseudolines intersect. Therefore, we can surround each lifted pseudoline by a triangulated sphere which we call a *tunnel*; see also Figure 5(a). The tunnel T_i^+ of ℓ_i is formed by $2n + 3 + i$ sections; later, we will be particularly interested in a part of a tunnel, denoted by T_i , in which the first two and last two sections are removed. Each section consists of six triangles forming a triangulated triangular prism as illustrated in Figure 5(b). We close the tunnel with triangles at the ends and think of the *bottom* side of the prism to lie in the plane $z = i - 1/2$ (for now). The remaining part of the tunnel, i.e., the tunnel without its bottom, constitutes the *roof*, see Figure 5(c). The roof contains three disjoint paths of length $2n + 4 + i$. The edges and vertices on the boundary of the bottom and the roof form the *left roof path*, respectively, when deleting the edges of the closing triangles. The remaining vertices induce the *central roof path*. The three roof paths are thickened in Figures 5(a) and 5(c).

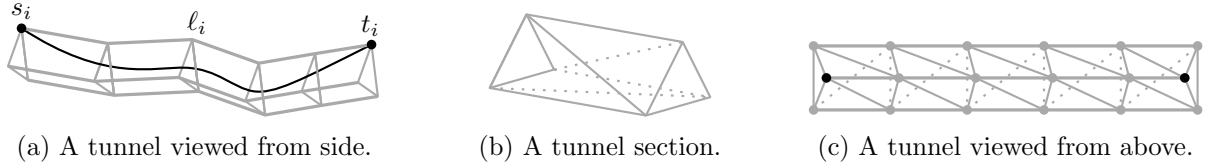


Figure 5: First step in the construction of the complex C – tunnel construction.

Note that we do not add a tunnel for the helper triangle. We distribute the sections along T_i^+ to edges and crossings of the crossing diagram as follows: Generally, we associate one section per edge and one section per crossing of two pseudolines. Moreover, we associate one extra section of T_j^+ to a crossing of ℓ_i and ℓ_j whenever $i < j$. In order to represent the pseudoline ℓ_i , we insert a *special edge* e_i between the two top vertices on either end of the tunnel; for later reference, we denote the start vertex by s_i and the end vertex by t_i . The special edge e_i is intended to lie inside the tunnel.

In the second step, we identify parts of the tunnels. To this end, consider the tunnel sections assigned to a crossing of a pseudoline ℓ_i with ℓ_j , $i < j$. Recall that we assigned one section of T_i^+

and two sections of T_j^+ to the crossing. We identify the four triangles in the bottom of the two sections of T_j^+ with the four triangles in the roof of one section of T_i^+ as indicated in Figure 6. Note that we hereby identify six vertices, four of which belong to a left or right roof path of both, T_i^+ and T_j^+ .

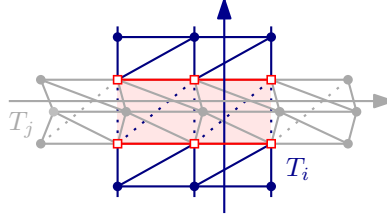


Figure 6: Second step in the construction of the complex C : gluing of tunnel parts.

In the third step, we add a new vertex to the construction that we call the *apex* and denote it by u . We think of u as the observer high above (at infinity) and insert a triangle defined by u and the vertices of every tunnel edge that is *visible* from u . Note that every roof section that is neither glued in a crossings nor hidden by the helper triangle is visible. Moreover, no bottom of any tunnel is visible in the intended geometric embedding.

In the fourth step, we enhance the 1-complex induced by the neighborhood $N(u)$ of the apex u such that it corresponds to an essentially 3-connected planar graph G^+ . We call a graph *essentially 3-connected* if it is a subdivision of a 3-connected graph. With the description so far, the 1-complex corresponds to the graph H depicted in black in Figure 7. To construct G^+ , we make use of the following fact.

Claim 1. *For every plane graph $G_1 = (V_1, E_1)$, there exists an essentially 3-connected plane graph $G_2 = (V_2, E_2)$ such that G_1 is a subgraph of G_2 and any straight-line drawing D_1 of G_1 in the plane can be extended to a straight-line drawing of G_2 . Moreover, if the maximum face degree of G_1 is k , then the size of G_2 can be bounded by $|V_2| + |E_2| \leq O(k|V_1|)$.*

Proof. In order to construct G_2 , we start with a drawing of G_1 . First, we ensure that G_1 has at least 4 vertices. Second, we guarantee 2-connectedness by inserting edges; the edges are represented by potentially non-straight curves. To this end, we iteratively insert edges between different connected components; this ensures connectedness. Afterwards we insert one vertex in each face and insert one edge to each incident vertex of this face; this ensures 2-connectedness. Third, we triangulate each face by repeating the last step: we insert a vertex into each face and insert an edge to every vertex incident to this face. By 2-connectedness, we obtain a triangulation T on at least four vertices and hence, a 3-connected plane graph. Lastly, we subdivide each new edge so that the number of subdivision vertices equals to the degree of the face in G_1 in which the edge has been inserted. The resulting graph is G_2 and by construction essentially 3-connected. Note that T has $O(|V_1|)$ vertices. Therefore, G_2 has $O(k|V_1|)$ edges.

It remains to show that any straight-line drawing D_1 of the plane graph G_1 can be extended to a straight-line drawing of the planar graph G_2 . Because D_1 and G_1 have the same set of faces, we can insert the additional edges of T by polylines. Following the face boundary, the number of bends on each edge is upper bounded by the degree of the face of G_1 in which it is inserted. Hence, we can easily extend D_1 to a straight-line drawing of G_2 . \square

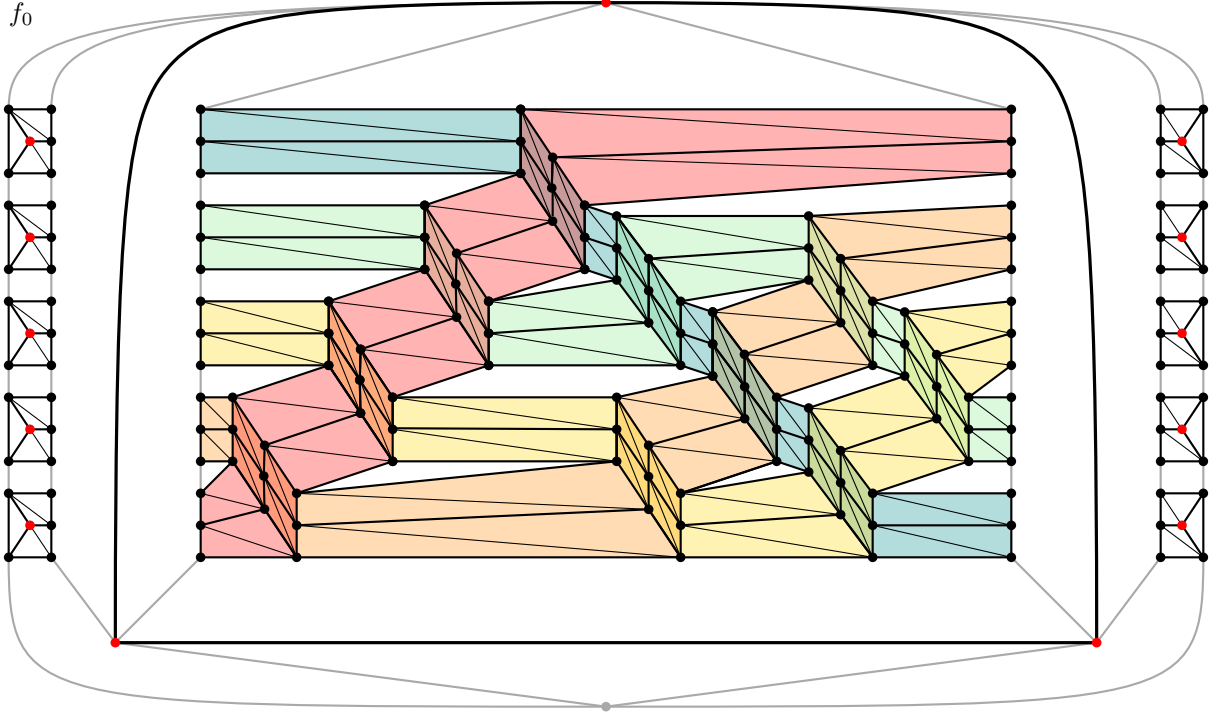


Figure 7: Third and fourth step in the construction of the complex C : neighborhood of the apex u . The graph H depicted in black after the third step. Together with the gray edges, the graph is a candidate for the essentially 3-connected plane graph G^+ and its subgraph G inside Δ .

Let $G^+ := G_2$ be an essentially 3-connected plane graph guaranteed by Claim 1 for the case that $G_1 = H$. Note that G_1 has $O(n^2)$ vertices and edges, and every face has degree $O(n)$. Hence, the size of G_2 is in $O(n^3)$. We denote the outer face of G_2 by f_0 . The reader is invited to think about the far more sparse graph depicted in Figure 7, which also serves as a candidate for G^+ . Indeed, the depicted graph also fulfills all properties necessary for our construction; however, not all properties of Claim 1. For example, the depicted graph is even 3-connected. The proof of this is straightforward, but a bit tedious. Thus, we leave it as an exercise to the interested reader to check that the graph remains connected even after the deletion of any two vertices or alternatively, that any pair of vertices is connected by three disjoint paths.

Later, the subgraph G of G^+ that is induced by all vertices of $\bigcup_i T_i$ will be of particular interest; in Figure 7, these vertices (and their convex hull) lie inside the helper triangle Δ . Recall that T_i denote the part of the tunnel T_i^+ obtained by deleting the first two and last two sections.

It is a well-known fact that all (straight-line or topological) planar drawings of a 3-connected planar graph on the sphere are equivalent [32]; for a definition of equivalent drawings consult Section 1.2. Consequently, the result extends to *essentially* 3-connected graphs as it also holds for topological drawings. For later reference, we note the following.

Claim 2. *The planar graph G^+ is essentially 3-connected. Therefore, all crossing-free drawings of G^+ on a sphere are equivalent. Furthermore, any straight-line drawing of H in the plane can be extended to a straight-line drawing of G^+ .*

We ensure that the neighborhood complex of u is the underlying planar graph of G^+ , i.e, for each edge of G^+ not present in H , we insert a triangle formed by the vertices of this edge together with u and call the resulting complex \overline{C} .

In the fifth and last step, our final complex C consist of two copies of \overline{C} in which the apex vertices are identified. We will later use these two copies in order to guarantee that in any geometric embedding the apex lies outside of all tunnels for one copy of \overline{C} . This finishes the construction of the complex C .

Time Complexity. In order to verify that the construction shows $\exists\mathbb{R}$ -hardness, we argue that it has a running time that is polynomial in the size of the input. To this end, note that a pseudoline arrangement with n pseudolines can be described by the order of the $O(n^2)$ crossings. Thus, the input size is $N = O(n^2)$. After adding the helper triangle and ℓ_0 , the crossing diagram still has a size in $O(n^2)$. It is easy to see that our construction has a size proportional to $N^{3/2}$: For each segment and crossing of the diagram, we build a constant size construction involving the apex. Moreover, we add a triangle for every (additional) edge in G^+ ; recall that G^+ has size $O(n^3)$. Consequently, the total construction has size $O(n^3) = O(N^{3/2})$. We remark, that a more careful choice of G^+ , as in Figure 7, yields a construction that is linear in N .

Correctness. It remains to show that the pseudoline arrangement L is stretchable if and only if C has a geometric embedding in \mathbb{R}^3 .

Correctness I: Stretchability implies Embedability. If L is stretchable, it is relatively straight-forward to construct a geometric embedding of C .

Claim 3. *If L is stretchable, then C has a geometric embedding.*

Proof. The construction of the geometric embedding goes along the same lines as the construction of C . Consider the stretched line arrangement L' equivalent to the pseudoline arrangement L , we construct L'' , by adding ℓ_0 to the left of all crossings of L' .

Next we show how to add the helper triangle \triangle such that the intersection pattern is as depicted in the crossing diagram. Note that we merely have to construct the triangle in a way that all intersections of L'' are contained in it and the vertices of the helper triangle lie in the correct faces. For an illustration of the following argument see Figure 8. The helper triangle has three

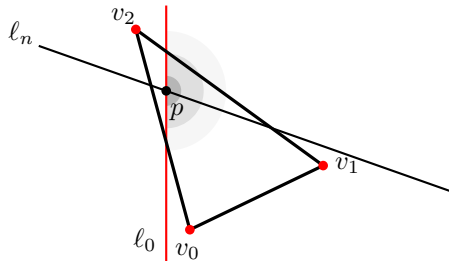


Figure 8: Illustration for the proof of Claim 2 concerning the insertion of \triangle . By scaling L'' while keeping point p and the helper triangle fixed, we can ensure that all intersections lie inside the helper triangle eventually.

vertices v_0, v_1, v_2 . By construction, the two vertices v_0, v_1 are supposed to lie in the bottom cell of L^* which is bounded by ℓ_0 and ℓ_n and potentially other pseudolines. Similarly, v_2 is in the upper cell bounded exactly by ℓ_0 and ℓ_n . Let us fix some triangle Δ that contains the intersection point p of ℓ_0 and ℓ_n with the properties above. Consider the (gray) region that contains all intersections of L'' . By scaling L'' while fixing p and the triangle Δ , the gray region becomes arbitrarily small and is eventually contained in Δ . This shows that the helper triangle can be added as wished.

Afterwards, we lift the lines and the helper triangle in 3-space, construct the tunnels, and enhance H to G^+ . For the latter step, we use the fact that we can extend any straight-line drawing of H to a straight-line drawing of G^+ by Claim 2. (In case that we want to use G^+ as in Figure 7, we need to use the fact that the drawing of H comes from the stretched line arrangement and thus the faces are convex.) Afterwards, we insert the apex u (high enough) above, make a second copy by taking the mirror image, and identify the apices. This yields a geometric embedding of C . \square

Correctness II: Embeddability implies Stretchability. The reverse direction is more involved. Let φ denote a geometric embedding of C . To show that L is stretchable, we start with a collection of crucial properties.

By definition, each tunnel forms a closed topological sphere; all of which are pairwise disjoint. By a generalization of the Jordan curve theorem, also known as the Jordan–Brouwer separation theorem, any topological embedding of a $(d - 1)$ -sphere in \mathbb{R}^d splits the space into two components [65]; we refer to the bounded component of a tunnel or any other topological sphere as its *inside* and to the unbounded component as its *outside*.

Claim 4. *There exists a copy \bar{C} in C such that the apex u lies outside all tunnels of $\varphi(\bar{C})$.*

Proof. Note that the apex u lies inside at most one tunnel of C ; otherwise, among all tunnels containing u , the innermost separates u from the outermost. A contradiction to the fact that shares an edge with the vertices of all tunnels. Consequently, for at least one copy of \bar{C} contained in C , the apex u lies outside of each tunnel. \square

From now on, we focus on the geometric embedding $\varphi(\bar{C})$ of this \bar{C} and do not make further use of the other copy.

Claim 5. *In $\varphi(\bar{C})$, the special edge e_i lies inside its tunnel T_i^+ for all i .*

Proof. Consider the vertex s_i of e_i and the tunnel T_i^+ forming a closed topological sphere incident to s_i . By construction, all roof edges of the first section of T_i^+ are visible by the apex u and thus form triangles with u . Note that the edges non-incident to s_i contain a cycle, see Figure 9(a).

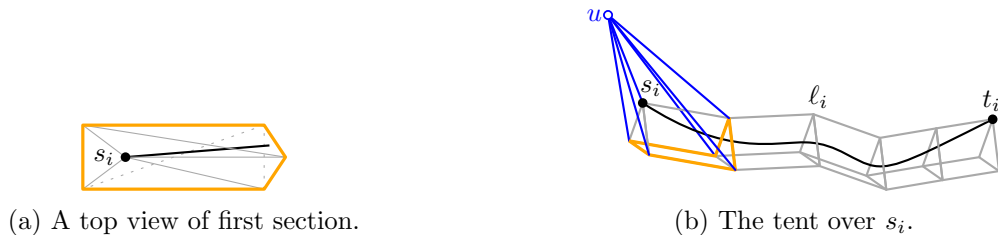


Figure 9: Illustration for the proof of Claim 5: the spheres surrounding s_i .

The triangles between the apex u and this cycle form a *tent* on top of T_i^+ , see Figure 9(b). In particular, the tent together with the tunnel roof of the first section form another sphere incident to s_i . By Claim 4, u is outside the tunnel. If e_i started towards the outside of the tunnel, then it would be *trapped* inside the tent-sphere which is not incident to t_i . Consequently, e_i lies inside the tunnel. \square

Consider a special edge e_i and the wedge W_i defined by rays originating at u through points in e_i , see Figure 10. We say a point p of $(W_i \setminus e_i)$ lies *above* e_i if the segment up does not intersect e_i ; we write $p > e_i$. Similarly, p lies *below* e_i if the segment up does intersect e_i ; we write $p < e_i$.

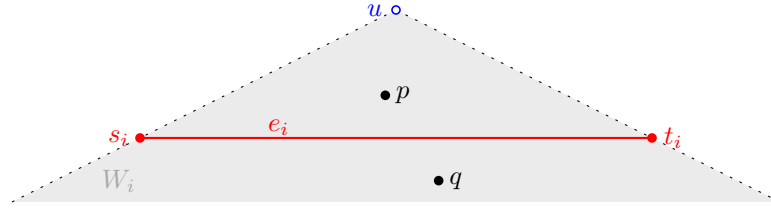


Figure 10: The wedge W_i of e_i containing a point p above e_i and a point q below e_i .

Claim 6. Consider a special edge e_i . For every two points p, q on $W_i \setminus e_i$ that belong to a common triangle in T_i^+ , the above/below-relation is consistent, i.e., $p < e_i \iff q < e_i$.

Proof. Suppose for a contradiction that $p > e_i$ and $q < e_i$. By visibility of s_i and t_i , p lies neither on the segment us_i nor on the segment ut_i . Hence, the segment pq intersects e_i in an inner point, i.e., it intersects $e_i - \{s_i, t_i\}$, see Figure 10. However, no triangle of C contains e_i and hence $e_i - \{s_i, t_i\}$ and pq must have an empty intersection. A contradiction. \square

Now, we consider a small sphere S around the apex u which has no other vertex inside, see Figure 11(a). For each triangle in $\varphi(\overline{C})$ containing u , the intersection with the sphere S yields an arc of a great circle. Consequently, we obtain a crossing-free drawing D^+ of the essentially 3-connected planar graph G^+ (the neighborhood complex of u) with arcs of great circles on S . Moreover, for each special edge e_i , we consider the projection a_i of the (artificial) triangle $\{s_i, t_i, u\}$ onto the sphere S ; note that $\{s_i, t_i, u\}$ is not a triangle of the complex, see Figure 11(b). This

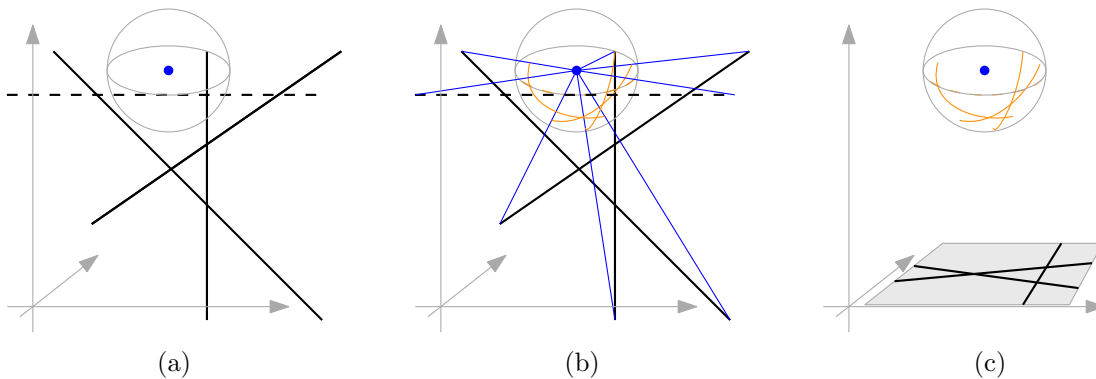


Figure 11: Obtaining a line arrangement from a geometric embedding.

yields a set of arcs \mathcal{A} . In the remainder, we show that \mathcal{A} has the same combinatorics/intersection pattern as the pseudoline arrangement L^* , i.e., the order of intersections along each arc/pseudoline are the same. Afterwards, we project \mathcal{A} to a plane as illustrated in Figure 11(c) and obtain a line arrangement proving that L^* (and thus L) is stretchable.

Let D denote the restriction of the drawing D^+ to the graph G . By Claim 2, all crossing-free drawings of G^+ on the sphere are equivalent. This implies the following fact:

Claim 7. *In D^+ , the helper triangle separates the subdrawing D from the vertices $\{s_j, t_j \mid j = 0, \dots, n\}$. In particular, by convexity, the convex hull of any vertex subset U in D is contained in the helper triangle.*

For a tunnel, we call the intersection of two consecutive sections a *tunnel loop*. Note that each tunnel loop consists of three edges, two of which belong to the roof of the tunnel and form a *rafter*; the remaining edge belongs to the bottom of the tunnel.

Claim 8. *$D \cup \mathcal{A}$ fulfills the following properties for all i :*

- i) The arc a_i intersects each rafter of T_i exactly one time.*
- ii) The intersection points appear in the correct order along a_i , i.e., they follow the natural order along the tunnel.*
- iii) The faces of a roof section in T_i cover an interval of a_i .*

Proof. Consider a tunnel loop o of tunnel T_i^+ and its (projected) vertices in D . We first show that the arc a_i intersects at least one (potentially artificial) segment connecting the vertices of o in D . To this end, we may assume that o is disjoint from s_i and t_i and consider the restriction of $\varphi(\overline{C})$ to the plane P_i containing u and e_i . By Claims 4 and 5, the apex u is outside of the tunnel T_i^+ while e_i is inside T_i^+ . Hence, there exist curves $\gamma_a, \gamma_b \subset T_i^+$ from s_i to t_i on P_i that together enclose e_i and separate e_i from u , see Figure 12.

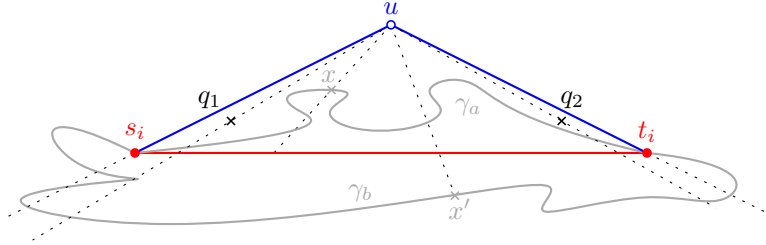


Figure 12: An illustration for the proof of Claim 8: the curve γ_a in P_i separates e_i from u .

Because s_i and t_i are visible from u , the curves intersect the segments $s_i u$ and $t_i u$ exactly in s_i and t_i , respectively. We denote the curve contained in the triangle of u, s_i, t_i by γ_a . The curve γ_a contains a point $x \in o$ because s_i and t_i lie in different components of $T_i^+ - o$. Hence, the ray supporting ux intersects e_i , i.e., $x > e_i$.

Next we show that if o belong to T_i , then the curve γ_b intersects o in a point below e_i . Analogous to γ_a , the curve γ_b also contains a point $x' \in o$. It remains to show that x' is below e_i . Note that this does not hold, only in the case that x' is to the left of the ray us_i or to the right of the ray ut_i ; otherwise, γ_b is below e_i . By Claim 7, the projection of the helper triangle separates D from the vertices s_i, t_i in D^+ . Consequently, the helper triangle Δ and P_i intersect in two points q_1 and q_2 .

The points q_1 and q_2 must be above e_i , because u forms a triangle with every edge of Δ . By Claim 7 and the fact that the vertices of o are contained in D , the vertices of o and their convex hull are inside Δ in D , see Figure 13. Note that o is not completely contained in D because it contains an edge of the tunnel bottom.

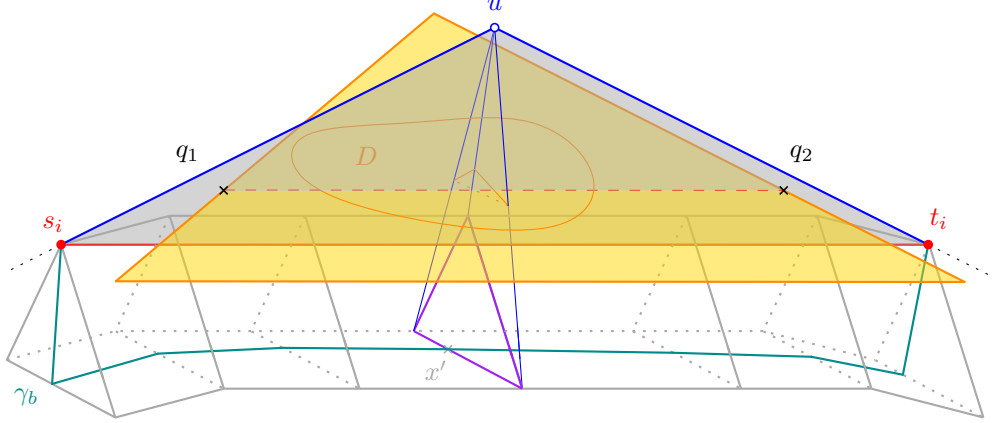


Figure 13: An illustration for the proof of Claim 8. The wedge W_i and the helper triangle Δ (yellow) intersect in the segment q_1q_2 . We drew the projection of D onto the the helper triangle. In D , the vertices of the sloop o (purple) are contained in Δ . Therefore, the point x' lies between the rays uq_1 and uq_2 on W_i and thus below e_i .

This implies that x' is in a part of γ_b that is bounded by the rays uq_1 and uq_2 , see Figure 12. Therefore, x' lies below e_i and is thus invisible from u . It follows that x' belongs to an invisible edge of o , i.e., x' is contained in the edge of o belonging to the tunnel bottom; we denote it by b_o . By Claim 6, no point of b_o lies above e_i . Consequently, $x > e_i$ belongs to the tunnel roof, i.e., the rafter of o .

Moreover, no triangle of T_i incident to b_o lies above e_i . Note that for each triangle in the bottom of T_i , there exists a tunnel loop o such that b_o is incident to it. Hence, no point of the bottom of T_i lies above e_i , i.e., we obtain the following *Property 1*: the curve γ_a does not contain points of the bottom of the tunnel T_i .

It remains to argue that there exists no further intersection point if $o \subset T_i$. We show that $(\gamma_a \cup \gamma_b) \cap o = \{x, x'\}$. Suppose that there exists a further point $x'' \in (\gamma_a \cup \gamma_b) \cap o$. Then o must be contained in P_i and we may assume that x, x'' are vertices of o . However, then the edge xx'' intersects the closed curve formed by concatenating the segments $us_i, s_it_i = e_i, t_iu$ – a contradiction to the properties of a geometric embedding. Consequently, $(\gamma_a \cup \gamma_b) \cap o = \{x, x'\}$. In particular, we obtain *Property 2*: γ_a intersects each rafter of T_i exactly once. This implies that a_i intersects each rafter in D exactly once and thus proves i).

Together, Properties 1 and 2 imply that γ_a visits any section of T_i exactly once. Consequently, the part of a_i covered by the roof faces of any section forms an interval on a_i . This proves iii).

It remains to prove ii). By Property 2, γ_a intersects each rafter exactly once. We denote the intersection points on the curve γ_a with the rafters by p'_1, \dots, p'_k in the order they appear on γ_a , i.e., p'_j denotes the intersection of γ_a with the j -th rafter of T_i . Let p_j denote the projection of p'_j onto S , i.e., the intersection point of the j -th rafter with a_i in $D \cup \mathcal{A}$. We show that p_1, \dots, p_k appear in this order along a_i in $D \cup \{a_i\}$, i.e., p_j lies before p_{j+1} for all j . To this end, we restrict our attention

to the wedge $W_i \subset P_i$ bounded by the rays uq_1 and uq_2 . For simplicity of the presentation, we transform W_i such that (the interesting part of) e_i is on the x -axis, s_i left of t_i , and that the apex u at infinity above, see also Figure 14(a).

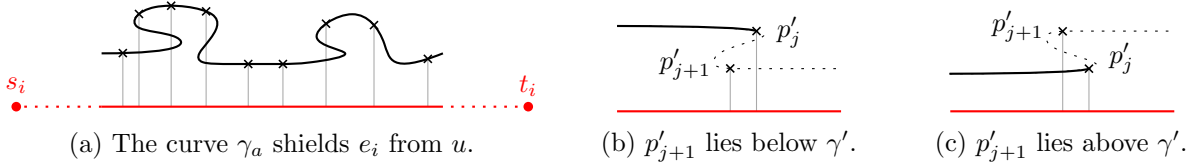


Figure 14: Illustration for the proof of Claim 8 ii).

Suppose for the purpose of a contradiction that p_{j+1} lies left of p_j , see also Figures 14(b) and 14(c). Note that the initial subcurve of γ_a until p'_j , denoted by γ'_a , shields the initial part of e_i until p_j . If p'_{j+1} lies below γ'_a as depicted in Figure 14(b), then p'_{j+1} is not visible from u . However, by construction, all rafters and thus all points p'_1, \dots, p'_k are visible from the apex u . A contradiction. If p'_{j+1} lies above γ'_a as depicted in Figure 14(c), then p'_{j+1} is not visible from u because of the subcurve $\gamma_a - \gamma'_a$, a contradiction. Consequently, p_j is left of p_{j+1} for all j . \square

In order to define a notion of *left* and *right* of a_i in $D \cup \{a_i\}$, we enhance $D \cup \{a_i\}$ by the projection of Δ . The arc a_i partitions the interior of (the projection of) Δ into two regions. Considering the orientation of a_i , it is easy to determine if a point within one of these two regions lies *left* or *right*.

Every drawing on the sphere can be transformed to a drawing in the plane where any face can be chosen as the outer face. In order to relate notions such as clockwise and counter-clockwise with our drawing on the sphere, we fix the face f_0 as the outer face of D^+ such that v_0, v_1, v_2 is a counter clockwise cycle. For D , these notions are then inherited because we consider a drawing D that is an induced subdrawing of D^+ .

Claim 9. *In $D \cup \{a_i\}$, the vertices of the left and right roof paths of T_i are left and right of a_i .*

Proof. Let u_1, \dots, u_k denote the vertices of the left roof path, v_1, \dots, v_k the vertices of the central roof path, and w_1, \dots, w_k the vertices of the right roof path of T_i in $D \cup \{a_i\}$. The value of k depends on the number of tunnel sections of T_i , namely, $k = 2n - 4 + i$. For each $j = 1, \dots, k$, it holds that u_j, v_j, w_j is a rafter which a_i intersects exactly once by Claim 8 i). Consequently, if u_j lies left of a_i then w_j lies right of a_i and vice versa.

We consider a visible section of T_i . To this end, let $u_j u_{j+1}$ be a visible edge. We show that both u_j and u_{j+1} lie left of a_i implying that w_j and w_{j+1} lie right of a_i . Because each vertex of the left or right roof path belongs to some visible section, this property implies the claim.

First consider the case that u_j and u_{j+1} lie right of a_i . By Claim 8 ii), the intersection x_j of the rafter u_j, v_j, w_j with a_i lies before the intersection x_{j+1} of the rafter $u_{j+1}, v_{j+1}, w_{j+1}$ with a_i . However, the cycle $u_j, u_{j+1}, v_{j+1}, w_{j+1}, w_j, v_j$ is flipped, i.e., it has the outer face f_0 to its right, however it is supposed to lie to its left, see Figures 15(a) and 15(b). A contradiction.

Hence it remains to consider the case in which u_j and u_{j+1} lie on different sides. We consider the case that u_j lies left and u_{j+1} right of a_i as illustrated in Figure 15(c). By crossing-freeness, exactly one of the edges $u_j u_{j+1}$ and $w_j w_{j+1}$ crosses a_i between x_j and x_{j+1} , while the other edge crosses before x_j or after x_{j+1} as depicted in Figure 15(c). However, then the roof faces of this section do not cover a consecutive part of a_i . A contradiction to Claim 8 iii). \square

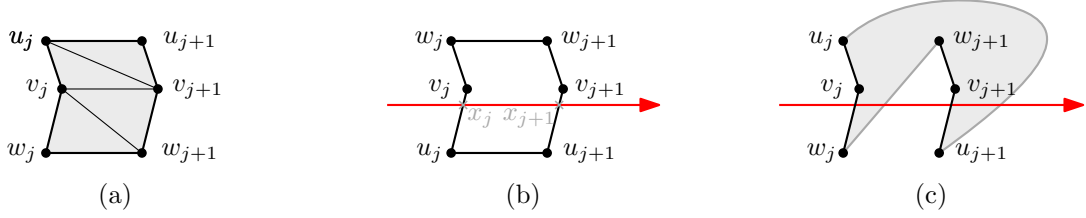
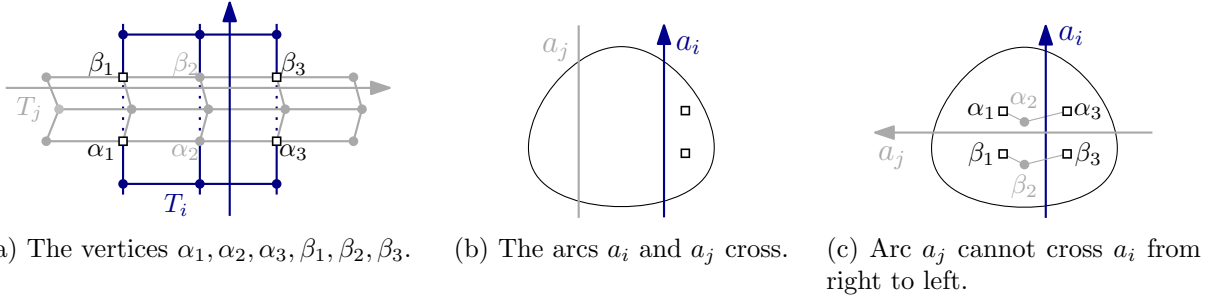


Figure 15: Illustration for the proof of Claim 9.

Claim 10. In \mathcal{A} , for all i, j with $i < j$, the arc a_j crosses the arc a_i from left to right.

Proof. We restrict our attention to the vertices of the left and right roof paths of T_i and T_j . Let $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ denote the vertices of two rafters of T_i such that they are contained in the right and left roof path of T_j , respectively. For an illustration, consider Figure 16(a). We call their induced rafters, the α -rafter and the β -rafter. By Claim 9, the vertices of a left/right roof path lie on the left/right of its corresponding arc.



(a) The vertices $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$. (b) The arcs a_i and a_j cross. (c) Arc a_j cannot cross a_i from right to left.

Figure 16: Illustration for the proof of Claim 10.

Suppose the arcs a_i and a_j do not cross. By symmetry, we may assume that a_j lies left of a_i , see Figure 16(b). Recall that the vertices α_3 and β_3 lie to the right side of a_i , but on different sides of a_j . A contradiction.

Now, suppose a_j crosses a_i from right to left as in Figure 16(c). Then each vertex among $\alpha_1, \alpha_3, \beta_1, \beta_3$ lies in one of the four regions defined by a_i and a_j . Moreover a_j separates the α -rafter of T_i containing α_1, α_2 , and α_3 from the β -rafter containing β_1, β_2 , and β_3 . This implies that a_i intersects the β -rafter before the α -rafter. A contradiction to Claim 8 ii). \square

Claim 11. In \mathcal{A} , the order of intersections on each arc a_i is the same as for l_i in L^* .

Proof. We consider any three arcs a_i, a_j, a_k with $i < j < k$. By Claim 10, the vertices s_i, s_j, s_k, t_i, t_j , and t_k lie on the correct sides for all three arcs. It is therefore the case that the order of intersections is correct on one arc of a triple if and only if it is correct on all three arcs. Suppose for a contradiction that order of intersections on a_i is not correct, i.e., the intersection of a_j and a_k lies on the wrong side of a_i . By symmetry, we consider the case that the intersection is left of a_i while it is supposed to lie right of a_i , see Figure 17.

Let $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ denote the vertices of two rafters of T_i such that they are contained in the left roof path of T_j and in the right roof path of T_k , respectively. For an illustration, consider Figure 17(a). Consequently, by Claim 9, $\alpha_1, \alpha_2, \alpha_3$ lie left of a_j and $\beta_1, \beta_2, \beta_3$ lie right of a_k . This

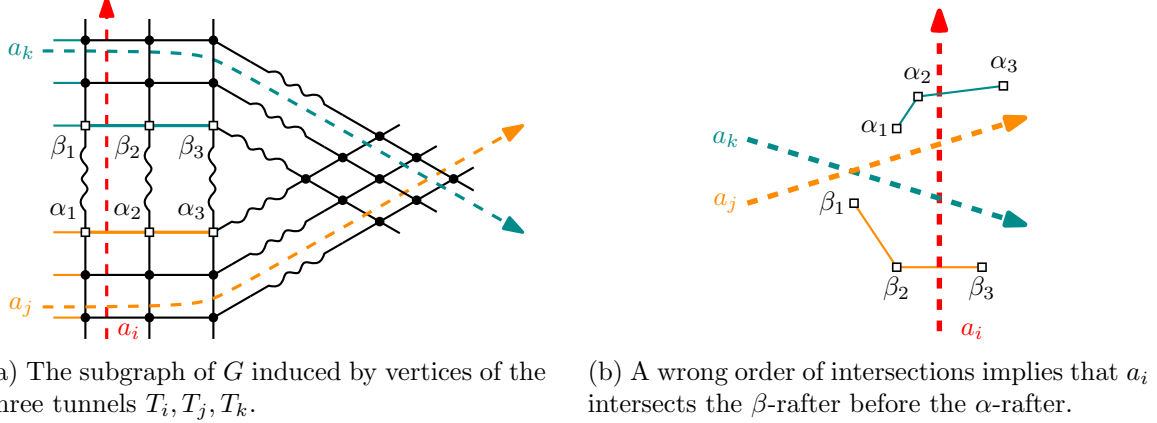


Figure 17: Illustration for the proof of Claim 11.

implies that the intersection of the β -rafter with a_i lies before the intersection of the α -rafter with a_i . A contradiction to Claim 8 ii). \square

Claim 12. *In \mathcal{A} , all intersections lie in one hemisphere of S bounded by a_0 .*

Proof. Consider the plane P through the arc a_0 and apex u and assume without loss of generality that P is horizontal. By construction and Claim 11, the first crossing on each arc a_i is with the arc a_0 ; all other crossings come afterwards. Moreover, Claim 10 ensures that each arc a_i crosses a_0 from left to right. Consequently, all arcs start in one hemisphere of S cut by P ; let us suppose it is the top hemisphere. Then, the intersection of any two arcs a_i and a_j lie on the bottom hemisphere; otherwise the arcs contain both intersections of the supporting great circles. \square

Let P' denote a plane obtained by shifting the plane P through a_0 and u to the bottom, see also Figure 11(c). Then, we 'project' the arc arrangement onto the plane P' by considering the intersection of P' with the plane supporting the arc a_i . Clearly, this intersection yields a line. Hence, we obtain a line arrangement in the plane P' that inherits the combinatorics of \mathcal{A} and thus, proves the stretchability of L . This finishes the proof of Theorem 4. \square

Fattening the Complex. The constructed 2-complex in the proof of Theorem 4 was not pure. Specifically, the special edges are not contained in any triangle. We can obtain a pure 2-complex by adding one new vertex to each special edge such that it forms a special triangle. Similarly, we can add a private vertex to each triangle to form a pure 3-complex C' . Given a geometric embedding of C , these new vertices can easily be added close enough to their defining set in C . Hence, C has a geometric embedding if and only if C' has a geometric embedding in \mathbb{R}^3 . Together with Theorem 4, this very small modification shows hardness of $\text{GEM}_{3 \rightarrow 3}$.

Lemma 5. *The decision problem $\text{GEM}_{3 \rightarrow 3}$ is $\exists\mathbb{R}$ -hard.*

2.3 Dimension Reduction

In order to show hardness for all remaining cases of Theorem 1, we establish the following dimension reduction. For dimension reductions in the context of PL embeddings, we refer to [24, 46, 47, 55].

Lemma 6. *The decision problem $\text{GEM}_{k \rightarrow d}$ reduces to $\text{GEM}_{k+1 \rightarrow d+1}$.*

The idea is to add two apices to a k -complex C in order to obtain a $(k+1)$ -complex C^+ . We will then argue that C has a geometric embedding in \mathbb{R}^d if and only if C^+ has a geometric embedding in \mathbb{R}^{d+1} . More formally, for a complex $C = (V, E)$ and a disjoint vertex set U , $C * U$ denotes the *join* complex $(V \cup U, E')$ where $E' := \{e \cup u \mid e \in E, u \in U\}$. The following claim immediately implies Lemma 6.

Claim 13. *Let $C = (V, E)$ be a complex, $a, b \notin V$ two new vertices, and $C^+ := C * \{a, b\}$ their join complex. Then C has a geometric embedding in \mathbb{R}^d if and only if C^+ has a geometric embedding in \mathbb{R}^{d+1} .*

Proof. Let φ be a geometric embedding of C in \mathbb{R}^d . Then, we define for $v \in V \cup \{a, b\}$,

$$\varphi'(v) = \begin{cases} (\varphi(v), 0) & \text{if } v \in V, \\ (0, \dots, 0, +1) & \text{if } v = a, \\ (0, \dots, 0, -1) & \text{if } v = b. \end{cases}$$

It is easy to check that φ' is a geometric embedding of C^+ in \mathbb{R}^{d+1} : While a and b are well separated in the last coordinate, all other potential intersections happen in the d -dimensional subspace induced by the first d coordinates. Hence φ implies the correctness of the geometric embedding.

For the reverse direction, consider a geometric embedding φ of C^+ in \mathbb{R}^{d+1} . Let $\varphi_a := \varphi(a)$ and $\varphi_b := \varphi(b)$. Without loss of generality, we assume that $\varphi_a - \varphi_b$ is orthogonal to the first d coordinates, i.e., $\varphi_a - \varphi_b$ is parallel to the $(d+1)$ -st coordinate axis. Let $\overline{\varphi}(C) := \bigcup_{e \in E} \overline{\varphi}(e)$ denote the induced geometric subrepresentation of C . We claim that the function $f : \overline{\varphi}(C) \rightarrow \mathbb{R}^d$ obtained by restricting to the first d coordinates is injective. Thus $\varphi' := f \circ \varphi$ yields a representation of C in \mathbb{R}^d .

For the purpose of a contradiction, suppose that f is not injective. Then there exist two distinct points $p = (p_1, \dots, p_{d+1})$ and $q = (q_1, \dots, q_{d+1})$ with $p, q \in \overline{\varphi}(C)$ such that $(p_1, \dots, p_d) = (q_1, \dots, q_d)$ and $p_{d+1} \neq q_{d+1}$. Without loss of generality, we may assume that $p_{d+1} > q_{d+1}$. Consider the plane P spanned by φ_a, φ_b, p . Note that $q \in P$. For an illustration, see Figure 18(a). Let us denote with $e_p \in E$ and $e_q \in E$ any choice of hyperedges such that $p \in \overline{\varphi}(e_p)$ and $q \in \overline{\varphi}(e_q)$. Consider the two open segments $\text{seg}^\circ(\varphi_a, q) \in \overline{\varphi}(e_q \cup a)$ and $\text{seg}^\circ(\varphi_b, p) \in \overline{\varphi}(e_p \cup b)$. Clearly, these open segments intersect in a point x , as illustrated Figure 18(a). Because φ is a geometric embedding, it holds that

$$x \in \overline{\varphi}(e_q \cup a) \cap \overline{\varphi}(e_p \cup b) = \overline{\varphi}(e_q \cap e_p) = \overline{\varphi}(e_q) \cap \overline{\varphi}(e_p).$$

In particular, this implies that $x \in \overline{\varphi}(e_q)$ and thus that $x \in \text{seg}^\circ(\varphi_a, q) \cap \overline{\varphi}(e_q)$. However, because $\overline{\varphi}(e_q \cup a)$ is a simplex, φ_a does not lie in the $\text{span}(\overline{\varphi}(e_q))$ and thus $\text{seg}^\circ(\varphi_a, q) \cap \overline{\varphi}(e_q) = \emptyset$. A contradiction. \square



Figure 18: Illustration for the proof of Claim 13. The geometric embedding φ of C^+ , gives a monotone embedding of C , otherwise we can find an intersection.

3 Conclusion

We established the computational complexity of $\text{GEM}_{k \rightarrow d}$ for all $d \geq 3$ and $k \in \{d-1, d\}$. In particular, we showed that for these values it is complete for $\exists\mathbb{R}$ to distinguish PL embeddable k -complexes in \mathbb{R}^d from geometrically embeddable ones. Arguably, $\text{GEM}_{2 \rightarrow 3}$ is the most interesting case.

Investigating the computational complexity for the remaining open entries in Table 2 remains for future work. We strengthen the conjecture of Skopenkov [57] as follows.

Conjecture. *The problem $\text{GEM}_{k \rightarrow d}$ is $\exists\mathbb{R}$ -complete for all k, d such that $\max\{3, k\} \leq d \leq 2k$.*

Acknowledgements.

We thank Arkadiy Skopenkov for his kind and swift help with acquiring literature. We thank Martin Tancer for pointing out a mistake in a previous version of this manuscript. Mikkel Abrahamsen is part of Basic Algorithms Research Copenhagen (BARC), generously supported by the VILLUM Foundation grant 16582. Linda Kleist is generously supported by a postdoc fellowship of the German Academic Exchange Service (DAAD). Tillmann Miltzow is generously supported by the Netherlands Organisation for Scientific Research (NWO) under project no. 016.Veni.192.250.

Table 3: A glossary for notions used in the proof of Theorem 4.

symbol	meaning
$L = \{\ell_1, \dots, \ell_n\}$	pseudoline arrangement
n	number of pseudolines in L , Figure 4(a)
ℓ_0	additional pseudoline
$L^* := L \cup \{\ell_0\}$	L together with ℓ_0
Δ	the <i>helper triangle</i> , all intersections of L^* are contained inside
$C = (V, E)$	the simplicial complex that we construct
T_i^+	tunnel around pseudoline ℓ_i , Figure 5(a)
T_i	tunnel T_i^+ without first and last two sections
section	part of tunnel, Figure 5(b)
tunnel bottom	the part of tunnel T_i^+ that lies in the plane $z = i$ in step 1
tunnel roof	the part of tunnel which is not in the bottom, see Figure 5(c)
left/right roof path	tunnel paths shared by roof and bottom.
central roof path	tunnel path that is disjoint from left/right roof path
$e_i = (s_i, t_i)$	<i>special edge</i> of C that is meant to represent ℓ_i
u	apex (taking the role of an observer high above)
H	graph in the third step of the construction, see black graph in Figure 7
G^+	an essentially 3-connected planar graph induced by the neighborhood of u
f_0	outer face of G^+
G	subgraph of G^+ that is inside the helper triangle
\overline{C}	the complex C consists of two copies of \overline{C} with the apex identified
φ	geometric embedding of C
tent	Figure 9(b)
tunnel loop o	triangle that is shared by two sections
rafter	two roof edges of a tunnel loop
wedge W_i	defined by the apex u and e_i , Figure 10
plane P_i	defined by the apex u and e_i
$p > e_i$	segment pu does not cross e_i
$p < e_i$	segment pu does cross e_i
S	sphere around apex u
D, D^+	projection of $\varphi(G^+)$ onto S yielding a drawing of G and G^+
a_i	projection of $\varphi(e_i)$ onto S
\mathcal{A}	arc arrangement of all a_i Figure 11(b)

References

- [1] Zachary Abel, Erik Demaine, Martin Demaine, Sarah Eisenstat, Jayson Lynch, and Tao Schardl. “Who Needs Crossings? Hardness of Plane Graph Rigidity”. In: *International Symposium on Computational Geometry (SoCG)*. 2016, 3:1–3:15. DOI: [10.4230/LIPIcs.SoCG.2016.3](https://doi.org/10.4230/LIPIcs.SoCG.2016.3).
- [2] Mikkel Abrahamsen. “Covering Polygons is Even Harder”. In: *Foundations on Computer Science (FoCS)*. to appear. 2021. arXiv: [2106.02335](https://arxiv.org/abs/2106.02335).
- [3] Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. “The art gallery problem is $\exists\mathbb{R}$ -complete”. In: *Symposium on Theory of Computing (STOC)*. 2018, pp. 65–73. DOI: [10.1145/3188745.3188868](https://doi.org/10.1145/3188745.3188868).
- [4] Mikkel Abrahamsen, Linda Kleist, and Tillmann Miltzow. “Training Neural Networks is $\exists\mathbb{R}$ -complete”. In: *Conference on Neural Information Processing Systems (NeurIPS)*. to appear. 2021. arXiv: [2102.09798](https://arxiv.org/abs/2102.09798).
- [5] Mikkel Abrahamsen, Tillmann Miltzow, and Nadja Seiferth. “Framework for ER-Completeness of Two-Dimensional Packing Problems”. In: *Foundations on Computer Science (FoCS)*. IEEE. 2020, pp. 1014–1021. DOI: [10.1109/FOCS46700.2020.00098](https://doi.org/10.1109/FOCS46700.2020.00098).
- [6] JL Ramírez Alfonsín. “Knots and links in spatial graphs: a survey”. In: *Discrete mathematics* 302.1-3 (2005), pp. 225–242. DOI: [10.1002/jgt.3190070410](https://doi.org/10.1002/jgt.3190070410).
- [7] Vittorio Bilò and Marios Mavronicolas. “A catalog of EXISTS-R-complete decision problems about Nash equilibria in multi-player games”. In: *Symposium on Theoretical Aspects of Computer Science (STACS)*. 2016. DOI: [10.4230/LIPIcs.STACS.2016.17](https://doi.org/10.4230/LIPIcs.STACS.2016.17).
- [8] R. H. Bing. “An Alternative Proof that 3-Manifolds Can be Triangulated”. In: *Annals of Mathematics* 69.1 (1959), pp. 37–65. ISSN: 0003486X. DOI: [10.2307/1970092](https://doi.org/10.2307/1970092).
- [9] Jürgen Bokowski and A. Guedes de Oliveira. “On the generation of oriented matroids”. In: *Discrete & Computational Geometry (DCG)* 24.2 (2000), pp. 197–208. DOI: [10.1007/s004540010027](https://doi.org/10.1007/s004540010027).
- [10] Ulrich Brehm. “A nonpolyhedral triangulated Möbius strip”. In: *Proceedings of the American Mathematical Society* 89.3 (1983), pp. 519–522. DOI: [10.1090/S0002-9939-1983-0715878-1](https://doi.org/10.1090/S0002-9939-1983-0715878-1).
- [11] Ulrich Brehm and Karanbir S. Sarkaria. “Linear vs. Piecewise-Linear Embeddability of Simplicial complexes”. In: *Technical Report MPI Bonn* (1992). available at <http://kssarkaria.org/docs/Linear>, pp. 1–15.
- [12] J. L. Bryant. “Approximating embeddings of polyhedra in codimension three”. In: *Transactions of the American Mathematical Society* 170 (1972), pp. 85–95. DOI: [10.1090/S0002-9947-1972-0307245-7](https://doi.org/10.1090/S0002-9947-1972-0307245-7).
- [13] Martin Čadek, Marek Krčál, Jiří Matoušek, Francis Sergeraert, Lukáš Vokřínek, and Uli Wagner. “Computing all maps into a sphere”. In: *Journal of the ACM (JACM)* 61.3 (2014), pp. 1–44. DOI: [10.1145/2597629](https://doi.org/10.1145/2597629).
- [14] Martin Čadek, Marek Krčál, Jiří Matoušek, Lukáš Vokřínek, and Uli Wagner. “Extendability of continuous maps is undecidable”. In: *Discrete & Computational Geometry* 51.1 (2014), pp. 24–66. DOI: [10.1007/s00454-013-9551-8](https://doi.org/10.1007/s00454-013-9551-8).
- [15] Martin Čadek, Marek Krčál, Jiří Matoušek, Lukáš Vokřínek, and Uli Wagner. “Time computation of homotopy groups and Postnikov systems in fixed dimension”. In: *SIAM Journal of Computing (SICOMP)* 43.5 (2014), pp. 1728–1780. DOI: [10.1137/120899029](https://doi.org/10.1137/120899029).
- [16] Martin Čadek, Marek Krčál, and Lukáš Vokřínek. “Algorithmic solvability of the lifting-extension problem”. In: *Discrete & Computational Geometry (DCG)* 57.4 (2017), pp. 915–965. DOI: [10.1007/s00454-016-9855-6](https://doi.org/10.1007/s00454-016-9855-6).
- [17] John Canny. “Some algebraic and geometric computations in PSPACE”. In: *Symposium on Theory of Computing (STOC)*. ACM. 1988, pp. 460–467. DOI: [10.1145/62212.62257](https://doi.org/10.1145/62212.62257).

- [18] Jean Cardinal. “Computational Geometry Column 62”. In: *SIGACT News* 46.4 (2015), pp. 69–78. DOI: [10.1145/2852040.2852053](https://doi.org/10.1145/2852040.2852053).
- [19] Jean Cardinal and Udo Hoffmann. “Recognition and complexity of point visibility graphs”. In: *Discrete & Computational Geometry (DCG)* 57.1 (2017), pp. 164–178. DOI: [10.1007/s00454-016-9831-1](https://doi.org/10.1007/s00454-016-9831-1).
- [20] Johannes Carmesin. *Embedding simply connected 2-complexes in 3-space – I. A Kuratowski-type characterisation*. arXiv preprint. 2019. arXiv: [1709.04642](https://arxiv.org/abs/1709.04642).
- [21] Jean Dieudonné. *A history of algebraic and differential topology, 1900-1960*. Springer, 2009.
- [22] Michael Gene Dobbins, Andreas Holmsen, and Tillmann Miltzow. *A Universality Theorem for Nested Polytopes*. arXiv preprint. 2019. arXiv: [1908.02213](https://arxiv.org/abs/1908.02213).
- [23] Jeff Erickson, Ivor van der Hoog, and Tillmann Miltzow. “Smoothing the gap between NP and ER”. In: *Foundations on Computer Science (FoCS)*. IEEE, 2020, pp. 1022–1033. DOI: [10.1109/FoCS46700.2020.00099](https://doi.org/10.1109/FoCS46700.2020.00099).
- [24] Marek Filakovský, Uli Wagner, and Stephan Zhechev. “Embeddability of Simplicial Complexes is Undecidable”. In: *Symposium on Discrete Algorithms (SODA)*. 2020, pp. 767–785. DOI: [10.1137/1.9781611975994.47](https://doi.org/10.1137/1.9781611975994.47).
- [25] Antonio Flores. “Über n-dimensionale Komplexe, die im \mathbb{R}_{2n+1} absolut selbstverschlungen sind”. In: *Ergeb. Math. Kolloq.* Vol. 34. 1933, pp. 4–6.
- [26] Michael H. Freedman, Vyacheslav S. Krushkal, and Peter Teichner. “Van Kampen’s embedding Obstruction is incomplete for 2-Complexes in \mathbb{R}^4 ”. In: *Mathematical Research Letters* 1.2 (1994), pp. 167–176.
- [27] Jugal Garg, Ruta Mehta, Vijay V. Vazirani, and Sadra Yazdanbod. “ $\exists\mathbb{R}$ -Completeness for Decision Versions of Multi-Player (Symmetric) Nash Equilibria”. In: *ACM Transactions on Economics and Computation* 6.1 (2018), 1:1 –1:23. ISSN: 2167-8375. DOI: [10.1145/3175494](https://doi.org/10.1145/3175494).
- [28] Branko Grünbaum. “Imbeddings of simplicial complexes”. In: *Commentarii Mathematici Helvetici* 44.1 (1969), pp. 502–513. DOI: [10.5169/seals-33795](https://doi.org/10.5169/seals-33795).
- [29] Branko Grünbaum. “Polytopes, graphs, and complexes”. In: *Bulletin of the American Mathematical Society* 76.6 (1970), pp. 1131–1201. DOI: [10.1090/S0002-9904-1970-12601-5](https://doi.org/10.1090/S0002-9904-1970-12601-5).
- [30] Anna Gundert. *On the Complexity of Embeddable Simplicial Complexes*. arXiv preprint. diploma thesis. 2018. arXiv: [1812.08447](https://arxiv.org/abs/1812.08447).
- [31] John Hopcroft and Robert Tarjan. “Efficient planarity testing”. In: *Journal of the ACM (JACM)* 21.4 (1974), pp. 549–568. DOI: [10.1145/321850.321852](https://doi.org/10.1145/321850.321852).
- [32] Wilfried Imrich. “On Whitney’s theorem on the unique embeddability of 3-connected planar graphs”. In: *Recent advances in graph theory: Proceedings of the Symposium held in Prague*. Vol. 1974. 1975, pp. 303–306.
- [33] Marek Krčál, Jiří Matoušek, and Francis Sergeraert. “Polynomial-time homology for simplicial Eilenberg–MacLane spaces”. In: *Foundations of Computational Mathematics (FoCM)* 13.6 (2013), pp. 935–963. DOI: [10.1007/s10208-013-9159-7](https://doi.org/10.1007/s10208-013-9159-7).
- [34] Anna Lubiw, Tillmann Miltzow, and Debajyoti Mondal. “The Complexity of Drawing a Graph in a Polygonal Region”. In: *International Symposium on Graph Drawing and Network Visualization (GD)*. Springer. 2018, pp. 387–401. DOI: [10.1007/978-3-030-04414-5_28](https://doi.org/10.1007/978-3-030-04414-5_28).
- [35] Jiří Matoušek. *Intersection graphs of segments and $\exists\mathbb{R}$* . ArXiv preprint. 2014. arXiv: [1406.2636](https://arxiv.org/abs/1406.2636).
- [36] Jiří Matoušek, Eric Sedgwick, Martin Tancer, and Uli Wagner. “Embeddability in the 3-sphere is decidable”. In: *Journal of the ACM (JACM)* 65.1 (2018), pp. 1–49. DOI: [10.1145/2582112.2582137](https://doi.org/10.1145/2582112.2582137).

- [37] Jiří Matoušek, Martin Tancer, and Uli Wagner. “Hardness of embedding simplicial complexes in \mathbb{R}^d ”. In: *Journal of the European Mathematical Society (JEMS)* 13.2 (2011), pp. 259–295. DOI: [10.4171/JEMS/252](https://doi.org/10.4171/JEMS/252).
- [38] Colin McDiarmid and Tobias Müller. “Integer realizations of disk and segment graphs”. In: *Journal of Combinatorial Theory, Series B* 103.1 (2013), pp. 114–143. DOI: [10.1016/j.jctb.2012.09.004](https://doi.org/10.1016/j.jctb.2012.09.004).
- [39] Karl Menger. *Dimensionstheorie*. 1st ed. Vieweg+Teubner Verlag, 1928. ISBN: 978-3-663-15484-6, 978-3-663-16056-4. DOI: [10.1007/978-3-663-16056-4](https://doi.org/10.1007/978-3-663-16056-4).
- [40] Arnaud de Mesmay, Yo’av Rieck, Eric Sedgwick, and Martin Tancer. “Embeddability in \mathbb{R}^3 is NP-hard”. In: *Journal of the ACM (JACM)* 67.4 (2020), 20:1–20:29. DOI: [10.1145/3396593](https://doi.org/10.1145/3396593).
- [41] Tillmann Miltzow and Reinier F. Schmiermann. *On Classifying Continuous Constraint Satisfaction problems*. arXiv preprint. arXiv: [2106.02397](https://arxiv.org/abs/2106.02397).
- [42] Nicolai Mnëv. “The universality theorems on the classification problem of configuration varieties and convex polytopes varieties”. In: *Topology and geometry – Rohlin seminar*. Ed. by Oleg Y. Viro. Springer, 1988, pp. 527–543. DOI: [10.1007/BFb0082792](https://doi.org/10.1007/BFb0082792).
- [43] Isabella Novik. “A note on geometric embeddings of simplicial complexes in a Euclidean space”. In: *Discrete & Computational Geometry (DCG)* 23.2 (2000), pp. 293–302. DOI: [10.1007/s004549910019](https://doi.org/10.1007/s004549910019).
- [44] Patrice Ossona deMendez. “Realization of Posets”. In: *Journal of Graph Algorithms and Applications (JGAA)* 6.1 (2002), pp. 149–153. DOI: [10.7155/jgaa.00048](https://doi.org/10.7155/jgaa.00048).
- [45] Christos Papakyriakopoulos. “A new proof of the invariance of the homology groups of a complex”. In: *Bulletin of the Greek Mathematical Society* 22 (1943), pp. 1–154.
- [46] S. Parsa and A. Skopenkov. *On embeddability of joins and their ‘factors’*. arXiv preprint. 2020. arXiv: [2003.12285](https://arxiv.org/abs/2003.12285).
- [47] Salman Parsa. “On the Links of Vertices in Simplicial d-Complexes Embeddable in the Euclidean 2 d-Space”. In: *Discrete & Computational Geometry (DCG)* 59.3 (2018), pp. 663–679. DOI: [10.1007/s00454-017-9936-1](https://doi.org/10.1007/s00454-017-9936-1).
- [48] Jürgen Richter-Gebert. *Realization spaces of polytopes*. Vol. 1643. Lecture notes in mathematics. Springer, 1996. DOI: [10.1007/BFb0093761](https://doi.org/10.1007/BFb0093761).
- [49] Jürgen Richter-Gebert and Günter M. Ziegler. “Realization spaces of 4-polytopes are universal”. In: *Bulletin of the American Mathematical Society* 32.4 (1995), pp. 403–412. DOI: [10.1090/S0273-0979-1995-00604-X](https://doi.org/10.1090/S0273-0979-1995-00604-X).
- [50] Marcus Schaefer. “Complexity of some geometric and topological problems”. In: *International Symposium on Graph Drawing (GD)*. LNCS. Springer, 2009, pp. 334–344. DOI: [10.1007/978-3-642-11805-0_32](https://doi.org/10.1007/978-3-642-11805-0_32).
- [51] Egon Schulte and Ulrich Brehm. “Polyhedral Maps”. In: *Handbook of Discrete and Computational Geometry, Third Edition*. Ed. by Csaba D. Toth, Jacob E. Goodman, and Joseph O’Rourke. Chapman and Hall/CRC, 2017, pp. 533–548. DOI: [10.1201/9781315119601](https://doi.org/10.1201/9781315119601).
- [52] Arnold Shapiro. “Obstructions to the imbedding of a complex in a Euclidean space.: I. The first obstruction”. In: *Annals of Mathematics* (1957), pp. 256–269. DOI: [10.2307/1969998](https://doi.org/10.2307/1969998).
- [53] Yaroslav Shitov. *A universality theorem for nonnegative matrix factorizations*. ArXiv preprint. 2016. arXiv: [1606.09068](https://arxiv.org/abs/1606.09068).
- [54] Peter Shor. “Stretchability of pseudolines is NP-hard”. In: *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift*. Ed. by Peter Gritzmann and Bernd Sturmfels. DIMACS – Series in Discrete Mathematics and Theoretical Computer Science. AMS, 1991, pp. 531–554.
- [55] Arkadiy Skopenkov. “A short exposition of Salman Parsa’s theorems on intrinsic linking and non-realizability”. In: *Discrete & Computational Geometry* (2019), pp. 1–2. DOI: [10.1007/s00454-019-00158-y](https://doi.org/10.1007/s00454-019-00158-y).

- [56] Arkadiy Skopenkov. *Extendability of simplicial maps is undecidable*. arXiv preprint. 2020. arXiv: [2008.00492](https://arxiv.org/abs/2008.00492).
- [57] Arkadiy Skopenkov. “Invariants of graph drawings in the plane”. In: *Arnold Mathematical Journal* 6 (2020), pp. 21–55. DOI: [10.1007/s40598-019-00128-5](https://doi.org/10.1007/s40598-019-00128-5).
- [58] Arkadiy Skopenkov. *Realizability of hypergraphs and Ramsey link theory*. arXiv preprint. 2014. arXiv: [1402.0658](https://arxiv.org/abs/1402.0658).
- [59] Arkadiy Skopenkov and Martin Tancer. “Hardness of almost embedding simplicial complexes in \mathbb{R}^d ”. In: *Discrete & Computational Geometry (DCG)* 61.2 (2019), pp. 452–463. DOI: [10.1007/s00454-018-0013-1](https://doi.org/10.1007/s00454-018-0013-1).
- [60] Mikhail Skopenkov. *Embedding products of graphs into Euclidean spaces*. arXiv preprint. 2016. arXiv: [0808.1199](https://arxiv.org/abs/0808.1199).
- [61] Dagmar Timmreck. “Necessary conditions for geometric realizability of simplicial complexes”. In: *Discrete Differential Geometry*. Ed. by A.I. Bobenko, P. Schröder, J.M. Sullivan, and G.M. Ziegler. Vol. 38. Oberwolfach Seminars. Birkhäuser Basel, 2008, pp. 215–233. DOI: [10.1007/978-3-7643-8621-4_11](https://doi.org/10.1007/978-3-7643-8621-4_11).
- [62] Dagmar Ingrid Timmreck. “Realization Problems for Point Configurations and Polyhedral Surfaces”. PhD thesis. Freie Universität Berlin, 2015. DOI: [10.17169/refubium-14465](https://doi.org/10.17169/refubium-14465).
- [63] Brian R. Ummel. “The Product of Nonplanar Complexes does not Imbed in 4-Space”. In: *Transactions of the American Mathematical Society* 242 (1978), pp. 319–328. DOI: [10.2307/1997741](https://doi.org/10.2307/1997741).
- [64] Egbert R Van Kampen. “Komplexe in euklidischen Räumen”. In: *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*. Vol. 9. 1. Springer. 1933, pp. 72–78.
- [65] Wikipedia. *Jordan Curve Theorem*. https://en.wikipedia.org/wiki/Jordan_curve_theorem. accessed 18-10-2021.