THE LOCALIZATION OF ORTHOGONAL CALCULUS WITH RESPECT TO HOMOLOGY

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Abstract. For a set of maps of based spaces S we construct a version of Weiss' orthogonal calculus which only depends on the S-local homotopy type of the functor involved. We show that S-local homogeneous functors of degree n are equivalent to levelwise S-local spectra with an action of the orthogonal group $O(n)$ via a zigzag of Quillen equivalences between appropriate model categories. Our theory specialises to homological localizations and nullifications at a based space. We give a variety of applications including a reformulation of the Telescope Conjecture in terms of our local orthogonal calculus and a calculus version of Postnikov sections.

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1. Introduction

Motivation. Building on work of Adams, Bousfield [17] established that on the category of (based) spaces there is a well defined localization functor L_E which sends a space X to a space X_E which contains all the E_*-homology of X but disregards any of the other homotopical information of X. For example, for rational homology, the space $X_{\mathbb{Q}}$ is the rationalisation of X in the sense of Quillen [49]. Similar localizations exist for spectra, see [18].

The theory of localizations at homology theories are ubiquitous and have had wide applications; of particular note is *chromatic homotopy theory* which among other things gives a spectrum level interpretation for the periodic families appearing in the stable homotopy groups of spheres. An extensive amount of effort has been geared toward understanding how localization at homology theories–particularly the chromatic localizations–interact with Goodwillie's calculus of functors [2, 38–40], see e.g., [41] for a survey. This article is concerned with the analogous question in orthogonal calculus although our approach is noticeably different than those applied to Goodwillie calculus.

Overview. The classical theory of orthogonal calculus (and more generally functor calculus) is a homotopy theoretic tool for studying 'geometric' objects by constructing a filtration on the object the 'quotients' of which are typically easier to understand and manipulate. The spaces under consideration in orthogonal calculus naturally take the form of a $\mathcal{J}$-space: a functor from the category of real vector spaces to the category of (based) spaces. The calculus is constructed in such a way that it not only constructs a filtration on each level of the $\mathcal{J}$-space, but these constructions are natural; a linear map induces a map between filtrations; given a $\mathcal{J}$-space F the calculus assigns a tower of $\mathcal{J}$-spaces under F

$$\begin{array}{c} F \\ \cdots \longrightarrow T_nF \longrightarrow T_{n-1}F \longrightarrow \cdots \longrightarrow T_1F \longrightarrow T_0F \end{array}$$

called the *Weiss tower* for F. The n-th term T_nF acts as a categorification of polynomial of degree less than or equal n functions from classical calculus. The n-th layer of the tower is the homotopy fibre D_nF of the map $T_nF \to T_{n-1}F$, and acts as the 'quotients' of the filtration. Of all the flavours of functor calculus, orthogonal calculus is synonymous with being the most computationally challenging due to the interaction between the highly 'geometric' nature of the objects of study and the highly homotopical constructions.

Given a set S of maps of based spaces we produce an *S-local Weiss tower* of the following form:

$$\begin{array}{ccccccccc} & & & & F & & & & \\ \cdots \longrightarrow & T_nL_SF & \longrightarrow & T_{n-1}L_SF & \longrightarrow \cdots \longrightarrow & T_1L_SF & \longrightarrow & T_0L_SF \\ & \uparrow & & \uparrow & & \uparrow & & \\ & D_n^SF & & D_{n-1}^SF & & D_1^SF & & \end{array}$$

To understand the S-local Weiss tower, we utilise the interpretation of localizations in terms of a model structure on the category of bases spaces in which the localization functor L_S is a fibrant replacement. In the case of homological localizations the weak equivalences are the homology isomorphisms. We begin by constructing a model structure, denoted $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\,\mathsf{Top}_*)$, which captures the homotopy theory of functors which are S-locally polynomial of degree less than or equal n. From this model structure we construct a new model structure, denoted $\mathsf{Homog}^n(\mathcal{J}_0, L_S \mathsf{Top}_*)$, which captures the homotopy theory of functors which are S-locally homogeneous of degree n. In particular, when the localization functor L_S is "nice" (in a sense which is made precise below) the S-local n-homogeneous model structure contains the n-th layer of the S-local Weiss tower as a bifibrant object.

We consider three classes of localizations, giving three levels of results:

(1) we give general existence theorems for these S-local model structures for *any* set S of maps of based spaces;
(2) we obtain more computationally auspicious results when we assume the class of local objects is closed under homotopy colimits[1]; and,
(3) we restrict to the case when the localization is a *nullification* which further advances the applicability of the S-local calculus to computations.

Through a zigzag of Quillen equivalences we characterise the S-local n-homogeneous functors as levelwise S-local spectra with an action of $O(n)$.

Theorem (Corollary 8.4.2)**.** *Let S be a set of maps of based spaces and $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, L_S \mathsf{Top}_*) \simeq_Q \mathsf{Sp}(L_S \mathsf{Top}_*)[O(n)].$$

On the derived level, we obtain a computationally accessible classification theorem for S-local homogeneous of degree n functors.

Theorem (Theorem 8.6.1)**.** *Let S be a set of maps of based spaces and $n \geq 1$. Any S-local homogeneous of degree n functor F is (up to homotopy) of the form*

$$V \longmapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)}],$$

where $\partial_n^S F$ is a levelwise S-local spectrum with an action of $O(n)$.

Applications. We envision that the applications of this local version of orthogonal calculus are vast. For example, computations of rational derivatives of functors in orthogonal calculus have been of interest in differential topology recently, see e.g., [37], and a full understanding of the Weiss tower of $\mathsf{BO}(-)$ has been achieved in v_n-periodic homotopy theory through complex computations of Arone [1] using compuations of Arone and Mahowald [2]. A compelling application of the local orthogonal calculus would be to recast these computations in a new light. In the last part of this paper we give a number of intial applications of the theory.

By considering the acyclics of a localization with respect to a homology theory, Bousfield [18] described a lattice of localizations built from *Bousfield classes*–equivalences classes of acyclics. A similar lattice exists for nullity classes of nullifications at based spaces, see, e.g., [29]. This has a number of repercussions for the localization of orthogonal calculus. We prove a more general result that the stated theorem below, see Theorem 9.1.1.

Theorem (Corollary 9.1.2 & Corollary 9.1.3)**.**

(1) Let E and E' be spectrum. The E-local orthogonal calculus is equivalent to the E'-local orthogonal calculus if and only if E and E' have the same Bousfield class.

[1] The requirement of the class of local objects to be closed under homotopy colimits is reminiscent of *smashing* localizations in the stable setting.

(2) *Let W and W' be based spaces. The W-local orthogonal calculus is equivalent to the W'-local orthogonal calculus if and only if W and W' have the same nullity class.*

A major application of our study of Bousfield classes is in relation to the Telescope Conjecture of Ravenel, [50, Conjecture 10.5]. Fix a prime p, and work p-locally. Denote by $T(n)$ the telescope of any v_n-self map on a finite complex of type n, and denote by $K(n)$ the n-th Morava K-theory.

Theorem (Theorem 9.2.3)**.** *The height n Telescope Conjecture holds if and only if the $K(n)$-local orthogonal calculus and the $T(n)$-local orthogonal calculus are equivalent.*

From a computational perspective we obtain the following relation between the Telescope Conjecture and the local Weiss spectral sequences.

Theorem (Lemma 9.2.4)**.** *If the height n Telescope Conjecture holds, then for all $r \geq 0$, the r-th page of the $T(n)$-local Weiss spectral sequence is isomorphic to the r-th page of the $K(n)$-local Weiss spectral sequence.*

By considering nullifications with respect to the spheres we obtain a theory of Postnikov sections in orthogonal calculus, which is a particular example of Postnikov sections in arbitrary model categories. For example, we prove that our S^{k+1}-local projective model structure on the category of orthogonal functors is identical to the model structure of k-types in the category of orthogonal functors in the sense of [31, §4].

Theorem (Proposition 10.2.1)**.** *Let $k \geq 0$. The model structure of k-types in orthogonal functors is identical to the S^{k+1}-local model structure, that is, there is an equality of model structures,*

$$P_k\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) := L_{W_k}\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) = \mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*).$$

As an application we produce a tower of model categories

$$\cdots \longrightarrow \mathsf{Homog}^n(\mathcal{J}_0, P_k\,\mathsf{Top}_*) \longrightarrow \cdots \longrightarrow \mathsf{Homog}^n(\mathcal{J}_0, P_1\,\mathsf{Top}_*) \longrightarrow \mathsf{Homog}^n(\mathcal{J}_0, P_0\,\mathsf{Top}_*),$$

by nullification at the spheres S^{k+1} as k varies. By applying the theory of homotopy limits of model categories, we show that the n-homogeneous model structure of Barnes and Oman [6, Proposition 6.9] is the homotopy limit of this tower, in the following sense.

Theorem (Corollary 10.7.3)**.** *There is a Quillen equivalence*

$$\mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) \simeq_Q \underset{k}{\mathrm{holim}}\ \mathsf{Homog}^n(\mathcal{J}_0, P_k\,\mathsf{Top}_*).$$

Relation to other work. This work is intimately related to the rational orthogonal calculus developed by Barnes [4]. Replacing our generalised homology theory E_* with rational homology recovers the theory developed by Barnes.

Unstable chromatic homotopy theory can be described algebraically, via Heuts' [32] *algebraic model* for v_n-periodic spaces via an equivalence (of ∞-categories) with Lie algebras in $T(n)$-local spectra. This model indicated that there is likely a relationship between v_n-periodic orthogonal calculus and orthogonal calculus of Heuts' Lie algebra models. Such an equivalence at chromatic height zero suggests a relationship between rational orthogonal calculus and the algebraic models for rational homotopy theory of Sullivan and Quillen [49, 51]. In particular this together with Barnes' [4] model for rational n-homogeneous functors as torsion models over the rational cohomology ring of $\mathsf{BSO}(n)$ suggests the existence of *algebraic model calculi*. We plan to return to this in future work.

There is a strong connection between the calculus and chromatic homotopy theory. Ravenel [50], made several conjectures relating to the structure of chromatic homotopy theory including the

Nilpotence and Periodicity Theorems of Devinatz, Hopkins and Smith [24,34]. These conjectures have be resolved except for one, the Telescope Conjecture, which is trivial at height $n = 0$, has been verified at height $n = 1$ by Bousfield [18], Mahowald [43] and Miller [47], but in general is widely believed to be false. The validity of the Telescope Conjecture would imply an equivalence between $K(n)$-local orthogonal calculus and and $T(n)$-local orthogonal calculus.

This work also forms part of an extensive program to go "beyond orthogonal calculus" which was initiated in the Ph.D. thesis of the author [53], together with a series of articles exploring extensions of the orthogonal calculus and the relations between these, [52, 54–56]. The hopes of this extensive project is to illuminate our understanding of orthogonal calculus which (at least relative to Goodwillie calculus) remains largely unexplored.

The future applications of the homological localization of orthogonal calculus are abound. For example in recent work of Beaudry, Bobkova, Pham and Xu [13], the authors compute the tmf-homology of $\mathbb{R}P^2$, where tmf denotes the connective spectrum of topological modular forms. It would be interesting to investigate whether their computation for $\mathbb{R}P^2$ and the tmf-local Weiss tower for the functor $V \mapsto \mathbb{R}P(V)$ yield a calculation of the tmf-homology of $\mathbb{R}P^k$ for all k. Such a connection would, for example, feed into a chromatic understanding of block structures, see e.g., [42].

Conventions. We work extensively with model categories and refer the reader to the survey article [27] and the textbooks [33,35] for a detailed account of the theory. We further assume the reader has familiarity with orthogonal calculus, references for which include [6,58].

The category Top_* will always denote the category of based compactly generated weak Hausdorff spaces, and we will, for brevity, call the objects of this category "based spaces". The category of based spaces will always be equipped with the Quillen model structure unless specified otherwise. The weak equivalences are the weak homotopy equivalences and fibrations are Serre fibrations. This is a cellular, proper and topological model category with sets of generating cofibrations and acyclic cofibrations denoted by I and J, respectively.

Unless otherwise stated the word "spectra" is synonymous with the phrase "orthogonal spectra", details of which can be found in [46] in the non-equivariant case, and [45] in the equivariant situation.

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Part 1. Preliminaries

2. The localization of topological spaces

2.1. Left Bousfield localization.

Given a model category $\mathcal{C}$ there is a systematic way of constructing a new model category which has the same cofibrations but more weak equivalences, yielding a finer homotopy theory. This process is called left Bousfield localization.

Give objects X and Y in a model category $\mathcal{C}$ we denote by $\mathsf{map}_{\mathcal{C}}(X, Y)$ the homotopy function complex in the sense of [33, Chapter 17]. In many cases our model structures will be simplicial or topological in which case the homotopy function complex may be replaced by suitably 'derived' version of the internal hom arising from the simplicial or topological structure, see e.g., [33, Example 17.1.4].

Definition 2.1.1. Let $\mathcal{C}$ be a model category and S a class of maps in $\mathcal{C}$.

(1) An object Z in $\mathcal{C}$ is *S-local* if for every map $f\colon A \to B$ in S the map on homotopy function complexes
$$\mathsf{map}_{\mathcal{C}}(B, Z) \longrightarrow \mathsf{map}_{\mathcal{C}}(A, Z),$$
induced by precomposition with f is a weak homotopy equivalence of simplicial sets.

(2) a map $g\colon X \to Y$ is an *S-local equivalence* if for every S-local object Z the map on homotopy function complexes
$$\mathsf{map}_{\mathcal{C}}(Y, Z) \longrightarrow \mathsf{map}_{\mathcal{C}}(X, Z),$$
induced by precomposition with g is a weak homotopy equivalence of simplicial sets.

Under mild conditions on the model category $\mathcal{C}$ there exists a model structure on $\mathcal{C}$ with weak equivalences the S-local equivalences and fibrant objects the S-local objects which are fibrant in $\mathcal{C}$. We will sometimes refer to such objects as being *S-fibrant*.

Theorem 2.1.2 ([33, Theorem 4.1.1.])**.** *Let $\mathcal{C}$ be a model category and S a set of maps in $\mathcal{C}$. If the model structure on $\mathcal{C}$ is left proper and cellular, then the left Bousfield localization of $\mathcal{C}$ at the set S exists. The weak equivalences are the S-local equivalences, the cofibrations are the cofibrations of $\mathcal{C}$ and the fibrant objects are the S-local objects which are fibrant in $\mathcal{C}$. This model structure is cellular, left proper and topological whenever the underlying model structure on $\mathcal{C}$ is topological. We denote this model structure by $L_S\mathcal{C}$.*

As an example, consider homological localization with respect to a homology theory E_*. Bousfield [17] showed that there exists a set J_E of maps of based spaces such that the J_E-local model structure on based spaces captures the homotopy theory of based spaces up to E_*-isomorphism.

Theorem 2.1.3 ([17, Theorem 10.2])**.** *Let E_* be a generalised homology theory. There is a model category structure on the category of based spaces with weak equivalences the E_*-isomorphisms, cofibrations the cofibrations of the Quillen model structure on based spaces and the fibrant objects are the E-local spaces. This model structure is cellular, left proper and topological. We denote this model structure by $L_E\,\mathsf{Top}_*$.*

2.2. **Right Bousfield localization.** There is a dual theory in which one adds more weak equivalences but fixes the class of fibrations. This is called a Right Bousfield localization.

Definition 2.2.1. Let $\mathcal{C}$ be a model category and $\mathcal{K}$ a set of cofibrant objects in $\mathcal{C}$.

(1) A map $g\colon X \to Y$ is an *$\mathcal{K}$-colocal equivalence* if for every $K \in \mathcal{K}$ the map on homotopy function complexes
$$\mathsf{map}_{\mathcal{C}}(K, X) \longrightarrow \mathsf{map}_{\mathcal{C}}(K, Y),$$
induced by postcomposition with g is a weak homotopy equivalence of simplicial sets.

(2) An object Z in $\mathcal{C}$ is *S-colocal* if for every $\mathcal{K}$-colocal equivalence $f\colon A \to B$ the map on homotopy function complexes
$$\mathsf{map}_{\mathcal{C}}(Z, A) \longrightarrow \mathsf{map}_{\mathcal{C}}(Z, B),$$
induced by postcomposition with f is a weak homotopy equivalence of simplicial sets.

Under mild conditions on the model category $\mathcal{C}$ there exists a model structure on $\mathcal{C}$ with weak equivalences the $\mathcal{K}$-colocal equivalences and cofibrant objects the $\mathcal{K}$-colocal objects which are cofibrant in $\mathcal{C}$. We will sometimes call the cofibrant objects $\mathcal{K}$-cofibrant.

Theorem 2.2.2 ([33, Theorem 5.1.1.]). *Let $\mathcal{C}$ be a model category and $\mathcal{K}$ a set of cofibrant objects in $\mathcal{C}$. If the model structure on $\mathcal{C}$ is right proper and cellular, then the right Bousfield localization of $\mathcal{C}$ at the set $\mathcal{K}$ exists. The weak equivalences are the $\mathcal{K}$-colocal equivalences, the fibrations are the fibrations of $\mathcal{C}$ and the cofibrant objects are the $\mathcal{K}$-colocal objects which are cofibrant in $\mathcal{C}$. This model structure is cellular, right proper and topological whenever the underlying model structure on $\mathcal{C}$ is topological. We denote this model structure by $R_{\mathcal{K}}\mathcal{C}$.*

Example 2.2.3. If $\mathsf{Ch}(\mathbb{Z})$ is the category of (unbounded) chain complexes of abelian groups equipped with the projective model structure, then derived completion at a prime p may be modelled by the right Bousfield localization at the complex $\mathbb{Z}/p$.

2.3. **Left Bousfield localization and right properness.**

Remark 2.3.1. The process of left Bousfield localization can interfere with other model categorical properties, for instance left Bousfield localization need not preserve right properness. For example if $E = H\mathbb{Q}$, then the $H\mathbb{Q}$-local model structure on based spaces is not right proper since there is a pullback square

$$\begin{array}{ccc} K(\mathbb{Q}/\mathbb{Z},0) & \longrightarrow & P \\ \downarrow & & \downarrow \\ K(\mathbb{Z},1) & \xrightarrow{\simeq_{H\mathbb{Q}}} & K(\mathbb{Q},1) \end{array}$$

in which the right hand vertical map is a fibration, P is contractible and the lower horizontal map is a $H\mathbb{Q}$-equivalence but the left hand vertical map is not. Another example is provided by Quillen in [49, Remark 2.9].

The property of being right proper has many advantages including the ability to right Bousfield localize. As such we investigate when the S-local model structure is right proper. It suffices to examine when the f-local model structure is right proper for some map $f\colon X \to Y$ of based spaces.

The following has motivation in [22, Remark 9.11], in which Bousfield remarks that the f-local model structure cannot be right proper unless the localization functor L_f is equivalent to a nullification. We extend Bousfield's remark by showing that his nullification condition is both necessary and sufficient in a stronger sense than originally proposed by Bousfield. This result depends on two constructions also due to Bousfield; the first is the construction of a based space $A(f)$ associated to a map $f\colon X \to Y$ of based spaces, see [21, Theorem 4.4], the second is a nullification functor $P_W\colon \mathsf{Top}_* \to \mathsf{Top}_*$ associated to any based space W, see [19, Theorem 2.10]. This nullification functor has two key properties which we would also like to highlight; firstly, when W is connected P_W preserves disjoint unions and secondly, P_W is contractible when W is not connected. For example, if f is the map which induces localization with respect to integral homology, then $P_{A(f)}$ is Quillen's plus construction, see e.g., [29, 1.E.5].

Proposition 2.3.2. *Let $f\colon X \to Y$ be a map of based spaces. The f-local model structure on based spaces is right proper if and only if there exists a based space $A(f)$ and and equality of model structures*

$$L_f \mathsf{Top}_* = P_{A(f)} \mathsf{Top}_*,$$

where $P_{A(f)} \mathsf{Top}_*$ is the Bousfield-Friedlander localization [22, Theorem 9.3]*, at the nullification endofunctor*

$$P_{A(f)}\colon \mathsf{Top}_* \to \mathsf{Top}_*.$$

Proof. By [21, Theorem 4.4], there exists a based space $A(f)$ such that the classes of $A(f)$-acyclic and f-acyclic spaces agree, and every $P_{A(f)}$-equivalence is an f-local equivalence.

Assume that the f-local model structure is right proper. For a connected based space X, the path fibration over L_fX is an f-local fibration, hence the homotopy fibre of the map $X \to L_fX$ is f-acyclic, and hence $A(f)$-acyclic. It follows by [19, Corollary 4.8(i)], the map $X \to L_fX$ is a $P_{A(f)}$-equivalence, hence every f-local equivalences of connected spaces is a $P_{A(f)}$-equivalence. Since the functor $P_{A(f)}$ on based spaces comes from a functor on unbased spaces which preserves disjoint unions when $A(f)$ is connected and which takes contractible values when $A(f)$ is not connected, every f-local equivalence must be a $P_{A(f)}$-equivalence. It follows that the class of f-local equivalences agrees with the class of $P_{A(f)}$-equivalences. The equality of the model structures follows immediately since both model structures have the same cofibrations inherited from the Quillen model structure on the category of based spaces.

For the converse, assume that the f-local model structure agrees with the $A(f)$-local model structure. The latter model structure is right proper by [22, Theorem 9.9], and since both model structures have the same weak equivalences and fibrations, the f-local model structure must also be right proper. □

Remark 2.3.3. The property of being right proper is completely determined by the weak equivalence class of the model structure; if two model structures have the the same weak equivalences, then one is right proper if and only if the other is, see e.g., [3, Remark 2.5.6].

We now provide some examples of localizations which can be written as a nullification.

Examples 2.3.4.

(1) Localization of spaces with respect to the map $u_{n+1} \colon * \to S^{n+1}$ gives the Postnikov section, that is for a space X, $P_nX = P_{S^{n+1}}X = L_{u_{n+1}}X$, see e.g., [29, Example E.1].
(2) Extensive work of Bousfield [19, 22] exhibits that v_n-periodic homotopy theory, or *unstable chromatic homotopy theory* can be expressed as a nullification.

3. The Borel stablisation of local spaces with a G-action

We investigate the stablisation of S-local spaces for S a set of maps of based spaces and the relationship between E-local spectra and the stablisation of E-local spaces for a homology theory E_*. This will be particularly useful since homogeneous functors of degree n are classified by the *Borel stablisation* of spaces with an $O(n)$-action, see e.g., [58, Theorem 7.3], and S-local homogeneous functors of degree n will be classified by the Borel stablisation of S-local spaces with an $O(n)$-action, see Section 8.

3.1. The Borel stablisation of local spaces with a G-action.

Let G be a compact Lie group. Recall from [46] that orthogonal spectra may be described as a category of topologically enriched functors $\mathsf{Fun}(\mathcal{J}, \mathsf{Top}_*)$. Furthermore, spectra with a G-action is the category of G-objects in spectra. Spectra with a G-action is the 'naive' stabilisation of G-spaces, which we prefer to call the *Borel stablisation*. This stablisation is captured by the underlying model structure on spectra with a G-action. We are particularly interested in spectra with a G-action, when G is the orthogonal group $O(n)$. Following the procedure in the non-equivariant setting [46], or the equivariant setting on a trivial universe [45], a left Bousfield localization of the projective model structure on spectra with a G-action yields the stable model structure.

Lemma 3.1.1. *The category $\mathsf{Sp}[G]$ of spectra with a G-action may be equipped with a model structure with weak equivalences the π_*-isomorphisms and a map $f\colon X \to Y$ is a fibration if it is levelwise* *fibration and for each $U \in \mathcal{I}$, the square*

$$\begin{array}{ccc} X(U) & \longrightarrow & \Omega X(U) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Omega f} \\ Y(U) & \longrightarrow & \Omega Y(U) \end{array}$$

is a homotopy pullback. This model structure is cellular, proper, stable and topological.

Analogous to spectra, we can preform a suitable left Bousfield localization of the category of spectra with a G-action to obtain a model for the Borel stablisation of S-local spaces with a G-action.

Proposition 3.1.2. *Let S a set of maps of based spaces. There is a model structure on the category of spectra with a G-action such that the cofibrations are the cofibrations of the stable model structure and the fibrant objects are the levelwise S-local Ω-spectra. This model structure is cellular, left proper and topological. We call this model structure the (Borel) stablisation of S-local spaces with an $O(n)$-action, and it $\mathsf{Sp}(L_S \mathsf{Top}_*)[G]$.*

3.2. **E-local spectra with a G-action.** For a homology theory E_*, it is also possible to construct a model structure on spectra such that the weak equivalences are detected by E_*-homology isomorphisms.

Proposition 3.2.1 (Bousfield [18])**.** *Let E be a spectrum. There is a model structure on the category of spectra such that the weak equivalences are the E_*-isomorphisms, the cofibrations are the cofibrations of the stable model structure and the fibrant objects are the E-local Ω-spectra. This model structure is cellular, proper, stable and topological. We call this model structure the E-local model structure and denote it Sp_E.*

Proof. Existence of the model structure follows from Hovey's Recognition Principle for model categories, [35, Theorem 2.1.19], see, e.g. [9, Theorem 7.3.3] for details and [9, Corollary 7.3.6] for the characterisation of the fibrant objects. This model structure is stable by [8, Example 4.3 & Proposition 4.6], and right proper by [8, Proposition 4.7]. □

This model structure can be transferred to spectra with a G-action.

Proposition 3.2.2. *There is a model structure on the category $\mathsf{Sp}[G]$ of spectra with a G-action where the weak equivalences are the E_*-isomorphisms, the cofibrations are the cofibrations of the projective model structure and the fibrant objects are the E-local Ω-spectra. This model structure is cellular, right proper, stable and topological. We denote this model structure by $\mathsf{Sp}_E[G]$.*

Proof. This is the right transfer of the E-local model structure on spectra through the adjunction

$$G_+ \wedge (-) : \mathsf{Sp}_E \rightleftarrows \mathsf{Sp}_E[G] : i^* .$$ □

3.3. **"Stably E-local" versus "stable and E-local".** We end this section with some remarks on the relationship between E-local spectra and the stablisation of E-local spaces. Various aspects of the relationship between these two constructions have been considered by various groups of authors including; Mahowald and Thompson for p-adic K-theory [44], Barthel and Bousfield for completion at a prime [11], and by Barnes and Roitzheim in their work on local framings [7].

Lemma 3.3.1 ([7, Lemma 8.3])**.** *Let E be a spectrum. The adjoint pair*

$$\mathbb{1} : \mathsf{Sp}(L_E \, \mathsf{Top}_*) \rightleftarrows \mathsf{Sp}_E : \mathbb{1} \ ,$$

is a Quillen adjunction between the stablisation of E-local spaces and E-local spectra.

Remark 3.3.2. This Quillen adjunction is not always a Quillen equivalence. For example, in the case of completion at a prime p, the suspension spectrum of a p-complete nilpotent space is p-complete if and only the homotopy groups of the space are bounded p-torsion [11, Theorem 4.7], hence the derived adjunction is not an equivalence of homotopy categories.

Barnes and Roitzheim give a large class of examples for which the adjunction is a Quillen equivalence. In fact, they give a class of examples for which the model structures become identical.

Lemma 3.3.3 ([7, Lemma 8.6])**.** *Let R be a subring of the rationals. There is an equality of model structures*

$$\mathsf{Sp}(L_{HR} \, \mathsf{Top}_*) = \mathsf{Sp}_{HR}.$$

Part 2. Local orthogonal calculus

With the preliminaries in place, we now construct the S-local orthogonal calculus for S a set of maps of based spaces. Our approach is motivated by the model categorical perspective on orthogonal calculus of Barnes and Oman [6], and Barnes' rational orthogonal calculus [4].

4. Input functors

4.1. **Input functors.**

We first recall the 'input functors' for the calculus. These are the base functors which one wishes to study through the lens of orthogonal calculus.

Definition 4.1.1. Define $\mathcal{J}$ to be the category with objects finite-dimensional inner product subspaces of $\mathbb{R}^\infty$ and morphisms linear isometries. Define $\mathcal{J}_0$ to be the category with the same objects and $\mathcal{J}_0(U,V) = \mathcal{J}(U,V)_+$.

The morphism set $\mathcal{J}(U,V)$ may be topologised as the Stiefel manifold of $\dim(U)$-frames in V. As such, $\mathcal{J}$ is a topologically enriched category, and $\mathcal{J}_0$ is enriched in based spaces.

The category of topological[2] functors from $\mathcal{J}_0$ to Top_* is the category of input functors for orthogonal calculus. We will refer to such functors as *orthogonal functors* and denote the category of such functors by $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$. Examples of orthogonal functors are abound in geometry, topology and homotopy theory, and include:

(1) the one-point compactification functor $\mathbb{S}\colon V \mapsto S^V$;
(2) the functor $\mathsf{BO}(-)\colon V \mapsto \mathsf{BO}(V)$ which sends an inner product space to the classifying space of its orthogonal group;
(3) the functor $\mathsf{BTOP}(-)\colon V \mapsto \mathsf{BTOP}(V)$, which sends an inner product space V to $\mathsf{BTOP}(V)$, the classifying space of the space of self-homeomorphisms of V;
(4) the functor $\mathsf{BDiff}^{\mathsf{b}}(M \times -)\colon V \mapsto \mathsf{BDiff}^{\mathsf{b}}(M \times V)$, which for a fixed smooth and compact manifold M sends an inner product space V to the classifying space of the group of bounded diffeomorphisms from $M \times V$ to $M \times V$ which are the identity on $\partial M \times V$; and,

[2] By which we mean "enriched in based spaces".

(5) the restriction of an endofunctor on spaces spaces to evaluation on spheres [3]

The category of orthogonal functors may be equipped with a projective model structure.

Proposition 4.1.2. *There is a model category structure on the category of orthogonal functors* $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$ *with weak equivalences and fibrations defined levelwise. The generating cofibrations are of the form*

$$\mathcal{J}_0(U, -) \wedge S^{n-1}_+ \longrightarrow \mathcal{J}_0(U, -) \wedge D^n_+,$$

for $U \in \mathcal{J}_0$ *and* $n \geq 0$. *The generating acyclic cofibrations are of the form*

$$\mathcal{J}_0(U, -) \wedge D^n_+ \longrightarrow \mathcal{J}_0(U, -) \wedge (D^n \times [0, 1])_+,$$

for $U \in \mathcal{J}_0$ *and* $n \geq 0$. *This model structure is cellular, proper and topological.*

4.2. **Local input functors.** The category of topological functors from $\mathcal{J}_0$ to Top_* is also the category of input functors for orthogonal calculus localized at a set of maps of based spaces S. The 'base' model structure for the S-local orthogonal calculus will be the S-local model structure on the category of orthogonal functors.

Proposition 4.2.1. *Let* S *be a set of maps of based spaces. There is model structure on the category of topological functors from* $\mathcal{J}_0$ *to* Top_* *such that a map* $f\colon X \to Y$ *is a weak equivalence or fibration if for all* $V \in \mathcal{J}_0$, *the induced map* $f(V)\colon X(V) \to Y(V)$ *is a* S*-local equivalence or a* S*-local fibration of based spaces, respectively. This model structure is cellular, left proper and topological. We call this model structure the* S*-local projective model structure and it by* $\mathsf{Fun}(\mathcal{J}_0, L_S \mathsf{Top}_*)$.

Proof. This model structure is an instance of a projective model structure on a category of functors, see e.g., [33, Theorem 11.6.1]. □

As a particular example, we obtain the following model structure for the localization with respect to a homology theory E_*.

Corollary 4.2.2. *Let* E *be a spectrum. There is a model structure on the category of topological functors from* $\mathcal{J}_0$ *to* Top_* *such that a map* $f\colon X \to Y$ *is a weak equivalence or fibration if for all* $V \in \mathcal{J}_0$ *the induced map* $f(V)\colon X(V) \to Y(V)$ *is a* E_**-isomorphism or a fibration in the model category of* E*-local based spaces, respectively.*

5. Polynomial functors

5.1. **Polynomial functors.** Polynomial functors behave in many ways like polynomial functions from classical calculus, e.g., a functor which is polynomial of degree less than or equal n, is polynomial of degree less than or equal $n + 1$. We give only the necessary details here and refer the reader to [58] or [6] for more details on polynomial functors in orthogonal calculus.

Definition 5.1.1. An orthogonal functor F is *polynomial of degree less than or equal* n if for each $U \in \mathcal{J}_0$, the canonical map

$$F(U) \longrightarrow \underset{0 \neq V \subseteq \mathbb{R}^{n+1}}{\mathrm{holim}} F(U \oplus V) =: \tau_n F(U),$$

is a weak homotopy equivalence. Functors which are polynomial of degree less than or equal n will sometimes be referred to as n*–polynomial* functors.

[3] Endofunctors of based spaces are particularly interesting from a homotopy theoretic point of view when you restrict to the values on spheres, see e.g., [1, 2, 14]. In particular for F the identity functor, the Weiss tower of $F \circ \mathbb{S} = \mathbb{S}$ and the Goodwillie tower for F agree up to weak equivalence [5], hence orthogonal calculus is intimately related to understanding the (stable) homotopy groups of spheres.

Remark 5.1.2. Given an orthogonal functor F and an inner product space U we can restrict the orthogonal functor $F(U \oplus -)$ to a functor

$$F(U \oplus -) \colon \mathcal{P}(\mathbb{R}^{n+1}) \longrightarrow \mathsf{Top}_*,$$

where $\mathcal{P}(\mathbb{R}^{n+1})$ is the poset of finite-dimensional inner product subspaces of $\mathbb{R}^{n+1}$. Such functors are deserving of the name $\mathbb{R}^{n+1}$-*cubes* by analogy with cubical homotopy theory. The orthogonal functor F being n-polynomial is equivalent to asking that for each U this restricted functor is homotopy cartesian. Informally speaking, orthogonal calculus can be thought of a calculus built from $\mathbb{R}^n$-*cubical homotopy theory* in a similar way to how Goodwillie calculus is built from cubical homotopy theory, see e.g., [48].

There is a functorial assignment of a universal (up to homotopy) n-polynomial functor to any orthogonal functor F. It is the *n-polynomial approximation* of F, and is defined as

$$T_nF(U) = \text{hocolim}(F(U) \longrightarrow \tau_nF(U) \longrightarrow \cdots \longrightarrow \tau_n^kF(U) \longrightarrow \cdots).$$

In [6, Proposition 6.5 & Proposition 6.6], Barnes and Oman construct a localization of the projective model structure on the category of orthogonal functors which captures the homotopy theory of n-polynomial functors. The n-polynomial approximation functor is a fibrant replacement in this model structure, which implies that any map $F \to F'$ with F' n-polynomial factors (up to homotopy) through the n-polynomial approximation of F. There are two equivalent ways to consider this model structure; as the Bousfield-Friedlander localization of $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$ at the n-polynomial approximation endofunctor

$$T_n \colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \longrightarrow \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*),$$

or, as the left Bousfield localization at the set of maps

$$\mathcal{S}_n = \{S\gamma_{n+1}(U,V)_+ \longrightarrow \mathcal{J}_0(U,V) \ \mid \ U,V \in \mathcal{J}_0\},$$

where $S\gamma_{n+1}(U,V)$ is the sphere bundle of the $(n+1)$-st complement bundle $\gamma_{n+1}(U,V)$ over the space of linear isometries $\mathcal{J}(U,V)$. The fibre over a linear isometry f in the $(n+1)$-st complement bundle is $\mathbb{R}^{n+1} \otimes f(U)^\perp$, where $f(U)^\perp$ is the orthogonal complement of $f(U)$ in V.

Proposition 5.1.3 ([6, Proposition 6.5])**.** *There is a model category structure on the category of orthogonal functors with weak equivalences the T_n-equivalences*[4] *and fibrations those levelwise fibrations $f \colon X \to Y$ such the square*

$$\begin{array}{ccc} X & \longrightarrow & T_nX \\ \downarrow & & \downarrow \\ Y & \longrightarrow & T_nY \end{array}$$

is a homotopy pullback. This model structure is cellular, proper and topological. We call this the n-polynomial model structure and denote it by $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, \mathsf{Top}_*)$.

5.2. **Local polynomial functors.**

Definition 5.2.1. Let S be a set of maps of based spaces. An orthogonal functor is *S-locally n-polynomial* if it is levelwise S-local and n-polynomial.

We have the following useful reduction using the S-local Whitehead's Theorem, [33, Theorem 3.2.13(1)].

[4]A map $f \colon X \to Y$ is a T_n-equivalence if $T_n(f) \colon T_nX \to T_nY$ is a levelwise weak equivalence.

Lemma 5.2.2. *Let S be a set of maps of based spaces, and F an orthogonal functor. If F is levelwise S-local, then F is n-polynomial if and only if the canonical map*

$$F(U) \longrightarrow \underset{0 \neq V \subseteq \mathbb{R}^{n+1}}{\text{holim}} F(U \oplus V),$$

is a S-local equivalence.

In our search for a model structure which captures the homotopy theory of S-locally n-polynomial functors we start with a general existence theorem which is an iterated left Bousfield localization involving the set $\mathcal{S}_n$ and the set

$$J_S = \{\mathcal{J}_0(U, -) \wedge j \mid U \in \mathcal{J}, , j \in J_{L_S \mathsf{Top}_*}\}.$$

Without extra assumptions on our localizing set (or homology theory) we are unable to obtain clear descriptions of the weak equivalences.

Proposition 5.2.3. *Let S be a set of maps of based spaces. There is model category structure on the category of orthogonal functors with cofibrations the projective cofibrations, and fibrant objects the S-locally n-polynomial functors. This model structure is cellular, left proper and topological. We call this model structure the S-local n-polynomial model structure and denote it by $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S \mathsf{Top}_*)$.*

Proof. The process of left Bousfield localizations may be iterated and it follows that the J_S-localization of the n-polynomial model structure on and the $\mathcal{S}_n$-localization of the S-local projective model structure are identical, and have as cofibrations the projective cofibrations. It remains to characterise the fibrant objects.

For the fibrant objects, notice that the model structure is equivalently described as the left Bousfield localization of the projecitve model structure with respect to the set of maps $\mathcal{S}_n \cup J_S$. By definition an object X is $\mathcal{S}_n \cup J_S$-local if and only if it is both $\mathcal{S}_n$-local and J_S-local, and hence the fibrant objects are precise those S-locally n-polynomial functors. □

The S-local n-polynomial model structure behaves precisely like the S-localization of the n-polynomial model structure in the following sense.

Lemma 5.2.4. *Let S be a set of maps of based spaces. The adjoint pair*

$$\mathbb{1} : \mathsf{Poly}^{\leq n}(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S \mathsf{Top}_*) : \mathbb{1} \ ,$$

is a Quillen adjunction.

Proof. Since the cofibrations of both model structures are the projective cofibrations the identity functor

$$\mathbb{1} \colon \mathsf{Poly}^{\leq n}(\mathcal{J}_0, \mathsf{Top}_*) \to \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S \mathsf{Top}_*),$$

preserves cofibrations. On the other hand, since an S-locally n-polynomial functor is in particular n-polynomial, the identity functor

$$\mathbb{1} \colon \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S \mathsf{Top}_*) \to \mathsf{Poly}^{\leq n}(\mathcal{J}_0, \mathsf{Top}_*),$$

preserves fibrant objects. □

Our focus now turns to understanding the S-local n-polynomial model structure better. The weak equivalences of the S-local n-polynomial model structure are the $\mathcal{S}_n \cup S$-local equivalences, i.e., detected by objects which are both $\mathcal{S}_n$-local and S-local, hence the class of which contains both the T_n-equivalences and S-local equivalences but may contain maps which are neither. To understand these equivalences better we place a condition on the local objects analogous to the condition that a homological localization (on spectra) is smashing, see e.g., [9, Proposition 7.4.3].

Proposition 5.2.5. *Let S be a set of maps of based spaces. If the class of S-local objects is closed under sequential homotopy colimits, then the weak equivalences of the S-local n-polynomial model structure are those maps $f\colon X \to Y$ such that the induced map*

$$T_n L_S f\colon T_n L_S X \to T_n L_S Y,$$

is a S-local equivalence. In particular, The composite $T_n L_S$ is a functorial fibrant replacement in the S-local n-polynomial model structure.

Proof. We show (1), the proof of (2) is similar. We apply [4, Lemma 5.5] which shows that a map $f\colon X \to Y$ is weak equivalence in the iterated left Bousfield localization if and only if $L_S f\colon L_S X \to L_S Y$ is a $\mathcal{S}_n$-local equivalence. This last is equivalent to $L_S f\colon L_S X \to L_S Y$ being a T_n-equivalence, i.e., $T_n L_S f\colon T_n L_S X \to T_n L_S Y$ being a levelwise weak equivalence. Since both domain and codomain of this map are S-local checking this map is a levelwise weak equivalence is equivalent to checking that it is an S-local equivalence by the S-local Whitehead's Theorem. □

Remark 5.2.6. Let S be a set of maps of based spaces. To ease notation, we will denote the composite $T_n L_S$ by T_n^S. In particular, for E a spectrum we denote the composite functor $T_n L_E$ by T_n^E. In general, T_n^S need not be S-local, but will be when the class of S-local objects is closed under sequential homotopy colimits.

Examples 5.2.7.

1. (1) For a finite cell complex W, $T_n^W F$ is W-local (or W-periodic) for all orthogonal functors F.
2. (2) For localization at the Eilenberg-Maclane spectrum associated to a subring R of the rationals, $T_n^{HR} F$ is HR-local for all orthogonal functors F.

5.3. **Characterisations for nullifications.** Bousfield [19, 20], and Farjoun [29], among others, see e.g., [23], have extensively studied the nullifcation of the category of based spaces at a based space W. This nullification is functorial giving a functor

$$P_W\colon \mathsf{Top}_* \longrightarrow \mathsf{Top}_*,$$

which extends levelwise to a functor

$$P_W\colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \longrightarrow \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*).$$

The nullified model structure on spaces (see, e.g., [22, §§9.8]) extends in a canonical way to give the Bousfield-Frielander localization of the category of orthogonal functors at the functor P_W, which we denote by $\mathsf{Fun}(\mathcal{J}_0, P_W\,\mathsf{Top}_*)$. We show that the W-local n-polynomial model structure is precisely the model structure obtained by Bousfield-Friedlander localization at the composite

$$T_n \circ P_W\colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \longrightarrow \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*).$$

We begin with a lemma which deals with fibrant objects in the Bousfield-Friedlander localization of orthogonal functors at the endofunctor P_W, which we call the W-periodic projective model structure.

Lemma 5.3.1. *For a finite cell complex W and an orthogonal functor F, the functor $T_n P_W F$ is fibrant in the Bousfield-Friedlander localization of the category of orthogonal functors at the functor P_W. In particular, the map*

$$\omega_{T_n P_W F}\colon T_n P_W F \longrightarrow P_W T_n P_W F,$$

is a levelwise weak homotopy equivalence.

Proof. The Bousfield-Friedlander localization of based spaces at the endofunctor P_W is identical to the left Bousfield localization of based spaces at the map $* \to W$, since both model structures have the same cofibrations and fibrant objects. It follows that the Bousfield-Friedlander localization of the category of orthogonal functors at the endofunctor P_W is identical to the W-local projective model structure. In particular, we see that $P_W F$ is fibrant and hence $\tau_n P_W F$ is also fibrant, since the class of W-local objects is closed under homotopy limits. The result follows since local objects for a nullification are closed under sequential homotopy colimits by [29, 1.D.6]. □

Proposition 5.3.2. *For a finite cell complex W the Bousfield-Friedlander localization of the category of orthogonal functors at the endofunctor*

$$T_n \circ P_W \colon \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \longrightarrow \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*),$$

exists. This model structure is proper and topological. We call this the W-periodic n-polynomial model structure and denote it by $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, P_W \mathsf{Top}_*)$.

Proof. We verify the axioms of [22, Theorem 9.3]. First note that since P_W and T_n both preserve levelwise weak equivalences so does their composite, hence verifying [22, Theorem 9.3(A1)].

The natural transformation from the identity to the composite $T_n \circ P_W$ is given in components as the composite

$$F \xrightarrow{\omega_F} P_W F \xrightarrow{\eta_{P_W F}} T_n P_W F,$$

where $\omega \colon \mathbb{1} \to P_W$ and $\eta \colon \mathbb{1} \to T_n$, hence at $T_n P_W F$, we obtain the composite

$$T_n P_W F \xrightarrow{\omega_{T_n P_W F}} P_W T_n P_W F \xrightarrow{\eta_{P_W T_n P_W F}} T_n P_W T_n P_W F.$$

Since the domain is fibrant in the W-periodic projective model structure the first map in the composite is a levelwise weak equivalence, see Lemma 5.3.1. The second map is also a weak equivalence. To see this, note that since $T_n P_W F$ is polynomial of degree less than or equal n, the functor $P_W T_n P_W F$ is also polynomial of degree less than or equal n by the commutativity of the diagram

$$\begin{array}{ccc} T_n P_W F & \longrightarrow & \tau_n T_n P_W F \\ \downarrow & & \downarrow \\ P_W T_n P_W F & \longrightarrow & \tau_n P_W T_n P_W F \end{array}$$

and the fact that homotopy limits preserve levelwise weak equivalences. It follows that the natural transformation $\eta \colon T_n P_W F \to T_n P_W T_n P_W F$ is a levelwise weak equivalence, as a composite of two levelwise weak equivalences.

The map $T_nP_W(\eta)\colon T_nP_WF \to T_nP_WT_nP_WF$ is also a levelwise weak equivalence. To see this, note that there is a commutative diagram,

$$\begin{array}{ccccc}
F & \xrightarrow{\quad\omega_F\quad} & P_WF & \xrightarrow{\quad\eta_{P_WF}\quad} & T_nP_WF \\
{\scriptstyle\omega_F}\big\downarrow & (1) & {\scriptstyle\omega_{P_WF}}\big\downarrow & (2) & \big\downarrow{\scriptstyle\omega_{T_nP_WF}} \\
P_WF & \xrightarrow[P_W\omega_F]{} & P_WP_WF & \xrightarrow[P_W\eta_{P_WF}]{} & P_WT_nP_WF \\
{\scriptstyle\eta_{P_WF}}\big\downarrow & (3) & {\scriptstyle\eta_{P_WP_WF}}\big\downarrow & (4) & \big\downarrow{\scriptstyle\eta_{P_WT_nP_WF}} \\
T_nP_WF & \xrightarrow[T_nP_W\omega_F]{} & T_nP_WP_WF & \xrightarrow[T_nP_W\eta_{P_WF}]{} & T_nP_WT_nP_W
\end{array}$$

in which, the required map is given by the lower horizontal composite. Since P_W is a homotopically idempotent functor, $P_W\omega_F$ is a levelwise weak equivalence. It follows that the bottom horizontal map

$$T_nP_W\omega_F\colon T_nP_WF \longrightarrow T_nP_WP_WF,$$

of (3) is a weak equivalence sine T_n preserves weak equivalences.

Moreover, P_W being homotopically idempotent yields that the vertical map

$$\omega_{P_WF}\colon P_WF \longrightarrow P_WP_WF$$

in (2) is a levelwise weak equivalence. The right-hand vertical map in this square is also an equivalence by Lemma 5.3.1. By [58, Theorem 6.3], the top right hand horizontal map

$$\eta_{P_WF}\colon P_WF \longrightarrow T_nP_WF,$$

is an approximation of order n in the sense of [58, Definition 5.16]. By commutativity of (2), the lower horizontal map

$$P_W\eta_{P_WF}\colon P_WP_WF \longrightarrow P_WT_nP_WF,$$

is an approximation of order n. The proof of [58, Theorem 6.3] also demonstrates that the vertical maps in (4) are approximations of order n, and since three out of the four maps in the lower right square are approximations of order n, so too is the lower right hand horizontal map

$$T_nP_W\eta_{P_WF}\colon T_nP_WP_WF \longrightarrow T_nP_WT_nP_WF.$$

An application of [58, Theorem 5.15] yields that this map is a levelwise weak equivalence as both source and target are polynomial of degree less than or equal n. This concludes the proof that the map

$$T_nP_W(\eta)\colon T_nP_WF \longrightarrow T_nP_WT_nP_WF,$$

is a levelwise weak equivalence, and verifying [22, Theorem 9.3(A2)].

Finally we verify [22, Theorem 9.3(A3)]. Let

$$\begin{array}{ccc}
A & \xrightarrow{k} & B \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
C & \xrightarrow[h]{} & D
\end{array}$$

be a pullback square with f a levelwise fibration between W-local n-polynomial functors, and $T_n P_W h \colon T_n P_W C \to T_n P_W D$ a levelwise weak equivalence. By [22, Theorem 9.9], we see that the fibre of k is P_W-acyclic, i.e. $P_W(\mathsf{fib}(k))$ is levelwise weakly contractible. Since T_n preserves levelwise weak equivalences, we see that $T_n P_W(\mathsf{fib}(k))$ is levelwise weakly contractible, and hence k is a $T_n P_W$-equivalence.

The fact that the resulting model structure is topological follows from [22, Theorem 9.1]. □

This Bousfield-Friedlander localization results in an identical model structure to the W-local n-polynomial model structure of Proposition 5.2.3

Proposition 5.3.3. *For a finite cell complex W there is an equality of model structures*

$$\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_W \mathsf{Top}_*) = \mathsf{Poly}^{\leq n}(\mathcal{J}_0, P_W \mathsf{Top}_*),$$

that is, the W-local n-polynomial model structure and the W-periodic n-polynomial model structure agree. In particular, these model structures are cellular, proper and topological.

Proof. Both model structures have the same cofibrations, namely the projective cofibrations. It suffices to show that they share the same fibrant objects. Working through the definition of a fibrant object in the Bousfield-Friedlander localization we see that an orthogonal functor F is fibrant if and only if the canonical map $F \to T_n P_W F$ is a levelwise weak equivalence. It follows that F must be W-local and n-polynomial, hence fibrant in the W-local n-polynomial model structure. Conversely, if F is fibrant in the W-local n-polynomial model structure, then the map $F \to P_W F$ is a levelwise weak equivalence and there is a commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & P_W F \\ \downarrow & & \downarrow \\ T_n F & \longrightarrow & T_n P_W F \end{array}$$

in which three out of the four arrows are levelwise weak equivalences, hence so to is the right-hand vertical arrow. It follows that F is fibrant in the Bousfield-Friedlander localization. □

Remark 5.3.4. The nullification condition here is necessary. The above lemma does not hold in general. To see this, consider the (smashing) localization at the spectrum $E = H\mathbb{Q}$. The $H\mathbb{Q}$-local model structure is not right proper, (see Remark 2.3.1) yet if this were expressible as a Bousfield-Friedlander localization it would necessarily be right proper, [22, Theorem 9.3].

Corollary 5.3.5. *For a finite cell complex W a map $f \colon X \to Y$ is a fibration is the W-local n-polynomial model structure if and only if f is a fibration in the projective model structure and the square*

$$\begin{array}{ccc} X & \longrightarrow & T_n P_W X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & T_n P_W Y \end{array}$$

is a homotopy pullback square in the projective model structure on $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$.

Remark 5.3.6. It is highly unlikely that this result holds in more general localizations than nullifications. Let $\mathcal{C}$ be a model category and S a set of maps in $\mathcal{C}$ such that the left Bousfield localization of $\mathcal{C}$ at S exists. By [33, Proposition 3.4.8(1)] right properness of $\mathcal{C}$ and $L_S\mathcal{C}$ is sufficient for a map $f\colon X \to Y$ being a fibration in $L_S\mathcal{C}$ if and only if f is a fibration in $\mathcal{C}$ and the square

$$\begin{array}{ccc} X & \xrightarrow{j_X} & \hat{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ Y & \xrightarrow[j_Y]{} & \hat{Y} \end{array}$$

is a homotopy pullback square, where $\hat{f}\colon \hat{X} \to \hat{Y}$ is a S-localization of f in the sense of [33, Definition 3.2.16]. In our situation, Proposition 2.3.2 guarantees that a homological localization is right proper if and only if it is a nullification. However, it is not clear in general if right properness of the base model category and the localized model category is a necessary condition for the above description of the fibrations in $L_S\mathcal{C}$.

6. Homogeneous functors

6.1. **Homogeneous functors.** The layers of the Weiss tower associated to an orthogonal functor F are the homotopy fibres of maps $T_nF \to T_{n-1}F$ and have two interesting properties: firstly, they are polynomial of degree less than or equal to n and secondly, their $(n-1)$-polynomial approximation is trivial. We denote the n-th layer of the Weiss tower of F by D_nF.

Definition 6.1.1. For $n \geq 0$, an orthogonal functor F is said to be *n-reduced* if its $(n-1)$-polynomial approximation is levelwise weakly contractible. An orthogonal functor F is said to be *homogeneous of degree n* if it is both polynomial of degree less than or equal n and n-reduced. We will sometimes refer to a functor which is homogeneous of degree n as being *n-homogeneous*.

There is a model structure on the category of orthogonal functors which contains the n-homogeneous functors as the bifibrant objects.

Proposition 6.1.2 ([6, Proposition 6.9])**.** *There is a model category structure on the category of orthogonal functors with weak equivalences the D_n-equivalences and fibrations the fibrations of the n-polynomial model structure. The cofibrant objects are the n-reduced projectively cofibrant objects and the fibrant objects are the n-polynomial functors. In particular, cofibrant-fibrant objects of this model structure are the projectively cofibrant n-homogeneous functors. This model structure is cellular, proper, stable and topological. We call this the n-homogeneous model structure and denote it by* $\mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*)$.

Remark 6.1.3. The model structure of [6, Proposition 6.9] has as weak equivalences those maps which induce levelwise weak equivalences on the n-th derivatives of their n-polynomial approximations. We showed in [52, Proposition 8.2], that the class of such equivalences is precisely the class of D_n-equivalences.

The n-homogeneous model structure is (zigzag) Quillen equivalent to spectra with an action of $O(n)$.

Proposition 6.1.4 ([6, Proposition 8.3 & Theorem 10.1])**.** *Let $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) \simeq_Q \mathsf{Sp}[O(n)].$$

On the homotopy category level the Barnes-Oman zigzag of Quillen equivalences recovers Weiss' characterisation of homogeneous functors of degree n.

Proposition 6.1.5 ([58, Theorem 7.3]). *Let $n \geq 1$. An n-homogeneous functor F is determined by and determines a spectrum $\partial_n F$ with an $O(n)$-action. In particular, an n-homogeneous functor F is levelwise weak homotopy equivalent to the functor*

$$V \longmapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n F)_{hO(n)}],$$

and any functor of the above form is homogeneous of degree n.

6.2. **Local homogeneous functors.**

Definition 6.2.1. Let S be a set of maps of based spaces. An orthogonal functor F is *S-locally homogeneous of degree n* if it is levelwise S-local and n-homogeneous.

Our prototypical example of an S-local n-homogeneous functor is the homotopy fibre of the map $T_n(L_S X) \to T_{n-1}(L_S X)$, when localization with respect to S is "nice", as made precise below.

Lemma 6.2.2. *Let S be a set of maps of based spaces, and F and orthogonal functor. For $n \geq 1$, there is a homotopy fibre sequence*

$$D_n^S F \longrightarrow T_n(L_S F) \longrightarrow T_{n-1}(L_S F),$$

in which $D_n^S(F)$ is

(1) homogeneous of degree n; and,
(2) S-locally n-homogeneous if, in addition, the class of S-local spaces is closed under sequential homotopy colimits.

Proof. By [58, Lemma 5.5] the homotopy fibre of a map between two n-polynomial functors is n-polynomial, hence $D_n^S F$ is n-polynomial. Applying T_{n-1} to the homotopy fibre sequence, yields that the $(n-1)$-polynomial approximation of $D_n^S F$ is levelwise weakly contractible, proving (1).

For (2), observe that the homotopy fibre of a map between S-local objects is S-local and when the class of S-local spaces is closed under sequential homotopy colimits, $T_n L_S F$ is S-local for all n. □

Examples 6.2.3.

(1) For homological localization at the Eilenberg-Maclane spectrum associated to a subring R of the rationals, $D_n^{HR} F$ is HR-locally n-homogeneous.
(2) For nullification at a finite cell complex W, $D_n^W F$ is W-locally n-homogeneous.

In most versions of functor calculus the n-homogeneous model structure is a right Bousfield localization of the n-polynomial model structure, see e.g., [16, Theorem 6.4], [6, Proposition 6.9] or [56, Proposition 3.2]. In the local picture, the n-polynomial model structure need not be right proper, hence one must find an alternative way to construct an n-homogeneous model structure. We will return to the search for a local n-homogeneous model structure once we have discussed the derivatives in orthogonal calculus.

7. The derivatives

7.1. **The derivatives.** For each $n \geq 0$, sitting over the space of linear isometries $\mathcal{J}(U, V)$ is a vector bundle $\gamma_n(U, V)$ with fibre over a linear isometry $f\colon U \to V$ given by $\mathbb{R}^n \otimes f(U)^\perp$, where $f(U)^\perp$ denotes the orthogonal complement of $f(U)$ in V.

Definition 7.1.1. For $n \geq 0$ define $\mathcal{J}_n$ to be the category with the same objects as $\mathcal{J}$ and morphism space $\mathcal{J}_n(U, V)$ given as the Thom space of $\gamma_n(U, V)$.

For $n = 0$ this recovers the category $\mathcal{J}_0$ of Definition 4.1.1.

Remark 7.1.2. The standard action of $O(n)$ on $\mathbb{R}^n$ via the regular representation induces an action on the vector bundles that is compatible with the composition, hence $\mathcal{J}_n$ is naturally enriched over based spaces with an $O(n)$-action.

Definition 7.1.3. For $n \geq 0$ define the *n-th intermediate category* to be the category of $O(n)\,\mathsf{Top}_*$-enriched functors from $\mathcal{J}_n$ to $O(n)\,\mathsf{Top}_*$. We will denote this category by $\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*)$.

Let $0 \leq m \leq n$. The inclusion $i^n_m \colon \mathbb{R}^m \to \mathbb{R}^n$ induces a functor $i^n_m \colon \mathcal{J}_m \to \mathcal{J}_n$. Postcomposition with i^n_m induced a topological functor

$$\mathsf{res}^n_m \colon \mathsf{Fun}(\mathcal{J}_\mathsf{n}, \mathsf{Top}_*) \longrightarrow \mathsf{Fun}(\mathcal{J}_\mathsf{m}, \mathsf{Top}_*),$$

which by [58, Proposition 2.1] has a right adjoint

$$\mathsf{ind}^n_m \colon \mathsf{Fun}(\mathcal{J}_\mathsf{m}, \mathsf{Top}_*) \longrightarrow \mathsf{Fun}(\mathcal{J}_\mathsf{n}, \mathsf{Top}_*),$$

given by

$$\mathsf{ind}^n_m F(U) = \mathsf{nat}_m(\mathcal{J}_n(U, -), F),$$

where $\mathsf{nat}_m(-,-)$ denotes the (based) space of natural transformations in $\mathsf{Fun}(\mathcal{J}_\mathsf{m}, \mathsf{Top}_*)$, and $\mathcal{J}_n(U,-)$ is considered as an object of $\mathsf{Fun}(\mathcal{J}_\mathsf{m}, \mathsf{Top}_*)$ by restriction. Combining the restriction and induction functors with change of group adjunctions from [45], we obtain an adjoint pair

$$\mathsf{res}^n_m /O(n-m) : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Fun}_{O(m)}(\mathcal{J}_m, O(m)\,\mathsf{Top}_*) : \mathsf{ind}^n_m\mathsf{CI}\ ,$$

see [6, §4].

Definition 7.1.4. Let F be an orthogonal functor. For $n \geq 0$, the *n-th derivative of* F is given by $\mathsf{ind}^n_0\,\mathsf{CI}F$. In which case, we write $\mathsf{ind}^n_0\,\varepsilon^*F$ or $F^{(n)}$.

The n-th intermediate category is equivalent to the category of modules over the monoid

$$n\mathbb{S} \colon \mathcal{I} \longrightarrow O(n)\,\mathsf{Top}_*,$$
$$V \longmapsto S^{\mathbb{R}^n \otimes V},$$

in the category of $O(n)$-equivariant $\mathcal{I}$-spaces, where $\mathcal{I}$ is the category of finite-dimensional inner product subspaces of $\mathbb{R}^\infty$ and linear isometric isomorphisms, see [6, Proposition 7.4]. As such it may be equipped with a stable model structure reminiscent of the stable model structure on spectra. Given an object F of the n-th intermediate category, we define the n-homotopy groups of F as

$$n\pi_k F = \operatorname*{colim}_q \pi_{nq+k}(F(\mathbb{R}^q)),$$

for $k \in \mathbb{Z}$. The weak equivalences of the n-stable model structure are then the $n\pi_*$-isomorphisms; those maps $f \colon X \to Y$ such that for each integer k, the induced map

$$n\pi_k(f) \colon n\pi_k(X) \longrightarrow n\pi_k(Y),$$

is an isomorphism.

Proposition 7.1.5 ([6, Proposition 7.14])**.** *There is a model category structure on the n-th intermediate category with weak equivalences the $n\pi_*$-isomorphisms and fibrations the levelwise fibrations $X \to Y$ such that the square*

$$\begin{array}{ccc} X(U) & \longrightarrow & \Omega^{\mathbb{R}^n \otimes V} X(U \oplus V) \\ \downarrow & & \downarrow \\ Y(U) & \longrightarrow & \Omega^{\mathbb{R}^n \otimes V} Y(U \oplus V) \end{array}$$

is a homotopy pullback for all $U, V \in \mathcal{J}_n$. The fibrant objects are the $n\Omega$-spectra, i.e., those objects F such that

$$F(U) \to \Omega^{\mathbb{R}^n \otimes V} F(U \oplus V),$$

is a weak homotopy equivalence for all $U, V \in \mathcal{J}_n$. This model structure is cellular, proper, stable and topological. We call this the n-stable model structure and denote it by $\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*)$.

7.2. **The local n-stable model structure.** We now equip the n-th intermediate category with an S-local model structure which will be intermediate in our classification of S-local n-homogeneous functors as levelwise S-local spectra with an action of $O(n)$.

Proposition 7.2.1. *Let S be a set of maps of based spaces. There is a model category structure on the n-th intermediate category with cofibrations the cofibrations of the n-stable model structure and fibrant objects the $n\Omega$-spectra which are levelwise S-local. This model structure is cellular, left proper and topological. We call this the S-local n-stable model structure and denote it by $L_S\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*)$.*

Proof. This model structure is the left Bousfield localization of the n-stable model structure at the set

$$\mathcal{Q}_n = \{O(n)_+ \wedge \mathcal{J}_n(U, -) \wedge j \mid U \in \mathcal{J}, j \in J_{L_S\,\mathsf{Top}_*}\}.$$

□

We record the following fact which will prove useful later.

Lemma 7.2.2. *Let S be a set of maps of based spaces. If F is an S-local functor, then $F^{(n)} = \mathsf{ind}_0^n F$ is S-local.*

Proof. The objectwise smash product

$$(-) \wedge (-) \colon \mathsf{Fun}(\mathcal{J}_n, L_S\,\mathsf{Top}_*) \times L_S\,\mathsf{Top}_* \longrightarrow \mathsf{Fun}(\mathcal{J}_n, L_S\,\mathsf{Top}_*),$$

is a Quillen bifunctor, and the result follows from the definition of $\mathsf{ind}_0^n F$. □

7.3. **The derivatives as spectra.** The n-th derivative of an orthogonal functor may be thought of as a spectrum, in particular, the n-th intermediate category and the category of spectra with an $O(n)$-action are Quillen equivalent via the following adjunction

$$(\alpha_n)_! : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^*\ ,$$

see e.g., [6, §8]. We now prove that this result holds S-locally for any set S of maps of based spaces.

Theorem 7.3.1. *Let S be a set of maps of based spaces. The adjoint pair*

$$(\alpha_n)_! : L_S\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}(L_S\,\mathsf{Top}_*)[O(n)] : (\alpha_n)^*\ ,$$

is a Quillen equivalence between the S-local model structures.

Proof. For the Quillen adjunction apply [33, Theorem 3.3.20(1)], noting that there is an isomorphism

$$(\alpha_n)_!(O(n)_+ \wedge \mathcal{J}_n(U, -) \wedge j) \cong O(n)_+ \wedge \mathcal{J}_1(\mathbb{R}^n \otimes U, -) \wedge j,$$

for j a generating acyclic cofibration for the S-local model structure on based spaces.

By [6, Proposition 8.3] the adjoint pair

$$(\alpha_n)_! : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^*\ ,$$

is a Quillen equivalence. To show that the adjunction is a Quillen equivalence, it suffices by [36, Proposition 2.3] to show that if Y is fibrant in $\mathsf{Sp}[O(n)]$ such that $(\alpha_n)^*Y$ is fibrant in the S-local n-stable model structure, then Y is fibrant in the S-local model structure on $\mathsf{Sp}[O(n)]$. This follows readily from the definitions of fibrant objects in both model structures. □

In the case of homological localizations we obtain the following Quillen equivalence.

Corollary 7.3.2. *Let E be a spectrum. The adjoint pair*

$$(\alpha_n)_! : L_E\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}(L_E\,\mathsf{Top}_*)[O(n)] : (\alpha_n)^* \ ,$$

is a Quillen equivalence between the E-local model structures.

8. The classification of local homogeneous functors

8.1. **The local n-homogeneous model structure.** We now return to the question of a suitable model structure on the category of orthogonal functors which caputures the homotopy theory of local n-homgoeneous functors. The case of the homological localization $E = H\mathbb{Q}$ was discussed by Barnes in [4, §6]

Proposition 8.1.1. *Let S be a set of maps of based spaces. There is model category structure on the category of orthogonal functors with cofibrations the cofibrations of the n-homogeneous model structure and fibrant objects the n-polynomial functors whose n-th derivative is levelwise S-local. This model structure is cellular, left proper and topological. We call this the S-local n-homogeneous model structure and denote it by $\mathsf{Homog}^n(\mathcal{J}, L_S\,\mathsf{Top}_*)$.*

Proof. We left Bousfield localize the n-homogeneous model structure at the set of maps

$$\mathcal{K}_n = \{\mathcal{J}_n(U,-)\wedge j \mid U \in \mathcal{J}, j \in J_{L_S\,\mathsf{Top}_*}\},$$

where $\mathcal{J}_n(U,V)$ is the Thom space of the n-th complement vector bundle $\gamma_n(U,V)$ sitting over the space of linear isometries $\mathcal{J}(U,V)$. This left Bousfield localization exists since the n-homogeneous model structure is cellular and left proper by [4, Lemma 6.1]. The description of the cofibrations follows immediately.

The fibrant objects are the $\mathcal{K}_n$-local objects which are fibrant in the n-homogeneous model structure, i.e., those n-polynomial functors Z for which the induced map

$$[\mathcal{J}_n(U,-)\wedge B, Z] \longrightarrow [\mathcal{J}_n(U,-)\wedge A, Z],$$

is an isomorphism for all maps $\mathcal{J}_n(U,-)\wedge A \to \mathcal{J}_n(U,-)\wedge B$ in $\mathcal{K}_n$. A straightforward adjunction argument, and the definition of the n-th derivative of an orthogonal functor yield the required characterisation of the fibrant objects. □

Corollary 8.1.2. *Let S be a set of maps of based spaces. The cofibrant objects of the S-local n-homogeneous model structure are the projectively cofibrant functors which are n-reduced.*

Proof. The S-local n-homogeneous model structure is a particular left Bousfield localization of the n-homogeneous model structure, hence has the same cofibrant objects. The result follows by the orthogonal calculus version of [52, Corollary 8.6]. □

We now relate the S-local n-homogeneous model structure to the S-local n-polynomial model structure, exhibiting that the S-local n-homogeneous model structure behaves like a right Bousfield localization of the S-local n-polynomial model structure.

Lemma 8.1.3. *Let S be a set of maps of based spaces. The adjoint pair*

$$\mathbb{1} : \mathsf{Homog}^n(\mathcal{J}_0, L_S\,\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\,\mathsf{Top}_*) : \mathbb{1}\ ,$$

is a Quillen adjunction.

Proof. To demonstrate that the left adjoint preserves cofibrations it suffices by [35, Lemma 2.1.20] to show that the identity functor sends the generating cofibrations of the S-local n-homogeneous model structure to cofibrations of the S-local n-polynomial model structure.

The cofibrations of the S-local n-homogeneous model structure are the cofibrations of the n-homogeneous model structure, which are contained in the cofibrations of the n-polynomial model structure, which in turn are precisely the cofibrations of the S-local n-polynomial model structure, hence

$$\mathbb{1} \colon \mathsf{Homog}^n(\mathcal{J}_0, L_S\,\mathsf{Top}_*) \longrightarrow \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S\,\mathsf{Top}_*),$$

preserves cofibrations.

On the other hand, to show that the right adjoint is right Quillen it suffices to show that the identity functor sends fibrant objects in the S-local n-polynomial model structure to fibrant objects in the S-local n-homogeneous model structure. This follows from Lemma 7.2.2 since the fibrant objects in the S-local n-polynomial model structure are the S-locally n-polynomial functors by Proposition 5.2.3 and the fibrant objects of the S-local n-homogeneous model structure are the n-polynomial functors with S-local n-th derivative by Proposition 8.1.1. □

8.2. **Characterisations for homological localizations.** In the case of a homological localization, we obtain a characterisation of the fibrations. A more general result holds when the localizing set S is stable in the sense of [8, Definition 4.2], i.e., when the class of S-local spaces is closed under suspension.

Proposition 8.2.1. *If S is a set of maps of based spaces which is stable, then the fibrations of the S-local n-homogeneous model structure are those maps $f\colon X \to Y$ which are fibrations in the n-polynomial model structure such that*

$$X^{(n)} \longrightarrow Y^{(n)},$$

is a levelwise fibration of S-local spaces.

Proof. We first given an explicit characterisation of the acyclic cofibrations since the fibrations are characterised by the right lifting property against these maps. The maps in $\mathcal{K}_n$ are cofibrations between cofibrant objects since $\mathcal{J}_n(U,-)$ is cofibrant in $\mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*)$ and the maps in $J_{L_S\,\mathsf{Top}_*}$ are cofibrations of S-local spaces. Moreover, since the localizing set S is stable, it follows the set of generating acyclic cofibrations $J_{L_S\,\mathsf{Top}_*}$ is stable and in turn that the set $\mathcal{K}_n$ is stable. Hence by [8, Theorem 4.11], the generating acyclic cofibrations are given by the set $J_{\mathsf{Homog}^n} \cup \Lambda(\mathcal{K}_n)$, where J_{Homog^n} is the set of the generating acyclic cofibrations of the n-homogeneous model structure and $\Lambda(\mathcal{K}_n)$ the set of horns on $\mathcal{K}_n$ in the sense of [33, Definition 4.2.1]. As horns in topological model categories are given by pushouts and $\mathcal{K}_n$ is a set of cofibrations between cofibrant objects it suffices to use the set $J_{\mathsf{Homog}^n} \cup \mathcal{K}_n$ as the generating acyclic cofibrations of the S-local n-homogeneous model structure.

If $f\colon X \to Y$ is a map with the right lifting property with respect to $J_{\mathsf{Homog}^n} \cup \mathcal{K}_n$, then f has the right lifting property with respect to J_{Homog^n} and the right lifting property with respect to $\mathcal{K}_n$ independently. Having the right lifting property with respect to J_{Homog^n} is equivalent to being a fibration in the n-polynomial model structure. On the other hand, a map in $\mathcal{K}_n$ is of the form

$\mathcal{J}_n(U,-) \wedge A \to \mathcal{J}_n(U,-) \wedge B$ for $A \to B$ a generating acyclic cofibration of the S-local model structure on based spaces. A lift in the diagram

$$\begin{array}{ccc} \mathcal{J}_n(U,-) \wedge A & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \mathcal{J}_n(U,-) \wedge B & \longrightarrow & Y \end{array}$$

(indicated by the dotted arrow) exists if and only if the lift in the diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathsf{nat}_0(\mathcal{J}_n(U,-), X) \\ \big\downarrow & \nearrow & \big\downarrow f_* \\ B & \longrightarrow & \mathsf{nat}_0(\mathcal{J}_n(U,-), Y) \end{array}$$

exists, which is equivalent to the statement that $X^{(n)} \to Y^{(n)}$ is a levelwise fibration of S-local spaces, see §7.1. □

This specialises to homological localizations.

Corollary 8.2.2. *Let E be a spectrum. The fibrations of the E-local n-homogeneous model structure are those maps $f\colon X \to Y$ which are fibrations in the n-polynomial model structure such that*

$$X^{(n)} \longrightarrow Y^{(n)},$$

is a levelwise fibration of E-local spaces.

Proof. Combine Proposition 8.2.1 with [8, Example 4.3]. □

Remark 8.2.3. For the case $E = H\mathbb{Q}$, Barnes [4, Theorem 6.2], states that the fibrant objects of this model structure are the n-homogenous functors F with $F^{(n)}$ levelwise $H\mathbb{Q}$-local. This is a typographical error; it is the *bifibrant* objects which admit this characterisation.

Corollary 8.2.4. *Let E be a spectrum. An orthogonal functor F is fibrant in the E-local n-homogeneous model structure if and only if F is n-polynomial and $F^{(n)}$ is levelwise E-local. In particular, F is bifibrant if F is projectively cofibrant, n-homogeneous and $X^{(n)}$ is levelwise E-local.*

Proof. Apply Corollary 8.2.2 to the map $F \to *$. □

8.3. **Characterisations for nullifications.** In the case of a nullification with respect to a based space W Proposition 8.1.1 is not the only way of constructing a model structure which deserves the title of the W-local n-homogeneous model structure. Since the W-local model structure on based spaces is right proper, so too is the W-local n-polynomial model structure and hence we can also follow the classical procedure and preform a right Bousfield localization at the set

$$\mathcal{K}'_n = \{\mathcal{J}_n(U,-) \mid U \in \mathcal{J}\},$$

to obtain a local n-homogeneous model category structure.

Proposition 8.3.1. *For a finite cell complex W there exists a model structure on the category of orthogonal functors with weak equivalences those maps $X \to Y$ such that*

$$(T_n P_W X)^{(n)} \longrightarrow (T_n P_W X)^{(n)},$$

is a levelwise weak equivalence and with fibrations the fibrations of the W-local n-polynomial model structure. This model structure cellular, proper, stable and topological. We call this the W-periodic n-homogeneous model structure and denote it $\mathsf{Homog}^n(\mathcal{J}_0, P_W\,\mathsf{Top}_*)$.

Proof. This is the right Bousfield localization of the W-local n-polynomial model structure. The proof of which follows exactly as in [6, Proposition 6.9]. Note that this right Bousfield localization exists since the W-local n-polynomial model structure in right proper and cellular when the localization is a nullification, see Proposition 5.3.3. □

This right Bousfield localization behaves like a left Bousfield localization of the n-homogeneous model structure in the following sense.

Lemma 8.3.2. *For a finite cell complex W the adjoint pair*

$$\mathbb{1} : \mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*) : \mathbb{1} \ ,$$

is a Quillen adjunction.

Proof. Since the acyclic cofibrations of the n-homogeneous model structure are precisely the acyclic cofibrations of the n-polynomial model structure and similarly, the acyclic cofibrations of W-periodic n-homogeneous model structure are precisely the acyclic cofibrations of the W-local n-polynomial model structure, the identity functor preserves acyclic cofibrations by Lemma 5.2.4.

On the other hand, by [33, Proposition 3.3.16(2)], cofibrations between cofibrant objects in a right Bousfield localization are cofibrations in the underlying model structure, hence Lemma 5.2.4 shows that the identity functor preserves cofibrations between cofibrant objects. The result follows by [25, Corollary A.2]. □

Remark 8.3.3. We will see in Corollary 8.5.2 that although the W-local n-homogeneous model structure and the W-periodic n-homogeneous model structure are not identical, the identity functor yields a Quillen equivalence between them and hence they have canonically equivalent homotopy categories.

8.4. **Differentiation as a Quillen functor.** The n-th derivative of a functor F is a right Quillen functor as part of a Quillen equivalence between the n-homogeneous model structure and the intermediate category; the adjunction

$$\mathsf{res}_0^n /O(n) : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) : \mathsf{ind}_0^n \varepsilon^* \ ,$$

is a Quillen equivalence, [6, Theorem 10.1]. We now show that this extends to the S-local situation with respect to a set of maps of based spaces.

Theorem 8.4.1. *Let S be a set of maps of based spaces. The adjoint pair*

$$\mathsf{res}_0^n /O(n) : L_S\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, L_S \mathsf{Top}_*) : \mathsf{ind}_0^n \varepsilon^* \ ,$$

is a Quillen equivalence between the S-local model structures.

Proof. The left adjoint applied to the localizing set of the S-local n-stable model structure is precisely the localization set of the S-local n-homogeneous model structure, hence the result follows from [33, Theorem 3.3.20(1)]. □

Corollary 8.4.2. *Let S be a set of maps of based spaces, and $n \geq 0$. There is a zigzag of Quillen equivalences*

$$\mathsf{Homog}^n(\mathcal{J}_0, L_S \mathsf{Top}_*) \simeq_Q \mathsf{Sp}(L_S \mathsf{Top}_*)[O(n)].$$

Example 8.4.3. Let R be a subring of the rationals. Then there is a zigzag of Quillen equivalences

$$\mathsf{Homog}^n(\mathcal{J}_0, L_{HR} \mathsf{Top}_*) \simeq_Q \mathsf{Sp}_{HR}[O(n)],$$

between HR-local n-homogeneous functors and HR-local spectra with an action of $O(n)$.

8.5. **Characterisations for nullifications.** An analogous Quillen equivalence is obtained between the W-local intermediate category and the W-periodic n-homogeneous model structure of Proposition 8.3.1 which recall is obtained as a right Bousfield localization of the W-local n-polynomial model structure.

Theorem 8.5.1. *For a finite cell complex W the adjoint pair*

$$\mathsf{res}_0^n/O(n) : L_W \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_W\,\mathsf{Top}_*) : \mathsf{ind}_0^n \varepsilon^* ,$$

is a Quillen equivalence.

Proof. By [6, Lemma 9.2], the adjoint pair

$$\mathsf{res}_0^n/O(n) : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, \mathsf{Top}_*) : \mathsf{ind}_0^n \varepsilon^* ,$$

is a Quillen adjunction, hence the adjunction

$$\mathsf{res}_0^n/O(n) : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, P_W\,\mathsf{Top}_*) : \mathsf{ind}_0^n \varepsilon^* ,$$

is also a Quillen adjunction. This Quillen adjunction extends to give a Quillen adjunction

$$\mathsf{res}_0^n/O(n) : L_W \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, P_W\,\mathsf{Top}_*) : \mathsf{ind}_0^n \varepsilon^* ,$$

by [33, Theorem 3.1.6(1) & Proposition 3.3.18(1)], since the n-th derivative of a W-local n-polynomial functor is W-local by Lemma 7.2.2, and an $n\Omega$-spectrum by [58, Corollary 5.12]. An application of [33, Theorem 3.1.6(2) & Proposition 3.3.18(2)] extends the above Quillen adjunction to a Quillen adjunction

$$\mathsf{res}_0^n/O(n) : L_W \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_W\,\mathsf{Top}_*) : \mathsf{ind}_0^n \varepsilon^* ,$$

since a map f is a weak equivalence between fibrant objects in the W-periodic n-homogeneous model structure if and only if $\mathsf{ind}_0^n \varepsilon^*(f)$is a levelwise weak equivalence.

We now move on to showing that the adjunction is a Quillen equivalence. First note that the right adjoint reflects weak equivalences, since if $f\colon X \to Y$ is a map between fibrant orthogonal functors in the W-local n-homogeneous model structure, such that

$$\mathsf{ind}_0^n \varepsilon^* T_n P_W(f)\colon\ \mathsf{ind}_0^n \varepsilon^* T_n P_W(X) \longrightarrow \mathsf{ind}_0^n \varepsilon^* T_n P_W(Y)$$

is a W-local equivalence in $L_W \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*)$, then $\mathsf{ind}_0^n \varepsilon^* T_n P_W(f)$ is a levelwise weak equivalence by the W-local Whitehead's Theorem and [6, Theorem 10.1]. It follows that f is a weak equivalence in $\mathsf{Homog}^n(\mathcal{J}_0, P_W\,\mathsf{Top}_*)$.

To prove the Quillen equivalence, by [35, Corollary 1.3.16], it is left to show that the derived unit is a weak equivalence. Let F be a cofibrant object of the W-local n-stable model structure. As in [6, Theorem 10.1], there is a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\ 1\ } & \widehat{c}((\alpha_n)^*\Psi) \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 3} \\ \mathsf{ind}_0^n \varepsilon^* T_n P_W\, \mathsf{res}_0^n F/O(n) & \xrightarrow[\ 4\ ]{} & \mathsf{ind}_0^n \varepsilon^* T_n P_W\, \mathsf{res}_0^n(\widehat{c}((\alpha_n)^*\Psi)) \end{array}$$

in which, Ψ is a fibrant replacement of $(\alpha_n)_! F$ in the Borel stablisation of W-local spaces with an $O(n)$-action and $\widehat{c}$ denotes cofibrant replacement in the W-local n-stable model structure. By *loc. cit.* the maps labelled "1" and "2" are n-stable equivalences, hence W-local n-stable equivalences, thus it suffices to show that the map labelled "3" is a W-local n-stable equivalence. This follows from the proof of [6, Theorem 10.1], and the observation that a T_n-equivalence is a T_nP_W-equivalence by the commutativity of the diagram

$$\begin{array}{ccc} T_nA & \xrightarrow{T_n(f)} & T_nB \\ {\scriptstyle T_n\omega_A}\downarrow & & \downarrow{\scriptstyle T_n\omega_B} \\ T_nP_WA & \xrightarrow[T_nP_W(f)]{} & T_nP_WB. \end{array}$$

□

Proposition 8.1.1 and Proposition 8.3.1 provide two different model structures which both capture the homotopy theory of W-locally n-homogeneous functors. However, these model structures are not identical. For instance, the W-local model structure of Proposition 8.1.1 has fibrant objects the n-polynomial functors which have W-local n-th derivative, whereas the fibrant objects of the W-periodic n-homogeneous model structure (Proposition 8.3.1) are the W-local n-polynomial functors. These model structures are Quillen equivalent via the identity functor. Recall that the W-local n-homogeneous model structure, which is a left Bousfield localization of the n-homogeneous model structure, see Proposition 8.1.1, is denoted by $\mathsf{Homog}^n(\mathcal{J}_0, L_W \mathsf{Top}_*)$, and the W-periodic n-homogeneous model structure is denoted by $\mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*)$.

Corollary 8.5.2. *For a finite cell complex W the adjoint pair*

$$\mathbb{1} : \mathsf{Homog}^n(\mathcal{J}_0, L_W \mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*) : \mathbb{1} \ ,$$

is a Quillen equivalence.

Proof. Since cofibrations between cofibrant objects in $\mathsf{Homog}^n(\mathcal{J}_0, L_W \mathsf{Top}_*)$ are projective cofibrations which are T_n-equivalences, and the cofibrations between cofibrant objects of $\mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*)$ are the projective cofibrations, it follows that the identity functor

$$\mathbb{1} \colon \mathsf{Homog}^n(\mathcal{J}_0, L_W \mathsf{Top}_*) \longrightarrow \mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*),$$

necessarily preserves cofibrations between cofibrant objects. On the other hand, the identity functor

$$\mathbb{1} \colon \mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*) \longrightarrow \mathsf{Homog}^n(\mathcal{J}_0, L_W \mathsf{Top}_*),$$

preserves fibrant objects since if X is levelwise W-local, $\mathsf{ind}_0^n X$ is levelwise W-local, by Lemma 7.2.2. It follows that the adjunction is a Quillen adjunction. To see that the adjunction is a Quillen equivalence, there is a commutative square

$$\begin{array}{ccc} L_W\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) & \underset{\mathsf{ind}_0^n\,\varepsilon^*}{\overset{\mathsf{res}_0^n\,/O(n)}{\rightleftarrows}} & \mathsf{Homog}^n(\mathcal{J}_0, L_W \mathsf{Top}_*) \\ {\scriptstyle \mathbb{1}}\downarrow\ \uparrow{\scriptstyle \mathbb{1}} & & {\scriptstyle \mathbb{1}}\downarrow\ \uparrow{\scriptstyle \mathbb{1}} \\ L_W\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) & \underset{\mathsf{ind}_0^n\,\varepsilon^*}{\overset{\mathsf{res}_0^n\,/O(n)}{\rightleftarrows}} & \mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*) \end{array}$$

of Quillen adjunctions, in which three-out-of-four are Quillen equivalences by Theorem 8.4.1 and Theorem 8.5.1. Hence the remaining Quillen adjunction must also be a Quillen equivalence. □

It follows that there is a zigzag of Quillen equivalences

$$\mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*) \simeq_Q \mathsf{Sp}(L_W \mathsf{Top}_*)[O(n)],$$

whenever both model structures exist.

8.6. **The classification.** As in the classical theory, any S-locally n-homogeneous functor may be expressed concretely in terms of a levelwise S-local spectrum with an action of $O(n)$. The proof of which follows as in the classical setting, [58, Theorem 7.3] and can be realised through the derived equivalence of homotopy categories provided by our zigzag of Quillen equivalences.

Theorem 8.6.1. *Let S be a set of maps of based spaces and $n \geq 1$. An S-local n-homogeneous functor F is determined by and determines a levelwise S-local spectrum with an $O(n)$-action, denoted $\partial_n^S F$. In particular, an S-local n-homogeneous functor F is levelwise weak homotopy equivalent to the functor*

$$V \longmapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n^S F)_{hO(n)}],$$

and any functor of the above form is levelwise S-local and n-homogeneous.

8.7. **The classification for homology theories.** This let's us characterise the n-th layer of the E-local Weiss tower for a "nice" localizations since by Lemma 6.2.2 the n-th layer is E-locally n-homogeneous.

Corollary 8.7.1. *Let E be a spectrum such that the class of local spaces is closed under sequential homotopy colimits. If F is an orthogonal functor, then for $n \geq 1$, $D_n^E F$ is levelwise weakly equivalent to the functor given by*

$$V \longmapsto \Omega^\infty[(S^{\mathbb{R}^n \otimes V} \wedge \partial_n^E F)_{hO(n)}].$$

Part 3. Applications

Before moving on to the initial applications of our theory of local orthogonal calculus we briefly recall the various constructions from Part 2, particularly the various model structures and how they relate to each other.

Let S be a set of maps of based spaces and let $L_S \mathsf{Top}_*$ denote the S-local model structure on based spaces. This model structure may be transferred (Proposition 4.2.1) to the category of orthogonal functors to produce a model structure $\mathsf{Fun}(\mathcal{J}_0, L_S \mathsf{Top}_*)$ in which the weak equivalences and fibrations are the levelwise S-local weak equivalences and fibrations respectively. A left Bousfield localization (Proposition 5.2.3) resulted in the S-local n-polynomial model structure $\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S \mathsf{Top}_*)$, in which the fibrant objects were the S-local n-polynomial functors. In nice cases such as nullification at a finite cell complex, we demonstrated (Proposition 5.3.3) that a fibrant replacement in this model structure is given by the composite of the localization functor L_S with the n-polynomial approximation functor T_n.

To understand the layers of the S-local Weiss tower better we began with the n-homogenous model structure of Barnes and Oman [6], which contains the homogeneous of degree n functors as the bifibrant objects and preformed a left Bousfield localization (Proposition 8.1.1) to obtain the S-local n-homogeneous model structure $\mathsf{Homog}^n(\mathcal{J}_0, L_S \mathsf{Top}_*)$ which contains the n-homogeneous functors with S-local n-th derivative as the bifibrant objects.

In the case of a nullification at a finite cell complex W, the W-local n-homogeneous model structure can chararcterised (up to Quillen equivalence via the identity functor, see Corollary 8.5.2) as the W-periodic n-homogeneous model structure $\mathsf{Homog}^n(\mathcal{J}_0, P_W \mathsf{Top}_*)$ in which the bifibrant objects are the W-local n-homogeneous functors.

For a general set of maps of based spaces S we showed (Theorem 8.4.1) that taking the n-th derivative realises a Quillen equivalence between the S-local n-homogenous model structure and the S-local n-th intermediate category and that this latter category is canonically Quillen equivalent

(Theorem 7.3.1) to the category of spectra in S-local spaces with an $O(n)$-action. This produces the following zigzag of Quillen equivalences

$$\mathsf{Homog}^n(\mathcal{J}_0, L_S\,\mathsf{Top}_*) \underset{\mathsf{ind}_0^n\,\varepsilon^*}{\overset{\mathsf{res}_0^n\,/O(n)}{\leftrightarrows}} L_S\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \underset{(\alpha_n)_*}{\overset{(\alpha_n)_!}{\rightleftarrows}} \mathsf{Sp}(L_S\,\mathsf{Top}_*)[O(n)].$$

For the case of a nullification with respect to a finite cell complex W, we summarise our model structures in Figure 1. Note that the Quillen adjunctions forming the lower "⌜" shape are Quillen equivalences.

$$\begin{array}{ccc}
\mathsf{Fun}(\mathcal{J}_0, L_W\,\mathsf{Top}_*) & & \\
\mathbb{1}\big\downarrow \ (5.2.3)\ \big\uparrow\mathbb{1} & & \\
\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_W\,\mathsf{Top}_*) & \overset{(5.3.3)}{=\!=\!=} & \mathsf{Poly}^{\leq n}(\mathcal{J}_0, P_W\,\mathsf{Top}_*) \\
\mathbb{1}\big\uparrow \ (8.1.1)\ \big\downarrow\mathbb{1} & & \\
\mathsf{Homog}^n(\mathcal{J}_0, L_W\,\mathsf{Top}_*) & \underset{\mathbb{1}}{\overset{\mathbb{1}}{\rightleftarrows}}\ (8.5.2) & \mathsf{Homog}^n(\mathcal{J}_0, P_W\,\mathsf{Top}_*) \\
\mathsf{res}_0^n\,/O(n)\big\uparrow \ (8.4.1)\ \big\downarrow\mathsf{ind}_0^n\,\varepsilon^* & & \\
L_W\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) & & \\
(\alpha_n)_!\big\downarrow \ (7.3.1)\ \big\uparrow(\alpha_n)^* & & \\
\mathsf{Sp}(L_W\,\mathsf{Top}_*)[O(n)] & &
\end{array}$$

FIGURE 1. Diagram of Quillen adjunctions for W-periodic orthogonal calculus

In the case of a homological localization with respect to a homology theory E_*, we summarise our model structures in Figure 2. In this case, the lower two Quillen adjunctions are Quillen equivalences.

$$\begin{array}{ccc}
\mathsf{Fun}(\mathcal{J}_0, L_E\,\mathsf{Top}_*) & & \\
\mathbb{1}\big\downarrow \ (5.2.3)\ \big\uparrow\mathbb{1} & & \\
\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_E\,\mathsf{Top}_*) & & \\
\mathbb{1}\big\uparrow \ (8.1.1)\ \big\downarrow\mathbb{1} & & \\
\mathsf{Homog}^n(\mathcal{J}_0, L_E\,\mathsf{Top}_*) & & \\
\mathsf{res}_0^n\,/O(n)\big\uparrow \ (8.4.1)\ \big\downarrow\mathsf{ind}_0^n\,\varepsilon^* & & \\
L_E\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) & & \\
(\alpha_n)_!\big\downarrow \ (7.3.1)\ \big\uparrow(\alpha_n)^* & & \\
\mathsf{Sp}(L_E\,\mathsf{Top}_*)[O(n)] & \underset{(E=HR,\,R\subseteq\mathbb{Q})}{\overset{(3.3.3)}{=\!=\!=}} & \mathsf{Sp}_{HR}[O(n)]
\end{array}$$

FIGURE 2. Diagram of Quillen adjunctions for E-local orthogonal calculus

9.1. **Bousfield classes.** The concept of Bousfield classes were introduced (in the stable setting) by Bousfield in [18], and extensively studied by Ravenel in [50]. For a spectrum E, the *Bousfield class* of E, denoted $\langle E\rangle$, is the equivalence class of of all spectra E' such that the class of E'-acyclic spectra is the class of E-acyclic spectra. If $\langle E\rangle = \langle E'\rangle$, then the classes of E_*-isomorphisms and E'_*-isomorphisms agree and hence the localization functors (on spaces or spectra) agree. The collection of all Bousfield classes forms a lattice, with partial ordering $\langle E\rangle \leq \langle E'\rangle$ given by reverse containment, i.e., if and only if the class of E'-acyclic spectra is contained in the class of E-acyclic spectra.

A similar story remains true unstably. Given a based space W, Bousfield, [19, §9] and Farjoun [29], introduced the *unstable Bousfield class* of W, or the *nullity class* of W. This is the equivalence class $\langle W\rangle$ of all spaces W' such that the class of W-local spaces agrees with the class of W'-local spaces. There is a partial ordering $\langle W\rangle \leq \langle W'\rangle$ given by reverse containment, i.e., if and only if every W'-periodic space is W-periodic. In particular, the relation $\langle W\rangle \leq \langle W'\rangle$ implies that every W-local equivalence is a W'-local equivalence and there is a natural transformation $P_W \to P_{W'}$, which is a W'-localization.

Theorem 9.1.1. *Let S and S' be sets of maps of based spaces. The class of S-local spaces agrees with the class of S' local spaces if and only if for every orthogonal functor F, the S-local Weiss tower of F is levelwise weakly equivalent to the S'-local Weiss tower of F.*

Proof. If the class of S-local spaces agrees with the class of S'-local spaces, then the localization functors L_S and $L_{S'}$ agree on the level of spaces and hence on the level of orthogonal functors. In particular, for every orthogonal functor F, the canonical map[5] $L_SF \to L_{S'}F$ is a levelwise weak equivalence. Now, consider the commutative diagram

$$\begin{array}{ccccc} D_n^S F & \longrightarrow & T_n^S F & \longrightarrow & T_{n-1}^S F \\ \downarrow & & \downarrow & & \downarrow \\ D_n^{S'} F & \longrightarrow & T_n^{S'} F & \longrightarrow & T_{n-1}^{S'} F \end{array}$$

in which the rows are homotopy fibre sequences. For each $n \geq 0$, the map $T_n^S F \to T_n^{S'} F$ is a levelwise weak equivalence since polynomial approximation preserves levelwise weak equivalences. It follows that the left-most vertical arrow is also a levelwise weak equivalence and that the S-local Weiss tower is levelwise weakly equivalent to the S'-local Weiss tower.

The converse is immediate from specialising for every based space A, to the constant functor at A. □

As corollaries, we obtain a relationship between Bousfield classes and the local Weiss towers.

Corollary 9.1.2. *Let W and W' be based spaces. For every orthogonal functor F the W-local Weiss tower of F and the W'-local Weiss tower of F agree if and only if $\langle W\rangle = \langle W'\rangle$.*

Corollary 9.1.3. *Let E and E' be spectra. For every orthogonal functor F the E-local Weiss tower of F and the E'-local Weiss tower of F agree if and only if $\langle E\rangle = \langle E'\rangle$.*

On the model category level, we have the following.

[5] This map is induced from the S-local objects being contained in the S'-local objects. We could also use the canonical $L_{S'}F \to L_SF$ since the S-local objects also contained the S'-local objects.

Theorem 9.1.4. *Let S and S' be sets of maps of based spaces. If the class of S-local spaces agrees with the class of S'-local spaces, then the following are equalities of model structures*

$$\begin{aligned}\mathsf{Fun}(\mathcal{J}_0, L_S \mathsf{Top}_*) &= \mathsf{Fun}(\mathcal{J}_0, L_{S'} \mathsf{Top}_*),\\ \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_S \mathsf{Top}_*) &= \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S'} \mathsf{Top}_*),\\ \mathsf{Homog}^{n}(\mathcal{J}_0, L_S \mathsf{Top}_*) &= \mathsf{Homog}^{n}(\mathcal{J}_0, L_{S'} \mathsf{Top}_*).\end{aligned}$$

Proof. First note that the S-local model structure on based spaces agrees with the S'-local model structure on based spaces since these model structures have the same cofibrations and the same fibrant objects, since a space is S-local if and only if it is S'-local. This equality lifts to the local projective model structures on the category of orthogonal functors since a functor is levelwise S-local if and only if it is levelwise S'-local under our assumption.

As left Bousfield localization does not alter the cofibrations, the cofibrations of the S-local n-polynomial model structure agree with the cofibrations of the S'-local n-polynomial model structure. These model structures also have the same fibrant objects since a functor is S-locally n-polynomial if and only if it is S'-local n-polynomial under our assumption.

For the local n-homogeneous model structures, recall that these are certain left Bousfield localizations of the n-homogeneous model structure (see Proposition 8.1.1), hence have the same cofibrations. As before, these model structures have the same fibrant objects since our assumption together with Lemma 7.2.2 implies that the n-th derivative of a functor is S-local if and only if it is S'-local, and the fibrant objects are the n-polynomial functors with local derivatives, see Proposition 8.1.1. □

With respect to the partial ordering on Bousfield classes, we obtain the following more general result. The proof of which relies on the fact that if the class of S'-local spaces is contained in the class of S-local spaces, then the induced map $L_S \to L_{S'}$ is an S'-local equivalence.

Lemma 9.1.5. *Let S and S' be sets of maps of based spaces and F an orthogonal functor. If the class of S'-local spaces is contained in the class of S-local spaces then,*

1. *there is an S'-local equivalence $D_n^S F \to D_n^{S'} F$; and,*
2. *if F is reduced, then the S-local Weiss tower of F is S'-locally equivalent to the S'-local Weiss tower of F.*

Proof. For (1), note that the map on derivatives $\partial_n^S F \to \partial_n^{S'} F$ induced by the natural transformation $L_S \to L_{S'}$ is an S'-local equivalence, hence the n-homogeneous functors which correspond to these spectra are S'-locally equivalent, i.e., the map $D_n^S F \to D_n^{S'} F$ is an S'-local equivalence. For (2), since F is reduced [58, Corollary 8.3] implies that there is a commutative diagram

$$\begin{array}{ccccc} T_n^S F & \longrightarrow & T_{n-1}^S F & \longrightarrow & R_n^S F \\ \downarrow & & \downarrow & & \downarrow \\ T_n^{S'} F & \longrightarrow & T_{n-1}^{S'} F & \longrightarrow & R_n^{S'} F \end{array}$$

in which both rows are homotopy fibre sequences. The map $R_n^S F \to R_n^{S'} F$ is an S'-local equivalence by part (1), and the map $T_0^S F \to T_0^{S'} F$ is also an S'-local equivalence since F is reduced. An induction argument on the degree of polynomials yields the result. □

We obtain corollaries for both stable and unstable Bousfield classes.

Lemma 9.1.6. *Let E and E' be spectra and F an orthogonal functor. If $\langle E \rangle \leq \langle E' \rangle$, then*

(1) *there is an E'-local equivalence $D_n^E F \to D_n^{D'} F$; and,*
(2) *if F is reduced, then the E-local Weiss tower of F is E'-locally equivalent to the E'-local Weiss tower of F.*

Lemma 9.1.7. *Let W and W' be based spaces and F an orthogonal functor. If $\langle W\rangle \leq \langle W'\rangle$, then*

(1) *there is an W'-local equivalence $D_n^W F \to D_n^{W'} F$; and,*
(2) *if F is reduced, then the W-local Weiss tower of F is W'-locally equivalent to the W'-local Weiss tower of F.*

9.2. **The Telescope Conjecture.** The height n Telescope Conjecture relates the $T(n)$-localization and $K(n)$-localization of spectra. There are numerous equivalent formalisations of the conjecture see e.g., [10, Proposition 3.6] and we choose the following as it best suits any possible interaction with the calculus.

Conjecture 9.2.1 (The height n Telescope Conjecture)**.** *Let $n \geq 0$. The Bousfield class of $T(n)$ agrees with the Bousfield class of $K(n)$.*

Lemma 9.2.2. *Let $n \geq 0$. The validity of the height n Telescope Conjecture implies an equality of model structures*

$$\mathsf{Fun}(\mathcal{J}_0, L_{K(n)}\,\mathsf{Top}_*) = \mathsf{Fun}(\mathcal{J}_0, L_{T(n)}\,\mathsf{Top}_*),$$

$$\mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{K(n)}\,\mathsf{Top}_*) = \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{T(n)}\,\mathsf{Top}_*),$$

$$\mathsf{Homog}^{n}(\mathcal{J}_0, L_{K(n)}\,\mathsf{Top}_*) = \mathsf{Homog}^{n}(\mathcal{J}_0, L_{T(n)}\,\mathsf{Top}_*).$$

Proof. The Telescope Conjecture implies that the Bousfield class of $T(n)$ and the Bousfield class of $K(n)$, agree, hence the result follows by Theorem 9.1.4. □

The following is an immediate corollary to Theorem 9.1.1.

Theorem 9.2.3. *Let $n \geq 0$. The height n Telescope Conjecture is equivalent to the statement that for every orthogonal functor F the $K(n)$-local Weiss tower of F and the $T(n)$-local Weiss tower of F agree.*

This provides new insight into the the height n Telescope Conjecture. For example, to find a counterexample it now suffices to find an orthogonal functor such that one corresponding term in the $K(n)$-local and $T(n)$-local Weiss towers disagree. This can also be seen through the spectral sequences associated to the local Weiss towers. The $K(n)$-local and $T(n)$-local Weiss towers of an orthogonal functor F produce two spectral sequences,

$$\pi_{t-s} D_s^{K(n)} F(V) \cong \pi_{t-s}((S^{\mathbb{R}^s \otimes V} \wedge \partial_s^{K(n)} F)_{hO(n)}) \Rightarrow \pi_* \operatorname*{holim}_d T_d^{K(n)} F(V),$$

and,

$$\pi_{t-s} D_s^{T(n)} F(V) \cong \pi_{t-s}((S^{\mathbb{R}^s \otimes V} \wedge \partial_s^{T(n)} F)_{hO(n)}) \Rightarrow \pi_* \operatorname*{holim}_d T_d^{T(n)} F(V),$$

These are closely related to the telescope conjecture as follows.

Lemma 9.2.4. *Let F be an orthogonal functor. If the height n Telescope Conjecture holds, then for all $r \geq 1$, the E_r-page of the $T(n)$-local Weiss spectral sequence is isomorphic to the E_r-page of the $K(n)$-local Weiss spectral sequence. In particular, the homotopy limit of the $T(n)$-local Weiss tower is levelwise weakly equivalent to the homotopy limit of the $K(n)$-local Weiss tower.*

Proof. It suffices to prove the claim for $r = 1$. The validity of the height n Telescope Conjecture implies that there is a natural transformation $L_{K(n)} \to L_{T(n)}$. This natural transformation induces a map $D_d^{K(n)}F \to D_d^{T(n)}F$, which by Theorem 9.2.3 is an levelwise weak equivalence. It hence suffices to show that the natural map $D_d^{K(n)}F \to D_d^{T(n)}F$ induces a map on the E_1-pages of the spectral sequences, that is, we have to show that the induced diagram

$$\begin{array}{ccc} \pi_{t-s}D_s^{K(n)}F(V) & \xrightarrow{d_1^{K(n)}} & \pi_{t-s+1}D_{s+1}^{K(n)}F(V) \\ \downarrow & & \downarrow \\ \pi_{t-s}D_s^{T(n)}F(V) & \xrightarrow[d_1^{T(n)}]{} & \pi_{t-s+1}D_{s+1}^{T(n)}F(V) \end{array}$$

commutes for all s and t. This follows from the commutativity of the induced diagram of long exact sequences induced by the diagram of homotopy fibre sequences,

$$\begin{array}{ccccc} D_s^{K(n)}F(V) & \longrightarrow & T_s^{K(n)}F(V) & \longrightarrow & T_{s-1}^{K(n)}F(V) \\ \downarrow & & \downarrow & & \downarrow \\ D_s^{T(n)}F(V) & \longrightarrow & T_s^{T(n)}F(V) & \longrightarrow & T_{s-1}^{T(n)}F(V) \end{array}$$

and the construction of the d_1-differential in the homotopy spectral sequence associated to a tower of fibrations. □

10. Postnikov sections

The classical theory of Postnikov sections of based spaces is obtained by the nullification with respect to the spheres, that is, given a based space A, the k-th Postnikov section of A is the nullification of A at S^{k+1}, i.e., $P_kA = P_{S^{k+1}}A$. Given a diagram of (simplicial, left proper, combinatorial) model categories, Barwick [12, Section 5 Application 1] and Bergner [15] develop a general machinery for producing a model structure which captures the homotopy theory of the homotopy limit of the diagram of model categories. Gutiérrez and Roitzheim [30, Section 4] applied this to the study of Postnikov sections for model categories, which recovers the classical theory when $\mathcal{C}$ is the Kan-Quillen model structure on simplicial sets. We consider the relationship between Postnikov sections and orthogonal calculus via our local calculus.

10.1. **A combinatorial model for calculus.** The current theory of homotopy limits of model categories requires that the model categories in question be combinatorial, i.e., locally presentable and cofibrantly generated. Since the category of based compactly generated weak Hausdorff spaces is not locally presentable the Quillen model structure is not combinatorial and hence none of our model categories for orthogonal functors are either. We invite the reader to take for granted that all of our cellular model categories may be replaced by combinatorial model categories by starting with a combinatorial model for the Quillen model structure on based spaces, and hence skip directly to Subsection 10.2.

We spell out the details of these combinatorial replacements here. We replace compactly generated weak Hausdorff spaces with Δ-*generated spaces*; a particular full subcategory of the category of topological spaces, which were developed by Vogt [57] and unpublished work of Smith, which are surveyed by Dugger in [26]. The category of Δ-generated spaces may be equipped with a model structure analogous to the Quillen model structure on compactly generated weak Hausdorff spaces.

Lemma 10.1.1. *There is a model category structure on the category of Δ-generated spaces with weak equivalences the weak homotopy equivalences and fibrations the Serre fibrations. This model structure is combinatorial, proper and topological.*

Proof. The existence of the model structure follows from [26, Subsection 1.9]. The locally presentable (and hence combinatorial) property follows from [28, Corollary 3.7] . □

The category of based Δ-generated spaces is a convenient model category for doing homotopy theory in the following sense, see [26, Subsection 1.9].

Lemma 10.1.2. *The model category of based Δ-generated spaces is Quillen equivalent to the Quillen model structure on based compactly generated weak Hausdorff spaces.*

The combinatorial model for spaces transfers to categories of functors and we obtain a projective model structure on the category of orthogonal functors which is Quillen equivalent to our original projective model structure but is now combinatorial.

A left or right Bousfield localization of a combinatorial model category is again combinatorial, hence the n-polynomial, n-homogeneous and local versions of these model categories are all combinatorial when we begin with the combinatorial model for the projective model structure on orthogonal fucntors.

Hypothesis 10.1.3. *For the remainder of this section, we will assume that all our model structures are combinatorial, since they are all Quillen equivalent to combinatorial model categories using the combinatorial model for based spaces.*

10.2. **The model structure of k-types in orthogonal functors.** Denote by I the set of generating cofibrations of the projective model structure of orthogonal functors, and denote by W_k the set $I\square\{S^{k+1} \to D^{k+2}\}$, that is, the set of maps of the form

$$B \wedge S^{k+1} \coprod_{A\wedge S^{k+1}} A \wedge D^{k+2} \longrightarrow B \wedge D^{k+2},$$

where $A \to B$ is a map in I. The model category of k-types in $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$ is the left Bousfield localization of the projective model structure at $I\square\{S^{k+1} \to D^{k+2}\}$ used by Gutiérrez and Roitzheim [30] to model Postnikov sections.

Proposition 10.2.1. *Let $k \geq 0$. Under Hypothesis 10.1.3, the model structure of k-types in orthogonal functors is identical to the S^{k+1}-local model structure, that is, there is an equality of model structures,*

$$P_k\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) := L_{W_k}\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) = \mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\,\mathsf{Top}_*).$$

Proof. It suffices to show that both model structures have the same fibrant objects since the cofibrations in both model structures are identical. To see this, note that by examining the pushout product we can rewrite the set W_k as

$$W_k = \{\mathcal{J}_0(U, -) \wedge S^{n+k+1}_+ \longrightarrow \mathcal{J}_0(U, -) \wedge D^{n+k+2}_+ \mid n \geq 0, U \in \mathcal{J}_0\}.$$

It follows by an adjunction argument that an orthogonal functor Z is W_k-local if and only if $\pi_i Z(U)$ is trivial for all $i \geq k+1$ and all $U \in \mathcal{J}_0$. This last condition is equivalent to being levelwise S^{k+1}-local. □

10.3. **The model structure of k-types in spectra.**

Proposition 10.3.1. *Let $k \geq 0$. Under Hypothesis 10.1.3, there is an equality of model structures between the model category of k-types in spectra, and the stablisation of S^{k+1}-local spaces, that is,*

$$P_k\mathsf{Sp} := L_{W_k}\mathsf{Sp} = \mathsf{Sp}(L_{S^{k+1}}\,\mathsf{Top}_*).$$

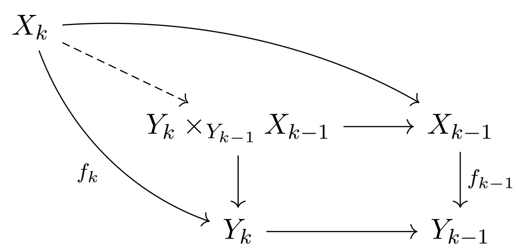
Proof. Both model structures can be described as particular left Bousfield localizations of the stable model structure on spectra, hence have the same cofibrations. The proof reduces to the fact that the model structures have the same fibrant objects. To see this, note that the fibrant objects of $P_k\mathsf{Sp}$ are the k-truncated Ω-spectra, and the fibrant objects of $\mathsf{Sp}(L_{S^{k+1}}\,\mathsf{Top}_*)$ are the levelwise k-truncated Ω-spectra. Since both fibrant objects are Ω-spectra a connectivity style argument yields that an Ω-spectrum is k-truncated if and only if it is levelwise k-truncated, and hence both model structures have the same fibrant objects. □

10.4. **Postnikov reconstruction of orthogonal functors.** The collection of S^{k+1}-local model structures on the category of orthogonal functors assembles into a tower of model categories[6]

$$\begin{aligned} \mathsf{P}_\bullet\colon \mathbb{N}^{\mathrm{op}} &\longrightarrow \mathsf{MCat}, \\ k &\longmapsto \mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\,\mathsf{Top}_*), \end{aligned}$$

where MCat denotes the category of model categories and left Quillen functors. The homotopy limit of this tower of model categories recovers the projective model structure on orthogonal functors. The existence of a model structure which captures the homotopy theory of the limit of these model categories follows from [30, Proposition 2.2]. In particular, the homotopy limit model structure is a model structure on the category of sections[7] of the diagram $\mathsf{P}_\bullet$ formed by right Bousfield localizing the injective model structure in which a map of sections is a weak equivalence or cofibration if it is a levelwise weak equivalence or cofibration respectively.

Lemma 10.4.1 ([30, Theorem 1.3 & Proposition 2.2])**.** *There is a combinatorial model structure on the category of sections of $\mathsf{P}_\bullet$ where a map $f_\bullet\colon X_\bullet \to Y_\bullet$ is a fibration if and only if f_0 is a fibration in $\mathsf{Fun}(\mathcal{J}_0, L_{S^1}\,\mathsf{Top}_*)$ and for every $k \geq 1$ the induced map*



indicated by a dotted arrow in the above diagram is a fibration in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\,\mathsf{Top}_*)$. A section $X_\bullet$ is cofibrant if and only if X_n is cofibrant in $\mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*)$ and for every $k \geq 0$, the map $X_{k+1} \to X_k$ is a weak equivalence in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\,\mathsf{Top}_*)$. A map of cofibrant sections is a weak equivalence if and only if the map is a weak equivalence in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\,\mathsf{Top}_*)$ for each $k \geq 0$. We will refer to this model structure as the homotopy limit model structure and denote it by $\operatorname{holim} \mathsf{P}_\bullet$.

[6]A tower of model categories is a special instance of a left Quillen presheaf, that is a diagram of the form $F\colon \mathcal{J}^{\mathrm{op}} \to \mathsf{MCat}$ for some small indexing category $\mathcal{J}$.

[7]A section $X_\bullet$ of the tower $\mathsf{P}_\bullet$ is a sequence

$$\cdots \longrightarrow X_k \longrightarrow X_{k+1} \longrightarrow \cdots \longrightarrow X_0,$$

of orthogonal functors, and a morphism of sections $f\colon X_\bullet \to Y_\bullet$ is given by maps of orthogonal functors $f_k\colon X_k \to Y_k$ for all $k \geq 0$ subject to a commutative ladder condition.

Proposition 10.4.2. *Under Hypothesis 10.1.3 the adjoint pair*

$$\mathrm{const} : \mathsf{Fun}(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \mathrm{holim}\, \mathsf{P}_\bullet : \lim$$

is a Quillen equivalence.

Proof. The adjoint pair exists, and is a Quillen adjunction by [30, Lemma 2.4].

To see that the adjoint pair is a Quillen equivalence let $X_\bullet$ be a cofibrant and fibrant section in the homotopy limit model structure. Showing that $\mathrm{const}\lim \mathrm{X}_\bullet \to \mathrm{X}_\bullet$ is a weak equivalence is equivalent to showing that the map $\lim X_\bullet \to X_k$ is a weak equivalence in $\mathsf{Fun}(\mathcal{J}_0, L_{S^{k+1}}\mathsf{Top}_*)$ for all $k \geq 0$. This is in turn, equivalent to the map $(\lim X_\bullet)(U) \to X_k(U)$ being a weak equivalence in $L_{S^{k+1}}\mathsf{Top}_*$ for all $k \geq 0$. Since limits in functor categories are computed levelwise, the fact that the unit is a weak equivalence follows from [30, Theorem 2.5]. A similar argument, shows that the counit is also a weak equivalence. □

10.5. **Postnikov reconstruction for spectra with an $O(n)$-action.** The aim is to show that similar reconstruction theorems may be obtained for the n-homogeneous functors. We first start by investigating analogous theorems for spectra and show that such reconstructions are compatible with the zigzag of Quillen equivalences between spectra with an $O(n)$-action and the n-homogeneous model structure.

Lemma 10.5.1. *The functor*

$$\begin{aligned} \mathsf{P}^{\mathsf{Sp}}_\bullet : \mathbb{N}^{\mathrm{op}} &\longrightarrow \mathsf{MCat}, \\ k &\longmapsto \mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*), \end{aligned}$$

defines a left Quillen presheaf.

Proof. This follows from Proposition 10.3.1 and [30, Subsection 2.1] since the stablisation of S^{k+1}-local spaces is precisely the model structure of k-types in spectra. □

Remark 10.5.2. Alternatively Lemma 10.5.1 may be proved by exhibiting that the adjoint pair

$$\mathbb{1} : \mathsf{Sp}(L_{S^{k+2}}\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}(L_{S^{k+1}}\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction. This fact follows from the facts that both model structures have the same cofibrations and a S^{k+1}-local space is S^{k+2}-local as $\langle \Sigma W \rangle \leq \langle W \rangle$ for all based spaces W, see e.g., [19, §9.9].

This left Quillen presheaf is 'convergent' in the following sense.

Proposition 10.5.3. *Under Hypothesis 10.1.3 the adjoint pair*

$$\mathrm{const} : \mathsf{Sp} \rightleftarrows \mathrm{holim}\ \mathsf{P}^{\mathsf{Sp}}_\bullet : \lim$$

is a Quillen equivalence.

Proof. The fact that the adjoint pair is a Quillen adjunction follows from [30, Lemma 2.4].

The left adjoint reflects weak equivalences between cofibrant objects. Indeed, if $X \to Y$ is a map between cofibrant spectra X and Y, such that

$$\mathrm{const}(X) \longrightarrow \mathrm{const}(Y),$$

is a weak equivalence in $\mathrm{holim}\ \mathsf{P}^{\mathsf{Sp}}_\bullet$, then

$$\mathrm{const}(X) \longrightarrow \mathrm{const}(Y),$$

is a weak equivalence in $\mathrm{Sect}(\mathbb{N}, \mathsf{P}_\bullet^{\mathsf{Sp}})$ by the colocal Whitehead's theorem and the fact that the left adjoint is left Quillen and thus preserves cofibrant objects. It follows that for each $k \in \mathbb{N}$, the induced map

$$\mathrm{const}(X)_k \longrightarrow \mathrm{const}(Y)_k,$$

is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)$, that is, $X \to Y$ is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)$ for all k. Unpacking the definition of a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)$ and using the fact that the right adjoint is a right Quillen functor and hence preserves weak equivalences between fibrant objects, we see that the induced map

$$\lim\ P_k X \longrightarrow \lim\ P_k Y,$$

is a weak equivalence in Sp, and hence, so is the map $X \to Y$.

It is left to show that the derived counit is an isomorphism. Let $Y_\bullet$ be bifibrant in holim $\mathsf{P}_\bullet^{\mathsf{Sp}}$. The condition that the counit applied to $Y_\bullet$ is a weak equivalence is equivalent to asking for the map

$$\lim_{\geq k}\ P_k Y_\bullet \longrightarrow Y_k,$$

to be a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)$ for all $k \in \mathbb{N}$. The structure maps of $Y_\bullet$ induce a map of towers

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Y_j & \longrightarrow \cdots \longrightarrow & Y_{k+3} & \longrightarrow & Y_{k+2} & \longrightarrow & Y_{k+1} \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow{=} \\
\cdots \longrightarrow & Y_{k+1} & \longrightarrow \cdots \longrightarrow & Y_{k+1} & \longrightarrow & Y_{k+1} & \longrightarrow & Y_{k+1}
\end{array}$$

in which each vertical arrow is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)$. This map of towers induces a map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \lim^1_{\geq k}\ \pi_{i+1}(Y_\bullet) & \longrightarrow & \pi_i(\lim_{\geq k}\ Y_\bullet) & \longrightarrow & \lim_{\geq k}\ \pi_i(Y_\bullet) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \lim^1_{\geq k}\ \pi_{i+1}(Y_{k+1}) & \longrightarrow & \pi_i(\lim_{\geq k}\ Y_{k+1}) & \longrightarrow & \lim_{\geq k}\ \pi_i(Y_{k+1}) & \longrightarrow & 0
\end{array}$$

of short exact sequences. For $0 \leq i < n$ the left and right hand side maps are isomorphisms hence the map

$$\lim_{\geq k}\ Y_\bullet \longrightarrow Y_{k+1},$$

is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)$ for all k, and it follows that the required map

$$\lim_{\geq k}\ Y_\bullet \longrightarrow Y_{k+1} \longrightarrow Y_k,$$

is a weak equivalence in $\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)$ for all k. □

A similar justification to Lemma 10.5.1 provides a left Quillen presheaf

$$\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]} \colon \mathbb{N}^{\mathrm{op}} \longrightarrow \mathsf{MCat},$$
$$k \longmapsto \mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)[O(n)],$$

where $\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)[O(n)]$ is the category of $O(n)$-objects in the category of k-types in spectra. This is equivalent to the category of k-types in spectra with an $O(n)$-action.

Lemma 10.5.4. *Let $k, n \geq 0$. The model structure of the Borel stablisation of S^{k+1}-local spaces with an $O(n)$-action is identical to the model structure of k-types in the category of spectra with an $O(n)$-action, that is, there is an equality of model structures*

$$\mathsf{Sp}(L_{S^{k+1}} \mathsf{Top}_*)[O(n)] = P_k(\mathsf{Sp}[O(n)]).$$

Proof. Both model structures are identical to model structures transferred through the same adjunction from identical model structures. □

As a corollary to Proposition 10.5.3, we obtain that the induced left Quillen presheaf on spectra with an $O(n)$-action is also suitably convergent.

Corollary 10.5.5. *Under Hypothesis 10.1.3 the adjoint pair*

$$\text{const} : \mathsf{Sp}[\mathrm{O(n)}] \rightleftarrows \text{holim } \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]} : \lim$$

is a Quillen equivalence.

10.6. **Postnikov reconstruction for the intermediate categories.** Our attention now turns to the intermediate categories. We construct an analogous left Quillen presheaf and show that it is also convergent in a fashion which interacts well with the convergent left Quillen presheaf for spectra with an $O(n)$-action.

Lemma 10.6.1. *The functor*

$$\begin{aligned} \mathsf{P}_\bullet^{\mathcal{J}_n} : \mathbb{N}^{\mathrm{op}} &\longrightarrow \mathsf{MCat}, \\ k &\longmapsto L_{S^{k+1}}\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*), \end{aligned}$$

defines a left Quillen presheaf.

Proof. As before, it suffices to show that there is an equality of model structures between the S^{k+1}-local n-stable model structure and the model structure of k-types in $\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*)$. The proof of which is completely analogous to the case for spectra, see Lemma 10.5.1. □

Remark 10.6.2. Since the S^{k+1}-local n-stable model structure agrees with the model structure of k-types, we will denote both model structure by $P_k\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*)$.

The homotopy limit of the left Quillen presheaf of Lemma 10.6.1 agrees with the homotopy limit of the left Quillen presheaf of Lemma 10.5.1, in the sense that the homotopy limit model categories are Quillen equivalent. In detail, the adjunction

$$(\alpha_n)_! : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^* \ ,$$

of [6, §8] induces an adjunction

$$(\alpha_n)_!^{\mathbb{N}} : \mathsf{Fun}(\mathbb{N}, \mathsf{Fun}_{\mathsf{O(n)}}(\mathcal{J}_\mathsf{n}, \mathsf{O(n)}\,\mathsf{Top}_*)) \rightleftarrows \mathsf{Fun}(\mathbb{N}, \mathsf{Sp}[\mathsf{O(n)}]) : (\alpha_\mathsf{n}^*)^{\mathbb{N}} \ ,$$

where $(\alpha_n^*)^{\mathbb{N}} = (\alpha_n)^* \circ (-)$. This adjunction in turn induces an adjunction

$$(\alpha_n)_!^{\mathbb{N}} : \text{holim } \mathsf{P}_\bullet^{\mathcal{J}_n} \rightleftarrows \text{holim } \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]} : (\alpha_n^*)^{\mathbb{N}} \ .$$

Proposition 10.6.3. *Under Hypothesis 10.1.3 the adjoint pair*

$$(\alpha_n)_!^{\mathbb{N}} : \text{holim } \mathsf{P}_\bullet^{\mathcal{J}_n} \rightleftarrows \text{holim } \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]} : (\alpha_n^*)^{\mathbb{N}} \ ,$$

is a Quillen equivalence.

Proof. Fibrations of the homotopy limit model structure of $\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]}$ are precisely the fibrations of the injective model structure on the category of sections of $\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]}$ since the homotopy limit model structure is a right Bousfield localization of the injective model structure. A similar characterisation holds for the left Quillen presheaf $\mathsf{P}_\bullet^{\mathcal{J}_n}$, hence to show that the right adjoint preserves fibrations it suffices to show that the left adjoint preserves acyclic cofibrations of the injective model structure on the categories of sections. To see this, note that the adjunction

$$(\alpha_n)_! : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathsf{Sp}[O(n)] : (\alpha_n)^* ,$$

is a Quillen adjunction and hence, so to is the induced adjunction on the injective model structures on the categories of sections.

To show that the left adjoint preserves cofibrations it suffices to show that cofibrations between cofibrant objects are preserved. As the homotopy limit model structures are right Bousfield localizations [33, Proposition 3.3.16(2)] implies that cofibrations between cofibrant objects are cofibrations of the injective model structures on the categories of sections which by the analogous reasoning as above are preserved by the left adjoint. This yields that the adjunction in question is a Quillen adjunction.

To show that the adjunction is a Quillen equivalence notice that the right adjoint reflects weak equivalences between cofibrant objects by the colocal Whitehead's Theorem [33, Theorem 3.2.13(2)], and the fact that the induced adjunction on the injective model structures on the categories of sections is a Quillen equivalence since for $B_\bullet \in \mathsf{Sect}(\mathbb{N}, \mathsf{P}_\bullet^{\mathcal{J}_n})$ and $X_\bullet \in \mathsf{Sect}(\mathbb{N}, \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]})$, a map $B_\bullet \to (\alpha_n^*)^{\mathbb{N}} X_\bullet$ is a weak equivalence if and only if for each $k \in \mathbb{N}$, the map $B_k \to (\alpha_n^*)^{\mathbb{N}} X_k$ is a weak equivalence of spectra, which in turn happens if and only if the adjoint map $(\alpha_n)_! B_k \to X_k$ is an n-stable equivalence, which is precisely the condition that the adjoint map $(\alpha_n)_!^{\mathbb{N}} B_\bullet \to X_\bullet$ is a weak equivalence.

It is left to show that the derived counit is an isomorphism. Let $Y_\bullet$ be bifibrant in the homotopy limit model structure of the left Quillen presheaf $\mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]}$. Then the derived counit

$$(\alpha_n)_!^{\mathbb{N}}\, \widehat{c}\, ((\alpha_n^*)^{\mathbb{N}} Y_\bullet) \longrightarrow Y_\bullet,$$

is a map between cofibrant objects, hence a weak equivalence in the homotopy limit model structure if and only if a weak equivalence in the injective model structure on the category of sections i.e., if and only if for each $k \in \mathbb{N}$, the induced map

$$(\alpha_n)_!(\alpha_n)^* Y_k \longrightarrow Y_k,$$

is a weak equivalence. This last is always a weak equivalence by [6, Proposition 8.3]. □

As a corollary, we see that the left Quillen presheaf $\mathsf{P}_\bullet^{\mathcal{J}_n}$ is convergent.

Corollary 10.6.4. *Under hypothesis 10.1.3 the adjoint pair*

$$\mathrm{const} : \mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) \rightleftarrows \mathrm{holim}\ \mathsf{P}_\bullet^{\mathcal{J}_n} : \lim ,$$

is a Quillen equivalence.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
\mathsf{Fun}_{O(n)}(\mathcal{J}_n, O(n)\,\mathsf{Top}_*) & \underset{(\alpha_n)^*}{\overset{(\alpha_n)_!}{\rightleftarrows}} & \mathsf{Sp}[O(n)] \\
\mathrm{const}\downarrow\ \uparrow\lim & & \mathrm{const}\downarrow\ \uparrow\lim \\
\mathrm{holim}\ \mathsf{P}_\bullet^{\mathcal{J}_n} & \underset{(\alpha_n^*)^{\mathbb{N}}}{\overset{(\alpha_n)_!^{\mathbb{N}}}{\rightleftarrows}} & \mathrm{holim}\ \mathsf{P}_\bullet^{\mathsf{Sp}[O(n)]}
\end{array}$$

of Quillen adjunctions in which three out of the four adjoint pairs are Quillen equivalences by [6, Proposition 8.3], Corollary 10.5.5 and Proposition 10.6.3. It follows since Quillen equivalences satisfy the 2-out-of-3 property, that the remaining Quillen adjunction is a Quillen equivalence. □

10.7. **Postnikov reconstruction for homogeneous functors.** Using the same approach as we have just employed from moving from spectra with an $O(n)$-action to the intermediate categories we obtain similar results for the homogeneous model structures. We choose to model S^{k+1}-local n-homogeneous functors by the S^{k+1}-periodic n-homogeneous model structures of Proposition 8.3.1.

Lemma 10.7.1. *The functor*

$$\begin{aligned}\mathsf{P}_\bullet^{\mathsf{Homog}^n} : \mathbb{N}^{\mathrm{op}} &\longrightarrow \mathsf{MCat}, \\ k &\longrightarrow \mathsf{Homog}^n(\mathcal{J}_0, P_{S^{k+1}}\,\mathsf{Top}_*),\end{aligned}$$

defines a left Quillen presheaf.

Proof. It suffices to show that the adjoint pair

$$\mathbb{1} : \mathsf{Homog}^n(\mathcal{J}_0, P_{S^{k+2}}\,\mathsf{Top}_*) \rightleftarrows \mathsf{Homog}^n(\mathcal{J}_0, P_{S^{k+1}}\,\mathsf{Top}_*) : \mathbb{1}\ ,$$

is a Quillen adjunction. The adjoint pair

$$\mathbb{1} : \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S^{k+2}}\,\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S^{k+1}}\,\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction since the composite of Quillen adjunctions is a Quillen adjunction so the adjunction

$$\mathbb{1} : \mathsf{Fun}(\mathcal{J}_0, L_{S^{k+2}}\,\mathsf{Top}_*) \rightleftarrows \mathsf{Poly}^{\leq n}(\mathcal{J}_0, L_{S^{k+1}}\,\mathsf{Top}_*) : \mathbb{1}$$

is a Quillen adjunction, and by [33, Proposition 3.3.18(1) & Theorem 3.1.6(1)], this composite Quillen adjunction extends to the S^{k+2}-local n-polynomial model structure since S^{k+1}-local n-polynomial functors are S^{k+2}-locally n-polynomial.

An application of [33, Theorem 3.3.20(2)(a)] yields the desired result about the n-homogeneous model structures. □

Similar proofs to Proposition 10.6.3 and Corollary 10.6.4 yeild the following results relating the n-homogeneous model structure to the homotopy limit of the tower of S^{k+1}-local n-homogeneous model structures.

Proposition 10.7.2. *Under Hypothesis 10.1.3 the adjunction*

$$(\mathsf{res}_0^n/O(n))^{\mathbb{N}} : \mathrm{holim}\ \mathsf{P}_\bullet^{\mathsf{Homog}^n} \rightleftarrows \mathrm{holim}\ \mathsf{P}_\bullet^{\mathcal{J}_n} : (\mathsf{ind}_0^n\,\varepsilon^*)^{\mathbb{N}}\ ,$$

is a Quillen equivalence.

Corollary 10.7.3. *Under Hypothesis 10.1.3 the adjunction*

$$\mathrm{const} : \mathsf{Homog}^n(\mathcal{J}_0, \mathsf{Top}_*) \rightleftarrows \mathrm{holim}\ \mathsf{P}_\bullet^{\mathsf{Homog}^n} : \lim\ ,$$

is a Quillen equivalence.

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