

# Steiner Trees for Hereditary Graph Classes: a Treewidth Perspective\*

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**Abstract.** We consider the classical problems (EDGE) STEINER TREE and VERTEX STEINER TREE after restricting the input to some class of graphs characterized by a small set of forbidden induced subgraphs. We show a dichotomy for the former problem restricted to  $(H_1, H_2)$ -free graphs and a dichotomy for the latter problem restricted to  $H$ -free graphs. We find that there exists an infinite family of graphs  $H$  such that VERTEX STEINER TREE is polynomial-time solvable for  $H$ -free graphs, whereas there exist only two graphs  $H$  for which this holds for EDGE STEINER TREE. We also find that EDGE STEINER TREE is polynomial-time solvable for  $(H_1, H_2)$ -free graphs if and only if the treewidth of the class of  $(H_1, H_2)$ -free graphs is bounded (subject to  $P \neq NP$ ). To obtain the latter result, we determine all pairs  $(H_1, H_2)$  for which the class of  $(H_1, H_2)$ -free graphs has bounded treewidth.

**Keywords:** Steiner Tree; hereditary graph class; treewidth.

## 1 Introduction

Let  $G = (V, E)$  be a connected graph and  $U \subseteq V$  be a set of *terminal* vertices. A *Steiner tree* for  $U$  (of  $G$ ) is a tree in  $G$  that contains all vertices of  $U$ . An *edge weighting* of  $G$  is a function  $w_E : E \rightarrow \mathbb{R}^+$ . For a tree  $T$  in  $G$ , the *edge weight*  $w_E(T)$  of  $T$  is the sum  $\sum_{e \in E(T)} w_E(e)$ . We consider the classical problem:

### EDGE STEINER TREE

*Instance:* a connected graph  $G = (V, E)$  with an weighting  $w_E$ , a subset  $U \subseteq V$  of terminals and a positive integer  $k$ .

*Question:* does  $G$  have a Steiner tree  $T_U$  for  $U$  with  $w_E(T_U) \leq k$ ?

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This is often known simply as STEINER TREE, but we wish to distinguish it from a closely related problem. A *vertex weighting* of  $G$  is a function  $w_V : V \rightarrow \mathbb{R}^+$ . For a tree  $T$  in  $G$ , the *vertex weight*  $w_V(T)$  of  $T$  is the sum  $\sum_{v \in V(T)} w(v)$ . The following problem is sometimes known as NODE-WEIGHTED STEINER TREE.

**VERTEX STEINER TREE**

*Instance:* a connected graph  $G = (V, E)$  with a vertex weighting  $w_V$ , a subset  $U \subseteq V$  and a positive integer  $k$ .

*Question:* does  $G$  have a Steiner tree  $T_U$  for  $U$  with  $w_V(T_U) \leq k$ ?

Note that we defined the problem for connected inputs, as for general inputs a Steiner tree exists only if the vertices of  $U$  all belong to the same connected component. Note also that EDGE STEINER TREE is a generalization of the SPANNING TREE problem (set  $U = V(G)$ ). We refer to the textbooks of Du and Hu [11] and Prömel and Steger [20] for further background information on Steiner trees.

We consider the problems EDGE STEINER TREE and VERTEX STEINER TREE separately so that, for any graph under consideration, we have either an edge or vertex weighting but not both, so we will generally denote weightings by  $w$  without any subscript. Moreover, when we use the following terminology there is no ambiguity. We say that a Steiner tree of least possible weight is *minimum*, and that an instance of a problem is *unweighted* if the weighting is constant. It is well known that the unweighted versions of EDGE STEINER TREE and VERTEX STEINER TREE are NP-complete [17,12], and we note that these unweighted problems are polynomially equivalent. We denote instances of the weighted problems by  $(G, w, U, k)$  and of the unweighted problems by  $(G, U, k)$ .

**Our Focus** We focus on the computational complexity of EDGE STEINER TREE and VERTEX STEINER TREE for *hereditary* graph classes, which are those graph classes that are closed under vertex deletion. We do this from a *systematic* point of view. It is well known, and readily seen, that a graph class  $\mathcal{G}$  is hereditary if and only if it can be characterized by a set  $\mathcal{H}$  of forbidden induced subgraphs. That is, a graph  $G$  belongs to  $\mathcal{G}$  if and only if  $G$  has no induced subgraph isomorphic to some graph in  $\mathcal{H}$ . We normally require  $\mathcal{H}$  to be minimal, in which case it is unique and we denote it by  $\mathcal{H}_{\mathcal{G}}$ . We note that  $\mathcal{H}_{\mathcal{G}}$  may have infinite size; for example, if  $\mathcal{G}$  is the class of bipartite graphs, then  $\mathcal{H}_{\mathcal{G}} = \{C_3, C_5, \dots\}$ , where  $C_r$  denotes the cycle on  $r$  vertices. For a systematic complexity study of a graph problem, we may first consider *monogenic graph classes* or *bigenic* graph classes, which are classes  $\mathcal{G}$  with  $|\mathcal{H}_{\mathcal{G}}| = 1$  or  $|\mathcal{H}_{\mathcal{G}}| = 2$ , respectively. This is the approach we follow here.

**Our Results** We prove a dichotomy for EDGE STEINER TREE for bigenic graph classes in Section 2 and a dichotomy for VERTEX STEINER TREE for monogenic graph classes in Section 3. We denote the *disjoint union* of two vertex-disjoint graphs  $G$  and  $H$  by  $G + H = (V(G) \cup V(H), E(G) \cup E(H))$ , and the disjoint union of  $s$  copies of  $G$  by  $sG$ . A *linear forest* is a disjoint union of paths. For a graph  $H$ , a graph is *H-free* if it has no induced subgraph isomorphic to  $H$ . For a set of graphs  $\{H_1, \dots, H_p\}$ , a graph is  $(H_1, \dots, H_p)$ -free if it is  $H_i$ -free

for every  $i \in \{1, \dots, p\}$ . We let  $K_r$  and  $P_r$  denote the complete graph and path on  $r$  vertices, respectively. The *complete bipartite* graph  $K_{s,t}$  is the graph whose vertex set can be partitioned into two sets  $S$  and  $T$  of size  $s$  and  $t$ , respectively, such that for any two distinct vertices  $u, v$ , we have  $uv \in E$  if and only if  $u \in S$  and  $v \in T$ . We call  $K_{1,3}$  the *claw*. In the first dichotomy, the roles of  $H_1$  and  $H_2$  are interchangeable.

**Theorem 1.** *Let  $H_1$  and  $H_2$  be two graphs. EDGE STEINER TREE is polynomial-time solvable for  $(H_1, H_2)$ -free graphs if and only if the treewidth of  $(H_1, H_2)$ -free graphs is bounded if and only if*

1.  $H_1 = K_r$  for some  $r \in \{1, 2\}$
2.  $H_1 = K_3$  and  $H_2 = K_{1,3}$
3.  $H_1 = K_r$  for some  $r \geq 3$  and  $H_2 = P_3$
4.  $H_1 = K_r$  for some  $r \geq 3$  and  $H_2 = sP_1$  for some  $s \geq 1$ ,

and otherwise it is NP-complete.

**Theorem 2.** *Let  $H$  be a graph. For every  $s \geq 0$ , VERTEX STEINER TREE is polynomial-time solvable for  $H$ -free graphs if  $H$  is an induced subgraph of  $sP_1 + P_4$ ; otherwise even unweighted VERTEX STEINER TREE is NP-complete.*

We make the following observations about these two results:

**1.** In Theorem 1 we show that EDGE STEINER TREE can be solved in polynomial time for  $(H_1, H_2)$ -free graphs if and only if the treewidth of the class of  $(H_1, H_2)$ -free graphs is bounded. It is known that VERTEX STEINER TREE is polynomial-time solvable for graph classes of bounded mim-width [1] and thus also for graph classes of bounded treewidth. However, the 1-to-1 correspondence does not hold for VERTEX STEINER TREE for treewidth or mim-width. To see this, observe that complete graphs, and hence  $P_4$ -free graphs, have unbounded treewidth, whereas co-bipartite graphs, and hence  $3P_1$ -free graphs, have unbounded mim-width. In Section 4 we discuss this connection between EDGE STEINER TREE and treewidth further. In the same section we also pose a number of open problems.

**2.** Theorem 1 also provides a dichotomy for boundedness of treewidth of  $(H_1, H_2)$ -free graphs. For the more powerful width parameter clique-width (or equivalently boolean-width, rank-width, module-width, or NLC-width [7,16,19,21]) or the even more powerful width parameter mim-width, such dichotomies have not yet been established for  $(H_1, H_2)$ -free graphs. We refer to [10] and [6] for state-of-the-art summaries for clique-width and mim-width, respectively.

**3.** The restriction of Theorem 1 to monogenic graph classes yields only two (trivial) graphs  $H$ , namely  $H = P_1$  or  $H = P_2$ , for which the restriction of EDGE STEINER TREE to  $H$ -free graphs can be solved in polynomial time. In contrast, by Theorem 2, VERTEX STEINER TREE can, when restricted to  $H$ -free graphs, be solved in polynomial time for an infinite family of linear forests  $H$ , namely  $H = sP_1 + P_4$  ( $s \geq 0$ ).

**4.** Theorem 2 is also a dichotomy for the unweighted VERTEX STEINER TREE problem. Moreover, as the unweighted versions of EDGE STEINER TREE and

VERTEX STEINER TREE are polynomially equivalent, Theorem 2 is also a classification of the unweighted version of EDGE STEINER TREE.

## 2 The Proof of Theorem 1

In this section we give a proof for our first dichotomy, which is for EDGE STEINER TREE for  $(H_1, H_2)$ -free graphs. We note that this is not the first systematic study of EDGE STEINER TREE. For example, Renjitha and Sadagopan [22] proved that unweighted EDGE STEINER TREE is NP-complete for  $K_{1,5}$ -free split graphs, but can be solved in polynomial time for  $K_{1,4}$ -free split graphs. We present a number of other results from the literature, which we collect in Section 2.1, together with some lemmas that follow from these results. Then in Section 2.2 we discuss the notion of treewidth; as we shall see, this notion will play an important role. We then use these results to prove Theorem 1 in Section 2.3.

### 2.1 Preliminaries

The NP-completeness of EDGE STEINER TREE on complete graphs follows from the result [17] that the general problem is NP-complete: to obtain a reduction add any missing edges and give them sufficiently large weight such that they will never be used in any solution. Bern and Plasman proved the following stronger result.

**Lemma 1 ([2]).** *EDGE STEINER TREE is NP-complete for complete graphs where every edge has weight 1 or 2.*

To subdivide an edge  $e = uv$  means to delete  $e$  and add a vertex  $w$  and edges  $uw$  and  $wv$ . Let  $r$  be a positive integer. To say that  $e$  is subdivided  $r$  times means that  $e$  is replaced by a path  $P_e = uw_1 \cdots w_r v$  of  $r + 1$  edges. The  $r$ -subdivision of a graph  $H$  is the graph obtained from  $H$  after subdividing each edge exactly  $r$  times. If we say that a graph is a *subdivision* of  $H$ , then we mean it can be obtained from  $H$  using subdivisions (the number of subdivisions can be different for each edge and some edges might not be subdivided at all). A graph  $G$  contains a graph  $H$  as a *subdivision* if  $G$  contains a subdivision of  $H$  as a subgraph.

**Proposition 1.** *If EDGE STEINER TREE is NP-complete on a class  $\mathcal{C}$  of graphs, then, for every  $r \geq 0$ , it is so on the class of  $r$ -subdivisions of graphs in  $\mathcal{C}$ .*

*Proof.* Let  $(G, w, U, k)$  be an instance of EDGE STEINER TREE where  $G \in \mathcal{C}$ . Let  $G'$  be the  $r$ -subdivision of  $G$  and for each edge  $e$  in  $G$ , let  $P_e$  be the corresponding path on  $r + 1$  edges in  $G'$ . We define an edge weighting  $w'$  for  $G'$  by letting  $w'(e') = w(e)/(r + 1)$ . In any minimum Steiner tree  $T'$  for  $U$  of  $(G', w')$ , for each  $e \in E$ , either all or no edges of  $P_e$  are in  $T'$ . Then it is easy to see that there is a bijection that preserves weight between minimum Steiner trees of  $(G, w)$  for  $U$  and minimum Steiner trees of  $(G', w')$  for  $U$ .  $\square$

We make the following observation.

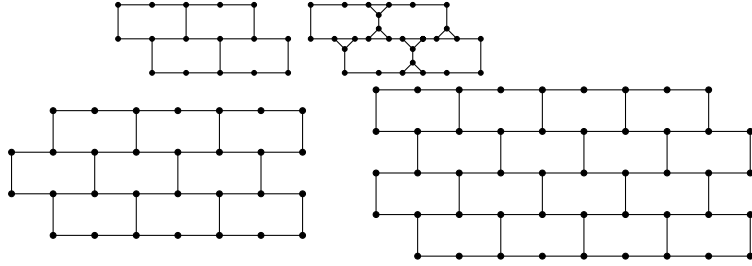


Fig. 1: A wall of height 2, its wye-net transformation, walls of height 3 and 4.

**Lemma 2.** *EDGE STEINER TREE is NP-complete for complete bipartite graphs.*

*Proof.* As EDGE STEINER TREE is NP-complete in general, and the 1-subdivision of any graph is bipartite, the problem remains NP-complete for bipartite graphs by Proposition 1. We describe a reduction from the problem on bipartite graphs to the problem on complete bipartite graphs. Let  $(G, w, U, k)$  be an instance of EDGE STEINER TREE where  $G$  is bipartite. Let  $M$  be the sum of the weights of all the edges of  $G$  and note that if  $k > M$  we can redefine it as  $k = M$  without changing the problem. Let  $G'$  be the complete bipartite graph obtained from  $G$  by adding all missing edges that preserve the bipartition. Let  $w'$  be an edge weighting for  $G'$  where  $w'(e) = w(e)$  if  $e$  belongs to  $G$  and  $w'(e) = M + 1$  otherwise. Thus  $(G', w', U, k)$  is an instance of EDGE STEINER TREE for complete bipartite graphs and clearly  $G$  has a Steiner tree  $T_U$  of weight at most  $k$  if and only if  $G'$  has a Steiner tree  $T'_U$  of weight at most  $k$ .  $\square$

The following follows by inspection of the reduction of Garey and Johnson for RECTILINEAR STEINER TREE [14]. Let  $n$  and  $m$  be positive integers. An  $n \times m$  grid graph has vertex set  $\{v^{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$  and  $v^{i,j}$  has neighbours  $v^{i-1,j}$  (if  $i > 1$ ),  $v^{i+1,j}$  (if  $i < n$ ),  $v^{i,j-1}$  (if  $j > 1$ ), and  $v^{i,j+1}$  (if  $j < m$ ).

**Theorem 3 ([14]).** *Unweighted EDGE STEINER TREE is NP-complete for grid graphs.*

A *wall* is a graph which can be thought of as a hexagonal grid. See Fig. 1 for three examples of walls of different *heights*. We refer to [9] for a formal definition. Note that walls of height at least 2 have maximum degree 3. From a wall of height  $h$  we obtain a *net-wall* by doing the following for each wall vertex  $u$  with three neighbours  $v_1, v_2, v_3$ : replace  $u$  and its incident edges with three new vertices  $u_1, u_2, u_3$  and edges  $u_1v_1, u_2v_2, u_3v_3, u_1u_2, u_1u_3, u_2u_3$ . We call this a *wye-net transformation*, reminiscent of the well-known wye-delta transformation. Note that a net-wall is  $K_{1,3}$ -free but contains an induced *net*, which is the graph obtained from a triangle on vertices  $a_1, a_2, a_3$  and three new vertices  $b_1, b_2, b_3$  after adding the edge  $a_i b_i$  for  $i = 1, 2, 3$ .

We have two results on these classes.

**Lemma 3.** *For every  $r \geq 0$ , EDGE STEINER TREE is NP-complete for  $r$ -subdivisions of walls.*

*Proof.* We reduce from unweighted EDGE STEINER TREE on grid graphs, which is NP-hard by Theorem 3. Let  $(G, U, k)$  be an instance of unweighted EDGE STEINER TREE where  $G$  is an  $n \times m$  grid graph. Think of  $v^{1,1}$  as the top-left corner of the grid, and in  $v^{i,j}$ ,  $i$  indicates the row of the grid containing the vertex, while  $j$  indicates the column.

From  $G$ , we obtain a graph  $W$  as follows. Two vertices of  $G$  are exceptional:  $v^{n,1}$  is always exceptional,  $v^{1,m}$  is exceptional if  $n$  is even, and  $v^{1,1}$  is exceptional if  $n$  is odd. For every vertex  $v^{i,j}$  of  $G$  that is not exceptional,  $W$  contains vertices  $v_{\uparrow}^{i,j}$  and  $v_{\downarrow}^{i,j}$  that are joined by an edge. We call these edges *new*. We also add to  $W$  vertices  $v_{\uparrow}^{n,1}$ , and  $v_{\downarrow}^{1,m}$  (if  $v^{1,m}$  is exceptional) or  $v_{\downarrow}^{1,1}$  (otherwise). We add an edge from  $v_{\downarrow}^{i,j}$  to  $v_{\uparrow}^{i+1,j}$ , for  $1 \leq i \leq n-1$ ,  $1 \leq j \leq m$ . For  $1 \leq i \leq n$ ,  $1 \leq j \leq m-1$ , if  $i$  is odd and  $n$  is even or if  $i$  is even and  $n$  is odd, we add an edge from  $v_{\downarrow}^{i,j}$  to  $v_{\uparrow}^{i,j+1}$ , and otherwise, we add an edge from  $v_{\uparrow}^{i,j}$  to  $v_{\downarrow}^{i,j+1}$ . The edges that are not new are *original*.

We note that  $W$  is a wall obtained from  $G$  by splitting each vertex in two (except the exceptional vertices that lie in a corner of the grid), and that there is a bijection between the original edges of  $W$  and the edges of  $G$ . We define an edge weighting  $w'$  for  $W$  by letting the weight of each original edge be 1 and the weight of each new edge be  $\varepsilon$ , where  $\varepsilon > 0$  is chosen so that the sum of the weights of all new edges is less than 1. We define a set of terminals  $U'$  for  $W$ : if  $v^{i,j}$  is in  $U$ , then  $U'$  contains each of  $v_{\downarrow}^{i,j}$  and  $v_{\uparrow}^{i,j}$  that exists (one or other will not exist if  $v^{i,j}$  is exceptional).

We claim that there is a Steiner tree of  $k$  edges in  $G$  for terminal set  $U$  if and only if there is a Steiner tree of weight  $k + \delta$  in  $(W, w')$  for terminal set  $U'$ , where  $0 \leq \delta < 1$ . Indeed, any Steiner tree  $T$  in  $G$  for terminal set  $U$  of  $k$  edges corresponds naturally to a Steiner tree  $T'$  for  $U'$  in  $(W, w')$  of weight less than  $k + 1$  by adding all new edges to  $T$  and letting  $T'$  be a spanning tree of the component of the resulting subgraph of  $W$  that contains  $U'$ . Conversely, any Steiner tree  $T'$  for  $U'$  in  $(W, w')$  of weight  $k + \delta$ ,  $0 \leq \delta < 1$ , corresponds naturally to a Steiner tree  $T$  for  $U$  in  $G$  of  $k$  edges by removing all new edges from  $T'$  and letting  $T$  be a spanning tree of the resulting subgraph of  $G$ . Effectively, this mimics the splitting and contraction operations which can be seen as the way in which we obtain  $W$  from  $G$  and vice versa.

The lemma now follows immediately from Proposition 1. □

The next lemma has a similar proof.

**Lemma 4.** *For every  $r \geq 0$ , EDGE STEINER TREE is NP-complete for  $r$ -subdivisions of net-walls.*

*Proof.* We reduce from EDGE STEINER TREE on walls, which is NP-complete by Lemma 3. Consider an instance  $(W, w, U, k)$  of EDGE STEINER TREE where  $W$  is a wall. We now apply a wye-net transformation to  $W$  to obtain a *net-wall*  $N$ .

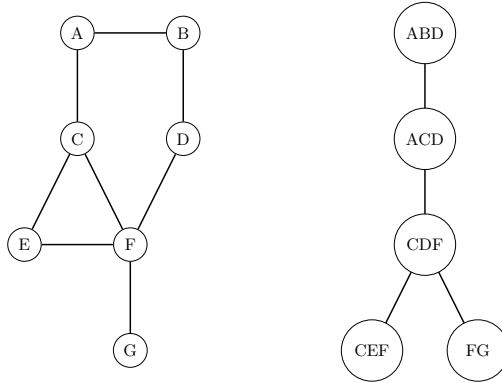


Fig. 2: A graph, and a tree decomposition of the graph with width 2.

The edges of the triangles added through the wye-net transformations are called *new* and all other edges of  $N$  are *original* (since they admit a bijection to the edges of  $W$ ). We create an edge weighting  $w'$  on  $N$ . For each original edge  $e$ ,  $w'(e) = w(e)$ . Let  $m$  be the smallest weight of an edge of  $(W, w)$ . Let the weight of each new edge be  $\varepsilon$ , where  $\varepsilon > 0$  is chosen so that the sum of the weights of all new edges is less than  $m$ . We define a set of terminals  $U'$  for  $N$ . We note that each vertex  $v$  of  $W$  corresponds to either a single vertex of  $N$  or a set of three vertices of  $N$  that form a triangle. If  $v$  is in  $U$ , then, according to which case we have, we add to  $U'$  either the single vertex or all three vertices.

We claim that there is a Steiner tree of weight  $k$  in  $(W, w)$  for terminal set  $U$  if and only if there is a Steiner tree of weight  $k + \delta$  in  $(N, w')$  for terminal set  $U'$ , where  $0 \leq \delta < m$ . Indeed, any Steiner tree  $T$  in  $W$  for terminal set  $U$  of weight at most  $k$  corresponds naturally to a Steiner tree  $T'$  for  $U'$  in  $(N, w')$  of weight less than  $k + m$  by adding all new edges to  $T$  and letting  $T'$  be a spanning tree of the component of the resulting subgraph of  $N$  that contains  $U'$ . Conversely, any Steiner tree  $T'$  for  $U'$  in  $(N, w')$  of weight  $k + \delta$ ,  $0 \leq \delta < m$ , corresponds naturally to a Steiner tree  $T$  for  $U$  in  $(W, w)$  of weight at most  $k$  by removing all new edges from  $T'$  and letting  $T$  be a spanning tree of the resulting subgraph of  $W$ .

The lemma now follows immediately from Proposition 1. □

## 2.2 Treewidth and Implications

A *tree decomposition* of a graph  $G = (V, E)$  is a tree  $T$  whose vertices, which are called *nodes*, are subsets of  $V$  and has the following properties: for each  $v \in V$ , the nodes of  $T$  that contain  $v$  induce a non-empty connected subgraph, and, for each edge  $vw \in E$ , there is at least one node of  $T$  that contains  $v$  and  $w$ . We refer to Figure 2 for an example.

The sets of vertices of  $G$  that form the nodes of  $T$  are called *bags*. The *width* of  $T$  is one less than the size of its largest bag. The *treewidth* of  $G$  is the minimum width of its tree decompositions. A graph class  $\mathcal{G}$  has *bounded treewidth* if there exists a constant  $c$  such that each graph in  $\mathcal{G}$  has treewidth at most  $c$ ; otherwise  $\mathcal{G}$  has *unbounded treewidth*. As trees with at least one edge form exactly the class of graphs with treewidth 1, the treewidth of a graph can be seen as a measure that indicates how close a graph is to being a tree. Many discrete optimization problems can be solved in polynomial time on every graph class of bounded treewidth. The EDGE STEINER TREE problem is an example of such a problem (see, for instance, [8] or, for a faster algorithm [4]).

**Lemma 5 ([4,8]).** *EDGE STEINER TREE can be solved in polynomial time on every graph class of bounded treewidth.*

We also need the well-known Robertson-Seymour Grid-Minor Theorem (also called the Excluded Grid Theorem), which can be formulated for walls.

**Theorem 4 ([23]).** *For every integer  $h$ , there exists a constant  $c_h$  such that a graph has treewidth at least  $c_h$  if and only if it contains a wall of height  $h$  as a subdivision.*

We will use two lemmas, both of which follow immediately from Theorem 4.

**Lemma 6.** *For every  $r \geq 0$ , the class of  $r$ -subdivided walls has unbounded treewidth.*

**Lemma 7.** *For every  $r \geq 0$ , the class of  $r$ -subdivided net-walls has unbounded treewidth.*

We now prove the part of Theorem 1 on the classification of the boundedness of treewidth for  $(H_1, H_2)$ -free graphs (in which we may exchange the roles of  $H_1$  and  $H_2$ ).

**Lemma 8.** *Let  $H_1$  and  $H_2$  be two graphs. Then the class of  $(H_1, H_2)$ -free graphs has bounded treewidth if and only if*

1.  $H_1 = K_r$  for some  $r \in \{1, 2\}$
2.  $H_1 = K_3$  and  $H_2 = K_{1,3}$
3.  $H_1 = K_r$  for some  $r \geq 3$  and  $H_2 = P_3$
4.  $H_1 = K_r$  for some  $r \geq 3$  and  $H_2 = sP_1$  for some  $s \geq 1$ .

*Proof.* We first prove that in each of the Cases 1–4, the class of  $(H_1, H_2)$ -free graphs has bounded treewidth. Let  $G$  be an  $(H_1, H_2)$ -free graph. First suppose that  $H_1 = K_r$  for some  $r \in \{1, 2\}$ . Then  $G$  has no edges and so has treewidth 0. If  $H_1 = K_3$  and  $H_2 = K_{1,3}$ , then  $G$  has maximum degree at most 2, that is,  $G$  is the disjoint union of paths and cycles. Hence  $G$  has treewidth at most 2. If  $H_1 = K_r$  for some  $r \geq 3$ , and  $H_2 = P_3$ , then  $G$  is the disjoint union of complete graphs, each of size at most  $r - 1$ . Hence  $G$  has treewidth at most  $r - 1$ . Finally if  $H_1 = K_r$ , for some  $r \geq 3$ , and  $H_2 = sP_1$ , for some  $s \geq 1$ , then, by Ramsey's



Theorem, the number of vertices of  $G$  is bounded by some constant  $R(r, s)$ . Hence  $G$  has treewidth at most  $R(r, s)$ .

We will now show that the class of  $(H_1, H_2)$ -free graphs has unbounded treewidth if Cases 1–4 do not apply. First suppose that neither  $H_1$  nor  $H_2$  is a complete graph. Then the class of  $(H_1, H_2)$ -free graphs contains the class of all complete graphs. As the treewidth of a complete graph  $K_r$  is readily seen to be equal to  $r - 1$ , the class of complete graphs, and thus the class of  $(H_1, H_2)$ -free graphs, has unbounded treewidth. From now on, assume that  $H_1 = K_r$  for some  $r \geq 1$ . As Case 1 does not apply, we find that  $r \geq 3$ .

Suppose that  $H_2$  contains a cycle  $C_s$  as an induced subgraph for some  $s \geq 1$ . As  $H_1 = K_r$  for some  $r \geq 3$ , the class of  $(H_1, H_2)$ -free graphs contains the class of  $(C_3, C_s)$ -free subgraphs. As the latter graph class contains the class of  $(s + 1)$ -subdivided walls, which have unbounded treewidth due to Lemma 6, the class of  $(H_1, H_2)$ -free graphs has unbounded treewidth.

Note that if  $H_2$  contains a cycle as a subgraph, then it also contains a cycle as an induced subgraph. So now suppose that  $H_2$  contains no cycle, that is,  $H_2$  is a forest. First assume that  $H_2$  contains an induced  $P_1 + P_2$ . Recall that  $H_1 = K_r$  for some  $r \geq 3$ . Then the class of  $(H_1, H_2)$ -free graphs contains the class of complete bipartite graphs. As this class has unbounded treewidth, the class of  $(H_1, H_2)$ -free graphs has unbounded treewidth. From hereon we assume that  $H_2$  is a  $(P_1 + P_2)$ -free forest.

Suppose that  $H_2$  has a vertex of degree at least 3. In other words, as  $H_2$  is a forest, the claw  $K_{1,3}$  is an induced subgraph of  $H_2$ . Recall that  $H_1 = K_r$  for some  $r \geq 3$ . First assume that  $r = 3$ . As Case 2 does not apply,  $H_2$  properly contains an induced  $K_{1,3}$ . As  $H_2$  is a forest, this means that  $H_2$  contains an induced  $P_1 + P_2$ , which is not possible. We conclude that  $r \geq 4$ . Then the class of  $(H_1, H_2)$ -free graphs contains the class of net-walls. As the latter graph class has unbounded treewidth due to Lemma 7, the class of  $(H_1, H_2)$ -free graphs has unbounded treewidth.

From the above we may assume that  $H_2$  does not contain any vertex of degree 3. This means that  $H_2$  is a linear forest, that is, a disjoint union of paths. As Case 4 does not apply,  $H_2$  has an edge. Every  $(P_1 + P_2)$ -free linear forest with an edge is either a  $P_2$  or a  $P_3$ . However, this is not possible, as Case 1 (with the roles of  $H_1$  and  $H_2$  reversed) and Case 3 do not apply. We conclude that this case cannot happen.  $\square$

### 2.3 The Proof

We are now ready to complete the proof of Theorem 1.

**Theorem 1 (restated).** *Let  $H_1$  and  $H_2$  be two graphs. EDGE STEINER TREE is polynomial-time solvable for  $(H_1, H_2)$ -free graphs if and only if the treewidth of  $(H_1, H_2)$ -free graphs is bounded if and only if*

1.  $H_1 = K_r$  for some  $r \in \{1, 2\}$
2.  $H_1 = K_3$  and  $H_2 = K_{1,3}$
3.  $H_1 = K_r$  for some  $r \geq 3$  and  $H_2 = P_3$

4.  $H_1 = K_r$  for some  $r \geq 3$  and  $H_2 = sP_1$  for some  $s \geq 1$ ,

and otherwise it is NP-complete.

*Proof.* Let  $H_1$  and  $H_2$  be two graphs. We already showed that the treewidth of  $(H_1, H_2)$ -free graphs is bounded if and only if one of Cases 1–4 applies. Hence it remains to prove that EDGE STEINER TREE is polynomial-time solvable for  $(H_1, H_2)$ -free graphs if and only if one of Cases 1–4 applies. We show this below by using the same outline approach as in the proof of Lemma 8.

If one of Cases 1–4 applies, the class of  $(H_1, H_2)$ -free graphs has bounded treewidth due to Lemma 8, and we can apply Lemma 5. Now suppose that none of these four cases apply. We will prove that EDGE STEINER TREE is NP-complete for the class of  $(H_1, H_2)$ -free graphs.

First suppose that neither  $H_1$  nor  $H_2$  is a complete graph. Then the class of  $(H_1, H_2)$ -free graphs contains the class of all complete graphs, and we can apply Lemma 1. From now on, assume that  $H_1 = K_r$  for some  $r \geq 1$ . As Case 1 does not apply, we find that  $r \geq 3$ .

Suppose that  $H_2$  contains a cycle  $C_s$  as an induced subgraph for some  $s \geq 1$ . As  $H_1 = K_r$  for some  $r \geq 3$ , the class of  $(H_1, H_2)$ -free graphs contains the class of  $(C_3, C_s)$ -free subgraphs. As the latter graph class contains the class of  $(s + 1)$ -subdivided walls, we can apply Lemma 3.

Note that if  $H_2$  contains a cycle as a subgraph, then it also contains a cycle as an induced subgraph. So now suppose that  $H_2$  contains no cycle, that is,  $H_2$  is a forest. First assume that  $H_2$  contains an induced  $P_1 + P_2$ . Recall that  $H = K_r$  for some  $r \geq 3$ . Then the class of  $(H_1, H_2)$ -free graphs contains the class of complete bipartite graphs, and we can apply Lemma 2. From now on assume that  $H_2$  is  $(P_1 + P_2)$ -free.

Suppose that  $H_2$  has a vertex of degree at least 3. In other words, as  $H$  is a forest, the claw  $K_{1,3}$  is an induced subgraph of  $H_2$ . Recall that  $H = K_r$  for some  $r \geq 3$ . First assume that  $r = 3$ . As Case 2 does not apply,  $H_2$  properly contains an induced  $K_{1,3}$ . As  $H_2$  is a forest, this means that  $H_2$  contains an induced  $P_1 + P_2$ , which is not possible. We conclude that  $r \geq 4$ . Then the class of  $(H_1, H_2)$ -free graphs contains the class of net-walls, and we can apply Lemma 4.

From the above we may assume that  $H_2$  does not contain any vertex of degree 3. This means that  $H_2$  is a linear forest. As Case 4 does not apply,  $H_2$  has an edge. Every  $(P_1 + P_2)$ -free linear forest with an edge is either a  $P_2$  or a  $P_3$ . However, this is not possible, as Case 1 (with the roles of  $H_1$  and  $H_2$  reversed) and Case 3 do not apply. We conclude that this case cannot happen.  $\square$

### 3 The Proof of Theorem 2

In this section we give a proof of our second dichotomy. We state useful past results in Section 3.1 followed by some new results for  $P_4$ -free graphs in Section 3.2. Then, in Section 3.3, we show how to combine these results to obtain the proof of Theorem 2.

### 3.1 Known Results

The first result we need is due to Brandstädt and Müller. A graph is *chordal bipartite* if it has no induced cycles of length 3 or of length at least 5; that is, a graph is chordal bipartite if it is  $(C_3, C_5, C_6, \dots)$ -free.

**Theorem 5 ([5]).** *The unweighted VERTEX STEINER TREE problem is NP-complete for chordal bipartite graphs.*

The second result that we need is due to Farber, Pulleyblank and White. A graph is *split* if its vertex set can be partitioned into a clique and an independent set. It is well known that the class of split graphs coincides with the class of  $(2P_2, C_4, C_5)$ -free graphs [13].

**Theorem 6 ([12]).** *The unweighted VERTEX STEINER TREE problem is NP-complete for split graphs.*

### 3.2 New Results

We start with the following lemma.

**Lemma 9.** *The unweighted VERTEX STEINER TREE problem is NP-complete for line graphs.*

*Proof.* By Theorem 3, unweighted EDGE STEINER TREE is NP-hard. Let  $(G, U, k)$  be an instance of this problem. From  $G$  we construct a new graph  $G'$  by introducing a new vertex  $v_u$  for each terminal  $u \in U$ , which we make only adjacent to  $u$ . We let  $U'$  consist of all these new vertices. We observe that  $G'$  has a Steiner tree  $T'$  for  $U'$  with at most  $k + |U|$  edges if and only if  $G$  has a Steiner tree  $T$  for  $U$  with at most  $k$  edges.

We now consider the line graph  $L(G')$  with set of terminals  $U^* = \{uv_u \mid u \in U\}$ ; this is a set of edges in  $G'$  and a set of vertices in  $L(G')$ . To complete the proof, we show that  $G'$  has a Steiner tree for  $U'$  on, say,  $\ell$  edges if and only if  $L(G')$  has a Steiner tree for  $U^*$  on  $\ell$  vertices. We first note that the edge set  $E'$  of a Steiner tree for  $U'$  of  $G'$  must contain the set  $U^*$ . Further,  $E'$ , considered as a set of vertices of  $L(G')$ , induces a connected subgraph and has  $|E'| = \ell$  vertices. Conversely, if there is a Steiner tree for  $U^*$  in  $L(G')$  on  $\ell$  vertices, then these vertices, considered as edges in  $G'$ , form a Steiner tree for  $U'$  in  $G'$ .  $\square$

Recall that a subgraph  $G'$  of a graph  $G$  is spanning if  $V(G') = V(G)$ . Let  $G_1$  and  $G_2$  be two graphs. The *join* operation adds an edge between every vertex of  $G_1$  and every vertex of  $G_2$ . The *disjoint union* operation takes the disjoint union of  $G_1$  and  $G_2$ . A graph  $G$  is a *cograph* if  $G$  can be generated from  $K_1$  by a sequence of join and disjoint union operations. A graph is a cograph if and only if it is  $P_4$ -free. This implies the following well-known lemma.

**Lemma 10.** *Every connected  $P_4$ -free graph on at least two vertices has a spanning complete bipartite subgraph.*

Let  $G$  be a graph. For a set  $S$ , the graph  $G[S] = (S, \{uv \in E(G) \mid u, v \in S\})$  denotes the subgraph of  $G$  induced by  $S$ . Note that  $G[S]$  can be obtained from  $G$  by deleting every vertex of  $V(G) \setminus S$ . If  $G$  has a vertex weighting  $w$ , then  $w(S) = \sum_{u \in S} w(u)$  denotes the *weight* of  $S$ .

**Lemma 11.** *For every  $s \geq 0$ , VERTEX STEINER TREE can be solved in time  $O(n^{2s^2-s+5})$  for connected  $(sP_1 + P_4)$ -free graphs on  $n$  vertices.*

*Proof.* Let  $s \geq 0$  be an integer. Let  $G = (V, E)$  be a connected  $(sP_1 + P_4)$ -free graph with a vertex weighting  $w : V \rightarrow \mathbb{R}^+$  and set of terminals  $U$ . We show how to solve the optimization version of VERTEX STEINER TREE on  $G$ . Let  $R \subseteq V \setminus U$  be such that  $G[U \cup R]$  is connected and, subject to this condition,  $U \cup R$  has minimum weight  $w(U \cup R)$ . Thus any spanning tree of  $G[U \cup R]$  is an optimal solution. Let us consider the possible size of  $R$ .

First suppose that  $G[U \cup R]$  is  $P_4$ -free. Then, by Lemma 10,  $G[U \cup R]$  has a spanning complete bipartite subgraph. That is, there is a bipartition  $(A, B)$  of  $U \cup R$  such that every vertex in  $A$  is joined to every vertex in  $B$  (and neither  $A$  nor  $B$  is the empty set). If  $U$  intersects both  $A$  and  $B$ , then  $G[U]$  is connected and  $|R| = 0$ . So let us assume that  $U \subseteq A$ , and so  $R \supseteq B$ . Then  $R \cap A = \emptyset$  since  $G[U \cup B]$  is connected. As we know that every vertex in  $A = U$  is joined to every vertex in  $B = R$ , we find that  $|R| = 1$ .

Suppose instead that  $G[U \cup R]$  contains an induced path  $P$  on four vertices. We call the connected components of  $G[U]$  *bad* if they do not intersect  $P$  or the neighbours of  $P$  in  $G$ . There are at most  $s - 1$  bad components; else,  $G$  contains an  $sP_1 + P_4$ . Let  $U^*$  be a subset of  $U$  that includes one vertex from each of these bad components. Then each vertex of  $G[U \cup R]$  belongs either to  $U$  or  $P$  or is an internal vertex of a shortest path in  $G[U \cup R]$  from  $P$  to a vertex of  $U^*$ . The number of internal vertices in such a shortest path is at most  $2s + 1$ ; else, the path contains an induced  $sP_1 + P_4$ . As  $R$  is a subset of  $P$  and these internal vertices, we find that  $|R| \leq 4 + (2s + 1)(s - 1) = 2s^2 - s + 3$ .

So in all cases  $R$  contains at most  $2s^2 - s + 3$  vertices and our algorithm is just to consider every such set  $R$  and check, in each case, whether  $G[U \cup R]$  is connected. Our solution is the smallest set found that satisfies the connectivity constraint. As there are  $O(n^{2s^2-s+3})$  sets to consider, and checking connectivity takes  $O(n^2)$  time, the algorithm requires  $O(n^{2s^2-s+5})$  time.  $\square$

### 3.3 The Proof

We are now ready to prove our second dichotomy.

**Theorem 2 (restated)** *Let  $H$  be a graph. For every  $s \geq 0$ , VERTEX STEINER TREE is polynomial-time solvable for  $H$ -free graphs if  $H$  is an induced subgraph of  $sP_1 + P_4$ ; otherwise even unweighted VERTEX STEINER TREE is NP-complete.*

*Proof.* If  $H$  has a cycle, then we apply Theorem 5 or Theorem 6. Hence, we may assume that  $H$  has no cycle, so  $H$  is a forest. If  $H$  contains a vertex of degree at least 3, then the class of  $H$ -free graphs contains the class of claw-free graphs,

which in turn contains the class of line graphs. Hence, we can apply Lemma 9. Thus we may assume that  $H$  is a linear forest. If  $H$  contains a connected component with at least five vertices or two connected components with at least two vertices each, then the class of  $H$ -free graphs contains the class of  $2P_2$ -free graphs. Hence, we can apply Theorem 6. It remains to consider the case where  $H$  is an induced subgraph of  $sP_1 + P_4$  for some  $s \geq 0$ , for which we can apply Lemma 11.  $\square$

## 4 Conclusions

We presented complexity dichotomies both for EDGE STEINER TREE restricted to  $(H_1, H_2)$ -free graphs and for VERTEX STEINER TREE for  $H$ -free graphs. The latter dichotomy also holds for the unweighted variant, in which case the problems EDGE STEINER TREE and VERTEX STEINER TREE are polynomially equivalent. In particular, we observed that EDGE STEINER TREE can be solved in polynomial time for  $(H_1, H_2)$ -free graphs if and only if the class of  $(H_1, H_2)$ -free graphs has bounded treewidth. This correspondence is not true in general.

**Theorem 7.** *There exists a hereditary graph class  $\mathcal{G}$  of unbounded treewidth for which EDGE STEINER TREE can be solved in polynomial time.*

*Proof.* Let  $\mathcal{G}$  consist of graphs  $G$  of maximum degree at most 3 such that every path between any two degree-3 vertices in  $G$  has at least  $2^r$  vertices, where  $r$  is the number of degree-3 vertices in  $G$ . As deleting a vertex neither increases the maximum degree of a graph nor decreases the number of vertices on paths between degree-3 vertices,  $\mathcal{G}$  is hereditary (note that  $\mathcal{G}$  is also closed under taking edge deletion). As  $\mathcal{G}$  contains subdivided walls of arbitrarily large height, the treewidth of  $\mathcal{G}$  is unbounded due to Theorem 4.

We solve EDGE STEINER TREE on an instance  $(G, w, U, k)$  with  $G \in \mathcal{G}$  as follows. If  $G$  has at most one vertex of degree 3, then  $G$  has treewidth at most 2, so we can apply Lemma 5. Otherwise, we apply the following rules, while possible.

**Rule 1.** There is a non-terminal  $x$  of degree 2. Let  $xy$  and  $xz$  be its two incident edges. We contract  $xy$  and give the new edge weight  $w(xy) + w(xz)$ . If there was already an edge between  $y$  and  $z$ , then we remove one with largest weight.

**Rule 2.** There is a terminal  $x$  of degree 2 and its neighbours  $y$  and  $z$  are also terminals. Assume  $w(xy) \leq w(xz)$ . We observe that there is an optimal solution that includes the edge  $xy$ . Hence, we may contract  $xy$  and decrease  $k$  by  $w(xy)$ .

**Rule 3.** There is a vertex  $x$  of degree 1. Let  $y$  be its neighbour. If  $x$  is not a terminal, then remove  $x$ . Otherwise, contract  $xy$  and decrease  $k$  by  $w(xy)$ .

Let  $(G', w', U', k')$  be the resulting instance, which is readily seen to be equivalent to  $(G, w, U, k)$ . Then  $G'$  has  $r$  vertices of degree 3 and each vertex of degree at most 2 has a neighbour of degree 3; otherwise, one of Rules 1–3 applies. So,  $G'$  has at most  $4r$  vertices and thus  $O(r)$  edges. It remains to solve EDGE STEINER TREE on  $(G', w', U', k)$ . We do this in  $r \cdot 2^{O(r)}$  time by guessing for each edge

in  $G'$  if it is in the solution and then verifying the resulting candidate solution. As  $r \geq 2$ , we have  $|V(G)| \geq 2^r$ . So, the running time is polynomial in  $|V(G)|$ .  $\square$

As the hereditary graph class  $\mathcal{G}$  in Theorem 7 has an infinite family  $\mathcal{H}_{\mathcal{G}}$  of forbidden induced subgraphs, we pose the following open problem.

**Open Problem 1** *Is EDGE STEINER TREE polynomial-time solvable for any finitely defined hereditary graph class  $\mathcal{G}$  if and only if  $\mathcal{G}$  has bounded treewidth?*

So far, we have not found any counterexample to Open Problem 1, and to increase our understanding we first aim to consider classes of  $(H_1, H_2, H_3)$ -free graphs. The graph  $S_{h,i,j}$ , for  $1 \leq h \leq i \leq j$ , is the *subdivided claw*, which is the tree with one vertex  $x$  of degree 3 and exactly three leaves, which are of distance  $h$ ,  $i$  and  $j$  from  $x$ , respectively. Note that  $S_{1,1,1} = K_{1,3}$  and that both walls and net-walls may contain arbitrarily large subdivided claws. Note also that complete graphs are  $C_4$ -free and complete bipartite graphs are  $C_3$ -free. As such we pose the following open problem.

**Open Problem 2** *For every subdivided claw  $S$ , does the class of  $(C_3, C_4, S)$ -free graphs have bounded treewidth?*

We also propose to consider VERTEX STEINER TREE and unweighted VERTEX STEINER TREE for  $(H_1, H_2)$ -free graphs as future research. To obtain a dichotomy, we need to answer several open problems, including the next ones.

**Open Problem 3** *Does there exist a pair  $(H_1, H_2)$  such that VERTEX STEINER TREE and unweighted VERTEX STEINER TREE have different complexities for  $(H_1, H_2)$ -free graphs?*

For Open Problem 3 we note the following. If there exists such a pair  $(H_1, H_2)$ , then the mim-width of  $(H_1, H_2)$ -free graphs must be unbounded, as VERTEX STEINER TREE is polynomial-time solvable for graph classes of bounded mim-width [1].

**Open Problem 4** *For every integer  $t$ , determine the complexity of VERTEX STEINER TREE for  $(K_{1,3}, P_t)$ -free graphs.*

For Open Problem 4 we note that VERTEX STEINER TREE is polynomial-time solvable for  $P_4$ -free graphs by Theorem 2. It is known that  $(K_{1,3}, P_5)$ -free graphs have unbounded mim-width [6]. Hence, the first open case is where  $t = 5$ . To obtain an answer to Open Problem 4, we need new insights into the structure of  $(K_{1,3}, P_t)$ -free graphs. These insights may also be useful to obtain new results for other problems, such as the GRAPH COLOURING problem restricted to  $(K_{1,3}, P_t)$ -free graphs (see [15,18]).

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