Compartment model on the circle

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August 10, 2021

Abstract

We study the emergence of a giant component in the configuration model subject to additional constraints on the possible connections in the network. In particular, we partition a circle into compartments, and only allow edges between vertices of neighbouring compartments. We prove that under similar conditions on the degree sequence as for the standard configuration model, a giant component emerges provided the number of vertices per compartment grows quickly enough. We demonstrate the difference from the standard configuration model by providing an example with fixed compartment size where no giant component emerges, while the conditions on the degree sequence lead to a giant component in the standard configuration model.

Keywords: Configuration model, giant component, multitype branching process, concentration inequalities, geometric networks

MSC2020 Classes: 05C80, 60J80

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1 Introduction

Since the classical random graph model was first introduced by Erdős and Rényi, many alternative models were studied by adding constraints to this random graph. For instance, the configuration model allowing one to control the degree sequence [MR98; JL09; BR15] was showed to feature a phase transition in the global connectivity which can be deduced from the degree distribution. That is, the model may or may not contain a giant connected component that has a positive fraction of all vertices. Random geometric graphs, in which the vertices have coordinates defined by a given point process and are connected based on their proximity, also feature a similar phase transition [Pen03]. This phase transition seems to be a property of the embedding metric space, although such a space also induces a certain degree distribution which one cannot control independently. In general, even though both models feature phase transition-like behaviour, there are only a few results allowing to study random geometric graphs that have a given degree distribution. One approach was suggested in the small world graphs [WS98; BR01], where a regular circular lattice or a continuous circle is randomly rewired by adding shortcuts to obtain an object that retains some of the original geometric properties while having a controlled degree distribution.

Random graph models in which both the degree distribution and geometrical features can be controlled are relevant when modelling real networks having some spatial content. These networks naturally arise, for example, in epidemiology [BN03; DJ07] and wireless communications [GGD16]. In materials science, such graphs pertain to a well-known problem of polymer gelation [Kry16].

Our aim is to provide a simple geometric generalisation of the configuration model by additionally forbidding some pairs of vertices to be connected, hence inducing a notion of a metric. We study the following model: We consider $k \in \mathbb{N}$ compartments on a circle and distribute the vertices equally over these compartments. Every compartment has two neighbouring compartments. We then only allow an edge to connect pairs of vertices belonging to the same or neighbouring compartments. This means that connections can only be made locally on the circle, which may only make it more difficult for a giant component to emerge. This model is furthermore motivated by studying networks with geometric constraints. Since we are only allowed to connect vertices from neighbouring compartments, such construction may be viewed as a random geometric graph on $\mathbf{Z}_k$ that has a given degree distribution. When $\mathbf{Z}_k$ is embedded in the circle, the larger k is, the closer connected vertices are together. However useful $\mathbf{Z}_k$ model is for applications, we also hope that the techniques used in this study will in future inspire investigation of the classical random geometric graphs, for example in $\mathbb{R}^n$.

Our method relies on the idea that the exploration of components in the random graph can be linked to a branching process. However, compared to the standard setting, this connection is only valid for a small number of exploration steps. As a consequence, this only allows us to prove that locally-large components emerge. To show a giant component emerges, we prove that all these local components connect together with high probability.
To do this, it is important to track how each explored component spreads through the different compartments. Therefore, we perform the exploration with a multitype branching process. This allows us to prove that these locally-large components spread through a significant amount of compartments, occupying a positive fraction of the vertices in each of these. The proof is then completed by showing that with high probability there are sufficiently many of such local components that connect together to form one giant component with high probability.

This article is structured as follows. In Section 2 we introduce the model we are studying and state our main theorem. As with the standard configuration model, the proof of our main theorem relies on the connection to branching processes, which in our case will be multitype branching processes. We introduce these in Section 3, where we also derive some relevant properties. With all preparations done, Section 4 is dedicated to proving our main theorem. Since the proof is rather involved, we break down the proof in a number of propositions. Finally, in Section 5 we provide an example which shows that the geometric constraints give rise to different behaviour compared to the configuration model without geometric constraints.

2 Compartment model on the circle

For every $n \in \mathbb{N}$ we consider $k(n)$ compartments $C_1, \ldots, C_{k(n)}$, each containing $m(n)$ vertices. Define $V_n = \bigcup_{i=1}^{k(n)} C_i$ as the set of vertices. Our aim is to study graphs on V_n satisfying two types of constraints:

1. *Constraint on allowed connections:* Vertices $x, y \in V_n$ can only be connected if $x \in C_i, y \in C_j$ with $|i - j| \leq 1$. Here, we identify $k(n)$ with 0, allowing also edges between $C_{k(n)}$ and C_1, which results in the circle structure.

2. *Degree constraint*: The vertices have prescribed degrees, given by a sequence $d_n = \{d(1), \ldots, d(k(n)m(n))\}$ of non-negative integers. We will refer to d_n as the *degree sequence*.

We construct a random graph $G_n = (V_n, E_n)$ satisfying the above two constraints as follows. The degree of a vertex $x \in V_n$ is represented by $d_n(x)$

half-edges. At each iteration we choose uniformly a pair of half-edges which are allowed to be connected together. We repeat this until no matches can be made anymore. We refer to this model as the *compartment model on the circle*.
First of all, note that G_n is in general a multi-graph since we do not exclude self-loops or multi-edges. Furthermore, it might happen that we do not satisfy the full degree sequence, even if we assume the sum of the degrees is even. However, when the construction terminates, at most one half-edge per compartment will be unmatched. This will be no problem, since we will be assuming that $m(n)$, the amount of vertices per compartment, tends to infinity.

The compartment model on the circle is closely related to a random geometric graph on the circle. Indeed, the parameter $k(n)$, i.e. the number of compartments, is related to the distance between vertices that can be connected. However, in the compartment model, the neighbourhoods of the vertices are homogenized, in the sense that each vertex of a compartment has the same neighbours it can be connected to.

2.1 Main theorem

Our main result is concerned with providing sufficient conditions under which the random graphs G_n asymptotically contain a giant component with high probability. Moreover, we will also determine its size. In this section we collect all of our assumptions.

First of all, we assume that V_n asymptotically contains n vertices. Noting that $|V_n| = k(n)m(n)$ we therefore assume that

$$\lim_{n\to\infty} \frac{k(n)m(n)}{n} = 1.$$

Second, we will assume that $\lim_{n\to\infty} k(n) = \infty$. On the one hand, this reflects the idea that vertices are only allowed to be connected when they are very close together. On the other hand, this assures that our model is clearly distinguished from the standard configuration model. Indeed, if we only have finitely many compartments, then it should be possible to deduce from the standard configuration model a giant component emerges locally. It them remains to show that (finitely many) of those connect together with high probability.
In Section 5 we will see that if the number of vertices per compartment becomes fixed, then a giant component not necessarily emerges, even if the degree sequence satisfied the conditions of our main theorem. Therefore, we will assume that $m(n)$, the number of vertices per compartment, tends to infinity. In particular, we will assume that

$$\lim_{n\to\infty} \frac{n}{m(n)^k} = 0.$$

Apart from assumptions on the graph structure, we also need assumptions on the degree sequences d_n. These are the same for the standard configuration model, see e.g. [BR15; Hof17; Dur07]. In what follows, we denote by $n_j(d_n, C_i^n)$ the amount of vertices of degree j in compartment C_i^n. Furthermore, we define $\mu_n(d_n, C_i^n)$ by

$$\mu_n(d_n, C_i^n) := \frac{1}{2} \sum_{x \in C_i^n} d(x) = \frac{1}{2} \sum_{j=1}^{\infty} j n_j(d_n, C_i^n).$$

Using this notation, we make the following assumption on the convergence of the degree sequence d_n.

Assumption 2.1 (Convergent degree sequence)**.** *The degree sequence d_n converges to a distribution D in the following sense:*

1. *For every $\varepsilon > 0$ there exists an N such that for all $n \geq N$ and all $i = 1, \ldots, k(n)$ we have*

$$\left| \frac{n_j(d_n, C_i^n)}{m(n)} - \mathbb{P}(D = j) \right| < \varepsilon$$

   for all j.

2. *For every $\varepsilon > 0$ there exists an N such that for all $n \geq N$ and all $i = 1, \ldots, k(n)$ we have*

$$\left| \frac{\mu_n(d_n, C_i^n)}{m(n)} - \frac{\mathbb{E}(D)}{2} \right| < \varepsilon.$$

We are now ready to state the main theorem.

Theorem 2.2. *Consider the compartment model on the circle with $k(n)$ compartments with $m(n)$ vertices each, so that $\lim_{n\to\infty} \frac{m(n)k(n)}{n} = 1$. Assume that $\lim_{n\to\infty} k(n) = \infty$ and that there exists $k \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{n}{m(n)^k} = 0.$$

Furthermore, for every n let d_n be a degree sequence on $m(n)k(n)$ vertices satisfying Assumption 2.1 with distribution D. Assume D is bounded and $E(D(D-2)) > 0$. If we denote by $L_1(G_n)$ the largest component in G_n, then there exists a $\rho \in [0, 1)$ such that

$$\lim_{n\to\infty} \frac{L_1(G_n)}{n} = 1 - \rho$$

in probability. Furthermore, with high probability, there is no other cluster of size more than $\beta \log m(n)$ for some $\beta > 0$.

Remark 2.3. The constant ρ in Theorem 2.2 can be determined from the distribution D. More precisely, we define the distribution D^* by $\mathbb{P}(D^* = i) = \frac{i\mathbb{P}(D=i)}{\mathbb{E}(D)}$, the so called *size-biased degree distribution*. We can then interpret ρ as the extinction probability of the Galton-Watson tree where the root has offspring distribution D, and all other individuals have offspring distribution $Z_D = D^* - 1$. The condition $E(D(D-2)) > 0$ implies that $\mathbb{E}(Z_D) > 1$. In particular, this implies that the Galton-Watson tree survives with positive probability, implying that $\rho < 1$.

3 Branching processes

Studying components in random graphs is intimately related to studying branching processes. This occurs when we explore components of a graph from a given vertex. The next generation of the branching process then resembles the neighbours in the graph of the current generation. During this exploration, we are also interested in which compartments the vertices lie. Therefore, we will make use of a multitype branching process. In this section we will shortly introduce these processes, and collect some necessary results. For a more thorough treatment, see e.g. [AN72; AL06].

3.1 Galton-Watson tree

The prototypical example of a branching process is the Galton-Watson tree, which models the evolution of a population in which every individual of a generation gets a random number of children. Furthermore, it is assumed that the number of children of different individuals are independent, and follow the same distribution.

More precisely, let D be a probability distribution on the nonnegative integers and denote by Z_n the number of individuals in generation n. For every n, let $X_1^n, \dots, X_{Z_n}^n$ be independent random variables with distribution D. Then

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_i^n.$$

An important question regarding such processes is whether they become extinct or grow on indefinitely. We define the *extinction probability* by

$$\rho(D) = \lim_{n\to\infty} \mathbb{P}(Z_n = 0).$$

If $\mathbb{E}(D) < 1$ then the process becomes almost surely extinct, i.e. $\rho(D) = 1$. If $\mathbb{E}(D) > 1$ then the process has a positive probability to grow on indefinitely. Moreover, this probability can be computed from the generating function of the distribution D. In particular, the extinction probability is the largest solution in $[0, 1]$ of the equation

$$x = \sum_{i=0}^{\infty} \mathbb{P}(D = i)x^i.$$

3.2 Multitype branching processes

For our purposes, it is not sufficient to understand how large components grow. We also need information on how components spread through different compartments. In order to study this, we consider a branching process with types, where type $I \subset \mathbb{Z}$ of a vertex represents its compartment. We denote generation n of the branching process by a vector Z_n of length $|I|$, where $Z_n(i)$ is the number of individuals of type i in generation n. We denote by $|Z_n|$ the size of generation n, i.e.,

$$|Z_n| = \sum_{i \in I} Z_n(i).$$

For every type $i \in I$ we have an offspring distribution D_i, which is now a distribution on vectors representing the types of the offspring. For every n and every $i \in I$, let $X_1^{n,i}, \ldots, X_{Z_n(i)}^{n,i}$ be independent random variables with distribution D_i. We then have that

$$Z_{n+1} = \sum_{i \in I} \sum_{j=1}^{Z_n(i)} X_j^{n,i}.$$

When I is finite, one looks at the matrix M of expected offspring to study the extinction of such processes. If we for instance assume that M^k has only positive entries for some k large enough, then the largest eigenvalue ρ_{max} of M determines whether extinction occurs almost surely or whether there is some positive probability that the tree grows indefinitely, see e.g. [Har63; Dur07]. When I is countably infinite, the conditions for extinction are much more subtle, and we refer to [Moy64; HLN13] among others.

3.2.1 Assigning types independently

We are specifically interested in the case where each offspring of a vertex is independently assigned a type according to some distribution. In this case, the offspring distribution is a multinomial distribution. Our claim is that the distribution of individuals over the types in generation n of such a multitype branching process can be found by running a number of n-step independent random walks equal to the size of the n-th generation.

More precisely, let $I \subset \mathbb{Z}$ be the state space. Let N be a random variable taking values in the nonnegative integers, denoting the number of children an individual will have. Furthermore, for $i \in I$, let $p^i = (p_j^i)_{j \in I}$ be a probability distribution on I. Let D_i denote a multinomial distribution with N trials and probability vector p^i, which we will take as offspring distribution of a type i individual. Finally, we denote by $(Z_n)_n$ the associated multitype branching process.

Let us now define the inhomogeneous random walk with which we want to compare this branching process, which we will denote by $(S_n)_n$. Since the walk is inhomogeneous, we will construct it recursively. Let S_0 be distributed according to a uniformly random individual of Z_0. Now, if S_n is given, we define S_{n+1} as the random variable with distribution p^{S_n}. The following proposition relates this random walk to the branching process with multinomial offspring distribution.

Proposition 3.1. *Let $(Z_n)_n$ be a multitype branching process with multinomial offspring distribution. Let $(S_n)_n$ be the associated random walk defined above,* *and let $S^1, S^2, \ldots, S^{|Z_n|}$ be independent copies of S_n. For $i \in I$, let e_i be the vector such that $e_i(j) = \delta_{ij}$. Then Z_n is in distribution equal to*

$$\mathcal{S}_n = \sum_{i=1}^{|Z_n|} e_{S^i}.$$

Proof. We will prove this using induction on n. First of all, note that $\mathcal{S}_0$ is equal in distribution to Z_0, since S_0 is distributed according to a uniformly random individual of Z_0.
Now suppose that $\mathcal{S}_n$ has the same distribution as Z_n. Observe that by definition, the random variables $Z_n(i)$ for $i \in I$ are independent. Therefore, if $X_1, \ldots, X_{|Z_{n+1}|}$ are independent samples taken uniformly from the population Z_{n+1}, then

$$Z_{n+1} = \sum_{j=1}^{|Z_{n+1}|} e_{X_j}$$

in distribution. Now note that the distribution of X_j is equal to p^{Y_n} where Y_n is a uniform sample from the population Z_n. Since Z_n is equal in distribution to $\mathcal{S}_n$, this means that X_j has distribution p^{S_n}. As a consequence, we find that X_j is equal in distribution to S_{n+1}. Putting everything together, we conclude that $Z_{n+1} = \mathcal{S}_{n+1}$ in distribution. □

The above identification of the multitype branching process as sum of random walks is useful in deriving properties of the distribution of its n-th generation. In particular, we consider the specific case where $I = \mathbb{Z}$ and $p^i = \frac{1}{3}(e_{i-1}+e_i+e_{i+1})$. One can show that in generation n, all types that are at most at distance $\sqrt{n}$ from the starting type $Z_0 = e_0$ are present with a significant fraction. Before we can turn this in a rigorous statement, we first need the following result on the associated random walk.

Lemma 3.2. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{3}$. Define $S_n = \sum_{i=1}^n X_i$. Then there exists a $\delta > 0$ such that for n large enough we have*

$$\mathbb{P}(S_{n^2} = n) \geq \delta \mathbb{P}(S_{n^2} = 0).$$

Moreover, for n large enough and $-\sqrt{n} \leq k \leq \sqrt{n}$ we have

$$\mathbb{P}(S_n = k) \geq \delta \mathbb{P}(S_n = 0).$$

Proof. By the Kolmogorov-Rogozin inequality ([Kol58; Rog61]) there exists a constant $C > 0$ such that

$$\mathbb{P}(S_{n^2} = 0) \leq \frac{C}{n}.$$

We are done once we show that

$$\mathbb{P}(S_{n^2} = n) \geq \frac{c}{n}$$

for some $c > 0$. To this end, note that $\mathbb{E}(X_1) = 0$ and $\mathrm{Var}(X_1) = \frac{2}{3}$. Therefore, by the central limit theorem we find that

$$\frac{1}{n} S_{n^2} \Rightarrow N\left(0, \frac{2}{3}\right).$$

This implies that asymptotically we have

$$\mathbb{P}(n \leq S_{n^2} \leq (1+\varepsilon)n) \geq \frac{\sqrt{3}\varepsilon}{\sqrt{4\pi}} \exp\left(-\frac{9}{8}(1+\varepsilon)^2\right).$$

Because $\mathbb{P}(S_{n^2} = k)$ is decreasing in k (when $k \geq 0$), we find that

$$\begin{aligned}\mathbb{P}(S_{n^2} = n) &\geq \frac{1}{\varepsilon n} \mathbb{P}(n \leq S_{n^2} \leq (1+\varepsilon)n) \\ &\geq \frac{\sqrt{3}}{n\sqrt{4\pi}} \exp\left(-\frac{9}{8}(1+\varepsilon)^2\right).\end{aligned}$$

This shows we can take

$$c = \frac{\sqrt{3}}{\sqrt{4\pi}} \exp\left(-\frac{9}{8}(1+\varepsilon)^2\right),$$

which proves the first statement.
The second statement now follows from the observation that $\mathbb{P}(S_n = k)$ is decreasing when $|k|$ is increasing. □

Using concentration inequalities, we now show that if we start with a type 0 individual, then in generation n all types between $-\sqrt{n}$ and $\sqrt{n}$ are a positive fraction of the total size of generation n. We have the following proposition.

Proposition 3.3. *Let Z_n be a multitype branching process with types $I = \mathbb{Z}$. Assume the offspring distribution is multinomial with parameters N and $p = (p^i)$, where $p^i = \frac{1}{3}(e_{i-1}+e_i+e_{i+1})$. Suppose M_n is such that $\mathbb{P}(|Z_n| \geq M_n) > 0$. Then there exists a $\delta > 0$ such that for n large enough we have*

$$\mathbb{P}\left(Z_n(k) \geq \delta \frac{M_n}{\sqrt{n}} \text{ for } -\sqrt{n} < k < \sqrt{n} \,\middle|\, |Z_n| \geq M_n\right) \geq 1 - 2(2n+1)\exp\left(-\frac{\delta^2 M_n}{8n}\right).$$

Proof. Let $X_1, X_2, \ldots, X_{|Z_n|}$ be independent copies of the random walk S_n in Lemma 3.2. By Proposition 3.1 we have that Z_n is equal in distribution to

$$\mathcal{S}_n = \sum_{i=1}^{|Z_n|} e_{X_i}.$$

Now define

$$\overline{\mathcal{S}}_n = \sum_{i=1}^{M_n} e_{X_i}.$$

Note that $\mathcal{S}_n$ is larger in distribution than $\overline{\mathcal{S}}_n$. As a consequence, we find that

$$\begin{aligned}\mathbb{P}&\left(Z_n(k) \geq \delta\frac{M_n}{\sqrt{n}} \text{ for } -\sqrt{n} < k < \sqrt{n} \middle| |Z_n| \geq M_n\right)\\ &= \mathbb{P}\left(\mathcal{S}_n(k) \geq \delta\frac{M_n}{\sqrt{n}} \text{ for } -\sqrt{n} < k < \sqrt{n} \middle| |Z_n| \geq M_n\right)\\ &\geq \mathbb{P}\left(\overline{\mathcal{S}}_n(k) \geq \delta\frac{M_n}{\sqrt{n}} \text{ for } -\sqrt{n} < k < \sqrt{n}\right).\end{aligned}$$

Now define the function $F : \mathbb{R}^{M_n} \to \mathbb{R}^{2n+1}$ given by

$$F(x_1, \ldots, x_{M_n}) = \sum_{i=1}^{M_n} e_{x_i}$$

If we change a variable x_i, then one entry in the image of F is increased by 1 while another is decreased by 1. Therefore, we can apply Theorem A.2 with $c_i = 1$ for all i. This gives us that

$$\mathbb{P}\left(||\overline{\mathcal{S}}_n - \mathbb{E}(\overline{\mathcal{S}}_n)||_\infty \geq \varepsilon\right) \leq 2(2n+1)\exp\left(-\frac{\varepsilon^2}{2M_n}\right).$$

for every $\varepsilon > 0$.
Now we can compute $\mathbb{E}(\overline{\mathcal{S}}_n(k)) = M_n\mathbb{P}(S_n = k)$. Therefore, by Lemma 3.2 there exists a $\delta > 0$ such that

$$\mathbb{E}(\overline{\mathcal{S}}_n(k)) \geq \delta M_n \mathbb{P}(S_n = 0).$$

for n large enough and $-\sqrt{n} \leq k \leq \sqrt{n}$. By then central limit theorem and the fact that $\mathbb{P}(S_n = k)$ is largest when $k = 0$, we find that

$$\mathbb{P}(S_n = 0) \geq c\frac{1}{\sqrt{n}}$$

for some $c > 0$. Therefore, by shrinking δ sufficiently, we find that

$$\mathbb{E}(\overline{\mathcal{S}}_n(k)) \geq \delta\frac{M_n}{\sqrt{n}}.$$

Now

$$\begin{aligned}\mathbb{P}&\left(\overline{\mathcal{S}}_n(k) \geq \frac{\delta}{2}\frac{M_n}{\sqrt{n}} \text{ for } -\sqrt{n} < k < \sqrt{n}\right)\\ &\geq \mathbb{P}\left(\left|\overline{\mathcal{S}}_n(k) - \mathbb{E}(\overline{\mathcal{S}}_n(k))\right| \leq \frac{\delta}{2}\frac{M_n}{\sqrt{n}} \text{ for } -\sqrt{n} < k < \sqrt{n}\right)\\ &\geq \mathbb{P}\left(||\overline{\mathcal{S}}_n - \mathbb{E}(\overline{\mathcal{S}}_n)||_\infty \leq \frac{\delta}{2}\frac{M_n}{\sqrt{n}}\right)\\ &= 1 - \mathbb{P}\left(||\overline{\mathcal{S}}_n - \mathbb{E}(\overline{\mathcal{S}}_n)||_\infty > \frac{\delta}{2}\frac{M_n}{\sqrt{n}}\right)\\ &\geq 1 - 2(2n+1)\exp\left(-\frac{\delta^2 M_n}{8n}\right)\end{aligned}$$

which concludes the proof. □

4 Proof of Theorem 2.2

In this section we prove Theorem 2.2. The proof follows a similar idea as the proof in [Dur07] for the standard configuration model. To study the components of the random graph G_n from the compartment model, we use an exploration process. Contrary to the standard case, we can only use this to find large components locally, because the compartment structure restricts the neighbours of vertices we are exploring.
To study this exploration process, we connect it to a branching process. In particular, we need to consider a multitype branching process to keep track of how the exploration process searches through different compartments. This is important for the next step, where we aim to connect local large components to form one large component that spread through all compartments of the circle. To show that this happens with high probability, we need to carefully analyse the probability that large components arise locally.
Finally, the proof is concluded in a similar way as for the standard configuration model. We show that the large component found above is actually a giant component by determining its size.

4.1 Exploration process

In order to study the growth of components in the graph $G_n = (V_n, E_n)$ of the compartment model, we will explore them iteratively. To do this, we start at a vertex $v \in V_n$ and reveal its neighbours. After that, we consider each of these newly revealed vertices and reveal their neighbours, and so on. In particular, in the exploration, we keep track of the compartment to which each vertex belongs. Let us now define this process rigorously.

Recall that the graph $G_n = (V_n, E_n)$ consists of $k(n)$ compartments $C_1^n, \dots, C_{k(n)}^n$, each containing $m(n)$ vertices. Let now $v \in C_j^n$ be some vertex in the graph G_n. The *exploration process* started at v is a sequence of tuples (R_l, A_l, U_l) of vectors of length $k(n)$ constructed recursively. Here $R_l(i)$ denotes the set of explored vertices in compartment C_i^n, $A_l(i)$ the set of active vertices in compartment C_i^n, i.e., those that we have already revealed, but not yet explored, and $U_l(i)$ are the other, yet unseen vertices in compartment C_i^n. We initialize the process by setting $R_0(i) = \emptyset$ for all i, $A_0(j) = \{v\}$ and $A_0(i) = \emptyset$ for all $i \neq j$ and $U_0(j) = C_j^n - \{v\}$ and $U_0(i) = C_i^n$ for $i \neq j$. Now, at every iteration, we define A_{l+1} to be all neighbours of vertices in A_l which are in U_l. We then set $R_{l+1} = R_l \cup A_l$ and $U_{l+1} = U_l \setminus A_{l+1}$, where the set operations have to be interpreted elementwise.

Using this exploration process, we want to analyse how large the component we explore grows. In order to do this, we need to find a lower bound on the size of the active set A_l. To this end, we introduce the following notation:

$$|A_l| = (|A_l(1)|, \dots, |A_l(k(n))|)$$

and

$$||A_l|| = \sum_{i=1}^{k(n)} |A_l(i)|.$$

In the upcoming proposition we prove that we can use a multitype branching process as pointwise stochastic lower bound for $|A_l|$. For real-valued random variables X and Y we say that X is a lower bound for Y if for all $a \in \mathbb{R}$ we have $\mathbb{P}(X \geq a) \leq \mathbb{P}(Y \geq a)$. Furthermore, we call a sequence $r = (r_k)_{k\geq 0}$ a *distribution* if $\sum_{k=0}^{\infty} r_k = 1$ and $r_k \geq 0$ for all k. For every such sequence, we let $W_r : (0,1) \to \mathbb{N}$ be a non-decreasing function satisfying

$$|\{\omega | W_r(\omega) = k\}| = r_k.$$

It follows that if we remove mass η from the distribution r and normalize, then this will be stochastically larger than $W_r^\eta = (W_r(\omega) | \omega < 1 - \eta)$. Indeed, the latter removes mass η, starting from the largest values of W_r. Using all this, we can state and prove the following proposition.

Proposition 4.1. *Let the assumptions of Theorem 2.2 be satisfied. Let $v \in C_1^n$ and denote by (R_l, A_l, U_l) the exploration process started at v. Let $\eta \in (0,1)$ and assume that at most $\eta m(n)$ vertices of each compartment have already been exposed. Then for every $\delta > 0$ there exist a multitype branching process $(S_l)_l$, such that until $\delta m(n)$ vertices in at least one compartment have been exposed, we have that $|A_l|$ is stochastically bounded from below by S_l.*
Moreover, the offspring distribution of $(S_l)_l$ can be chosen to be multinomial with parameters N and $p = (p^i)$ where $p^i = \frac{1}{3}(e_{i-1} + e_i + e_{i+1})$. Furthermore, for δ small enough, N can be chosen such that $\mathbb{E}(N) > 1$.

Proof. We argue the existence by constructing a suitable multitype branching process. To this end, we first argue what happens when exploring a single vertex $w \in C_j^n$. Assume that at most $\eta m(n)$ vertices have been exposed of every compartment. Let D^* be the size-biased degree distribution (see Remark 2.3) and set $Z_D = D^* - 1$. We define the distribution $q = (q_k)$ by $q_k = \mathbb{P}(Z_D = k)$. Since at most a fraction η of the vertices has been exposed, together with the fact that d_n converges to D as in Assumption 2.1, it follows that for n large enough the amount of new neighbours found while exploring w is bounded from below by $W_q^{2\eta}$.
By symmetry, these $W_q^{2\eta}$ new vertices are equally likely to be in any of the neighbouring compartments of w, i.e., in the compartments C_{j-1}^n, C_j^n and C_{j+1}^n. Therefore, we consider the random variables

$$\bar{W}_{q,j}^{2\eta} \sim \text{Mult}\left(W_q^{2\eta}, \frac{1}{3}(e_{j-1} + e_j + e_{j+1})\right),$$

which is a multinomial distribution. Here $\{e_1, \ldots, e_{k(n)}\}$ denote the standard basis of $\mathbb{R}^{k(n)}$.

Finally, we need to take into account that the new vertices may already have been exposed before. Because of the degree constraints, we remove these vertices from the active set. Note that $\bar{W}^{2\eta}_{q,j}(i)$ takes values in the set $\{0, 1, \dots, L\}$, where $L = W_q(1 - 2\eta)$. Therefore, if at most $\delta m(n)$ vertices have been exposed from any compartment C^n_i, then there are at most $L\delta m(n)$ possible half-edges connected to active vertices in C^n_i. On the other hand, there are at least $\mu_n(d_n, C^n_i) - L\delta m(n)$ half-edges left which are not connect to an active vertex. Therefore, the probability of choosing an active neighbour in that compartment is at most

$$\frac{L\delta m(n)}{\mu_n(d_n, C^n_i) - L\delta m(n)}.$$

From Assumption 2.1 it follows that $\frac{\mu_n(d_n,C^n_i)}{m(n)}$ converges to $\frac{1}{2}\mathbb{E}(D)$ uniformly over the compartments. Therefore, given $\varepsilon > 0$, for n large enough the above is smaller than

$$\gamma_\delta := \frac{L\delta}{\frac{1}{2}\mathbb{E}(D) - \varepsilon - L\delta}$$

provided $\delta > 0$ is small enough.

Collecting everything, we see that the number of new vertices found while exploring $w \in C^n_j$ is bounded from below by

$$X_j \sim \bar{W}^{2\eta}_{q,j} - 2\sum_{t=j-1}^{j+1} \text{Bin}(|\bar{W}^{2\eta}_{q,j}(t)|, \gamma_\delta)e_t.$$

From this we can conclude that X_j follows a multinomial distribution with parameters N and $p^j = \frac{1}{3}(e_{j-1} + e_j + e_{j+1})$. In particular, for N we have

$$N \sim W^{2\eta}_q - 2\text{Bin}(W^{2\eta}_q, \gamma_\delta).$$

From this it follows that

$$\mathbb{E}(N) = \mathbb{E}(W^{2\eta}_q)(1 - 2\gamma_\delta).$$

Now note that $\lim_{\eta\to 0} \mathbb{E}(W^{2\eta}_q) = \mathbb{E}(W_q) = \mathbb{E}(Z_D)$. Since by assumption $\mathbb{E}(Z_D) > 1$ (see Remark 2.3), we find that for η small enough we have $\mathbb{E}(W^{2\eta}_q) > 1$. Furthermore, note that γ_δ tends to 0 as δ tends to 0. Combining the above, we find that we can choose δ and η small enough so that $\mathbb{E}(N) > 1$. □

Our next aim is to prove that if $||A_l||$ grows to size $\beta \log m(n)$, then it actually grows to size $m(n)^{\frac{2}{3}}$ with high probability. For this, we will use the lower bound we found in Proposition 4.1. Before we can show this, we first need a lemma.

Lemma 4.2. *Let X be a random variable such that $X \geq -1$, $\mathbb{P}(X = -1) > 0$ and $\mathbb{E}(X) > 0$. Define S_n by*

$$S_n = S_0 + \sum_{i=1}^{n} Y_i,$$

where $Y_i = \sum_{j=1}^{S_{i-1}} X_j$ *with* $X_1, \dots X_{S_{i-1}}$ *i.i.d. with distribution* X*. Suppose* $S_0 = x > 0$ *and define*

$$T(x) := \inf\{n||S_n| = 0\}.$$

Then there exists a $\lambda > 0$ *such that*

$$\mathbb{P}(T(x) < \infty) \leq e^{-\lambda x}.$$

Proof. It suffices to prove the statement for X bounded from above, since this only increases $T(x)$. Let $M(t) = \mathbb{E}(e^{tX})$ be the moment generating function of X. Since X is bounded from below, we have that $M(t)$ is defined for all $t \leq 0$. Note that $M(0) = 1$, $M'(0) = \mathbb{E}(X) > 0$ and

$$\lim_{t\to-\infty} M(t) \geq \lim_{t\to-\infty} \mathbb{P}(X = -1)e^{-t} = \infty.$$

From this, together with the continuity of $M(t)$, it follows that there exists a $\lambda > 0$ such that $M(-\lambda) = 1$. This implies that $Z_n = e^{-\lambda S_n}$ is a martingale. From the optional stopping theorem, we find that

$$\mathbb{E}\left(Z_{T(x)}\right) = \lim_{n\to\infty} \mathbb{E}\left(Z_{n\wedge T(x)}\right) = \mathbb{E}(Z_0) = e^{-\lambda x}.$$

On the other hand,

$$\mathbb{E}\left(Z_{T(x)}\right) \geq \mathbb{P}(T < \infty),$$

and hence we find that $\mathbb{P}(T(x) < \infty) \leq e^{-\lambda x}$. □

Using this lemma, we can show that if the active set grows to size $\beta \log m(n)$, then the probability that the exploration process does not explore a large cluster is small. More precisely, we have the following proposition.

Proposition 4.3. *Let the assumptions in Proposition 4.1 be satisfied. Suppose* $\sum_{i=0}^{l} ||A_i|| \geq \beta \log m(n)$ *for some* l*. Define*

$$T := \inf\{n|||A_n|| = 0\}.$$

Then for β *large enough we have*

$$\mathbb{P}(T < \infty) \leq 2m(n)^{-k}.$$

Proof. Let S_n be the lower bound for $|A_n|$ from Proposition 4.1. Then $|S_n|$ is a lower bound for $||A_n||$, and in particular,

$$\Sigma_l = \sum_{i=0}^{l} |S_i|$$

is a lower bound for $\sum_{i=0}^{l} ||A_i||$.
Now assume that $\Sigma_l \geq \beta \log m(n)$ and define

$$\tilde{T} = \inf\{n||S_n| = 0\}.$$

Then $\mathbb{P}(T < \infty) \leq \mathbb{P}(\tilde{T} < \infty)$.
Note that we can write

$$|S_{l+1}| = \sum_{i=1}^{|S_l|} \tilde{X}_i,$$

where $\tilde{X}_1, \ldots, \tilde{X}_{|S_l|}$ are independent and distributed like N as in Proposition 4.1. By telescoping, this implies that

$$|S_{l+1}| = \sum_{j=1}^{\Sigma_l} X_j,$$

where $X_1, \ldots, X_{\Sigma_l}$ are independent and equal in distribution to $N-1$.
Let λ be as in Lemma 4.2 for the random variable X_1. From Chernoff's bound it follows that

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^{\sigma} X_j \leq \frac{k}{\lambda}\log m(n)\right) &\leq e^{\theta\frac{k}{\lambda}\log m(n)}\mathbb{E}(e^{-\theta X_1})^{\sigma} \\ &= \exp\left(\theta\frac{k}{\lambda}\log m(n) + \sigma\log M(-\theta)\right)\end{aligned}$$

for all $\theta > 0$, where $M(t) = \mathbb{E}(e^{tX_1})$. Since $M(0) = 1, M(-\lambda) = 1$ and $M'(0) = \mathbb{E}(X_1) > 0$, there exists a $\lambda' \in (0, \lambda)$ such that $M(-\lambda') < 1$. In particular, this implies that $\log M(-\lambda') < 0$. From this it follows that for $\sigma \geq \beta \log m(n)$ we have

$$\mathbb{P}\left(\sum_{j=1}^{\sigma} X_j \leq \frac{k}{\lambda}\log m(n)\right) \leq \exp\left(\left(\lambda'\frac{k}{\lambda} + \beta\log M(-\lambda')\right)\log m(n)\right).$$

Using that $\log M(-\lambda') < 0$, we can take β enough such that

$$\lambda'\frac{k}{\lambda} + \beta\log M(-\lambda') \leq -k$$

so that

$$\mathbb{P}\left(\sum_{j=1}^{\sigma} X_j \leq \frac{k}{\lambda}\log m(n)\right) \leq m(n)^{-k}.$$

From this we conclude that if $\Sigma_l \geq \beta \log m(n)$ for large enough β, we have that

$$\mathbb{P}\left(|S_{l+1}| < \frac{k}{\lambda}\log m(n)\right) \leq m(n)^{-k}.$$

From this, we obtain that

$$\mathbb{P}(\tilde{T} < \infty)$$

$$\leq \mathbb{P}\left(|S_{l+1}| \leq \frac{k}{\lambda}\log m(n)\right) + \mathbb{P}\left(\tilde{T} < \infty \middle| |S_{l+1}| \geq \frac{k}{\lambda}\log m(n)\right)$$
$$\leq 2m(n)^{-k}.$$

Here, we applied Lemma 4.2 to bound the second term. Since $\mathbb{P}(T < \infty) \leq \mathbb{P}(\tilde{T} < \infty)$, this proves the claim. ☐

We conclude this part by proving that if we repeatedly start the exploration process at a vertex in C_1^n, then with high probability we find a component of size at least $m(n)^{\frac{2}{3}}$ before $\delta m(n)$ vertices have been exposed. This follows from the fact that with high probability, each failed attempt uses at most $\beta \log m(n)$ vertices.

Proposition 4.4. *The probability that the exploration process started (repeatedly) at a point in C_1^n finds a component of size at least $m(n)^{\frac{2}{3}}$ before a total of $\delta m(n)$ vertices have been exposed is at least*

$$\left(1 - 2m(n)^{-k}\right)^{\frac{\delta m(n)}{\beta \log m(n)}} \left(1 - (1-p)^{\frac{\delta m(n)}{\beta \log m(n)}}\right).$$

Here, $p > 0$ is the probability that the branching process $(S_n)_n$ in Proposition 4.1 survives indefinitely.

Proof. Let G denote the number of tries it takes before S_n grows to size $m(n)^{\frac{2}{3}}$. Then G is geometrically distributed with parameter $\tilde{p} \geq p > 0$. Let $R_1, R_2, \ldots$ be a sequence of i.i.d. random variables representing the number of vertices exposed in a failed attempt. We need to prove that

$$\mathbb{P}\left(\sum_{i=1}^{G} R_i \leq \delta m(n)\right) \geq \left(1 - 2m(n)^{-k}\right)^{\frac{\delta m(n)}{\beta \log m(n)}} \left(1 - (1-p)^{\frac{\delta m(n)}{\beta \log m(n)}}\right).$$

Now
$$\mathbb{P}\left(\sum_{i=1}^{G} R_i \leq \delta m(n)\right) = \sum_{g=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{g} R_i \leq \delta m(n)\right) \mathbb{P}(G = g).$$

For $g \leq \frac{\delta m(n)}{\beta \log m(n)}$ we have

$$\mathbb{P}\left(\sum_{i=1}^{g} R_i \leq \delta m(n)\right) \geq (1 - 2m(n)^{-k})^g \geq \left(1 - 2m(n)^{-k}\right)^{\frac{\delta m(n)}{\beta \log m(n)}},$$

where the first inequality follows from Proposition 4.3. Using this, we find that

$$\mathbb{P}\left(\sum_{i=1}^{G} R_i \leq \delta m(n)\right) \geq \left(1 - 2m(n)^{-k}\right)^{\frac{\delta m(n)}{\beta \log m(n)}} \mathbb{P}\left(G \leq \frac{\delta m(n)}{\beta \log m(n)}\right)$$
$$= \left(1 - 2m(n)^{-k}\right)^{\frac{\delta m(n)}{\beta \log m(n)}} \left(1 - (1-\tilde{p})^{\frac{\delta m(n)}{\beta \log m(n)}}\right).$$

The desired bound now follow because $\tilde{p} \geq p$. ☐

4.2 From local to global

In Proposition 4.4 we have seen that if we start exploring from a vertex in C_1^n, with high probability we find a component of size at least $m(n)^{\frac{2}{3}}$ at some point. In this section, we will first show that such a component spreads equally through all compartments near C_1^n. Since we could have started equally well from any other compartment, the idea is to show that with high probability, many of such local large components exist which together cover all compartments. It then remains to prove that these components are all connected with high probability, forming a large component which spreads through every compartment.

4.2.1 Spreading through compartments

To see how the explored component spreads through neighbouring compartments, we use the multitype branching process found in Proposition 4.1. We will first show that, provided the branching process grows to a certain size, it actually does so exponentially fast with high probability. The following proposition is closely related to the large deviation results in [Ath94]. However, we need more precise information on the growth of the involved constants.

Proposition 4.5. *Let $T = (T_l)_l$ be a Galton-Watson tree with bounded offspring distribution N satisfying $\mathbb{E}(N) > 1$ and $T_0 = 1$. Suppose there exists an l such that $T_l \geq M_n$. Define*

$$p(n) = \inf\{p | T_p \geq M_n\} < \infty.$$

Then for any $k \in \mathbb{N}$ there exist constants $\alpha, \beta > 0$ such that

$$\mathbb{P}(p(n) \geq \alpha \log M_n) \geq 1 - \beta M_n^{-k}.$$

Furthermore, there exists an $L > 0$ such that

$$\mathbb{P}\left(p(n) \leq L M_n^{\tau}\right) \geq 1 - \beta e^{-k M_n^{\tau}}$$

for every $\tau > 0$.

Proof. Because there exists and l such that $T_l \geq M_n$, we know that every generation contains at least one vertex which survives until the tree grows to size M_n. We call such a vertex immortal. Every immortal vertex has at least one child that is also immortal. Moreover, since $E(N) > 1$, the probability of having only one immortal child is less than 1. Denote by $\overline{N}$ the offspring distribution N conditioned to be at least 1. Then $\mathbb{E}(\overline{N}) > 1$ and $\overline{N}$ is bounded since N is bounded.
Denote by (Z_l) the Galton-Watson tree with offspring distribution $\overline{N}$. Since $\overline{N} \geq 1$, Z_l is non-decreasing in l and therefore we have

$$\mathbb{P}(p(n) > \alpha \log M_n) \geq \mathbb{P}\left(Z_{\alpha \log M_n} < M_n\right).$$

We will first show that there exist $L, \beta > 0$ such that for every n we have

$$\mathbb{P}(Z_{Ln} \leq n) \leq \beta e^{-nk}. \tag{4.1}$$

To this end, note that Z_l is larger than $\sum_{i=1}^{l} B_i$, where $B_1, \ldots, B_l$ are i.i.d. Bernoulli random variables with parameter $p = \mathbb{P}(\overline{N} > 1) > 0$. This implies that

$$\mathbb{P}(Z_l \leq n) \leq \mathbb{P}\left(\sum_{i=1}^{l} B_i \leq n\right)$$

Choosing $L > 0$ such that $Lp = L\mathbb{E}(B_1) > 1$, we have

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{Ln} B_i \leq n\right) &\leq \mathbb{P}\left(\left|\sum_{i=1}^{Ln} B_i - Lnp\right| > (Lp-1)n\right) \\ &\leq 2\exp\left(-2L^{-1}n(Lp-1)^2\right).\end{aligned}$$

Here, the last line follows from Hoeffding's inequality. The claim now follows by taking L sufficiently large.

From (4.1) we find that

$$\mathbb{P}\left(Z_{LM_n^\tau} \geq M_n^\tau\right) \geq 1 - \beta e^{-kM_n^\tau}.$$

From this it follows that

$$\mathbb{P}\left(p(n) \leq LM_n^\tau\right) \geq \mathbb{P}\left(Z_{LM_n^\tau} \geq M_n^\tau\right) \geq 1 - \beta e^{-kM_n^\tau},$$

which proves the second estimate.

We now prove the first estimate. Take $c > \mathbb{E}(\tilde{N})$. From Hoeffding's inequality we find

$$\begin{aligned}&\mathbb{P}(Z_{Ln+1} \geq cZ_{Ln}) \\ &\quad = \mathbb{P}\left(\frac{1}{Z_{Ln}}\sum_{i=1}^{Z_{Ln}} \tilde{N}_i \geq c\right) \\ &\quad \leq \mathbb{P}\left(\left|\frac{1}{Z_{Ln}}\sum_{i=1}^{Z_{Ln}} \tilde{N}_i - \mathbb{E}(\tilde{N})\right| \geq c - \mathbb{E}(\tilde{N})\right) \\ &\quad = \sum_z \mathbb{P}\left(\left|\frac{1}{z}\sum_{i=1}^{z} \tilde{N}_i - \mathbb{E}(\tilde{N})\right| \geq c - \mathbb{E}(\tilde{N})\right)\mathbb{P}(Z_{Ln} = z) \\ &\quad \leq 2\mathbb{P}(Z_{Ln} \leq n) + \sum_{z>n} 2\exp\left(-\frac{2z(c-\mathbb{E}(\tilde{N}))^2}{||N||_\infty}\right)\mathbb{P}(Z_{Ln} = z) \\ &\quad \leq 2\beta e^{-nk} + 2\exp\left(-\frac{2n(c-\mathbb{E}(\tilde{N}))^2}{||N||_\infty}\right).\end{aligned}$$

Here we used inequality (4.1) in the last line. Furthermore, $||N||_\infty$ is the bound of N. By taking c sufficiently large, we conclude that there exist a $\beta > 0$ such that

$$\mathbb{P}(Z_{Ln+1} \geq cZ_{Ln}) \leq \beta e^{-kn}$$

For $l \in \mathbb{N}$ we now find that

$$\begin{aligned}
\mathbb{P}(Z_{Ln+1} < c^{n-l}) &\geq \mathbb{P}\left(\bigcap_{j=l}^{n}\{Z_{Lj+1} < cZ_{Lj}\}\right) \\
&\geq \mathbb{P}\left(\bigcap_{j=l}^{\infty}\{Z_{Lj+1} < cZ_{Lj}\}\right) \\
&= 1 - \mathbb{P}\left(\bigcup_{j=l}^{\infty}\{Z_{Lj+1} \geq cZ_{Lj}\}\right) \\
&\geq 1 - \sum_{j=l}^{\infty}\mathbb{P}(Z_{Lj+1} \geq cZ_{Lj}) \\
&\geq 1 - \beta\sum_{j=l}^{\infty}e^{-kj} \\
&= 1 - \frac{\beta}{1-e^{-k}}e^{-kl}.
\end{aligned}$$

In particular, there is some constant $\tilde{\beta} > 0$ such that

$$\mathbb{P}(Z_{Ln+1} < c^{n-l}) \geq 1 - \tilde{\beta}e^{-kl}.$$

Hence, if we take $\alpha = L(1 + \frac{1}{\log c})$ and $l = \log M_n$, the above implies that

$$\mathbb{P}\left(Z_{\alpha \log M_n} < M_n\right) \geq 1 - \tilde{\beta}M_n^{-k},$$

which concludes the proof. □

We are now ready to prove how local large components we find while exploring spread through the compartments. We have the following.

Proposition 4.6. *Let the assumptions of Proposition 4.1 be satisfied and denote by C a component explored by the exploration process started at a vertex $v \in C_1^n$. Assume the active set of the exploration process reaches size $m(n)^{\frac{2}{3}}$. Then for every $k \in \mathbb{N}$ and $\delta > 0$ sufficiently small, there exist $\varepsilon, \alpha, \beta, B > 0$ such that*

$$\begin{aligned}
&\mathbb{P}\left(|C \cap C_l| \geq \varepsilon m(n)^{\frac{2}{3}-\tau} \text{ for } |l-1| < \sqrt{\alpha \log m(n)}\right) \\
&\quad\geq \left(1 - Bm(n)^{2\tau}\exp\left(-\frac{1}{8}\varepsilon^2 m(n)^{\frac{2}{3}-2\tau}\right)\right)\left(1 - \beta m(n)^{-k}\right)
\end{aligned}$$

for every $\tau \in (0, \frac{2}{3})$ and $m(n)$ large enough. Here we consider $|l-1|$ to be modulo $k(n)$ to respect the circle structure.

Proof. Let $(S_n)_n$ be the multitype branching process from Proposition 4.1. Assume there exists an l such that $|S_l| \geq m(n)^{\frac{2}{3}}$. Set

$$p(n) := \inf\left\{p \,\middle|\, |S_p| \geq m(n)^{\frac{2}{3}}\right\} < \infty.$$

Then $|C \cap C_l| \geq S_{p(n)}(l)$. Therefore, it suffices to find a lower bound for

$$\mathbb{P}\left(S_{p(n)}(l) \geq \varepsilon m(n)^{\frac{2}{3}-\tau} \text{ for } |l-1| < \sqrt{\alpha \log m(n)}\right).$$

Note that $p(n)$ is a random variable. We have

$$\begin{aligned}\mathbb{P}&\left(S_{p(n)}(l) \geq \varepsilon m(n)^{\frac{2}{3}-\tau} \text{ for } |l-1| < \sqrt{\alpha \log m(n)}\right)\\ &= \sum_p \mathbb{P}\left(S_p(l) \geq \varepsilon m(n)^{\frac{2}{3}-\tau} \text{ for } |l-1| < \sqrt{\alpha \log m(n)}\right)\mathbb{P}(p(n)=p).\end{aligned}$$

By Proposition 4.5 there exist constants $\alpha, \beta, \eta > 0$ such that

$$\mathbb{P}(p(n) \geq \alpha \log m(n)) \geq 1 - \beta m(n)^{-k}$$

and

$$\mathbb{P}(p(n) \leq \eta^2 m(n)^{2\tau}) \geq 1 - \beta e^{-km(n)^{2\tau}}.$$

In particular, this implies that

$$\begin{aligned}\mathbb{P}(\alpha \log m(n) \leq p(n) \leq \eta^2 m(n)^{2\tau}) &\geq 1 - \beta m(n)^{-k} - \beta e^{-km(n)^{2\tau}}\\ &\geq 1 - 2\beta m(n)^{-k},\end{aligned}$$

where the last line holds as along as $m(n)$ is large enough.
Now, for $\alpha \log m(n) \leq p \leq \eta^2 m(n)^{2\tau}$ we have

$$\begin{aligned}\mathbb{P}&\left(S_p(l) \geq \varepsilon m(n)^{\frac{2}{3}-\tau} \text{ for } |l-1| < \sqrt{\alpha \log m(n)}\right)\\ &\geq \mathbb{P}\left(S_p(l) \geq \varepsilon\eta \frac{m(n)^{\frac{2}{3}}}{\sqrt{p}} \text{ for } |l-1| < \sqrt{p}\right)\\ &\geq 1 - 2(2p+1)\exp\left(-\frac{(\varepsilon\eta)^2 m(n)^{\frac{2}{3}}}{8p}\right)\\ &\geq 1 - 2(2\eta^2 m(n)^{2\tau}+1)\exp\left(-\frac{1}{8}\varepsilon^2 m(n)^{\frac{2}{3}-2\tau}\right).\end{aligned}$$

Here, the third line follows from Proposition 3.3. We conclude that there exists a constant $B > 0$ such that

$$\begin{aligned}\mathbb{P}&\left(S_p(l) \geq \varepsilon m(n)^{\frac{2}{3}-\tau} \text{ for } |l-1| < \sqrt{\alpha \log m(n)}\right)\\ &\geq 1 - Bm(n)^{2\tau}\exp\left(-\frac{1}{8}\varepsilon^2 m(n)^{\frac{2}{3}-2\tau}\right)\end{aligned}$$

for all $\alpha \log m(n) \leq p \leq \eta^2 m(n)^{2\tau}$.
If we now collect everything, we find that

$$\mathbb{P}\left(S_{p(n)}(l) \geq \varepsilon m(n)^{\frac{2}{3}-\tau} \text{ for } |l-1| < \sqrt{\alpha \log m(n)}\right)$$

$$\geq \left(1 - Bm(n)^{2\tau} \exp\left(-\frac{1}{8}\varepsilon^2 m(n)^{\frac{2}{3}-2\tau}\right)\right) \mathbb{P}(\alpha \log m(n) \leq p(n) \leq \eta^2 m(n)^{2\tau})$$
$$\geq \left(1 - Bm(n)^{2\tau} \exp\left(-\frac{1}{8}\varepsilon^2 m(n)^{\frac{2}{3}-2\tau}\right)\right) \left(1 - 2\beta m(n)^{-k}\right).$$

This concludes the proof. □

4.2.2 Connecting local components

Proposition 4.4 and 4.6 together give a lower bound on the probability that there exists a component of at least size $m(n)^{\frac{2}{3}}$ which spreads through approximately $\sqrt{\log m(n)}$ compartments, each compartment containing at least $\varepsilon m(n)^{\frac{2}{3}-\tau}$ vertices of the component.

Set

$$N(n) = \left\lfloor \frac{k(n)}{\sqrt{\alpha \log m(n)}} \right\rfloor \tag{4.2}$$

and define the indices $l_i = 1 + i\lfloor\sqrt{\alpha \log m(n)}\rfloor$ for $i = 0, \ldots, N(n)$. Denote by I_{l_i} the indicator random variable of the event that there exists a component as in Section 4.2.1, where the exploration is started in compartment C_{l_i}. It follows from Proposition 4.4 and 4.6 that $\mathbb{P}(I_{l_i} = 1) = a_n b_n$, where

$$a_n = \left(1 - 2m(n)^{-k}\right)^{\frac{\delta m(n)}{\beta \log m(n)}} \left(1 - (1-p)^{\frac{\delta m(n)}{\beta \log m(n)}}\right)$$

and

$$b_n = \left(1 - Bm(n)^{2\tau} \exp\left(-\frac{1}{8}\varepsilon^2 m(n)^{\frac{2}{3}-2\tau}\right)\right) \left(1 - \beta m(n)^{-k}\right).$$

However, the random variables I_{l_i} are not (necessarily) independent. We have

$$\begin{aligned}
\mathbb{P}(I_{l_i} = 1 \text{ for all } i = 1, \ldots, N(n)) &= 1 - \mathbb{P}(I_{l_i} = 0 \text{ for some } i = 1, \ldots, N(n)) \\
&\geq 1 - \sum_{i=1}^{N(n)} \mathbb{P}(I_{l_i} = 0) \\
&= 1 - N(n)\mathbb{P}(I_1 = 0) \\
&= 1 - N(n)(1 - a_n b_n).
\end{aligned}$$

Since $|l_{i+1} - l_i| \leq \sqrt{\alpha \log m(n)}$, the components $C_{l_{i+1}}$ and C_{l_i} have a common compartment in which they both have at least $\varepsilon m(n)^{\frac{2}{3}-\tau}$ vertices. If these sets of vertices intersect, then surely there is a connection between the components. If not, then the probability that there is no edge between the two sets is smaller than (see e.g. [BR15, Lemma 20])

$$\exp\left(-C\varepsilon^2 m(n)^{\frac{1}{3}-2\tau}\right)$$

for some constant $C > 0$ depending on the bound of the degree distribution. Hence, the probability that two neighbouring components are connected is more than

$$c_n := 1 - \exp\left(-C\varepsilon^2 m(n)^{\frac{1}{3}-2\tau}\right).$$

Collecting everything, we find that there exists a component C with $|C \cap C_k^n| \geq \varepsilon m(n)^{\frac{2}{3}-\tau}$ for all $k = 1, \ldots, k(n)$ with probability at least

$$(1 - N(n)(1 - a_n b_n))(1 - N(n)c_n),$$

where $N(n)$ is as in (4.2). From this discussion, we obtain the following.

Proposition 4.7. *For every $\tau \in (0, \frac{1}{6})$ we have that with high probability there exists a component C in G_n such that for all $k = 1, \ldots, k(n)$ we have*

$$|C \cap C_k^n| \geq \varepsilon m(n)^{\frac{2}{3}-\tau}.$$

Proof. Following the reasoning above, it remains to show that

$$\lim_{n\to\infty} (1 - N(n)(1 - a_n b_n))(1 - N(n)c_n) = 1,$$

where

$$N(n) = \left\lfloor \frac{k(n)}{\sqrt{\alpha \log m(n)}} \right\rfloor.$$

In order to do this, we observe that is suffices to prove that

$$\lim_{n\to\infty} a_n^{N(n)} = \lim_{n\to\infty} b_n^{N(n)} = \lim_{n\to\infty} (1 - c_n)^{N(n)} = 1 \tag{4.3}$$

Indeed, suppose $0 < x_n < 1$ and assume $\lim_{n\to\infty} x_n^{N(n)} = 1$. It then holds that $\lim_{n\to\infty} N(n) \log(x_n) = 0$. But $\log(x_n) \leq x_n - 1 \leq 0$, and hence, by the squeeze theorem we find that $\lim_{n\to\infty} N(n)(x_n - 1) = 0$ from which it follows that $\lim_{n\to\infty} 1 - N(n)(1 - x_n) = 1$.

Let us prove that (4.3) holds. We will only show this for a_n, the result for b_n and c_n being proven similarly (the conditions on τ being needed there to have the desired decay).

Since by assumption $\lim_{n\to\infty} \frac{k(n)m(n)}{n} = 1$, we have for n large that

$$\left\lfloor \frac{k(n)}{\sqrt{\alpha \log m(n)}} \right\rfloor \approx \frac{n}{m(n)\sqrt{\alpha \log m(n)}}.$$

Therefore, we have asymptotically

$$a_n^{N(n)} \approx \left(1 - 2m(n)^{-k}\right)^{\frac{\delta n}{\beta \log m(n)\sqrt{\alpha \log m(n)}}} \left(1 - (1-p)^{\frac{\delta m(n)}{\beta \log m(n)}}\right)^{\frac{n}{m(n)\sqrt{\alpha \log m(n)}}}.$$

For the first factor, taking logarithms, we have

$$\frac{\delta n}{\beta \log m(n)\sqrt{\alpha \log m(n)}} \log\left(1 - 2m(n)^{-k}\right) \approx -\frac{\delta n}{\beta m(n)^k \log m(n)\sqrt{\alpha \log m(n)}}$$

where we used that $\log(1-x) \approx -x$. Since by assumption k is such that $\lim_{n\to\infty} nm(n)^{-k} = 0$, the above converges to 0 and therefore

$$\lim_{n\to\infty} \left(1 - 2m(n)^{-k}\right)^{\frac{\delta n}{\beta \log m(n)\sqrt{\alpha \log m(n)}}} = 1.$$

In a similar way, the second factor converges to 1 if

$$\lim_{n\to\infty} \frac{n}{m(n)\sqrt{\alpha \log m(n)}} (1-p)^{\frac{\delta m(n)}{\beta \log m(n)}} = 0.$$

This again follows from the assumptions that $\lim_{n\to\infty} nm(n)^{-k} = 0$. This concludes the proof. □

4.3 The size of the giant component

So far, we have shown that with high probability there exists a large component spreading through all compartments. It remains to show that there is only one such component, and that its size is asymptotically $(1-\rho)n$, where ρ is the extinction probability of the Galton-Watson tree as explained in Remark 2.3.

From Proposition 4.3 we obtain the following identification of the largest component in the compartment model G_n. Note that this also proves the final statement of Theorem 2.2.

Proposition 4.8. *Let the assumptions of Theorem 2.2 be satisfied. Then with high probability the largest component in G_n is equal to*

$$\{x \| C_x| \geq \beta \log m(n)\},$$

where C_x denotes the component of G_n containing x.

Proof. By Proposition 4.7 we know that $C \subset \{x \| C_x| \geq \beta \log m(n)\}$ with high probability. The claim now follows once we show that

$$\mathbb{P}\left(\{x \| C_x| \geq \beta \log m(n)\} \subset C\right)$$

goes to 1. For this, it suffices to prove that

$$\mathbb{P}(|C_x| \geq \beta \log m(n) \text{ and } x \notin C \text{ for some } x)$$

goes to 0. Note that there are at most $\frac{n}{\beta \log m(n)}$ components of size larger than $\beta \log m(n)$. Therefore, the above probability is bounded above by

$$\frac{n}{\beta \log m(n)} \mathbb{P}\left(|\tilde{C}| \geq \beta \log m(n) \text{ and } \tilde{C} \cap C = \emptyset\right).$$

By conditioning we have

$$\mathbb{P}\left(|\tilde{C}| \geq \beta \log m(n) \text{ and } \tilde{C} \cap C = \emptyset\right)$$

$$\leq \mathbb{P}\left(|\tilde{C}| < m(n)^{\frac{2}{3}} \middle| |\tilde{C}| \geq \beta \log m(n)\right) + \mathbb{P}\left(\tilde{C} \cap C = \emptyset \middle| |\tilde{C}| \geq m(n)^{\frac{2}{3}}\right).$$

From Proposition 4.3 it follows that

$$\mathbb{P}\left(|\tilde{C}| \geq \beta \log m(n) \text{ and } \tilde{C} \cap C = \emptyset\right) \leq m(n)^{-k}.$$

Furthermore, an argument similar to the proof of [BR15, Lemma 20] gives us that

$$\mathbb{P}\left(\tilde{C} \cap C = \emptyset \middle| |\tilde{C}| \geq m(n)^{\frac{2}{3}}\right) \leq \exp\left(-Cm(n)^{\frac{1}{3}-\tau}\right).$$

Here we used that the component C contains at least $\varepsilon m(n)^{\frac{2}{3}-\tau}$ vertices from each compartment and the fact that the degree sequence is bounded.
Since by assumption $\lim_{n\to\infty} nm(n)^{-k} = 0$, it follows that

$$\lim_{n\to\infty} \frac{n}{\beta \log m(n)} \left(m(n)^{-k} + \exp\left(-Cm(n)^{\frac{1}{3}-\tau}\right)\right) = 0$$

as long as we take τ small enough. This completes the proof. □

From Proposition 4.8 it follows that we are done once we show that

$$\frac{1}{n}|\{x||C_x| \geq \beta \log m(n)\}| \to 1-\rho$$

in probability. To this end, we first show that

$$\lim_{n\to\infty} \mathbb{P}(|C| \leq \beta \log m(n)) = \rho. \tag{4.4}$$

For this, we need the following result, where we show that until depth $\beta \log m(n)$, the component $|C|$ is locally a tree with high probability.

Proposition 4.9. *Let the assumptions of Theorem 2.2 be satisfied. A component C in G_n with $|C| \leq \beta \log m(n)$ is with high probability a tree.*

Proof. Note that with high probability, the degree distribution of the graph G_n is close to D. In particular, the degree sequence is bounded, say by B. Let us explore C from a vertex $v \in C$. We first reveal all neighbours of v by going through the half-edges at v. We then consider a neighbour of v and expose the neighbours attached to all its half-edges, and so on.
If we have explored t vertices in this way, we have exposed at most Bt vertices, which together have at most B^2t half edges. From the convergence of the degree sequence, it follows that for n large enough, there are at least $cm(n)$ half-edges a vertex can be attached to for some $c > 0$. Therefore, when exploring a vertex at iteration t, there are at least $cm(n) - B^2(t+1)$ half-edges to choose from. Of these, at most $B^2(t+1)$ are incident with already exposed vertices. Therefore, the probability that a half-edge we are exploring is attached to an already exposed vertex is at most

$$\frac{B^2(t+1)}{cm(n) - B^2(t+1)}$$

As a consequence, the probability that we only find unexposed vertices is at least

$$\left(1 - \frac{B^2(t+1)}{cm(n) - B^2(t+1)}\right)^B.$$

From this it follows that the probability that there exists a $t \leq \beta \log m(n)$ for which we find an already exposed vertex during the exploration is at most

$$\sum_{t=1}^{\beta \log m(n)} 1 - \left(1 - \frac{B^2(t+1)}{cm(n) - B^2(t+1)}\right)^B$$
$$\leq \beta \log m(n) \left(1 - \left(1 - \frac{B^2(\beta \log m(n) + 1)}{cm(n) - B^2(\beta \log m(n) + 1)}\right)^B\right).$$

Using a binomial expansion, one can prove that the latter converges to 0 since $m(n)$ tends to infinity. We conclude that the probability of only finding new vertices during the exploration until $\beta \log m(n)$ vertices have been explored goes to 1. This proves that the probability that C is a tree converges to 1. □

In particular, we find that C is with high probability equal to a Galton-Watson tree where the root has offspring distribution D, while all other vertices have offspring distribution $Z_D = D^* - 1$, the size-biased off-spring distribution (see Remark 2.3). Since ρ is the extinction probability of this Galton-Watson tree, Proposition 4.9 implies (4.4).

4.4 Proof of Theorem 2.2

With all preparations done, we are finally ready to prove Theorem 2.2.

Proof of Theorem 2.2. From Proposition 4.8 it follows that we are done once we show that

$$\lim_{n\to\infty} \frac{1}{n} |\{x| |C_x| \geq \beta \log m(n)\}| = 1 - \rho$$

in probability.
To this end, define the random variables Y_x^n, where $Y_x^n = 1$ if $|C_x| \leq \beta \log m(n)$ and 0 otherwise. Then

$$|\{x| |C_x| \leq \beta \log m(n)\}| = \sum_{x=1}^{k(n)m(n)} Y_x.$$

Note that by (4.4) we have

$$\lim_{n\to\infty} \mathbb{E}(Y_x^n) = \lim_{n\to\infty} \mathbb{P}(|C_x| \leq \beta \log m(n)) = \rho.$$

Therefore, we find that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{n}|\{x||C_x| \geq \beta \log m(n)\}| - \rho\right| > \varepsilon\right) = \lim_{n\to\infty} \mathbb{P}\left(\left|\sum_{x=1}^{k(n)m(n)} Y_x - k(n)m(n)\rho\right| \geq n\varepsilon\right).$$

By Chebyshev's inequality, we have

$$\mathbb{P}\left(\left|\sum_{x=1}^{k(n)m(n)} Y_x - k(n)m(n)\rho\right| \geq n\varepsilon\right) \leq \frac{\mathrm{Var}\left(\sum_{x=1}^{k(n)m(n)} Y_x\right)}{\varepsilon^2 n^2}.$$

Note that

$$\mathrm{Var}\left(\sum_{x=1}^{k(n)m(n)} Y_x\right) \leq k(n)m(n) + \sum_{x\neq y} \mathrm{Cov}(Y_x, Y_y) \leq k(n)m(n) + k(n)^2 m(n)^2 \mathrm{Cov}(Y_1, Y_2).$$

We can compute

$$\mathrm{Cov}(Y_1, Y_2) = \mathbb{P}(Y_1 = 1, Y_2 = 1) - \mathbb{P}(Y_1 = 1)\mathbb{P}(Y_2 = 1).$$

To estimate this, we consider two independent exploration processes starting at vertex 1 and 2 where we couple them once they meet. It can be shown that the above covariance is then bounded by twice the probability that the two exploration processes meet (compare to [Dur07, Lemma 2.3.4]). Since we consider the event where both processes will not find more than $\beta \log m(n)$ vertices, the above is bounded by twice the probability that two sets of $\beta \log m(n)$ vertices are connected. Following a reasoning similar to [BR15, Lemma 20], we therefore find that

$$\mathrm{Cov}(Y_1, Y_2) \leq \frac{2B^2(\beta \log m(n))^2}{n}.$$

Altogether, we obtain

$$\mathrm{Var}\left(\sum_{x=1}^{k(n)m(n)} Y_x\right) \leq Ck(n)m(n)(\log m(n))^2$$

for some constant $C > 0$. Plugging this into the equation above and using that $\lim_{n\to\infty} \frac{k(n)m(n)}{n} = 1$, we find that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\sum_{x=1}^{k(n)m(n)} Y_x - k(n)m(n)\rho\right| \geq n\varepsilon\right) = 0.$$

Putting everything together, we obtain

$$\frac{1}{n}|\{x||C_x| \leq \beta \log m(n)\}| \to \rho$$

in probability, which implies that

$$\frac{1}{n}|\{x||C_x| \geq \beta \log m(n)\}| \to 1 - \rho$$

in probability as desired. □

5 Difference with standard configuration model

We conclude by considering an example to see the difference between the compartment model on the circle and the standard configuration model. This example also shows that the condition that the number of vertices $m(n)$ per compartments tends to infinity cannot be removed without altering the conditions on the degree sequence.

More precisely, let D be random variable taking values in the non-negative integers. Assume that $\mathbb{P}(D \leq 1) = p > 0$ and $\mathbb{E}(D(D-2)) > 0$. Let d_n be a degree sequence on n vertices converging to D in the sense of Assumption 2.1 (without the compartments). Let $G(d_n)$ be the random graph obtain from the standard configuration model on n vertices with degree sequence d_n. Then (see e.g. [BR15])

$$\lim_{n\to\infty} \frac{L_1(G(d_n))}{n} = 1 - \rho$$

in probability, where ρ is the extinction probability of the Galton-Watson tree associated to D as in Remark 2.3. In particular, because $\mathbb{E}(D(D-2)) > 0$ it holds that $\rho < 1$. We thus see that with high probability the graph $G(d_n)$ contains a giant component.

We will now prove that under the same conditions, the compartment model on the circle does not contain a giant component with high probability if we assume the compartment contain a fixed number of vertices. This is caused only by the assumption that $\mathbb{P}(D \leq 1) > 0$.

Proposition 5.1. *Let D be a random variable taking values in the non-negative integers such that $\mathbb{P}(D \leq 1) = p > 0$. Let d_n be a degree sequence sample independently and uniformly from D. Let G_n be the compartment model on the circle with degree sequence d_n and assume that $m(n) = \lambda \geq 1$ for all n. Then*

$$\lim_{n\to\infty} \frac{L_1(G_n)}{n} = 0$$

in probability.

Proof. Observe that if all vertices in a compartment have degree 0 or 1, then no component can cross this compartment. As a consequence, the size of components is bounded by the maximum distance between such compartments multiplied by λ.
Note that with probability p^λ a compartment contains only degree 0 or 1 vertices. Since for different compartments these events are independent, the distance between such compartments is geometrically distributed with parameter p^λ. Moreover, since $\lambda \geq 1$, we have at most n such intervals.
Let $X_1, \ldots, X_n$ be independent random variables with a geometric distribution with parameter p^λ. By the above, it follows that the size of the largest component is bounded by $\max_{i=1}^n X_i$.
Now let $\varepsilon > 0$. Then

$$\mathbb{P}\left(\max_{i=1}^{n} X_i \leq \varepsilon n\right) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq \varepsilon n) = \left(1 - (1-p^\lambda)^{\varepsilon n}\right)^n$$

Now,

$$\lim_{n\to\infty} \left(1 - (1-p^\lambda)^{\varepsilon n}\right)^n = 1.$$

To see this, note that

$$\log\left(\left(1 - (1-p^\lambda)^{\varepsilon n}\right)^n\right) = n \log\left(1 - (1-p^\lambda)^{\varepsilon n}\right) \approx n(1-p^\lambda)^{\varepsilon n},$$

which goes to 0 since $0 < (1-p^\lambda)^\varepsilon < 1$.
Using the above, we find that

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{i=1}^{n} X_i \leq \varepsilon n\right) = 1$$

Collecting everything, it follows that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(\frac{L_1(G_n)}{n} > \varepsilon\right) &\leq \lim_{n\to\infty} \mathbb{P}\left(\max_{i=1}^{n} X_i > n\varepsilon\lambda^{-1}\right) \\
&= 1 - \lim_{n\to\infty} \mathbb{P}\left(\max_{i=1}^{n} X_i \leq n\varepsilon\lambda^{-1}\right) \\
&= 0.
\end{aligned}$$

We conclude that

$$\lim_{n\to\infty} \frac{L_1(G_n)}{n} = 0$$

in probability. ◻

Remark 5.2. Following the same reasoning, one can actually show that Proposition 5.1 also holds when $m(n) = \lambda \log n$ as long as $\lambda < -\frac{1}{\log p}$. This proves that at least for some degree distributions D it is actually necessary for $m(n)$ to tend to infinity in order to see a giant component. This also underpins the idea that there is an interplay between the assumptions on the degree sequence and compartment for the emergence of a giant component.

Remark 5.3 (Percolation). As a consequence of the results in this section, we find that percolation for the circle model with fixed size compartments looks very unusual. Indeed, if we independently keep edges with probability $p < 1$, then the probability that a vertex in the resulting graph has degree at most 1 is greater than 0. The argument above then shows that this graph does not have a giant component. From this it follows that the percolation threshold is $p^* = 1$.

A Appendix: Concentration inequalities

In this appendix we obtain the vector-valued extension of the classical result on concentration inequalities by McDiarmid. This is a special case of the results in [Kat+21]. Since we do not need such generality, we state McDiarmid's theorem ([McD+89]) for completeness and derive the vector-valued extension from this.

Theorem A.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $X_1, \ldots, X_n$ independent real-valued random variables. Let $c_1, \ldots, c_n$ be constants such that*

$$\sup_{x_1,\ldots,x_i,x_i',\ldots,x_n} f(x_1, \ldots, x_i, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n) \leq c_i.$$

Then for every $\varepsilon > 0$ we have

$$\mathbb{P}(|f(X_1, \ldots, X_n) - \mathbb{E}(f(X_1, \ldots, X_n))| \geq \varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{2\sum_{i=1}^n c_i^2}\right).$$

We will prove a similar estimate when F is vector-valued. For $x \in \mathbb{R}^d$ we denote by $||x||_\infty$ the sup-norm of x, i.e.,

$$||x||_\infty = \max_{i=1,\ldots,n} |x_i|.$$

We obtain the following extension of McDiarmid's theorem.

Theorem A.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^d$ be a function and $X_1, \ldots, X_n$ independent, real-valued random variables. Let $c_1, \ldots, c_n$ be constants such that*

$$\sup_{x_1,\ldots,x_i,x_i',\ldots,x_n} ||F(x_1, \ldots, x_i, \ldots, x_n) - F(x_1, \ldots, x_i', \ldots, x_n)||_\infty \leq c_i.$$

Then for every $\varepsilon > 0$ we have

$$\mathbb{P}(||F(X_1, \ldots, X_n) - \mathbb{E}(F(X_1, \ldots, X_n))||_\infty \geq \varepsilon) \leq 2d \exp\left(-\frac{\varepsilon^2}{2\sum_{i=1}^n c_i^2}\right).$$

Proof. For every $j = 1, \ldots, d$ we can apply Theorem A.1 to $F_j : \mathbb{R}^n \to \mathbb{R}$, the j-th component of F. This gives us that

$$\mathbb{P}(|F_j(X_1, \ldots, X_n) - \mathbb{E}(F_j(X_1, \ldots, X_n))| \geq \varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{2\sum_{i=1}^n c_i^2}\right).$$

We can now estimate

$$\begin{aligned}
&\mathbb{P}(||F(X_1,\dots,X_n) - \mathbb{E}(F(X_1,\dots,X_n))||_\infty \geq \varepsilon) \\
&\qquad = \mathbb{P}\left(|F_j(X_1,\dots,X_n) - \mathbb{E}(F_j(X_1,\dots,X_n))| \geq \varepsilon \text{ for some } j = 1,\dots,d\right) \\
&\qquad \leq \sum_{j=1}^{d} \mathbb{P}(|F_j(X_1,\dots,X_n) - \mathbb{E}(F_j(X_1,\dots,X_n))| \geq \varepsilon) \\
&\qquad \leq 2d \exp\left(-\frac{\varepsilon^2}{2\sum_{i=1}^{n} c_i^2}\right),
\end{aligned}$$

which completes the proof. □

Acknowledgement This research was sponsored by the Army Research Office and was accomplished under Cooperative Agreement Number W911NF-20-2-0187. The views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Office or the U.S. Government. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation herein.

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