Finding Large Matchings in 1-Planar Graphs of Minimum Degree 3

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Abstract. A *matching* is a set of edges without common endpoint. It was recently shown that every *1-planar graph* (i.e., a graph that can be drawn in the plane with at most one crossing per edge) that has minimum degree 3 has a matching of size at least $\frac{n+12}{7}$, and this is tight for some graphs. The proof did not come with an algorithm to find the matching more efficiently than a general-purpose maximum-matching algorithm. In this paper, we give such an algorithm. More generally, we show that any matching that has no augmenting paths of length 9 or less has size at least $\frac{n+12}{7}$ in a 1-planar graph with minimum degree 3.

1 Introduction

The *matching problem* (i.e., finding a large set of edges in a graph such that no two chosen edges have a common endpoint) is one of the oldest problem in graph theory and graph algorithms, see for example [3,19] for overviews.

To find a maximum matching in a graph $G = (V, E)$, the fastest algorithm is the one by Hopcroft and Karp if G is bipartite [16], and the one by Micali and Vazirani otherwise ([20], see also [25] for further clarifications). As pointed out in [25], for a graph with n vertices and m edges the run-time of the algorithm by Micali and Vazirani is $O(m\sqrt{n})$ in the RAM model and $O(m\sqrt{n}\alpha(m,n))$ in the pointer model, where $\alpha(\cdot)$ is the inverse Ackerman function. For *planar graphs* (graphs that can be drawn without crossing in the plane) there exists a linear-time approximation scheme for maximum matching [1], and it can easily be generalized to so-called H-minor-free graphs [10] and k-planar graphs [14].

For many graph classes, specialized results concerning matchings and matching algorithms have been found. To name just a few, every bipartite d-regular graph has a *perfect matching* (a matching of size $n/2$) [15] and it can be found in $O(m)$ time [9]. Every 3-regular biconnected graph has a perfect matching [22]

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and it can be found in linear time for planar graphs and in near-linear time for arbitrary graphs [4]. Every graph with a Hamiltonian path has a *near-perfect matching* (of size $\lceil (n-1)/2 \rceil$); this includes for example the 4-connected planar graphs [24] for which the Hamiltonian path (and with it the near-perfect matching) can be found in linear time [8].

For graphs that do not have near-perfect matchings, one possible avenue of exploration is to ask for guarantees on the size of matchings. One of the first results in this direction is due to Nishizeki and Baybars [21], who showed that every planar graph with minimum degree 3[1] has a matching of size at least $\frac{n+4}{3}$. (This bound is tight for some planar graphs with minimum degree 3.) The proof relies on the Tutte-Berge theorem and does not give an algorithm to find such a matching (or at least, none faster than any maximum-matching algorithm). Over 30 years later, a linear-time algorithm to find a matching of this size in planar graphs of minimum degree 3 was finally developed by Franke, Rutter, and Wagner [13]. The latter paper was a major inspiration for our current work.

In recent years, there has been much interest in *near-planar* graphs, i.e., graphs that may be required to have crossings but that are "close" to planar graphs in some sense. We are interested here in *1-planar graphs*, which are those that can be drawn with at most one crossing per edge. (Detailed definitions can be found in Sect. 2.) See a recent annotated bibliography [18] for an overview of many results known for 1-planar graphs. The first author and Wittnebel [6] gave matching-bounds for 1-planar graphs of varying minimum degrees, and showed that any 1-planar graph with minimum degree 3 has a matching of size at least $\frac{n+12}{7}$. (This bound is again tight.)

The proof in [6] is again via the Tutte-Berge theorem and does not give rise to a fast algorithm to find a matching of this size. This is the topic of the current paper. We give an algorithm that finds, for any 1-planar graph with minimum degree 3, a matching of size at least $\frac{n+12}{7}$ in linear time in the RAM model and time $O(n\alpha(n))$ in the pointer-model. The algorithm consists simply of running the algorithm by Micali and Vazirani for a limited number of rounds (and in particular, does not require that a 1-planar drawing of the graph is given). The bulk of the work consists of the analysis, which states that if there are no augmenting paths of length 9 or less, then the matching has the desired size for graphs with minimum degree 3. Along the way, we prove some bounds obtained for graphs with higher minimum degree, though these are not tight.

The paper is structured as follows. After reviewing some background in Sect. 2, we state the algorithm in Sect. 3. The analysis proceeds in multiple steps in Sect. 4. We first delete short flowers from the graph (and account for free vertices in them directly). The remaining graph is basically bipartite, and we can use bounds known for independent sets in 1-planar graphs to obtain matching-bounds that are very close to the desired goal. Closing this gap requires non-trivial modifications; we give a sketch of the involved techniques in Sect. 5 and refer to the full paper for the technical details.

[1] In this paper, 'minimum degree k' stands for 'minimum degree at least k'; of course the bounds also hold if all degrees are higher.

2 Background

We assume familiarity with graphs and graph algorithms, see for example [11,23]. Throughout the paper, G is a simple graph with n vertices and m edges. A *matching* of G is a subset M of its edges without common endpoints; we say that $e = (x, y) \in M$ is *matched* and x and y are *matching-partners*. $V(M)$ denotes the endpoints of edges in M; we call $v \in V(M)$ *matched* and all other vertices *free*. An *alternating walk* of M in G is a walk that alternates between unmatched and matched edges. An *augmenting path* of M in G is an alternating walk that repeats no vertices and begins and ends at a free vertex; we use *k-augmenting path* for an augmenting path with at most k edges. If P is an augmenting path of M (and viewed as an edge-set), then $(M \setminus P) \cup (P \setminus M)$ is also a matching and has one edge more than M.

A *drawing* Γ of a graph consists of assigning points in $\mathbb{R}^2$ to vertices and simple curves to each edge such that curves of edges end at the points of its endpoints. We usually identify the graph-theoretic object (vertex, edge) with the geometric object (point, curve) that it has been assigned to. We only consider *good* drawings (see [23] for details) that avoid degeneracies such as an edge going through the point of a non-incident vertex or two edges intersecting in more than one point. The connected sets of $\mathbb{R}^2 \setminus \Gamma$ are called the *regions* of the drawing.

A *crossing* c of Γ is a pair of two edges (v, w) and (x, y) that have a point in their interior in common. A drawing Γ is called *k-planar* (or *planar* for $k = 0$) if every edge has at most k crossings. A graph is called *k-planar* if it has a k-planar drawing. While planarity can be tested in linear time [7,17], testing 1-planarity is NP-complete [14].

Fix a 1-planar drawing Γ and consider a crossing c between edges (v_0, v_2) and (v_1, v_3). Then we could draw edge (v_i, v_{i+1}) (for $i = 0, \dots, 3$ and addition modulo 4) without crossing by walking "very close" to crossing c. We call the pair (v_i, v_{i+1}) a *potential kite-edge* and note that if we inserted (v_i, v_{i+1}) in the aforementioned manner, then it would be consecutive with the crossing edges in the cyclic orders of edges around v_i and v_{i+1} in Γ.

3 Finding the Matching

Our algorithm to find a large matching is a one-liner: repeatedly extend the matching via 9-augmenting paths (i.e., of length at most 9) until there are no more such paths. Note that the algorithm does not depend on the knowledge that the graph is 1-planar and does not require having a 1-planar drawing at hand. It could be executed on any graph; our contribution is to show (in the next section) that if it is executed on a 1-planar graph G with minimum degree 3 then the resulting matching M has size at least $\frac{n+12}{7}$.

Running Time. Finding a matching M in G such that there is no k-augmenting path can be done in time $O(k|E|)$ in the RAM model using the algorithm by Micali and Vazirani [20]. (We state all run-time bounds here in the RAM model;

for the pointer model add a factor of $\alpha(|E|,|V|)$.) This algorithm runs in phases, each of which has a running time of $O(|E|)$ and increases the length of the minimum-length augmenting path by at least two. See for example the paper by Bast et al. [2] for a more detailed explanation. Since for 1-planar graphs we have $|E| \in O(|V|)$ we get a linear time algorithm in the number of vertices of G to find a matching without 9-augmenting paths.

4 Analysis

Assume that M is a matching without augmenting paths of length at most 9, and let F be the free vertices; $|F| = n - 2|M|$. To analyze the size of M, we proceed in three stages. First we remove some vertices and matching-edges that belong to short flowers (defined below); these are "easy" to account for. Next we split the remaining vertices by their distance (measured along alternating paths) to free vertices. Since short flowers have been removed, no edges can exist between vertices of even small distance; they hence form an independent set. Using a crucial lemma from [6] on the size of independent sets in 1-planar graphs, this shows that $|M| \geq \frac{7}{50}(n+12)$, which is very close to the desired bound of $\frac{n+12}{7}$. The last stage (which does the improvement from $\frac{7}{50}$ to $\frac{1}{7}$) will require non-trivial effort and is done mostly out of academic interest; a sketch is in Sect. 5 and details are in the full paper [5].

Flowers. A *flower*[2] is an alternating walk that begins and ends at the same free vertex; we write *k-flower* for a flower with at most k edges. We only consider 7-flowers; Fig. 1 illustrates all possible such flowers. Note that such short flowers split into a path (called *stem*) and an odd simple cycle (the *blossom*); we call a flower a *cycle-flower* if the stem is empty.

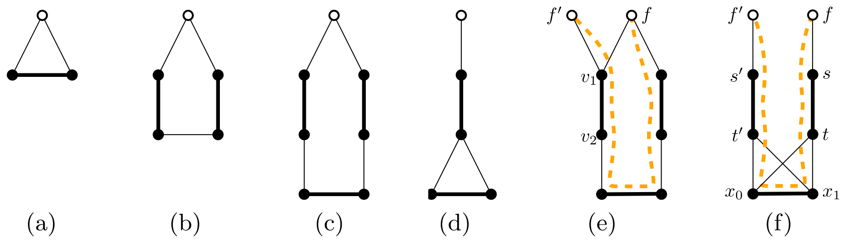

Fig. 1. (a–d) All possible 7-flowers. Free vertices are white, matched edges are thick. (e-f) Augmenting paths found in the proofs of (e) Claim 1 and (f) Claim 2.

Let V_C (the "C" reminds of "cycle") be all vertices that belong to some 7-cycle-flower, let F_C be all free vertices in V_C, and let M_C be all matching-edges within V_C, i.e., all edges with both endpoints in V_C.

[2] Our terminology follows the one in Edmonds' famous blossom-algorithm [12].

Claim 1. *Let M be a matching in a 1-planar graph G with minimum degree 3 such that there is no 9-augmenting path and let F_C and M_C be defined as above. Then $|F_C| \leq |M_C|$.*

Proof. For every $f \in F_C$ there exists some 7-cycle-flower f-v_1-v_2-…-v_k-f with $k \in \{2, 4, 6\}$. Assign f to edge (v_1, v_2). We claim that f is the only vertex in F_C assigned to (v_1, v_2), otherwise there would be an augmenting path of length less than 9. Since $(v_1, v_2) \in M_C$, this then proves the claim. So assume for contradiction that another vertex $f' \in F_C$ was also assigned to (v_1, v_2). Then f' is adjacent to one of v_1, v_2. If it is v_2, then f'-v_2-v_1-f is a 3-augmenting path. If it is v_1, then f'-v_1-…-v_k-f is a 7-augmenting path, see Fig. 1(e). □

From now on we will only study the graph $G \setminus V_C$. Observe that M restricted to this graph is again a matching without augmenting paths up to length 9. All following definitions are only for vertices and edges in $G \setminus V_C$. Let F_B (the "B" reminds of "blossom") be all those free vertices f that are not in F_C and that belong to a 7-flower. By $f \notin F_C$ this flower has a non-empty stem, which is possible only if its length is exactly 7 and the stem has two edges f-s-t while the blossom is a 3-cycle t-x_0-x_1-t. Furthermore (s, t) and (x_0, x_1) are matching-edges. Let M_B be the set of such matching-edges (x_0, x_1) i.e., matching-edges that belong to the blossom of such a 7-flower. We do *not* include the matching-edge (s, t) in M_B (unless it belongs to a different 7-flower where it is in the blossom). Let T_B be the set of such vertices t, i.e., vertices that belong to a 7-flower and belong to both the stem and the blossom. Set $V_B = T_B \cup V(M_B)$ (see also Fig. 2).

Claim 2. *Let M be a matching in a 1-planar graph G with minimum degree 3 such that there is no 9-augmenting path and let T_B and M_B be defined as above. Then $|T_B| \leq |M_B|$.*

Proof. We argue similarly to the proof of Claim 1, i.e., assign each $t \in T_B$ to an edge in M_B and argue that no two vertices are assigned to the same edge unless there is a 9-augmenting path. Choose for each $t \in T_B$ a matching-edge $(x_0, x_1) \in M_B$ that is within the same blossom of some 7-flower of $G \setminus V_C$. Assume for contradiction that some other vertex $t' \in T_B$ is also assigned to (x_0, x_1). Let t-s-f and t'-s'-f' be the stems of the 7-flowers containing t and t', and note that $s \neq s'$ since they are matching-partners of $t \neq t'$. This gives an alternating path f-s-t-x_0-x_1-t'-s'-f', see Fig. 1(f). Depending on whether $f = f'$ this is a 7-augmenting path or 7-cycle-flower; the former contradicts the choice of M and the latter that $x_0, x_1 \in G \setminus V_C$. □

The Auxiliary Graph H. For any vertex $v \in G \setminus V_C \setminus V_B$, let the *distance to a free vertex* be the number of edges in a shortest alternating path from a free vertex to v. Let D_k be the vertices of distance k to a free vertex. Since there are no 9-agumenting paths, one can easily see:

Observation 1. *In graph $G \setminus V_C \setminus V_B$, there are no matching-edges within D_k for $k = 1$ and $k = 3$, and no edges at all within D_k for $k = 0$ and $k = 2$.*

Proof. If there was such an edge (v, v'), then it, together with the alternating paths of length k that lead from free vertices to v, v', form a 7-augmenting path or a 7-flower. □

From now on, we will only study the subgraph H induced by $D_0 \cup \cdots \cup D_3$, noting again that this does *not* include the vertices in $V_C \cup V_B$. For ease of referring to them, we rename the vertices of H as follows (see also Fig. 2):

- $F_H = F \setminus F_C = D_0$ are the free vertices in H.
- $S = D_1$ are the vertices in H that are adjacent to F_H.
- $T_H = D_2$ are the vertices in H that have matching-partners in S and are not in S.
- $U = D_3$ are the vertices in H that are adjacent to T_H and not in $F \cup S \cup T_H$.

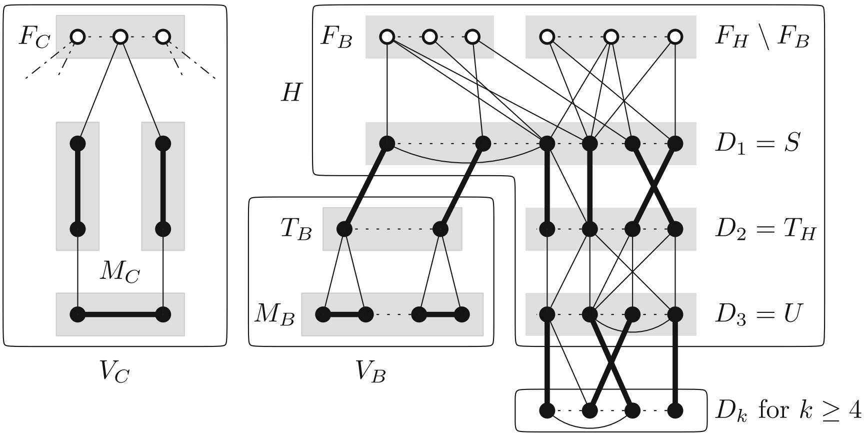

Fig. 2. Illustration of the partitioning of edges and vertices and graph H.

The following shortcuts will be convenient. For any vertex sets A, B, an *A-vertex* is a vertex in A, and an *AB-edge* is an edge between an A-vertex and a B-vertex. For any vertex v an *A-neighbour* is a neighbour of v in A. Using Observation 1 and the definition of V_C (which includes the *entire* flower) and V_B (which includes *both* ends of the matching-edge) one easily verifies the following:

Observation 2. – *There are no matching-edges within S or within U.*
- *There are no edges within F_H or within T_H.*
- *The matching-partner of an S-vertex is in $T_H \cup T_B$.*
- *The matching-partner of a U-vertex is not in H.*
- *All neighbours of an F_H-vertex belong to S or are not in H.*
- *All neighbours of a T_H-vertex belong to $S \cup U$ or are not in H.*

Let M_S be the set of matching-edges incident to S. Let M_U be the matching-edges incident to U. Since there are no matching-edges within S or U, we have $|S| = |M_S|$ and $|U| = |M_U|$.

We stated earlier that any neighbour of F_H is either in S or not in H. The latter is actually impossible (though this is non-trivial), and likewise for T_H.

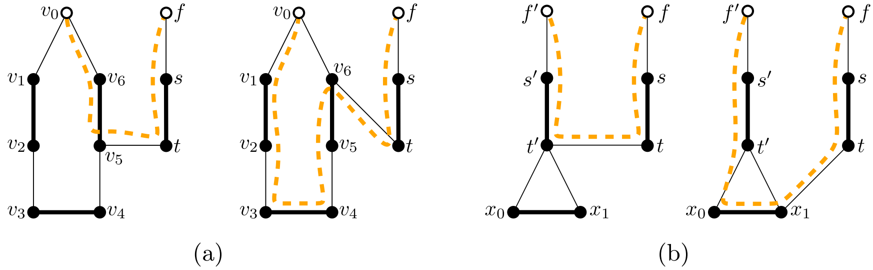

Fig. 3. Augmenting paths found in the proofs of (a) Lemma 1, $t \in T_H$ has a neighbour in V_C. (b) Lemma 1, $t \in T_H$ has a neighbour in V_B.

Lemma 1. *No vertex in $F_H \cup T_H$ has a neighbour in G that is outside H.*

Proof. First observe that no edge can connect a vertex in $F_H \cup T_H = D_0 \cup D_2$ with a vertex $z \in D_k$ for $k \geq 4$ since z would have been added to $D_1 = S$ or $D_3 = U$ instead. So we must only show that no vertex in $F_H \cup T_H$ has a neighbour in $V_C \cup V_B$. We show this only for $t \in T_H$; the proof is similar (and even easier) for $f \in F_H$ by replacing the path t-s-f defined below with just f.

Consider Fig. 3(a). Fix some $t \in T_H$, let $s \in S$ be its matching-partner and let $f \in F_H$ be an arbitrary free vertex incident to s. Assume for contradiction that t has a neighbour v_i in V_C, so v_i belongs to some 7-cycle-flower v_0-v_1-…-v_k-v_0 where $k \in \{2, 4, 6\}$ and $v_0 \in F$. Note that $v_0 \neq f$ since $v_0 \in F_C$ while $f \in F_H$. If i is odd then f-s-t-v_i-…-v_k-v_0 is a 9-augmenting path, and if i is even then f-s-t-v_i-v_{i-1}-…-v_1-v_0 is a 9-augmenting path; both are impossible.

Now consider some $(x_0, x_1) \in M_B$ that belongs to a 7-flower f'-s'-t'-x_0-x_1-t'-s'-f' where (s', t') is a matching-edge and $t' \in T_B$. Note that $t' \neq t$ (hence $s' \neq s$) since $t' \in T_B$ while $t \in T_H$. If t and t' are adjacent, then f-s-t-t'-s'-f' is a 5-augmenting path or a 5-cycle-flower. If t and x_i are adjacent for $i \in \{0, 1\}$, then f-s-t-x_i-x_{1-i}-t'-s'-f' is a 7-augmenting path or 7-cycle-flower. See Fig. 3(b). Both are impossible since $t \notin T_C$. □

In particular, if a vertex in $F_H \cup T_H$ had degree d in G, then it also has degree d in H; this will be important below.

Minimum Degree 3. With this, we can prove our first matching-bound. We need the following lemma by Biedl and Wittnebel, which is derived via (quite complicated) graph-augmentation and edge-counting:

Lemma 2 ([6]). *Let G be a simple 1-planar graph. Let A be a non-empty independent set of G where all vertices in A have degree 3 or more in G. Let A_d be the vertices of degree d in A. Then $2|A_3| + \sum_{d>3}(3d-6)|A_d| \leq 12|V \setminus A| - 24$.*

Lemma 3. *We have (i) $|F_H| \leq 6|S| - 12$ and (ii) $|F_H| + |T_H| \leq 6|S| + 6|U| - 12$.*

Proof. Consider first the subgraph of H induced by F_H and S. By Observation 2 and Lemma 1 any vertex in F_H has degree at least 3 in this subgraph, and they form an independent set. Consider the inequality of Lemma 2. Any vertex in F_H contributes at least 2 units to the left-hand side while the right-hand side is $12|S| - 24$. This proves Claim (i) after dividing.

Now consider the full graph H. By Observation 2 and Lemma 1 any vertex in $F_H \cup T_H$ has degree at least 3 in H, and they form an independent set. Claim (ii) now follows from Lemma 2 as above. □

Corollary 1. *If the minimum degree is 3, then $|M| \geq \frac{7}{50}(n + 12)$.*

Proof. Adding Lemma 3(ii) six times to Lemma 3(i) gives

$$7\,|F_H| + 6|T_H| \leq 42\,|S| + 36|U| - 84 \leq 42|M_S| + 36|M_U| - 84.$$

Adding Claim 1 seven times and Claim 2 six times gives

$$7\,|F_C| + 7\,|F_H| + 6|T_B| + 6|T_H| \leq 42|M_S| + 36|M_U| + 7|M_C| + 6|M_B| - 84.$$

Since $|S| = |M_S| = |T_H| + |T_B|$, this simplifies to

$$7\,|F| = 7\,|F_H| + 7\,|F_C| \leq 36|M_S| + 36|M_U| + 7|M_C| + 6|M_B| - 84 \leq 36|M| - 84.$$

Therefore $2|M| = n - |F| \geq n + 12 - \frac{36}{7}|M|$ which gives the bound after rearranging. □

It is worth pointing out that this result (as well as Theorem 2 below) does not use 1-planarity of the graph except when using the bound in Lemma 2. Hence, similar bounds could be proved for any graph class where the size of independent sets can be upper-bounded relative to its minimum degree.

Doing the improvement from $\frac{7}{50}$ to $\frac{1}{7}$ will be done by improving Lemma 3(ii) slightly. We will show the following in Sect. 5:

Lemma 4. *$|F_H| + |T_H| \leq 6|S| + 5|U| - 12$.*

This then gives our main result:

Theorem 1. *Let G be a 1-planar graph with minimum degree 3, and let M be a matching in G that has no augmenting path of length 9 or less. Then $|M| \geq \frac{n+12}{7}$.*

Proof. Using $|S| = |M_S|$ and $|U| = |M_U|$ we have

$$\begin{aligned}
|F_H| + |T_H| &\leq 6|M_S| + 5|M_U| - 12 && \text{from Lemma 4}\\
|F_C| &\leq |M_C| && \text{from Claim 1}\\
|T_B| &\leq |M_B| && \text{from Claim 2.}
\end{aligned}$$

Since $|T_H| + |T_B| = |M_S|$ this gives $|F| + |M_S| \leq |M_C| + |M_B| + 6|M_S| + 5|M_U| - 12$, therefore $|F| \leq 5|M| - 12$. This implies $2|M| = n - |F| \geq n - 5|M| + 12$ or $7|M| \geq n + 12$. □

Higher Minimum Degree. Since the bound for independent sets in 1-planar graphs gets smaller when the minimum degree is larger, we can prove better matching-bounds for higher minimum degree. The following is proved exactly like Lemma 3:

Lemma 5. *If the minimum degree is $\delta > 3$, then*

$$(i)\ |F_H| \le \tfrac{4}{\delta-2}(|S|-2) \quad \textit{and} \quad (ii)\ |F_H| + |T_H| \le \tfrac{4}{\delta-2}(|S|+|U|-2).$$

Theorem 2. *Let G be a 1-planar graph with minimum degree δ. Let M be any matching in G without 9-augmenting path. Then*

- $|M| \ge \frac{3}{10}(n+12)$ *for* $\delta = 4$,
- $|M| \ge \frac{1}{3}(n+12)$ *for* $\delta \ge 5$.

Proof. Set $c = \frac{4}{\delta-2}$, so $|F_H| \le c(|S|-12)$ and $|F_H|+|T_H| \le c(|S|+|U|-12)$. Taking the former inequality once and adding the latter one c times gives

$$(c{+}1)|F_H| + c|T_H| \le (c^2{+}c)|S| + c^2|U| - (c{+}1)12 = (c^2{+}c)|M_S| + c^2|M_U| - (c{+}1)12.$$

Adding Claim 1 $c+1$ times and Claim 2 c times gives

$$\begin{aligned}(c{+}1)(|F_C|{+}|F_H|) + c(|T_B|{+}|T_H|)\\ \le (c^2{+}c)|M_S| + c^2|M_U| + (c{+}1)|M_C| + c|M_B| - (c{+}1)12. \qquad (1)\end{aligned}$$

For $\delta = 4$ we have $c = 2$, and with $|T_B| + |T_H| = |M_S|$ Eq. 1 simplifies to

$$3\,|F| \le 4|M_S| + 4|M_U| + 3|M_C| + 2|M_B| - 36 \le 4|M| - 36.$$

Therefore $2|M| = n - |F| \ge n + 12 - \frac{4}{3}|M|$. For $\delta \ge 5$ we have $c^2 < c+1$ and so can only simplify Eq. 1 to $(c+1)(|F_C| + |F_H|) \le (c+1)|M| - (c+1)12$, hence $2|M| = n - |F| \ge n + 12 - |M|$. The bounds follow after rearranging. ◻

For $\delta = 4, 5$ these are close to the bounds of $\frac{1}{3}(n{+}4)$ (for $\delta = 4$) and $\frac{1}{5}(2n{+}3)$ (for $\delta = 5$) that we know to be the tight lower bounds on the maximum matching size [6]. Unfortunately we do not know how to improve Theorem 2 for $\delta > 3$; the techniques of Sect. 5 do not work for higher minimum degree.

Stopping Earlier? Currently we remove all augmenting paths up to length 9. Naturally one wonders whether one could stop earlier? We can show that it suffices to remove only 7-augmenting paths by inspecting the analysis. The details are not difficult but tedious and require even more notation; we omit them.

On the other hand, it is not enough to remove only 3-augmenting paths. Figure 4 shows a matching in a 1-planar graph that has no 3-augmenting paths, but only size $\frac{n+12}{8}$. We can show that this is tight.

Theorem 3. *Let G be a 1-planar graph with minimum degree 3 and let M be a matching without 3-augmenting paths. Then $|M| \ge \frac{n+12}{8}$.*

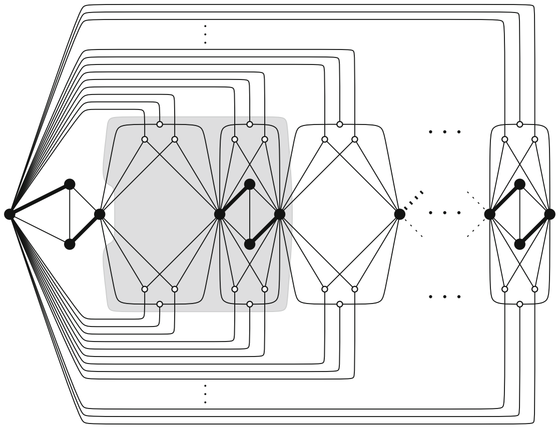

Fig. 4. A graph with a matching marked in thick edges of size $\frac{n+12}{8}$. No 3-augmenting path exists for the chosen matching, but there are 5-augmenting paths. The gray area marks an example of 16 vertices such that only 2 matching edges exist. Repeating this configuration gives the example for arbitrary n.

Proof. The proof is very similar to the one of Theorem 2 in [13] except that we use Lemma 2 rather than the edge-bound for planar bipartite graphs. We repeat it here for completeness, mimicking their notation. Let M_c be all those matching-edges (x, y) for which some free vertex $f \in F$ is adjacent to both x and y, and let F_c be all such free vertices. Vertex f is necessarily the only F-neighbour of x and y, else there would be a 3-augmenting path. Hence $|F_c| \leq |M_c|$.

Let M_o and F_o be the remaining matching-edges and free vertices. For each edge (x, y) in M_o, at most one of the ends can have F-neighbours, else (x, y) would be in M_c or there would be a 3-augmenting path. Let S be the ends of edges in M_o that have F-neighbours, and let G' be the auxiliary graph induced by F_o and S. Then $|F_o| \leq 6|S| - 12 \leq 6|M_o| - 12$ by Lemma 2.

Putting both together, $2|M| = n-|F| \geq n+12-|M_c|-6|M_o| \geq n+12-6|M|$ and the bound follows after rearranging. $\square$

5 Proof of Lemma 4

(Sketch; details are in the full paper [5].) Fix an arbitrary 1-planar drawing of H. We obtain a 1-planar drawing H^+ from H by inserting any potential kite-edge (t, x) with $t \in T_H$ and $x \in S \cup U$ that does not exist yet. If (t, x) exists, but has a crossing, then re-route it to become *uncrossed* (i.e., without crossing).

We split T_H-vertices and assign them as follows. If $t \in T_H$ has an uncrossed edge to a U-neighbour u, then assign t to u. Else, if t has three or more S-neighbours, then add t to a vertex set T_σ. Else assign t to an arbitrary U-neighbour u. In the first and third case we call (t, u) the *assignment-edge*. Let $U^{(d)}$ be the set of all those vertices $u \in U$ that have d incident assignment-edges.

Let $T^{(d)}$ be all those vertices in $T_H \setminus T_\sigma$ that have been assigned to a vertex in $U^{(d)}$. Since $|T^{(d)}| = d|U^{(d)}|$, we have:

Observation 3. $|T_0| = 0$ *and* $\sum_{d=1}^{5} |T^{(d)}| \leq 5 \sum_{d=0}^{5} |U^{(d)}|$.

Transform drawing H^+ as follows:

- Delete all vertices in $U^{(0)} \cup \cdots \cup U^{(5)}$ and $T^{(1)} \cup \cdots \cup T^{(5)}$ and all SU-edges.
- For any remaining $t \in T_H$, delete all edges to U-neighbours except the assignment-edge (if any).
- While there exists a vertex $t \in T_H \setminus T_\sigma$ for which either the assignment-edge (t, u) or the matching-edge (s, t) is uncrossed: Delete t and insert edge (s, u). Normally (s, u) is routed along the path s-t-u, which has at most one crossing. But if this leads to a crossing of (s, u) with an edge that ends at s or u, then instead draw (s, u) as a kite-edge of that crossing so that the drawing remains good.

For this proof sketch, let us assume that *all* vertices in $T_H \setminus T_\sigma$ get deleted. (This is not always the case, and those "remaining" vertices of T_H are a major difficulty to overcome; see [5].)

Assuming this to be the case, we have in the resulting drawing J the independent set $F_H \cup T_\sigma \cup \bigcup_{d \geq 6} U^{(d)}$ and the vertices of S_H. All vertices in $F_H \cup T_\sigma$ have degree at least 3. Vertex $u \in U^{(d)}$ (for $d \geq 6$) has degree at least d in J, because it was assigned to d T_H-vertices and therefore inherits edges to their d distinct matching-partners. Lemma 4 now holds by applying Lemma 2 to drawing J and combining it with Observation 3 as follows:

$$\begin{aligned} 12|S_H| - 24 &\geq 2|F_H| + 2|T_\sigma| + \sum_{d\geq 6}(3d-6)|U^{(d)}| \geq 2|F_H| + 2|T_\sigma| + \sum_{d \geq 6}(2d-10)|U^{(d)}| \\ &\geq 2|F_H| + 2|T_\sigma| + 2\sum_{d\geq 6}|T^{(d)}| - 10\sum_{d\geq 6}|U^{(d)}| + 2\sum_{d\leq 5}|T^{(d)}| - 10\sum_{d\leq 5}|U^{(d)}| \\ &\geq 2|F_H| + 2|T_H| - 10|U| \end{aligned}$$

and hence $|F_H| + |T_H| \leq 6|S_H| + 5|U| - 12$.

6 Summary and Outlook

In this paper, we considered how to find a large matching in a 1-planar graph with minimum degree 3. We argued that any matching without augmenting paths of length up to 9 has size at least $\frac{n+12}{7}$, which is also the largest matching one can guarantee to exist in any 1-planar graph with minimum degree 3. Such a matching can easily be found in linear time, even if no 1-planar drawings is known, by stopping the matching algorithm by Micali and Vazirani after a constant number of rounds.

It remains open how to find large matchings in 1-planar graphs with minimum degree $\delta > 3$ that match the upper bounds. It would also be interesting to study other near-planar graph classes such as k-planar graphs (for $k > 1$); here we do not even know what tight matching-bounds exist and much less how to find matchings of that size in linear time.

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