∞-categories with duality and hermitian multiplicative ∞-loop space machines

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Abstract

We show that any preadditive ∞-category with duality gives rise to a direct sum hermitian K-theory spectrum. This assignment is lax symmetric monoidal, thereby producing E_∞-ring spectra from preadditive symmetric monoidal ∞-categories with duality. To have examples of preadditive symmetric monoidal ∞-categories with duality we show that any preadditive symmetric monoidal ∞-category, in which every object admits a dual, carries a canonical duality. Moreover we classify and twist the dualities in various ways and apply our definitions for example to finitely generated projective modules over E_∞-ring spectra.

Contents

1 Introduction

K-theory has by now a longstanding history. Vector bundles on topological spaces or schemes give rise to the K_0-functor and also higher K-theory ([12]). Algebraic K-theory can be defined for any stable ∞-category ([3]) or Waldhausen (∞-) category ([18], [1]).

Direct sum K-theory of a symmetric monoidal ∞-category $\mathcal{C}$ can be defined to be the spectrum $K(\mathcal{C})$ associated to the group completion of the E_∞-space $\mathcal{C}^\simeq$, the maximal subspace of $\mathcal{C}$ ([4]). It was shown in loc. cit. that the functor K is in fact lax symmetric monoidal, using a symmetric monoidal structure on the ∞-category of small symmetric monoidal ∞-categories resembling the one on E_∞-spaces (defined in [11] for Γ-spaces and studied in [15]).

Instead of vector bundles one can consider vector bundles equipped with non-degenerate symmetric or anti-symmetric bilinear forms giving rise to the Grothendieck-Witt group and hermitian K-theory. Motivated by direct sum K-theory we consider in this article a version of direct sum hermitian K-theory. Given a symmetric monoidal category $\mathcal{C}$ with duality, e.g. finitely generated projective R-modules $\mathcal{P}(R)$ over a commutative ring R with the direct sum as symmetric monoidal structure and the usual duality, one can consider the symmetric monoidal groupoid of the symmetric monoidal category C_h of hermitian objects of $\mathcal{C}$ and the spectrum associated to its group completion. This version of hermitian K-theory coincides with the common definitions in the case of $\mathcal{P}(R)$ (see [13] and [14, Theorem A.1] if 2 is invertible in R and [6, Theorem A] in general, compare also to [7, 1.6. Theoreme]).

Direct sum K-theory is also meaningful in the case of modules over E_∞-ring spectra, e.g. the direct sum K-theory of finitely generated projective R-modules for R a connective E_∞-ring spectrum is the connective algebraic K-theory of R (see e.g. [9, Theorem 5]). We expect a similar behavior in the case of hermitian K-theory. In this article we exhibit perfect or finitely generated projective modules over an E_∞-ring spectrum as an ∞-category with duality together with the relevant symmetric monoidal structures, to which direct sum hermitian K-theory can be applied, thereby producing an E_∞-ring spectrum.

Precisely, we define ∞-categories with duality as homotopy fixed points of the action of the group on two elements C_2 on the ∞-category Cat_∞ of small ∞-categories, where the non-trivial element of C_2 sends an ∞-category $\mathcal{C}$ to its opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ (Definition 2.3). Similarly we define symmetric monoidal ∞-categories with duality as homotopy C_2-fixed points of the induced C_2-action on symmetric monoidal ∞-categories, which are equivalently given by commutative monoids of ∞-categories with duality. For these symmetric monoidal ∞-categories with duality we construct a hermitian K-theory that carries a multiplicative structure: We construct a closed symmetric monoidal structure on symmetric monoidal ∞-categories with duality (Proposition 3.6) and promote hermitian K-theory to a lax symmetric monoidal functor from symmetric monoidal ∞-categories with duality to spectra (Proposition 4.1). Moreover we refine hermitian K-theory to direct sum real K-theory KR that assigns to any symmetric monoidal ∞-category with duality a genuine C_2-spectrum, whose fixed points are hermitian K-theory, and promote KR to a lax symmetric monoidal functor (Proposition 4.2). As a consequence real K-theory gives an genuine C_2-E_∞-ring spectrum for any commutative algebra with respect to the closed symmetric monoidal structure on symmetric monoidal ∞-categories with duality, which we call bimonoidal ∞-categories with duality. To produce such we identify preadditive symmetric monoidal ∞-categories with duality, whose tensor product preserves sums in each component, with preadditive bimonoidal ∞-categories with duality, whose underlying symmetric monoidal structures are the given one and the one, for which the tensor product is the sum (Theorem 3.5). So real K-theory gives an genuine C_2-E_∞-ring spectrum for any preadditive symmetric monoidal ∞-category with duality, whose tensor product preserves sums in each component.

In section 5 we classify and twist certain dualities: We classify dualities on a fixed symmetric monoidal ∞-category $\mathcal{C}$ with duality that are compatible with the action of $\mathcal{C}$ on itself and the given duality on $\mathcal{C}$. We prove that such dualities are classified by invertible objects in $\mathcal{C}$ that are homotopically fixed by the C_2-action on $\mathcal{C}$ twisted by the C_2-action inverting an object, in other words by $\mathrm{Pic}(\mathcal{C})^{hC_2}$, where $\mathrm{Pic}(\mathcal{C})$ carries the respective C_2-action (Theorem 5.2). Besides that we classify symmetric monoidal dualities: We prove that the C_2-action on symmetric monoidal ∞-categories sending a symmetric monoidal ∞-category to its opposite restricts to the trivial C_2-action on rigid symmetric monoidal ∞-categories (Theorem 5.11). To show this theorem we prove that any rigid symmetric monoidal ∞-category $\mathcal{C}$ carries a canonical symmetric monoidal duality sending an object $X \in \mathcal{C}$ to its dual. To organize the assignment $\mathcal{C} \ni X \mapsto X^\vee$ to a (homotopy-coherent) duality we introduce another model for dualities: We prove that a duality on an ∞-category $\mathcal{C}$ is the same datum as a presheaf on $(\mathcal{C} \times \mathcal{C})_{hC_2}$, whose restriction to $\mathcal{C} \times \mathcal{C}$ classifies a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathsf{Spc}$ adjoint to a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Spc})$, where Spc denotes the ∞-category of spaces, that factors through the Yoneda-embedding $\mathcal{C} \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Spc})$ and induces an equivalence $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{C}$ (Theorem 5.9). More generally given a tensor invertible object $\mathbb{L}$ of $\mathcal{C}$ we construct a duality on $\mathcal{C}$ sending an object $X \in \mathcal{C}$ to $X^\vee \otimes \mathbb{L}$.

In section 6 we discuss applications e.g. to finitely generated projective modules over E_∞-ring spectra making use of our twisting techniques to also define symplectic versions of hermitian K-theory.

Moreover we classify the dualities on compact spectra. Using an argument from equivariant homotopy theory provided to us by Niko Naumann (Lemma 6.3) we deduce that there are uncountably many inequivalent dualities on the ∞-category of compact spectra (Proposition 6.4).

In the paper [16] hermitian and real K-theory for stable ∞-categories with duality is developed using an ∞-categorical version of the hermitian $S_\bullet$-construction thereby obtaining a suitable theory which goes beyond direct sum hermitian K-theory. In the paper [5] real K-theory for Waldhausen ∞-categories with genuine duality is developed producing a real K-theory functor universal among additive theories.

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2 ∞-categories with duality

In this section we introduce the notion of ∞-category with duality and symmetric monoidal ∞-category with duality.

We first fix some notation: For a group G and an ∞-category $\mathcal{C}$ denote $\mathcal{C}[\mathrm{G}] := \mathrm{Fun}(\mathrm{BG}, \mathcal{C})$ the ∞-category of objects of $\mathcal{C}$ with G-action, where BG is the ∞-category with connected space of objects and G as automorphisms of any object. Pulling back along the map $\mathrm{BG} \to *$ induces a functor $\mathrm{tr} : \mathcal{C} \to \mathcal{C}[\mathrm{G}]$ that endows an object of $\mathcal{C}$ with trivial G-action.

For $X \in \mathcal{C}[G]$ we denote by $X^{hG} := \lim X \in \mathcal{C}$ the homotopy fixed points of the G-action on X if this limit exists. If all BG-shaped limits exist in $\mathcal{C}$, the functor $\mathrm{tr} : \mathcal{C} \to \mathcal{C}[G]$ admits a right adjoint $(-)^{hG}$ that sends X to X^{hG}.

For $\mathcal{C}$ an ∞-category, $X \in \mathcal{C}$ and a space K we denote by $X^K \in \mathcal{C}$ the cotensor if it exists. For $X \in \mathcal{C}[G]$ with trivial G-action we have $X^{hG} \simeq X^{BG}$.

We set

$$C_2 := \mathbb{Z}/2\mathbb{Z}.$$

We call objects of $\mathsf{Cat}_\infty[C_2]$ ∞-categories with C_2-action.

In the following we equip the ∞-category Cat_∞ of small ∞-categories with a canonical C_2-action that sends a small ∞-category to its opposite ∞-category. Then we define small ∞-categories with duality as C_2-homotopy fixed points with respect to this C_2-action on Cat_∞.

We start with the following proposition:

Proposition 2.1. *There is a unique non-trivial C_2-action on Cat_∞. Precisely, the fiber $\{\mathsf{Cat}_\infty\} \times_{\widehat{\mathsf{Cat}}_\infty} \widehat{\mathsf{Cat}}_\infty[C_2]$ is a discrete space with two connected components.*

Proof. By Proposition 2.2 the monoidal ∞-category $\mathrm{Aut}(\mathsf{Cat}_\infty)$ of autoequivalences of Cat_∞ is equivalent as monoidal ∞-category to C_2. Hence the space of C_2-actions on Cat_∞, i.e. the fiber $\{\mathsf{Cat}_\infty\} \times_{\widehat{\mathsf{Cat}}_\infty} \widehat{\mathsf{Cat}}_\infty[C_2]$, is equivalent to the set of group endomorphisms of C_2, which is a set of cardinality 2.

□

We used the following proposition:

Proposition 2.2. *(Toën) The monoidal ∞-category $\mathrm{Aut}(\mathsf{Cat}_\infty)$ of autoequivalences of Cat_∞ is equivalent as monoidal ∞-category to C_2.*

Proof. This is due to [17, Th. 6.3] (see also [2, Theorem 8.12]): Every autoequivalence of Cat_∞ restricts to an autoequivalence of $\Delta \subset \mathsf{Cat}_\infty$. So restriction defines a map ϕ of grouplike A_∞-spaces $\mathrm{Aut}(\mathsf{Cat}_\infty) \to \mathrm{Aut}(\Delta)$. As Cat_∞ is a localization of $\mathcal{P}(\Delta)$, the map ϕ is an embedding. One has an isomorphism of groups $\mathrm{Aut}(\Delta) \cong C_2$. So ϕ is an equivalence.

□

Definition 2.3. *We define the ∞-category of small ∞-categories with duality resp. small symmetric monoidal ∞-categories with duality by*

$$\mathsf{Cat}_\infty^{hC_2} \text{ resp. } \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2} \simeq \mathrm{Cmon}(\mathsf{Cat}_\infty^{hC_2}),$$

where we use the unique non-trivial C_2-action on Cat_∞ of Proposition 2.1.

Note that the C_2-action on Cat_∞ restricts to the full subcategory Spc of spaces. This restricted C_2-action is trivial and the only C_2-action on Spc as $\mathrm{Aut}(\mathsf{Spc})$ is contractible. So we have

$$\mathsf{Spc}^{hC_2} \simeq \mathsf{Spc}[C_2]$$

and a canonical left and right adjoint embedding

$$\mathsf{Spc}[C_2] \simeq \mathsf{Spc}^{hC_2} \subset \mathsf{Cat}_\infty^{hC_2}.$$

3 Symmetric monoidal and bimonoidal ∞-categories with duality

We let

$$\mathrm{Cmon}(\mathsf{Cat}_\infty)$$

be the ∞-category of commutative monoid objects in Cat_∞, which we identify with small symmetric monoidal ∞-categories and symmetric monoidal functors.

Denote Cat_∞^Π the subcategory of Cat_∞ of small ∞-categories with finite products and finite products preserving functors. We let

$$\mathrm{Cmon}(\mathsf{Cat}_\infty^\Pi) \subset \mathrm{Cmon}(\mathsf{Cat}_\infty)$$

be the ∞-category of commutative monoid objects in Cat_∞^Π (which will turn out to be equivalent to Cat_∞^Π), which we identify with the full subcategory of cartesian symmetric monoidal ∞-categories.

Denote $\mathsf{Pr}^L \subset \widehat{\mathsf{Cat}}_\infty$ the subcategory of presentable ∞-categories and left adjoint functors. The ∞-category Pr^L carries a closed symmetric monoidal structure. Denote $\mathrm{Calg}(\mathsf{Pr}^L)$ the ∞-category of commutative algebras in Pr^L, which have the following description: There is a lax symmetric monoidal inclusion $\mathsf{Pr}^L \subset \widehat{\mathsf{Cat}}_\infty$ that yields a subcategory inclusion $\mathrm{Calg}(\mathsf{Pr}^L) \subset \mathrm{Cmon}(\widehat{\mathsf{Cat}}_\infty)$ that identifies $\mathrm{Calg}(\mathsf{Pr}^L)$ with the subcategory of presentable closed symmetric monoidal ∞-categories and left adjoint symmetric monoidal functors (see [10, Proposition 5.8.1.15]).

We start with the following proposition:

Proposition 3.1. *The unique non-trivial C_2-action on Cat_∞ is compatible with the cartesian symmetric monoidal structure.*

Precisely, if we endow Cat_∞ with the cartesian symmetric monoidal structure (that is closed), the fiber

$$\{\mathsf{Cat}_\infty\} \times_{\mathrm{Calg}(\mathsf{Pr}^L)} \mathrm{Calg}(\mathsf{Pr}^L)[C_2] \simeq \{\mathsf{Cat}_\infty\} \times_{\mathrm{Cmon}(\widehat{\mathsf{Cat}}_\infty)} \mathrm{Cmon}(\widehat{\mathsf{Cat}}_\infty)[C_2]$$

is a discrete space with two connected components.

Proof. We have a canonical equivalence

$$\{\mathsf{Cat}_\infty\} \times_{\widehat{\mathsf{Cat}}_\infty^\Pi} \widehat{\mathsf{Cat}}_\infty^\Pi[C_2] \simeq \{\mathsf{Cat}_\infty\} \times_{\widehat{\mathsf{Cat}}_\infty} \widehat{\mathsf{Cat}}_\infty[C_2] \simeq * \sqcup *.$$

By Proposition 3.3 the forgetful functor

$$\mathrm{Cmon}(\widehat{\mathsf{Cat}}_\infty^\Pi) \to \widehat{\mathsf{Cat}}_\infty^\Pi$$

is an equivalence. So we get a canonical equivalence

$$* \sqcup * \simeq \{\mathsf{Cat}_\infty\} \times_{\mathrm{Cmon}(\widehat{\mathsf{Cat}}_\infty^\Pi)} \mathrm{Cmon}(\widehat{\mathsf{Cat}}_\infty^\Pi)[C_2] \simeq$$

$$\{\mathsf{Cat}_\infty\} \times_{\mathrm{Cmon}(\widehat{\mathsf{Cat}}_\infty)} \mathrm{Cmon}(\widehat{\mathsf{Cat}}_\infty)[C_2].$$

□

We used the following lemma, which uses the notion of preadditive ∞-category. We call an ∞-category $\mathcal{C}$ preadditive if it has a zero object and for any objects $\mathrm{X}, \mathrm{Y} \in \mathcal{C}$ the canonical morpism $\mathrm{X} \coprod \mathrm{Y} \to \mathrm{X} \times \mathrm{Y}$ is an equivalence.

Lemma 3.2. *The ∞-category Cat_∞^Π is preadditive.*

Proof. The ∞-category Cat_∞^Π admits finite products (in fact all small limits) that are preserved by the inclusion $\mathsf{Cat}_\infty^\Pi \subset \mathsf{Cat}_\infty$. Let $\mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathsf{Cat}_\infty^\Pi$. Denote $\mathrm{Fun}^\Pi(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by the functors preserving finite products, whose maximal subspace is the mapping space in Cat_∞^Π from $\mathcal{C}$ to $\mathcal{D}$.

If $*$ denotes the final object of Cat_∞^Π, the ∞-category $\mathrm{Fun}^\Pi(*, \mathcal{D})$ is contractible so that $*$ is a zero object of Cat_∞^Π. So we have canonical functors $\mathcal{B} \to \mathcal{B} \times \mathcal{C}, \mathcal{C} \to \mathcal{B} \times \mathcal{C}$ that are the identity respectively the zero morphism on each factor. We need to see that the canonical functor

$$\psi_{\mathcal{D}} : \mathrm{Fun}^\Pi(\mathcal{B} \times \mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^\Pi(\mathcal{B}, \mathcal{D}) \times \mathrm{Fun}^\Pi(\mathcal{C}, \mathcal{D})$$

is an equivalence. For any ∞-category K we have that $\mathrm{Fun}(\mathrm{K}, \psi_{\mathcal{D}})$ is canonically equivalent to $\psi_{\mathrm{Fun}(\mathrm{K},\mathcal{D})}$. So by Yoneda we may reduce to check that ψ induces a bijection on equivalence classes. An inverse of $\psi_{\mathcal{D}}$ is given by the functor

$$\mathrm{Fun}^\Pi(\mathcal{B}, \mathcal{D}) \times \mathrm{Fun}^\Pi(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^\Pi(\mathcal{B} \times \mathcal{C}, \mathcal{D} \times \mathcal{D}) \to \mathrm{Fun}^\Pi(\mathcal{B} \times \mathcal{C}, \mathcal{D}).$$

□

Corollary 3.3. *The forgetful functor*

$$\mathrm{Cmon}(\mathsf{Cat}_\infty^\Pi) \to \mathsf{Cat}_\infty^\Pi$$

is an equivalence.

In this section we define bimonoidal ∞-categories with duality, which are ∞-categories endowed with two symmetric monoidal structures and a duality all compatible among each other.

To define bimonoidal ∞-categories with duality we use the following symmetric monoidal localizations $\mathcal{L}$ on Pr^L from [4]:

$$\mathcal{C} \mapsto \mathrm{Cmon}(\mathcal{C}),\ \mathcal{C} \mapsto \mathrm{Grp}(\mathcal{C}),\ \mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$$

with local objects the preadditive, additive respectively stable presentable ∞-categories. Being symmetric monoidal the localization $\mathcal{L}$ gives rise to a localization

$$\mathrm{Calg}(\mathsf{Pr}^L)[C_2] \xrightarrow{\mathcal{L}} \mathrm{Calg}(\mathsf{Pr}^L)[C_2] \tag{1}$$

on commutative algebras with C_2-action.

So we get $\mathcal{L}(\mathsf{Cat}_\infty) \in \mathrm{Calg}(\mathsf{Pr}^L)[C_2]$ and especially

$$\mathrm{Cmon}(\mathsf{Cat}_\infty) \in \mathrm{Calg}(\mathsf{Pr}^L)[C_2].$$

Now we are able to define bimonoidal ∞-categories with duality.

We have a functor

$$\mathrm{Calg}(\mathsf{Pr}^L) \to \mathsf{Pr}^L,\ \mathcal{C} \mapsto \mathrm{Calg}(\mathcal{C})$$

that yiels a functor $\mathrm{Calg}(\mathsf{Pr}^L)[C_2] \to \mathsf{Pr}^L[C_2]$.

We define a functor

$$\mathrm{Rig} : \mathrm{Calg}(\mathsf{Pr}^L)[C_2] \xrightarrow{\mathrm{Cmon}} \mathrm{Calg}(\mathsf{Pr}^L)[C_2] \xrightarrow{\mathrm{Calg}} \mathsf{Pr}^L[C_2].$$

We call objects of $\mathrm{Rig}(\mathsf{Cat}_\infty)$ bimonoidal ∞-categories and objects of $\mathrm{Rig}(\mathsf{Spc})$ bimonoidal spaces. The embedding $\mathsf{Spc} \subset \mathsf{Cat}_\infty$ in $\mathrm{Calg}(\mathsf{Pr}^L)[C_2]$ yields an embedding $\mathrm{Rig}(\mathsf{Spc}) \subset \mathrm{Rig}(\mathsf{Cat}_\infty)$ in $\mathsf{Pr}^L[C_2]$.

Definition 3.4. *A bimonoidal ∞-category with duality is an object of*

$$\mathrm{Rig}(\mathsf{Cat}_\infty)^{hC_2}.$$

We have two C_2-equivariant forgetful functors

$$((-),+) : \mathrm{Rig}(\mathsf{Cat}_\infty) = \mathrm{Calg}(\mathrm{Cmon}(\mathsf{Cat}_\infty)) \to \mathrm{Cmon}(\mathsf{Cat}_\infty)$$

by forgetting the commutative algebra structure with respect to the closed symmetric monoidal structure on $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ and

$$((-),\bullet) : \mathrm{Rig}(\mathsf{Cat}_\infty) = \mathrm{Calg}(\mathrm{Cmon}(\mathsf{Cat}_\infty)) \to \mathrm{Calg}(\mathsf{Cat}_\infty) \simeq \mathrm{Cmon}(\mathsf{Cat}_\infty),$$

induced by the lax symmetric monoidal C_2-equivariant forgetful functor

$$\mathrm{Cmon}(\mathsf{Cat}_\infty) \to \mathsf{Cat}_\infty$$

right adjoint to the symmetric monoidal free functor $\mathsf{Cat}_\infty \to \mathrm{Cmon}(\mathsf{Cat}_\infty)$ that is the unit of the localization 1.

We call a bimonoidal ∞-category $\mathcal{C}$ (with duality) preadditive if $\mathcal{C}$ is preadditive and $(\mathcal{C},+)$ is cartesian symmetric monoidal. We call a symmetric monoidal ∞-category (with duality) preadditive if its underlying ∞-category is preadditive and its tensor product preserves finite sums in both variables.

Denote $\mathsf{Cat}_\infty^{\mathrm{preadd}} \subset \mathsf{Cat}_\infty^{\Pi}$ the full subcategory of preadditive ∞-categories. By Corollary 3.7 the ∞-category $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ is a presentable symmetric monoidal ∞-category with C_2-action and the inclusion $\mathsf{Cat}_\infty^{\mathrm{preadd}} \subset \mathsf{Cat}_\infty$ is a lax symmetric monoidal C_2-equivariant functor. So we get an induced inclusion

$$\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \subset \mathrm{Calg}(\mathsf{Cat}_\infty)^{hC_2} \simeq \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}$$

that identifies $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2}$ with the subcategory of preadditive symmetric monoidal ∞-categories with duality and symmetric monoidal functors preserving the duality and finite sums.

Now we are ready to state the main theorem of this section:

Theorem 3.5. *There is a canonical localization*

$$\mathrm{Rig}(\mathsf{Cat}_\infty)^{hC_2} \to \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2},$$

whose fully faithful right adjoint identifies preadditive symmetric monoidal ∞-categories with duality with preadditive bimonoidal ∞-categories with duality.

Proposition 3.6. *The ∞-categories $\mathsf{Cat}_\infty^{\mathrm{preadd}}, \mathsf{Cat}_\infty^{\Pi}$ are presentable symmetric monoidal ∞-categories and the embedding*

$$\mathsf{Cat}_\infty^{\mathrm{preadd}} \subset \mathsf{Cat}_\infty^{\Pi}$$

admits a symmetric monoidal left adjoint.

Proof. Let $\mathcal{C} \in \mathsf{Cat}_\infty^{\mathrm{preadd}}$ and $\mathcal{D} \in \mathsf{Cat}_\infty^{\Pi}$. The forgetful functor $\mathrm{Cmon}(\mathcal{C}) \to \mathcal{C}$ is an equivalence and the functor

$$\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cmon}(\mathcal{D})) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{D})$$

is inverse to the functor

$$\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}^{\Pi}(\mathrm{Cmon}(\mathcal{C}),\mathrm{Cmon}(\mathcal{D})) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C},\mathrm{Cmon}(\mathcal{D})).$$

Here Fun^{Π} refers to finite products preserving functors. As $\mathrm{Cmon}(\mathcal{D})$ is preadditive, this shows that the embedding $\mathsf{Cat}_\infty^{\mathrm{preadd}} \subset \mathsf{Cat}_\infty^{\Pi}$ is left adjoint to the functor

$$\mathrm{Cmon}(-) : \mathsf{Cat}_\infty^{\Pi} \to \mathsf{Cat}_\infty^{\mathrm{preadd}}$$

that preserves filtered colimits. The full subcategory $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ is closed under small limits in $\mathsf{Cat}_\infty^{\Pi}$. This implies that $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ is presentable as $\mathsf{Cat}_\infty^{\Pi}$ is presentable and that the embedding $\mathsf{Cat}_\infty^{\mathrm{preadd}} \subset \mathsf{Cat}_\infty^{\Pi}$ admits a left adjoint.

The ∞-category $\mathsf{Cat}_\infty^{\Pi}$ carries a closed symmetric monoidal structure with internal hom Fun^{Π} such that the inclusion $\mathsf{Cat}_\infty^{\Pi} \subset \mathsf{Cat}_\infty$ admits a symmetric monoidal left adjoint ([10, Corollary 4.8.1.4]).

Thus to see that $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ carries a closed symmetric monoidal structure such that the embedding $\mathsf{Cat}_\infty^{\mathrm{preadd}} \subset \mathsf{Cat}_\infty^{\Pi}$ admits a symmetric monoidal left adjoint it is enough to check that local equivalences are stable under the tensor product or equivalently that for every $\mathrm{X} \in \mathsf{Cat}_\infty^{\Pi}$ and $\mathrm{Z} \in \mathsf{Cat}_\infty^{\mathrm{preadd}}$ the internal hom $\mathrm{Fun}^{\Pi}(\mathrm{X},\mathrm{Z})$ is preadditive, which follows from the fact that finite products in $\mathrm{Fun}^{\Pi}(\mathrm{X},\mathrm{Z})$ are formed objectwise.

□

By Lemma 3.2 the ∞-category $\mathsf{Cat}_\infty^{\Pi}$ is preadditive and so also its full subcategory $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ closed under finite products. So the forgetful functor

$$\mathrm{Cmon}(\mathsf{Cat}_\infty^{\mathrm{preadd}}) \to \mathsf{Cat}_\infty^{\mathrm{preadd}}$$

is an equivalence.

Corollary 3.7. *The ∞-category $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ is a presentable symmetric monoidal ∞-category with C_2-action and the symmetric monoidal left adjoint ξ of the inclusion $\mathsf{Cat}_\infty^{\mathrm{preadd}} \subset \mathsf{Cat}_\infty$ is C_2-equivariant.*

Proof. The symmetric monoidal left adjoint ξ yields a symmetric monoidal localization

$$\psi : \mathrm{Cmon}(\mathsf{Cat}_\infty) \to \mathrm{Cmon}(\mathsf{Cat}_\infty^{\mathrm{preadd}}).$$

Thus with $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ also the full subcategory $\mathrm{Cmon}(\mathsf{Cat}_\infty^{\mathrm{preadd}}) \simeq \mathsf{Cat}_\infty^{\mathrm{preadd}}$ is a presentable symmetric monoidal ∞-category with C_2-action such that ψ is a C_2-equivariant symmetric monoidal functor. The symmetric monoidal left adjoint ξ factors as

$$\mathsf{Cat}_\infty \xrightarrow{\mathrm{free}} \mathrm{Cmon}(\mathsf{Cat}_\infty) \xrightarrow{\psi} \mathrm{Cmon}(\mathsf{Cat}_\infty^{\mathrm{preadd}}) \simeq \mathsf{Cat}_\infty^{\mathrm{preadd}}.$$

□

Corollary 3.8. *We get a symmetric monoidal localization $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2} \to (\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2}$ so that the right adjoint $(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \to \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}$ is lax symmetric monoidal.*

Proof of Theorem 3.5. As $\mathsf{Cat}_\infty^{\mathrm{preadd}}$ is preadditive, the forgetful functor

$$\mathrm{Rig}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \to \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2}$$

is an equivalence.

By Corollary 3.7 the symmetric monoidal left adjoint ξ of the inclusion $\mathsf{Cat}_\infty^{\mathrm{preadd}} \subset \mathsf{Cat}_\infty^{\Pi} \subset \mathsf{Cat}_\infty$ is C_2-equivariant and so yields a localization

$$\mathrm{Rig}(\mathsf{Cat}_\infty)^{hC_2} \to \mathrm{Rig}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \simeq \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2},$$

whose right adjoint identifies preadditive symmetric monoidal ∞-categories with duality with preadditive bimonoidal ∞-categories with duality.

□

3.1 Hermitian objects

Denote $e : \Delta \to \Delta$ the functor $[n] \mapsto [n] * [n]^{\mathrm{op}} \simeq [2n+1]$, which we call edgewise subdivision. The functor e is C_2-equivariant if its source carries the trivial action and its target carries the unique non-trivial action. For every $[n] \in \Delta$ we have natural maps $[n] \to [n] * [n]^{\mathrm{op}}$, $[n]^{\mathrm{op}} \to [n] * [n]^{\mathrm{op}}$, i.e. natural transformations $\mathrm{id} \to e, (-)^{\mathrm{op}} \to e$.

Given a simplicial space $\mathcal{C}$ we define a map of simplicial spaces

$$\mathrm{Tw}(\mathcal{C}) \coloneqq \mathcal{C} \circ e^{\mathrm{op}} \to (\mathcal{C} \circ \mathrm{id}) \times (\mathcal{C} \circ (-)^{\mathrm{op}}) = \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

By functoriality of taking presheaves the induced functor $\mathrm{Tw} \coloneqq e^* : \mathcal{P}(\Delta) \to \mathcal{P}(\Delta)$ is C_2-equivariant if its target carries the trivial action and its source carries the induced non-trivial action. Passing to homotopy C_2-fix points Tw yields a functor $\mathcal{P}(\Delta)^{hC_2} \to \mathcal{P}(\Delta)[C_2]$, also denoted by Tw. We define $\mathcal{H}^{\mathrm{lax}}$ as the composition

$$\mathcal{P}(\Delta)^{hC_2} \xrightarrow{\mathrm{Tw}} \mathcal{P}(\Delta)[C_2] \xrightarrow{(-)^{hC_2}} \mathcal{P}(\Delta).$$

The functor $\mathcal{H}^{\mathrm{lax}}$ restricts to a functor $\mathsf{Cat}_\infty^{hC_2} \to \mathsf{Cat}_\infty$, which we denote by the same name. Here we view Cat_∞ as the full subcategory of $\mathcal{P}(\Delta)$ spanned by the complete Segal spaces, on which the C_2-action on $\mathcal{P}(\Delta)$ restricts giving the unique non-trivial C_2-action.

Definition 3.9. *For $\mathcal{C} \in \mathsf{Cat}_\infty^{hC_2}$ we call an object of $\mathcal{H}^{\mathrm{lax}}(\mathcal{C})$ a lax hermitian object of $\mathcal{C}$.*

We call a lax hermitian object of $\mathcal{C}$ a hermitian object if its image under the functor $\mathcal{H}^{\mathrm{lax}}(\mathcal{C}) \to \mathrm{Tw}(\mathcal{C})$ corresponds to an equivalence in $\mathcal{C}$.

We write

$$\mathcal{H}(\mathcal{C}) \subset \mathcal{H}^{\mathrm{lax}}(\mathcal{C})$$

for the full subcategory spanned by the hermitian objects of $\mathcal{C}$.

By the next Lemma 3.10 for every $\mathcal{C} \in \mathcal{P}(\Delta)^{hC_2}$ the map $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ of simplicial spaces is C_2-equivariant, where the C_2-action on $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ switches the factors and applies the dualities on $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$. So if $\mathcal{C}$ is a small ∞-category with duality, the C_2-action on $\mathcal{C}\times\mathcal{C}^{\mathrm{op}}$ sends (X,Y) to $(Y^\vee, X^\vee)$. Hence the map $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}\times\mathcal{C}^{\mathrm{op}}$ of simplicial spaces yields a map of simplicial spaces

$$\mathcal{H}^{\mathrm{lax}}(\mathcal{C}) = \mathrm{Tw}(\mathcal{C})^{hC_2} \to (\mathcal{C}\times\mathcal{C}^{\mathrm{op}})^{hC_2} \simeq \mathcal{C}.$$

If $\mathcal{C}$ is an ∞-category, $\mathrm{Tw}(\mathcal{C})$ is a complete Segal space and the map $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}\times\mathcal{C}^{\mathrm{op}}$ is a right fibration. So also the induced map $\mathcal{H}^{\mathrm{lax}}(\mathcal{C}) \to \mathcal{C}$ is a right fibration. In this case the fiber of the right fibration $\mathcal{H}^{\mathrm{lax}}(\mathcal{C}) \to \mathcal{C}$ over some object X of $\mathcal{C}$ is the space of homotopy C_2-fixed points of the fiber of the C_2-equivariant functor $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}\times\mathcal{C}$ over the fixed point $(X, X^\vee)$, which is $\mathrm{map}_{\mathcal{C}}(X, X^\vee)$ with the following C_2-action: A morphism $f : X \to X^\vee$ is sent to $f^\vee : X \simeq (X^\vee)^\vee \to X^\vee$.

So a lax hermitian object in $\mathcal{C}$ is roughly an object X of $\mathcal{C}$ together with a morphism $f : X \to X^\vee$ in $\mathcal{C}$ that is homotopic to $f^\vee : X \simeq (X^\vee)^\vee \to X^\vee$ and is a hermitian object if the morphism $f : X \to X^\vee$ is an equivalence.

Lemma 3.10. *The canonical natural transformation* $\mathrm{Tw} \to \mathrm{id}\times(-)^{\mathrm{op}}$ *is* C_2*-equivariant.*

Proof. By the naturality of the Yoneda-equivalence $\mathrm{Tw} = e^* : \mathcal{P}(\Delta) \to \mathcal{P}(\Delta)$ factors C_2-equivariantly as the Yoneda-embedding $\mathcal{P}(\Delta) \to \mathrm{Fun}(\mathcal{P}(\Delta)^{\mathrm{op}}, \mathsf{Spc})$ followed by restriction along the C_2-equivariant functor $\Delta \xrightarrow{e} \Delta \xrightarrow{\mathsf{y}} \mathcal{P}(\Delta)$, where y denotes the Yoneda-embedding.

The natural transformation $\mathrm{id} \to e$ yields a natural transformation $\mathsf{y} \to \mathsf{y}\circ e$ of functors $\Delta \to \mathcal{P}(\Delta)$ that is adjoint to a C_2-equivariant natural transformation $\alpha : \mathsf{y} \coprod (\mathsf{y}\circ(-)^{\mathrm{op}}) \to \mathsf{y}\circ e$ of functors $\Delta \to \mathcal{P}(\Delta)$, whose source carries the trivial action and whose target carries the induced non-trivial action.

Note that the natural transformation $\mathsf{y}\circ(-)^{\mathrm{op}} \to \mathsf{y}\circ e$ is induced by the natural transformation $(-)^{\mathrm{op}} \to e$.

Composing the C_2-equivariant Yoneda-embedding $\mathcal{P}(\Delta) \to \mathrm{Fun}(\mathcal{P}(\Delta)^{\mathrm{op}}, \mathsf{Spc})$ with restriction along α, i.e. the C_2-equivariant natural transformation

$$\alpha^* : (\mathsf{y}\circ e)^* \to \mathsf{y}^* \times (-)^{\mathrm{op}} \circ \mathsf{y}^* : \mathrm{Fun}(\mathcal{P}(\Delta)^{\mathrm{op}}, \mathsf{Spc}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Spc}),$$

we get a C_2-equivariant natural transformation $\mathrm{Tw} \to \mathrm{id}\times(-)^{\mathrm{op}}$.

□

4 Direct sum real K-theory

We denote by

$$(-)^{\simeq} : \mathsf{Cat}_\infty \to \mathsf{Spc}$$

the right adjoint of the embedding $\mathsf{Spc} \subset \mathsf{Cat}_\infty$.

The direct sum K-theory functor ([4, Definition 8.3]) is the composition

$$\mathrm{K} : \mathrm{Cmon}(\mathsf{Cat}_\infty) \xrightarrow{(-)^{\simeq}} \mathrm{Cmon}(\mathsf{Spc}) \to \mathrm{Grp}(\mathsf{Spc}) \to \mathrm{Sp}$$

of lax symmetric monoidal functors [4, Theorem 8.6].

Passing to commutative algebras we get a functor

$$\mathrm{Rig}(\mathsf{Cat}_\infty) \xrightarrow{(-)^{\simeq}} \mathrm{Rig}(\mathsf{Spc}) \to \mathrm{Ring}(\mathsf{Spc}) \to \mathrm{Calg}(\mathrm{Sp})$$

with $\mathrm{Ring}(\mathsf{Spc}) := \mathrm{Calg}(\mathrm{Grp}(\mathsf{Spc}))$.

Mimicking direct sum K-theory we define direct sum hermitian K-theory K_{h} as the K-theory of the hermitian objects:

$$\mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2} \xrightarrow{(-)^{\simeq}} \mathrm{Cmon}(\mathsf{Spc})[C_2] \xrightarrow{(-)^{hC_2}} \mathrm{Cmon}(\mathsf{Spc}) \to \mathrm{Grp}(\mathsf{Spc}) \to \mathrm{Sp},$$

which is the composition

$$\mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2} \xrightarrow{\mathcal{H}} \mathrm{Cmon}(\mathsf{Cat}_\infty) \xrightarrow{\mathrm{K}} \mathrm{Sp}.$$

Proposition 4.1. K_{h} *is lax symmetric monoidal.*

Proof. The first functor in the composition is right adjoint to the symmetric monoidal embedding $\mathrm{Cmon}(\mathsf{Spc})[C_2] \subset \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}$ and so lax symmetric monoidal. The second functor is right adjoint to the symmetric monoidal diagonal functor

$$\mathrm{Cmon}(\mathsf{Spc}) \to \mathrm{Cmon}(\mathsf{Spc})[C_2].$$

The third and fourth functors in the definition of K_{h} are symmetric monoidal. □

Passing to commutative algebras we get a functor

$$\mathrm{Rig}(\mathsf{Cat}_\infty)^{hC_2} \to \mathrm{Calg}(\mathrm{Sp}).$$

In the following we will lift direct sum hermitian K-theory to genuine C_2-spectra, which we define as next:

For any ∞-category $\mathcal{B}$ denote $\mathrm{Tw}(\mathcal{B})$ its twisted arrow ∞-category. For any ∞-category $\mathcal{C}$ we define the ∞-category $\mathrm{Span}(\mathcal{C})$ of spans in $\mathcal{C}$ as the ∞-category corresponding to the simplicial space

$$[n] \mapsto \mathsf{Cat}_\infty(\mathrm{Tw}(n), \mathcal{C})$$

that turns out to be a complete Segal space.

The functor $\mathrm{Span} : \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ preserves finite products and for any ∞-category $\mathcal{C}$ the natural transformation $\mathrm{Tw}(-) \to \mathrm{id}$ gives rise to a functor $\mathcal{C} \to \mathrm{Span}(\mathcal{C})$. Especially for any ∞-category $\mathcal{D}$ the canonical functor

$$\mathrm{Span}(\mathrm{Fun}(\mathcal{C},\mathcal{D})) \times \mathcal{C} \to \mathrm{Span}(\mathrm{Fun}(\mathcal{C},\mathcal{D})) \times \mathrm{Span}(\mathcal{C})$$

$$\to \mathrm{Span}(\mathrm{Fun}(\mathcal{C},\mathcal{D}) \times \mathcal{C}) \to \mathrm{Span}(\mathcal{D})$$

is adjoint to a functor

$$\theta : \mathrm{Span}(\mathrm{Fun}(\mathcal{C},\mathcal{D})) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Span}(\mathcal{D}))$$

that preserves finite products.

Denote Fin the category of finite sets. A genuine C_2-spectrum is a finite products preserving functor

$$\mathrm{Span}(\mathsf{Fin}[C_2]) \to \mathrm{Sp}.$$

We write

$$\mathrm{Sp}^{C_2} := \mathrm{Fun}^{\Pi}(\mathrm{Span}(\mathsf{Fin}[C_2]), \mathrm{Sp})$$

for the ∞-category of genuine C_2-spectra.

The ∞-category $\mathrm{Span}(\mathsf{Fin})$ is the free preadditive ∞-category on the singleton, i.e. for any preadditive ∞-category $\mathcal{C}$ evaluation at the image of the singleton under the functor $\mathsf{Fin} \to \mathrm{Span}(\mathsf{Fin})$ defines an equivalence

$$\mathrm{Fun}^{\Pi}(\mathrm{Span}(\mathsf{Fin}), \mathcal{C}) \simeq \mathcal{C}.$$

We define direct sum real K-theory KR as the composition:

$$\mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2} \simeq \mathrm{Fun}^{\Pi}(\mathrm{Span}(\mathsf{Fin}), \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}) \to$$

$$\mathrm{Fun}^{\Pi}(\mathrm{Span}(\mathsf{Fin})[C_2], \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}[C_2]) \xrightarrow{\mathrm{Fun}^{\Pi}(\theta,t)}$$

$$\mathrm{Fun}^{\Pi}(\mathrm{Span}(\mathsf{Fin}[C_2]), \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}) \xrightarrow{\mathrm{Fun}^{\Pi}(\mathrm{Span}(\mathsf{Fin}[C_2]), \mathrm{K}_{\mathsf{h}})} \mathrm{Sp}^{C_2} =$$
$$\mathrm{Fun}^{\Pi}(\mathrm{Span}(\mathsf{Fin}[C_2]), \mathrm{Sp}),$$

where t is the functor

$$\mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}[C_2] \simeq \mathrm{Cmon}(\mathsf{Cat}_\infty)^{h(C_2 \times C_2)} \to \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2},$$

where the last functor restricts the action via the diagonal.

For any symmetric monoidal ∞-category $\mathcal{C}$ with duality we have

$$\mathrm{KR}(\mathcal{C})^{\mathrm{u}} \simeq \mathrm{K}(\mathcal{C}), \quad \mathrm{KR}(\mathcal{C})^{C_2} \simeq \mathrm{K}(\mathcal{H}(\mathcal{C})),$$

where the superscript u denotes the underlying spectrum.

Proposition 4.2. KR *is lax symmetric monoidal.*

Proof. The functor $\theta : \mathrm{Span}(\mathsf{Fin}[C_2]) \to \mathrm{Span}(\mathsf{Fin})[C_2]$ preserves finite products, the functor $t : \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}[C_2] \to \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}$ is symmeric monoidal. So all functors in the composition of KR are symmetric monoidal except $\mathrm{Fun}^{\Pi}(\mathrm{Span}(\mathsf{Fin}[C_2]), \mathrm{K}_{\mathsf{h}})$, which is lax symmetric monoidal as K_{h} is (Proposition 4.1). □

We can compose KR with the lax symmetric monoidal embedding

$$(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \hookrightarrow \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}$$

of Corollary 3.8 to obtain a lax symmetric monoidal functor

$$\mathrm{KR}\colon (\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \to \mathrm{Sp}^{C_2}.$$

The effect of KR on bimonoidal ∞-categories with duality is the functor

$$\mathrm{KR} : \mathrm{Rig}(\mathsf{Cat}_\infty)^{hC_2} \to \mathrm{Calg}(\mathrm{Sp})^{C_2}.$$

Restricting KR to the full subcategory $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \subset \mathrm{Rig}(\mathsf{Cat}_\infty)^{hC_2}$ of Theorem 3.5 we get a functor

$$\mathrm{KR} : \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \to \mathrm{Calg}(\mathrm{Sp})^{C_2}.$$

So we can associate to any preadditive symmetric monoidal ∞-category with duality a real K-theory spectrum.

In the next section we endow any (preadditive) symmetric monoidal ∞-category, in which every object admits a dual, with a canonical duality sending an object to its dual to get a real K-theory spectrum from any (preadditive) symmetric monoidal ∞-category.

5 Symmetric monoidal and linear dualities

In this section we "classify" the following two types of dualities:

1. Symmetric monoidal ∞-categories carrying a duality (compatible with the symmetric monoidal structure), i.e. homotopy C_2-fixed points of $\mathrm{Cmon}(\mathsf{Cat}_\infty)$, in which every object has a dual.
2. Dualities on a fixed symmetric monoidal ∞-category $\mathcal{C}$ with duality compatible with the action of $\mathcal{C}$ on itself and the duality, i.e. objects of

$$\{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty^{hC_2}).$$

We start with 2. We need the following

Proposition 5.1. *Let G be a group, $\mathcal{D} \in \mathrm{Cmon}(\widehat{\mathsf{Cat}}_\infty)[G]$ such that $\mathcal{D}$ has small colimits that are preserved by the tensor product in both variables. Let $A \in \mathrm{Calg}(\mathcal{D})^{hG} \simeq \mathrm{Calg}(\mathcal{D}^{hG})$.*

Then there is a natural G-action on the symmetric monoidal ∞-category $\mathrm{Mod}_A(\mathcal{D})$ and an equivalence

$$\mathrm{Mod}_A(\mathcal{D}^{hG}) \simeq \mathrm{Mod}_A(\mathcal{D})^{hG}$$

of symmetric monoidal ∞-categories.

Proof. The commutative algebra A in $\mathcal{D}^{hG}$ uniquely lifts to a commutative algebra in $\mathrm{Calg}(\mathcal{D}^{hG})$ equipped with the cocartesian symmetric monoidal structure. So A defines a lax symmetric monoidal functor $* \to \mathrm{Calg}(\mathcal{D})^{hG}$ corresponding to a lax symmetric monoidal G-equivariant functor $* \to \mathrm{Calg}(\mathcal{D})$.

We have a pullback square

$$\begin{array}{ccc} \mathrm{Mod}_A(\mathcal{D}) & \longrightarrow & \mathrm{Mod}(\mathcal{D}) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Calg}(\mathcal{D}) \end{array}$$

of symmetric monoidal ∞-categories with G-action and lax symmetric monoidal G-equivariant functors. Note that the right vertical functor is a cocartesian fibration of symmetric monoidal ∞-categories with G-action as $\mathcal{D}$ has small colimits that are preserved by the tensorproduct in both variables. This implies that $\mathrm{Mod}_A(\mathcal{D})$ is a symmetric monoidal ∞-category with G-action.

Taking G-homotopy fixed points yields the pullback square

$$\begin{array}{ccc} \mathrm{Mod}_A(\mathcal{D})^{hG} & \longrightarrow & \mathrm{Mod}(\mathcal{D}^{hG}) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Calg}(\mathcal{D}^{hG}) \end{array}$$

of symmetric monoidal ∞-categories and lax symmetric monoidal functors inducing an equivalence

$$\mathrm{Mod}_A(\mathcal{D})^{hG} \simeq \mathrm{Mod}_A(\mathcal{D}^{hG})$$

of symmetric monoidal ∞-categories. □

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with duality. By Proposition 5.1 the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)$ carries an induced $\mathcal{C}_2$-action, whose homotopy C_2-fixed points are $\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty^{hC_2})$.

We call the E_∞-space

$$\{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)^{hC_2} \simeq \{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty^{hC_2})$$

the E_∞-space of $\mathcal{C}$-linear dualities on $\mathcal{C}$.

The symmetric monoidal duality on $\mathcal{C}$ yields a C_2-action on the E_∞-space $\mathcal{C}^{\simeq}$ that yields a C_2-action on the grouplike E_∞-space $\mathrm{Pic}(\mathcal{C})$.

We will prove the following theorem:

Theorem 5.2. *Let $\mathcal{C} \in \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}$. There is a canonical equivalence*

$$\mathrm{Pic}(\mathcal{C})^{hC_2} \simeq \{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty^{hC_2})$$

of E_∞-spaces sending a tensor invertible object $\mathbb{L} \in \mathrm{Pic}(\mathcal{C})^{hC_2}$ to a $\mathcal{C}$-linear duality $\mathrm{X} \mapsto \mathrm{X}^\vee \otimes \mathbb{L}$ on $\mathcal{C}$, where the C_2-action on $\mathrm{Pic}(\mathcal{C})$ is the diagonal action of the $C_2 \times C_2$-action induced by the C_2-action on $\mathrm{Pic}(\mathcal{C})$ coming from the duality on $\mathcal{C}$ and the C_2-action on $\mathrm{Pic}(\mathcal{C})$ that takes the inverse with respect to the group structure on $\mathrm{Pic}(\mathcal{C})$.

We deduce this from the following Lemmas:

Lemma 5.3. *The C_2-equivariant functor*

$$(-) \otimes_{\mathrm{Pic}(\mathcal{C})} \mathcal{C} : \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty) \to \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)$$

induces an equivalence

$$\{\mathrm{Pic}(\mathcal{C})\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Spc})} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Spc})^{hC_2} \simeq$$

$$\{\mathrm{Pic}(\mathcal{C})\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2} \simeq$$

$$\{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)^{hC_2}.$$

Proof. The C_2-equivariant adjunction

$$(-) \otimes_{\mathrm{Pic}(\mathcal{C})} \mathcal{C} : \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty) \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty) \tag{2}$$

induces an adjunction

$$\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2} \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)^{hC_2}$$

covering the adjunction 2.

For every $\mathrm{X} \in \{\mathrm{Pic}(\mathcal{C})\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2}$ the unit

$$\eta : \mathrm{X} \to \mathrm{X} \otimes_{\mathrm{Pic}(\mathcal{C})} \mathcal{C}$$

in $\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2}$ is cartesian over the canonical inclusion $\theta : \mathrm{Pic}(\mathcal{C}) \subset \mathrm{Pic}(\mathcal{C}) \otimes_{\mathrm{Pic}(\mathcal{C})} \mathcal{C} \simeq \mathcal{C}$ (like any map in $\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2}$ lying over θ).

For any $\mathrm{Y} \in \{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2}$ there is a cartesian lift $\mathrm{Y}' \to \mathrm{Y}$ of θ in $\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2}$. So we get a functor

$$\theta^* : \{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2} \to$$

$$\{\mathrm{Pic}(\mathcal{C})\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2}.$$

For any $\mathrm{Z} \in \{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)^{hC_2}$ the cartesian lift $\mathrm{Z}' \to \mathrm{Z}$ of θ in $\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2}$ induces a map $\mathrm{Z}' \otimes_{\mathrm{Pic}(\mathcal{C})} \mathcal{C} \to \mathrm{Z}$ in $\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)^{hC_2}$ (lying over the canonical equivalence $\mathrm{Pic}(\mathcal{C}) \otimes_{\mathrm{Pic}(\mathcal{C})} \mathcal{C} \simeq \mathcal{C}$ in $\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)$) and so is an equivalence.

Hence the induced functor $(-) \otimes_{\mathrm{Pic}(\mathcal{C})} \mathcal{C}$:

$$\{\mathrm{Pic}(\mathcal{C})\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2} \to \{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)^{hC_2}$$

is inverse to the composition

$$\{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)^{hC_2} \xrightarrow{\text{forget}} \{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2}$$

$$\xrightarrow{\theta^*} \{\mathrm{Pic}(\mathcal{C})\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Cat}_\infty)^{hC_2}.$$

□

The C_2-action on $\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Spc})$ restricts to the full subcategory $\mathrm{BPic}(\mathcal{C})$ spanned by $\mathrm{Pic}(\mathcal{C})$.

So we get a canonical equivalence

$$\{\mathrm{Pic}(\mathcal{C})\} \times_{\mathrm{BPic}(\mathcal{C})} (\mathrm{BPic}(\mathcal{C}))^{hC_2} \simeq \{\mathrm{Pic}(\mathcal{C})\} \times_{\mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Spc})} \mathrm{Mod}_{\mathrm{Pic}(\mathcal{C})}(\mathsf{Spc})^{hC_2}.$$

Set $X := \mathrm{BPic}(\mathcal{C}) \in \mathrm{Cmon}(\mathsf{Spc})[C_2]$ the E_∞-space with C_2-action.

Lemma 5.4. *Let $\mathcal{D}$ be a category with small limits and a zero object and $\mathrm{X} \in \mathcal{D}[C_2]$.*

We have a fiber sequence

$$\Omega(X)^{hC_2} \to X^{hC_2} \to X,$$

where the C_2-action on $\Omega(X)$ is the diagonal action of the $C_2 \times C_2$-action induced by the canonical C_2-action on $\Omega(X)$ and the C_2-action on $\Omega(X)$ that takes the inverse with respect to the group structure on $\Omega(X)$.

Proof. The map of sets $[0] \coprod [0] \to [0]$ is C_2-equivariant if $[0] \coprod [0]$ carries the C_2-action switching the summands. So taking cotensors in $\mathcal{D}[C_2]$ the diagonal morphism $\delta : X \to X \times X$ in $\mathcal{D}[C_2]$ is C_2-equivariant, i.e. a morphism of $\mathcal{D}[C_2][C_2]$, when X carries additionally the trivial action and $X \times X$ carries additionally the C_2-action switching the factors. Taking the diagonal action we get a C_2-equivariant morphism $\delta' : X \to X \times X$ in $\mathcal{D}$, whose fiber is $\Omega(X)$ with the C_2-action described above. Hence $\Omega(X)^{hC_2}$ is the fiber of the induced morphism $\delta'^{hC_2} : X^{hC_2} \to (X \times X)^{hC_2} \simeq \mathrm{X}$ that is the canonical morphism.

□

Corollary 5.5. *For $\mathcal{C} \in \mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}$ we have a canonical exact triangle*

$$\mathrm{pic}(\mathcal{C}) \to \mathrm{pic}(\mathcal{C})^{\tilde{h}C_2}_{\geq 0} \to (\mathrm{pic}(\mathcal{C})[1])^{hC_2}_{\geq 0} \to \mathrm{pic}(\mathcal{C})[1]$$

in Sp*, where $\tilde{h}$ means taking homotopy fixed points with respect to the action on* $\mathrm{pic}(\mathcal{C})$ *coming from the duality on $\mathcal{C}$ twisted by the action "multiplication by -1" and the first map in the triangle is induced by the norm map (i.e. sends* *and* $\mathbb{L} \in \mathrm{Pic}(\mathcal{C})$ *to* $\mathbb{L} \otimes \tau(\mathbb{L})$ *with the canonical fixed point datum, where* τ *is the action by the non-trivial element of* C_2*).*

The infinite loop space of the third entry in this exact triangle $(\mathrm{BPic}(\mathcal{C}))^{hC_2} \simeq \mathrm{Pic}(\mathcal{C})^{\tilde{h}C_2}/\mathrm{Pic}(\mathcal{C})$ *is the subgroupoid of* $\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty^{hC_2})$ *of all modules over* $\mathcal{C}$ *which are equivalent to* $\mathcal{C}$ *after forgetting the duality. Especially we have lax symmetric monoidal functors*

$$\mathrm{Pic}(\mathcal{C})^{\tilde{h}C_2} \to (\mathrm{BPic}(\mathcal{C}))^{hC_2} \to \mathsf{Cat}_\infty^{hC_2}.$$

To compose the last composition in Corollary 5.5 with the real K-theory functor we let $\mathcal{C} \in \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2}$.

Then as in Theorem 5.2 we have

$$\mathrm{Pic}(\mathcal{C})^{\tilde{h}C_2} \simeq \{\mathcal{C}\} \times_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty^{\mathrm{preadd}})} ((\mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2}),$$

thus we get lax symmetric monoidal functors

$$\mathrm{Pic}(\mathcal{C})^{\tilde{h}C_2} \to (\mathrm{BPic}(\mathcal{C}))^{hC_2} \to (\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2}$$

which we can compose with KR to obtain the lax symmetric monoidal functors

$$\mathrm{KR}_{\mathcal{C}}\colon \mathrm{Pic}(\mathcal{C})^{\tilde{h}C_2} \to (\mathrm{BPic}(\mathcal{C}))^{hC_2} \to \mathrm{Sp}^{C_2}.$$

Because of this factorization of $\mathrm{KR}_{\mathcal{C}}$ we get the following

Corollary 5.6. *Let* $\mathcal{C} \in \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2}$*. For* $\mathbb{L} \in \mathrm{Pic}(\mathcal{C})^{\tilde{h}C_2}$ *and* $\mathcal{M} \in \mathrm{Pic}(\mathcal{C})$ *the* $\mathrm{KR}(\mathcal{C})$*-module spectra* $\mathrm{KR}_{\mathcal{C}}(\mathbb{L})$ *and* $\mathrm{KR}_{\mathcal{C}}(\mathbb{L} \otimes \widetilde{(\mathcal{M} \otimes \tau(\mathcal{M}))})$*, where the tilde denotes the canonical fixed point datum, are canonically equivalent.*

Now we come to 1:

We call a symmetric monoidal ∞-category rigid if all its objects admit a dual. Given a rigid symmetric monoidal ∞-category $\mathcal{C}$ and an object $\mathbb{L} \in \mathcal{C}$ we will construct a lax duality (see below) on $\mathcal{C}$ sending Y to $\mathrm{Y}^\vee \otimes \mathbb{L}$:

The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is C_2-equivariant if the source carries the action switching the factors and the target carries the trivial action, and so is adjoint to a functor $(\mathcal{C} \times \mathcal{C})_{h\Sigma_2} \to \mathcal{C}$. The object $\mathbb{L} \in \mathcal{C}$ gives rise to a right fibration $\mathcal{C}_{/\mathbb{L}} \to \mathcal{C}$. So we may form the right fibration

$$(\mathcal{C} \times \mathcal{C})_{h\Sigma_2} \times_{\mathcal{C}} \mathcal{C}_{/\mathbb{L}} \to (\mathcal{C} \times \mathcal{C})_{h\Sigma_2},$$

whose pullback along the canonical functor $\mathcal{C} \times \mathcal{C} \to (\mathcal{C} \times \mathcal{C})_{h\Sigma_2}$ is the right fibration $(\mathcal{C} \times \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/\mathbb{L}} \to \mathcal{C} \times \mathcal{C}$ classifying the functor

$$\alpha : \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathsf{Spc},\ (\mathrm{X}, \mathrm{Y}) \mapsto \mathrm{map}_{\mathcal{C}}(\mathrm{X} \otimes \mathrm{Y}, \mathbb{L}) \simeq \mathrm{map}_{\mathcal{C}}(\mathrm{X}, \mathrm{Y}^\vee \otimes \mathbb{L}).$$

Hence the functor α is adjoint to a functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Spc}),\ \mathrm{Y} \mapsto \mathrm{map}_{\mathcal{C}}(-, \mathrm{Y}^\vee \otimes \mathbb{L})$$

that factors as $\beta : \mathcal{C}^{\mathrm{op}} \to \mathcal{C},\ \mathrm{Y} \mapsto \mathrm{Y}^\vee \otimes \mathbb{L}$ through the Yoneda-embedding. If $\mathbb{L}$ is tensor-invertible, we have

$$(\beta^{\mathrm{op}} \circ \beta)(\mathrm{Y}) = (\mathrm{Y}^\vee \otimes \mathbb{L})^\vee \otimes \mathbb{L} \simeq (\mathrm{Y}^\vee)^\vee \otimes \mathbb{L}^\vee \otimes \mathbb{L} \simeq \mathrm{Y}$$

(and so also $\beta \circ \beta^{\mathrm{op}} \simeq \mathrm{id}$) so that β is inverse to β^{op}.

In the following we will show that there is an equivalence between dualities on $\mathcal{C}$ and right fibrations $\phi : \mathcal{D} \to (\mathcal{C} \times \mathcal{C})_{hC_2}$ as the constructed one, i.e. right fibrations with the following property: The composition

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}})_{hC_2} \to \mathsf{Spc}$$

of the canonical functor with the functor classified by ϕ is adjoint to a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Spc})$ that induces an equivalence $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{C}$. By abuse of terminology we call such right fibrations $\phi : \mathcal{D} \to (\mathcal{C} \times \mathcal{C})_{hC_2}$ dualities on $\mathcal{C}$. If the functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Spc})$ only induces a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ (not neccessarily an equivalence), we call $\phi : \mathcal{D} \to (\mathcal{C} \times \mathcal{C})_{hC_2}$ a lax duality on $\mathcal{C}$.

Denote $\mathcal{R} \subset \mathrm{Fun}([1], \mathsf{Cat}_\infty)$ the full subcategory spanned by the right fibrations. We consider the pullback

$$\mathsf{Cat}_\infty \times_{\mathsf{Cat}_\infty} \mathcal{R}$$

of evaluation at the target $\mathcal{R} \to \mathsf{Cat}_\infty$ along the endofunctor of Cat_∞ sending $\mathcal{C}$ to $(\mathcal{C} \times \mathcal{C})_{hC_2}$. So an object of $\mathsf{Cat}_\infty \times_{\mathsf{Cat}_\infty} \mathcal{R}$ is a small ∞-category $\mathcal{C}$ and a right fibration $\phi : \mathcal{D} \to (\mathcal{C} \times \mathcal{C})_{hC_2}$.

The following theorem follows from the next lemma and theorem:

Theorem 5.7. *There is a canonical (non full) subcategory inclusion*

$$\mathsf{Cat}_\infty^{hC_2} \subset \mathsf{Cat}_\infty \times_{\mathsf{Cat}_\infty} \mathcal{R}$$

over Cat_∞*. The image precisely consists of the dualities.*

Lemma 5.8. *Taking homotopy C_2-coinvariants defines an equivalence*

$$\mathsf{Cat}_\infty \times_{\mathsf{Cat}_\infty[C_2]} \mathcal{R}[C_2] \simeq \mathsf{Cat}_\infty \times_{\mathsf{Cat}_\infty} \mathcal{R},$$

where the pullback on the left hand side is formed over the functor $\mathsf{Cat}_\infty \to \mathsf{Cat}_\infty[C_2]$ *that sends* $\mathcal{C}$ *to* $\mathcal{C} \times \mathcal{C}$ *equipped with the cofree C_2-action that switches the factors.*

Proof. As $\mathrm{B}C_2$ is a space, we have a canonical equivalence $\mathsf{Cat}_\infty[C_2] \simeq \mathsf{Cat}_{\infty/\mathrm{B}C_2}$.

Hence we have a right fibration $\mathsf{Cat}_\infty[C_2] \simeq \mathsf{Cat}_{\infty/\mathrm{B}C_2} \to \mathsf{Cat}_\infty$ that sends a small ∞-category with C_2-action $\mathcal{D}$ classified by a functor $X \to \mathrm{B}C_2$ to X, where X has the universal property of the coinvariants $\mathcal{D}_{hC_2}$. The right fibration $\mathsf{Cat}_\infty[C_2] \to \mathsf{Cat}_\infty$ yields an equivalence

$$\mathrm{Fun}([1], \mathsf{Cat}_\infty[C_2]) \simeq \mathsf{Cat}_\infty[C_2] \times_{\mathsf{Cat}_\infty} \mathrm{Fun}([1], \mathsf{Cat}_\infty)$$

over $\mathsf{Cat}_\infty[C_2]$ that restricts to an equivalence $\mathcal{R}[C_2] \simeq \mathsf{Cat}_\infty[C_2] \times_{\mathsf{Cat}_\infty} \mathcal{R}$ over $\mathsf{Cat}_\infty[C_2]$.

□

Theorem 5.9. *Sending a small ∞-category $\mathcal{C}$ with duality to its twisted arrow right fibration* $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ *defines a (non full) subcategory inclusion*

$$\mathsf{Cat}_\infty^{hC_2} \subset \mathsf{Cat}_\infty \times_{\mathsf{Cat}_\infty[C_2]} \mathcal{R}[C_2]$$

over Cat_∞*, where the pullback on the right hand side is formed over the functor* $\mathsf{Cat}_\infty \to \mathsf{Cat}_\infty[C_2]$ *that sends* $\mathcal{C}$ *to* $\mathcal{C}\times\mathcal{C}$ *equipped with the (cofree)* C_2*-action that switches the factors.*

The image are those C_2*-equivariant right fibrations* $\mathcal{B} \to \mathcal{C}\times\mathcal{C}$ *classifying a* C_2*-equivariant functor* $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}^{\mathrm{op}} \to \mathsf{Spc}$*, whose adjoint* $\mathcal{C}^{\mathrm{op}} \to \mathcal{P}(\mathcal{C})$ *factors as an equivalence* $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{C}$ *through the Yoneda-embedding.*

Proof. Given two objects

$$\mathrm{X} = (\mathcal{A},\mathcal{B},\phi:\mathcal{C}\to\mathcal{A}\times\mathcal{B}),\ \mathrm{Y} = (\mathcal{A}',\mathcal{B}',\psi:\mathcal{C}'\to\mathcal{A}'\times\mathcal{B}') \in (\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty)\times_{\mathsf{Cat}_\infty}\mathcal{R}$$

we set

$$\mathcal{Q} := \mathrm{Fun}(\mathcal{A},\mathcal{A}')\times_{\mathrm{Fun}(\mathcal{A},\mathcal{P}(\mathcal{B}'^{\mathrm{op}}))}\mathrm{Fun}^{\mathrm{rep}}(\mathcal{P}(\mathcal{B}^{\mathrm{op}}),\mathcal{P}(\mathcal{B}'^{\mathrm{op}})),$$

$$\mathcal{Z} := (\mathrm{Fun}(\mathcal{A},\mathcal{A}')\times\mathrm{Fun}^{\mathrm{rep}}(\mathcal{P}(\mathcal{B}^{\mathrm{op}}),\mathcal{P}(\mathcal{B}'^{\mathrm{op}})))\times_{\mathrm{Fun}(\{0,1\},\mathrm{Fun}(\mathcal{A},\mathcal{P}(\mathcal{B}'^{\mathrm{op}})))} \mathrm{Fun}([1],\mathrm{Fun}(\mathcal{A},\mathcal{P}(\mathcal{B}'^{\mathrm{op}}))),$$

where $\mathrm{Fun}(\mathcal{B}^{\mathrm{op}},\mathcal{B}'^{\mathrm{op}}) \simeq \mathrm{Fun}^{\mathrm{rep}}(\mathcal{P}(\mathcal{B}^{\mathrm{op}}),\mathcal{P}(\mathcal{B}'^{\mathrm{op}})) \subset \mathrm{Fun}(\mathcal{P}(\mathcal{B}^{\mathrm{op}}),\mathcal{P}(\mathcal{B}'^{\mathrm{op}}))$ denotes the full subcategory spanned by the left adjoint functors preserving representables. The diagonal embedding

$$\mathrm{Fun}(\mathcal{A},\mathcal{P}(\mathcal{B}'^{\mathrm{op}})) \subset \mathrm{Fun}([1],\mathrm{Fun}(\mathcal{A},\mathcal{P}(\mathcal{B}'^{\mathrm{op}})))$$

yields an embedding $\mathcal{Q}\subset\mathcal{Z}$. We consider the pullback

$$(\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty)\times_{\mathsf{Cat}_\infty}\mathcal{R}$$

of evaluation at the target $\mathcal{R}\to\mathsf{Cat}_\infty$ with trivial C_2-action along the C_2-equivariant functor $\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty\to\mathsf{Cat}_\infty$ that takes the product, where $\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty$ carries the C_2-action switching the factors. The space of maps from X to Y in $(\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty)\times_{\mathsf{Cat}_\infty}\mathcal{R}$ is canonically equivalent to $\mathcal{Z}^\simeq$ and so contains $\mathcal{Q}^\simeq\subset\mathcal{Z}^\simeq$.

Denote

$$\Theta\subset(\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty)\times_{\mathsf{Cat}_\infty}\mathcal{R}$$

the subcategory with

- objects the right fibrations $\mathcal{C}\to\mathcal{A}\times\mathcal{B}$ that classify a functor $\alpha:\mathcal{A}^{\mathrm{op}}\times\mathcal{B}^{\mathrm{op}}\to\mathsf{Spc}$ adjoint to a functor $\mathcal{A}^{\mathrm{op}}\to\mathcal{P}(\mathcal{B})$ that factors as an equivalence $\mathcal{A}^{\mathrm{op}}\simeq\mathcal{B}$ through the Yoneda-embedding,
- with morphisms the maps $\mathrm{X}\to\mathrm{Y}$ in $(\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty)\times_{\mathsf{Cat}_\infty}\mathcal{R}$ that belong to $\mathcal{Q}$ (that are closed under composition).

The C_2-action on $(\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty)\times_{\mathsf{Cat}_\infty}\mathcal{R}$ restricts to Θ. By definition the mapping space in Θ between two perfect right fibrations encoding adjoint equivalences $\mathcal{A}\simeq\mathcal{B}^{\mathrm{op}}$, $\mathcal{A}'\simeq\mathcal{B}'^{\mathrm{op}}$ is canonically equivalent to the maximal subspace in

$$\mathrm{Fun}(\mathcal{A},\mathcal{A}')\simeq\mathrm{Fun}(\mathcal{A},\mathcal{A}')\times_{\mathrm{Fun}(\mathcal{A},\mathcal{B}'^{\mathrm{op}})}\mathrm{Fun}(\mathcal{B}^{\mathrm{op}},\mathcal{B}'^{\mathrm{op}})\simeq\mathrm{Fun}(\mathcal{B}^{\mathrm{op}},\mathcal{B}'^{\mathrm{op}}).$$

Hence the forgetful functors

$$\gamma_1,\gamma_2:\Theta\subset(\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty)\times_{\mathsf{Cat}_\infty}\mathcal{R}\to\mathsf{Cat}_\infty\times\mathsf{Cat}_\infty\to\mathsf{Cat}_\infty$$

that project to the first respectively second component, are fully faithful.

The natural transformation $\alpha : \mathrm{Tw} \to \mathrm{id} \times (-)^{\mathrm{op}}$ of functors $\mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ is C_2-equivariant if the source carries the unique non-trivial C_2-action and the target carries the trivial C_2-action. α defines a C_2-equivariant functor

$$\rho : \mathsf{Cat}_\infty \to \Theta \subset (\mathsf{Cat}_\infty \times \mathsf{Cat}_\infty) \times_{\mathsf{Cat}_\infty} \mathcal{R}$$

that is a section of γ_1. Especially γ_1 is essentially surjective and so an equivalence. Thus ρ is an equivalence and so gives rise to an equivalence

$$\zeta : \mathsf{Cat}_\infty^{hC_2} \simeq \Theta^{hC_2} \subset ((\mathsf{Cat}_\infty \times \mathsf{Cat}_\infty) \times_{\mathsf{Cat}_\infty} \mathcal{R})^{hC_2} \simeq \mathsf{Cat}_\infty \times_{\mathsf{Cat}_\infty[C_2]} \mathcal{R}[C_2]$$

over Cat_∞. □

Remark 5.10. *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories. We have a canonical equivalence*

$$\mathrm{Fun}((\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}})_{hC_2}, \mathrm{Fun}(\mathcal{D}, \mathsf{Spc})) \simeq \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}((\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}})_{hC_2}, \mathsf{Spc})),$$

under which the following objects correspond:

- *functors $(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}})_{hC_2} \xrightarrow{\psi} \mathcal{D}^{\mathrm{op}} \subset \mathrm{Fun}(\mathcal{D}, \mathsf{Spc})$ such that for any $\mathrm{Z} \in \mathcal{C}$ the functor*

  $$\{\mathrm{Z}\} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C} \to (\mathcal{C} \times \mathcal{C})_{hC_2} \xrightarrow{\psi^{\mathrm{op}}} \mathcal{D}$$

  admits a right adjoint.

- *functors $\mathcal{D} \to \{$lax dualities on $\mathcal{C}\} \subset \mathrm{Fun}((\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}})_{hC_2}, \mathsf{Spc})$ such that the composition*

  $$\mathcal{D} \to \{\text{lax dualities on } \mathcal{C}\} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{C})$$

  is adjoint to a functor $\beta : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{C}$ such that for any $\mathrm{Z} \in \mathcal{C}$ the functor

  $$\{\mathrm{Z}\} \times \mathcal{D} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\beta} \mathcal{C}$$

  admits a left adjoint.

In other words if we fix a functor $\alpha : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ adjoint to a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{C}$ (respectively a functor $\beta : \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{C})$) via an adjunction of two variables

$$\mathrm{map}_{\mathcal{D}}(\alpha(\mathrm{X}, \mathrm{Y}), \mathrm{Z}) \simeq \mathrm{map}_{\mathcal{C}}(\mathrm{X}, \beta(\mathrm{Y}, \mathrm{Z}))$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Spc}$, then an extension of α to $(\mathcal{C} \times \mathcal{C})_{hC_2}$ corresponds to a lift of β to lax dualities on $\mathcal{C}$.

Now we are ready to construct a functor sending a rigid symmetric monoidal ∞-category $\mathcal{C}$ and a tensor-invertible object $\mathbb{L} \in \mathcal{C}$ to a duality $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{C}$, $\mathrm{Y} \mapsto \mathrm{Y}^\vee \otimes \mathbb{L}$. Denote

$$\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}} \subset \mathrm{Cmon}(\mathsf{Cat}_\infty)$$

the full subcategory of rigid symmetric monoidal ∞-categories. As the opposite of a rigid symmetric monoidal ∞-category is rigid, the non-trivial C_2-action on $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ restricts to $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$ giving the following C_2-action:

Theorem 5.11. *The C_2-action on* $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$ *induced by the non-trivial C_2-action on* Cat_∞ *is trivial.*

Proof. We will construct a canonical section α of the forgetful functor

$$(\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}})^{hC_2} \to \mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}.$$

Such a section α is adjoint to a C_2-equivariant functor from the trivial C_2-action on $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$ to the C_2-action on $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$ induced by the non-trivial C_2-action on Cat_∞, whose underlying functor (after forgetting the equivariance) is the identity.

We start with constructing the section α: Note that evaluation at the target $\mathcal{R} \to \mathsf{Cat}_\infty$ is a cartesian fibration classifying a functor

$$\mathsf{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathsf{Cat}}_\infty,\ \mathcal{C} \mapsto \mathcal{R}_{\mathcal{C}} \simeq \mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Spc}).$$

Thus a natural transformation $\theta : \mathrm{F} \to \mathrm{G}$ of functors $\mathrm{Cmon}(\mathsf{Cat}_\infty) \to \mathsf{Cat}_\infty$ yields a map of cartesian fibrations $\mathrm{Cmon}(\mathsf{Cat}_\infty) \times_{\mathsf{Cat}_\infty} \mathcal{R} \to \mathrm{Cmon}(\mathsf{Cat}_\infty) \times_{\mathsf{Cat}_\infty} \mathcal{R}$ over $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ that induces on the fiber over any $\mathcal{C} \in \mathrm{Cmon}(\mathsf{Cat}_\infty)$ the functor

$$\theta_{\mathcal{C}}^* : \mathcal{P}(\mathrm{G}(\mathcal{C})) \to \mathcal{P}(\mathrm{F}(\mathcal{C})).$$

We apply this to the canonical map from the forgetful functor $\mathrm{Cmon}(\mathsf{Cat}_\infty) \to \mathsf{Cat}_\infty$ followed by the functor $\mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$, $\mathcal{C} \mapsto (\mathcal{C} \times \mathcal{C})_{hC_2}$ to the forgetful functor $\mathrm{Cmon}(\mathsf{Cat}_\infty) \to \mathsf{Cat}_\infty$ to obtain a map of cartesian fibrations

$$\Psi : \mathrm{Cmon}(\mathsf{Cat}_\infty) \times_{\mathsf{Cat}_\infty} \mathcal{R} \to \mathrm{Cmon}(\mathsf{Cat}_\infty) \times_{\mathsf{Cat}_\infty} \mathcal{R}$$

over $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ that induces on the fiber over any $\mathcal{C} \in \mathrm{Cmon}(\mathsf{Cat}_\infty)$ the canonical functor

$$\mathcal{P}(\mathcal{C}) \to \mathcal{P}((\mathcal{C} \times \mathcal{C})_{hC_2}).$$

Denote

$$\mathrm{Pic} \subset \mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}} \times_{\mathsf{Cat}_\infty} (\mathsf{Cat}_\infty)_*$$

the full subcategory spanned by the $\mathcal{C} \in \mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$ equipped with an object $\mathbb{L} \in \mathcal{C}$ that is tensor-invertible. The functor Ψ over $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ restricts to a functor

$$\Phi : \mathrm{Pic} \to \mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}} \times_{\mathsf{Cat}_\infty} \mathsf{Cat}_\infty^{hC_2}$$

over $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$ using the inclusion $(\mathsf{Cat}_\infty)_* \subset \mathcal{R}$ and the inclusion $\mathsf{Cat}_\infty^{hC_2} \subset \mathsf{Cat}_\infty \times_{\mathsf{Cat}_\infty} \mathcal{R}$ of Theorem 5.7. The functor Φ turns a tensor-invertible object $\mathbb{L}$ of some rigid symmetric monoidal ∞-category $\mathcal{C}$ to a duality $\mathcal{C}^{\mathrm{op}} \simeq \mathcal{C},\ \mathrm{Y} \mapsto \mathrm{Y}^\vee \otimes \mathbb{L}$ following the recipe at the beginning of section 5.

The functor Φ over $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$ preserves finite products (in fact small limits) as the pullbacks $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}} \times_{\mathsf{Cat}_\infty} (\mathsf{Cat}_\infty)_*, \mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}} \times_{\mathsf{Cat}_\infty} \mathsf{Cat}_\infty^{hC_2}$ are pullbacks of ∞-categories with small limits and functors preserving such and Pic is closed under small limits in $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}} \times_{\mathsf{Cat}_\infty} (\mathsf{Cat}_\infty)_*$. Preserving finite products Φ gives rise to the functor

$$\alpha : \mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}} \simeq \mathrm{Cmon}(\mathrm{Pic}) \to \mathrm{Cmon}(\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}} \times_{\mathsf{Cat}_\infty} \mathsf{Cat}_\infty^{hC_2})$$

$$\simeq (\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}})^{hC_2}$$

over $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$.

□

We complete this section by combining Theorem 5.11 with direct sum real K-theory:

Denote $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}} \subset \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})$ the full subcategory spanned by the rigid preadditive symmetric monoidal ∞-categories. The non-trivial C_2-action on $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ restricts to $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}$. By Theorem 5.11 the C_2-action on $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ restricts to the trivial C_2-action on $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$ and so also restricts to the trivial C_2-action on $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}$.

So we have a canonical equivalence

$$\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}[C_2] \simeq (\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}})^{hC_2}$$

over $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}$.

Via real K-theory we get a functor

$$\mathrm{KR}: \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}[C_2] \simeq (\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}})^{hC_2}$$

$$\subset \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2} \xrightarrow{\mathrm{KR}} \mathrm{Calg}(\mathrm{Sp})^{C_2}.$$

We have a canonical functor

$$\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}[C_2] \to \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2},$$

thus for $\mathcal{C} \in \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}[C_2]$ we obtain a lax symmetric monoidal functor

$$\mathrm{KR}_{\mathcal{C}}: \mathrm{Pic}(\mathcal{C})^{\tilde{h}C_2} \to (\mathrm{BPic}(\mathcal{C}))^{hC_2} \to \mathrm{Sp}^{C_2}.$$

Remark 5.12. *Let $\mathcal{C} \in \mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}$ and equip it with the trivial C_2-action. Then $\mathrm{Pic}(\mathcal{C})^{\tilde{h}C_2} \simeq \mathrm{Pic}(\mathcal{C})[C_2] \simeq \mathrm{Pic}(\mathcal{C}) \times A$, where A is the fiber of the map $\mathrm{Pic}(\mathcal{C})[C_2] \to \mathrm{Pic}(\mathcal{C})$ which forgets the C_2-action. Thus we can twist the duality on the corresponding object of $(\mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}})^{hC_2}$ by objects of A. Note that the underlying space of A is the space of A_∞-maps from C_2 to the space of (homotopy) automorphisms of the tensor unit of $\mathrm{Pic}(\mathcal{C})$.*

Remark 5.13. *Let $\mathcal{C} \in \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}$ and equip it with the trivial C_2-action. For $\mathbb{L} \in \mathrm{Pic}(\mathcal{C})$ the norm of $\mathbb{L}$ is given by $(\mathbb{L}^{\otimes 2}, a) \in \mathrm{Pic}(\mathcal{C}) \times A \simeq \mathrm{Pic}(\mathcal{C})[C_2]$ for an object $a \in A$. Then we see that*

$$\mathrm{KR}_{\mathcal{C}}(\mathbb{L}^{\otimes 2}) \simeq \mathrm{KR}_{\mathcal{C}}((\mathbf{1}, a^{\otimes(-1)}))$$

in $\mathrm{Mod}_{\mathrm{KR}(\mathcal{C})}(\mathrm{Sp}^{C_2})$.

6 Applications

Assigning to a commutative ring R (possibly with C_2-action) its symmetric monoidal category of finitely generated projective R-modules yields a functor

$$\mathsf{CRing}[C_2] \to \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}[C_2],$$

which we can prolong with the functor KR to obtain real K-theory. This recovers the usual connective hermitian K-theory of R (see the corresponding references in the introduction).

Similarly we can assign to an E_∞-ring spectrum R (possibly with C_2-action) its ∞-category of finitely generated projective modules $\mathcal{P}(R)$ (which is the full subcategory of the ∞-category of R-modules $\mathrm{Mod}(R)$ which are retracts of modules of the form R^n for some $n \in \mathbb{N}$) which inherits a rigid symmetric monoidal structure from $\mathrm{Mod}(R)$.

Thus, again by prolonging with KR, we obtain a functor

$$\mathrm{Calg}(\mathrm{Sp})[C_2] \to \mathrm{Calg}(\mathrm{Sp}^{C_2})$$

which should give a reasonable real K-theory if R is connective. The underlying K-theory spectrum is connective algebraic K-theory (thereby carrying a C_2-action) if R is connective ([9, Theorem 5]).

Remark 6.1. *If we use instead of $\mathcal{P}(R)$ all perfect R-modules we obtain direct sum (hermitian) K-theory of this category, where in the 0-th homotopy group triangles do not split into sums. Better constructions of hermitian and real K-theory in this case can be found in [16] and [5].*

Remark 6.2. *Given any presentably stable symmetric monoidal ∞-category $\mathcal{C}$ with compact tensor unit the full subcategory of dualizable objects $\mathcal{C}^d$ belongs to $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{preadd}})^{\mathrm{rig}}$ and so gives rise to a direct sum E_∞-hermitian K-theory spectrum. As in Remark 6.1 one can use the constructions in loc. cit. to obtain more suitable versions. We can apply the above machinery e.g. to the dualizable objects in stable motivic homotopy categories or in categories of triangulated motives. If the base scheme is a field of characteristic 0 these dualizable objects are just the compact objects.*

Let $\mathcal{C} \in \mathrm{Cmon}(\mathsf{Cat}_\infty)^{\mathrm{rig}}[C_2]$ and suppose $\mathcal{C}$ is stable. Then $\mathcal{C}^{hC_2}$ is also stable, and we have the object $K := \mathbf{1}[1] \in \mathrm{Pic}(\mathcal{C})^{hC_2}$ at our disposal. The object K gives rise to the object $\mathcal{M} := sq(K) \otimes K^{\otimes(-2)} \in \mathrm{Pic}(\mathcal{C})^{hC_2}$, where sq denotes the norm. If $\mathcal{C}$ has trivial C_2-action, then $\mathrm{Pic}(\mathcal{C})^{hC_2} \simeq \mathrm{Pic}(\mathcal{C}) \times A$ (see Remark 5.12), and the first component of $\mathcal{M}$ is 0, so $\mathcal{M}$ can be considered as an object of A.

In particular for $R \in \mathrm{Calg}(\mathrm{Sp})$ we can specialize to $\mathcal{C} = \mathrm{Perf}(R)$ (perfect R-modules). Note that if we write $\mathrm{Pic}(\mathrm{Perf}(R))[C_2] \simeq \mathrm{Pic}(\mathrm{Perf}(R)) \times A$ then we have $\mathrm{Pic}(\mathcal{P}(R))[C_2] \simeq \mathrm{Pic}(\mathcal{P}(R)) \times A$, and we have the object $\mathcal{M} \in A$ ($\mathcal{M}$ can be viewed as a C_2-enhancement of the automorphism "multiplication by -1" on the tensor unit of $\mathcal{P}(R)$). So we can consider $T_{\mathcal{P}(R)}(\mathcal{M}) \in (\mathsf{Cat}_\infty^{\mathrm{preadd}})^{hC_2}$ and $\mathrm{KR}_{\mathcal{P}(R)}(\mathcal{M})^{C_2} \in \mathrm{Sp}$ ($\mathrm{KR}_{\mathcal{P}(R)}(\mathcal{M})^{C_2}$ is also a $\mathrm{KR}_{\mathcal{P}(R)}(\mathbf{1})^{C_2}$-module).

If we let R vary, the objects $\mathcal{M}$ are compatible, thereby we get a functor

$$\mathrm{Calg}(\mathrm{Sp}) \to \mathrm{Sp}$$

which can be considered as symplectic (hermitian) K-theory (if we restrict to connective E_∞-spectra).

In the following we will specialize the results from section 5 to certain stable ∞-categories.

We first consider dualities on stable ∞-categories in general.

We denote by $\mathsf{Cat}_\infty^{\mathrm{st}}$ the subcategory of Cat_∞ of (small) stable ∞-categories and exact functors. The canonical C_2-action on Cat_∞ restricts to a C_2-action on $\mathsf{Cat}_\infty^{\mathrm{st}}$ and $(\mathsf{Cat}_\infty^{\mathrm{st}})^{hC_2}$ is the subcategory of $\mathsf{Cat}_\infty^{hC_2}$ on the ∞-categories with duality whose underlying ∞-category is stable, and similarly for the morphisms.

The ∞-category $\mathsf{Cat}_\infty^{\mathrm{st}}$ has a canonical symmetric monoidal structure compatible with colimits and with the C_2-action, thus $(\mathsf{Cat}_\infty^{\mathrm{st}})^{hC_2}$ is also closed symmetric monoidal. We have $\mathrm{Calg}((\mathsf{Cat}_\infty^{\mathrm{st}})^{hC_2}) \simeq \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{st}})^{hC_2}$, and this ∞-category is a subcategory of $\mathrm{Cmon}(\mathsf{Cat}_\infty)^{hC_2}$. The tensor unit of $(\mathsf{Cat}_\infty^{\mathrm{st}})^{hC_2}$ is Sp^ω equipped with its unique symmetric monoidal duality.

Now let $\mathcal{C}$ be an object in $\mathrm{Calg}((\mathsf{Cat}_\infty^{\mathrm{st}})^{hC_2})$, i.e. a stable symmetric monoidal ∞-category with duality.

We have the following pullback square

$$\begin{array}{ccc} \mathrm{Mod}_{\mathcal{C}}((\mathsf{Cat}_\infty^{\mathrm{st}})^{hC_2}) & \longrightarrow & \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty^{hC_2}) , \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty^{\mathrm{st}}) & \longrightarrow & \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty) \end{array}$$

which after pullback to $\mathrm{BPic}(\mathcal{C}) \subset \mathrm{Mod}_{\mathcal{C}}(\mathsf{Cat}_\infty)$ has equivalences as horizontal arrows, whereas the vertical arrow is $(\mathrm{BPic}(\mathcal{C}))^{hC_2} \to \mathrm{BPic}(\mathcal{C})$.

Specializing to $\mathcal{C} = \mathrm{Sp}^\omega$ it follows that the preimage of $\mathrm{BPic}(S) \simeq \mathrm{BAut}(\mathrm{Sp}^\omega) \subset \mathsf{Cat}_\infty$ with respect to the map $\mathsf{Cat}_\infty^{hC_2} \to \mathsf{Cat}_\infty$ is also $(\mathrm{BPic}(S))^{hC_2}$.

We have the cofiber sequence

$$\mathrm{pic}(S) \to \mathrm{pic}(S)^{\mathrm{B}C_2}_{\geq 0} \to (\mathrm{pic}(S)[1])^{hC_2}_{\geq 0} \to \mathrm{pic}(S)[1]$$

in Sp, thus the space of dualities on Sp^ω is $\mathrm{Pic}(S)^{\mathrm{B}C_2}$, and the corresponding (Picard) ∞-groupoid in $\mathsf{Cat}_\infty^{hC_2}$ (the subgroupoid of ∞-categories with duality whose underlying ∞-category is equivalent to Sp^ω) is

$$\mathrm{Pic}(S)^{\mathrm{B}C_2}/\mathrm{Pic}(S) \simeq (\mathrm{BPic}(S))^{hC_2}.$$

We learned the proof of the following lemma from Niko Naumann:

Lemma 6.3. *The map $\mathbb{Z} \oplus \mathbb{Z}_2 \cong \pi_0(S^{BC_2}) \to \pi_0(\mathrm{pic}(S)^{BC_2})$ (the first isomorphism is [8, Theorem 1.1]) induced by the map $S \to \mathrm{pic}(S)$ sending 1 to the element $S[1]$ is injective.*

Proof. We first show that the map $\mathbb{Z} \oplus \mathbb{Z} \cong \pi_0(\mathrm{Pic}(\mathrm{Sp}^{C_2})) \to \pi_0(\mathrm{Pic}(\mathrm{Sp}[C_2]))$ is injective (here the first map sends (a,b) to $S^{a+b\sigma}$, where σ is the sign representation). Suppose (a,b) is sent to 0. Then we have an isomorphism $b(S^{a+b\sigma}) \cong b(S)$ in Sp^{C_2}, where b denotes Borel completion. Since $\widetilde{E}_{C_2} \simeq \mathrm{colim}_n S^{n\sigma}$ we get $\widetilde{E}_{C_2} \wedge b(S^a) \cong \widetilde{E}_{C_2} \wedge b(S)$. Taking C_2-fixed points we see that $tS[a] \simeq tS$ in Sp, where tS is the Tate construction of S. Since tS is non-trivial and bounded below we get $a = 0$, and thus also $b = 0$.

Writing $\mathrm{Pic}(S)^{\mathrm{B}C_2} \simeq \mathrm{Pic}(S) \oplus A$, the map in question restricted to $\mathbb{Z}_2$ factors through $\pi_0(A)$ and is injective on $\mathbb{Z} \subset \mathbb{Z}_2$. But $\pi_0(A)$ is a pro-finite abelian group (more precisely a limit over a diagram indexed by ω^{op} of finite abelian groups) from which the claim follows. □

As a corollary we get the following proposition:

Corollary 6.4. *The cardinality of $\pi_0((\mathrm{BPic}(S))^{hC_2})$ is the continuum, in particular there are uncountably many inequivalent ∞-categories with duality with underlying ∞-category Sp^ω.*

Remark 6.5. *The $\mathcal{M}$ from above is in the case of $R = S$ the sphere spectrum is the image of $S^{0,1} \simeq S^{\sigma-1} \in \mathrm{Sp}^{C_2}$ in $\mathrm{Sp}[C_2]$ (here we use motivic notation in $S^{0,1}$). Thus $\mathcal{M}$ has infinite order in this case. For general R one can consider tensor powers $\mathcal{M}^{\otimes n}$, $n \in \mathbb{Z}$, of $\mathcal{M}$. If $R \in \mathsf{CRing}$ then $\mathcal{M}$ has order at most 2 which recovers the known periodicity of hermitian K-theory. If R is n-truncated, then it follows at least that the order of $\mathcal{M}$ is of the form 2^k for $0 \leq k \leq n+1$: It is sufficient to show this for $R = S_{\leq n}$, in which case it follows by induction on n and the fact that for an abelian group G the group $\pi_0((HG[i])^{\mathrm{B}C_2})$ is an $\mathbb{F}_2$-vector space for $i > 0$.*

The map $sq: S \to S^{\mathrm{B}C_2} \simeq S \oplus A$ sends 1 to $(2, a) \in \pi_0(S \oplus A) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Considering the composition $S^{\mathrm{B}C_2} \to \mathrm{pic}(S)^{\mathrm{B}C_2} \to \mathrm{pic}(\mathbb{Z})^{\mathrm{B}C_2}$ we see that $a \in \pi_0(A)$ is a free generator of the $\mathbb{Z}_2$-module $\pi_0(A)$.

By specializing the above considerations to $R = \mathbb{Z}$ we see that

$$\pi_0((\mathrm{BPic}(\mathbb{Z}))^{hC_2}) \cong (\mathbb{Z} \oplus \mathbb{Z}/2)/(2,1) \cong \mathbb{Z}/4$$

(we also have the groups $\pi_1((\mathrm{BPic}(\mathbb{Z}))^{hC_2}) \cong \mathbb{Z}/2$, $\pi_2((\mathrm{BPic}(\mathbb{Z}))^{hC_2}) \cong \mathbb{Z}/2$ and $\pi_i((\mathrm{BPic}(\mathbb{Z}))^{hC_2}) \cong 0$ for $i > 2$).

The real K-theory of [16] can be enhanced to a lax symmetric monoidal functor $(\mathsf{Cat}_\infty^{\mathrm{st}})^{hC_2} \to \mathrm{Sp}^{C_2}$ by results of [5], thus for R as above we obtain a lax symmetric monoidal functor

$$(\mathrm{BPic}(R))^{hC_2} \to \mathrm{Sp}^{C_2}$$

which sends the tensor unit to $\mathrm{KR}(R)$ (in the sense of [16]). In the case $R \in \mathsf{CRing}$ this endows the known 4-periodic theory (at least if 2 is invertible in R) with an E_∞-multiplication in the sense that there is a lax symmetric monoidal functor

$$(\mathrm{BPic}(\mathbb{Z}))^{hC_2} \to \mathrm{Sp}^{C_2}$$

which sends the elements of $\pi_0((\mathrm{BPic}(\mathbb{Z}))^{hC_2})$ to the four classical versions of hermitian K-theory after taking C_2-fixed points.

For general R one gets in particular a lax symmetric monoidal functor

$$(\mathrm{BPic}(S))^{hC_2} \to \mathrm{Sp}^{C_2}.$$

Note that if we invert 2 in $\pi_0(S)$ the situation drastically simplifies: we have $\pi_0((\mathrm{BPic}(S[\frac{1}{2}]))^{hC_2}) \cong \mathbb{Z}/4$.

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