Stability of Phase Retrieval Problem

Palina Salanevich
University of California, LA
Email: psalanevich@math.ucla.edu.

***Abstract*—Phase retrieval is a non-convex inverse problem of signal reconstruction from intensity measurements with respect to a measurement frame. One of the main problems in phase retrieval is to determine for which frames the associated phaseless measurement map is injective and stable. In this paper we address the question of stability of phase retrieval for two classes of random measurement maps, namely, frames with independent frame vectors satisfying bounded fourth moment assumption and frames with no independence assumptions. We propose a new method based on the frame order statistics, which can be used to establish stability of the measurement maps for other classes of frames.**

I. Introduction

The phase retrieval problem arises naturally in many applications within a variety of fields in science and engineering, where the only available information about a signal of interest is the set of magnitudes of its frame coefficients with respect to a measurement frame. Among such applications are optics [19], astronomical imaging [12], quantum mechanics [9], and speech recognition [3].

Phase retrieval problem can be formulated as follows. Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame, that is, a (possibly over-complete) spanning set of $\mathbb{C}^M$. We consider the measurement map $\mathcal{A}_\Phi : \mathbb{C}^M \to \mathbb{R}^N$ defined by $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$. For a given vector of measurements $b \in \mathbb{R}^N$, we address the following non-convex inverse problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & \mathcal{A}_\Phi(x) = b. \end{array}$$

The phase retrieval problem can be also formulated in the real case, when $\Phi \subset \mathbb{R}^M$ and $x \in \mathbb{R}^M$.

Since $\mathcal{A}_\Phi(x) = \mathcal{A}_\Phi(e^{i\theta}x)$ for any $\theta \in [0, 2\pi)$, the initial signal x can be reconstructed in the best case only up to a global phase factor. Thus, we identify each $x \in \mathbb{C}^M$ with its up-to-a-global-phase equivalence class and consider the measurement map $\mathcal{A}_\Phi$ to be defined on the set of equivalence classes $\mathbb{C}^M/_\sim$ in the sequel.

Not every frame has the injective associated measurement map, so reconstruction is not always possible. One of the main research directions in phase retrieval therefore is to determine when is the map $\mathcal{A}_\Phi$ injective. In the case when phaseless measurements can be corrupted by noise, injectivity is not enough to guarantee accurate reconstruction of a signal, and the measurement map has to satisfy some stronger assumptions. More precisely, we want to ensure that, if for two signals x and y the measurements $\mathcal{A}_\Phi(x)$ and $\mathcal{A}_\Phi(y)$ are close, then x and y are also close up to a global phase factor. This leads to the notion of *stability* of the measurement map.

The rest of this paper is organized as follows. In Section II, we give an overview of the state of art results on injectivity and stability of phaseless measurement maps. In Section III, we introduce the notion of frame order statistics and discuss their connection to stability of the measurement maps. Using this connection, we analyze measurement map stability for two classes of random frames. Namely, in Section III-A we show stability for a more general compared to the previously known results class of random frames with independent frame vectors having a bounded fourth moment, and in Section III-B we discuss stability for frames with frame vectors that are not assumed to be independent. In this paper, we focus on the complex case phase retrieval, but similar results can be obtained in the real case as well, for the adjusted notion of stability.

II. Injectivity and stability of phase retrieval

In the investigation of the injectivity of phaseless measurement maps, the question about the minimal number of measurements required receives the most attention. In the real case, when we consider the restriction of $\mathcal{A}$ to $\mathbb{R}^M$, the following result is shown by Balan, Casazza, and Edidin [3].

Theorem II.1. [3] *For any dimension M and a frame $\Phi \subset \mathbb{R}^M$, the following holds for the measurement map $\mathcal{A}_\Phi : \mathbb{R}^{\mathbb{M}} \to \mathbb{R}^{|\Phi|}$ give by $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi\rangle|^2\}_{\varphi\in\Phi}$*

(i) *If $|\Phi| < 2M - 1$, then $\mathcal{A}_\Phi$ is not injective.*
(ii) *If $N \geq 2M - 1$, then $\mathcal{A}_\Phi$ is injective for a generic Φ with $|\Phi| = N$.*

For the complex case, no similar result is known to the date. The following conjecture has been proposed by Bandeira, Cahill, Mixon, and Nelson in 2014 [4].

Conjecture II.2. (The 4M-4 Conjecture.) *For any $M \geq 2$, consider a frame $\Phi \subset \mathbb{C}^M$. Then the following holds:*

(i) *If $|\Phi| < 4M - 4$, then $\mathcal{A}_\Phi$ is not injective.*
(ii) *If $N \geq 4M - 4$, then $\mathcal{A}_\Phi$ is injective for a generic Φ with $|\Phi| = N$.*

Over the last decade the following progress has been achieved on this conjecture.

- In 2006 Balan, Casazza, and Edidin showed that if $N \geq 4M-2$ then $\mathcal{A}_\Phi$ is injective for a generic Φ [3].
- In 2011 Heinosaari, Mazzarella, and Wolf showed that if $|\Phi| < (4+o(1))M$, then $\mathcal{A}_\Phi$ is not injective [13].
- In 2014 several examples of frames with cardinality $4M-4$ and injective measurement maps were constructed [5], [11].
- Later in 2014 Conca, Edidin, Hering, and Vinzant and, independently, Király and Ehler showed that if $N \geq 4M-4$, then $\mathcal{A}_\Phi$ is injective for a generic Φ with $|\Phi| = N$, which proves part (ii) of the conjecture [8] [15].
- In 2015 Vinzant disproved part (i) of the conjecture for $M = 4$. She constructed a frame with 11 vectors and showed the injectivity of $\mathcal{A}$ for this frame [22].

The example of an injective measurement frame proposed in [22] is not unique. In fact, this paper shows that the set of injective frames is of full dimension in $\mathbb{C}^{4\times 11}$. Even though this certainly disproves part (i) of the $4M-4$ Conjecture in the case $M=4$, Vinzant conjectured that it is asymptotically true in the following probabilistic sense.

Conjecture II.3. (Vinzant's Refined Injectivity Conjecture.) *Draw Φ uniformly from the Grassmannian of M-dimensional subspaces of $\mathbb{C}^{4M-5}$. Let p_M denote the probability that the measurement map $\mathcal{A}_\Phi$ is injective. Then $p_M < 1$ for all M, and $\lim\limits_{M\to\infty} p_M = 0$.*

While part (ii) of the $4M-4$ Conjecture, proven by Conca, Edidin et. al. (and, independently, by Király and Ehler), guarantees that for a randomly selected frame Φ with $|\Phi| \geq 4M-4$ the measurement map is injective with probability 1, it does not provide any method to check whether the measurement map of a concrete frame is injective. Since in practice the particular structure of the frame is often dictated by the application considered, it is also important to study injectivity of $\mathcal{A}$ for some particular classes of frames.

As such, the injectivity property of the full Gabor frames where studied by Bojarovska and Flinth [6]. In particular, they found the following easily checkable sufficient condition for injectivity.

Theorem II.4. [6] *Let $g \in \mathbb{C}^M$ be a window, such that for any $\lambda \in \mathbb{Z}_M \times \mathbb{Z}_M$, $\langle g, \pi(\lambda) g\rangle \neq 0$. Then the measurement map $\mathcal{A}_{(g,\mathbb{Z}_M\times\mathbb{Z}_M)}$, corresponding to the full Gabor frame $(g, \mathbb{Z}_M \times \mathbb{Z}_M)$, is injective.*

Remark II.5. Note, that the number of measurements considered in Theorem II.4 is $|(g, \mathbb{Z}_M \times \mathbb{Z}_M)| = M^2$. Finding a condition for injectivity of phaseless measurements with respect to a Gabor frame (g, Λ) with $|\Lambda| < M^2$ remains an important open question.

Another important research task in phase retrieval is to find conditions on the measurement frame Φ to ensure *stable uniqueness* of the reconstruction of a signal x from its phaseless measurements $\mathcal{A}_\Phi(x)$. Eldar and Mendelson proposed the following notion of phaseless measurement map stability [10].

Definition II.6. Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ be a frame. The measurement map $\mathcal{A}_\Phi : \mathbb{C}^M \to \mathbb{R}^N$ given by $\mathcal{A}_\Phi(x) = \{|\langle x, \varphi_j\rangle|^2\}_{j=1}^N$ is called *stable with a constant C in a set $T \subset \mathbb{C}^M$* if for every $x, y \in T$,

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq C \min_{\theta\in[0,2\pi)} ||x - e^{i\theta}y||_2 ||x + e^{i\theta}y||_2.$$

Note that stability in a set is a much stronger property than injectivity up to a global phase factor, as it provides a quantitative bound on how $\mathcal{A}_\Phi(x)$ and $\mathcal{A}_\Phi(y)$ differ for different (up to a global phase) x and y.

To date, the following is known about stability:

- In 2014, Eldar and Mendelson showed that for a frame Φ of cardinality $O(M)$, such that $\varphi_j(m)$ are independent L-subgaussian random variables, the mapping $\mathcal{A}_\Phi$ is stable in $\mathbb{C}^M$ under the additional *small ball assumption* on the distribution of $\varphi_j(m)$ [10].
- In 2016, Krahmer and Liu showed that the small ball assumption can be dropped to show stability in the set of μ-flat vectors $T_\mu = \{x \in \mathbb{R}^M, ||x||_\infty \leq \mu ||x||_2\}$ [16].
- In 2016, Kabanava, Kueng, Rauhut, and Terstiege showed stability of a measurement map $\mathcal{A}_\Phi$ when frame vectors are independently uniformly sampled from Gaussian distribution or from an approximate 4-design [14].
- In 2016, Kueng, Zhu, and Gross showed stability of a measurement map $\mathcal{A}_\Phi$ when when frame vectors are independently uniformly sampled from Clifford orbit [17].

The question of stability is also discussed in [2, Lemma 3.2]. We propose a new method based on the frame order statistics, which can be used to establish stability of the measurement maps for other classes of frames, including frame with correlated frame vectors.

III. Stability of phase retrieval using frame order statistics

If frame vectors are well spread in $\mathbb{C}^M$, one should expect that, for each one-dimensional subspace of $\mathbb{C}^M$, there are not too many frame vectors that are almost colinear or almost orthogonal to it. To formalize this idea we introduce frame order statistics [21]. Here and in the sequel, $\mathbb{S}^{M-1} = \{x \in \mathbb{C}^M, \text{ s.t. } ||x||_2 = 1\}$ denotes the unit sphere.

Definition III.1. Let $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{S}^{M-1}$ be a unit norm frame and consider a vector $x \in \mathbb{S}^{M-1}$.

(i) For $\alpha \leq N$, the *α-smallest frame order statistics* of Φ is given by

$$\mathcal{S}_{FOS}(\Phi, \alpha, x) = \max_{\substack{J \subseteq \{1,\dots,N\},\\ |J| \geq \alpha}} \min_{j \in J} |\langle x, \varphi_j \rangle|.$$

(ii) For $\beta \leq N$, the *β-largest frame order statistics* of Φ is given by

$$\mathcal{L}_{FOS}(\Phi, \beta, x) = \min_{\substack{J \subseteq \{1,\dots,N\},\\ |J| \geq \beta}} \max_{j \in J} |\langle x, \varphi_j \rangle|.$$

As follows from the definition, if we delete $\lfloor N - \alpha \rfloor$ smallest and $\lfloor N - \beta \rfloor$ largest in modulus frame coefficients, then the remaining ones satisfy

$$\mathcal{S}_{FOS}(\Phi, \alpha, x) \leq |\langle x, \varphi_j \rangle| \leq \mathcal{L}_{FOS}(\Phi, \beta, x).$$

The study of frame order statistics is not only of interest in frame theory, but it also plays an important role in various areas of signal processing, such as phase retrieval [1], [20] and quantization [18], [7]. The following result shows that a frame with bounded frame order statistic has an injective associated measurement map.

Theorem III.2. *Let $\Phi \subset \mathbb{C}^M$ be a frame. Suppose that, for each fixed $\alpha < 1 - \frac{1}{2C_0}$, there exist constant $c \in (0,1)$, such that*

$$\min_{x \in \mathbb{S}^{M-1}} \mathcal{S}_{FOS}(\Phi, \alpha N) \geq \frac{c}{\sqrt{M}}.$$

Then there exists a constant $L > 0$, such that the phaseless measurement map $\mathcal{A}_\Phi$ is stable with constant $C \geq L\frac{|\Phi|}{M}$ in $\mathbb{C}^M$. That is, for any $x, y \in \mathbb{C}^M$,

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq L\frac{|\Phi|}{M} \min_{\theta \in [0, 2\pi)} ||x - e^{i\theta}y||_2 ||x + e^{i\theta}y||_2.$$

Proof. Let $|\Phi| = N$ and $\Phi = \{\varphi_i\}_{i=1}^N$. For any $x, y \in \mathbb{C}^M$, we have

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 = \sum_{i=1}^N ||\langle x, \varphi_i \rangle|^2 - |\langle y, \varphi_i \rangle|^2|$$
$$= \sum_{i=1}^N |\left(|\langle x, \varphi_i \rangle| - |\langle y, \varphi_i \rangle|\right)\left(|\langle x, \varphi_i \rangle| + |\langle y, \varphi_i \rangle|\right)|.$$

Let $\theta_x, \theta_y \in [0, 2\pi)$ be such that

$$\begin{aligned}
&|\left(|\langle x, \varphi_i \rangle| - |\langle y, \varphi_i \rangle|\right)\left(|\langle x, \varphi_i \rangle| + |\langle y, \varphi_i \rangle|\right)| \\
&= |e^{i\theta_x}\langle x, \varphi_i \rangle - e^{i\theta_y}\langle y, \varphi_i \rangle||e^{i\theta_x}\langle x, \varphi_i \rangle + e^{i\theta_y}\langle y, \varphi_i \rangle| \\
&= |\langle x - e^{i(\theta_y - \theta_x)}y, \varphi_i \rangle||\langle x + e^{i(\theta_y - \theta_x)}y, \varphi_i \rangle|.
\end{aligned}$$

Then we have

$$\begin{aligned}
&||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \\
&= ||x - e^{i(\theta_y - \theta_x)}y||_2 ||x + e^{i(\theta_y - \theta_x)}y||_2 \\
&\cdot \sum_{i=1}^N \left|\left\langle \tfrac{x - e^{i(\theta_y - \theta_x)}y}{||x - e^{i(\theta_y - \theta_x)}y||_2}, \varphi_i \right\rangle\right| \left|\left\langle \tfrac{x + e^{i(\theta_y - \theta_x)}y}{||x + e^{i(\theta_y - \theta_x)}y||_2}, \varphi_i \right\rangle\right|.
\end{aligned}$$

Let us fix some $\frac{1}{2} < \alpha < 1 - \frac{1}{2C_0}$. Then assumptions of the theorem imply that there exist constants c, such that, for every unit norm vector $u \in \mathbb{S}_\mathbb{R}^{M-1}$, there exists a set of indices $J_u \subset \{1, \dots, N\}$ with $|J_u| \geq \alpha N$ and $|\langle u, \varphi_j \rangle| \geq \frac{c}{\sqrt{M}}$ for all $j \in J_u$.

In particular, for unit vectors $u = \frac{x - e^{i(\theta_y - \theta_x)}y}{||x - e^{i(\theta_y - \theta_x)}y||_2}$ and $v = \frac{x + e^{i(\theta_y - \theta_x)}y}{||x + e^{i(\theta_y - \theta_x)}y||_2}$, there exist $J_u, J_v \subset \{1, \dots, N\}$, such that $|J_u|, |J_v| \geq \alpha N$, and $|\langle u, \varphi_i \rangle| |\langle v, \varphi_i \rangle| \geq \frac{c^2}{M}$ for all $j \in J_u \cap J_v$. Then, since $|J_u \cap J_v| \geq (2\alpha - 1)N > 0$, we have

$$\begin{aligned}
&\sum_{i=1}^N \left|\left\langle \tfrac{x - e^{i(\theta_y - \theta_x)}y}{||x - e^{i(\theta_y - \theta_x)}y||_2}, \varphi_i \right\rangle\right| \left|\left\langle \tfrac{x + e^{i(\theta_y - \theta_x)}y}{||x + e^{i(\theta_y - \theta_x)}y||_2}, \varphi_i \right\rangle\right| \\
&\geq \sum_{i \in J_u \cap J_v} |\langle u, \varphi_i \rangle||\langle v, \varphi_i \rangle| \geq \frac{c^2(2\alpha - 1)N}{M}.
\end{aligned}$$

That is, for all pairs $x, y \in \mathbb{C}^M$,

$$\begin{aligned}
&||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \\
&\geq c^2(2\alpha - 1)\frac{N}{M}||x - e^{i(\theta_y - \theta_x)}y||_2 ||x + e^{i(\theta_y - \theta_x)}y||_2.
\end{aligned}$$

□

A. Stability for random frames with bounded fourth moment

For random frames with independent frame vectors, such that entries of the frame vectors safisfy a bounded fourth moment assumption, we have the following uniform bounds on the frame oder statistics [21].

Theorem III.3. *Consider a frame $\Phi = \{\varphi_j\}_{j=1}^N \subset \mathbb{C}^M$ with M large enough, such that $\varphi_j(m)$ are independent identically distributed centered random variables, normalized so that $\mathrm{Var}(\varphi_j(m)) = \frac{1}{M}$. Assume further that $\mathbb{E}(|\varphi_i(m)|^4) \leq \frac{B}{M^2}$, for some constant $B \geq 1$, and $N \geq C_0 M \log M$, for some constant C_0. Then*

(a) *For each fixed $\alpha < 1 - \frac{1}{2C_0}$,*

$$\min_{x \in \mathbb{S}^{M-1}} \mathcal{S}_{FOS}(\Phi, \alpha N) \geq \frac{c}{\sqrt{M}}$$

with probability at least $1 - e^{-c_1 M \log M}$, where constants $c, c_1 > 0$ depend only on B, α, and C_0.

(b) *For each fixed $\beta < 1 - \frac{1}{2C_0}$,*

$$\max_{x \in \mathbb{S}^{M-1}} \mathcal{L}_{uFOS}(\Phi, \beta N) \leq \frac{K}{\sqrt{M}}$$

with probability at least $1 - e^{-c_1 M \log M}$, where constants $K, c_1 > 0$ depend only on B, β, and C_0.

Using the uniform bounds on the frame order statistics obtained in Theorem III.3 and Theorem III.2, we obtain the following result that shows stability of the phaseless measurement map for a random frame with independent frame vectors under bounded fourth moment assumption.

Corollary III.4. *Let the frame $\Phi \subset \mathbb{C}^M$ satisfy assumptions of Theorem III.3. Then there exists a numerical constant $L > 0$, such that, with overwhelming probability, the measurement map $\mathcal{A}_\Phi$ is stable with constant $C \geq L\log(M)$ in $\mathbb{C}^M$. That is, for any $x, y \in \mathbb{C}^M$,*

$$||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||_1 \geq L\log(M) \min_{\theta\in[0,2\pi)} ||x - e^{i\theta}y||_2||x + e^{i\theta}y||_2.$$

At the cost of slight increase of the measurement frame cardinality, Theorem III.3 allows to show stability of $\mathcal{A}_\Phi$ in $\mathbb{C}^M$ for a larger class of random frames Φ than considered before, and without any additional restrictions on the set T of the measured signals.

B. Stability without independence of frame vectors

While all the previous results on the stability of the phaseless measurement map are obtained for frames with independent frame vectors, this assumption is often not realistic for particular signal processing applications where the phase retrieval problem arises. The following result gives bounds of the frame order statistics for a wide class of frames with (possibly dependent) identically distributed frame vectors. This class of frames includes, in particular, Gabor frames with a random window. This result can be also found in [20], [21].

Theorem III.5. *Fix $x \in \mathbb{S}^{M-1}$ and consider a frame Φ with frame vectors uniformly distributed on $\mathbb{S}^{M-1}$. Then*

(a) *For any $c \in (0,1)$ and $k > 0$, with probability at least $1 - \frac{1}{k^2}$, we have*

$$\mathcal{S}_{FOS}\left(\Phi, |\Phi|(1 - c^2 + kc), x\right) \geq \frac{c}{\sqrt{M}}.$$

(b) *For any $C > 1$ and $k > 0$, with probability at least $1 - \frac{1}{k^2}$, we have*

$$\mathcal{L}_{FOS}\left(\Phi, |\Phi|\left(1 - \frac{8}{\pi}e^{-\frac{C^2}{2}} + k\frac{2\sqrt{2}}{\sqrt{\pi}}e^{-\frac{C^2}{4}}\right), x\right) \leq \frac{C}{\sqrt{M}}.$$

Note that, unlike Theorem III.3, Theorem III.5 is a non-uniform result in the sense that the proven bounds hold with high probability for each individual signal x. It implies the following non-uniform stability result.

Corollary III.6. *Let Φ be a frame with frame vectors uniformly distributed on $\mathbb{S}^{M-1}$. Then, for any $k > \sqrt{2}$, there exists a constant $C = \frac{1}{27k^2}\left(1 - \frac{2}{3k^2}\right)$, such that for each pair $x, y \in \mathbb{C}^M$ the following holds with probability at least $1 - \frac{2}{k^2}$*

$$||\mathcal{A}_\Lambda(x) - \mathcal{A}_\Lambda(y)||_1 \geq C||x - y||_2||x + y||_2.$$

Obtaining a uniform version of Theorem III.5 would imply stability, and thus also injectivity, of $\mathcal{A}_\Phi$ for a wide range of frames whose frame vectors are not independent, including Gabor frames. This would be a big step forward in the study of phase retrieval problem.

IV. References

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