

ON RATIONALLY CONNECTED VARIETIES OVER C_1 FIELDS OF CHARACTERISTIC 0

MARTA PIEROPAN

ABSTRACT. We use birational geometry to show that the existence of rational points on proper rationally connected varieties over fields of characteristic 0 is a consequence of the existence of rational points on terminal Fano varieties. We discuss several consequences of this result, especially in relation to the C_1 -conjecture. We also provide evidence that supports the conjecture in dimension 3 for C_1 fields of characteristic 0.

CONTENTS

1. Introduction	1
2. Notation and basic properties	4
3. Reformulation of the C_1 -conjecture	5
4. Del Pezzo surfaces	7
5. Fano threefolds	8
6. Field of definition of rational points	15
7. Varieties with points over C_1 fields	16
8. Terminal Fano varieties of large index	24
9. Terminal Fano threefolds	24
References	27

1. INTRODUCTION

A field k is called C_1 (or quasi algebraically closed) if every hypersurface of degree at most n in \mathbb{P}_k^n has a k -rational point. Quasi algebraically closed fields were introduced by Artin and first extensively studied by Tsen (see [DKT99]) and Lang [Lan52]. Smooth hypersurfaces of degree at most n in \mathbb{P}^n are Fano and rationally chain connected [KMM92, Cam92]. In characteristic 0 they are also rationally connected. Hence, it is natural to study rational points on rationally (chain) connected varieties over C_1 fields. In an unpublished paper of 2000 Lang formulated the following conjecture, also known as C_1 -conjecture.

Conjecture 1.1 (Lang, 2000). Every smooth proper separably rationally connected variety over a C_1 field has a rational point.

At the time it was known that smooth proper rational curves and surfaces have points over C_1 fields [CT87]. Soon after it was formulated, the conjecture has been proven to hold for the following C_1 fields: finite fields [Esn03], function fields of curves over algebraically closed fields [CPP02, GHS03, dJS03], fields of formal power series over algebraically closed fields [CT11]. It is still open for the maximal

Date: August 6, 2021.

2010 Mathematics Subject Classification. 14M22 (14G05, 14J45).

Key words and phrases. Rationally connected varieties, rational points, Fano varieties, C_1 fields.

unramified extensions \mathbb{Q}_p^{nr} of p -adic fields. Positive evidence in support of the conjecture for the latter class of fields is given by [DK17], which proves that for sufficiently large primes p smooth projective rationally connected varieties over \mathbb{Q}_p^{nr} with fixed Hilbert polynomial have a rational point. As a consequence of our first main result we manage to improve [DK17]: we replace the dependence on the Hilbert polynomial by dependence only on the dimension, and we remove the smoothness and properness assumptions.

The goal of this paper is to use birational geometry in characteristic 0 to reduce the C_1 -conjecture to the case of Fano varieties and to provide evidence for the existence of rational points on rationally connected varieties of dimension 3. Along the way, we also show that for proper rationally connected varieties of fixed dimension over an arbitrary field of characteristic 0 there is a uniform upper bound for the degree of the minimal field extensions where the set of rational points becomes nonempty.

1.1. Birational geometry. The first main result of this paper reduces the C_1 -conjecture for C_1 fields of characteristic 0 to the following conjecture.

Conjecture 1.2. Every terminal \mathbb{Q} -factorial Fano variety of Picard rank 1 over a C_1 field of characteristic 0 has a rational point.

More precisely, birational geometry and induction on dimension are used to prove the following statement.

Theorem 1.3. *Let k be a field of characteristic 0. For every positive integer n the following statements are equivalent:*

- (i) *Every smooth proper rationally connected k -variety of dimension $\leq n$ has a k -point.*
- (ii) *Every terminal \mathbb{Q} -factorial Fano variety of dimension $\leq n$ and Picard rank 1 over k has a k -point.*

The proof of Theorem 1.3 rests upon the birational invariance of rational connectedness and the fact that the existence of rational points is a birational invariant among proper smooth varieties [Nis55]. We recall that the use of the Minimal Model Program in dimension ≥ 3 produces singular birational models in general. Results on degenerations of rationally connected varieties [HX09] and induction on dimension are used to transfer the existence of rational points among birationally equivalent varieties with mild singularities. The restriction on the characteristic is due to the use of the Minimal Model Program, of resolution of singularities and of [HX09]. None of these are known to hold in positive characteristic, except for some results in low dimension.

Boundedness of terminal Fano varieties [Bir21] assures that there are only finitely many deformation families of terminal Fano varieties in each fixed dimension over algebraically closed fields of characteristic 0. As an application of this fact we prove the following corollary of Theorem 1.3, which gives a generalization of [DK17].

Corollary 1.4. *For every positive integer n there exists a finite set $S(n)$ of prime numbers such that for all prime numbers $p \notin S(n)$, every rationally connected variety of dimension n over \mathbb{Q}_p^{nr} has a \mathbb{Q}_p^{nr} -point.*

During a workshop in Edinburgh in November 2018, it was brought to the author's attention that the strategy of the proof of Theorem 1.3 is similar to the one used in the paper [PS16], which proves, among other things, a uniform bound depending only on dimension for the indices of subgroups of finite groups of automorphisms such that the subgroup acts with fixed points. The corresponding statement for rational points is the following theorem, which is proven by combining the proof of Theorem 1.3 with boundedness of terminal Fano varieties.

Theorem 1.5. *For every positive integer n there exists a positive integer $d(n)$ such that for every field k of characteristic 0 and every proper rationally connected k -variety X of dimension n there exists a field extension k' of k of degree $\leq d(n)$ such that $X(k') \neq \emptyset$.*

For every positive integer n , let $\tilde{d}(n)$ be the smallest positive integer such that for every field k of characteristic 0 and every proper rationally connected k -variety X of dimension n there exists a field extension k' of k of degree $\leq \tilde{d}(n)$ such that $X(k') \neq \emptyset$. We prove the following effective bounds.

Theorem 1.6. *We have $\tilde{d}(1) = 2$, $\tilde{d}(2) = 6$, $\tilde{d}(3) \leq 144$.*

The proof of Theorem 1.5 gives a bound

$$\tilde{d}(n) \leq \tilde{d}(n-1) \max\{d'(n), \max_{1 \leq m \leq n-1} \tilde{d}(m)\tilde{d}(n-m)\},$$

where $d'(n)$ is an upper bound for the degree of the minimal field extensions where terminal Fano varieties of dimension n acquire rational points. The existence of $d'(n)$ is a consequence of boundedness of terminal Fano varieties. The effective computability of $d'(n)$ depends on the classification of the Fano varieties that appear in Theorem 1.3(ii). In dimension 4 or higher, the classification is not complete. In dimension 2 the inequality only gives an upper bound $\tilde{d}(2) \leq 12$. The sharp bound $\tilde{d}(2) = 6$ is obtained in Section 4 using the Enriques–Iskovskih–Manin classification of surfaces over nonclosed fields [Isk79]. In Section 5 we recall the classification of the terminal Fano threefolds that appear in Theorem 1.3(ii), and we prove that $d'(3) \leq 24$. We do not expect the bound for $\tilde{d}(3)$ to be sharp.

1.2. Classification of Fano varieties. Further evidence for Conjecture 1.1 that can be found in the literature is given by the fact that the following rationally connected varieties have rational points over C_1 fields that admit normic forms of arbitrary degree: complete intersections in weighted projective spaces (see [Kol96] for example) and hypersurfaces of split toric varieties [Gui14]. We refer to [Lan52] for the condition about normic forms and we recall that it is satisfied by fields that admit finite extensions of every degree. In Section 7 we prove similar results for varieties that after extension of the base field to the algebraic closure belong to the following classes: toric varieties, complete intersections in products of weighted projective spaces, some Fano cyclic coverings, some special Fano varieties of dimension 3.

As a consequence of Theorem 1.3 combined with boundedness of terminal Fano varieties [Bir21], the verification of Conjecture 1.1 is reduced to finding rational points on finitely many geometric families of Fano varieties in each given dimension. In general, the \mathbb{Q} -factoriality and the Picard rank 1 conditions in Conjecture 1.2 and in Theorem 1.3 are not preserved under base field extensions. So in order to verify Conjecture 1.2 one needs to find rational points also on some Fano varieties that are not geometrically \mathbb{Q} -factorial and on some Fano varieties of geometric Picard rank > 1 .

We recall that the Gorenstein index of a log terminal Fano variety X is the smallest positive integer $I(X)$ such that $I(X)K_X$ is Cartier. The \mathbb{Q} -Fano index of X is then the largest positive integer $q(X)$ such that $-I(X)K_X$ is linearly equivalent to $q(X)H$ for some Cartier divisor H on X . We write $r(X) = q(X)/I(X)$. In general these indices do not need to be invariant under field extension. In Section 5 we show that $r(X) = r(X_{\bar{k}})$ for X defined over a C_1 field of characteristic 0 with algebraic closure \bar{k} . In Section 8, we use Fujita’s and Sano’s classification results [Fuj82, San96] to find rational points on Fano varieties of large index.

Theorem 1.7. *If X is a terminal Fano variety of dimension $n \geq 2$ and index $r(X) > n - 2$ over a C_1 field k of characteristic 0 that admits normic forms of arbitrary degree, then $X(k) \neq \emptyset$.*

The classification of terminal Fano varieties is far from known. However, the current literature on terminal Gorenstein Fano threefolds [Muk02, Pro13b] makes the following result possible.

Theorem 1.8. *Let X be a Gorenstein terminal \mathbb{Q} -factorial Fano threefold over a C_1 field k of characteristic 0 that admits normic forms of arbitrary degree. Assume that $\mathrm{rk} \mathrm{Pic}(X) = 1$. Let $g(X) = \frac{1}{2}(-K_X)^3 + 1$. Let \bar{k} be an algebraic closure of k . Then $X(k) \neq \emptyset$ except, possibly, in the following cases:*

- (i) $r(X) = 1$, $\mathrm{rk} \mathrm{Pic}(X_{\bar{k}}) = 1$ and $g(X) \in \{6, 7, 8, 9, 10\}$;
- (ii) $r(X) = 1$, $\mathrm{rk} \mathrm{Pic}(X_{\bar{k}}) = 1$ and $|-K_{X_{\bar{k}}}|$ has a movable decomposition.

The varieties in part (i) of the theorem are complete intersections of sections of certain vector bundles on some Grassmannians. To the best of the author's knowledge, Fano varieties as in (ii) are not classified. At the beginning of Section 5.1.5 we recall the definition of movable decomposition from [Muk02].

To date there is no complete classification of non-Gorenstein terminal Fano threefolds of index $r \leq 1$. There is, however, a classification of the possible configurations of non-Gorenstein singularities for geometrically \mathbb{Q} -factorial non-Gorenstein terminal Fano threefolds of geometric Picard rank 1. See the Graded Ring Database [B⁺]. Such Fano varieties X are studied using a Fano index defined as the largest positive integer $f(X)$ such that $-K_X$ is linearly equivalent to $f(X)A$ for some Weil divisor A on X [Suz04]. In Section 9.2 we prove that if $f(X_{\bar{k}}) \geq 5$ the variety has a rational points over C_1 fields of characteristic 0. We also discuss to what extent the same proof applies to varieties of lower Fano index.

To the author's knowledge there is no classification of non-Gorenstein terminal Fano threefolds of geometric Picard rank > 1 except for [San95]. The varieties in [San95] have rational points over C_1 fields of characteristic 0 by Theorem 9.1.

We recall that the condition about the existence of normic forms of arbitrary degree is satisfied by the fields $\mathbb{Q}_p^{\mathrm{nr}}$, as they admit finite extensions of every degree. Their algebraic extensions, however, do not need to satisfy the condition on normic forms, but if Conjecture 1.1 holds for $\mathbb{Q}_p^{\mathrm{nr}}$, then it holds also for all its algebraic extensions, as Weil restriction under finite separable field extensions preserves rational connectedness, smoothness and properness, and the Weil restriction of a variety X from a finite extension k to $\mathbb{Q}_p^{\mathrm{nr}}$ has a $\mathbb{Q}_p^{\mathrm{nr}}$ -point if and only if X has a k -point.

Theorem 1.3 and Corollary 1.4 are proven in Section 3, Theorems 1.5 and 1.6 are proven in Section 6, Theorem 1.7 is proven in Section 8, Theorem 1.8 is proven in Section 9.

Acknowledgements. This work was partially supported by grant ES 60/10-1 of the Deutsche Forschungsgemeinschaft. The author is grateful to H. Esnault for introducing her to the C_1 -conjecture. The author wishes to thank J. Blanc, Y. Gongyo, A. Höring, Z. Patakfalvi, Y. Prokhorov, R. Svaldi for useful discussions, J.-L. Colliot-Thélène and the anonymous referee for their remarks that led to significant improvements of the paper, and K. Shramov for informing her about the paper [PS16] and for suggesting the topic of Theorem 1.5. The author acquainted herself to the Minimal Model Program at the *Introductory Workshop on MMP* held in Hannover in February 2016.

2. NOTATION AND BASIC PROPERTIES

Let k be a field of characteristic 0, and \bar{k} an algebraic closure of k . We denote by $\mathrm{Br}(k)$ the Brauer group of k .

For us a k -variety is a separated scheme of finite type over k . We use the words curve, surface, threefold to denote a variety of dimension 1, 2, 3, respectively.

We say that a k -variety X is rationally connected if $X_{\bar{k}}$ is integral and rationally connected in the sense of [Kol96, §IV.3]. We recall that if X is rationally connected, $X \dashrightarrow Y$ is a dominant rational map and Y is proper, then Y is rationally connected. In particular, being rationally connected is a birational invariant of proper varieties.

Weil divisors on an integral normal variety X correspond to Cartier divisors on its smooth locus U . The reflexive sheaf $\mathcal{O}_X(D)$ associated to a Weil divisor D on X is the push-forward of $\mathcal{O}_U(D|_U)$ under the inclusion $U \subseteq X$. A canonical divisor K_X on X is the Weil divisor corresponding to a canonical divisor on U . We say that X is Gorenstein if its canonical divisor K_X is Cartier, \mathbb{Q} -Gorenstein if a positive multiple of K_X is Cartier. We say that X is \mathbb{Q} -factorial if every Weil divisor on X has a positive multiple that is Cartier. We recall that \mathbb{Q} -factoriality is not invariant under field extension (see [GNT19, Remark 2.7]).

We refer to [Kol13] for the singularities of the minimal model program. We recall that (log) terminal varieties are normal and \mathbb{Q} -Gorenstein by definition. We also recall that the notion of (log) terminal singularities is invariant under separable field extension.

A (log) terminal variety X is Fano if $-K_X$ has a positive multiple which is Cartier and ample. We refer to [IP99] for the theory of Fano varieties.

We recall that, given a field k and an algebraic closure \bar{k} of k , the k -forms of a k -variety Y are k -varieties Y' such that Y'_k is isomorphic to $Y_{\bar{k}}$, the k -models of a \bar{k} -variety Z are k -varieties Z' such that Z'_k is isomorphic to Z .

3. REFORMULATION OF THE C_1 -CONJECTURE

In this section we prove that over a C_1 field of characteristic 0 the C_1 -conjecture is equivalent to Conjecture 1.2. We start by observing that in characteristic 0 the C_1 -conjecture is equivalent to both the stronger version obtained by removing the smoothness assumption and to the weaker version obtained by replacing the properness assumption by projectivity, as the following lemma shows.

Lemma 3.1. *Let k be a field of characteristic 0 and n a positive integer. If all smooth projective rationally connected k -varieties of dimension n have k -points, then every proper rationally connected k -variety of dimension n has a k -point.*

Proof. Let X be a proper rationally connected k -variety of dimension n . Since X is of finite type over k , there exists an open subset U of X which is isomorphic to an affine k -variety. Since resolutions of singularities exist for all varieties in characteristic 0 by [Hir64], we can find a smooth projective compactification \bar{U} of U over k . Then \bar{U} is rationally connected because it is birationally equivalent to X . So $\bar{U}(k) \neq \emptyset$ by assumption, and we conclude that $X(k) \neq \emptyset$ by [Nis55]. \square

If the base field is large in the sense of [Pop96, Proposition 1.1] we can remove also the properness assumption.

Lemma 3.2. *Let k be a large field of characteristic 0 and n a positive integer. If all smooth projective rationally connected k -varieties of dimension n have k -points, then every rationally connected k -variety of dimension n has a k -point.*

Proof. For every rationally connected k -variety X we can find a smooth projective compactification X' of a smooth affine open subset of X as in the proof of Lemma 3.1. Since $X'(k) \neq \emptyset$ by assumption, the set $X'(k)$ is dense in X' as k is large, hence $X(k) \neq \emptyset$. \square

In the proof of Theorem 1.3 we use the following form of the Minimal Model Program, which is a consequence of [BCHM10, Corollary 1.3.3]. See [Pro13c, Proposition 2.3] for a proof.

Proposition 3.3. *Let k be a field of characteristic 0. Let X be a smooth projective rationally connected k -variety. Then there exist a birational map $X \dashrightarrow X'$ and a projective dominant morphism $f : X' \rightarrow Y$ such that X' is a projective \mathbb{Q} -factorial terminal k -variety, Y is a projective \mathbb{Q} -factorial log terminal k -variety, $\dim Y < \dim X'$, $\mathrm{rk} \mathrm{Pic}(Y) = \mathrm{rk} \mathrm{Pic}(X') - 1$, and the generic fiber of f is a terminal Fano variety.*

We can now prove the main theorem of this section.

Proof of Theorem 1.3. For every positive integer n , the implication (i) \Rightarrow (ii) holds by Lemma 3.1 because every terminal Fano variety is rationally connected by [Zha06].

To prove the reverse implication, we fix a positive integer n and assume that (ii) holds. Let X be a smooth proper rationally connected k -variety of dimension $\leq n$. We prove that X has a k -rational point by induction on the dimension of X . If $\dim X = 0$, $X = \mathrm{Spec} k$. Assume that $\dim X \geq 1$ and that every smooth proper rationally connected k -variety of dimension $< \dim X$ has a k -point. By Lemma 3.1 we can assume, without loss of generality, that X is projective. Let $f : X' \rightarrow Y$ be the fibration with X' birationally equivalent to X provided by Proposition 3.3.

We first prove that $X'(k) \neq \emptyset$. If $\dim Y = 0$, then X' is a \mathbb{Q} -factorial terminal Fano variety of Picard rank 1. So $X'(k) \neq \emptyset$ by (ii). If $\dim Y > 0$, then Y is rationally connected and has dimension $< \dim X$. Hence, $Y(k) \neq \emptyset$ by the induction hypothesis combined with Lemma 3.1. Let $y \in Y(k)$. By [HX09, Theorem 1.2] the fiber of f over y contains a projective rationally connected k -subvariety Z . Since $\dim Y > 0$, $\dim Z < \dim X' = \dim X$. So $Z(k) \neq \emptyset$ by the induction hypothesis combined with Lemma 3.1. Then $X'(k) \neq \emptyset$.

Let $x \in X'(k)$. Let $h : X'' \rightarrow X'$ be a resolution of singularities. By [HX09, Theorem 1.3], the fiber of h over x contains a rationally connected k -subvariety W . Since h is a proper birational morphism, W is a proper rationally connected k -variety with $\dim W < \dim X'' = \dim X$. Thus $W(k) \neq \emptyset$ by the induction hypothesis combined with Lemma 3.1. Hence, $X''(k) \neq \emptyset$, and $X(k) \neq \emptyset$ by [Nis55]. \square

3.1. A consequence of boundedness of Fano varieties. We show that Birkar's boundedness of Fano varieties [Bir21] implies boundedness of Hilbert polynomials for terminal Fano varieties over nonclosed fields of characteristic 0.

Proposition 3.4. *Let k be a field of characteristic 0 and n a positive integer. Then there exist a positive integer N and finitely many polynomials $f_1, \dots, f_s \in \mathbb{Q}[t]$ such that for every terminal Fano variety X of dimension n there exists an embedding $X \subseteq \mathbb{P}_k^N$ such that X has Hilbert polynomial f_i for some $i \in \{1, \dots, s\}$.*

Proof. Let \bar{k} be an algebraic closure of k . By [Bir21, Theorem 1.1] there are finitely many projective morphisms of \bar{k} -varieties, say $\{V^i \rightarrow T^i\}_{i \in I}$ for a finite set I , such that for every terminal Fano k -variety X of dimension n there exists $i \in I$, a point $t \in T^i(\bar{k})$ and an isomorphism $V_t^i \cong X_{\bar{k}}$ of \bar{k} -varieties. Up to replacing each T^i by a suitable finite stratification, we can assume that T^i is smooth, $V^i \rightarrow T^i$ is flat, $K_{V^i}|_{V_t^i} = K_{V_t^i}$ for all $t \in T^i(\bar{k})$ and there exists $m \in \mathbb{Z}_{>0}$ such that $-mK_{V^i}$ is Cartier and relatively very ample over T^i for all $i \in I$. We can use the complete linear system $| -mK_{V^i} |$ to embed V^i into a projective space $\mathbb{P}_{T^i}^{N_i}$ over T^i for all $i \in I$, where N_i is the projective dimension of the linear system $| -mK_{V^i} |$ for all

$i \in I$. Then for every $i \in I$ the Hilbert polynomial f_i of V_t^i with respect to the projective embedding given by $|-mK_{V_t^i}|$ is independent of the choice of $t \in T^i$ by [Har77, Corollary III.9.13]. Let s be the cardinality of I and write $I = \{1, \dots, s\}$. Let $N = \max_{1 \leq i \leq s} N_i$. Then for every terminal Fano variety X of dimension n over k , there exists $i \in \{1, \dots, s\}$ such that X has Hilbert polynomial f_i under the embedding $X \subseteq \mathbb{P}_k^{N_i}$ defined by the complete linear system $|-mK_X|$. Choose a linear embedding $\mathbb{P}_k^{N_i} \subseteq \mathbb{P}_k^N$. Then X has Hilbert polynomial f_i under the induced embedding $X \subseteq \mathbb{P}_k^N$. \square

3.2. Application to rationally connected varieties over \mathbb{Q}_p^{nr} . We apply the previous results to prove Corollary 1.4.

Proposition 3.5. *Let $f \in \mathbb{Q}[t]$ and $N \in \mathbb{N}$. Then there exists a finite set $S(f, N)$ of prime numbers such that for every prime number $p \notin S(f, N)$, every projective rationally connected variety $X \subseteq \mathbb{P}_{\mathbb{Q}_p^{\text{nr}}}^N$ with Hilbert polynomial f has a \mathbb{Q}_p^{nr} -point.*

Proof. Replace [Kol96, Theorem IV.3.11] by [dFF13, Proposition 2.6] in the proof of [DK17, Theorem 1.3]. \square

Proof of Corollary 1.4. Let n be a positive integer. Combining Propositions 3.4 and 3.5 we obtain a finite set $S(n)$ of prime numbers such that for every prime number $p \notin S(n)$ every terminal Fano variety of dimension n over \mathbb{Q}_p^{nr} has a \mathbb{Q}_p^{nr} -point. Then by Theorem 1.3 every smooth proper rationally connected variety of dimension n over \mathbb{Q}_p^{nr} with $p \notin S(n)$ has a \mathbb{Q}_p^{nr} -point. We conclude by Lemma 3.2 as \mathbb{Q}_p and \mathbb{Q}_p^{nr} are large fields by the Implicit Function Theorem over local fields [Ser06, p.73] and [Pop96, Proposition 1.2], respectively. \square

4. DEL PEZZO SURFACES

In this section we collect some properties related to the existence of rational points on rationally connected curves and surfaces. We recall that terminal varieties of dimension ≤ 2 are smooth [KM98, Corollary 5.18], and that the birational classification of smooth rationally connected curves and surfaces is well understood.

Remark 4.1. We recall that isomorphism classes of Severi-Brauer varieties of dimension n over k are in bijection with $(n+1)$ -torsion elements of $\text{Br}(k)$ [GS06, Theorem 5.2.1], and the projective space corresponds to the neutral element. Moreover, a Severi-Brauer variety over k is a projective space if and only if it has a k -rational point [GS06, Theorem 5.1.3]. In particular, a Severi-Brauer variety of dimension n acquires a rational point over a base field extension of degree at most $n+1$.

Smooth curves are rationally connected if and only if they are Fano. Hence, terminal rationally connected curves are precisely the smooth conics.

Lemma 4.2 ([GS06, §1.3, Example 5.2.4]). *For a field k , the following are equivalent:*

- (1) $\text{Br}(k)$ has nontrivial 2-torsion;
- (2) there is a conic C over k with $C(k) = \emptyset$.

Smooth surfaces are rationally connected if and only if they are rational. Hence, terminal rationally connected surfaces are rational conic bundles or del Pezzo surfaces [Isk79].

Lemma 4.3. *Let X be a rational conic bundle surface over a field k . Then*

- (i) *there exists a finite extension k' of k of degree at most 4 such that X has a k' -point.*

- (ii) *There exists a rational conic bundle over \mathbb{Q} that has no rational points on any finite extension of \mathbb{Q} of degree at most 3.*

Proof. The conic bundle structure is given by a morphism $f : X \rightarrow C$ where C and the fibers of f are conics. Then C acquires a rational point over a suitable quadratic extension of k and the fiber of f over that point acquires a rational point over a suitable further quadratic extension. Moreover, given two conics C_1 and C_2 without rational points over \mathbb{Q} and with distinct splitting fields, for example, $C_1 = \{x^2 + y^2 + z^2 = 0\}$ and $C_2 = \{x^2 + 2y^2 + 5z^2 = 0\}$ by [Lam05, Example III.2.13], then $C_1 \times C_2$ has no rational points on any finite extension of \mathbb{Q} of degree ≤ 3 . \square

Del Pezzo surfaces are the Fano varieties of dimension 2, and they are classified according their degree, i.e., the self intersection number of the canonical class, which is an integer between 1 and 9.

Lemma 4.4. *Let X be a del Pezzo surface of degree d over a field k . Then*

- (i) *X acquires a rational point over a suitable finite extension of k of degree at most d .*
- (ii) *If $d = 9$, then X acquires rational points over a suitable finite extension of k of degree at most 3.*
- (iii) *If $X_{\bar{k}} \cong \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$, then X acquires rational points over a suitable finite extension of k of degree at most 4.*
- (iv) *If $d = 5$, or $d = 7$, or $d = 8$ and $X_{\bar{k}} \not\cong \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$, then X is k -rational and $X(k) \neq \emptyset$.*
- (v) *There exists a del Pezzo surface of degree 6 over \mathbb{Q} that has no rational points on any finite extension of \mathbb{Q} of degree smaller than 6.*

Proof. Part (i) is a consequence of the fact that the base locus of the anticanonical linear system has dimension at most 0. In (ii) X is a Severi-Brauer surface, and we conclude by Remark 4.1. Part (iii) follows from Lemma 4.3 if $\text{rk Pic}(X) = 2$ and from Lemma 5.10 if $\text{rk Pic}(X) = 1$. Part (iv) follows from [SD72, Sko93] for $d = 5$, and from [Man66, Theorem (3.7)] together with [VA13, §1.4] for the other cases.

For (v), let $K = \mathbb{Q}(\sqrt{29})$ and $L = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, where ζ_7 is a primitive 7th root of unity. Let E be the smallest number field containing both K and L . Then K , L and E are cyclic Galois extensions of \mathbb{Q} of degree 2, 3 and 6, respectively. Let $b = (\sqrt{29} - 5)/2$ and $q = (\zeta_7 + \zeta_7^{-1})^2 - 2$. For $F \in \{K, L, \mathbb{Q}\}$ we denote by $N_{E/F}$ the norm of E over F and by $\text{Br}(E/F)$ the Brauer group of F relative to E . Since the extensions are cyclic, by [GS06, Corollary 4.4.10] we can write $\text{Br}(E/F) = F^\times / N_{E/F}(E^\times)$. Under this identification, the corestriction maps $\text{Br}(E/K) \rightarrow \text{Br}(E/\mathbb{Q})$ and $\text{Br}(E/L) \rightarrow \text{Br}(E/\mathbb{Q})$ are induced by the norms $N_{K/\mathbb{Q}}$ and $N_{L/\mathbb{Q}}$, respectively, see [EKM08, Example 99.6]. Let $B \in \text{Br}(E/K)$ be the class of b and $Q \in \text{Br}(E/L)$ be the class of q . Computations (e.g., via the software Sage [The19]) show that

$$b \notin N_{E/K}(E^\times), \quad q \notin N_{E/L}(E^\times), \quad N_{K/\mathbb{Q}}(b), N_{L/\mathbb{Q}}(q) \in N_{E/\mathbb{Q}}(E^\times).$$

Let X be the del Pezzo surface of degree 6 associated to (B, Q, E) by [Blu10, Theorem 3.4]. By [Blu10, Corollary 3.5] X has a rational point on an extension k' of \mathbb{Q} if and only if both B and Q are split over k' . This happens if and only if $E \subseteq k'$, which implies $[k' : \mathbb{Q}] \geq 6$. \square

5. FANO THREEFOLDS

We recall the classification of terminal Fano threefolds that are mentioned in Conjecture 1.2 and in Theorem 1.3(ii), and we investigate some of their properties related to the existence of rational points for the proof of Theorem 1.6.

Unless stated otherwise, in this section X denotes a terminal Fano threefold over a field k of characteristic 0. We distinguish between Gorenstein (for example, smooth) and non-Gorenstein varieties. We recall from §1.2 that Fano varieties are classified according to the index $r(X_{\bar{k}})$, which is a positive rational number up to $\dim X + 1$, see [IP99, Corollary 2.1.13]. In the Gorenstein case $r(X_{\bar{k}})$ is a positive integer. We denote by H a fundamental divisor of $X_{\bar{k}}$, i.e., $H \in \text{Pic}(X_{\bar{k}})$ such that $-K_{X_{\bar{k}}} = r(X_{\bar{k}})H$.

The classification of Fano threefolds uses the following invariants:

- the geometric Picard rank $\text{rk}(\text{Pic}(X_{\bar{k}}))$;
- the index $r(X_{\bar{k}})$;
- the degree H^3 , i.e., the top self intersection of the fundamental divisor H .

We recall that $(-K_{X_{\bar{k}}})^3 = r(X_{\bar{k}})^3 H^3$.

We consider first Gorenstein Fano threefolds, which are well studied and mostly classified, while the classification of the non-Gorenstein ones is very far from being complete.

Lemma 5.1. *Let X be a Gorenstein terminal Fano threefold over a field k of characteristic 0.*

- (i) *If the base locus of $|-K_X|$ is nonempty, then $r(X_{\bar{k}}) = 1$ and X acquires a rational point over a suitable base field extension of degree at most 2.*
- (ii) *If $|-K_X|$ is base point free, then X acquires a rational point over a suitable base field extension of degree at most $(-K_X)^3$.*

Proof. Part (i) follows from the fact that if nonempty, the base locus of $|-K_X|$ either consists of one k -point or it is a smooth conic by [Shi89, Theorem 0.5]. Part (ii) is a consequence of Bertini's theorem. \square

The rest of the section is devoted to improve the bound in Lemma 5.1(ii). We start by collecting a few facts about indices of Fano varieties.

Lemma 5.2. *Let X be a log terminal Fano variety over a field k of characteristic 0. Let \bar{k} be a separable closure of k . Let $H \in \text{Pic}(X_{\bar{k}})$ such that $-K_{X_{\bar{k}}} = r(X_{\bar{k}})H$. Then $H \in \text{Pic}(X_{\bar{k}})^{\text{Gal}(\bar{k}/k)}$. Moreover,*

- (i) *$I(X) = I(X_{\bar{k}})$ and $q(X) \mid q(X_{\bar{k}})$;*
- (ii) *if $\text{Br}(k) = 0$ or if $X(k) \neq \emptyset$, then $r(X) = r(X_{\bar{k}})$.*

Proof. Let $q := q(X_{\bar{k}})$ and $I := I(X_{\bar{k}})$, so that $r(X_{\bar{k}}) = q/I$. Since the canonical divisor class is invariant under field extension, IK_X is a Cartier divisor class on X . Thus $I(X) = I$ and $qH = -IK_X \in \text{Pic}(X)$. Let $g \in \text{Gal}(\bar{k}/k)$. Since $qH = gqH = qgH$, the element $H - gH$ of $\text{Pic}(X_{\bar{k}})$ is torsion. But $\text{Pic}(X_{\bar{k}})$ is free by [IP99, Proposition 2.1.2]. Hence, $gH = H$. Thus H is invariant under the action of $\text{Gal}(\bar{k}/k)$ on $\text{Pic}(X_{\bar{k}})$. The divisibility condition $q(X) \mid q$ comes from the definition of $q(X)$ and $q(X_{\bar{k}})$ and the fact that $\text{Pic}(X_{\bar{k}})$ is free. If $\text{Br}(k) = 0$ or $X(k) \neq \emptyset$, the exact sequence [CTS87, (1.5.0)] gives $\text{Pic}(X) = \text{Pic}(X_{\bar{k}})^{\text{Gal}(\bar{k}/k)}$. \square

Lemma 5.3. *Let X be a projective geometrically integral variety over a field k . Let \bar{k} be a separable closure of k . Let $H \in \text{Pic}(X_{\bar{k}})^{\text{Gal}(\bar{k}/k)}$. Then there exists a field extension k'/k of degree at most $\dim_{\bar{k}} H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(H))$ such that $H \in \text{Pic}(X_{k'})$.*

Proof. By [Kol16, (7.4), Aside 32], the exact sequence [Kol16, Proposition 69] associates to H a Severi-Brauer k -variety P of dimension $\dim_{\bar{k}} H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(H)) - 1$ such that P splits over a finite extension k' of k contained in \bar{k} if and only if $H \in \text{Pic}(X_{k'})$. Then we conclude by Remark 4.1. \square

5.1. Gorenstein of geometric Picard rank 1. In this subsection let X be a Gorenstein terminal Fano threefold of geometric Picard rank 1 over a field k of characteristic 0.

5.1.1. *Index 4.* If $r(X_{\bar{k}}) = 4$, then $H^3 = 1$, $(-K_{X_{\bar{k}}})^3 = 64$, and X is a Severi Brauer variety of dimension 3. Hence, X is smooth and acquires a rational point over a suitable field extension of degree at most 4 by Remark 4.1.

5.1.2. *Index 3.* If $r(X_{\bar{k}}) = 3$, then $H^3 = 2$ and $(-K_{X_{\bar{k}}})^3 = 54$.

Lemma 5.4. *Let X be a Gorenstein terminal Fano threefold over a field k of characteristic 0. Assume that $\text{rk}(\text{Pic}(X_{\bar{k}})) = 1$ and $r(X_{\bar{k}}) = 3$. Then X is a quadric hypersurface in $\mathbb{P}_{\bar{k}}^4$. In particular, X acquires a rational point over a suitable base field extension of degree at most 2.*

Proof. By [Fuj82, Theorem 0] (cf. [IP99, Theorem 3.1.14]) we know that $X_{\bar{k}} \xrightarrow{|H|} \mathbb{P}_{\bar{k}}^4$ is a quadric hypersurface. By Lemma 5.2, H is invariant under the $\text{Gal}(\bar{k}/k)$ action on $\text{Pic}(X_{\bar{k}})$. Hence, by [Liel7, Theorem 3.4] there is a Severi-Brauer k -variety P of dimension 4 and a morphism $X \rightarrow P$ that is a k -model of the inclusion $X_{\bar{k}} \subseteq \mathbb{P}_{\bar{k}}^4$. Let α_H be the image of H in $\text{Br}(k)$ under the morphism α from [Kol16, (69.1)]. Since $X_{\bar{k}}$ is a quadric in $\mathbb{P}_{\bar{k}}^4$, we see that $2\alpha_H = 0$, but also $5\alpha_H = 0$ (e.g., by [Kol16, Proposition 44]), from which we conclude that $\alpha_H = 0$ and $P \cong \mathbb{P}_k^4$. Then X is a quadric hypersurface in \mathbb{P}_k^4 . \square

5.1.3. *Index 2.* If $r(X_{\bar{k}}) = 2$, then H^3 is an integer between 1 and 5, and $(-K_{X_{\bar{k}}})^3 = 8H^3$. We recall the classification from [Shi89, Corollary 0.8] (cf. [IP99, Theorems 3.2.5 and 3.3.1] and [Pro13a, Theorem 1.7]).

- If $H^3 = 1$, $X_{\bar{k}}$ is a sextic hypersurface in the weighted projective space $\mathbb{P}_{\bar{k}}(3, 2, 1, 1, 1)$.
- If $H^3 = 2$, $X_{\bar{k}} \xrightarrow{|H|} \mathbb{P}_{\bar{k}}^3$ is a double cover ramified along a smooth quartic surface.
- If $H^3 = 3$, $X_{\bar{k}} \xrightarrow{|H|} \mathbb{P}_{\bar{k}}^4$ is a smooth cubic hypersurface.
- If $H^3 = 4$, $X_{\bar{k}} \xrightarrow{|H|} \mathbb{P}_{\bar{k}}^5$ is a smooth complete intersection of two quadrics.
- If $H^3 = 5$, $X_{\bar{k}} \xrightarrow{|H|} \mathbb{P}_{\bar{k}}^6$ is an intersection of five quadrics. If X is smooth, $X_{\bar{k}}$ is a section of the Grassmannian $\text{Gr}(2, 5) \subseteq \mathbb{P}_{\bar{k}}^9$ by a linear space of codimension 3 under the Plücker embedding.

Lemma 5.5. *Let X be a Gorenstein terminal Fano threefold over a field k of characteristic 0. Assume that $\text{rk}(\text{Pic}(X_{\bar{k}})) = 1$ and $r(X_{\bar{k}}) = 2$. Then X acquires a rational point over a suitable field extension of degree at most $d_1(H^3)$, for*

$$(H^3, d_1(H^3)) \in \{(1, 1), (2, 8), (3, 3), (4, 16), (5, 1)\}.$$

Proof. By [IP99, Remarks 3.2.2(ii)] we know that $\dim_{\bar{k}} H^0(X_{\bar{k}}, H) = H^3 + 2$. Hence, by Lemmas 5.2 and 5.3 there exists a finite extension k' of k of degree at most $H^3 + 2$ such that $H \in \text{Pic}(X_{k'})$. By [IP99, Proposition 3.2.3] the general member of $|H|$ is a del Pezzo surface of degree H^3 over k' , and hence acquires a rational point over a further extension of degree H^3 by Lemma 4.4(i). If $H^3 = 5$, then $X_{k'}$ has a k' -point by Lemma 4.4(iv).

Let α_H be the image of H in $\text{Br}(k)$ under the morphism α from [Kol16, (69.1)]. Since $2H = -K_X \in \text{Pic}(X)$, we have $2\alpha_H = 0$ in $\text{Br}(k)$. But we have also $(H^3 + 2)\alpha_H = 0$ (e.g., by [Kol16, Proposition 44]). Hence, if $H^3 \in \{1, 3, 5\}$, we conclude that $\alpha_H = 0$, and we can choose $k' = k$. If $H^3 = 4$, we can choose k' of degree at most 4 over k by [Kol16, Theorem 53, Corollary 54]. \square

5.1.4. *Smooth of index 1.* If X is smooth and $r(X_{\bar{k}}) = 1$, then $H = -K_X \in \text{Pic}(X)$, and H^3 is an even integer between 2 and 22, $H^3 \neq 20$. Let $g := (-K_X)^3/2 + 1$. Then g is an integer between 2 and 12, $g \neq 11$. According to [Muk02], $|H|$ is base point free and it induces a morphism $X \rightarrow \mathbb{P}_k^{g+1}$. We recall the classification from [Muk02, Theorem 1.10], [IP99, Theorem 5.2.3, or §12.2]. See [Pie19] for the Galois descent of complete intersections.

- If $H^3 = 2$, then $g = 2$ and $X \xrightarrow{|H|} \mathbb{P}_k^3$ is a double cover ramified along a sextic surface.
- If $H^3 = 4$, then $g = 3$ and $X \xrightarrow{|H|} \mathbb{P}_k^4$ is either an embedding as quartic hypersurface, or a double cover of a quadric hypersurface in \mathbb{P}_k^4 .

If $g \geq 4$, H is very ample.

- If $H^3 = 6$, then $g = 4$ and $X \xrightarrow{|H|} \mathbb{P}_k^5$ is a complete intersection of a quadric and a cubic.
- If $H^3 = 8$, then $g = 5$ and $X \xrightarrow{|H|} \mathbb{P}_k^6$ is a complete intersection of three quadrics.
- If $H^3 = 10$, then $g = 6$ and $X \xrightarrow{|H|} \mathbb{P}_k^7$ is a k -form of a complete intersection of a cone over the Grassmannian $\text{Gr}(2, 5)$ under the Plücker embedding with a quadric and a linear space of codimension 3 (cf. [IP99, Examples 5.2.2(i)]).

If $g \geq 7$, a general hyperplane section of the embedding $X \xrightarrow{|H|} \mathbb{P}_k^{g+1}$ cuts a smooth K3 surface S in X by [Kol97, §7.7] and [Muk02, Proposition 7.8].

- If $H^3 = 12$, then $g = 7$. Denote by $\mathcal{N}_{S/\mathbb{P}_k^7}^*$ the dual of the normal bundle of $S \subseteq \mathbb{P}_k^7$. In the proof of [Muk02, Theorem 4.7], the vector bundle $\mathcal{N}_{S/\mathbb{P}_k^7}^* \otimes \mathcal{O}_S(2)$ determines a $\text{Gal}(\bar{k}/k)$ -equivariant embedding $X_{\bar{k}} \subseteq \text{Gr}(5, 10)$ whose image is a linear section of an orthogonal Grassmannian $\text{OGr}(5, 10, Q) \subseteq \mathbb{P}_{\bar{k}}^{15}$, where Q is a $\text{Gal}(\bar{k}/k)$ -invariant quadric hypersurface Q in $\mathbb{P}_{\bar{k}}^9$ by a version of [Muk95, Corollary 2.5] for S .

If $g \geq 8$, a general hyperplane section of $S \xrightarrow{|H|_S} \mathbb{P}_k^g$ is a smooth curve of genus g which determines a Mukai-Lazarsfeld bundle E on S (cf. [Apr13, §1.3]).

- If $H^3 = 14$, then $g = 8$ and E induces a $\text{Gal}(\bar{k}/k)$ -equivariant embedding of $X_{\bar{k}}$ in $\text{Gr}(2, 6)$ such that if $\text{Gr}(2, 6) \subset \mathbb{P}_{\bar{k}}^{14}$ is the Plücker embedding, $X_{\bar{k}}$ is a linear section of $\text{Gr}(2, 6)$ by a $\text{Gal}(\bar{k}/k)$ -invariant linear subspace of codimension 5.
- If $H^3 = 16$, then $g = 9$ and E induces a $\text{Gal}(\bar{k}/k)$ -equivariant embedding of $X_{\bar{k}}$ in $\text{Gr}(3, 6)$ such that if $\text{Gr}(3, 6) \subseteq \mathbb{P}_{\bar{k}}^{19}$ is the Plücker embedding, $X_{\bar{k}}$ is a complete intersection of $\text{Gr}(3, 6)$ with the zero locus of a $\text{Gal}(\bar{k}/k)$ -invariant global section of the second exterior power of the dual of the tautological bundle of $\text{Gr}(3, 6)$ and a $\text{Gal}(\bar{k}/k)$ -invariant linear subspace of codimension 3 (cf. [Muk02, Example 5.1]).
- If $H^3 = 18$, then $g = 10$ and E induces a $\text{Gal}(\bar{k}/k)$ -equivariant embedding of $X_{\bar{k}}$ in $\text{Gr}(5, 7)$ such that if $\text{Gr}(5, 7) \subseteq \mathbb{P}_{\bar{k}}^{20}$ is the Plücker embedding, $X_{\bar{k}}$ is a complete intersection of $\text{Gr}(5, 7)$ with the zero locus of a $\text{Gal}(\bar{k}/k)$ -invariant global section of the fourth exterior power of the dual of the tautological bundle of $\text{Gr}(5, 7)$ and a $\text{Gal}(\bar{k}/k)$ -invariant linear subspace of codimension 2 (cf. [Muk02, Example 5.2]).
- If $H^3 = 22$, then $g = 12$ and E induces a $\text{Gal}(\bar{k}/k)$ -equivariant embedding of $X_{\bar{k}}$ in $\text{Gr}(3, 7)$ such that $X_{\bar{k}}$ is the zero locus in $\text{Gr}(3, 7)$ of three linearly

independent $\text{Gal}(\bar{k}/k)$ -invariant global sections of the second exterior power of the dual of the tautological bundle of $\text{Gr}(3, 7)$ (cf. [Muk02, §5, p.15]).

Now we investigate some properties of linear sections of orthogonal Grassmannians that we can use to study rational points in the case $g = 7$.

Given a nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^9$, we denote by $\text{OGr}(5, 10, Q)$ the orthogonal Grassmannian of isotropic 4-dimensional linear subspaces of \mathbb{P}^9 . It is also known as spinor variety in \mathbb{P}^{15} . We refer to [RS00] and [Kuz18] for a detailed description.

Lemma 5.6. *Let k be a field of characteristic 0 and let \bar{k} be a separable closure of k . Let $Q \subseteq \mathbb{P}_k^9$ be a nonsingular quadric hypersurface. Let $Y \subseteq \mathbb{P}_k^{15}$ be a k -model of the spinor embedding $\text{OGr}(5, 10, Q) \subseteq \mathbb{P}_k^{15}$. Let $L \subseteq \mathbb{P}_k^{15}$ be a linear subspace of codimension 7 such that $X := L \cap Y$ is geometrically irreducible of dimension 3. Then X acquires a rational point over a suitable quadratic extension of k .*

Proof. We denote by \mathbb{P}_k^{15*} the dual projective space as in [Kuz18, Notation 3.1] and by $Y^* \subseteq \mathbb{P}_k^{15*}$ the dual of $Y \subseteq \mathbb{P}_k^{15}$. The dual L^* of L is a 6-dimensional linear subspace of \mathbb{P}_k^{15*} , hence $L^* \not\subseteq Y^*$ by [Kuz18, Theorem 3.2]. Let v be a k -point on L^* that is not contained in Y^* , and let $L_v \subseteq \mathbb{P}_k^{15}$ be the corresponding hyperplane. By [Kuz18, Lemma 5.10] the intersection $L_v \cap Y$ contains a quadric Q_v of dimension 6 defined over k . Then $L \cap Q_v$ is a nonempty k -subvariety of degree 2 in L , it is contained in X , and it acquires a rational point over a suitable extension of k of degree at most 2. \square

Lemma 5.7. *Let X be a smooth Fano threefold over a field k of characteristic 0. Assume that $\text{rk}(\text{Pic}(X_{\bar{k}})) = 1$, $r(X_{\bar{k}}) = 1$ and $H^3 = 22$. Then X acquires a rational point over a suitable field extension of degree at most 6.*

Proof. The Fano variety of conics of X is a Severi Brauer surface by [KPS18, Proposition B.4.1]. Hence, X contains smooth conics defined over a cubic extension of k by Remark 4.1, and has rational points over a further quadratic extension. \square

5.1.5. Singular Gorenstein of index 1.

Definition 5.8. The anticanonical linear system $| -K_X |$ of a Gorenstein Fano variety X of index 1 is said to have a movable decomposition if there are Weil divisors A and B such that the linear systems $|A|$ and $|B|$ have positive dimension and $A+B$ is linearly equivalent to $-K_X$. If $| -K_{X_{\bar{k}}} |$ has no movable decomposition, for an algebraic closure \bar{k} of the base field k , we say that X is indecomposable. Otherwise, we say that X is decomposable.

Indecomposable Gorenstein terminal Fano threefolds of index 1 and geometric Picard rank 1 are classified in [Muk02, Theorems 1.10, 6.5] (we remark that [Muk02, Proposition 7.8] and [Mel99, Theorem 1, Theorem 2.4] show that the classification is exhaustive), and the bounds from Section 5.1.4 apply.

Decomposable terminal Gorenstein Fano threefolds of index 1 and geometric Picard rank 1 are not completely classified. However, by [IP99, Proposition 4.1.12], [PCS05, Theorem 1.5] (cf. [PS17, Proposition 6.1.1]) we know that the cases with $g \in \{2, 3, 4, 5\}$ are completely classified in [Muk02, Theorem 6.5].

Lemma 5.9. *Let X be a Gorenstein terminal \mathbb{Q} -factorial Fano threefold over a field k of characteristic 0. Assume that $\text{rk}(\text{Pic}(X_{\bar{k}})) = 1$ and $r(X_{\bar{k}}) = 1$. Then X acquires a rational point over a finite extension of k of degree at most 18.*

Proof. By [Nam97, JR11], $X_{\bar{k}}$ admits a smoothing that preserves the Picard group and the degree $(-K_{X_{\bar{k}}})^3$. Hence, $(-K_X)^3 \in \{2, 4, 6, 8, 10, 12, 14, 16, 18, 22\}$. If $(-K_X)^3 = 22$, the only decomposable cases we are interested in are classified in

[Pro16, Theorem 1.3], and their singular locus consists of a rational point. We conclude by Lemmas 5.1 and 5.7. \square

5.2. Gorenstein of geometric Picard rank > 1 . In this subsection let X be a Gorenstein terminal Fano threefold of geometric Picard rank > 1 and Picard rank 1 over a field k of characteristic 0.

5.2.1. *Products of projective spaces.*

Lemma 5.10. *Let k be a field of characteristic 0, and let X be a twisted form of $\prod_{i=1}^n \mathbb{P}_k^m$ over k such that $\text{rk}(\text{Pic}(X)) = 1$. Then there is an extension \tilde{k} of k of degree at most $n!(m+1)$ such that $X_{\tilde{k}} \cong \prod_{i=1}^n \mathbb{P}_{\tilde{k}}^m$.*

Proof. Let e_1, \dots, e_n be the standard generators of $\text{Pic}(X_{\tilde{k}})$. From the exact sequence

$$1 \rightarrow \prod_{i=1}^n \mathbf{Aut}(\mathbb{P}_k^m) \rightarrow \mathbf{Aut}\left(\prod_{i=1}^n \mathbb{P}_k^m\right) \rightarrow S_n \rightarrow 1$$

and the induced exact sequence in étale cohomology, there exists a unique finite Galois extension k' of k of degree between n and $n!$ such that e_i is $\text{Gal}(\bar{k}/k')$ -invariant for all $i \in \{1, \dots, n\}$. For every $i \in \{1, \dots, n\}$, let α_i be the image of e_i in $\text{Br}(k')$ under the morphism α from [Kol16, (69.1)]. Then $\alpha_1, \dots, \alpha_n$ form an orbit under the conjugation action of $\text{Gal}(k'/k)$ (see [GS06, Construction 3.3.12]). Hence, they split over the same field extension \tilde{k} of k' , which has degree at most $m+1$ by Lemma 5.3. In particular, $e_1, \dots, e_n \in \text{Pic}(X_{\tilde{k}})$, and the morphism $X_{\tilde{k}} \rightarrow \prod_{i=1}^n \mathbb{P}_{\tilde{k}}^m$ induced by the product of the projections corresponding to e_1, \dots, e_n is an isomorphism. \square

5.2.2. *Smooth.* Let X be a smooth Fano threefold of geometric Picard rank > 1 and Picard rank 1. We recall the classification from [Pro13b, Theorem 1.2, §2].

If $\text{rk Pic}(X_{\tilde{k}}) = 2$, we have

- [Pro13b, Case 1.2.1 a), p. 421]: $r(X_{\tilde{k}}) = 1$, $H^3 = 12$, and $X_{\tilde{k}}$ is a divisor of bidegree $(2, 2)$ in $\mathbb{P}_{\tilde{k}}^2 \times \mathbb{P}_{\tilde{k}}^2$.
- [Pro13b, Case 1.2.1 b), p. 421]: $r(X_{\tilde{k}}) = 1$, $H^3 = 12$, and X is a double cover $\pi : X \rightarrow V$ of a variety V that belongs to Case (1.2.4) with branch locus a member of $|-K_V|$.
- [Pro13b, (1.2.2)]: $r(X_{\tilde{k}}) = 1$, $H^3 = 20$, and $X_{\tilde{k}}$ is a complete intersection of three divisors of bidegree $(1, 1)$ in $\mathbb{P}_{\tilde{k}}^3 \times \mathbb{P}_{\tilde{k}}^3$.
- [Pro13b, (1.2.3)]: $r(X_{\tilde{k}}) = 1$, $H^3 = 28$, and $X_{\tilde{k}}$ is a blow up of a quadric in $\mathbb{P}_{\tilde{k}}^4$ along a twisted quartic curve.

If $\text{rk Pic}(X_{\tilde{k}}) = 3$, we have

- [Pro13b, (1.2.4)]: $r(X_{\tilde{k}}) = 2$, $H^3 = 6$, and X is a k -model of a divisor of bidegree $(1, 1)$ in $\mathbb{P}_{\tilde{k}}^2 \times \mathbb{P}_{\tilde{k}}^2$.
- [Pro13b, (1.2.5)]: $r(X_{\tilde{k}}) = 1$, $H^3 = 12$, and $X_{\tilde{k}}$ is a double cover of $\mathbb{P}_{\tilde{k}}^1 \times \mathbb{P}_{\tilde{k}}^1 \times \mathbb{P}_{\tilde{k}}^1$ ramified along an element of $|-K_{\mathbb{P}_{\tilde{k}}^1 \times \mathbb{P}_{\tilde{k}}^1 \times \mathbb{P}_{\tilde{k}}^1}|$.
- [Pro13b, (1.2.6)]: $r(X_{\tilde{k}}) = 1$, $H^3 = 30$, and $X_{\tilde{k}}$ is a complete intersection of three divisors in $\mathbb{P}_{\tilde{k}}^2 \times \mathbb{P}_{\tilde{k}}^2 \times \mathbb{P}_{\tilde{k}}^2$ of tridegrees $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, respectively.
- [Pro13b, (1.2.7)]: $r(X_{\tilde{k}}) = 2$, $H^3 = 6$, and $X_{\tilde{k}} \cong \mathbb{P}_{\tilde{k}}^1 \times \mathbb{P}_{\tilde{k}}^1 \times \mathbb{P}_{\tilde{k}}^1$.

If $\text{rk Pic}(X_{\tilde{k}}) = 4$, we have

- [Pro13b, (1.2.8)]: $r(X_{\tilde{k}}) = 1$, $H^3 = 24$, and X is k -model of a divisor of multidegree $(1, 1, 1, 1)$ in $\mathbb{P}_{\tilde{k}}^1 \times \mathbb{P}_{\tilde{k}}^1 \times \mathbb{P}_{\tilde{k}}^1 \times \mathbb{P}_{\tilde{k}}^1$.

Lemma 5.11. *Let $n \in \{1, \dots, 8\}$, and let X be a smooth Fano threefold over a field k of characteristic 0 as in [Pro13b, (1.2.n)]. Then X acquires a rational point over a suitable base field extension of degree at most $d_2(n)$, for*

$$(n, d_2(n)) \in \{(2, 8), (3, 4), (4, 6), (6, 18), (7, 12)\}.$$

Proof. If $n = 2$, we have $\text{Pic}(X_{\bar{k}}) \cong \text{Pic}(\mathbb{P}_{\bar{k}}^3 \times \mathbb{P}_{\bar{k}}^3)$ by [Har70, Corollary IV.3.3] and $-K_X$ is an element of bidegree $(1, 1)$. By Lemma 5.10 there is an extension k' of k of degree at most 8 such that $X_{k'} \subseteq \mathbb{P}_{k'}^3 \times \mathbb{P}_{k'}^3$ is defined by three bihomogeneous polynomials f_1, f_2, f_3 of bidegree $(1, 1)$ with coefficients in k' in a set of variables $(x_0, x_1, x_2, x_3; y_0, y_1, y_2, y_3)$. Evaluating f_1, f_2, f_3 at $y_0 = y_1 = y_2 = y_3 = 1$ gives three linear forms in 4 variables, which have a nontrivial common zero $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \neq (0, 0, 0, 0)$ over k' . Then $((\tilde{x}_0 : \tilde{x}_1 : \tilde{x}_2 : \tilde{x}_3), (1 : 1 : 1 : 1))$ is a k' -point on X .

For $n = 3$, let $\varphi : X_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^4$ be the morphism induced by the blow-up and let H be the hyperplane class in $\text{Pic}(\mathbb{P}_{\bar{k}}^4)$. By [Pro13b, pp.426-427,432] there is a quadratic extension k' of k such that $\varphi^*(H)$ is invariant under the $\text{Gal}(\bar{k}/k')$ action on $\text{Pic}(X_{\bar{k}})$. By [Lie17, Theorem 3.4] there is a Severi-Brauer k' -variety P of dimension 4 and a morphism $\varphi' : X_{k'} \rightarrow P$ that is a k' -model of φ . Let α_H be the image of H in $\text{Br}(k')$ under the morphism α from [Kol16, (69.1)]. Since the image of φ is a quadric in $\mathbb{P}_{\bar{k}}^4$, we see that $2\alpha_H = 0$, but also $5\alpha_H = 0$, from which we conclude that $\alpha_H = 0$ and $P \cong \mathbb{P}_{k'}^4$. Then $X_{k'}$ is the blow up of a quadric Q in $\mathbb{P}_{k'}^4$ along a suitable quartic curve. In particular, there is a suitable quadratic extension k'' of k' such that Q has a k'' -point x and φ' is an isomorphism around x .

If $n = 4$, we have $\text{Pic}(X_{\bar{k}}) \cong \text{Pic}(\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2)$ by [Har70, Corollary IV.3.3] and H is an element of bidegree $(1, 1)$. By Lemma 5.10 there is an extension k' of k of degree at most 6 such that $X_{k'} \subseteq \mathbb{P}_{k'}^2 \times \mathbb{P}_{k'}^2$ is defined by a bihomogeneous polynomial f of bidegree $(1, 1)$ with coefficients in k' in a set of variables $(x_0, x_1, x_2; y_0, y_1, y_2)$. Evaluating f on $y_0 = y_1 = y_2 = 1$ yields a linear form in 3 variables, which has a nontrivial zero $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) \neq (0, 0, 0)$ over k' . Then $((\tilde{x}_0 : \tilde{x}_1 : \tilde{x}_2), (1 : 1 : 1))$ is a k' -point on X .

If $n = 6$, we have $\text{Pic}(X_{\bar{k}}) \cong \text{Pic}(\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2)$ by [Har70, Corollary IV.3.3] and H is an element of tridegree $(1, 1, 1)$. By Lemma 5.10 there is an extension k' of k of degree at most 18 such that $X_{k'} \subseteq \mathbb{P}_{k'}^2 \times \mathbb{P}_{k'}^2 \times \mathbb{P}_{k'}^2$ is defined by three trihomogeneous polynomials f_1, f_2, f_3 of tridegrees $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, respectively, with coefficients in k' in a set of variables $(x_0, x_1, x_2; y_0, y_1, y_2; z_0, z_1, z_2)$. Evaluating f_1 and f_2 at $z_0 = z_1 = z_2 = 1$ gives two linear forms f_1, f_2 in two distinct sets of variables. Let $(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2) \neq (0, 0, 0)$ be a nontrivial zero of f_1 over k' . Evaluating f_3 at $y_i = \tilde{y}_i$ for $i \in \{0, 1, 2\}$ gives a linear form \tilde{f}_3 . Let $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) \neq (0, 0, 0)$ be a nontrivial common zero of \tilde{f}_2 and \tilde{f}_3 over k' . Then $((\tilde{x}_0 : \tilde{x}_1 : \tilde{x}_2), (\tilde{y}_0 : \tilde{y}_1 : \tilde{y}_2), (1 : 1 : 1))$ is a k' -point on X .

For $n = 7$, we conclude by Lemma 5.10. \square

5.2.3. *Singular Gorenstein.* By [Pro13b, Theorem 6.6] singular Gorenstein terminal Fano threefolds of geometric Picard rank > 1 and Picard rank 1 have the same description (1.2.n) as in Section 5.2.2 for $n \in \{1, 2, 3, 5, 6, 8\}$.

Lemma 5.12. *Let $n \in \{1, \dots, 8\}$, and let X be a singular Gorenstein terminal Fano threefold over a field k of characteristic 0 of type (1.2.n) as in [Pro13b, Theorem 6.6]. Then X acquires a rational point over a suitable base field extension of degree at most $d_3(n)$, for*

$$(n, d_3(n)) \in \{(1, 12), (2, 20), (3, 18), (5, 12), (6, 17), (8, 17)\}.$$

Proof. Let \bar{k} be an algebraic closure of k , and let X' be a smoothing of $X_{\bar{k}}$. Since the Picard group and the degree $(-K_{X_{\bar{k}}})^3$ are preserved under smoothing by [Nam97, JR11], $X_{\bar{k}}$ contains at most

$$N(X) = 20 - \text{rk Pic}(X_{\bar{k}}) + h^{1,2}(X')$$

singular points by [PS17, (6.3.3)]. Combining Lemma 5.1 with the fact that the singular locus of $X_{\bar{k}}$ is defined over k , we conclude that X acquires a rational point over suitable base field extension of degree at most $\min\{N(X), (-K_X)^3\}$. We compute $N(X)$ by consulting [IP99, §§12.3–12.5]. \square

5.3. Non-Gorenstein terminal.

Lemma 5.13. *Let X be a non-Gorenstein terminal Fano threefold over a field k of characteristic 0. Then X acquires a rational point over a suitable field extension of degree at most 16.*

Proof. Let \bar{k} be an algebraic closure of k . The orbit of a non-Gorenstein \bar{k} -point on $X_{\bar{k}}$ under the action of $\text{Gal}(\bar{k}/k)$ has cardinality at most 16 (the computation can be found in the proof of [PS17, Lemma 4.2.1]). \square

6. FIELD OF DEFINITION OF RATIONAL POINTS

In this section we prove Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.5. For the first statement we proceed by induction on n . If $n = 0$ there is nothing to prove. Let $n > 0$. Since the degree of a projective variety is encoded in the Hilbert polynomial, by Proposition 3.4 there exists a positive integer d' such that for every terminal Fano variety X of dimension n there exists an embedding $X \subseteq \mathbb{P}_k^N$ of degree $\leq d'$. By Bertini's theorem n general hyperplanes of \mathbb{P}_k^N intersect X in a smooth subvariety of dimension 0 that acquires a rational point after a field extension of degree $\leq d'$. We retrace the proof of Theorem 1.3 to show that we can take

$$d(n) = d(n-1) \max\{d'(n), \max_{1 \leq m \leq n-1} d(m)d(n-m)\}, \quad (6.1)$$

where $d'(n)$ is a positive integer (for example d' above) such that for every field k of characteristic 0, every terminal \mathbb{Q} -factorial Fano variety of dimension n and Picard rank 1 over k acquires rational points over a suitable field extension of degree at most $d'(n)$ over k . Let X be a proper rationally connected variety of dimension n . By the argument in the proof of Lemma 3.1 we can assume without loss of generality that X is smooth and projective. Let $f : X' \rightarrow Y$ be the fibration with X' birationally equivalent to X provided by Proposition 3.3, and let $X'' \rightarrow X'$ be a resolution of singularities. Then there exists a field extension $k \subseteq k'$ of degree at most $\max\{d'(n), \max_{1 \leq m \leq n-1} d(m)d(n-m)\}$ such that $X'(k') \neq \emptyset$. Indeed, if $\dim Y = 0$, then X' acquires a rational point after a suitable field extension of degree at most $d'(n)$, as X' is a terminal \mathbb{Q} -factorial Fano k -variety of dimension n and Picard rank 1. If $\dim Y = m > 0$, then by induction hypothesis Y acquires a rational point after a suitable field extension of degree at most $d(m)$ of k and the fiber of $X' \rightarrow Y$ over such a point acquires a rational point after a further suitable field extension of degree at most $d(n-m)$ by [HX09, Theorem 1.2] and the induction hypothesis. By [HX09, Theorem 1.3] and the induction hypothesis there exists a finite field extension $k' \subseteq k''$ of degree $\leq d(n-1)$ such that $X''(k'') \neq \emptyset$. Then $X(k'') \neq \emptyset$ by [Nis55]. \square

Proof of Theorem 1.6. The bound $\tilde{d}(1) = 2$ is immediate, because all conics acquire rational points on suitable quadratic extensions and $\{x^2 + y^2 + z^2 = 0\}$ defines a conic without rational points over \mathbb{Q} . For $n = 2$, by resolution of singularities, [Nis55],

and the birational classification of proper smooth surfaces [Isk79], it suffices to find a bound for del Pezzo surfaces and rational conic bundles. Hence, $\tilde{d}(2) = 6$ follows from Lemmas 4.3 and 4.4.

For $n = 3$, let $d'(3)$ be the bound for terminal Fano threefolds of Picard rank 1. Then $d'(3) \leq 24$ by the results in Section 5, and (6.1) gives $\tilde{d}(3) \leq 6 \max\{24, 12\} = 144$. \square

7. VARIETIES WITH POINTS OVER C_1 FIELDS

In this section we find rational points on a number of rationally connected varieties over C_1 fields. The results in this section will be used for the proof of Theorem 1.8.

Remark 7.1. We recall that C_1 fields have cohomological dimension ≤ 1 and hence trivial Brauer group [Ser02, §II.3.2]. In particular, projective spaces have no nontrivial forms over C_1 fields [GS06, §5]. We also recall that algebraic extensions of C_1 fields are C_1 [Lan52, Corollary to Theorem 5].

7.1. Toric varieties. We prove that k -forms of toric varieties have rational points over C_1 fields k of characteristic 0.

Proposition 7.2. *Let k be a field of characteristic 0 of cohomological dimension ≤ 1 . Let \bar{k} be a separable closure of k . Let X be a k -variety such that $X_{\bar{k}}$ is isomorphic to a proper \mathbb{Q} -factorial toric \bar{k} -variety. Then X is an equivariant compactification of a k -torus and has a smooth k -point.*

Proof. Let Σ be a fan in \mathbb{Z}^n and Y the split toric k -variety associated to $\Sigma \subseteq \mathbb{Z}^n$. Assume that Y is proper and \mathbb{Q} -factorial (i.e., Σ is simplicial and $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$). By [Ser02, Proposition III.1.3.5], the k -forms of Y are classified by $H_{\text{ét}}^1(k, \mathbf{Aut}_Y)$ up to isomorphism. By [Cox95, Corollary 4.7] \mathbf{Aut}_Y is a k -linear algebraic group. Hence, $H_{\text{ét}}^1(k, \mathbf{Aut}_Y) \cong H_{\text{ét}}^1(k, \mathbf{Aut}_Y / \mathbf{Aut}_Y^0)$ by [Ser02, Corollary III.2.4.3]. By [Cox95, Corollary 4.7], there is a surjective homomorphism $\text{Aut}(\mathbb{Z}^n, \Sigma) \rightarrow \mathbf{Aut}_Y / \mathbf{Aut}_Y^0$, where $\text{Aut}(\mathbb{Z}^n, \Sigma)$ is the group of lattice automorphisms of \mathbb{Z}^n that preserve the fan Σ . Therefore, there is a surjective map $H_{\text{ét}}^1(k, \text{Aut}(\mathbb{Z}^n, \Sigma)) \rightarrow H_{\text{ét}}^1(k, \mathbf{Aut}_Y / \mathbf{Aut}_Y^0)$ by [Ser02, Corollary III.2.4.2]. Moreover, to every element of $H_{\text{ét}}^1(k, \text{Aut}(\mathbb{Z}^n, \Sigma))$ correspond an isomorphism class of normal k -varieties with a faithful action of a k -torus that has a dense orbit by [ELFST14, Theorems 3.2 and 3.4]. Hence, every k -form X of Y is a normal k -variety with a faithful action of a k -torus that has a dense orbit U . By [Vos82, Theorem 6], U is a principal homogeneous space under a torus T . Since tori are connected linear algebraic groups by [Bor91, p. 114], $H_{\text{ét}}^1(k, T) = 0$ by [Ser02, Corollary III.2.4.3]. Therefore, $U \cong T$ has a k -point (the unit element). Since tori are smooth, X has a smooth k -point. \square

Note that the assumption on the characteristic of k ensures that the automorphism group of the toric variety is smooth. The author does not know whether smoothness and the results in [Cox95, Corollary 4.7] hold in positive characteristics. The \mathbb{Q} -factoriality assumption can be removed using [MGSdSSdS18, Theorem 7.8].

7.2. Intersections of low degree in products of weighted projective spaces. We generalize [Lan52, Theorem 4] and [Kol96, Theorem IV.6.7] to orbit complete intersections (in the sense of [Pie19]) in forms of products of weighted projective spaces over C_1 fields of arbitrary characteristic.

We first study the forms of products of projective spaces over C_1 fields.

Lemma 7.3. *Let k be a field of cohomological dimension ≤ 1 . Let \bar{k} be a separable closure of k . Let N, n_1, \dots, n_N and $m_1 < \dots < m_N$ be positive integers. Then $H_{\acute{e}t}^1(k, \mathbf{Aut}_{\prod_{i=1}^N \prod_{j=1}^{n_i} \mathbb{P}_k^{m_i}}) = \prod_{i=1}^N H_{\acute{e}t}^1(k, S_{n_i})$.*

Proof. Since automorphisms preserve the effective cone in $\text{Pic}(Y_{\bar{k}})$ and the intersection product, $\text{Aut}(\prod_{i=1}^N \prod_{j=1}^{n_i} \mathbb{P}_k^{m_i}) = \prod_{i=1}^N \text{Aut}(\prod_{j=1}^{n_i} \mathbb{P}_k^{m_i})$ and for every $i \in \{1, \dots, N\}$ there is an exact sequence of groups

$$1 \rightarrow \prod_{j=1}^{n_i} \text{Aut}(\mathbb{P}_k^{m_i}) \rightarrow \text{Aut}\left(\prod_{j=1}^{n_i} \mathbb{P}_k^{m_i}\right) \rightarrow S_{n_i} \rightarrow 1.$$

This yields exact sequences

$$\prod_{j=1}^{n_i} H_{\acute{e}t}^1(k, \mathbf{Aut}_{\mathbb{P}_k^{m_i}}) \rightarrow H_{\acute{e}t}^1(k, \mathbf{Aut}_{\prod_{j=1}^{n_i} \mathbb{P}_k^{m_i}}) \rightarrow H_{\acute{e}t}^1(k, S_{n_i}).$$

for every $i \in \{1, \dots, N\}$, where the second arrow is surjective by [Ser02, Corollary III.2.4.2], and $H_{\acute{e}t}^1(k, \mathbf{Aut}_{\mathbb{P}_k^{m_i}}) = 0$ as $\text{Br}(k) = 0$ (cf. Remark 7.1). \square

Notation 7.4. For positive integers a_1, \dots, a_m , we denote by $\mathbb{P}_k(a_1, \dots, a_m)$ the $(m-1)$ -dimensional weighted projective space over k with weights a_1, \dots, a_m .

For every N -tuples of positive integers $\mathbf{n} = (n_1, \dots, n_N)$ and (m_1, \dots, m_N) and every $(\sum_{i=1}^N m_i)$ -tuple of positive integers $\mathbf{a} = (a_{i,l})_{\substack{1 \leq i \leq N \\ 1 \leq l \leq m_i}}$, we denote by $\mathbb{P}(\mathbf{n}, \mathbf{a})$ a k -variety such that

$$\mathbb{P}(\mathbf{n}, \mathbf{a})_{\bar{k}} \cong \prod_{i=1}^N \prod_{j=1}^{n_i} \mathbb{P}_{\bar{k}}(a_{i,1}, \dots, a_{i,m_i})$$

with a $\text{Gal}(\bar{k}/k)$ -action by permutation of the factors with exactly N orbits given by $\prod_{j=1}^{n_i} \mathbb{P}_{\bar{k}}(a_{i,1}, \dots, a_{i,m_i})_{\bar{k}}$ for all $i \in \{1, \dots, N\}$. That is, if we write $(x_{i,j,l})_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n_i \\ 1 \leq l \leq m_i}}$

for the coordinates (here $x_{i,j,1}, \dots, x_{i,j,m_i}$ are the coordinates in the j -th factor $\mathbb{P}_{\bar{k}}(a_{i,1}, \dots, a_{i,m_i})$ of the i -th orbit), then $\text{Gal}(\bar{k}/k)$ acts on the coordinates by permutations of the second index in a way that $\{x_{i,1,l}, \dots, x_{i,n_i,l}\}$ forms an orbit for all $i \in \{1, \dots, N\}$ and all $l \in \{1, \dots, m_i\}$.

Remark 7.5. Lemma 7.3 together with [Ser02, Proposition III.1.3.5] shows that all forms of products of projective spaces over a field k of cohomological dimension ≤ 1 are isomorphic to $\mathbb{P}(\mathbf{n}, \mathbf{a})$ for some \mathbf{n} and $\mathbf{a} = (1, \dots, 1)$.

Proposition 7.6. *Let k be a C_1 -field that admits normic forms of every degree. Let \bar{k} be a separable closure of k . Let \mathbf{n} and \mathbf{a} as in Notation 7.4. Let $X \subseteq \mathbb{P}(\mathbf{n}, \mathbf{a})$ be a subvariety such that $X_{\bar{k}}$ is an intersection of hypersurfaces H_1, \dots, H_s in $\mathbb{P}(\mathbf{n}, \mathbf{a})_{\bar{k}}$ of weighted multidegrees $(d_{i,j,t})_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n_i \\ 1 \leq t \leq m_i}}$ for $t \in \{1, \dots, s\}$, such that $\sum_{t=1}^s d_{i,j,t} < \sum_{l=1}^{m_i} a_{i,l}$ for all $i \in \{1, \dots, N\}$ and all $j \in \{1, \dots, n_i\}$. Assume that for some $1 \leq s_1 < \dots < s_r := s$, $s_0 := 0$, the hypersurfaces $H_{s_{i-1}+1}, \dots, H_{s_i}$ form an orbit under the $\text{Gal}(\bar{k}/k)$ -action on $\mathbb{P}(\mathbf{n}, \mathbf{a})_{\bar{k}}$ for all $i \in \{1, \dots, r\}$. Then X has a k -rational point.*

Proof. Let $R = \bar{k}[x_{i,j,l}]_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n_i \\ 1 \leq l \leq m_i}}$ be the weighted coordinate ring of $\mathbb{P}(\mathbf{n}, \mathbf{a})_{\bar{k}}$ (here the variable $x_{i,j,l}$ corresponds to the l -th coordinate in the j -th factor $\mathbb{P}_{\bar{k}}(a_{i,1}, \dots, a_{i,m_i})$ of the i -th orbit), then the group $\text{Gal}(\bar{k}/k)$ acts on the variables by permutations of the second index. Let $f_1, \dots, f_s \in R$ that define H_1, \dots, H_s , respectively, and

such that $\{f_{s_{i-1}+1}, \dots, f_{s_i}\}$ is an orbit under the $\text{Gal}(\bar{k}/k)$ -action on R for all $i \in \{1, \dots, r\}$.

Step 1. Assume that f_1, \dots, f_s are $\text{Gal}(\bar{k}/k)$ -invariant and $N = 1$. For each $l \in \{1, \dots, m_1\}$, the \bar{k} -vector space $\sum_{1 \leq j \leq n_1} x_{1,j,l} \bar{k}$ is a $\text{Gal}(\bar{k}/k)$ -invariant subset of R , hence it has a basis $z_{1,l}, \dots, z_{n_1,l}$ consisting of $\text{Gal}(\bar{k}/k)$ -invariant elements of degree $a_{1,l}$. Then a linear change of variables gives an isomorphism $\varphi : R \rightarrow \bar{k}[z_{j,l}]_{\substack{1 \leq j \leq n_1 \\ 1 \leq l \leq m_1}}$ of \bar{k} -algebras such that, for every $t \in \{1, \dots, s\}$, $\varphi(f_t)$ is a weighted homogeneous polynomial of weighted degree $\sum_{j=1}^{n_1} d_{1,j,t}$ in the variables $z_{1,1}, \dots, z_{n_1, m_1}$. Since $\sum_{\substack{1 \leq j \leq n_1 \\ 1 \leq t \leq s}} d_{1,j,t} < n_1 \sum_{l=1}^{m_1} a_{1,l}$, the system of equations $\varphi(f_1) = \dots = \varphi(f_s) = 0$ has a solution $(\tilde{z}_{1,1}, \dots, \tilde{z}_{n_1, m_1}) \in k^{n_1 m_1} \setminus \{0\}$ by [Kol96, Theorem IV.6.7]. Hence, the system $f_1 = \dots = f_s = 0$ has a $\text{Gal}(\bar{k}/k)$ -invariant solution $(\tilde{x}_{1,j,l})_{\substack{1 \leq j \leq n_1 \\ 1 \leq l \leq m_1}} \in \bar{k}^{\sum_{l=1}^{m_1} n_1 a_{1,l}}$ with $\tilde{x}_{1,\tilde{j},\tilde{l}} \neq 0$ for some $(\tilde{j}, \tilde{l}) \in \{1, \dots, n_1\} \times \{1, \dots, m_1\}$. Since $\{\tilde{x}_{1,1,\tilde{l}}, \dots, \tilde{x}_{1,n_1,\tilde{l}}\}$ is an orbit under the $\text{Gal}(\bar{k}/k)$ -action, then $\tilde{x}_{1,j,\tilde{l}} \neq 0$ for all $j \in \{1, \dots, n_1\}$. Thus $(\tilde{x}_{1,j,l})_{\substack{1 \leq j \leq n_1 \\ 1 \leq l \leq m_1}}$ defines a $\text{Gal}(\bar{k}/k)$ -invariant point in $X_{\bar{k}}(\bar{k})$, and hence a k -rational point on \bar{X} .

Step 2. Assume that f_1, \dots, f_s are $\text{Gal}(\bar{k}/k)$ -invariant and N is arbitrary. We proceed by induction on N . The case $N = 1$ is Step 1. Assume that $N > 1$. Let $F \subseteq \{f_1, \dots, f_s\}$ be the set of elements f_t such that $d_{N,j,t} = 0$ for all $j \in \{1, \dots, n_N\}$. We observe that F is invariant under the $\text{Gal}(\bar{k}/k)$ -action on R . If $F \neq \emptyset$, by hypothesis of induction the subvariety $V(F) \subseteq \prod_{i=1}^{N-1} \prod_{j=1}^{n_i} \mathbb{P}_{\bar{k}}(a_{i,1}, \dots, a_{i,m_i})$ contains a $\text{Gal}(\bar{k}/k)$ -invariant point with coordinates $(\tilde{x}_{i,j,l})_{\substack{1 \leq i \leq N-1 \\ 1 \leq j \leq n_i \\ 1 \leq l \leq m_i}} \in \bar{k}^{\sum_{i=1}^{N-1} n_i m_i}$. If $F = \emptyset$, let $\tilde{x}_{i,j,l} := 1$ for all $i \in \{1, \dots, N-1\}$, $j \in \{1, \dots, n_i\}$, $l \in \{1, \dots, m_i\}$. Evaluating f_1, \dots, f_s in $x_{i,j,l} = \tilde{x}_{i,j,l}$ for all $i \in \{1, \dots, N-1\}$, $j \in \{1, \dots, n_i\}$, $l \in \{1, \dots, m_i\}$, yields a system of $\text{Gal}(\bar{k}/k)$ -invariant forms f'_1, \dots, f'_s in $\bar{k}[x_{N,j,l}]_{\substack{1 \leq j \leq n_N \\ 1 \leq l \leq m_N}}$. By Step 1 the subvariety $V(f'_1, \dots, f'_s) \subseteq \prod_{j=1}^{n_N} \mathbb{P}_{\bar{k}}(a_{N,1}, \dots, a_{N,m_N})$ contains a $\text{Gal}(\bar{k}/k)$ -invariant point with coordinates $(\tilde{x}_{N,j,l})_{\substack{1 \leq j \leq n_N \\ 1 \leq l \leq m_N}} \in \bar{k}^{n_N m_N}$. Then $(\tilde{x}_{i,j,l})_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n_i \\ 1 \leq l \leq m_i}}$

defines a $\text{Gal}(\bar{k}/k)$ -invariant point in $X_{\bar{k}}(\bar{k})$, hence, a k -point on X .

Step 3. No restrictions on f_1, \dots, f_s . For $l = 1, \dots, r$, let $h_l := \prod_{t=s_{l-1}+1}^{s_l} f_t$. Then $h_l \in R$ is a $\text{Gal}(\bar{k}/k)$ -invariant weighted multihomogeneous element of R of weighted degree $(\sum_{t=s_{l-1}+1}^{s_l} d_{i,j,t})_{\substack{1 \leq i \leq N \\ 1 \leq j \leq n_i}}$. The subvariety of $Y_{\bar{k}}$ defined by $h_1 = \dots = h_r = 0$ has a $\text{Gal}(\bar{k}/k)$ -invariant \bar{k} -point \tilde{x} by Step 2. The point \tilde{x} belongs to at least one hypersurface in each orbit under the $\text{Gal}(\bar{k}/k)$ -action on $\{H_1, \dots, H_s\}$, hence it belongs to all. So \tilde{x} defines a k -point on X . \square

7.3. Some cyclic coverings. We study rational points over C_1 fields for cyclic coverings of complete intersections in forms of products of projective spaces that have Picard rank 1 over the base field. The assumption on the Picard rank is essential for the proof.

Proposition 7.7. *Let k be a C_1 field that admits normic forms of every degree. Let \bar{k} be a separable closure of k . Let Y be a k -variety such that $Y_{\bar{k}} \cong \prod_{i=1}^n \mathbb{P}_{\bar{k}}^m$ and $\text{rk Pic}(Y) = 1$. Let $V \subseteq Y$ be an intersection of r hypersurfaces of degrees $d_1(1, \dots, 1), \dots, d_r(1, \dots, 1)$ in $\text{Pic}(Y_{\bar{k}})$. Let $X \rightarrow V$ be a cyclic covering of degree d with reduced ramification divisor given by the restriction to V of a hypersurface of Y of degree $d_0(1, \dots, 1)$ in $\text{Pic}(Y_{\bar{k}})$. If $\sum_{i=0}^r d_i \leq m + \frac{d}{d}$, then $X(k) \neq \emptyset$.*

Proof. Let $R = \bar{k}[x_{1,0}, \dots, x_{n,m}]$ be the coordinate ring of $Y_{\bar{k}}$. By Remark 7.5 we can assume that $\text{Gal}(\bar{k}/k)$ acts by permutations of the first index. Since $\text{rk Pic}(Y) = 1$, the set $\{x_{1,j}, \dots, x_{n,j}\}$ is an orbit under the $\text{Gal}(\bar{k}/k)$ -action on R for all $j \in \{0, \dots, m\}$. Let $f_0, \dots, f_r \in R$ be $\text{Gal}(\bar{k}/k)$ -invariant homogeneous elements of degrees $d_0(1, \dots, 1), \dots, d_r(1, \dots, 1)$ such that f_1, \dots, f_r define V and f_0 defines the ramification divisor Z .

The Segre embedding $\alpha : Y_{\bar{k}} \subseteq \mathbb{P}_{\bar{k}}^{(m+1)^n-1}$ is defined over k . Let $S = \bar{k}[z_{\underline{j}} : \underline{j} \in \{0, \dots, m\}^n]$ be a coordinate ring of $\mathbb{P}_{\bar{k}}^{(m+1)^n-1}$ such that α corresponds to the morphism $\varphi : S \rightarrow R$ that sends a coordinate $z_{(j_1, \dots, j_n)}$ to $\prod_{i=1}^n x_{i,j_i}$. Let $I \subseteq S$ be the kernel of φ . Then S/I is isomorphic to a subring of R that contains f_0, \dots, f_r . Let $\tilde{f}_0, \dots, \tilde{f}_r$ be the corresponding elements in S/I . Then $X_{\bar{k}}$ is the subvariety of an $(m+1)^n$ -dimensional weighted projective space $\mathbb{P}_{\bar{k}}(1, \dots, 1, d_0/d)$ defined by the ideal $I + (\tilde{f}_1, \dots, \tilde{f}_r, t^d - \tilde{f}_0) \subseteq S[t]$ where t is a new variable of degree d_0/d .

Let $\mathbf{n} = (n)$ and $\mathbf{a} = (\underbrace{1, \dots, 1}_{m+1}, d_0/d)$. We consider the k -variety $\mathbb{P}(\mathbf{n}, \mathbf{a})$ with coordinates

$$((x_{1,0} : \dots : x_{1,m} : t_1), \dots, (x_{n,0} : \dots : x_{n,m} : t_n))$$

on $\mathbb{P}(\mathbf{n}, \mathbf{a})_{\bar{k}}$ and a $\text{Gal}(\bar{k}/k)$ -action that is compatible with the $\text{Gal}(\bar{k}/k)$ -action on $Y_{\bar{k}}$ under the embedding that identifies $Y_{\bar{k}}$ with the subvariety of $\mathbb{P}(\mathbf{n}, \mathbf{a})_{\bar{k}}$ defined by $t_1 = \dots = t_n = 0$. The coordinate ring of $\mathbb{P}(\mathbf{n}, \mathbf{a})_{\bar{k}}$ is $R' := R[t_1, \dots, t_n]$, where t_1, \dots, t_n are new variables with $\deg t_i = \frac{d_0}{d} \deg x_{i,0}$ for all $i \in \{1, \dots, n\}$. We observe that $f_1, \dots, f_r, (t_1 \cdots t_n)^d - f_0$ define a $\text{Gal}(\bar{k}/k)$ -invariant subvariety of $\mathbb{P}(\mathbf{n}, \mathbf{a})_{\bar{k}}$, which has a $\text{Gal}(\bar{k}/k)$ -invariant \bar{k} -point $\tilde{x} = (\tilde{x}_{1,0} : \dots : \tilde{x}_{n,m} : \tilde{t}_1 : \dots : \tilde{t}_n)$ by Proposition 7.6. If $\tilde{x}_{i,j} \neq 0$ for some $(i, j) \in \{1, \dots, n\} \times \{0, \dots, m\}$, then $\tilde{x}_{1,j}, \dots, \tilde{x}_{n,j} \neq 0$ as they form an orbit under the $\text{Gal}(\bar{k}/k)$ -action on \bar{k} . If $\tilde{x}_{1,0}, \dots, \tilde{x}_{n,m} = 0$, then $t_1, \dots, t_n \neq 0$ as \tilde{x} is a point of $\prod_{j=1}^n \mathbb{P}_{\bar{k}}^{m+1}$. Let $\tilde{t} := \tilde{t}_1 \cdots \tilde{t}_n$, and let $\tilde{z}_{\underline{j}}$ be the evaluation of $\varphi(z_{\underline{j}})$ at $(\tilde{x}_{1,0}, \dots, \tilde{x}_{n,m})$ for all $\underline{j} \in \{0, \dots, m\}^n$. The \bar{k} -point of $\mathbb{P}(1, \dots, 1, d_0/d)$ with coordinates $(\{\tilde{z}_{\underline{j}}\}_{\underline{j} \in \{0, \dots, m\}^n}, \tilde{t})$ is then a $\text{Gal}(\bar{k}/k)$ -point of $X_{\bar{k}}$. \square

7.4. A symmetric Cremona transformation. In this section we consider the varieties described in [Pro13b, (1.2.3)] (cf. Section 5.2). Some of the computations have been carried out using the web interface SageMathCell of the software [The19].

Notation 7.8. We denote the adjugate of an invertible matrix M by $Ad(M) := \det(M)M^{-1}$. We recall that, given an automorphism σ of a field k , a morphism $\theta : V \rightarrow W$ of k -vector spaces is called σ -linear if $\theta(av) = \sigma(a)\theta(v)$ for all $a \in k$ and $v \in V$. We denote by $\sigma : \mathbb{P}^n(k) \rightarrow \mathbb{P}^n(k)$ the σ -linear automorphism that sends $(y_0 : \dots : y_n)$ to $(\sigma^{-1}(y_0) : \dots : \sigma^{-1}(y_n))$.

Let k be a field of characteristic 0. Let $S \subseteq \mathbb{P}_k^5$ be the image of the Veronese embedding $v : \mathbb{P}_k^2 \rightarrow \mathbb{P}_k^5$,

$$v(x_0 : x_1 : x_2) = (x_0^2 : x_1^2 : x_2^2 : x_1x_2 : x_0x_2 : x_0x_1).$$

Denote by $(y_0 : \dots : y_5)$ the coordinates on \mathbb{P}_k^5 . Then S is defined by the quadrics

$$\begin{aligned} A_0 &:= y_1y_2 - y_3^2, & A_1 &:= y_0y_2 - y_4^2, & A_2 &:= y_0y_1 - y_5^2, \\ A_3 &:= y_4y_5 - y_0y_3, & A_4 &:= y_3y_5 - y_1y_4, & A_5 &:= y_3y_4 - y_2y_5, \end{aligned}$$

and A_0, \dots, A_5 is a basis of the linear system \mathcal{S} of quadrics in \mathbb{P}_k^5 containing S .

Let S_3 be the k -vector space of symmetric 3×3 matrices. We consider the isomorphism $\mathbb{P}_k^5 \rightarrow \mathbb{P}(S_3)$ induced by the isomorphism

$$\alpha : k^6 \rightarrow S_3, \quad (y_0, \dots, y_5) \mapsto \begin{pmatrix} y_0 & y_5 & y_4 \\ y_5 & y_1 & y_3 \\ y_4 & y_3 & y_2 \end{pmatrix}.$$

Every automorphism of \mathbb{P}_k^5 that preserves S belongs to the image of the embedding $\mathrm{PGL}(3, k) \subseteq \mathrm{PGL}(6, k)$ induced by the group homomorphism $\varphi : \mathrm{GL}(3, k) \subseteq \mathrm{GL}(6, k)$ that sends a matrix $M \in \mathrm{GL}(3, k)$ to the element of $\mathrm{GL}(6, k)$ defined by $\varphi(M)(y) = \alpha^{-1}(M\alpha(y)M^t)$ for all $y \in k^6$.

Let $A : \mathbb{P}_k^5 \dashrightarrow \mathbb{P}_k^5$ be the rational map defined by (A_0, \dots, A_5) . It is a symmetric Cremona transformation. For every matrix $M \in \mathrm{GL}(3, k)$ the birational map $\varphi(M) \circ A$ is a Cremona transformation of \mathbb{P}_k^5 with inverse $\varphi(M^t) \circ A$.

Let $f : Y \rightarrow \mathbb{P}_k^5$ be the blow up with center S . Then f resolves any Cremona transformation $\mathbb{P}_k^5 \dashrightarrow \mathbb{P}_k^5$ induced by the linear system \mathcal{S} (see [ESB89] for example). Let H be the pullback of a hyperplane in Y and E the exceptional divisor of f . Then $|2H - E|$ is the linear system of strict transforms of elements of \mathcal{S} under f .

Lemma 7.9. *Let L be a hyperplane in \mathbb{P}_k^5 such that $L \cap S$ spans L . Let Q be a quadric in \mathcal{S} . Let $X \subseteq Y$ be the strict transform of $L \cap Q$. Let σ be an automorphism of k . Then every σ -linear automorphism θ of X that induces a σ -linear isomorphism between the complete linear systems of $H|_X$ and $(2H - E)|_X$ is the restriction to X of a σ -linear automorphism θ_Y of Y that resolves a σ -linear Cremona transformation on \mathbb{P}_k^5 of the form $\sigma \circ \varphi(M) \circ A$ for some $M \in \mathrm{GL}(3, k)$. Moreover, if θ is an involution, so is θ_Y .*

Proof. We fix a basis z_0, \dots, z_5 of the linear system $|H|$ such that z_5 defines the hyperplane L . Let $C \in \mathrm{GL}(6, k)$ such that the basis y_0, \dots, y_5 of $|H|$ corresponding to the choice of coordinates on \mathbb{P}_k^5 can be written as $(y_0, \dots, y_5) = C(z_0, \dots, z_5)$. Let $B_0, \dots, B_4, \tilde{B}_0, \dots, \tilde{B}_4 \in \mathcal{S}$ such that $B_0|_Q, \dots, B_4|_Q$ and $\tilde{B}_0|_Q, \dots, \tilde{B}_4|_Q$ are the bases of $\mathcal{S}|_Q$ image of the basis $z_0|_L, \dots, z_4|_L$ of $|H||_L$ under the isomorphisms induced by θ and θ^{-1} , respectively. Let $R, \tilde{R} \in \mathrm{GL}(6, k)$ such that

$$\sigma(C)(B_0, \dots, B_4, Q) = R(A_0, \dots, A_5), \quad \sigma(C)(\tilde{B}_0, \dots, \tilde{B}_4, Q) = \tilde{R}(A_0, \dots, A_5)$$

as bases of \mathcal{S} . Then θ induces via f a birational map $(\sigma \circ R \circ A)|_{Q \cap L} : Q \cap L \rightarrow Q \cap L$ with inverse $(\sigma \circ \tilde{R} \circ A)|_{Q \cap L}$. Since $\sigma(S) = S$ by construction and $(\sigma \circ \tilde{R} \circ A)|_{Q \cap L} = (A \circ R^{-1} \circ \sigma^{-1})|_{Q \cap L}$ is a birational map with exceptional locus $S \cap L = \sigma(R(S \cap L))$, we have $R(S \cap L) = S \cap \sigma^{-1}(L)$; that is, R induces an isomorphism of conics between $v^{-1}(S \cap L)$ and $v^{-1}(S \cap \sigma^{-1}(L))$. Hence there is a matrix $M \in \mathrm{GL}(3, k)$ such that $(v^{-1} \circ R \circ v)|_{v^{-1}(S \cap L)}$ is the restriction of the automorphism of \mathbb{P}_k^2 defined by M . Then $\varphi(M^{-1})|_{\sigma^{-1}(L)} = R^{-1}|_{\sigma^{-1}(L)}$ because they agree on $S \cap \sigma^{-1}(L)$ and $S \cap \sigma^{-1}(L)$ spans $\sigma^{-1}(L)$. So $(A \circ R^{-1} \circ \sigma^{-1})|_{Q \cap L} = (A \circ \varphi(M^{-1}) \circ \sigma^{-1})|_{Q \cap L}$, and the σ -linear Cremona transformations on \mathbb{P}_k^5 induced by R and $\varphi(M)$ have the same restriction to $Q \cap L$.

If $\theta \circ \theta$ is the identity on X , then the automorphism

$$\varphi(\sigma(M) \cdot \mathrm{Ad}(M)^t) = \varphi(\sigma(M)) \circ A \circ \varphi(M) \circ A$$

of \mathbb{P}_k^5 restricts to the identity on an open subset of $Q \cap L$, and hence on L . Then $\sigma(M) \cdot \mathrm{Ad}(M)^t$ is an automorphism of \mathbb{P}_k^2 that restricts to the identity on $v^{-1}(L \cap S)$. Since the conic $v^{-1}(L \cap S)$ spans \mathbb{P}_k^2 , we conclude that $\sigma(M) \cdot \mathrm{Ad}(M)^t$ is the identity on \mathbb{P}_k^2 , and so $\sigma \circ \varphi(M) \circ A$ is a symmetric σ -linear Cremona transformation. \square

Let

$$\beta : k^6 \rightarrow k^6, \quad (a_0, \dots, a_5) \mapsto (a_0, a_1, a_2, 2a_3, 2a_4, 2a_5).$$

Let $\psi : \mathrm{GL}(3, k) \rightarrow \mathrm{GL}(6, k)$ be the group homomorphism defined by $\psi(M) = \beta \circ \varphi(\mathrm{Ad}(M)) \circ \beta^{-1}$. Let $M \in \mathrm{GL}(3, k)$ and $a = (a_0, \dots, a_5) \in k^6$, then

$$\sum_{i=0}^5 a_i A_i(\varphi(M)(y)) = \sum_{i=0}^5 \psi_i(M)(a) A_i(y) \quad (7.1)$$

for all $y \in k^6$, where $(\psi_0(M)(a), \dots, \psi_5(M)(a)) = \psi(M)(a) = \varphi(\mathrm{Ad}(M)^t)^t(a)$. In particular,

$$A \circ \varphi(M) \circ A = \varphi(\mathrm{Ad}(M)^t) \quad (7.2)$$

as birational maps on \mathbb{P}_k^5 .

The secant variety V of S in \mathbb{P}_k^5 is the cubic hypersurface defined by

$$y_0 y_1 y_2 + 2y_3 y_4 y_5 - y_0 y_3^2 - y_1 y_4^2 - y_2 y_5^2 = 0.$$

The action of $\mathrm{PGL}(3, k)$ on \mathbb{P}_k^5 induced by φ has three orbits: S , $V \setminus S$ and $\mathbb{P}_k^5 \setminus V$, see [Ure18, §3.1] for example. Hence, the action of $\mathrm{PGL}(3, k)$ on \mathbb{P}_k^5 induced by ψ has three orbits: $\tilde{\beta}(S)$, $\tilde{\beta}(V)$ and $\mathbb{P}_k^5 \setminus \tilde{\beta}(V)$, where $\tilde{\beta}$ is the automorphism of \mathbb{P}_k^5 induced by β . Now, $\tilde{\beta}(S)$ is defined by $A_0 \circ \beta^{-1}, \dots, A_5 \circ \beta^{-1}$, and $\tilde{\beta}(V)$ is defined by $p = 0$, where

$$p = 4y_0 y_1 y_2 + y_3 y_4 y_5 - y_0 y_3^2 - y_1 y_4^2 - y_2 y_5^2.$$

Remark 7.10. If $a = (a_0, \dots, a_5) \in k^6$ and $q_a := \sum_{i=0}^5 a_i A_i$, the matrix J_a that defines the linear system

$$\frac{\partial q_a}{\partial y_0} = \dots = \frac{\partial q_a}{\partial y_5} = 0, \quad (7.3)$$

satisfies $\det J_a = -p(a)^2$ and every 5×5 minor of J_a is divisible by $p(a)$. So, if nonempty, the linear subvariety of \mathbb{P}_k^5 defined by (7.3) has dimension ≥ 1 , and hence it intersects the quadric $q_a = 0$. In particular, q_a is smooth if and only if $p(a) \neq 0$.

7.4.1. *Smooth case.*

Lemma 7.11. *Let k be an algebraically closed field. For $a_0, \dots, a_3, b_0, b_1, b_2 \in k$, let $g_3 := a_3 x^3 + a_2 x^2 + a_1 x + a_0$, $g_2 := b_2 x^2 + b_1 x + b_0 \in k[x]$, and*

$$\delta(g_3, g_2) := b_2 \delta_1^2 - b_1 \delta_1 \delta_2 + b_0 \delta_2^2,$$

where $\delta_1 := a_3 b_0 b_1 - a_2 b_0 b_2 + a_0 b_2^2$ and $\delta_2 := a_3 b_1^2 - a_3 b_0 b_2 - a_2 b_1 b_2 + a_1 b_2^2$. If $\delta(g_3, g_2) \neq 0$, then $b_2 \neq 0$, and g_3 and g_2 have no common zero in k .

Proof. If $b_2 = 0$, then $\delta(g_3, g_2) = 0$. Assume now that $b_2 \neq 0$ and that g_3 and g_2 have a common zero $c \in k$. Then $c^2 = -\frac{b_1}{b_2} c - \frac{b_0}{b_2}$ and substituting it twice in g_3 gives $\delta_2 c + \delta_1 = 0$. If $\delta_2 = 0$, then $\delta_1 = 0$. If $\delta_2 \neq 0$, we substitute $c = -\delta_1/\delta_2$ in g_2 and obtain $\delta(g_3, g_2) = 0$. \square

Lemma 7.12. *Let k be an algebraically closed field of characteristic 0. Let Q be a smooth quadric in the linear system \mathcal{S} . Let H be a hyperplane in \mathbb{P}_k^5 . Then for a general quadric Q' in \mathcal{S} , the singular locus $\mathrm{Sing}(Q \cap Q')$ is contained in the union of three lines of \mathbb{P}_k^5 , $\mathrm{Sing}(Q \cap Q') \cap S$ is nonempty of dimension 0, and $\mathrm{Sing}(Q \cap Q') \cap S \cap H = \emptyset$.*

Proof. By Remark 7.10, the open orbit under the action of $\mathrm{PGL}(3, k)$ on $\mathbb{P}^5(k)$ via ψ is the locus of $(a_0 : \dots : a_5) \in \mathbb{P}^5(k)$ such that the quadric $\sum_{i=0}^5 a_i A_i = 0$ is smooth. Hence, by (7.1), up to an automorphism of \mathbb{P}_k^5 that fixes S , we can assume that Q is defined by the quadratic form $q := A_0 + A_3$. Let $a := (1, 0, 0, 1, 0, 0)$. Let $q'(u) := \sum_{i=0}^5 u_i A_i \in k[u_0, \dots, u_5, y_0, \dots, y_5]$, where $u := (u_0, \dots, u_5)$. For

every $b \in k^6$, let $Q'_b \subseteq \mathbb{P}_k^5$ be the quadric defined by the quadratic form $q'_b := q'(b_0, \dots, b_5) \in k[y_0, \dots, y_5]$.

Let $L \subseteq \mathbb{G}_{m,k} \times \mathbb{P}_k^5 \times \mathbb{P}_k^5$ be the subvariety defined by

$$\frac{\partial(q - tq'(u))}{\partial y_0} = \dots = \frac{\partial(q - tq'(u))}{\partial y_5} = 0, \quad (7.4)$$

where $(t, (u_0 : \dots : u_5), (y_0 : \dots : y_5))$ denotes the coordinates on $\mathbb{G}_{m,k} \times \mathbb{P}_k^5 \times \mathbb{P}_k^5$. Let $J(t, u) \in \text{Mat}_{6 \times 6}(k[t, u_0, \dots, u_5])$ be the matrix that defines (7.4) as a linear system in the variables y_0, \dots, y_5 with coefficients in $k[t, u_0, \dots, u_5]$. Then $\det(J(t, u)) = -p(a - tu)^2$ and every 5×5 minor of $J_{t,u}$ is divisible by $p(a - tu)$.

For every $b \in k^6$, the singular locus of $Q \cap Q'_b$ is contained in the subvariety of \mathbb{P}_k^5 where the Jacobian matrix of (q, q'_b) has rank 1; that is, the union of the linear subspaces $L_{c,b}$, for $c \in k^\times$, where $L_{c,b} \subseteq \mathbb{P}_k^5$ is the subvariety defined by the linear system $J(c, b)(y_0, \dots, y_5) = 0$. We observe that $L_{c,b} \neq \emptyset$ if and only if $p(a - cb) = 0$ and in that case $\dim L_{c,b} \geq 1$. Moreover, $\dim L_{c,b} \geq 2$ if and only if all the 4×4 minors of $J(c, b)$ vanish.

The 4×4 minor $m(t, u)$ of $J(t, u)$ obtained by deleting the 4th and 6th rows and the 4th and 6th columns is $-4u_1^2 t^3 (A_1(\beta^{-1}(u))t - u_2)$. If $b \in k^6$ satisfies

$$b_1 A_1(\beta^{-1}(b)) p(A_1(\beta^{-1}(b))a + b_2 b) \neq 0, \quad (7.5)$$

then $m(t, b)$ and $p(a - tb)$ have no common zeros in k^\times . Hence, for general $b \in k^6$, the singular locus of $Q \cap Q'_b$ is contained in a union of three lines of \mathbb{P}_k^5 , as $p(a - tb) \in k[t]$ has degree ≤ 3 .

The variety L is defined by equations

$$y_i = l_{t,u,i}(y_3, y_5), \quad i \in \{0, 1, 2, 4\}, \quad (7.6)$$

where $l_{t,u,i}$ are linear forms in $k(t, u_0, \dots, u_5)[y_3, y_5]$ for $i \in \{0, 1, 2, 4\}$. If $b \in k^6$ satisfies $b_2 p(b_0 a - b) \neq 0$ and (7.5), then for every $c \in k^\times$ such that $L_{c,b} \neq \emptyset$, the line $L_{c,b}$ is defined by the linear equations $y_i = l_{c,b,i}(y_3, y_5)$, for $i \in \{0, 1, 2, 4\}$, obtained by evaluating (7.6) in $t = c$ and $u = b$. By substituting these equations into the equations that define S , we see that $L_{c,b} \cap S$ is the subvariety of $L_{c,b}$ defined by the quadratic equation

$$(1 - cb_0)y_3^2 + cb_4 y_3 y_5 - cb_2 y_5^2 = 0. \quad (7.7)$$

In particular, it is nonempty and of dimension 0.

Let $h_0, \dots, h_5 \in k$ such that $h := \sum_{i=0}^5 h_i y_i$ is a linear form that defines H . Let $H_3(t, u), H_5(t, u) \in k[t, u_0, \dots, u_5]$, such that $H_3(t, u)y_3 + H_5(t, u)y_5 \in k(t, u_0, \dots, u_5)[y_3, y_5]$ is the polynomial obtained by substituting (7.6) into $m(t, u)h$. If $b \in k^6$ satisfies (7.5), then $m(c, b) \neq 0$ for all $c \in k^\times$ such that $L_{c,b} \neq \emptyset$, so the intersection $L_{c,b} \cap H$ is the subvariety of $L_{c,b}$ defined by the equation $H_3(c, b)y_3 + H_5(c, b)y_5 = 0$.

We use Lemma 7.11 to show that for general $b \in k^6$ the polynomials $H_3(t, b)$, $H_5(t, b)$ and $p(a - tb)$ have no common solutions, that is, there is no $c \in k^\times$ such that $L_{c,b} \neq \emptyset$ and $L_{c,b} \subseteq H$. Indeed, we observe that $H_5(t, u)/t \in k[t, u_0, \dots, u_5]$ is a polynomial of degree 2 in t , and that $\delta(p(a - tu), H_5(t, u)/t)$ is a nonzero element in $k[u_0, \dots, u_5]$ if $h_0 \neq 0$ or $h_1 \neq 0$ or $h_2 \neq 0$ or $h_4 \neq 0$ or $h_5 \neq 0$. If $h_0 = h_1 = h_2 = h_4 = h_5 = 0$, we observe that $H_3(t, u)/t \in k[t, u_0, \dots, u_5]$ is a polynomial of degree 2 in t and $\delta(p(a - tu), H_3(t, u)/t)$ is a nonzero element in $k[u_0, \dots, u_5]$, as $h_3 \neq 0$.

Then, for general $b \in k^6$ and for every $c \in k^\times$ such that $L_{c,b} \neq \emptyset$, the intersection $L_{c,b} \cap H$ consists of a point z , and $L_{c,b} \cap H \cap S$ is defined by evaluating (7.7) in z . Computations show that since $p(a - cb) = 0$, if $m(c, b) \neq 0$ then $L_{c,b} \cap H \cap S$ is

nonempty if and only if $C(c, b) = 0$, where

$$\begin{aligned} C(t, u) := & w_0^2 h_0^2 + w_1^2 h_1^2 + w_2^2 h_2^2 + w_1 w_2 h_3^2 + w_0 w_2 h_4^2 + w_0 w_1 h_5^2 + (w_5^2 - 2w_0 w_1) h_0 h_1 \\ & + (w_4^2 - 2w_0 w_2) h_0 h_2 - (w_0 w_3 - w_4 w_5) h_0 h_3 + w_0 w_4 h_0 h_4 + w_0 w_5 h_0 h_5 \\ & + (w_3^2 - 2w_1 w_2) h_1 h_2 + w_1 w_3 h_1 h_3 - (w_1 w_4 - w_3 w_5) h_1 h_4 + w_1 w_5 h_1 h_5 + w_2 w_3 h_2 h_3 \\ & + w_2 w_4 h_2 h_4 - (w_2 w_5 - w_3 w_4) h_2 h_5 + w_2 w_5 h_3 h_4 + w_1 w_4 h_3 h_5 + w_0 w_3 h_4 h_5 \quad (7.8) \end{aligned}$$

with $w_i := a_i - t u_i \in k[t, u_0, \dots, u_5]$ for all $i \in \{0, \dots, 5\}$. Since $\delta(p(a - tu), C(t, u))$ is a nonzero element in $k[u_0, \dots, u_5]$, we conclude by Lemma 7.11 that $L_{c,b} \cap H \cap S = \emptyset$ for general $b \in k^6$ and for every $c \in k^\times$ such that $L_{c,b} \neq \emptyset$. \square

Proposition 7.13. *Let X be a variety [Pro13b, (1.2.3)] defined over a field k of characteristic 0 and cohomological dimension ≤ 1 . Then $X(k) \neq \emptyset$.*

Proof. By [Pro13b, pp.426–427], there exists a quadratic extension k' of k such that $X_{k'}$ is a blowing up of a smooth quadric $Q' \subseteq \mathbb{P}_{k'}^4$ along a twisted quartic curve Γ . Let $\pi : X_{k'} \rightarrow \mathbb{P}_{k'}^4$ be the induced morphism. We identify $\mathbb{P}_{k'}^4$ with a hyperplane L_1 of $\mathbb{P}_{k'}^5$ such that $L_1 \cap S = \Gamma$. Let $Q_1 \subseteq \mathbb{P}_{k'}^5$ be the unique quadric containing S such that $Q_1 \cap L_1 = Q'$. Then $X_{k'}$ embeds into $Y_{k'}$ as the strict transform of $Q_1 \cap L_1$ under f , and $\pi = f|_{X_{k'}}$. By Lemma 7.9 the action of $\text{Gal}(k'/k)$ on $X_{k'}$ is induced by an action of $\text{Gal}(k'/k)$ on $Y_{k'}$. Let σ be a generator of $\text{Gal}(k'/k)$. Then $f(\sigma(f^{-1}(L_1))) = Q_1$.

By construction, $v^{-1}(\Gamma)$ is a smooth conic, so $\Gamma(k')$ is nonempty and hence dense in Γ by Remark 7.1. Then the set of hyperplanes of $\mathbb{P}_{k'}^5$ that intersect $\Gamma(k')$ is dense in $\mathbb{P}(H^0(\mathbb{P}_{k'}^5, \mathcal{O}_{\mathbb{P}_{k'}^5}(1)))$. So we can choose a general hyperplane $L_2 \subseteq \mathbb{P}_{k'}^5$ that intersects $\Gamma(k')$, such that $Q_2 := f(\sigma(f^{-1}(L_2)))$ is a general member of \mathcal{S} , and hence $Q_1 \cap Q_2$ is smooth outside the base locus S of \mathcal{S} by Bertini [Kle98, Theorem (4.1)] and smooth at all k' -points in $L_1 \cap S$ by Lemma 7.12.

Let $x \in L_2 \cap \Gamma(k')$. Let Z be the strict transform of $Q_1 \cap Q_2$ under f . Then the morphism $f|_Z : Z \rightarrow Q_1 \cap Q_2$, which is the blowing up of $Q_1 \cap Q_2$ at S , is an isomorphism around x because S is a smooth divisor in $Q_1 \cap Q_2$ and x is a smooth point of $Q_1 \cap Q_2$. We observe that $f(\sigma(Z)) = L_1 \cap L_2$. Let $L_3, L_4, L_5 \subseteq \mathbb{P}_{k'}^5$ be hyperplanes such that $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5 = \{x\}$. Then

$$f|_Z^{-1}(x) = Z \cap \sigma(Z) \cap \bigcap_{i=3}^5 (f^{-1}(L_i) \cap \sigma(f^{-1}(L_i)))$$

consists of a $\text{Gal}(k'/k)$ -invariant point of $X_{k'}$. \square

7.4.2. Singular case.

Proposition 7.14. *Let X be a variety [Pro13b, Theorem 6.6(iii)] defined over a field k of characteristic 0 with $\text{Br}(k) = 0$. Then X has a k -rational point.*

Proof. By [Pro13b, p.432] there exists a quadratic extension k' of k , contained in an algebraic closure \bar{k} of k , such that $X_{k'}$ is a blowing up of a quadric cone $Q' \subseteq \mathbb{P}_{k'}^4$ with center a curve Γ such that Γ does not contain the vertex ν of Q' and $\Gamma_{\bar{k}}$ is the union of two conics $\Gamma^{(1)}$ and $\Gamma^{(2)}$ that intersect each other transversally. Let σ be the generator of $\text{Gal}(k'/k)$, and let σ_X be the induced σ -linear automorphism of $X_{k'}$. By [Pro13b, p.432] X contains at least 2 singular points. Since $f(\text{Sing}(X_{\bar{k}})) \setminus \Gamma = \{\nu\}$, the k' -point $\gamma := f(\sigma_X(f_1^{-1}(\nu)))$ belongs to $\text{Sing}(\Gamma) = \Gamma^{(1)} \cap \Gamma^{(2)}$. Let ℓ be the strict transform of the line between ν and γ . Since $\ell \cong \mathbb{P}_{k'}^1$, it suffices to show that ℓ is invariant under σ_X , because in that case ℓ contains a σ_X -invariant k' -point, as $\text{Br}(k) = 0$.

Let $\pi : X_{k'} \rightarrow \mathbb{P}_{k'}^4$ be the morphism induced by the blowing up. Let H' be the pullback of a hyperplane of $\mathbb{P}_{k'}^4$ under π . Let E' be the exceptional divisor of the blowing up. Then $\sigma_X(H') = 2H' - E'$ in $\text{Pic}(X_{k'})$ by [Pro13b, (5.2.1), Theorem 6.1, and p.432]. Let \mathcal{S} be the linear system of quadrics in $\mathbb{P}_{k'}^4$ that contain Γ . Since π is defined by the complete linear system $|H'|$, $|2H' - E'|$ is the linear system of strict transforms of elements of $\mathcal{S}|_{Q'}$. Thus $\dim \mathcal{S} - 1 = \dim \mathcal{S}|_{Q'} = \dim |2H' - E'| = \dim H' = 5$ as k' -vector spaces.

Let $P_i \subseteq \mathbb{P}_k^4$ be the plane spanned by $\Gamma^{(i)}$ for $i \in \{1, 2\}$. We claim that $P_1 \cap P_2 = \{\gamma\}$. Clearly $P_1 \cap P_2$ contains γ because it contains the intersection $\Gamma^{(1)} \cap \Gamma^{(2)}$. If $P_1 = P_2$ then $\dim \mathcal{S} \geq 9$ as k' -vector space, which is a contradiction. If $P_1 \cap P_2$ has dimension 1, the linear system \mathcal{S}' of quadrics containing $P_1 \cup P_2$ has dimension 6, hence $\mathcal{S} = \mathcal{S}'$. So the base locus of \mathcal{S} contains $(P_1 \cup P_2) \setminus \Gamma$, contradicting the fact that $\sigma_X(|H'|) = |2H' - E'|$ is base point free.

Now we show that X is a degeneration of a Fano threefold as in Proposition 7.13. Let k''/k' be a quadratic extension such that $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are defined on k'' . Let $u \in k''$ such that $k'' = k'(u)$ and $u^2 \in k'$. Without loss of generality, we can choose coordinates z_0, \dots, z_4 on $\mathbb{P}_{k'}^4$ such that $\Gamma^{(1)} = \{z_3 - uz_2 = z_1 - uz_4 = z_0z_2 - z_4^2\}$ and $\Gamma^{(2)} = \{z_3 + uz_2 = z_1 + uz_4 = z_0z_2 - z_4^2\}$. Let $w : \mathbb{P}_{k'}^4 \subseteq \mathbb{P}_{k'}^5$ be the embedding given by $(z_0 : \dots : z_4) \mapsto (z_0 : u^2z_2 : z_2 : z_3 : z_4 : z_1)$. Then the image of w is the hyperplane $L = \{y_1 - u^2y_2 = 0\}$, also $w(\Gamma) = L \cap S$ and $v^{-1}(L \cap S)$ is the degenerate conic $\{x_1^2 - u^2x_2^2 = 0\} \subseteq \mathbb{P}_{k'}^2$. Since $\dim \mathcal{S} = 6 = \dim \mathcal{S}'$ as k' -vector spaces and $\mathcal{S}'|_L = \mathcal{S}$, there exists a unique quadric Q in \mathcal{S}' such that $Q \cap L = w(Q')$. Then $X_{k'}$ embeds into $Y_{k'}$ as the strict transform of $Q \cap L$ under f , and $\pi = f|_{X_{k'}}$.

By Lemma 7.9 there is $M \in \text{GL}(3, k')$ such that the σ -linear Cremona transformation $\sigma \circ \varphi(M) \circ A$ defines a σ -linear involution of Y that restricts to σ_X on $X_{k'}$. The quadric Q is the image of L under $\sigma \circ \varphi(M) \circ A$ and has equation $q = (0, 1, -\sigma(u)^2, 0, 0, 0) \cdot \varphi(M) \cdot (A_0, \dots, A_5)$. Let $U := (Q \cap L) \setminus \pi(E' \cup \sigma_X(E'))$. Computations show that $\pi(\ell) \cap \pi(E' \cup \sigma_X(E')) = \{\gamma, \nu\}$, and $(\sigma \circ \varphi(M) \circ A)(\pi(\ell) \cap U) = \pi(\ell) \cap U$. Thus, ℓ is invariant under σ_X . \square

8. TERMINAL FANO VARIETIES OF LARGE INDEX

Proof of Theorem 1.7. We denote by \bar{k} an algebraic closure of k . We recall that $r(X) \leq n + 1$ by a result of Shokurov (cf. [IP99, Corollary 2.1.13]).

Gorenstein terminal Fano varieties of index $r(X) \geq n$ are classified in [Fuj82, Theorem 0] (cf. [IP99, Theorem 3.1.14]). If $r(X) = n + 1$, then $X_{\bar{k}} \cong \mathbb{P}_{\bar{k}}^n$, hence, $X \cong \mathbb{P}_k^n$ by Remark 7.1. If $r(X) = n$, then the linear system $|-\frac{1}{n}K_X|$ embeds X as a quadric hypersurface in \mathbb{P}_k^{n+1} . Hence $X(k) \neq \emptyset$ by definition of C_1 field.

Gorenstein terminal Fano varieties of index $r(X) = n - 1$ are del Pezzo varieties (see [IP99, Remarks 3.2.2]). Let X be a del Pezzo k -variety of dimension n . We prove that $X(k) \neq \emptyset$ by induction on n . If $n = 2$, $X_{\bar{k}}$ is a smooth rational surface, hence $X(k) \neq \emptyset$ by [CT87, Proposition 2]. Assume that $n > 2$. By [IP99, Proposition 3.2.3] the general element of the linear system $-\frac{1}{n-1}K_X$ is a del Pezzo k -variety of dimension $n - 1$, hence it has a k -rational point by inductive hypothesis. Thus $X(k) \neq \emptyset$.

Non-Gorenstein terminal Fano varieties of index $r(X) > n - 2$ are classified in [San96] and have a rational point by Proposition 7.2 and [Kol96, Theorem IV.6.7]. \square

9. TERMINAL FANO THREEFOLDS

9.1. Gorenstein case. We prove Theorem 1.8.

Proof of Theorem 1.8. We recall that the index $r(X)$ of a Gorenstein terminal Fano variety X is a positive integer. Terminal Fano threefolds of index ≥ 2 are covered by Theorem 1.7. Hence, we need to consider only the cases of index 1.

\mathbb{Q} -factorial Gorenstein terminal Fano threefolds of Picard rank 1 and geometric Picard rank > 1 are classified in [Pro13b, Theorem 1.2, §2, Theorem 6.6] (cf. Section 5.2).

The varieties [Pro13b, (1.2.4), (1.2.7)] have index 2 by [IP99, Remarks (vi) p. 217].

The varieties [Pro13b, Case 1.2.1 a), p. 421] and [Pro13b, (1.2.2), (1.2.8)] are k -models of complete intersections of divisors in products of projective spaces that have a k -rational point by [Pie19, Theorem 1.1] and Proposition 7.6 (cf. [Pro13b, Remark 2.1]).

The variety [Pro13b, Case 1.2.1 b), p. 421] is a double cover $\pi : X \rightarrow V$ of a variety V that belongs to [Pro13b, (1.2.4)] with branch locus a member of $|-K_V|$. The morphism $\pi : X_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$ is defined over k (see [Pro13b, proof of Theorem 6.6 (ii)]) and the $\text{Gal}(\bar{k}/k)$ -action on $V_{\bar{k}}$ is induced by the one on $\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$. We observe that V is smooth by [Pro13b, Theorem 6.6 (i)], and $\text{Pic}(V_{\bar{k}}) \cong \text{Pic}(\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2)$ by [Har70, Corollary IV.3.3]. Then $\text{rk Pic}(\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2)^{\text{Gal}(\bar{k}/k)} = \text{rk Pic}(V) = 1$. The variety $V_{\bar{k}}$ is a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$. Moreover, $-K_V = (2, 2)$ in $\text{Pic}(V_{\bar{k}})$. Hence, the branch locus of π is defined by the restriction to V of a hypersurface of degree $(2, 2)$ in $\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$. Then $X(k) \neq \emptyset$ by Proposition 7.7.

The varieties [Pro13b, (1.2.3)] have a k -rational point by Propositions 7.13 and 7.14.

The variety [Pro13b, (1.2.5)] is a k -model X of a double cover of $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ with branch locus a member of $|-K_{\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1}|$ (which is a divisor of tridegree $(2, 2, 2)$). By [Pro13b, Lemma 4.4], the composition with the first projection $f_1 : X_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^1$ is a del Pezzo bundle with general fiber F_1 , let $\{F_1, F_2, F_3\}$ be the $\text{Gal}(\bar{k}/k)$ -orbit of F_1 in $\text{Pic}(X_{\bar{k}})$. Then there are other two del Pezzo bundles f_2, f_3 conjugate to f_1 . Then product $f = f_1 \times f_2 \times f_3 : X_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ is a finite map and it is defined over k . Thus X is a double cover of a k -model Y of $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$. Since $\text{rk Pic}(X) = 1$, also $\text{rk Pic}(Y) = 1$ and hence $\text{Pic}(Y)$ is the subgroup of $\text{Pic}(\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1)$ generated by $(1, 1, 1)$. Then X has a k -point by Proposition 7.7.

The variety [Pro13b, (1.2.6)] is a k -model X of an intersection of divisors of tridegrees $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ in $\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$. Let $\pi_1, \pi_2, \pi_3 : \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2 \rightarrow \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$ be the three projections. The $\text{Gal}(\bar{k}/k)$ -action on $\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$ is by permutation of the factors by Lemma 7.3, and $\text{rk Pic}(\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2)^{\text{Gal}(\bar{k}/k)} = 1$. So, for every $i, j \in \{1, 2, 3\}$, there exists $g_{i,j} \in \text{Gal}(\bar{k}/k)$ such that $\pi_j = \pi_i \circ g_{i,j}$. Then $\pi_i(X) = \pi_i \circ g_{i,j}(X) = \pi_j(X)$ for all $i, j \in \{1, 2, 3\}$. By [Pro13b, Case 1.2.6, p. 422], $\pi_1(X_{\bar{k}}) \subseteq \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$ is a hypersurface of degree $(1, 1)$. So $X_{\bar{k}} = \pi_1^{-1}(\pi_1(X)) \cap \pi_2^{-1}(\pi_1(X)) \cap \pi_3^{-1}(\pi_1(X))$. Since $\pi_j^{-1}(\pi_1(X)) = g_{i,j}^{-1}(\pi_i^{-1}(\pi_1(X)))$ for every $i, j \in \{1, 2, 3\}$, $X_{\bar{k}}$ is a complete intersections of hypersurfaces H_1, H_2, H_3 in $\mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2 \times \mathbb{P}_{\bar{k}}^2$ of degrees $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ respectively, such that $\{H_1, H_2, H_3\}$ is a $\text{Gal}(\bar{k}/k)$ -invariant set under the action of $\text{Gal}(\bar{k}/k)$ over $X_{\bar{k}}$. Hence, $X(k) \neq \emptyset$ by Proposition 7.6.

Indecomposable Gorenstein terminal Fano threefolds of index 1 and geometric Picard rank 1 are classified in [Muk02, Theorems 1.10, 6.5] (cf. Sections 5.1.4 and 5.1.5). Let X be an indecomposable Gorenstein terminal Fano threefold of index 1 and geometric Picard rank 1. We use the notation introduced in Section 5.1.4.

If $g = 2$, then $X \subseteq \mathbb{P}(1, 1, 1, 1, 3)$ is a hypersurface of degree 6. Thus $X(k) \neq \emptyset$ by [Kol96, Theorem IV.6.7].

If $g = 3$ and X is a quartic hypersurface in \mathbb{P}_k^4 , then X has a k -rational point by definition of C_1 field. If $g = 3$ and X is a double cover of a quadric hypersurface Q in \mathbb{P}_k^4 , then it is ramified along the restriction to Q of a hypersurface of degree 4 (see [Pro04, (4.3.2)]). Thus $X \subseteq \mathbb{P}(1, 1, 1, 1, 1, 2)$ is a complete intersection of two hypersurfaces of degrees 2 and 4, and $X(k) \neq \emptyset$ by [Kol96, Theorem IV.6.7].

If $g \in \{4, 5\}$ the variety X has a k -rational point by [Lan52, Theorem 4].

If $g = 12$ and X is singular, then its singular locus consists of a k -point by [Pro16, Theorem 1.3]. If $g = 12$ and X is smooth, then the Fano variety of conics of X is isomorphic to \mathbb{P}_k^2 by [KPS18, Proposition B.4.1] and Remark 7.1. Hence, X contains smooth conics defined over k , which have k -points again by Remark 7.1. \square

As recalled in Section 5.1.5, decomposable terminal Gorenstein Fano threefolds of index 1 and geometric Picard rank 1 are not completely classified. However, the cases with $g \in \{2, 3, 4, 5\}$ are completely classified in [Muk02, Theorem 6.5]. Moreover, if $g = 12$ the only decomposable cases we are interested in are classified in [Pro16, Theorem 1.3]. These cases are covered by Theorem 1.8.

9.2. Non-Gorenstein case. We start by considering non-Gorenstein terminal Fano threefolds of index 1 with only cyclic quotient singularities, which are classified in [San95].

Theorem 9.1. *Let X be a non-Gorenstein terminal Fano threefold of index 1 with only cyclic quotient singularities over a C_1 field k of characteristic 0 that admits normic forms of arbitrary degree. Then $X(k) \neq \emptyset$.*

Proof. Let \bar{k} be an algebraic closure of k . By Lemma 5.2 the double covering in [San95, Theorem 1.1] is defined over k , we denote it by $Y \rightarrow X$. Moreover, Y is a smooth Fano threefold, and there are 14 possible cases for Y .

In the cases [San95, Theorem 1.1, No. 8, 12, 14] the variety Y has index 2, then $Y(k) \neq \emptyset$ by Theorem 1.7. In the remaining cases the variety Y has index 1. If Y has Picard rank 1 over k , then $Y(k) \neq \emptyset$ by Theorem 1.8. Therefore we can assume that Y has Picard rank ≥ 2 over k .

In the case [San95, Theorem 1.1, No. 3, 7] (see also [IP99, §12.3, No. 3, 10]), the variety Y is a blow-up of a smooth Fano threefold of index 2 that has a k -point by Theorem 1.7. Then $Y(k) \neq \emptyset$ by [Nis55].

In the cases [San95, Theorem 1.1, No. 4, 10, 13], by studying the automorphisms of $Y_{\bar{k}}$ as in [Bay94, §§6.5.1, 6.6.1, 6.6.2] we conclude that Y is isomorphic to $C \times_k S_d$, where C is a smooth conic and S_d is a smooth del Pezzo surface of degree $d \in \{2, 4, 6\}$. Since $C(k) \neq \emptyset$ by Remark 7.1 and $S_d(k) \neq \emptyset$ by [CT87, Proposition 2], we conclude that $Y(k) \neq \emptyset$.

In the case [San95, Theorem 1.1, No. 5] (see also [IP99, §12.4, No. 1]), the variety Y is a double covering of a k -form Z of $\mathbb{P}_k^1 \times \mathbb{P}_k^1 \times \mathbb{P}_k^1$ ramified along a divisor of tridegree $(2, 2, 2)$. Since $\text{rk Pic}(Y_{\bar{k}}) = 3$ we deduce that $\text{rk Pic}(Z) = \text{rk Pic}(Y) \geq 2$. By Lemma 7.3 this can only happen if Z is a product of a smooth conic C with a k -form of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. By [Pro13b, Lemma 4.4], the composition with the projection $Z \rightarrow C$ is a del Pezzo bundle structure on Y . By Remark 7.1 $C(k)$ is nonempty, and hence Zariski dense in C . Thus Y contains a smooth del Pezzo surface defined over k that has a k -point by [CT87, Proposition 2].

In the case [San95, Theorem 1.1, No. 6], since $\text{rk Pic}(Y) = 2$, the two extremal contractions from [IP99, Case (C_1, C_1) , p.140] are defined over k . Hence there is a dominant morphism $Y \rightarrow \mathbb{P}_k^2$ whose general fiber is a smooth conic, and hence contains k -points by Remark 7.1.

In the case [San95, Theorem 1.1, No. 9], Y is a k -model of a complete intersection of three divisors of tridegree $(1, 1)$ in $\mathbb{P}_k^3 \times \mathbb{P}_k^3$, hence Y is a complete intersection of three divisors of bidegree $(1, 1)$ in a k -form of $\mathbb{P}_k^3 \times \mathbb{P}_k^3$ by [Pie19, Theorem 1.1] and $Y(k) \neq \emptyset$ by Proposition 7.6.

In the case [San95, Theorem 1.1, No. 11], Y is a k -model of a divisor of multi-degree $(1, 1, 1, 1)$ in $\mathbb{P}_k^1 \times \mathbb{P}_k^1 \times \mathbb{P}_k^1 \times \mathbb{P}_k^1$, hence $Y(k) \neq \emptyset$ by Proposition 7.6. \square

The Graded Ring Database [B⁺] contains the list of possible baskets [BS07, §2] of non-Gorenstein singularities for geometrically \mathbb{Q} -factorial non-Gorenstein terminal Fano threefolds of geometric Picard rank 1 over an algebraically closed field of characteristic 0. Such varieties X are studied in [BS07] using the Fano index $f(X)$ defined in the introduction. The Fano index $f(X)$ divides $q(X)$ and they coincide if there is no torsion in the Weil divisor class group (see [Pro07, Corollary 2.3] and [Pro10, Lemma 3.2]). In particular, $f(X) \in \{1, \dots, 9, 11, 13, 17, 19\}$ by [Suz04].

Remark 9.2. The basket of singularities of a terminal Fano variety was introduced in [Rei87, (8.2), (10.2)]. It is a collection of quotient singularity germs. To each non-Gorenstein singular point of the variety there is an associated collection of quotient singularity germs. The disjoint union of such collections forms the basket of the variety. The collection of quotient singularity germs associated to a singular point is invariant under the automorphisms of the variety. Therefore, if a given quotient singularity germ appears only once in a basket (we say it has multiplicity one in the basket), then the corresponding singular point is a fixed point for all the automorphisms of the variety.

Theorem 9.3. *Let k be a field of characteristic 0 and \bar{k} an algebraic closure. Let X be a non-Gorenstein terminal Fano threefold over k such that $X_{\bar{k}}$ is \mathbb{Q} -factorial and $\text{rk Pic}(X_{\bar{k}}) = 1$.*

- (1) *If $f(X_{\bar{k}}) \geq 6$, then $X(k) \neq \emptyset$.*
- (2) *If $f(X_{\bar{k}}) = 5$ and k is C_1 and admits normic forms of arbitrary degree, then $X(k) \neq \emptyset$.*

Proof. By inspection in the Graded Ring Database [B⁺, Fano 3-folds], if $f(X_{\bar{k}}) \geq 6$ each possible basket of singularities for X contains a quotient singularity germ that appears with multiplicity one. Hence, the corresponding singular point on $X(\bar{k})$ is invariant under the action of $\text{Gal}(\bar{k}/k)$ by Remark 9.2. If $f(X_{\bar{k}}) = 5$ the same argument works for all possible baskets of singularities except one [B⁺, Fano 3-folds, ID 41439], which is realized by a hypersurface of degree 6 in $\mathbb{P}(1, 2, 2, 3, 3)$, and hence has a k -point by [Kol96, Theorem IV.6.7]. \square

The Graded Ring Database [B⁺] contains 1847 cases with Fano index $f \in \{2, 3, 4\}$ and 52646 cases with Fano index $f = 1$. The arguments used in Theorem 9.3 work for all except 9 cases if $f = 4$, for all except 13 cases if $f = 3$, for all except 109 cases if $f = 2$, and for at least 1581 cases if $f = 1$ (the basket has cardinality 1 in 1415 cases, the variety is a Fano complete intersection of codimension at most 2 in a weighted projective space in 166 further cases).

REFERENCES

- [Apr13] M. Aprodu. Lazarsfeld-Mukai bundles and applications. In *Commutative algebra*, pages 1–23. Springer, New York, 2013.
- [B⁺] G. Brown et al. Graded Ring Database. <http://www.grdb.co.uk/>.
- [Bay94] L. Bayle. Classification des variétés complexes projectives de dimension trois dont une section hyperplane générale est une surface d’Enriques. *J. Reine Angew. Math.*, 449:9–63, 1994.
- [BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

- [Bir21] C. Birkar. Singularities of linear systems and boundedness of Fano varieties. *Ann. of Math. (2)*, 193(2):347–405, 2021.
- [Blu10] M. Blunk. Del Pezzo surfaces of degree 6 over an arbitrary field. *J. Algebra*, 323(1):42–58, 2010.
- [Bor91] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [BS07] G. Brown and K. Suzuki. Computing certain Fano 3-folds. *Japan J. Indust. Appl. Math.*, 24(3):241–250, 2007.
- [Cam92] F. Campana. Connexit e rationnelle des vari et es de Fano. *Ann. Sci.  cole Norm. Sup. (4)*, 25(5):539–545, 1992.
- [Cox95] D. A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.
- [CPP02] F. Campana, T. Peternell, and A. V. Pukhlikov. The generalized Tsen theorem and rationally connected Fano fibrations. *Mat. Sb.*, 193(10):49–74, 2002.
- [CT87] J.-L. Colliot-Th el ene. Arithm etique des vari et es rationnelles et probl emes birationnels. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 641–653. Amer. Math. Soc., Providence, RI, 1987.
- [CT11] J.-L. Colliot-Th el ene. Vari et es presque rationnelles, leurs points rationnels et leurs d eg en escences. In *Arithmetic geometry*, volume 2009 of *Lecture Notes in Math.*, pages 1–44. Springer, Berlin, 2011.
- [CTS87] J.-L. Colliot-Th el ene and J.-J. Sansuc. La descente sur les vari et es rationnelles. II. *Duke Math. J.*, 54(2):375–492, 1987.
- [dFF13] T. de Fernex and D. Fusi. Rationality in families of threefolds. *Rend. Circ. Mat. Palermo (2)*, 62(1):127–135, 2013.
- [dJS03] A. J. de Jong and J. Starr. Every rationally connected variety over the function field of a curve has a rational point. *Amer. J. Math.*, 125(3):567–580, 2003.
- [DK17] B. Duesler and A. Knecht. Rationally connected varieties over the maximally unramified extension of p -adic fields. *Rocky Mountain J. Math.*, 47(8):2605–2617, 2017.
- [DKT99] S. Ding, M.-C. Kang, and E.-T. Tan. Chiungtze C. Tsen (1898–1940) and Tsen’s theorems. *Rocky Mountain J. Math.*, 29(4):1237–1269, 1999.
- [EKM08] R. Elman, N. Karpenko, and A. Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.
- [ELFST14] E. J. Elizondo, P. Lima-Filho, F. Sottile, and Z. Teitler. Arithmetic toric varieties. *Math. Nachr.*, 287(2-3):216–241, 2014.
- [ESB89] L. Ein and N. Shepherd-Barron. Some special Cremona transformations. *Amer. J. Math.*, 111(5):783–800, 1989.
- [Esn03] H. Esnault. Varieties over a finite field with trivial Chow group of 0-cycles have a rational point. *Invent. Math.*, 151(1):187–191, 2003.
- [Fuj82] T. Fujita. Classification of projective varieties of Δ -genus one. *Proc. Japan Acad. Ser. A Math. Sci.*, 58(3):113–116, 1982.
- [GHS03] T. Graber, J. Harris, and J. Starr. Families of rationally connected varieties. *J. Amer. Math. Soc.*, 16(1):57–67 (electronic), 2003.
- [GNT19] Y. Gongyo, Y. Nakamura, and H. Tanaka. Rational points on log Fano threefolds over a finite field. *J. Eur. Math. Soc.*, electronically published on August 13, 2019.
- [GS06] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology*, volume 101 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.
- [Gui14] R. Guilbot. Low degree hypersurfaces of projective toric varieties defined over C_1 fields have a rational point. arXiv:1407.6945, 2014.
- [Har70] R. Hartshorne. *Ample subvarieties of algebraic varieties*. Lecture Notes in Mathematics, Vol. 156. Springer-Verlag, Berlin-New York, 1970. Notes written in collaboration with C. Musili.
- [Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [Hir64] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)* **79** (1964), 109–203; *ibid.* (2), 79:205–326, 1964.
- [HX09] A. Hogadi and C. Xu. Degenerations of rationally connected varieties. *Trans. Amer. Math. Soc.*, 361(7):3931–3949, 2009.

- [IP99] V. A. Iskovskih and Yu. G. Prokhorov. Fano varieties. In *Algebraic geometry*, V, volume 47 of *Encyclopaedia Math. Sci.*, pages 1–247. Springer, Berlin, 1999.
- [Isk79] V. A. Iskovskih. Minimal models of rational surfaces over arbitrary fields. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):19–43, 237, 1979.
- [JR11] P. Jahnke and I. Radloff. Terminal Fano threefolds and their smoothings. *Math. Z.*, 269(3-4):1129–1136, 2011.
- [Kle98] S. L. Kleiman. Bertini and his two fundamental theorems. *Rend. Circ. Mat. Palermo (2) Suppl.*, (55):9–37, 1998. Studies in the history of modern mathematics, III.
- [KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
- [KMM92] J. Kollár, Y. Miyaoka, and S. Mori. Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom.*, 36(3):765–779, 1992.
- [Kol96] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.
- [Kol97] J. Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997.
- [Kol13] J. Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.
- [Kol16] J. Kollár. Severi-Brauer varieties; a geometric treatment. arXiv:1606.04368v1, 2016.
- [KPS18] A. G. Kuznetsov, Y. G. Prokhorov, and C. A. Shramov. Hilbert schemes of lines and conics and automorphism groups of Fano threefolds. *Jpn. J. Math.*, 13(1):109–185, 2018.
- [Kuz18] A. G. Kuznetsov. On linear sections of the spinor tenfold. I. *Izv. Ross. Akad. Nauk Ser. Mat.*, 82(4):53–114, 2018.
- [Lam05] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.
- [Lan52] S. Lang. On quasi algebraic closure. *Ann. of Math. (2)*, 55:373–390, 1952.
- [Lie17] C. Liedtke. Morphisms to Brauer-Severi varieties, with applications to del Pezzo surfaces. In *Geometry over nonclosed fields*, Simons Symp., pages 157–196. Springer, Cham, 2017.
- [Man66] Y. I. Manin. Rational surfaces over perfect fields. *Inst. Hautes Études Sci. Publ. Math.*, (30):55–113, 1966.
- [Mel99] M. Mella. Existence of good divisors on Mukai varieties. *J. Algebraic Geom.*, 8(2):197–206, 1999.
- [MGSdSSdS18] J. P. Moreno Gonzalez, C. Sancho de Salas, and M. T. Sancho de Salas. Automorphism group of a toric variety. arXiv:1809.09070v1, 2018.
- [Muk95] S. Mukai. Curves and symmetric spaces. I. *Amer. J. Math.*, 117(6):1627–1644, 1995.
- [Muk02] S. Mukai. New developments in the theory of Fano threefolds: vector bundle method and moduli problems [translation of Sūgaku 47 (1995), no. 2, 125–144; MR1364825 (96m:14059)]. *Sugaku Expositions*, 15(2):125–150, 2002. Sugaku expositions.
- [Nam97] Y. Namikawa. Smoothing Fano 3-folds. *J. Algebraic Geom.*, 6(2):307–324, 1997.
- [Nis55] H. Nishimura. Some remarks on rational points. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.*, 29:189–192, 1955.
- [PCS05] V. V. Przyjalkowski, I. A. Cheltsov, and K. A. Shramov. Hyperelliptic and trigonal Fano threefolds. *Izv. Ross. Akad. Nauk Ser. Mat.*, 69(2):145–204, 2005.
- [Pie19] M. Pieropan. On Galois descent of complete intersections. *Math. Res. Lett.*, to appear, arXiv:1905.00227v2, 2019.
- [Pop96] F. Pop. Embedding problems over large fields. *Ann. of Math. (2)*, 144(1):1–34, 1996.
- [Pro04] Y. G. Prokhorov. A remark on Fano threefolds with canonical Gorenstein singularities. In *The Fano Conference*, pages 647–657. Univ. Torino, Turin, 2004.
- [Pro07] Y. G. Prokhorov. The degree of \mathbb{Q} -Fano threefolds. *Mat. Sb.*, 198(11):153–174, 2007.

- [Pro10] Y. Prokhorov. \mathbb{Q} -Fano threefolds of large Fano index, I. *Doc. Math.*, 15:843–872, 2010.
- [Pro13a] Y. Prokhorov. G -Fano threefolds, I. *Adv. Geom.*, 13(3):389–418, 2013.
- [Pro13b] Y. Prokhorov. G -Fano threefolds, II. *Adv. Geom.*, 13(3):419–434, 2013.
- [Pro13c] Yu. G. Prokhorov. On birational involutions of \mathbb{P}^3 . *Izv. Ross. Akad. Nauk Ser. Mat.*, 77(3):199–222, 2013.
- [Pro16] Y. G. Prokhorov. Singular Fano manifolds of genus 12. *Mat. Sb.*, 207(7):101–130, 2016.
- [PS16] Y. Prokhorov and C. Shramov. Jordan property for Cremona groups. *Amer. J. Math.*, 138(2):403–418, 2016.
- [PS17] Y. Prokhorov and C. Shramov. Jordan constant for Cremona group of rank 3. *Mosc. Math. J.*, 17(3):457–509, 2017.
- [Rei87] M. Reid. Young person’s guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. Amer. Math. Soc., Providence, RI, 1987.
- [RS00] K. Ranestad and F.-O. Schreyer. Varieties of sums of powers. *J. Reine Angew. Math.*, 525:147–181, 2000.
- [San95] T. Sano. On classifications of non-Gorenstein \mathbb{Q} -Fano 3-folds of Fano index 1. *J. Math. Soc. Japan*, 47(2):369–380, 1995.
- [San96] T. Sano. Classification of non-Gorenstein \mathbb{Q} -Fano d -folds of Fano index greater than $d - 2$. *Nagoya Math. J.*, 142:133–143, 1996.
- [SD72] H. P. F. Swinnerton-Dyer. Rational points on del Pezzo surfaces of degree 5. In *Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer School in Math.)*, pages 287–290, 1972.
- [Ser02] J.-P. Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002.
- [Ser06] J.-P. Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition.
- [Shi89] K.-H. Shin. 3-dimensional Fano varieties with canonical singularities. *Tokyo J. Math.*, 12(2):375–385, 1989.
- [Sko93] A. N. Skorobogatov. On a theorem of Enriques-Swinnerton-Dyer. *Ann. Fac. Sci. Toulouse Math. (6)*, 2(3):429–440, 1993.
- [Suz04] K. Suzuki. On Fano indices of \mathbb{Q} -Fano 3-folds. *Manuscripta Math.*, 114(2):229–246, 2004.
- [The19] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.6)*, 2019. <https://www.sagemath.org>.
- [Ure18] C. Urech. On homomorphisms between Cremona groups. *Ann. Inst. Fourier (Grenoble)*, 68(1):53–100, 2018.
- [VA13] A. Várilly-Alvarado. Arithmetic of del Pezzo surfaces. In *Birational geometry, rational curves, and arithmetic*, Simons Symp., pages 293–319. Springer, Cham, 2013.
- [Vos82] V. E. Voskresenskii. Projective invariant Demazure models. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(2):195–210, 431, 1982.
- [Zha06] Qi Zhang. Rational connectedness of log \mathbb{Q} -Fano varieties. *J. Reine Angew. Math.*, 590:131–142, 2006.

MATHEMATICAL INSTITUTE, UTRECHT UNIVERSITY, BUDAPESTLAAN 6, 3584 CD UTRECHT, THE NETHERLANDS; EPFL SB MATH CAG, BÂT. MA, STATION 8, 1015 LAUSANNE, SWITZERLAND
Email address: m.pieropan@uu.nl