Balanced Independent and Dominating Sets on Colored Interval Graphs

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Abstract. We study two new versions of independent and dominating set problems on vertex-colored interval graphs, namely f*-Balanced Independent Set* (f-BIS) and f*-Balanced Dominating Set* (f-BDS). Let $G = (V, E)$ be an interval graph with a *color assignment* function $\gamma\colon V \to \{1, \dots, k\}$ that maps all vertices in G onto k colors. A subset of vertices $S \subseteq V$ is called f*-balanced* if S contains f vertices from each color class. In the f-BIS and f-BDS problems, the objective is to compute an independent set or a dominating set that is f-balanced. We show that both problems are NP-complete even on proper interval graphs. For the BIS problem on interval graphs, we design two FPT algorithms, one parameterized by (f, k) and the other by the vertex cover number of G. Moreover, for an optimization variant of BIS on interval graphs, we present a polynomial time approximation scheme (PTAS) and an $O(n \log n)$ time 2-approximation algorithm.

1 Introduction

A graph G is an interval graph if it has an intersection model consisting of intervals on the real line. Formally, $G = (V, E)$ is an interval graph if there is an assignment of an interval $I_v \subseteq \mathbb{R}$ for each $v \in V$ such that $I_u \cap I_v$ is nonempty if and only if $\{u, v\} \in E$. A *proper* interval graph is an interval graph that has an intersection model in which no interval properly contains another [10]. Consider an interval graph $G = (V, E)$ and additionally assume that the vertices of G are k-colored by a *color assignment*[1] $\gamma\colon V \to \{1, \dots, k\}$. We define and study

[1] We use the term *color assignment* instead of *vertex coloring* to avoid any confusion with the general notion of vertex coloring; in particular, a color assignment γ can map adjacent vertices to the same color.

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color-balanced versions of two classical graph problems: maximum independent set and minimum dominating set on vertex-colored (proper) interval graphs. In what follows, we define the problems formally and discuss their underlying motivation.

f-Balanced Independent Set (f-BIS): Let $G = (V, E)$ be an interval graph with a color assignment of the vertices $\gamma \colon V \to \{1, \ldots, k\}$. Find an *$f$-balanced independent set* of G, i.e., an independent set $L \subseteq V$ that contains exactly f elements from each color class.

The classic maximum independent set problem serves as a natural model for many real-life optimization problems and finds applications across fields, e.g., computer vision [2], information retrieval [18], and scheduling [20]. Specifically, it has been used widely in map-labeling problems [1,4,14,21], where an independent set of a given set of label candidates corresponds to a conflict-free and hence legible set of labels. To display as much relevant information as possible, one usually aims at maximizing the size or, in the case of weighted label candidates, the total weight of the independent set. This approach may be appropriate if all labels represent objects of the same category. In the case of multiple categories, however, maximizing the size or total weight of the labeling does not reflect the aim of selecting a good mixture of different object types. For example, if the aim was to inform a map user about different possible activities in the user's vicinity, labeling one cinema, one theater, and one museum may be better than labeling four cinemas. In such a setting, the f-BIS problem asks for an independent set that contains f vertices from each object type.

We initiate this study for interval graphs which is a primary step to understand the behavior of this problem on intersection graphs. Moreover, solving the problem for interval graphs gives rise to optimal solutions for certain labeling models, e.g., if every label candidate is a rectangle that is placed at a fixed position on the boundary of the map [11].

While there exists a simple greedy algorithm for the maximum independent set problem on interval graphs, it turns out that f-BIS is much more resilient and NP-complete even for proper interval graphs and $f = 1$ (Sect. 2.1). Then, in Sect. 3, we complement this complexity result with two FPT algorithms for interval graphs, one parameterized by (f, k) and the other parameterized by the vertex cover number. Section 4 introduces a polynomial time approximation scheme (PTAS) and an $O(n \log n)$ time 2-approximation algorithm for an optimization variant (1-MCIS) of BIS on interval graphs.

The second problem we discuss is defined as follows.

f-Balanced Dominating Set (f-BDS): Let $G = (V, E)$ be an interval graphs with a color assignment of the vertices $\gamma \colon V \to \{1, \ldots, k\}$. Find an *$f$-balanced dominating set*, i.e., a subset $D \subseteq V$ such that every vertex in $V \setminus D$ is adjacent to at least one vertex in D, and D contains exactly f elements from each color class.

The dominating set problem is another fundamental problem in theoretical computer science which also finds applications in various fields of science and engineering [6,12]. Several variants of the dominating set problem have been considered over the years: k-tuple dominating set [7], Liar's dominating set [3], independent dominating set [13], and more. The colored variant of the dominating set problem has been considered in parameterized complexity, namely, red-blue dominating set, where the objective is to choose a dominating set from one color class that dominates the other color class [9]. Instead, our f-BDS problem asks for a dominating set of a vertex-colored graph that contains f vertices of each color class. Similar to the independent set problem, we primarily study this problem on vertex-colored interval graphs, which can be of independent interest. In Sect. 2.2, we prove that f-BDS on vertex-colored proper interval graphs is NP-complete, even for $f = 1$. Due to space constraints, please refer to the appendix for missing proofs and detailed descriptions.

2 Complexity Results

In this section we show that f-BIS and f-BDS are NP-complete even if the given graph G is a proper interval graph and $f = 1$. Our reductions are from restricted, but still NP-complete versions of 3SAT, namely 3-bounded 3SAT [19] and 2P2N-3SAT (hardness follows from the result for 2P1N-SAT [22]). In the former 3SAT variant a variable is allowed to appear in at most three clauses and clauses have two or three literals, in the latter each variable appears exactly four times, twice as positive literal and twice as negative literal.

2.1 f-Balanced Independent Set

We first describe the reduction. Let $\phi(x_1, \ldots, x_n)$ be a 3-bounded 3SAT formula with variables $x_1, \ldots, x_n$ and clause set $\mathcal{C} = \{C_1, \ldots, C_m\}$. From ϕ we construct a proper interval graph $G = (V, E)$ and a color assignment γ of V as follows. We choose the set of colors to contain exactly m colors, one for each clause in $\mathcal{C}$ and we number these colors from 1 to m. We add a vertex $u_{i,j} \in V$ for each occurrence of a variable x_i in a clause C_j in ϕ. Furthermore, we insert an edge $\{u_{i,j}, u_{i,j'}\} \in E$ whenever $u_{i,j}$ was inserted because of a positive occurrence of x_i and $u_{i,j'}$ was inserted because of a negative occurrence of x_i. Finally, we color each vertex $u_{i,j} \in V$ with color j. See Fig. 1 for an illustration. It is clear that the construction is in polynomial time. The graph G created from ϕ is a proper interval graph as it consists only of disjoint paths of length at most three and can clearly be constructed in polynomial time and space.

Theorem 1. *The f-balanced independent set problem on a graph $G = (V, E)$ with a color assignment of the vertices $\gamma\colon V \to \{1, \ldots, k\}$ is NP-complete, even if G is a proper interval graph and $f = 1$.*

Proof. The problem is clearly in NP since for a given solution it can be checked in linear time if it is an independent set and contains f vertices of each color.

We already described the reduction. It remains to argue the correctness. Assume $G = (V, E)$ was constructed as above from a 3-bounded 3SAT formula

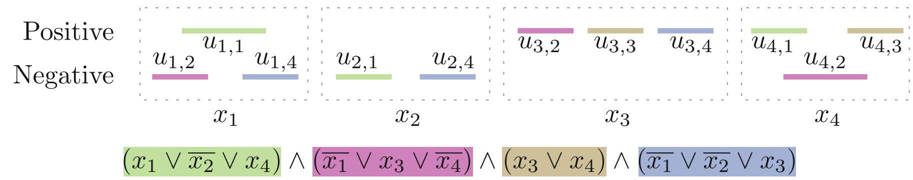

Fig. 1. The graph resulting from the reduction for 1-balanced independent set in Theorem 1 depicted as interval representation with the vertex colors being the colors of the intervals.

$\phi(x_1, \dots, x_n)$ and let $V' \subseteq V$ be a solution to the 1-balanced independent set problem on G.

We construct a variable assignment for $x_1, \dots, x_n$ as follows. By definition we find for each color j precisely one vertex $u_{i,j} \in V'$. If $u_{i,j}$ was inserted for a positive occurrence of x_i, then we set x_i to *true* and otherwise x_i to *false*. Moreover, all variables x_i with $i = 1, \dots, n$ for which we do not find a corresponding interval in V' are also set to *false*. Since V' is an independent set in G this assignment is well defined. Now assume it was not satisfying, then there exists a clause C_j for which none of its literals evaluates to *true*. Hence, none of the at most three vertices corresponding to the literals in C_j is in V'. Recall that there is a one-to-one correspondence between clauses and colors in the instance of 1-balanced independent set we created. Yet, V' does not contain a vertex of that color, a contradiction.

For the opposite direction assume we are given a satisfying assignment of the 3-bounded 3SAT formula $\phi(x_1, \dots, x_n)$ with clauses $\mathcal{C} = \{C_1, \dots, C_m\}$. Furthermore let $G = (V, E)$ be the graph with a color assignment of the vertices γ constructed from ϕ as described above. We find a 1-balanced independent set of G from the given assignment as follows. For each clause $C_j \in \mathcal{C}$ we choose one of its literals that evaluates to *true* and add the corresponding vertex $v \in V$ to the set of vertices V'. Since there is a one-to-one correspondence between the colors and the clauses and the assignment is satisfying, V' clearly contains one vertex per color. It remains to show that V' is an independent set of vertices in G. Assume for contradiction that there are two vertices $v_{i,j}, v_{i',j'} \in V'$ and $\{v_{i,j}, v_{i',j'}\} \in E$. Then, by construction of G, we know that $i = i'$ and further that $v_{i,j}, v_{i',j'}$ correspond to one positive and one negative occurrence of x_i in ϕ. By the construction of V' this implies a contradiction to the assignment being satisfying. $\square$

2.2 *f*-Balanced Dominating Set

We reduce from 2P2N-3SAT where each variable appears exactly twice positive and twice negative. Let $\phi(x_1, \dots, x_n)$ be a 2P2N-3SAT formula with variables $x_1, \dots, x_n$ and clause set $\mathcal{C} = \{C_1, \dots, C_m\}$. For variable x_i in ϕ we denote with $\mathcal{C}_{x_i} = \{C_t^1, C_t^2, C_f^1, C_f^2\}$ the four clauses x_i appears in, where C_t^1, C_t^2 are clauses with positive occurrences of x_i and C_f^1, C_f^2 are clauses containing negative occurrences of x_i.

We construct a graph $G = (V, E)$ from $\phi(x_1, \ldots, x_n)$ as follows. For each variable x_i we introduce six vertices t_1, t_2, f_1, f_2, h_t, and h_f and for each clause C_j with occurrences of variables x_{j_1}, x_{j_2}, and x_{j_3} we add up to three vertices c_k for each $k \in \{j_1, j_2, j_3\}$ (In case a clause has less than three literals we add only one or two vertices). If the connection to the variable is clear, we also write c_t^1, c_t^2, c_f^1, and c_f^2 for the vertices introduced for this variable's occurrences in the clauses C_t^1, C_t^2, C_f^1, and C_f^2, respectively. Furthermore, we add for each variable x_i the edges $\{h_t, t_1\}$, $\{h_t, t_2\}$, $\{h_f, f_1\}$, and $\{h_f, f_2\}$, as well as for each clause C_j all possible edges between the three vertices introduced for C_j. For each variable x_i we introduce five colors, namely z_t^1, z_t^2, z_f^1, z_f^2, and z_h. We set $\gamma(h_t) = \gamma(h_f) = z_h$. Finally, we set $\gamma(t_1) = \gamma(c_t^1) = z_t^1$. Equivalently for t_2, f_1, and f_2. See Fig. 2 for an example.

In total we create $6n + 3m$ many vertices and $4n + 3m$ many edges, thus the reduction is in polynomial time. All variable and clause gadgets are independent components and only consist of paths of length three and triangles, hence G is a proper interval graph. Furthermore, G can clearly be constructed in polynomial time and space.

To establish the correctness of our reduction for 1-BDS we first introduce a canonical type of solutions for the graphs produced by our reduction. We call V_D *canonical*, if for each variable x_i we either find $\{h_t, f_1, f_2, c_t^1, c_t^2\} \subset V_D$ or $\{h_f, t_1, t_2, c_f^1, c_f^2\} \subset V_D$. If for a variable $x \in X$ and a 1-balanced dominating set $V_D \subseteq V$ we find one of the two above sets in V_D, we say x is in *canonical form* in V_D. The next lemma shows that if G has a 1-balanced dominating set we can turn it into a canonical one.

Lemma 1. *Let $G = (V, E)$ be a graph generated from a 2P2N-3SAT formula $\phi(x_1, \ldots, x_n)$ with clause set $\mathcal{C} = \{C_1, \ldots, C_m\}$ as above and $V_D \subseteq V$ a* 1-*balanced dominating set, then V_D can be transformed into a canonical* 1-*balanced dominating set in $O(|V|)$ time.*

Proof. Let x be not in canonical form in V_D. Since V_D is a 1-balanced dominating set we know that either h_t or h_f of x is in V_D. Without loss of generality assume that $h_t \in V_D$. Consequently, we find that $f_1, f_2 \in V_D$ and $c_f^1, c_f^2 \notin V_D$. Now, we

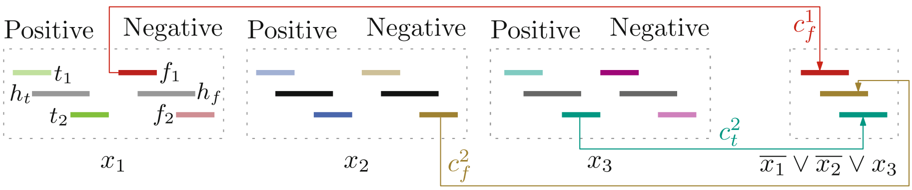

Fig. 2. Illustrations of three variable gadgets and a clause gadget from Theorem 2 as interval representations. Vertex colors correspond to interval colors.

obtain the set V'_D from V_D by removing any occurrence of t_1 or t_2 from V_D and inserting all missing elements of $\{c_t^1, c_t^2\}$. Clearly x is in canonical form in V'_D. We need to show that V'_D is still a 1-balanced dominating set. It is straight forward to verify that every color appears exactly once in V'_D if V_D was 1-balanced. Now assume there was a vertex $u \in V$ that is not dominated by any vertex in V'_D. Yet, we at most deleted t_1 and t_2 in V'_D but since $h_t \in V'_D$ both and all their neighbors are dominated. As our operations only affected vertices introduced for x and occurrences of x we can simply iterate this process for each variable until every variable x_i is in canonical form. □

Theorem 2. *The f-balanced dominating set problem on a graph $G = (V, E)$ with a color assignment of the vertices $\gamma : V \to \{1, \ldots, k\}$ is* NP*-complete, even if G is a proper interval graph and $f = 1$.*

Proof. The problem is clearly in NP as we can verify if a given set of vertices is an f-balanced dominating set by checking if it is a dominating set and if it contains f vertices of each color in linear time.

Let $G = (V, E)$ be constructed from a 2P2N-3SAT formula $\phi(x_1, \ldots, x_n)$ with clause set $\mathcal{C} = \{C_1, \ldots, C_m\}$ as above and let V_D be a 1-balanced dominating set of G. By Lemma 1 we can assume V_D is canonical. We construct an assignment of the variables in ϕ by setting x_i to *true* if its $h_t \in V_D$ and to *false* otherwise. Assume this assignment was not satisfying, i.e., there exists a clause $C_j \in \mathcal{C}$ such that none of the literals in C_j evaluates to *true*. For each positive literal of C_j we then get that the corresponding variable x_i was set to *false*. Hence, $h_f \in V_D$ for x_i and consequently $c_t^1, c_t^2 \notin V_D$. Equivalently for each negative literal we find $h_t \in V_D$ and $c_f^1, c_f^2 \notin V_D$. As a result we find that none of the vertices introduced for literals in C_j is in V_D and especially that none of them is dominated as they are each others only neighbors. Yet, V_D is a 1-balanced dominating set by assumption, a contradiction.

In the other direction, assume we are given a satisfying assignment of a 2P2N-3SAT formula $\phi(x_1, \ldots, x_n)$ with clause set $\mathcal{C} = \{C_1, \ldots, C_m\}$. Furthermore, let $G = (V, E)$ be the graph constructed from ϕ as above. We form a canonical 1-balanced dominating set $V_D \subseteq V$ of G in the following way. For every variable x_i that is set to *true* in the assignment we add $\{h_t, f_1, f_2, c_t^1, c_t^2\}$ to V_D and for every variable $x_{i'}$ that is set to *false* we add $\{h_f, t_1, t_2, c_f^1, c_f^2\}$. This clearly is a 1-balanced set and it is canonical. It remains to argue that it dominates G. For the vertices introduced for variables this is clear, since we pick either h_t or h_f, as well as f_1, f_2 or t_1, t_2 for every variable x_i. Now, assume there was a clause $C_j \in \mathcal{C}$ and none of the vertices introduced for literals in C_j was in V_D. Then, by construction of V_D, we find that for any positive (negative) occurrence of a variable x_i in C_j the variable x_i was set to *false* (*true*). A contradiction to the assignment being satisfying. □

3 Algorithmic Results for the Balanced Independent Set

In this section, we take a parameterized perspective on f-BIS and provide two FPT algorithms[2] with different parameters. The algorithms described in this section can be easily generalized to maximize the value of f in f-BIS.

3.1 An FPT Algorithm Parameterized by (f, k)

Assume we are given an instance of f-BIS with $G = (V, E)$ being an interval graph with a color assignment of the vertices $\gamma\colon V \to \{1, \ldots, k\}$. We can construct an interval representation $\mathcal{I} = \{I_1, \ldots, I_n\}$, $n = |V|$, from G in linear time [15]. Our algorithm is a dynamic programming based procedure that work as follows. Firstly, we sort the right end-points of the n intervals in $\mathcal{I}$ in ascending order. Next, we define a function $prev\colon V \to \{1, \ldots, n\}$. for each interval $I_i \in I$, the $prev(I_i)$ is the index of the rightmost interval with its right endpoint left to I_i's left endpoint. If no such interval exists for some interval I_i, we set $prev(I_i) = 0$.

For each color $\kappa \in \{1, \ldots, k\}$, let $\hat{e}_\kappa$ denote the k-dimensional unit vector of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the element at the κ-th position is 1 and the rest are 0. For a subset $\mathcal{I}' \subseteq \mathcal{I}$ we define a *cardinality vector* as the k-dimensional vector $C_{\mathcal{I}'} = (c_1, \ldots, c_k)$, where each element c_i represents the number of intervals of color i in $\mathcal{I}'$. We say $C_{\mathcal{I}'}$ is *valid* if all $c_i \le f$ and the set $\mathcal{I}'$ is independent.

The key observation here is that there are at most $O((f+1)^k)$ many different valid cardinality vectors as there are only k colors and we are interested in at most f intervals per color. In the following let U_j, $j \in \{1, \ldots, n\}$, be the union of all valid cardinality vectors of the first j intervals in $\mathcal{I}$. Let $U_0 = \{(0, \ldots, 0)\}$ in the beginning. To compute an f-balanced independent set the algorithm simply iterates over all right endpoints of the intervals in $\mathcal{I}$ and in the i-th step computes U_i as $U_i = \{u + \hat{e}_{\gamma(I_i)} \mid u \in U_{prev(I_i)}$ and $u + \hat{e}_{\gamma(I_i)}$ is a valid cardinality vector$\} \cup U_{i-1}$. Checking if a new cardinality vector is valid can be done easily by remembering for each $u \in U_{i-1}$ one representative interval set with u as its cardinality vector. Finally, we check the cardinality vectors in U_n and return *true* in case there is one cardinality vector $w \in U_n$ with entries being all f and *false* otherwise. Moreover, the representative interval set of w builds an f-balanced independent set.

Theorem 3. *Let $G = (V, E)$ be an interval graph with a color assignment of the vertices $\gamma : V \to \{1, \ldots, k\}$. We can compute an f-balanced independent set of G or determine that no such set exists in $O(n \log n + k(f+1)^k n)$ time.*

Proof. Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be an interval representation of G on which we execute our algorithm. For U_0 the set just contains the valid cardinality vector with all zeros which is clearly correct. Let U_{i-1} be the set of valid cardinality

[2] FPT is the class of parameterized problems that can be solved in time $O(g(k)n^{O(1)})$ for input size n, parameter k, and some computable function g.

vectors computed after step $i-1$. Now, in step $i \leq n$ we calculate the set U_i as the union of U_{i-1} and the potential new solutions based on independent sets of intervals containing I_i. Assume $\mathcal{I}_x \subseteq \{I_1, \ldots, I_i\}$ is an independent set of intervals such that its cardinality vector $C_{\mathcal{I}_x}$ is valid, but there is no valid cardinality vector $C_{\mathcal{I}'} \in U_i$ such that $C_{\mathcal{I}'}$ is larger or equal in every component than $C_{\mathcal{I}_x}$. Since U_{i-1} contained all valid cardinality vectors for the intervals in $\{I_1, \ldots, I_{i-1}\}$ we know that $C_{\mathcal{I}_x}$ is such that $I_i \in \mathcal{I}_x$. Yet, the set $U_{prev(I_i)}$ contained all valid cardinality vectors for the set of intervals $\{I_1, \ldots, I_{prev(I_i)}\}$. Since I_i has overlaps with all intervals in $\{I_{prev(I_i)+1}, \ldots, I_{i-1}\}$ and hence cannot be in any independent set with any such interval we can conclude that $C_{\mathcal{I}_x} - \hat{e}_{\gamma(I_i)} \in U_{prev(I_i)}$. Though, we also find $C_{\mathcal{I}_x} \in U_i$, a contradiction.

Next we consider the running time. The key observation is that there are at most $(f+1)^k$ different valid cardinality vectors. Checking the validity can be done in $O(1)$ time for each new vector as only one entry changes. Computing the sets U_i can be done in time $O(k(f+1)^k)$, by storing the cardinality vectors in lexicographic sorted order for each set. Keeping the sets in sorted order does not require any extra running time, as U_0 is clearly sorted in the beginning (it only contains one element) and we only increase the same entry for each vector in $U_{prev(I_i)}$ when forming the union, thus not changing their ordering. Hence, the set $U_{prev(I_i)}$ and U_{i-1} can be assumed to be sorted in lexicographic order. Consequently, by merging from smallest to largest element the set U_i is again lexicographically sorted after the union. Furthermore, we can easily discard double entries by comparing also against the vector we inserted last into U_i. Finally, we have to sort the intervals themselves. Using standard sorting algorithms this works in $O(n \log n)$ time. Altogether, this results in a running time of $O(n \log n + k(f+1)^k n)$. □

3.2 An FPT Algorithm Parameterized by the Vertex Cover Number

Here we will give an alternative FPT algorithm for f-BIS, this time parameterized by the *vertex cover number* $\tau(G)$ of G, i.e., the size of a minimum vertex cover of G.

Lemma 2. *Let $G = (V, E)$ be a graph. Consider a vertex cover V_c in G and its complement $V_{\text{ind}} = V \setminus V_c$. Then any maximal independent set M of G can be constructed from V_{ind} by adding the subset $M \cap V_c$ of V_c and removing its neighborhood in V_{ind}, namely $M = (V_{\text{ind}} \cup (M \cap V_c)) \setminus N(M \cap V_c)$.*

Proof. For a fixed but arbitrary maximal independent set M, in the following, we denote the set $(V_{\text{ind}} \cup (M \cap V_c)) \setminus N(M \cap V_c)$ as M_{swap}.

We first prove the independence of M_{swap}. Note that by the definition of a vertex cover V_{ind} is an independent set. Furthermore, the set $(M \cap V_c)$, as a subset of the independent set M, is also independent. Then, in the union $V_{\text{ind}} \cup (M \cap V_c)$ of these two independent sets, any adjacent pair of vertices must contain one vertex in $M \cap V_c$ and one in V_{ind}. Hence, after removing all the neighboring vertices of $M \cap V_c$, the set M_{swap} is independent.

Next we prove that $M \subseteq M_{\text{swap}}$. Assume there exists one vertex v_m in M but not in M_{swap}. Since $v_m \in M$ it must also be in the set $V_{\text{ind}} \cup (M \cap V_c)$. With the assumption that $v_m \notin M_{\text{swap}}$, we get that v_m must be in $N(M \cap V_c)$. Consequently, v_m is in the independent set M and is at the same time a neighbor of vertices in M, a contradiction.

Finally we prove $M = M_{\text{swap}}$. We showed above that M_{swap} is an independent set and also $M \subseteq M_{\text{swap}}$. Since M is a maximal independent set by assumption we get $M = M_{\text{swap}}$. □

Lemma 3. *Let $G = (V, E)$ be a graph with vertex cover number $\tau(G)$. There are $O(2^{\tau(G)})$ maximal independent sets of G.*

Proof. Consider a minimum vertex cover V_c in G and its complement $V_{\text{ind}} = V \setminus V_c$. Note that since V_c is a (minimum) vertex cover, V_{ind} is a (maximum) independent set. Furthermore, any maximal independent set M of G can be constructed from V_{ind} by adding $M \cap V_c$ and removing its neighborhood in V_{ind}, namely $M = (V_{\text{ind}} \cup (M \cap V_c)) \setminus N(M \cap V_c)$ by Lemma 2. Thus there are $O(2^{\tau(G)})$ maximal independent sets of G. □

Theorem 4. *Let $G = (V, E)$ be an interval graph with a color assignment of the vertices $\gamma \colon V \to \{1, \ldots, k\}$. We can compute an f-balanced independent set of G or determine that no such set exists in $O(2^{\tau(G)} \cdot n)$ time.*

Proof. According to Lemma 3, there are $O(2^{\tau(G)})$ maximal independent sets of G. The basic idea is to enumerate all the $O(2^{\tau(G)})$ maximal independent sets and compute their maximum balanced subsets. Enumerating all maximal independent sets of an interval graph takes $O(1)$ time per output [17]. Given an arbitrary independent set of G we can compute an f-balanced independent subset in $O(n)$ time or conclude that no such subset exists. Therefore, the running time of the algorithm is $O(2^{\tau(G)} \cdot n)$. □

4 Approximation Algorithms for the 1-Max-Colored Independent Set

Here we study a variation of the BIS, which asks for a maximally colorful independent set.

1-Max-Colored Independent Set (1-MCIS): Let $G = (V, E)$ be an interval graph with a color assignment of the vertices $\gamma \colon V \to \{1, \ldots, k\}$. The objective is to find a 1-max-colored independent set of G, i.e., an independent set $L \subseteq V$, whose vertices contain a maximum number of colors and L contains exactly 1 element from each color class.

We note that the NP-completeness of 1-BIS implies that 1-MCIS is an NP-hard optimization problem as well.

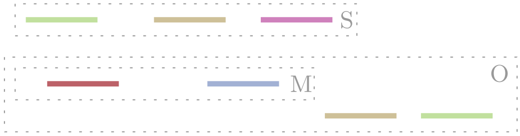

Fig. 3. Comparison of a solution S of the algorithm and an optimal solution O. Subset $M \subseteq O$ contains two colors (red and blue) missing from S, but each interval in M contains the right endpoint of a different interval from S. (Color figure online)

4.1 A 2-Approximation for the 1-Max-Colored Independent Set

In the following, we will show a simple sweep algorithm for 1-MCIS with approximation ratio 2.

First, we sort the intervals from left to right based on their right end-points. Then, our algorithm scans the intervals from left to right, and at each step selects greedily an interval of a distinct color such that no interval of the same color has been selected before. Moreover, we maintain a solution array S of size k to store the selected intervals.

For each interval I_i in this order, we check if the color of I_i is still missing in our solution (by checking if $S[\gamma(I_i)]$ is not yet occupied). If yes, we store I_i in $S[\gamma(i)]$ and remove all the remaining intervals overlapping I_i. Otherwise, if $S[\gamma(I_i)]$ is not empty, we remove I_i and continue scanning the intervals. This process is repeated until all intervals are processed. Then, by using a simple charging argument on the colors in an optimal solution that are missing in our greedy solution, we obtain the desired approximation factor.

Theorem 5. *Let $G = (V, E)$ be an interval graph with a color assignment of the vertices $\gamma \colon V \to \{1, \ldots, k\}$. In $O(n \log n)$ time, we can compute an independent set with at least $\lceil \frac{c}{2} \rceil$ colors, where c is the number of colors in a 1-max-colored independent set.*

Proof. It is clear from the above description that the greedy algorithm finds an independent set. We maintain a solution array S, and it is possible to check if an interval of a particular color is already available in S in constant time. Therefore, the entire algorithm runs in $O(n \log n)$ time.

In order to prove the approximation factor, we compare the solution S of our greedy algorithm with a fixed 1-max-colored independent set O (see Fig. 3). Let $M = \{I_i \in O \mid \nexists I_j \in S \text{ with } \gamma(I_j) = \gamma(I_i)\}$ be the subset of O consisting of intervals of missing colors in S. Now, consider an interval $I_m \in M$. There must be at least one interval $I_s \in S$, whose right endpoint is contained in the interval I_m. Otherwise, since there is no interval of the same color as I_m in S, the greedy algorithm would scan I_m as the interval with the leftmost right endpoint in the process and select it in S. Thus, the function ρ, which maps each interval I_m in M to an interval I_s in S such that I_s is the rightmost interval in S with its right endpoint is contained in I_m, is well-defined. Furthermore, ρ is an injective function because of the independence of the set M. Therefore, we can conclude that the cardinality of the set S is greater than or equal to the cardinality of M. Note that, $|M| + |S| \geq |O|$. Hence, S has size at least $\lceil \frac{c}{2} \rceil$. □

4.2 A PTAS for the 1-Max-Colored Independent Set

In this section, we present a polynomial time approximation scheme (PTAS) for 1-Max-Colored Independent Set (1-MCIS). Our algorithm is based on the careful usage and analysis of the local search technique. We prove that, this algorithm is, in fact, a PTAS for 1-MCIS on interval graphs. Let $\mathscr{L}$ be the solution of the local search algorithm. We aim to bound the size of an optimal solution $\mathscr{O}$ in terms of $|\mathscr{L}|$. To this end, we construct a bipartite planar conflict graph between the subsets of $\mathscr{L}$ and $\mathscr{O}$. Then, by applying a version of the planar separator theorem [8], we obtain the desired bounds. Mustafa and Ray [16] were the first to show the usefulness of local search to obtain a PTAS for the geometric hitting set problem. Here, we use an analysis that is similar to the one used by Chan and Har-Peled for the maximum independent set problem on pseudo-disks [5].

The Algorithm. Let $G = (V, E)$ be a vertex-colored interval graph with a k-coloring $\gamma\colon V \to \{1, \ldots, k\}$ for some $k \in \mathbb{N}$. Furthermore, set $n = |V|$ and $m = |E|$. For two subsets L, $L' \subseteq V$, we say L and L' are b-local neighbors if their differences are bounded by some $b \in \mathbb{R}$, i.e., $|L' \setminus L| \leq b$ and $|L \setminus L'| \leq b$. Let $N(L)$ be the set of all b-local neighbors of L. Observe, that for each subset $L \subseteq V$ we find $|N(L)| \leq O(\binom{n}{2b})$. We denote with $c(L)$ the number of different colors among the vertices in L. An independent set $L \subseteq V$ is *b-locally optimal* for the 1-MCIS problem on G if for each $L' \in N(L)$ we find that either L' is not an independent set or $c(L') \leq c(L)$.

Our algorithm first computes an initial solution L by executing the algorithm described in Sect. 4.1. In case $c(L) = k$ we return L as there is no chance to improve the solution. Assume $c(L) < k$. To turn L into a b-locally optimal solution for some fixed $b \in \mathbb{R}$ we perform a local search over the b-local neighbors of L. If a b-local neighbor L' of L is an independent set for G and $c(L') > c(L)$ we set L' as the current solution and restart the local search with L'. Once the local search terminates without finding such an L' or when $c(L) = k$ we return the current solution L.

Run-Time. Clearly, checking for a set $L \subseteq V$ if it is an independent set for G can be done in $O(m)$ time. Furthermore, we can compute $c(L)$ in time $O(|L|)$. Recall, that for $L \subseteq V$, the set $N(L)$ has at most $O(\binom{n}{2b})$ elements. It remains to bound the number of times we might swap the current solution in the second step of our algorithm. Let $L \subseteq V$ be the current solution in the second step, then we swap L for some $L' \in N(L)$ only if $c(L') > c(L)$. This happens at most k times as after k such improvements we would have found an optiomal solution to the 1-MCIS problem on G. Consequently, the running time of the whole algorithm is bounded by $O(k \cdot n^{2b+1})$.

Analysis. Let $\mathscr{L}$ be a $\frac{1}{\epsilon^2}$-locally optimal solution for 1-MCIS on G obtained by our local search approach for a fixed constant ϵ and $\mathscr{O}$ is a fixed but arbitrary optimal solution for the same problem on G. To ease the following analysis, we assume each solution set contains for each color at most one vertex. Let $L = \mathscr{L} \setminus \mathscr{O}$ and $O = \mathscr{O} \setminus \mathscr{L}$. Next, we construct a graph $H = (V_H, E_H)$ containing a vertex $u \in V_H$ for every interval in $L \cup O$ and there is an edge between two vertices $u, v \in V_H$ (with $u \neq v$) such that, either the corresponding intervals of u and v intersect or $\gamma(u) = \gamma(v)$. We distinguish between these two types of edges: the former edges are called *interval-edges* and the latter are called *color-edges*.

Observation 1. *Let $H = (V_H, E_H)$ be a graph constructed as above, then a vertex $u \in V_H$ is incident to at most one color-edge.*

Lemma 4. *Let graph $H = (V_H, E_H)$ be constructed as above, H is bipartite and planar.*

Proof. Let $V_H = L \cup O$ be the set as used in the definition of H. Since L and O are both independent sets in G and contain for each color at most one vertex it follows that H is bipartite.

It remains to show that H is also planar. We are going to show that H cannot contain a $K_{3,3}$ as subgraph. For the sake of contradiction, assume H did in fact contain a $K_{3,3}$ and let $V' \subseteq V_H$ be its vertices and $E' \subseteq E_H$ its edges. To make the following arguments easier we fix an arbitrary interval representation $\mathcal{I} = \{I_1, \ldots, I_n\}$ of G. Let $I_L = \{I_\ell^1, I_\ell^2, I_\ell^3\}$ be the set of three intervals corresponding to vertices in $V' \cap L$ and $I_O = \{I_o^1, I_o^2, I_o^3\}$ the set of three intervals corresponding to vertices in $V' \cap O$. Without loss of generality we assume that the intervals are ordered by their left endpoints. Since the corresponding vertices are part of the independent sets $\mathscr{L}$ and $\mathscr{O}$ we get that I_ℓ^1 is completely to the left of I_ℓ^2 which is in turn completely to the left of I_ℓ^3. The same holds for the I_o^i with $i = 1, 2, 3$.

We differentiate two cases, namely if there are nesting intervals in $I_L \cup I_O$ or not. First, assume there are no nesting intervals in $I_L \cup I_O$. Among the edges in E' at most three are color-edges by Observation 1. Hence, the other edges must be interval-edges and consequently, every interval in I_O has to intersect at least two intervals in I_L and vice versa. Furthermore, since the intervals in I_L are pairwise non-intersecting, no interval in I_O can intersect all three intervals in I_L. Consequently, every interval in I_L has to intersect two intervals in I_O and every interval in I_O has to intersect two intervals in I_L. An impossibility since no nestings are allowed. Second, assume that there are two intervals nesting and let $u \in E_H$ be the vertex corresponding to the nested interval. But then, u has degree at most two in $H[V']$. This is since the intervals in I_L and I_O are pairwise non-intersecting and by Observation 1 at most one of the edges is a color-edge. $\square$

Since H is planar, we can follow a similar analysis as in [5] using the following lemma. For a set $U \subseteq (O \cup L)$, let $\Gamma(U) \subseteq V_H$ be the set of neighbors of vertices in U.

Lemma 5. *([8]) There are constants c_1, c_2 and c_3, such that for any planar graph $G = (V, E)$ with n vertices, and a parameter r, one can find a set of $X \subseteq V$*

of size at most $\frac{c_1 n}{\sqrt{r}}$*, and a partition of* $V \setminus X$ *into* $\frac{n}{r}$ *sets* $V_1, \ldots, V_{\frac{n}{r}}$*, satisfying: (i)* $|V_i| \leq c_2 r$*, (ii)* $\Gamma(V_i) \cap V_j = \emptyset$*, for* $i \neq j$*, and (iii)* $|\Gamma(V_i) \cap X| \leq c_3\sqrt{r}$*.*

Now, we apply Lemma 5 to H with $r = \frac{1}{\epsilon^2(c_2+c_3)}$. Let $X \subseteq V_H$ be the separator set in Lemma 5 and $V_1, \ldots, V_{\frac{n}{r}}$ be the resulting vertex partition. For each $i \in \{1, \ldots, \frac{n}{r}\}$, let $L_i = V_i \cap L$, $O_i = V_i \cap O$. The following two lemmas equip us with the necessary bounds to show the result.

Lemma 6. *Let* O_i *be obtained from* H *as defined above, then* $|O_i| \leq |\Gamma(O_i)|$ *for every* $i = 1, \ldots, \frac{n}{r}$*.*

Proof. We show $|O_i| \leq |\Gamma(O_i)|$ by contradiction. First, with $r = \frac{1}{\epsilon^2(c_2+c_3)}$ we obtain that $|O_i| \leq |V_i| \leq c_2 r = \frac{c_2}{\epsilon^2(c_2+c_3)} \leq \frac{1}{\epsilon^2}$, and $|\Gamma(O_i)| \leq |V_i| + |\Gamma(V_i) \cap X| \leq c_2 r + c_3\sqrt{r} \leq (c_2+c_3)r = \frac{1}{\epsilon^2}$. Now, $\Gamma(O_i)$ contains exactly the intervals of L, which have the same color as or intersect with intervals in O_i. Thus, the set $L' = (L \cup O_i) \setminus \Gamma(O_i)$ is an independent set and contains each color at most once. If $|O_i| > \Gamma(O_i)$, then $|L'| > |L|$. Moreover, the sizes of O_i and $\Gamma(O_i)$ are both bounded by $\frac{1}{\epsilon^2}$. A contradiction to our assumption that $\mathscr{L}$ is a $\frac{1}{\epsilon^2}$ optimal solution and L is a subset of $\mathscr{L}$. □

Lemma 7. *Let* $H = (L \cup O, E_H)$ *be the graph constructed as above from sets* L *and* O*, then* $|L| \geq \frac{1-O(\epsilon)}{1+O(\epsilon)}|O|$*.*

Proof. Let $X \subseteq V$ be the set guaranteed to exist by Lemma 5 and the O_i for $i = 1, \ldots, \frac{n}{r}$ be constructed as above, then

$$
\begin{aligned}
|O| &\leq \sum_i |O_i| + |X| \\
&\leq \sum_i \Gamma(O_i) + |X| \quad \text{(by Lemma 6)} \\
&\leq \sum_i (|L_i| + |\Gamma(O_i) \cap X|) + |X| \\
&\leq |L| + \sum_i |\Gamma(V_i) \cap X| + |X| \\
&\leq |L| + c_3\sqrt{r} \cdot \frac{|O|+|L|}{r} + c_1 \cdot \frac{|O|+|L|}{\sqrt{r}} \\
&\leq |L| + (c_1+c_3) \cdot \frac{|O|+|L|}{\sqrt{r}} \\
&= |L| + (c_1+c_3)\epsilon\sqrt{c_2+c_3}(|O|+|L|) \\
&= |L| + O(\epsilon)(|O|+|L|).
\end{aligned}
$$

Now, rearranging the final inequality gives us $|L| \geq \frac{1-O(\epsilon)}{1+O(\epsilon)}|O|$ as desired. □

Using Lemma 7 we obtain the following theorem which implies that our algorithm is indeed a PTAS for 1-MCIS on interval graphs.

Theorem 6. *Let $G = (V, E)$ be a vertex-colored interval graph with a k-coloring $\gamma\colon V \to \{1, \dots, k\}$ for some $k \in \mathbb{N}$. For a sufficiently small parameter ϵ, each $\frac{1}{\epsilon^2}$-locally optimal solution $L \subseteq V$ to the 1-MCIS problem on G contains at least $\frac{1-O(\epsilon)}{1+O(\epsilon)} \cdot \mathrm{opt}$ distinct colors, where opt is the number of colors in an optimal solution to the 1-MCIS problem on G.*

Proof. From Lemma 7, $|\mathscr{L}| = |L| + |\mathscr{L} \cap \mathscr{O}| \geq \frac{1-O(\epsilon)}{1+O(\epsilon)}|O| + |\mathscr{L} \cap \mathscr{O}| \geq \frac{1-O(\epsilon)}{1+O(\epsilon)}$ $|\mathscr{O}|$. □

5 Conclusions

In this paper, we have studied the f-Balanced Independent and Dominating set problem for interval graphs. We proved that these problems are NP-complete and obtained algorithmic results for the f-Balanced Independent Set problem. An interesting direction is to obtain algorithmic results for f-Balanced Independent Set problem for other geometric intersection graphs, e.g., rectangle intersection graphs, unit disk graphs etc. Our results may help to tackle these problems since algorithms for computing (maximum weighted) independent sets of geometric objects in the plane often use algorithms for interval graphs as subroutines. Another interesting problem is to design approximation or parameterized algorithm for the f-Balanced Dominating Set problem for interval graphs.

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References

1. Agarwal, P.K., Van Kreveld, M.J., Suri, S.: Label placement by maximum independent set in rectangles. Comput. Geom. Theory Appl. **11**(3–4), 209–218 (1998). https://doi.org/10.1016/S0925-7721(98)00028-5
2. Balas, E., Yu, C.S.: Finding a maximum clique in an arbitrary graph. SIAM J. Comput. **15**(4), 1054–1068 (1986). https://doi.org/10.1137/0215075
3. Banerjee, S., Bhore, S.: Algorithm and hardness results on liar's dominating set and k-tuple dominating set. In: Colbourn, C.J., Grossi, R., Pisanti, N. (eds.) IWOCA 2019. LNCS, vol. 11638, pp. 48–60. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-25005-8_5
4. Been, K., Nöllenburg, M., Poon, S.H., Wolff, A.: Optimizing active ranges for consistent dynamic map labeling. Comput. Geome. Theory Appl. **43**(3), 312–328 (2010). https://doi.org/10.1016/j.comgeo.2009.03.006
5. Chan, T.M., Har-Peled, S.: Approximation algorithms for maximum independent set of pseudo-disks. In: Computational Geometry (SoCG 2009), pp. 333–340. ACM, New York (2009). https://doi.org/10.1145/1542362.1542420
6. Chang, G.J.: Algorithmic aspects of domination in graphs. In: Du, D.Z., Pardalos, P.M. (eds.) Handbook of Combinatorial Optimization, pp. 1811–1877. Springer, Boston (1998). https://doi.org/10.1007/978-1-4613-0303-9_28

7. Chellali, M., Favaron, O., Hansberg, A., Volkmann, L.: k-domination and k-independence in graphs: a survey. Graphs Comb. **28**(1), 1–55 (2012). https://doi.org/10.1007/s00373-011-1040-3
8. Federickson, G.N.: Fast algorithms for shortest paths in planar graphs, with applications. SIAM J. Comput. **16**(6), 1004–1022 (1987). https://doi.org/10.1137/0216064
9. Garnero, V., Sau, I., Thilikos, D.M.: A linear kernel for planar red-blue dominating set. Discrete Appl. Math. **217**, 536–547 (2017). https://doi.org/10.1016/j.dam.2016.09.045
10. Golumbic, M.C.: Algorithmic Graph Theory and Perfect Graphs. Elsevier, Amsterdam (2004)
11. Haunert, J.H., Hermes, T.: Labeling circular focus regions based on a tractable case of maximum weight independent set of rectangles. In: ACM SIGSPATIAL Workshop on Interacting with Maps (MapInteract 2014), pp. 15–21 (2014). https://doi.org/10.1145/2677068.2677069
12. Haynes, T.W., Hedetniemi, S., Slater, P.: Fundamentals of Domination in Graphs. Pure and Applied Mathematics, vol. 208. Dekker (1998)
13. Irving, R.W.: On approximating the minimum independent dominating set. Inf. Process. Lett. **37**(4), 197–200 (1991). https://doi.org/10.1016/0020-0190(91)90188-N
14. van Kreveld, M.J., Strijk, T., Wolff, A.: Point labeling with sliding labels. Comput. Geom. Theory Appl. **13**(1), 21–47 (1999). https://doi.org/10.1016/S0925-7721(99)00005-X
15. Lekkeikerker, C., Boland, J.: Representation of a finite graph by a set of intervals on the real line. Fundam. Math. **51**(1), 45–64 (1962). https://doi.org/10.4064/fm-51-1-45-64
16. Mustafa, N.H., Ray, S.: PTAS for geometric hitting set problems via local search. In: Computational Geometry (SoCG 2009), pp. 17–22. ACM, New York (2009). https://doi.org/10.1145/1542362.1542367
17. Okamoto, Y., Uno, T., Uehara, R.: Counting the number of independent sets in chordal graphs. J. Discrete Algorithms **6**(2), 229–242 (2008). https://doi.org/10.1016/j.jda.2006.07.006
18. Pardalos, P.M., Xue, J.: The maximum clique problem. J. Global Optim. **4**(3), 301–328 (1994). https://doi.org/10.1007/BF01098364
19. Tovey, C.A.: A simplified NP-complete satisfiability problem. Discrete Appl. Math. **8**(1), 85–89 (1984). https://doi.org/10.1016/0166-218X(84)90081-7
20. Van Bevern, R., Mnich, M., Niedermeier, R., Weller, M.: Interval scheduling and colorful independent sets. J. Sched. **18**(5), 449–469 (2015). https://doi.org/10.1007/s10951-014-0398-5
21. Wagner, F., Wolff, A.: A combinatorial framework for map labeling. In: Whitesides, S.H. (ed.) GD 1998. LNCS, vol. 1547, pp. 316–331. Springer, Heidelberg (1998). https://doi.org/10.1007/3-540-37623-2_24
22. Yoshinaka, R.: Higher-order matching in the linear lambda calculus in the absence of constants is NP-complete. In: Giesl, J. (ed.) RTA 2005. LNCS, vol. 3467, pp. 235–249. Springer, Heidelberg (2005). https://doi.org/10.1007/978-3-540-32033-3_18